Study Notes On Using Lie Symmetry For Solving Differential Equations

Nasser M. Abbasi

December 3, 2025 Compiled on December 3, 2025 at 1:56am

Contents

1 Lie symmetry method for solving first order ODE

Local contents

1.1 Introduction

The magic of Lie symmetries comes from the idea that if non trivial symmetry for an ode can be found, the symmetry makes solving the ode much simpler. For first order differential equation, the symmetry (or rather the tangent vectors associated with the symmetry) are used to convert the ode to canonical coordinates where the original ode becomes a quadrature ode solved by just integration, and this is regardless of how complicated the original ode was.

The ode in canonical coordinates is solved by integration and the solution is converted back to the original natural coordinates giving the solution to the original ode.

For higher order ode's, the symmetry is used to reduce the order of the ode by one. Making it easier to solve.

Let us start by looking at first order ode's first. Next section will consider second order ode's once I learn that subject more.

Given an ode in the space (x, y) which is called the natural coordinates to distinguish these from the canonical coordinates (X, Y) used later in which the ode becomes a quadrature. Assume the ode to solve is

$$\frac{dy}{dx} = \omega(x, y) \tag{1}$$

We look for one parameter group of symmetries that when applied to the above ode leaves it in same form. i.e. invariant. This is called Lie continuous group of symmetries which depends on one parameter. It is continuous group and not discrete symmetry group (as say with the case of symmetries for a triangle) because symmetries depend on parameter ε which is a real number and whose value is continuous. This Lie parameter is typically called ε but some books call it λ. These transformations are given by

$$\begin{aligned}\bar{x}(x,y) &= f(x,y;\varepsilon) \\ \bar{y}(x,y) &= g(x,y;\varepsilon)\end{aligned} \tag{2}$$

Where in the above f, g are independent functions of each others and continuous and analytic in ε. The ; before ε is used instead of comma, to indicate this is a parameter. Eq (2) is the symmetry transformation that when applied to ode (1) results in an ode of same form as (1) but using the new coordinates $(\bar{x}, \bar{y})$.

We are interested in non trivial transformation (2). This means a transformation that maps points on one solution curve of (1) to a different solution curve. If the transformation maps point (x, y) to $(\bar{x}, \bar{y})$ on the same solution curve, then it is called a trivial transformation.

The identity transformation in (2) is when $\varepsilon = 0$. Therefore

$$\begin{aligned}\bar{x} &= f(x,y;0) = x \\ \bar{y} &= g(x,y;0) = y\end{aligned} \tag{3}$$

Every Lie group must have an identity element and this is given by $\varepsilon = 0$. (The Lie group must also have a unique inverse element for each element in the set of transformations, and the set be closed under application of the transformation. But not all Lie groups are commutative or associative (i.e. Abelian). Consult more mathematical book on properties of Lie group if interested. These notes will concentrate on the algorithmic aspect.

Looking back at (2), since f, g are analytic in ε, they can be expanded in Taylor series near $\varepsilon = 0$ which results in

$$\begin{aligned}\bar{x} &= f(x,y;0) + \left.\frac{\partial f}{\partial \varepsilon}\right|_{\varepsilon=0} \varepsilon + O(\varepsilon^2) \\ \bar{y} &= g(x,y;0) + \left.\frac{\partial g}{\partial \varepsilon}\right|_{\varepsilon=0} \varepsilon + O(\varepsilon^2)\end{aligned} \tag{4}$$

Choosing ε to be very small, higher order terms $O(\varepsilon^2)$ can be ignored. Also since $f(x,y;0) = x$ and $g(x,y;0) = y$ because $\varepsilon = 0$ is the identity transformation, (4) simplifies to

$$\begin{aligned}\bar{x} &= x + \left.\frac{\partial f}{\partial \varepsilon}\right|_{\varepsilon=0} \varepsilon \\ \bar{y} &= y + \left.\frac{\partial g}{\partial \varepsilon}\right|_{\varepsilon=0} \varepsilon\end{aligned} \tag{5}$$

Typically, the term $\left.\frac{\partial f}{\partial \varepsilon}\right|_{\varepsilon=0}$ is called $\xi(x,y)$ and $\left.\frac{\partial g}{\partial \varepsilon}\right|_{\varepsilon=0}$ is called $\eta(x,y)$. Therefore (5) is written as

$$\begin{aligned}\bar{x} &= x + \varepsilon\xi(x,y) \\ \bar{y} &= y + \varepsilon\eta(x,y)\end{aligned} \tag{6}$$

The functions $\xi(x,y)\,,\eta(x,y)$ are the most important quantities in Lie group symmetry. These are called the Lie infinitesimals. Geometrically, these are the tangent vectors at the identity on the tangent plane which is the Lie algebra. If these tangent vectors can be found for the given ode, then the ode can be now transformed to quadrature in the canonical coordinates, regardless of how complicated the original ode was.

The transformation (6) must leave the ode invariant. This is called the Lie invariance condition. Therefore

$$\frac{d\bar{y}}{d\bar{x}} = \omega(\bar{x},\bar{y}) \tag{7}$$

Whenever

$$\frac{dy}{dx} = \omega(x,y) \tag{1}$$

Hence, starting with (7) it can be written as

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\frac{d\bar{y}}{dx}}{\frac{d\bar{x}}{dx}} = \omega(\bar{x},\bar{y}) \tag{8}$$

Where $\frac{d\bar{y}}{dx}$ is the total derivative with respect to x and $\frac{d\bar{x}}{dx}$is the total derivative with respect to x. But

$$\begin{aligned}
\frac{d\bar{y}}{dx} &= \frac{d}{dx}(y+\varepsilon\eta(x,y)) \\
&= \frac{dy}{dx} + \varepsilon\frac{d\eta(x,y)}{dx} \\
&= \frac{dy}{dx} + \varepsilon\left(\eta_x + \eta_y\frac{dy}{dx}\right)
\end{aligned}$$

But $\frac{dy}{dx} = \omega(x,y)$, therefore the above becomes

$$\frac{d\bar{y}}{dx} = \omega + \varepsilon(\eta_x + \omega\eta_y) \tag{9}$$

Similarly

$$\begin{aligned}
\frac{d\bar{x}}{dx} &= \frac{d}{dx}(x+\varepsilon\xi(x,y)) \\
&= 1 + \frac{d}{dx}(\varepsilon\xi(x,y)) \\
&= 1 + \varepsilon\left(\xi_x + \xi_y\frac{dy}{dx}\right) \\
&= 1 + \varepsilon(\xi_x + \omega\xi_y) \qquad (10)
\end{aligned}$$

Substituting (9,10) into (8) gives

$$\begin{aligned}
\frac{d\bar{y}}{d\bar{x}} &= \frac{\frac{d\bar{y}}{dx}}{\frac{d\bar{x}}{dx}} \\
&= \omega(\bar{x},\bar{y})
\end{aligned}$$

Hence the above becomes

$$\frac{\omega+\varepsilon(\eta_x+\omega\eta_y)}{1+\varepsilon\left(\xi_x+\omega\xi_y\right)} = \omega(\bar{x},\bar{y}) \tag{11}$$
$$= \omega(x+\varepsilon\xi, y+\varepsilon\eta)$$

Looking at left side of (11), the term $\frac{1}{1+\varepsilon(\xi_x+\omega\xi_y)}$ can be moved to numerator using binomial expansion. Since $\frac{1}{1+x}=(1+x)^{-1}$, then using

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots$$

Therefore, using $n=-1$ the term $(1+\varepsilon(\xi_x+\omega\xi_y))^{-1}$ becomes (using $x\equiv\varepsilon(\xi_x+\omega\xi_y)$ here)

$$(1+\varepsilon(\xi_x+\omega\xi_y))^{-1} = 1-\varepsilon(\xi_x+\omega\xi_y)+(\varepsilon(\xi_x+\omega\xi_y))^2-\cdots$$

But since ε is very small, all terms with $O(\varepsilon^2)$ and higher can be ignored. The above becomes

$$(1+\varepsilon(\xi_x+\omega\xi_y))^{-1} = 1-\varepsilon(\xi_x+\omega\xi_y)$$

Using the above into (11) gives

$$(\omega+\varepsilon(\eta_x+\omega\eta_y))\left(1-\varepsilon(\xi_x+\omega\xi_y)\right) = \omega(x+\varepsilon\xi, y+\varepsilon\eta)$$
$$\omega-\varepsilon\omega(\xi_x+\omega\xi_y)+\varepsilon(\eta_x+\omega\eta_y)-\varepsilon^2((\eta_x+\omega\eta_y)\left(\xi_x+\omega\xi_y\right)) = \omega(x+\varepsilon\xi, y+\varepsilon\eta)$$

But since ε is very small, the above simplifies to

$$\omega-\varepsilon\omega(\xi_x+\omega\xi_y)+\varepsilon(\eta_x+\omega\eta_y) = \omega(x+\varepsilon\xi, y+\varepsilon\eta) \tag{12}$$

Now we expand the right side of the above using Taylor series. Since $\omega(x,y)$ is function of two variables, we use Taylor series expansion for two variables. The expansion is made around the point (x,y) as shown in this diagram

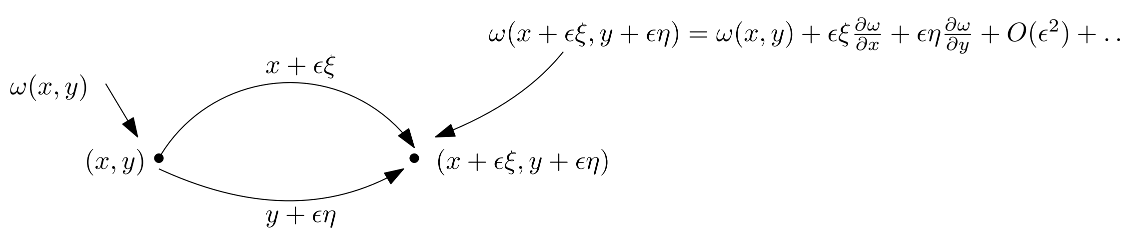

Figure 1: Expanding $\omega(x,y)$ in Taylor series

Hence approximation of the right side of (12) is

$$\omega(x+\varepsilon\xi, y+\varepsilon\eta) = \omega(x,y)+\varepsilon\xi\omega_x+\varepsilon\eta\omega_y+O\left(\varepsilon^2\right)+\cdots$$
$$= \omega(x,y)+\varepsilon(\omega_x\xi+\omega_y\eta)$$

Substituting the above in (12) results in

$$
\begin{aligned}
\omega-\varepsilon\omega(\xi_x+\omega\xi_y)+\varepsilon(\eta_x+\omega\eta_y) &= \omega+\varepsilon(\omega_x\xi+\omega_y\eta)\\
-\varepsilon\omega(\xi_x+\omega\xi_y)+\varepsilon(\eta_x+\omega\eta_y) &= \varepsilon(\omega_x\xi+\omega_y\eta)\\
-\omega(\xi_x+\omega\xi_y)+(\eta_x+\omega\eta_y) &= \omega_x\xi+\omega_y\eta\\
-\omega\xi_x-\omega^2\xi_y+\eta_x+\omega\eta_y-\omega_x\xi-\omega_y\eta=0
\end{aligned}
$$

Or

$$
\boxed{\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0} \tag{13}
$$

The above equation (13) is what is used to solve for ξ,η. It is the linearized symmetry condition. There is an additional constraint not mentioned above which is that we must have

$$
\bar{x}_x\bar{y}_y\neq\bar{x}_y\bar{y}_x
$$

PDE (13) is solved using ansatz. Examples below show how this is done.

OK, now let us assume that we have found the tangent vectors ξ,η by solving (13). What to do next? Here comes the main point of using Lie symmetry. To make any first ode $\frac{dy}{dx}=\omega(x,y)$ solvable by quadrature, we impose transformation of the form

$$
\begin{aligned}
X &= x\\
Y &= y+\varepsilon
\end{aligned} \tag{14}
$$

This transformation maps points on one solution curve to points on another solution curve, but by only changing the y coordinate. Hence it is vertical displacement or shift. The X,Y are called the canonical coordinates and these are functions of x,y. In other words, the above can be written as

$$
X(x,y)=x \tag{14(a)}
$$

$$
Y(x,y)=y+\varepsilon \tag{14(b)}
$$

From (14 a) we obtain

$$
\begin{aligned}
\left.\frac{\partial X}{\partial\varepsilon}\right|_{\varepsilon=0} &= 0\\
&= X_x\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0}+X_y\left.\frac{dy}{d\varepsilon}\right|_{\varepsilon=0}
\end{aligned} \tag{15}
$$

But what are $\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0}$ and $\left.\frac{dy}{d\varepsilon}\right|_{\varepsilon=0}$? These come from the earlier relation we had, which is

$$
\bar{x}=x+\varepsilon\xi
$$

Therefore

$$
\frac{d\bar{x}}{d\varepsilon}=\xi
$$

But at $\varepsilon = 0$, we have $\bar{x} = x$ since the identity. Therefore

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} = \xi$$

Similarly

$$\left.\frac{dy}{d\varepsilon}\right|_{\varepsilon=0} = \eta$$

Now (14 a) becomes

$$0 = X_x\xi + X_y\eta \tag{16}$$

Similarly (14 b) gives

$$\begin{aligned}\left.\frac{\partial Y}{\partial\varepsilon}\right|_{\varepsilon=0} &= 1\\ &= Y_x\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} + Y_y\left.\frac{dy}{d\varepsilon}\right|_{\varepsilon=0}\\ &= Y_x\xi + Y_y\eta\end{aligned}$$

Hence

$$1 = Y_x\xi + Y_y\eta \tag{17}$$

Equations (16,17) are now solved for $Y(x, y)$ and $X(x, y)$. Once these are found, we will find the ode in canonical coordinates to be of the form

$$\frac{dY}{dX} = G(X)$$

always. In other words, one that can be solved by just integration. Equations (16,17) are again the following

$$0 = X_x\xi + X_y\eta \tag{16}$$
$$1 = Y_x\xi + Y_y\eta \tag{17}$$

Each is solved by the standard method of characteristics in PDE. To apply this method, let us start with (16). We assume the characteristics variable is τ and say that $X(\tau) \equiv X(x(\tau), y(\tau))$. Hence

$$\frac{dX}{d\tau} = X_x\frac{dx}{d\tau} + X_y\frac{dy}{d\tau} \tag{16 a}$$

Comparing (16, 16a) shows that

$$\begin{aligned}\frac{dX}{d\tau} &= 0 \\ \frac{dx}{d\tau} &= \xi \\ \frac{dy}{d\tau} &= \eta\end{aligned} \tag{18}$$

And similarly, we let $Y(\tau) \equiv Y(x(\tau), y(\tau))$, hence

$$\frac{dY}{d\tau} = Y_x \frac{dx}{d\tau} + Y_y \frac{dy}{d\tau} \tag{17 a}$$

Comparing (17, 17a) shows that

$$\begin{aligned} \frac{dY}{d\tau} &= 1 \\ \frac{dx}{d\tau} &= \xi \\ \frac{dy}{d\tau} &= \eta \end{aligned} \tag{19}$$

Now we solve (18,19) for $Y(x,y), X(x,y)$ since ξ, η are known. Examples below show how this is done. At the end of this process, we will have Y, X and now we evaluate $\frac{dY}{dX}$ using

$$\begin{aligned} \frac{dY}{dX} &= \frac{\frac{dY}{dx}}{\frac{dX}{dx}} \\ &= \frac{Y_x + Y_y \frac{dy}{dx}}{X_x + X_y \frac{dy}{dx}} \\ &= \frac{Y_x + Y_y \omega}{X_x + X_y \omega} \end{aligned}$$

Everything on the RHS above is known, since we have solved for Y, X from (18,19). This complete the process. All what is left is to solve this resulting ode, which should always be

$$\frac{dY}{dX} = G(X)$$

This will give solution $Y(X)$. Now this is converted back to $y(x)$ using the known $X(x,y), Y(x,y)$ relations. This complete the algorithm using Lie symmetry for first order ode. Here is a summary of the steps.

1. First order ode $\frac{dy}{dx} = \omega(x,y)$ is given.
2. Solve the Lie linearized symmetry pde $\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0$ for the tangent vectors η, ξ.
3. Now that tangent vectors η, ξ are found, these are used to find canonical coordinates $X(x,y), Y(x,y)$. without knowing tangent vectors the canonical coordinates can not be found.
4. Impose transformation to canonical coordinates using $X = x, Y = y + \varepsilon$. Solve the resulting pair of pde which result due to this. These are $0 = X_x \xi + X_y \eta$ and $1 = Y_x \xi + Y_y \eta$.
5. Using method of method of characteristics, the above pair of pde's are solved for $X(x,y), Y(x,y)$.

6. Set up the ode $\frac{dY}{dX} = \frac{Y_x + Y_y \omega}{X_x + X_y \omega}$ and simplify the RHS. It should come out as function of X only.

7. Solve the ode $\frac{dY}{dX} = G(X)$ by integration to find $Y(X)$.

8. Replace all Y, X in the solution above by y, x to convert the solution back from canonical coordinates (X, Y) to natural coordinates (x, y).

1.2 Terminology used and high level introduction

1. Sophus Lie was inspired by Galois theory for solving algebraic equations.

2. The goal of Lie symmetry for solving first order ode is to transform the ode $\frac{dy}{dx} = \omega(x, y)$ to canonical coordinates $\frac{dY}{dX} = F(X)$ where it is easily solved by quadrature. Once the solution $Y(X)$ is found, it is converted back to $y(x)$ in the original x, y coordinates, thus obtaining the solution $y(x)$ to the original ode. This method works regardless of how complicated the original ode happened to be (linear or not). But this requires finding what is called the Lie infinitesimals $\xi(x, y), \eta(x, y)$ which requires solving a PDE using ansatz and this can be difficult. Different algorithms are designed to help find $\xi(x, y), \eta(x, y)$ using different forms of ansatz.

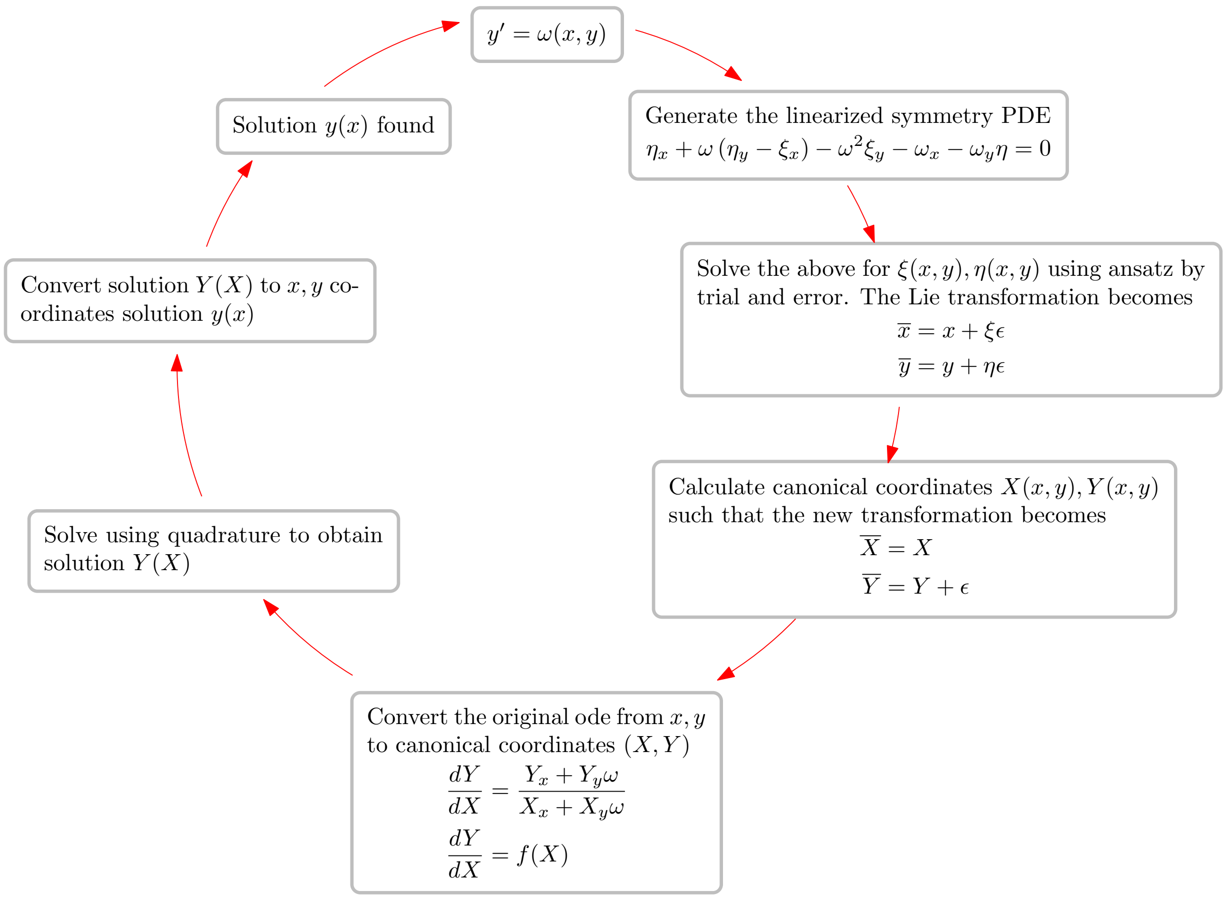

Figure 2: Lie symmetry for solving first order ode

3. x, y are the natural coordinates used in the input ode $\frac{dy}{dx} = \omega(x,y)$. For example $y' = x^2 + y^2$. Here $\omega = x^2 + y^2$.

4. $\bar{x}, \bar{y}$ are called the Lie group (local) transformation coordinates. The Lie transformation is one parameter transformation. Meaning it depends only on one parameter. Some books call this λ and some call it ε. Here ε is used. Hence we write $(\bar{x}, \bar{y}) = T(x, y; \varepsilon)$ to mean transformation T is applied on point (x, y) to obtain new point $(\bar{x}, \bar{y})$ and this transformation depends on the value used for ε. The parameter ε is real value. The ode $\frac{d\bar{y}}{d\bar{x}} = \omega(\bar{x}, \bar{y})$ must remain invariant (i.e. same shape) but with new letters $\bar{x}, \bar{y}$ that replace each of the letters x, y in the original ode. If the new ode does not have same exact form, then this is not valid Lie symmetry transformation that was used.

5. The group transformation is defined as $T_\varepsilon : \{\bar{x} = \varphi(x, y, \varepsilon), \bar{y} = \psi(x, y, \varepsilon)\}$. It is required that $\varphi(x, y, \varepsilon), \psi(x, y, \varepsilon)$ are independent of each others, which means the Jacobian do not vanish. Hence $\begin{vmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{vmatrix} \neq 0$. Lie group comes from the above transformation group when expanding $\varphi(x, y, \varepsilon), \psi(x, y, \varepsilon)$ in Taylor series near (x, y) and keep linear terms, which results in $\bar{x} \approx x + \varepsilon\xi(x, y), \bar{y} \approx y + \varepsilon\eta(x, y)$.

6. If given Transformation group T_ε defined as $\bar{x}=\varphi(x,y,\varepsilon)\,,\bar{y}=\psi(x,y,\varepsilon)$, then ξ,η are found as follows.

$$\xi = \left.\frac{d\bar{x}}{d\varepsilon}\right|_{\varepsilon=0}$$
$$\eta = \left.\frac{d\bar{y}}{d\varepsilon}\right|_{\varepsilon=0}$$

For example. Given transformation $\bar{x}=x\cos(\varepsilon)+y\sin(\varepsilon)\,,\bar{y}=y\cos(\varepsilon)-x\sin(\varepsilon)$, then the above gives $\xi=y,\eta=-x$ and the Lie transformation becomes

$$\bar{x}\approx x+\varepsilon y$$
$$\bar{y}\approx y-\varepsilon x$$

7. The quantities (ξ,η) define the tangent direction at (x,y) of the path that is taken to move (x,y) to $(\bar{x},\bar{y})$. In other words, starting from any point (x,y) and calculating (ξ,η) at (x,y), then the line going from (x,y) to the point $(x+\varepsilon\xi,y+\varepsilon\eta)$ for a very small ε value, will be tangent line to the path from (x,y) to $(\bar{x},\bar{y})$.

8. It is good to look at $\bar{x}=x+\varepsilon\xi(x,y)$ as in kinematics, where $x=x(0)+v_x t$, where now ε represents the time and $x(0)$ is initial position and x is final position and $\xi(x,y)$ is the speed v_x which is a function of position. Same for the y coordinate. $y=y(0)+v_y t$. In this view, as t increases the point moves more and all points covered in the path are the orbit of point (x,y).

9. The coordinates (X,Y) (some books use lower case r,s) are called the canonical coordinates in which the input ode becomes a quadrature and therefore easily solved by just integration. In other words, in canonical coordinates, Lie transformation is given by

$$\bar{X}=X$$
$$\bar{Y}=Y+\varepsilon$$

Comparing with the original coordinates which is

$$\bar{x}\approx x+\varepsilon\xi$$
$$\bar{y}\approx y+\varepsilon\eta$$

The ode is always solved in canonical coordinates (X,Y) and not in (x,y) since it is much simpler to solve it in those coordinates.

10. $\xi(x,y)\,,\eta(x,y)$ are called the Lie infinitesimals, also called tangent vectors. They are functions of (x,y). These are the core quantities of Lie symmetry method. These can be calculated knowing $\bar{x},\bar{y}$. Also $\bar{x},\bar{y}$ can be calculated given ξ,η. In practice, $\bar{x},\bar{y}$ are not given, and hence these have to be found using solving a PDE. It is ξ,η which are the most important quantities that need to be determined in order to find the canonical coordinates X,Y.

11. The tangent vectors ξ, η are calculated at $\epsilon = 0$. They are defined as $\xi = \left.\frac{dx}{d\epsilon}\right|_{\epsilon=0}, \eta = \left.\frac{dy}{d\epsilon}\right|_{\epsilon=0}$. The point $(\bar{x}, \bar{y})$ (orbit of (x, y)) is given by $\bar{x} = x + \xi\epsilon$ and $\bar{y} = y + \eta\epsilon$.

12. Given

$$\begin{aligned} \bar{x} &\equiv \bar{x}(x, y; \epsilon) \\ \bar{y} &\equiv \bar{y}(x, y; \epsilon) \end{aligned}$$

Expanding using Taylor series near $\epsilon = 0$ gives (see sections below for more details)

$$\begin{aligned} \bar{x} &= x + \left.\frac{\partial \bar{x}}{\partial \epsilon}\right|_{\epsilon=0} \epsilon + O(\epsilon^2) \\ &= x + \epsilon\xi(x, y) \\ \bar{y} &= y + \left.\frac{\partial \bar{y}}{\partial \epsilon}\right|_{\epsilon=0} \epsilon + O(\epsilon^2) \\ &= y + \epsilon\eta(x, y) \end{aligned}$$

The above shows the importance of $\xi(x, y), \eta(x, y)$. These (along with the specific value of ϵ) determine the orbit of each point (x, y).

13. The *orbit* of a point A given by natural coordinates (x, y) is the set of all possible points $(\bar{x}, \bar{y})$ that the point A transforms to for all possible value of ε.

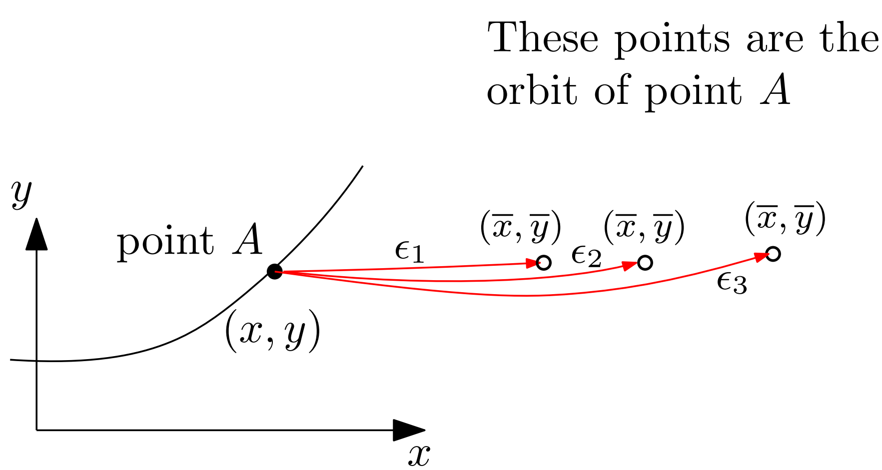

Figure 3: The orbit of point A using Lie transformation

14. The ultimate goal is write $\frac{dy}{dx} = \omega(x, y)$ in X, Y coordinates where symmetry have the ideal form $(\bar{X}, \bar{Y}) = (X, Y + \varepsilon)$ because this leads to ode of form $\frac{dY}{dX} = f(X)$. The right hand side should always be a function of X only in canonical coordinates.

15. The ideal transformation has the form $(\bar{X}, \bar{Y}) \to (X, Y + \varepsilon)$ as mentioned above, because with this transformation the ode becomes quadrature in the transformed

coordinates. But because not all ode's have this transformation as given, the ode is first transformed to canonical coordinates (X, Y) where the transformation is $(\bar{X}, \bar{Y}) \to (X, Y + \varepsilon)$ is imposed. If the transformation $(\bar{x}, \bar{y}) \to (x, y + \epsilon)$ is already present in original coordinates, then there will be no need for canonical coordinates (X, Y).

16. The main goal of Lie symmetry method is to determine X, Y and solve the ode $\frac{dY}{dX} = f(X)$ in that space instead in the natural coordinates (x, y). To be able to do this, the quantities ξ, η must be determined first.

17. The remarkable thing about Lie symmetry method, is that regardless of how complicated the original ode $\frac{dy}{dx} = \omega(x, y)$ is, if the similarity condition PDE can be solved for ξ, η, then X, Y can always be found and the ode becomes quadrature $\frac{dY}{dX} = f(X)$. The ode is then solved in canonical coordinates and the solution transformed back to x, y.

18. The quantity ϵ is called the Lie parameter. This is a real quantity which as it goes to zero, gives the identity transformation. In other words, when $\epsilon = 0$ then $(x, y) = (\bar{x}, \bar{y})$.

19. But there is no free lunch, even in Mathematics. The problem comes down to finding ξ, η. This requires solving a PDE. This is done using ansatz and trial and error. This reason possibly explains why the Lie symmetry method have not become standard in textbooks for solving ODE's as the algebra and computation needed to find ξ, η from the PDE become very complex to do by hand.

20. Total derivative operator: Given $f(x, y)$ then $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$ where it is assumed that $y(x)$ depends on x. Total derivative operator will be used extensively in all the derivations below, so good to practice this. It is written as $D_x = \partial_x + \partial_y y'$ for first order ode, and as $D_x = \partial_x + \partial_y y' + \partial_{y'} y''$ for second order ode and as $D_x = \partial_x + \partial_y y' + \partial_{y'} y'' + \partial_{y''} y'''$ for third order ode and so on.

21. The notation f_x means partial derivative. Hence $\frac{\partial f}{\partial x}$ is written as f_x. Total derivative will always be written as $\frac{df}{dx}$. It is important to distinguish between these two as the algebra will get messy with Lie symmetry. Sometimes we write f' to mean $\frac{df}{dx}$ but it is better to avoid f' and just write $\frac{df}{dx}$ when f is function of more than one variable.

22. Given first ode $\frac{dy}{dx} = \omega(x, y)$, where $\bar{y} \equiv \bar{y}(x, y)$ and $\bar{x} \equiv \bar{x}(x, y)$ then then$\frac{d\bar{y}}{d\bar{x}}$ is given by the following (using the total derivative operator)

$$\begin{aligned}\frac{d\bar{y}}{d\bar{x}} &= \frac{D_x \bar{y}}{D_x \bar{x}} \\ &= \frac{\bar{y}_x + \bar{y}_y y'}{\bar{x}_x + \bar{x}_y y'} \\ &= \frac{\bar{y}_x + \bar{y}_y \omega}{\bar{x}_x + \bar{x}_y \omega}\end{aligned}$$

23. Given second order ode $\frac{d^2y}{dx^2} = \omega(x, y, y')$ where $\bar{y} \equiv \bar{y}(x, y, y')$ and $\bar{x} \equiv \bar{x}(x, y, y')$ then

$\frac{d^2\bar{y}}{d\bar{x}^2}$ is given by

$$
\begin{aligned}
\frac{d^2\bar{y}}{d\bar{x}^2} &= \frac{D_x \frac{d\bar{y}}{d\bar{x}}}{D_x \bar{x}} \\
&= \frac{\bar{y}'_x + \bar{y}'_y y' + \bar{y}'_{y'} y''}{\bar{x}'_x + \bar{x}'_y y'}
\end{aligned}
$$

To simplify notation we have used $\bar{y}'$ for $\frac{d\bar{y}}{d\bar{x}}$ in the above. The above simplifies to

$$
\frac{d^2\bar{y}}{d\bar{x}^2} = \frac{\bar{y}'_x + \bar{y}'_y y' + \bar{y}'_{y'} \omega}{\bar{x}'_x + \bar{x}'_y y'}
$$

Keeping in mind that $(\circ)_x$ or $(\circ)_y$ mean partial derivative.

24. Given third order ode $\frac{d^3y}{dx^3} = \omega(x, y, y', y'')$ where $\bar{y} \equiv \bar{y}(x, y, y', y'')$ and $\bar{x} \equiv \bar{x}(x, y, y', y')$ then $\frac{d^3\bar{y}}{d\bar{x}^3}$ is given by

$$
\begin{aligned}
\frac{d^3\bar{y}}{d\bar{x}^3} &= \frac{D_x \frac{d^2\bar{y}}{d\bar{x}^2}}{D_x \bar{x}} \\
&= \frac{\bar{y}''_x + \bar{y}''_y y' + \bar{y}''_{y'} y'' + \bar{y}''_{y''} y'''}{\bar{x}'_x + \bar{x}'_y y'} \\
&= \frac{\bar{y}''_x + \bar{y}''_y y' + \bar{y}''_{y'} y'' + \bar{y}''_{y''} \omega}{\bar{x}'_x + \bar{x}'_y y'}
\end{aligned}
$$

To simplify notation we used $\bar{y}''$ for $\frac{d^2\bar{y}}{d\bar{x}^2}$ above. And so on for higher order ode's.

1.3 Additional notes

Given any first order ODE

$$
\frac{dy}{dx} = \omega(x, y) \tag{A}
$$

The first goal is to find a one parameter invariant Lie group transformation that keeps the ode invariant. The Lie parameter the transformation depends on is called ϵ. This means finding transformation of (x, y) to new coordinates $(\bar{x}, \bar{y})$ that keeps the ode the same form when written using $\bar{x}, \bar{y}$.

This view looks at the transformation on the ode itself. Another view is to look at the family of the solution curves of the ode instead. Looking at solution curves transformation is geometrical in nature and can lead to more insight.

What does the transformation mean when looking at solution curves instead of the ODE itself? It is the mapping of a point (x, y) on one solution curve to another point $(\bar{x}, \bar{y})$ on another solution curve. If the mapping sends point (x, y) to another point $(\bar{x}, \bar{y})$ on the same solution curve, then it is called a trivial mapping or trivial transformation.

As an example, given the ode $y' = 0$, this has solutions $y = c_1$. For any constant c_1 there is a solution curve. There are infinite number of solution curves. All solution curves are horizontal lines. The mapping $(x, y) \to (x + \epsilon, y)$ is trivial transformation as it moves the point (x, y) to another point $(\bar{x}, \bar{y})$ on the *same* solution curve.

The transformation $(x, y) \to (x, y + \epsilon)$ however is non trivial as it moves the point (x, y) to point $(\bar{x}, \bar{y})$ on another solution curve. Here $\bar{x} = x$ and $\bar{y} = y + \epsilon$. This can also be written $(x, y) \to (x, e^{\epsilon} y)$ which is the preferred way.

The transformation $(x, y) \to (x + \epsilon, y + \epsilon)$ is non trivial for this ode. The simplest non trivial transformation that map all points on one solution curve to another solution curve is selected. In canonical coordinates the transformation used has the form $\left(\bar{X}, \bar{Y}\right) \to (X, Y + \epsilon)$.

Another example is $y' = y$. This has solution curves given by $y = ce^{x}$. This is a plot showing two such curves for different c values.

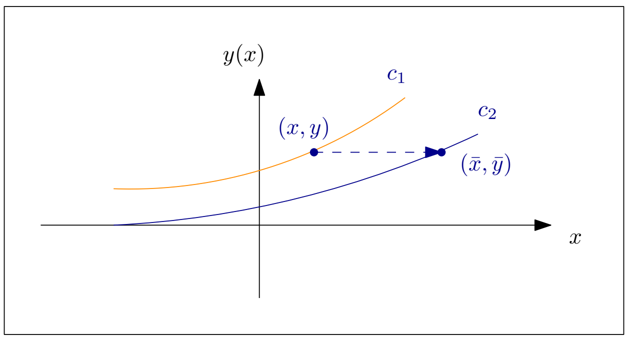

Figure 4: Point transformation example for $y' = y$

The above shows that a non trivial transformation is given by $\bar{x} = x + \epsilon, \bar{y} = y$. This can be found analytically by solving the symmetry condition as will be illustrated below using examples. For this case, the tangent vectors are $\xi = \frac{\partial \bar{x}}{\partial \epsilon}\Big|_{\epsilon=0} = 1$ and $\eta = \frac{\partial \bar{y}}{\partial \epsilon}\Big|_{\epsilon=0} = 0$. In Maple this is found using

```
ode:=diff(y(x),x)=y(x);
DEtools:-symgen(ode)
[_xi = 1, _eta = 0]
```

But the following transformation $\bar{x} = x, \bar{y} = y + \epsilon$ does not work

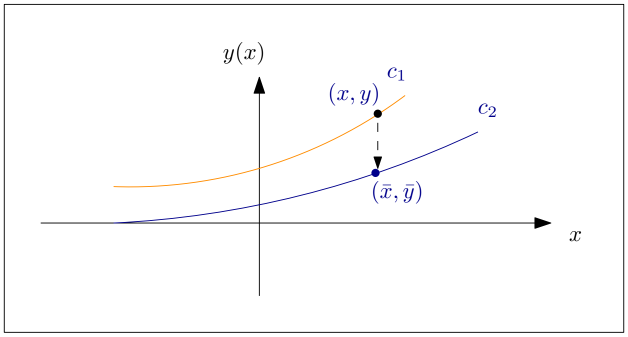

Figure 5: Possible Point transformation for $y' = y$

This is because it does not keep the original ode invariant because $\frac{d\bar{y}}{d\bar{x}} = \bar{y}$ becomes $\frac{\bar{y}_x + \bar{y}_y y'}{\bar{x}_x + \bar{x}_y y'} = \bar{y}$, where now $\bar{y}_x = 0, \bar{y}_y = 1, \bar{x}_x = 1, \bar{x}_y = 0, \bar{y} = y + \epsilon$, and hence $\frac{\bar{y}_x + \bar{y}_y y'}{\bar{x}_x + \bar{x}_y y'} = \bar{y}$ simplifies to $y' = y + \epsilon$ which is not the same ode. This shows that $\bar{x} = x, \bar{y} = y + \epsilon$ is not valid Lie point symmetry.

However $\bar{x} = x + \epsilon, \bar{y} = y$ leaves the ODE invariant. In this case $\bar{y}_x = 0, \bar{y}_y = 1, \bar{x}_x = 1, \bar{x}_y = 0, \bar{y} = y$ and hence $\frac{\bar{y}_x + \bar{y}_y y'}{\bar{x}_x + \bar{x}_y y'} = \bar{y}$ becomes $y' = y$ which is the same ode.

The transformation must keep the ode invariant as this is the main definition of symmetry transformation.

The goal is find the simplest transformation that move point (x, y) from one solution curve to another solution curve, such that the transformation also leaves the ode invariant (same form) in the new coordinates $(\bar{x}, \bar{y})$. In the above example, this was $\bar{x} = x + \epsilon, \bar{y} = y$.

In the above, the points on the path the point (x, y) travels on as it moves to $(\bar{x}, \bar{y})$ as ϵ changes are called the *orbit*. Each point (x, y) travels on its orbit during transformation.

In all such transformations, there is a parameter ϵ that the transformation depends on. This is why this is called the Lie one parameter symmetry transformation group. There are infinite number of such transformations.

Lie symmetry is hence called *point symmetry*, because of the above. It transforms points from one solution curve to points on another solution curve for the same ODE. The identity transformation is when $\epsilon = 0$, since then the point is transformed to itself.

An example using an ODE. The Clairaut ode of the form $y = xf(p)\,p + g(p)$ where $p \equiv y'$ and $f(p) = p$.

$$xp^2 - yp + m = 0 \tag{1}$$

$$y = xp + \frac{m}{p}$$

Where $f(p) = p$ and $g(p) = \frac{m}{p}$. Using the dilation transformation Lie group

$$\bar{x} \equiv \bar{x}(x, y; \epsilon) = e^{2\epsilon} x \tag{2}$$

$$\bar{y} \equiv \bar{y}(x, y; \epsilon) = e^{\epsilon} y \tag{3}$$

Eq. (1) is now expressed in the new coordinates $\bar{x}, \bar{y}$. If this results in *same ode* form but written in $\bar{x}, \bar{y}$ then the transformation is invariant. But how to find $\frac{d\bar{y}}{d\bar{x}}$? This is done as follows

$$\begin{aligned} \frac{d\bar{y}}{d\bar{x}} &= \frac{\frac{d\bar{y}}{dx}}{\frac{d\bar{x}}{dx}} \\ &= \frac{\bar{y}_x + \bar{y}_y \frac{dy}{dx}}{\bar{x}_x + \bar{x}_y \frac{dy}{dx}} \end{aligned}$$

In this example $\bar{y}_x = 0, \bar{y}_y = e^{\epsilon}, \bar{x}_x = e^{2\epsilon}, \bar{x}_y = 0$. The above now becomes

$$\begin{aligned} \frac{d\bar{y}}{d\bar{x}} &= \frac{e^{\epsilon} \frac{dy}{dx}}{e^{2\epsilon}} \\ &= e^{-\epsilon} \frac{dy}{dx} \end{aligned}$$

Writing (1) in terms of $\bar{x}, \bar{y}$ now gives

$$\bar{x} \left(\frac{d\bar{y}}{d\bar{x}} \right)^2 - \bar{y} \frac{d\bar{y}}{d\bar{x}} + m = 0 \tag{4}$$

$$\left(e^{2\epsilon} x\right) \left(e^{-\epsilon} \frac{dy}{dx} \right)^2 - \left(e^{\epsilon} y\right) e^{-\epsilon} \frac{dy}{dx} + m = 0$$

$$x \left(\frac{dy}{dx} \right)^2 - y \frac{dy}{dx} + m = 0 \tag{5}$$

Which gives the same ode. The above method starts by replacing the given ode by $\bar{x}, \bar{y}, \frac{d\bar{y}}{d\bar{x}}$ and finds if the result gives back the original ode in $x, y, \frac{dy}{dx}$. This is simpler than having to transform the original ode to $\bar{x}, \bar{y}, \frac{d\bar{y}}{d\bar{x}}$. This transformation can be verified in Maple as follows

```
ode:=x*diff(y(x),x)^2-y(x)*diff(y(x),x)+m=0;
the_tr:={x=X*exp(-2*s),y(x)=Y(X)*exp(-s)};
newode:=PDEtools:-dchange(the_tr,ode,{Y(X),X},'known'={y(x)},'uknown'={Y(X)});
diff(Y(X), X)^2*X - Y(X)*diff(Y(X), X) + m = 0
```

Comparing (4) to (5) shows that the ode form did not change, only the *letters changed* from x to $\bar{x}$ and y to $\bar{y}$. The resulting ode must never have the parameter ϵ shows in it as of course this will make it different form than the original ode which do not have ϵ in it.

The above shows how to verify that a transformation is invariant or not.

In Lie group transformation there is only one parameter ϵ and the transformation is obtained by evaluating the group as ϵ goes to zero.

But how does this help in solving the original ode? If the ode in x, y is hard to solve, then the ode written with $\bar{x}, \bar{y}$ will also be hard to solve since it is the same.

But Eq. (4) is not what is used to solve the ode. The above is just to verify the transformation is *invariant*. Similarity transformation is used to determine the tangent vectors ξ, η only. These are the most important quantities. These are then used to obtain the ode in canonical coordinates X, Y). In the canonical coordinates (X, Y) the ode becomes quadrature and solved by integration. The transformation found above is only one step toward finding (X, Y) and it is these canonical coordinates that are the goal.

1.4 Outline of the steps in solving a differential equation using Lie symmetry method

These are the steps in solving an ODE using Lie symmetry method.

1. Given an ode $y' = \omega(x, y)$ to solve in natural coordinates.
2. The tangent vector $\xi(x, y), \eta(x, y)$ are found. There are two options to find these.
   (a) If Lie group coordinates $(\bar{x}, \bar{y})$ are given, then it is easy to determine $\xi(x, y), \eta(x, y)$ using

   $$\xi(x, y) = \left.\frac{\partial \bar{x}}{\partial \epsilon}\right|_{\epsilon=0}$$
   $$\eta(x, y) = \left.\frac{\partial \bar{y}}{\partial \epsilon}\right|_{\epsilon=0}$$

   Lie group coordinates $(\bar{x}, \bar{y})$ must also satisfy

   $$\bar{x}_x \bar{y}_y - \bar{x}_y \bar{y}_x \neq 0$$

   (b) In practice Lie group coordinates $(\bar{x}, \bar{y})$ are not given and are not known. In this case $\xi(x, y), \eta(x, y)$ must be found by solving the similarity condition which results in a PDE (derivation is given below). The PDE for first order ode $y' = \omega(x, y)$ comes out to be

   $$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0$$

3. ξ, η are now used to determine the canonical coordinates (X, Y). In the canonical coordinates, only S translation is needed to make the ode quadrature. The transformation is $(\bar{X}, \bar{Y}) \to (X, Y + \varepsilon)$. This transforms the original ode $y' = \omega(x, y)$ to $\frac{dY}{dX} = f(X)$ which is then solved by only integration. This is the main advantage of moving to

canonical coordinates (X,Y). In canonical coordinates Y is like the y and X is like the x. i.e. dependent variable is $Y(X)$ like $y(x)$ in natural coordinates. The ODE in canonical coordinates is much simpler to solve than in natural coordinates.

4. The ODE is solved in (X,Y) space where $X \equiv X(x,y), Y \equiv Y(x,y)$. The transformation from (x,y) to (X,Y) is found by solving two set of PDEs using the characteristic method. After finding $X(x,y), Y(x,y)$ the ode will then be given by $\frac{dY}{dX} = \frac{Y_x + Y_y \frac{dy}{dx}}{X_x + X_y \frac{dy}{dx}}$ which will be quadrature. If this ode does not come out as $\frac{dY}{dX} = f(X)$ then something went wrong in the process. This ode is now solved for $Y(X)$. It is the symmetry of the form $(\bar{X}, \bar{Y}) \rightarrow (X, Y + \varepsilon)$ which is of the most interest in the Lie method. This is called a translation transformation along the Y axis (i.e. vertical). This is because this vertical transformation is what leads to an ode which is solved by just integration.

5. Solution is transformed from $Y(X)$ to $y(x)$.

6. An alternative to steps (3) to (5) (Which seems to be only applicable to first order odes) is to use ξ, η to determine an integrating factor $\mu(x,y)$ which is given by $\mu(x,y) = \frac{1}{\eta - \xi\omega}$ then the general solution to $y' = \omega(x,y)$ can be written directly as $\int \mu(x,y)(dy - \omega dx) = c_1$ or $\int \frac{dy - \omega dx}{\eta - \xi\omega} = c_1$ but this requires finding a function $F(x,y)$ whose *differential* is $dF = \frac{dy - \omega dx}{\eta - \xi\omega}$ and now the solution becomes $\int dF = c_1$ or $F = c_1$. If we can integrate this using $\int \mu dy - \int \mu\omega dx = c_1$ then this is the solution to the ode. It is implicit in $y(x)$. Currently my program does not implement Lie symmetry to find an integrating factor due to difficulty of finding dF that satisfies $dF = \frac{dy - \omega dx}{\eta - \xi\omega}$ or in carrying out the integration in all general cases but hope to add this soon as a backup algorithm if the main one fails. This method is similar to solving exact ode if the integrating factor can be found.

7. An important property, at least for first order ode's (I do not know now if this applies to higher order) is that given $\xi = f(x,y), \eta = g(x,y)$, then it is always possible to shift and use $\xi \equiv 0, \eta = g - \omega f$ where $y' = \omega(x,y)$. This means everything can be based on $\xi \equiv 0$ after this shift is done. This can simplify some parts of the computation. Of course if ξ was found to be zero initially, i.e. just after solving the linearized similarity PDE, then there is nothing more to do.

The *most difficult* step in all of the above is 2(b) which requires finding $\xi(x,y), \eta(x,y)$. In practice Lie group $\bar{x}, \bar{y}$ transformation is not given. Lie infinitesimal $\xi(x,y), \eta(x,y)$ have to be found directly from the linearized symmetry condition PDE using ansatz and by trial and error. The following diagram illustrates the above steps.

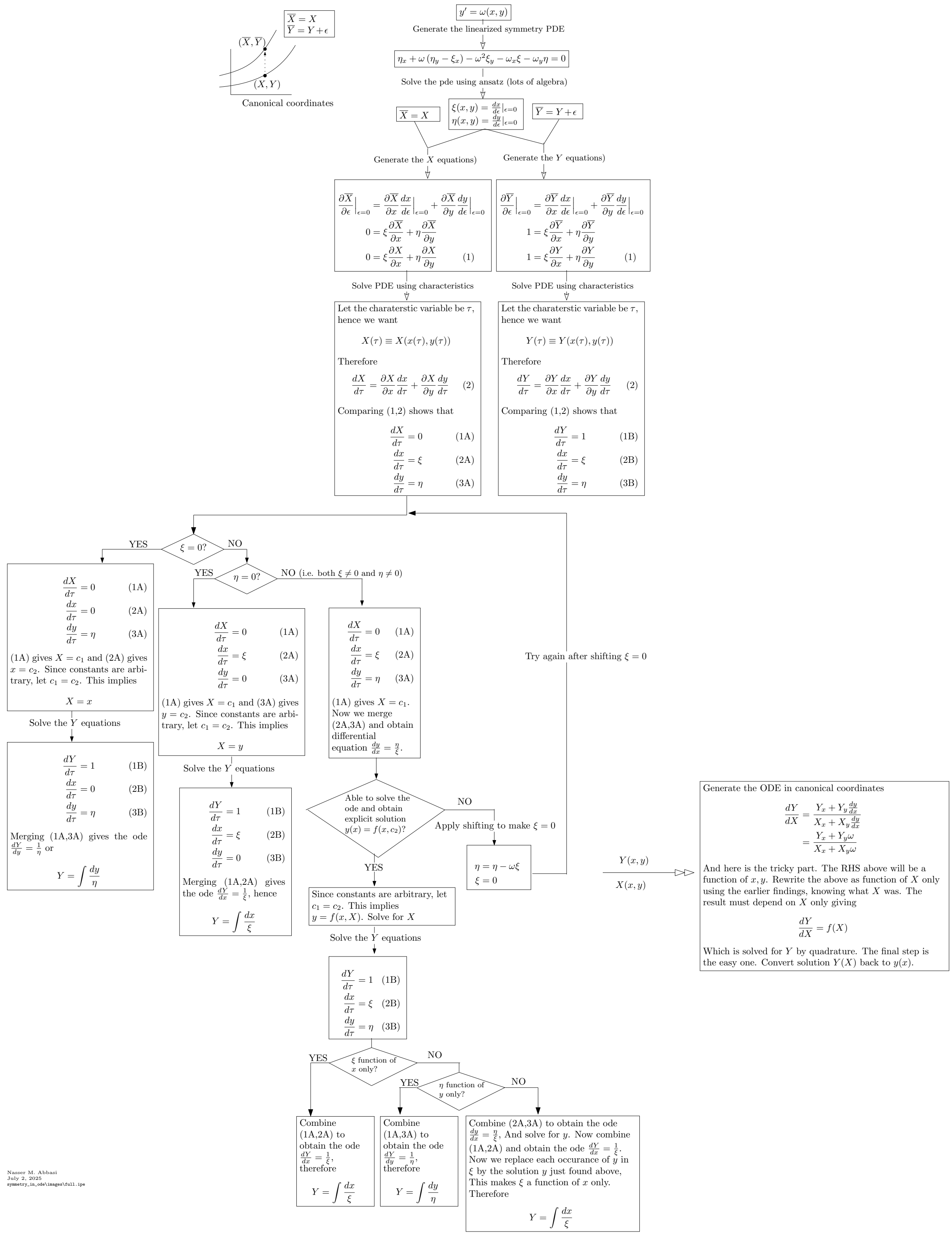

Figure 6: General steps for solving an ode using Lie symmetry method

1.5 Finding xi and eta knowing the first order ode type. Table lookup method.

There is a short cut to obtaining $\xi(x,y)\,,\eta(x,y)$ if the first order ode type is known or can be determined. (of course, if we know the ode type, then a direct method for solving the ode can be used which is much simpler, since the type is known and there is no need to use Lie symmetry), but still Lie symmetry can be useful in this case, and also it allows us to find the integrating factor quickly, which provides one more method to solve the ode. An example of a first order ode which does not have known type is

$$\left(x\cos y - e^{-\sin y}\right)y' + 1 = 0$$

The above can be solved using Lie symmetry but with functional form of ansatz $\xi = f(x)\,g(y)\,,\eta = 0$. which gives $\xi = e^{-\sin y}, \eta = 0$.

I am in the process of building table for ready to use infinitesimal based on the first ode type. The following small list is the current ones determined. For some first order ode such as linear $y' = f(x)\,y(x) + g(x)$ or separable $y' = f(x)\,g(y)$ the infinitesimals can be written directly (but again, for these simple ode's Lie method is not really needed but it provides good illustration on how to use it. Lie method is meant to be used for ode's which have no known type or difficult to solve otherwise). For an ode type not given in this list, an ansatz have to be used to solve the similarity PDE.

ode type	form	ξ	η	notes
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$	Notice that $g(x)$ does not affect the result
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$ 0	0 g	This works for any g function that depends on y only

Currently the above are the ones I am able to determine for known first order ode's. If I find more, will add them. The table lookup is much faster to use than having to solve the similarity PDE each time using ansatz in order to find ξ, η.

1.6 The linearized symmetry condition

This was derived in the introduction

$$\boxed{\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0} \tag{14}$$

The above equation (14) is what is used to determine ξ, η. It is the linearized symmetry condition. There is an additional constraint not mentioned above which is

$$\bar{x}_x \bar{y}_y \neq \bar{x}_y \bar{y}_x$$

The restricted form of (14) is

$$\chi_x + \chi_y \omega - \chi \omega_y = 0$$

An important property is the following. Given any

$$\xi = A, \eta = B$$

Then we can always write the above as

$$\xi = 0, \eta = B - \omega A$$

So that $\xi = 0$ can always be used if needed to simplify some things.

After finding ξ, η from (14), the question now becomes is how to use them to solve the original ODE?

1.7 Moving to canonical coordinates X, Y

The next step is to determine what is called the canonical coordinates (X, Y). In these canonical coordinates the ODE becomes a quadrature and solved by integration. Once solved, the solution is transformed back to (x, y). The canonical coordinates (X, Y) are found as follows. Selecting the transformation to be

$$\bar{X}(x, y) = X(x, y) \tag{15}$$

$$\bar{Y}(x, y) = Y(x, y) + \epsilon \tag{16}$$

Eq. (15) becomes

$$\left.\frac{\partial \bar{X}}{\partial \epsilon}\right|_{\epsilon=0} = \left.\left(\frac{\partial \bar{X}}{\partial x}\frac{dx}{d\epsilon}\right)\right|_{\epsilon=0} + \left.\left(\frac{\partial \bar{X}}{\partial y}\frac{dy}{d\epsilon}\right)\right|_{\epsilon=0}$$

But $\left.\frac{\partial \bar{X}}{\partial x}\right|_{\epsilon=0} = \frac{\partial X}{\partial x}$ and $\left.\frac{dx}{d\epsilon}\right|_{\epsilon=0} = \xi(x, y)$ and similarly $\left.\frac{\partial \bar{X}}{\partial y}\right|_{\epsilon=0} = \frac{\partial X}{\partial y}$ and $\left.\frac{dy}{d\epsilon}\right|_{\epsilon=0} = \eta(x, y)$. The above becomes

$$\left.\frac{\partial \bar{X}}{\partial \epsilon}\right|_{\epsilon=0} = \frac{\partial X}{\partial x}\xi + \frac{\partial X}{\partial y}\eta$$

But $\left.\frac{\partial \bar{X}}{\partial \epsilon}\right|_{\epsilon=0} = 0$ since $\bar{X} = X$. The above reduces to

$$0 = \frac{\partial X}{\partial x}\xi + \frac{\partial X}{\partial y}\eta$$

This PDE have solution using symmetry method given by

$$\frac{dX}{dt} = 0 \tag{15A}$$
$$\frac{dx}{dt} = \xi \tag{15B}$$
$$\frac{dy}{dt} = \eta \tag{15C}$$

The same procedure is applied to Eq. (16) which gives

$$\left.\frac{\partial \bar{Y}}{\partial \epsilon}\right|_{\epsilon=0} = \left.\left(\frac{\partial \bar{Y}}{\partial x}\frac{dx}{d\epsilon}\right)\right|_{\epsilon=0} + \left.\left(\frac{\partial \bar{Y}}{\partial y}\frac{dy}{d\epsilon}\right)\right|_{\epsilon=0}$$

But $\left.\frac{\partial \bar{Y}}{\partial x}\right|_{\epsilon=0} = \frac{\partial Y}{\partial x}$ and $\left.\frac{dx}{d\epsilon}\right|_{\epsilon=0} = \xi(x,y)$ and similarly $\left.\frac{\partial \bar{Y}}{\partial y}\right|_{\epsilon=0} = \frac{\partial Y}{\partial y}$ and $\left.\frac{dy}{d\epsilon}\right|_{\epsilon=0} = \eta(x,y)$. The above becomes

$$\left.\frac{\partial \bar{Y}}{\partial \epsilon}\right|_{\epsilon=0} = \frac{\partial Y}{\partial x}\xi + \frac{\partial Y}{\partial y}\eta$$

But $\left.\frac{\partial \bar{Y}}{\partial \epsilon}\right|_{\epsilon=0} = 1$ since $\bar{Y} = Y + \epsilon$. The above reduces to

$$1 = \frac{\partial Y}{\partial x}\xi + \frac{\partial Y}{\partial y}\eta$$

This PDE have solution using symmetry method given by

$$\frac{dY}{dt} = 1 \tag{16A}$$
$$\frac{dx}{dt} = \xi \tag{16B}$$
$$\frac{dy}{dt} = \eta \tag{16C}$$

Equations (15A,B,C) are used to solve for $X(x,y)$ and equations (16A,B,C) are used to solve for $Y(x,y)$. Starting with X. In the case when $\xi = 0$ the equations become

$$\frac{dX}{dt} = 0$$
$$\frac{dx}{dt} = 0$$
$$\frac{dy}{dt} = \eta$$

First equation above gives $X=c_1$. Second equation gives $x=c_2$. Letting $c_1=c_2$ then

$$X=x$$

If $\xi\neq 0$ then combining Eqs. (15B,15C) gives

$$\begin{aligned}\frac{dy}{dx}&=\frac{\eta}{\xi}\\ X&=c_1\end{aligned}$$

The ODE $\frac{dy}{dx}=\frac{\eta}{\xi}$ is solved first and the constant of integration is replaced by X. Hence X is now found. $Y(x,y)$ is found similarly using Eqs. (16A,B,C). If $\xi=0$ then

$$\begin{aligned}\frac{dY}{dt}&=1\\ \frac{dx}{dt}&=0\\ \frac{dy}{dt}&=\eta\end{aligned}$$

The first and third equations give

$$\begin{aligned}\frac{dY}{dy}&=\frac{1}{\eta}\\ Y&=\int\frac{1}{\eta}dy\end{aligned}$$

If $\xi\neq 0$ then using the second and third equation gives

$$\begin{aligned}\frac{dY}{dx}&=\frac{1}{\xi}\\ Y&=\int\frac{1}{\xi}dx\end{aligned}$$

Now that X,Y are found and the problem is solved. The ode in (X,Y) space is set up using

$$\frac{dY(X)}{dX}=\frac{Y_x+Y_y\frac{dy}{dx}}{X_x+X_y\frac{dy}{dx}} \tag{16}$$

Where $\frac{dy}{dx}=\omega(x,y)$ which is given. The solution $Y(X)$ is next converted back to $y(x)$.

Examples below illustrate how this done on a number of ODE's. Eq. (16) is solved by quadrature. This is the whole point of Lie symmetry method, is that the original ode is solved in canonical coordinates where it is much easier to solve and the solution is transformed back to natural coordinates.

The only way to understand this method well, is to workout some problems. To learn more about the theory of Lie transformation itself and why it works, there are many links in my links page on the subject.

1.8 Definitions and various notes

1. infinitesimal generator operator. $\mathbf{\Gamma} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$. Any first order ode has such generator. For instance, for the ode $y' = \omega(x,y)$ then $\mathbf{\Gamma}\omega = \xi\frac{\partial\omega}{\partial x} + \eta\frac{\partial\omega}{\partial y}$. The ode $y' = \omega(x,y) = \frac{y}{x} + x$ has solution $y = x^2 + xc_1$, therefore the solution family is $\phi(x,y) = \frac{y-x^2}{x} = c$. Using $\xi = 0, \eta = x$ then $\mathbf{\Gamma}\phi = x\frac{\partial\left(\frac{y-x^2}{x}\right)}{\partial y} = 1$. This is another example: using $\xi = x, \eta = 2y$, hence $\mathbf{\Gamma}\phi = x\frac{\partial\left(\frac{y-x^2}{x}\right)}{\partial x} + 2y\frac{\partial\left(\frac{y-x^2}{x}\right)}{\partial y} = x\left(-\frac{y}{x^2} - 1\right) + 2y\left(\frac{1}{x}\right) = -\frac{y}{x} - 1 + 2\frac{y}{x} = \frac{y}{x} - 1 \neq 1$. I must be not applying the symmetry generator correct as the result supposed to be 1. Need to visit this again. See book Bluman and Anco, page 109. Maybe some of the assumptions for using this generator are not satisfied for this ode.

2. $\omega(x,y)$ is invariant iff $\mathbf{\Gamma}\omega = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} = 0$.

3. The linearized PDE from the symmetry condition is $\omega\xi_x + \omega^2\xi_y + \omega_x\xi = \omega_y\eta + \eta_x + \omega\eta_y$. This is used to determine tangent vector $(\xi(x,y), \eta(x,y))$ which is one of the core parts of the algorithm to solve the ode using symmetry methods. There are infinite number of solutions and only one is needed.

4. Symmetries and first integrals are the two most important structures of differential equations. First integral is quantity that depends on x, y and when integrated over any solution curve is constant.

5. Lie symmetry allows one to reduce the order of an ode by one. So if we have third order ode and we know the symmetry for it, we can change the ode to second order ode. Then if apply the symmetry for this second order ode, its order is reduced to one now.

6. If ξ, η are known then the canonical coordinates R, S can now be found as functions of x, y. We just ξ, η to find R, S. Once R, S are known then $\frac{dS}{dR} = f(R)$ can be formulated. This ode is solved for S by quadrature. Final solution is found by replacing R, S back by x, y. I have functions and a solver now written and complete to do all of this but just for first order ode's only. I need to start on second order ode's after that. The main and most difficult step is in finding ξ, η. Currently I only use multivariable polynomial ansatz up to second order for ξ and multivariable polynomial ansatz up to third order for η and then try all possible combinations. This is not very efficient. But works for now. I need to add better and more efficient methods to finding ξ, η but need to do more research on this.

7. When using polynomial ansatz to find ξ, η do not mix x, y in both ansatz. For example if we use $\xi = p(x)$ then can use $\eta = q(x)$ or $\eta = q(x,y)$ polynomial ansatz to find η. But do not try $\xi = p(x,y)$ ansatz with $\eta = q(x,y)$ ansatz. In other words, if one ansatz polynomial is multivariable, then the other should be single variable. Otherwise results

will be complicated and this defeats the whole ides of using Lie symmetry as the ode generated will be as complicated or more than the original ode we are trying to solve. I found this the hard way. I was generating all permutations of ξ, η ansatz's but with both as multivariable polynomials. This did not work well.

8. Symmetries on the ode itself, is same as talking about symmetries on solution curves. i.e. given an ode $y' = \omega(x,y)$ with solution $y = f(x)$, then when we look for symmetry on the ode which leaves the ode looking the same but using the new variables $\bar{x}, \bar{y}$. This is the same as when we look for symmetry which maps any point (x,y) on solution curve $y = f(x)$ to another solution curve. In other words, the symmetry will map all solution curves of $y' = \omega(x,y)$ to the same solution curves. i.e. a specific solution curve $y = f(x, c_1)$ will be mapped to $y = f(x, c_2)$. All solution curves of $y' = \omega(x,y)$ will be mapped to the same of solution curves. But each curve maps to another curve within the same set. If the same curve maps to itself, then this is called invariant curve.

9. An orbit is the name given to the path the transformation moves the point (x,y) from one solution curve to another point on another solution curve due to the symmetry transformation.

10. A solution curve of $y' = \omega(x,y)$ that maps to itself under the symmetry transformation is called an invariant curve.

11. Not every first order ode has symmetry. At least according to Maple. For example $y' + y^3 + xy^2 = 0$ which is Abel ode type, it found no symmetries using way=all. May be with special hint it can find symmetry?

12. After trying polynomials ansatz, I find it is limited. Since it will only find symmetries that has polynomials form. A more powerful ansatz is the functional form. But these are much harder to work with but they are more general at same time and can find symmetries that can't be found with just polynomials. So I have to learn how to use functional ansatz's. Currently I only use Polynomials.

13. ξ, η are called Lie infinitesimal and $\bar{x}, \bar{y}$ are called the Lie group.

14. If we given the ξ, η then we can find Lie group $(\bar{x}, \bar{y})$. See example below.

15. If we are given Lie group $(\bar{x}, \bar{y})$ then we can find the infinitesimal using $\xi(x,y) = \frac{\partial}{\partial \epsilon}\bar{x}\Big|_{\epsilon=0}$ and $\eta(x,y) = \frac{\partial}{\partial \epsilon}\bar{y}\Big|_{\epsilon=0}$.

16. First order ode have infinite number of symmetries. Talking about symmetry of an ode is the same as talking about symmetry between solution curves of the ode itself. i.e. symmetry then becomes finding mapping that maps each solution curve to another one in the same family of solutions of the ode.

17. ξ, η can also be used to find the integrating factor for the first order ode. This is given by $\mu(x,y) = \frac{1}{\eta - \xi\omega}$ where the ode is $y'(x) = \omega(x,y)$. This gives an alternative approach

to solve the ode. I still need to add examples using $\mu(x, y)$.

18. For first order ode, to find Lie infinitesimal, we have to solve first order PDE in 2 variables. For second order ode, to find Lie infinitesimal, we have to solve second order PDE in 3 variables. For third order ode, to find Lie infinitesimal, we have to solve third order PDE in 4 variables and so on. Hence in general, for n^{th} order ode, we have to solve n^{th} order PDE in $n+1$ variables to find the required Lie infinitesimal. For first order, these variables are ξ, η and the PDE is $\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0$. Currently my program only handles first order odes. Once I am more familiar with Lie method for second order ode, will update these notes. See at the end a section on just second order ode that I started working on.

1.9 Closer look at orbits and tangent vectors

This section takes a closer look at orbits and tangent vectors ξ, η which are the core of Lie symmetry method. By definition

$$
\begin{aligned}
\xi(x, y) &= \left.\frac{d\bar{x}}{d\epsilon}\right|_{\epsilon=0} \\
\eta(x, y) &= \left.\frac{d\bar{y}}{d\epsilon}\right|_{\epsilon=0}
\end{aligned} \tag{1}
$$

Hence $\xi(x, y)$ shows how $\bar{x}$ changes as function of (x, y). And $\eta(x, y)$ shows how $\bar{y}$ changes as function of (x, y). This is because

$$
\begin{aligned}
\bar{x} &= x + \xi\epsilon \\
\bar{y} &= y + \eta\epsilon
\end{aligned} \tag{2}
$$

Comparing (2) to equation of motion where $\bar{x}$ represents final position and x is initial position, then ξ is the speed and ϵ is the time. When time is zero, initial and final position is the same. As time increases final position changes depending on the speed as time (here represented as ϵ) increases. So it helps to think of ξ, η as the rate at which $\bar{x}, \bar{y}$ change location depending on the value ϵ. ξ, η are calculated when ϵ is very small in the limit as it reaches zero.

As ϵ increases the point (x, y) moves closer to the final destination point $(\bar{x}, \bar{y})$. So these quantities ξ, η specify the orbit shape. The orbit is the path taken by point transformation from (x, y) to $(\bar{x}, \bar{y})$ and depends on ϵ such that the ode remain invariant in $\bar{x}, \bar{y}$ and points on solution curves are mapped to points on other solution curves.

Different ξ, η give different orbits between two solution curves. The following example shows this. Given the ode

$$
y' = \frac{x - y}{x + y}
$$

This is Abel type ode. Also Homogeneous class A.

It has two solutions. One solution is given by Mathematica as $y = -x - \sqrt{c_1 + 2x^2}$. A small program was now written that plots the orbit for 4 solutions ξ, η found for the similarity conditions. The similarity solution were found by Maple's symgen command.

```
> ode:=diff(y(x),x)=(x-y(x))/(x+y(x));
  DEtools:-odeadvisor(ode);
  DEtools:-symgen(ode,way=all)
```

$$ode := \frac{d}{dx} y(x) = \frac{x - y(x)}{x + y(x)}$$

[[_homogeneous, class A], _rational, [_Abel, 2nd type, class A]]

$$\left[\_\xi = \frac{1}{x}, \_\eta = -\frac{1}{x}\right], \left[\_\xi = 0, \_\eta = \frac{1}{x+y}\right], \left[\_\xi = 0, \_\eta = -\frac{x^2 - 2xy - y^2}{x+y}\right], \left[\_\xi = \frac{1}{x-y}, \_\eta = 0\right], \left[\_\xi = -\frac{x^2 - 2xy - y^2}{x-y}, \_\eta = 0\right], [\_\xi = x, \_\eta = y], [\_\xi = 2x + y, \_\eta = x], \left[\_\xi = \frac{x(x-2y)}{x-y}, \_\eta = \frac{y^2}{x+y}\right]$$

Figure 7: Command used to find ξ, η

The program starts from the same (x, y) point from one solution curve and determines $(\bar{x}, \bar{y})$ location on anther solution curve using each pair of ξ, η found. The same solution curves are used in order to compare the orbits. The following plot was generated showing the result

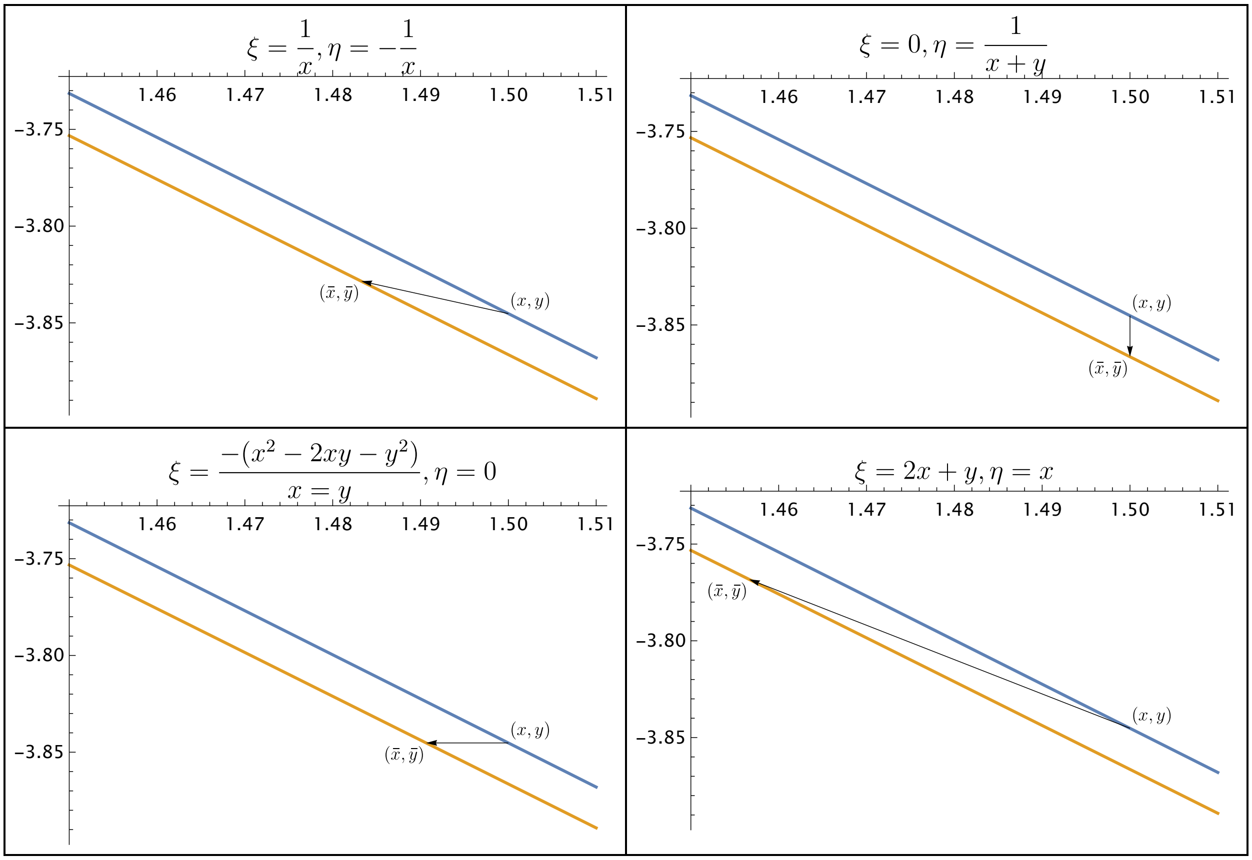

Figure 8: Different orbits using different ξ, η

The source code used to generate the above plot is

```
<<MaTeX`
ode=y'[x]==(x-y[x])/(x+y[x]);
ysol=DSolve[ode,y[x],x]
ysol=-x-Sqrt[C[1]+2 x^2];

x1 = 1.5;
y1 = ysol /. {C[1] -> 1, x -> x1};

ysol2=ysol/.C[1]->1.1

getSolutions[inf_List, titles_List, x_Symbol, ysol1_, ysol2_, x1_,
  y1_, from_, to_] :=
 Module[{xbar, ybar, eps, eq, soleps, p, data, n, xi, eta, texStyle},
  data = Table[0, {n, Length@inf}];
  texStyle = {FontFamily -> "Latin Modern Roman", FontSize -> 12};

  Do[
   xi = First[inf[[n]]];
```

```
    eta = Last[inf[[n]]];
    xbar = x1 + eps*xi ;
    ybar = y1 + eps*eta;
    eq = ybar == ysol2 /. x -> xbar;
    soleps = SolveValues[eq, eps];
    soleps = First@SortBy[soleps, Abs];
    ybar = ybar /. eps -> soleps;
    xbar = xbar /. eps -> soleps;
    p = Plot[{ysol1, ysol2}, {x, from, to},
      PlotLabel -> MaTeX[titles[[n]], Magnification -> 1.5],
      BaseStyle -> texStyle,
      Epilog -> {{Arrowheads[.02], Arrow[{{x1, y1}, {xbar, ybar}}]},
        Text[MaTeX["\\left( x,y \\right)"], {x1, y1}, {-1, -1}],
        Text[
         MaTeX["\\left( \\bar{x},\\bar{y}\\right)"], {xbar, ybar}, {1,
          1}]},
      ImageSize -> 400];
    data[[n]] = p
    ,
    {n, 1, Length@inf}
    ];

  data

  ];

inf = {{1/x1, -1/x1},
   {0, 1/(x1 + y1)},
   {-(x1^2 - 2*x1*y1 - y1^2)/(x1 - y1), 0},
   {2*x1 + y1, x1}
   };
titles = {"\\xi=\\frac{1}{x},\\eta=-\\frac{1}{x}",
   "\\xi=0,\\eta=\\frac{1}{x+y}",
   "\\xi=\\frac{-(x^2-2 x y-y^2)}{x=y},\\eta=0", "\\xi=2 x+y,\\eta=x"};
data = getSolutions[inf, titles, x, ysol /. C[1] -> 1, ysol2, x1, y1,
   1.45, 1.51];
p = Grid[Partition[data, 2], Frame -> All, Spacings -> {1, 1}]
```

1.10 Selection of ansatz to try

The following are selection of ansatz to try for solving the linearized PDE above generated from the symmetry condition in order to solve for $\xi(x,y)\,,\eta(x,y)$. These use the functional form. As a general rule, the simpler that ansatz that works, the better it is.

Functional form of ansatz is better than explicit polynomials but much harder to use and implement. Maple's symgen has 16 different algorithms that can be specified using HINT option to support functional forms. The following are possible cases to use.

1. $\xi = 0, \eta = f(x)$
2. $\xi = 0, \eta = f(y)$
3. $\xi = f(x)\,, \eta = 0$
4. $\xi = f(y)\,, \eta = 0$
5. $\xi = f(x)\,, \eta = xg(y)$. An example: applied to $y' = \frac{x+\cos(e^y+(1+x)e^{-x})}{e^{y+x}}$ should give $\xi = e^x, \eta = xe^{-y}$ which leads to solution $y = \ln\left(2\arctan\left(\frac{e^{-\left(c_1+e^{-x}\right)}-1}{e^{-\left(c_1+e^{-x}\right)}+1}\right) - (1+x)\,e^{-x}\right)$.
6. $\xi = f(x)\,, \eta = g(y)$
7. $\xi = 0, \eta = f(x)\,g(y)$. For example, applied to $y' = \frac{x\sqrt{1+y}+\sqrt{1+y}+1+y}{1+x}$ should give $f(x) = \sqrt{1+x}, g(y) = \sqrt{1+y}$.
8. $\xi = f(x)\,g(y)\,, \eta = 0$

1.11 To shift ξ or not to shift

This section looks again on the option if we should shift ξ to zero and what effect this can have. As we mentioned above, given an ode $\frac{dy}{dx} = \omega(x,y)$ and after finding the Lie symmetry tangent vectors ξ, η and if $\xi \neq 0$ then we can do the following

$$\eta_{new} = \eta_{old} - \omega\xi_{old}$$
$$\xi_{new} = 0$$

So now we have new tangent vectors where $\xi = 0$ which simplifies the steps that follows. Here we will show two examples showing what happens if we do the shift and compare the solution when we do not do the shift.

1.11.1 Example 1 $y' = xy^2 - \frac{2y}{x} - \frac{1}{x^3}$

$$\begin{aligned} y' &= xy^2 - \frac{2y}{x} - \frac{1}{x^3} \\ &= \omega(x,y) \end{aligned}$$

This has Lie tangents

$$\begin{aligned} \xi &= x \\ \eta &= -2y \end{aligned}$$

Let solve it first without shifting. To find X we solve

$$\begin{aligned} \frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{-2y}{x} \end{aligned}$$

Hence

$$y = \frac{c_1}{x^2}$$

Therefore $X = c_1$ or

$$X = yx^2$$

To find Y we evaluate

$$\begin{aligned} Y &= \int \frac{dx}{\xi} \\ &= \int \frac{dx}{x} \\ &= \ln x \end{aligned}$$

Hence the canonical coordinates are

$$\begin{aligned} X &= yx^2 \\ Y &= \ln x \end{aligned}$$

Now we need to find the ODE in the canonical coordinates space. This is found using

$$\frac{dY}{dX} = \frac{Y_x + \omega Y_y}{X_x + \omega X_y}$$

but $Y_x = \frac{1}{x}, Y_y = 0, X_x = 2yx, X_y = x^2$. The above simplifies to

$$\begin{aligned} \frac{dY}{dX} &= \frac{1}{x^4y^2 - 1} \\ &= \frac{1}{X^2 - 1} \end{aligned}$$

We see as expected that the ODE in canonical coordinates is always quadrature. Solving this gives

$$Y(X) = -\operatorname{arctanh}(X) + c_1$$

Converting back to natural coordinates x, y gives

$$\begin{aligned}
\ln x &= -\operatorname{arctanh}(yx^2) + c_1 \\
-\ln x + c_1 &= \operatorname{arctanh}(yx^2) \\
yx^2 &= \tanh(-\ln x + c_1) \\
y &= \frac{\tanh(-\ln x + c_1)}{x^2} \qquad (1)
\end{aligned}$$

Now we will solve the same ode using shifting. Hence, since $\xi = x, \eta = -2y$ then

$$\begin{aligned}
\eta &= \eta - \omega\xi \\
&= -2y - \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)x \\
&= -\frac{1}{x^2}(x^4y^2 - 1)
\end{aligned}$$

And

$$\xi = 0$$

Since $\xi = 0$ then

$$\begin{aligned}
X &= x \\
Y &= \int \frac{dy}{\eta} \\
&= -x^2 \int \frac{dy}{(x^4y^2 - 1)} \\
&= -x^2 \left(\frac{\ln(x^2y - 1) - \ln(x^2y + 1)}{2x^2}\right) \\
&= -\frac{1}{2}\left(\ln(x^2y - 1) - \ln(x^2y + 1)\right)
\end{aligned}$$

Now we need to find the ODE in the canonical coordinates space. This is found using

$$\frac{dY}{dX} = \frac{Y_x + \omega Y_y}{X_x + \omega X_y}$$

Where now

$$\begin{aligned}
Y_x &= -\frac{2xy}{x^4y^2 - 1} \\
Y_y &= -\frac{x^2}{x^4y^2 - 1} \\
X_x &= 1 \\
X_y &= 0
\end{aligned}$$

Hence the ode in canonical coordinates becomes

$$\begin{aligned}\frac{dS}{dR} &= \frac{-\frac{2xy}{x^4y^2-1} + \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)\left(-\frac{x^2}{x^4y^2-1}\right)}{1} \\ &= -\frac{1}{x} \\ &= \frac{-1}{R}\end{aligned}$$

Solving gives

$$S = -\ln R + c_1$$

Moving back to natural coordinates gives

$$\begin{aligned}-\frac{1}{2}\left(\ln\left(x^2y-1\right) - \ln\left(x^2y+1\right)\right) &= -\ln x + c_1 \\ \ln\left(x^2y-1\right) - \ln\left(x^2y+1\right) &= 2\ln x + c_2 \\ \ln\left(\frac{x^2y-1}{x^2y+1}\right) &= \ln x^2 + c_2 \\ y &= -\frac{1+c_3x^2}{x^2\left(c_3x^2-1\right)} \\ &= \frac{c_4+x^2}{x^2\left(c_4-x^2\right)} \qquad (2)\end{aligned}$$

In summary we have solution for $y' = xy^2 - \frac{2y}{x} - \frac{1}{x^3}$ as

solution using no shift	$y = \frac{\tanh(-\ln x + c_1)}{x^2}$
solution using shift	$y = \frac{c_4+x^2}{x^2(c_4-x^2)}$

Maple 2025 gives the first solution as default. But when asked to use Lie method, it does give the second solution, which means it used $\xi = 0$.

Both solutions are verified correct. The solution using shifting is simpler since it involves no functions at all.

1.11.2 Example 2 $y' = \sqrt{x-y}$

$$\begin{aligned}y' &= \sqrt{x-y} \\ &= \omega(x,y)\end{aligned}$$

This has Lie tangents

$$\begin{aligned}\xi &= 1 \\ \eta &= 1\end{aligned}$$

Let solve it first without shifting. To find R we solve

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= 1\end{aligned}$$

Hence

$$y = x + c_1$$

Therefore $X = c_1$ or

$$X = y - x$$

To find Y we evaluate

$$\begin{aligned}Y &= \int \frac{dx}{\xi}\\ &= \int dx\\ &= x\end{aligned}$$

Hence the canonical coordinates are

$$\begin{aligned}X &= y - x\\ Y &= x\end{aligned}$$

Now we need to find the ODE in the canonical coordinates space. This is found using

$$\frac{dY}{dX} = \frac{Y_x + \omega Y_y}{X_x + \omega X_y}$$

but $Y_x = 1, Y_y = 0, X_x = -1, X_y = 1$. The above simplifies to

$$\begin{aligned}\frac{dY}{dX} &= \frac{1}{-1 + \sqrt{x - y}}\\ &= \frac{1}{-1 + \sqrt{-(y - x)}}\\ &= \frac{1}{-1 + \sqrt{-X}}\end{aligned}$$

We see as expected that the ODE in canonical coordinates is always quadrature. Solving this gives

$$Y = -\ln(-X - 1) - 2\sqrt{-X} - \ln\left(-1 + \sqrt{-X}\right) + \ln\left(1 + \sqrt{-X}\right) + c_1$$

Converting back to natural coordinates x, y gives

$$\begin{aligned}x &= -\ln(-y + x - 1) - 2\sqrt{-(y - x)} - \ln\left(-1 + \sqrt{-(y - x)}\right) + \ln\left(1 + \sqrt{-(y - x)}\right) + c_1\\ x &= -\ln(-y + x - 1) - 2\sqrt{x - y} - \ln\left(-1 + \sqrt{x - y}\right) + \ln\left(1 + \sqrt{x - y}\right) + c_1 \qquad (1)\end{aligned}$$

Now we will solve the same ode using shifting. Hence, since $\xi = 1, \eta = 1$ then

$$\begin{aligned}\eta &= \eta - \omega\xi \\ &= 1 - \sqrt{x-y}\end{aligned}$$

And

$$\xi = 0$$

Since $\xi = 0$ then

$$\begin{aligned}X &= x \\ Y &= \int \frac{dy}{\eta} \\ &= \int \frac{dy}{1-\sqrt{x-y}} \\ &= 2\sqrt{x-y} + 2\ln\left(-1+\sqrt{x-y}\right)\end{aligned}$$

Now we need to find the ODE in the canonical coordinates space. This is found using

$$\frac{dY}{dX} = \frac{Y_x + \omega Y_y}{X_x + \omega X_y}$$

Where now

$$\begin{aligned}Y_x &= \frac{1}{\sqrt{x-y}} + \frac{1}{\left(-1+\sqrt{x-y}\right)\sqrt{x-y}} \\ Y_y &= -\frac{1}{\sqrt{x-y}} - \frac{1}{\left(-1+\sqrt{x-y}\right)\sqrt{x-y}} \\ X_x &= 1 \\ X_y &= 0\end{aligned}$$

Hence the ode in canonical coordinates becomes

$$\begin{aligned}\frac{dY}{dX} &= \frac{Y_x + \omega Y_y}{X_x + \omega X_y} \\ &= \left(\frac{1}{\sqrt{x-y}} + \frac{1}{\left(-1+\sqrt{x-y}\right)\sqrt{x-y}}\right) + \sqrt{x-y}\left(-\frac{1}{\sqrt{x-y}} - \frac{1}{\left(-1+\sqrt{x-y}\right)\sqrt{x-y}}\right) \\ &= -1\end{aligned}$$

Solving gives

$$Y = -X + c_1$$

Moving back to natural coordinates gives

$$2\sqrt{x-y} + 2\ln\left(-1+\sqrt{x-y}\right) = -x + c_1$$

In summary we have solution for $y' = \sqrt{x-y}$ as

solution using no shift	$x=-\ln(-y+x-1)-2\sqrt{x-y}-\ln(-1+\sqrt{x-y})+\ln(1+\sqrt{x-y})+c_1$
solution using shift	$2\sqrt{x-y}+2\ln(-1+\sqrt{x-y})=-x+c_1$

Both are valid solutions. We notice that when using shifting the algebra can become more complicated since $Y=\int\frac{dy}{\eta}$ where η in this case becomes more complicated. But both methods will produce valid solution.

1.12 Examples

1.12.1 Example 1 on how to find Lie group (x,y) given Lie infinitesimal xi and eta

Given $\xi=1,\eta=2x$ find Lie group $\bar{x},\bar{y}$. Since

$$\xi(x,y)=\left.\frac{\partial\bar{x}}{\partial\epsilon}\right|_{\epsilon=0}$$

Then

$$\begin{aligned}\frac{d\bar{x}}{d\epsilon}&=\xi(\bar{x},\bar{y})\\&=1\end{aligned}\tag{1}$$

Similarly, since

$$\eta(x,y)=\left.\frac{\partial\bar{y}}{\partial\epsilon}\right|_{\epsilon=0}$$

Then

$$\begin{aligned}\frac{d\bar{y}}{d\epsilon}&=\eta(\bar{x},\bar{y})\\&=2\bar{y}\end{aligned}\tag{2}$$

Where in both odes (1,2) we have the condition that at $\epsilon=0$ then $\bar{x}=x,\bar{y}=y$. Starting with (1), solving it gives

$$\bar{x}=\epsilon+c_1(x,y)$$

Where $c_1(x,y)$ is arbitrary function which acts like constant of integration since $\bar{x}(x,y)$ is function of two variables. At $\epsilon=0$ then $c_1(x,y)=x$. Hence the above is

$$\bar{x}=\epsilon+x\tag{3}$$

And from (2), solving give

$$\bar{y}=2\bar{x}\epsilon+c_2(x,y)$$

But at $\epsilon=0$, $\bar{y}=y,\bar{x}=x$ then the above gives $c_2=y$. Hence the above becomes

$$\bar{y}=2\bar{x}\epsilon+y$$

But $\bar{x} = \epsilon + x$ from (3), hence the above becomes

$$\begin{aligned}\bar{y} &= 2(\epsilon + x)\,\epsilon + y \\ &= 2\epsilon^2 + 2\epsilon x + y\end{aligned}$$

Therefore Lie group is

$$\begin{aligned}\bar{x} &= \epsilon + x \\ \bar{y} &= 2\epsilon^2 + 2\epsilon x + y\end{aligned}$$

1.12.2 Example how to find Lie group (x, y) given canonical coordinates X, Y

Given $X = x, Y = \frac{y}{x}$ find Lie group $\bar{x}, \bar{y}$. Solving for x, y from X, Y gives

$$\begin{aligned}x &= X \\ y &= YX\end{aligned}$$

Hence

$$\begin{aligned}\bar{x} &= \bar{X} \\ \bar{y} &= \bar{Y}\bar{X}\end{aligned}$$

But $\bar{Y} = Y + \epsilon$ by definition of canonical coordinates and $\bar{X} = X$ by definition of canonical coordinates. Hence the above becomes

$$\begin{aligned}\bar{x} &= X \\ \bar{y} &= (Y + \epsilon)\,X\end{aligned}$$

Using the values given for X, Y in terms of x, y the above becomes

$$\begin{aligned}\bar{x} &= x \\ \bar{y} &= \left(\frac{y}{x} + \epsilon\right) x \\ &= y + \epsilon x\end{aligned}$$

1.12.3 Example $y' = \frac{y}{x} + x$

This is linear first order which can be easily solved using integrating factor. But this is just to illustrate Lie symmetry method.

$$\begin{aligned}y' &= \frac{y}{x} + x \qquad (1) \\ y' &= \omega(x, y)\end{aligned}$$

The first step is to find ξ and η. Using lookup method, since this is linear ode of form $y' = f(x)\,y + g(x)$ then

$$\begin{aligned}\xi &= 0\\ \eta &= e^{\int f dx} = e^{\int \frac{1}{x} dx} = x\end{aligned}$$

The end of this problem shows also how to find these from the symmetry conditions. Therefore we write

$$\begin{aligned}\bar{x} &= x + \xi\epsilon\\ &= x\\ \bar{y} &= y + \eta\epsilon\\ &= y + \eta x \end{aligned} \tag{2}$$

The integrating factor is therefore

$$\begin{aligned}\mu(x,y) &= \frac{1}{\eta - \xi\omega}\\ &= \frac{1}{x}\end{aligned}$$

Before solving this, let us first verify that transformation (2) is invariant which means it leaves the ode in same form but using $\bar{x}, \bar{y}$. We do the same as in the above introduction.

$$\begin{aligned}\frac{d\bar{y}}{d\bar{x}} &= \frac{\frac{d\bar{y}}{dx}}{\frac{d\bar{x}}{dx}}\\ &= \frac{\bar{y}_x + \bar{y}_y \frac{dy}{dx}}{\bar{x}_x + \bar{x}_y \frac{dy}{dx}}\end{aligned}$$

But $\bar{y}_x = s, \bar{y}_y = 1, \bar{x}_x = 1, \bar{x}_y = 0$ and the above becomes

$$\begin{aligned}\frac{d\bar{y}}{d\bar{x}} &= \frac{\epsilon + \frac{dy}{dx}}{1}\\ &= \epsilon + \frac{dy}{dx}\end{aligned}$$

Substituting $\bar{x}, \bar{y}, \frac{d\bar{y}}{d\bar{x}}$ in the original ode gives

$$\begin{aligned}\frac{d\bar{y}}{d\bar{x}} &= \frac{\bar{y}}{\bar{x}} + \bar{x}\\ \epsilon + \frac{dy}{dx} &= \frac{y + \epsilon x}{x} + x\\ \epsilon + \frac{dy}{dx} &= \frac{y}{x} + \epsilon + x\\ \frac{dy}{dx} &= \frac{y}{x} + x\end{aligned}$$

Which is the original ODE. Therefore (2) are indeed an invariant Lie group transformation as it leaves the ODE unchanged. The next step is to determine what is called the canonical coordinates X, Y. Where X is the independent variable and Y is the dependent variable. So we are looking for $Y(X)$ function. This is done by using the standard characteristic equation by writing

$$
\begin{aligned}
\frac{dx}{\xi} &= \frac{dy}{\eta} = dS \\
\frac{dx}{0} &= \frac{dy}{x} = dS
\end{aligned}
\tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) Y(x,y) = 1$. Which is a first order PDE. This is solved for Y, which gives (1) using the method of characteristic to solve first order PDE which is standard method. In the special case when $\xi = 0$ and $\eta \neq 0$ these give

$$
\begin{aligned}
X &= x \\
Y &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{x} dy \\
&= \frac{y}{x} + c
\end{aligned}
$$

We are free to set $c = 0$, hence $Y = \frac{y}{x}$. Therefore the transformation to canonical coordinates is

$$
(x, y) \to (X, Y) = \left(x, \frac{y}{x}\right)
$$

The derivative in (X, Y) is found same as with $\frac{d\bar{y}}{d\bar{x}}$ giving

$$
\frac{dY}{dX} = \frac{Y_x + Y_y \frac{dy}{dx}}{X_x + X_y \frac{dy}{dx}}
$$

But $Y_x = -\frac{y}{x^2}, Y_y = \frac{1}{x}, X_x = 1, X_y = 0$ and the above becomes

$$
\begin{aligned}
\frac{dY}{dX} &= \frac{-\frac{y}{x^2} + \frac{1}{x}\frac{dy}{dx}}{1} \\
&= -\frac{y}{x^2} + \frac{1}{x}\frac{dy}{dx}
\end{aligned}
$$

But $\frac{dy}{dx} = \frac{y}{x} + x$ hence the above becomes

$$
\begin{aligned}
\frac{dY}{dX} &= -\frac{y}{x^2} + \frac{1}{x}\left(\frac{y}{x} + x\right) \\
&= 1
\end{aligned}
$$

Solving this gives

$$
Y = X + c_1
$$

But $Y = \frac{y}{x}, X = x$. Therefore the above becomes

$$\begin{aligned}\frac{y}{x} &= x + c_1\\ y &= x^2 + c_1 x\end{aligned}$$

Which is the solution to the original ode. Of course this was just an example showing how to use Lie symmetry method. The original ode is linear and can be easily solved using an integrating factor

$$\begin{aligned}y' - \frac{y}{x} &= x\\ I &= e^{-\int \frac{1}{x} dx}\\ &= e^{-\ln x}\\ &= \frac{1}{x}\end{aligned}$$

Multiplying the ode by I gives

$$\begin{aligned}\frac{d}{dx}(yI) &= Ix\\ \frac{y}{x} &= \int \frac{x}{x} dx\\ &= x + c_1\end{aligned}$$

Hence

$$y = x^2 + x c_1$$

Which is same solution. But Lie symmetry method works the same way for any given ode. And this is where it powers are. It can solve much more complicated odes than this using the same procedure. The main difficulty is in finding the infinitesimals for the group, which are ξ, η that leaves the ode invariant.

Finding Lie symmetries for this example

$$\begin{aligned}y' &= \frac{y}{x} + x\\ &= \omega(x, y)\end{aligned}$$

The condition of symmetry is a the linearized PDE given above in equation (14) as

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{14}$$

We first find the determining equation before solving for ξ, η. Since $\omega = \frac{y}{x} + x$ then $\omega_y =$

$\frac{1}{x}, \omega_x = -\frac{y}{x^2} + 1$. Hence the above becomes

$$
\begin{aligned}
\eta_x + \left(\frac{y}{x} + x\right)(\eta_y - \xi_x) - \left(\frac{y}{x} + x\right)^2 \xi_y - \left(-\frac{y}{x^2} + 1\right)\xi - \frac{1}{x}\eta = 0 \\
\eta_x + \left(\frac{y}{x} + x\right)(\eta_y - \xi_x) - \left(\frac{y^2}{x^2} + x^2 + 2y\right)\xi_y - \left(-\frac{y}{x^2} + 1\right)\xi - \frac{1}{x}\eta = 0 \\
\eta_x + \left(\frac{y}{x} + x\right)\eta_y - \xi_x\left(\frac{y}{x} + x\right) - \left(\frac{y^2}{x^2} + x^2 + 2y\right)\xi_y - \left(-\frac{y}{x^2} + 1\right)\xi - \frac{1}{x}\eta = 0
\end{aligned}
$$

Multiplying by x^2 to normalize gives

$$
x^2\eta_x + (yx + x^3)\,\eta_y - \xi_x(yx + x^3) - (y^2 + x^4 + 2yx^2)\,\xi_y - (-y + x^2)\,\xi - x\eta = 0 \tag{A}
$$

Equation (A) is called the determining equation. Using different ansatz can result in more solutions.

Trying ansatz

$$
\begin{aligned}
\xi &= 0 \\
\eta &= b_0 x
\end{aligned}
$$

Plugging these into (A) and comparing coefficients to solve for the unknown gives

$$
\begin{aligned}
x^2(b_0) - x\eta &= 0 \\
b_0x^2 - x(b_0x) &= 0 \\
b_0x^2 - b_0x^2 &= 0 \\
b_0(0) &= 0
\end{aligned}
$$

So any b_0 will work. Let $b_0 = 1$. Hence

$$
\begin{aligned}
\xi &= 0 \\
\eta &= x
\end{aligned}
$$

Now Trying ansatz as

$$
\begin{aligned}
\xi &= a_0 + a_1x \\
\eta &= b_0 + b_1y
\end{aligned}
$$

Then $\xi_x = a_1, \xi_y = 0, \eta_x = 0, \eta_y = b_1$ and the determining equation (A) becomes

$$
\begin{aligned}
(b_0 + b_1y)\,x + (a_0 + a_1x)\,(x^2 - y) + b_1(-yx - x^3) + a_1(yx + x^3) &= 0 \\
(b_0 + b_1y)\,x + (a_0 + a_1x)\,(x^2 - y) + (b_1 - a_1)\,(-yx - x^3) &= 0 \\
xb_0 - ya_0 + x^2a_0 + x^3(2a_1 - b_1) &= 0
\end{aligned}
$$

Setting each coefficient to zero gives

$$
\begin{aligned}
b_0 &= 0 \\
a_0 &= 0 \\
a_0 &= 0 \\
2a_1 - b_1 &= 0
\end{aligned}
$$

Hence the solution is $a_0 = 0, b_0 = 0, a_1 = \frac{b_1}{2}$. Using $b_1 = 2$ gives $a_1 = 1$ and therefore

$$
\begin{aligned}
\xi &= x \\
\eta &= 2y
\end{aligned}
$$

And Trying ansatz as

$$
\begin{aligned}
\xi &= a_0 + a_1 x + a_2 y \\
\eta &= b_0 + b_1 y + b_2 x
\end{aligned}
$$

Hence $\xi_x = a_1, \xi_y = a_2, \eta_x = b_2, \eta_y = b_1$ and the determining equation (A) becomes

$$
(b_0 + b_1 y + b_2 x)\, x + (a_0 + a_1 x + a_2 y)\left(x^2 - y\right) + b_1\left(-yx - x^3\right) + a_2\left(y^2 + x^4 + 2yx^2\right) + b_2\left(-x^2\right) + a_1\left(yx +
$$

$$
x^4(-a_2) + x^3(-2a_1) + x^2 y(-3a_2) + x^3(b_1) + x^2(-a_0) + y(a_0) - x(
$$

Setting each coefficient to zero gives

$$
\begin{aligned}
b_0 &= 0 \\
a_0 &= 0 \\
a_1 &= 0 \\
b_1 &= 0 \\
a_2 &= 0 \\
b_2 &= 0
\end{aligned}
$$

This shows there is no solution for this ansatz. There are more solutions depending on what ansatz we used. We just need one to obtain the final solution. In Maple, these solutions can be found as follows

```
ode:=diff(y(x),x)= y(x)/x+x;
DEtools:-symgen(ode,y(x),way=all)
[_xi = 0, _eta = x],
[_xi = 0, _eta = x],
[_xi = 0, _eta = x^2 - y],
[_xi = x, _eta = 2*y],
[_xi = 1, _eta = y/x],
[_xi = x^2 + y, _eta = 4*y*x],
[_xi = x^2 - 3*y, _eta = -4*y^2/x]
```

Trying ansatz using functional form. Let $\xi = 0, \eta = f(x)$ then $\xi_x = 0, \xi_y = 0, \eta_x = f'(x), \eta_y = 0$ and the determining equation (A) becomes

$$
\begin{aligned}
x^2\eta_x + (yx + x^3)\eta_y - \xi_x(yx + x^3) - (y^2 + x^4 + 2yx^2)\xi_y - (-y + x^2)\xi - x\eta = 0 \\
x^2 f'(x) - x f(x) = 0 \\
x f'(x) - f(x) = 0
\end{aligned}
$$

This is easily solved to give $f = cx$. Hence $\xi = 0, \eta = x$ by choosing $c = 1$. We see that this choice of ansatz was the easiest in this case, as the ode generated was linear. Let us try another and see what happens.

Trying ansatz as $\xi = 0, \eta = f(y)$ then $\xi_x = 0, \xi_y = 0, \eta_x = 0, \eta_y = f'(y)$ and the determining equation (A) becomes

$$
\begin{aligned}
(yx + x^3) f'(y) - x f(y) = 0 \\
(y + x^2) f'(y) - f(y) = 0
\end{aligned}
$$

This is separable and its solution is $f = c_1(x^2 + y)$. Hence $\xi = 0, \eta = (x^2 + y)$ by using $c_1 = 1$. But this is not function of y only. So this choice did not work. Trying $[\xi = f(x), \eta = 0], [\xi = f(y), \eta = 0]$ shows these also do not work.

ξ, η can be checked for validity by substituting them in the PDE. Maple's *symtest* command does this. These functional ansatz's lead to an ode which have to be solved.

1.12.4 Example $y' = xy^2 - \frac{2y}{x} - \frac{1}{x^3}$

Solve

$$
\begin{aligned}
y' &= xy^2 - \frac{2y}{x} - \frac{1}{x^3} \\
y' &= \omega(x, y)
\end{aligned}
\tag{1}
$$

For $x \neq 0$. Given dilation transformation

$$
\begin{aligned}
\bar{x} &= e^{\epsilon} x \\
\bar{y} &= e^{-2\epsilon} y
\end{aligned}
\tag{2}
$$

Hence

$$
\begin{aligned}
\xi(x, y) &= \left.\frac{d\bar{x}}{d\epsilon}\right|_{\epsilon=0} = x \\
\eta(x, y) &= \left.\frac{d\bar{y}}{d\epsilon}\right|_{\epsilon=0} = -2y
\end{aligned}
$$

(At end shows how to obtain these). The integrating factor is therefore

$$
\begin{aligned}
\mu(x,y) &= \frac{1}{\eta - \xi\omega} \\
&= \frac{1}{-2y - x\left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)} \\
&= -\frac{x^2}{x^4y^2 - 1}
\end{aligned}
$$

Now

$$
\begin{aligned}
\bar{x} &= x + \xi\epsilon = x + \epsilon x \\
\bar{y} &= y + \eta\epsilon = y - 2y\epsilon
\end{aligned} \tag{3}
$$

This transformation $\bar{x} = e^{\epsilon}x, \bar{y} = e^{-2\epsilon}y$ is now verified that it keeps the ode invariant.

$$
\frac{d\bar{y}}{d\bar{x}} = \frac{\bar{y}_x + \bar{y}_y\frac{dy}{dx}}{\bar{x}_x + \bar{x}_y\frac{dy}{dx}} = \frac{e^{-2\epsilon}\frac{dy}{dx}}{e^{\epsilon}} = e^{-3\epsilon}\frac{dy}{dx}
$$

Substituting $\bar{x}, \bar{y}, \frac{d\bar{y}}{d\bar{x}}$ in the original ode gives

$$
\begin{aligned}
\frac{d\bar{y}}{d\bar{x}} &= \bar{x}\bar{y}^2 - \frac{2\bar{y}}{\bar{x}} - \frac{1}{\bar{x}^3} \\
e^{-3\epsilon}\frac{dy}{dx} &= (e^{\epsilon}x)\left(e^{-2\epsilon}y\right)^2 - \frac{2(e^{-2\epsilon}y)}{(e^{\epsilon}x)} - \frac{1}{(e^{\epsilon}x)^3} \\
e^{-3\epsilon}\frac{dy}{dx} &= e^{-3\epsilon}xy^2 - \frac{2e^{-3\epsilon}y}{x} - \frac{e^{-3\epsilon}}{x^3} \\
\frac{dy}{dx} &= xy^2 - \frac{2y}{x} - \frac{1}{x^3}
\end{aligned}
$$

Which is the original ode. Hence the transformation (2) is invariant. It is important to use (2) and not (3) when doing the verification.

The next step is to determine what is called the canonical coordinates X, Y. Where X is the independent variable and Y is the dependent variable. So we are looking for $Y(X)$ function. This is done by using the standard characteristic equation by writing

$$
\begin{aligned}
\frac{dx}{\xi} &= \frac{dy}{\eta} = dY \\
\frac{dx}{x} &= \frac{dy}{-2y} = dY
\end{aligned} \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)Y(x,y) = 1$. Which is a first order PDE. This is solved for S, which gives (1) using the method of characteristic to solve first order PDE which is standard method. Starting with the first pair of ODE gives

$$
\frac{dy}{dx} = -\frac{2y}{x}
$$

Integrating gives $yx^2 = c$ where c is constant of integration. In this method X is always c. Hence

$$X = yx^2$$

$Y(x, y)$ is now found from the first equation in (1) and the last equation which gives

$$\begin{aligned}
dY &= \frac{dx}{\xi} \\
Y &= \int \frac{dx}{x} \\
Y &= \ln x
\end{aligned}$$

Now that $X(x, y), Y(x, y)$ are found, the ODE $\frac{dY}{dX} = f(X)$ is setup. The ODE comes out to be function of X only, so it is quadrature. This is the main idea of this method. But How to find $\frac{dY}{dX}$? There is an equation to determine this given by

$$\begin{aligned}
\frac{dY}{dX} &= \frac{\frac{dY}{dx} + \omega(x, y)\frac{dY}{dy}}{\frac{dX}{dx} + \omega(x, y)\frac{dX}{dy}} \\
&= \frac{Y_x + \omega(x, y) Y_y}{X_x + \omega(x, y) X_y}
\end{aligned}$$

Everything on the RHS is known. But

$$\begin{aligned}
Y_x &= \frac{1}{x} \\
Y_y &= 0 \\
X_x &= 2yx \\
X_y &= x^2
\end{aligned}$$

Substituting gives

$$\begin{aligned}
\frac{dY}{dX} &= \frac{\frac{1}{x} + \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)(0)}{2xy + \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)x^2} \\
&= \frac{\frac{1}{x}}{2xy + \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)x^2} \\
&= \frac{1}{x^4y^2 - 1}
\end{aligned}$$

But $X = yx^2$, hence the above becomes

$$\frac{dY}{dX} = \frac{1}{X^2 - 1}$$

This is just quadrature. Integrating gives

$$Y = -\operatorname{arctanh}(x) + c_1$$

This solution is converted back to x, y. Since $S = \ln x, R = yx^2$, the above becomes

$$\ln|x| = -\operatorname{arctanh}\left(yx^2\right) + c_1$$

Or

$$\begin{aligned}
-\ln|x| + c_1 &= \operatorname{arctanh}\left(yx^2\right) \\
yx^2 &= \tanh\left(-\ln|x| + c_1\right) \\
y &= \frac{\tanh\left(-\ln|x| + c_1\right)}{x^2}
\end{aligned}$$

Which is the solution to the original ODE.

The above shows the basic steps in this method. Let us solve more ODE's to practice this method more.

Finding Lie symmetries for this example

The condition of symmetry is given above in equation (14) as

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{14}$$

We now need to solve the above for ξ, η given a specific $\omega(x, y)$ for the ODE at hand. This PDE can not be solved as is for ξ, η without an ansatz. One common ansatz is to use $\xi = \alpha(x)$ and $\eta = \beta(x)\, y + \gamma(x)$ and plugging these into the above and then compare coefficients to solve for $\alpha(x), \beta(x), \gamma(x)$.

Another ansatz is to use a polynomials for ξ, η. And this is what we will start with.

Using polynomial as ansatz

We start with order 1 polynomials. Hence

$$\xi = a_0 + a_1 x \tag{1}$$
$$\eta = b_0 + b_1 y \tag{2}$$

If this does not generate solution, we will try higher order polynomials. Eq (14) becomes

$$\begin{aligned}
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta &= 0 \\
0 + \omega(b_1 - a_1) - \omega^2(0) - \omega_x(a_0 + a_1 x) - \omega_y(b_0 + b_1 y) &= 0
\end{aligned}$$

But in this ODE $\omega = xy^2 - \frac{2y}{x} - \frac{1}{x^3}$, hence $\omega_x = y^2 + \frac{2y}{x^2} + \frac{3}{x^4}$ and $\omega_y = 2yx - \frac{2}{x}$. The above becomes

$$\left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)(b_1 - a_1) - \left(y^2 + \frac{2y}{x^2} + \frac{3}{x^4}\right)(a_0 + a_1 x$$

$$xy^2 b_1 - \frac{2y}{x} b_1 - \frac{1}{x^3} b_1 - xy^2 a_1 + \frac{2y}{x} a_1 + \frac{1}{x^3} a_1 - y^2 a_0 - \frac{2y}{x^2} a_0 - \frac{3}{x^4} a_0 - xy^2 a_1 - a_1 \frac{2y}{x} - a_1 \frac{3}{x^3} - 2yx$$

$$xy^2(b_1 - a_1 - a_1 - 2b_1) + \frac{y}{x}(-2b_1 + 2a_1 - 2a_1 + 2b_1) + \frac{1}{x^3}(-b_1 + a_1 - 3a_1) + y^2(-a_0) + \frac{y}{x^2}(-2a_0) + \frac{1}{x^4}(-$$

Each coefficient to each monomial must be zero. Hence

$$
\begin{aligned}
-2a_1 - b_1 &= 0 \\
-b_1 - 2a_1 &= 0 \\
-2a_1 - 2b_1 &= 0 \\
a_0 &= 0 \\
b_0 &= 0
\end{aligned}
$$

These are overdetermined equations. Solving gives $a_1 = -\frac{1}{2}b_1$ and $a_0 = b_0 = 0$. Choosing $b_1 = -2$ gives $a_1 = 1$. Hence

$$
\begin{aligned}
\xi &= a_0 + a_1 x = x \\
\eta &= b_0 + b_1 y = -2y
\end{aligned}
$$

Which is what we wanted to show for this ODE. These are the values we used earlier to solve the ODE using symmetry method.

Using functions as ansatz

Now ξ, η are found using $\xi = \alpha(x)$ and $\eta = \beta(x)\,y + \gamma(x)$ as ansatz. Eq. (14) is

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{14}
$$

But

$$
\eta_x = \beta'(x)\,y + \gamma'(x)
$$

And

$$
\eta_y = \beta(x)
$$

And

$$
\begin{aligned}
\xi_y &= 0 \\
\xi_x &= \alpha'(x)
\end{aligned}
$$

Substituting the above into EQ. (14) gives

$$
\beta'(x)\,y + \gamma'(x) + \omega(\beta(x) - \alpha'(x)) - \omega_x \alpha(x) - \omega_y(\beta(x)\,y + \gamma(x)) = 0
$$

But in this ODE $\omega = xy^2 - \frac{2y}{x} - \frac{1}{x^3}$, hence $\omega_x = y^2 + \frac{2y}{x^2} + \frac{3}{x^4}$ and $\omega_y = 2yx - \frac{2}{x}$. The above becomes

$$
\beta' y + \gamma' + \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)(\beta - \alpha') - \left(y^2 + \frac{2y}{x^2} + \frac{3}{x^4}\right)\alpha - \left(2yx - \frac{2}{x}\right)(\beta y + \gamma) = 0
$$

Or

$$
\gamma' + y\beta' + \frac{2}{x}\gamma - \frac{1}{x^3}\beta - \frac{3}{x^4}\alpha - y^2\alpha + \frac{1}{x^3}\alpha' - 2xy\gamma - \frac{2}{x^2}y\alpha - xy^2\beta + \frac{2}{x}y\alpha' - xy^2\alpha' = 0
$$

Collecting on y gives

$$y^0\left(\gamma' + \frac{2}{x}\gamma - \frac{1}{x^3}\beta - \frac{3}{x^4}\alpha + \frac{1}{x^3}\alpha'\right) + y\left(\beta' - 2xy\gamma - \frac{2}{x^2}\alpha + \frac{2}{x}\alpha'\right) + y^2(-\alpha - x\beta - x\alpha') = 0$$

Each term above is zero. This gives the following equations

$$\begin{aligned}
\gamma'(x) + \frac{2}{x}\gamma(x) - \frac{1}{x^3}\beta(x) - \frac{3}{x^4}\alpha(x) + \frac{1}{x^3}\alpha'(x) &= 0\\
\beta'(x) - 2xy\gamma(x) - \frac{2}{x^2}\alpha(x) + \frac{2}{x}\alpha'(x) &= 0\\
-\alpha(x) - x\beta(x) - x\alpha'(x) &= 0
\end{aligned}$$

Solving these coupled ODE on the computer gives

$$\begin{aligned}
\alpha(x) &= \frac{1}{x}\left(c_3x^4 + c_1x^2 + c_2\right)\\
\beta(x) &= -4c_3x^2 - 2c_1\\
\gamma(x) &= -2c_3 - 2\frac{c_2}{x^4}
\end{aligned}$$

Where the c_1, c_2, c_3 above are constant of integration. Let $c_2 = c_3 = 0$. Hence

$$\begin{aligned}
\alpha(x) &= \frac{1}{x}\left(c_3x^4 + c_1x^2\right)\\
\beta(x) &= -4c_3x^2 - 2c_1\\
\gamma(x) &= 0
\end{aligned}$$

Let $c_3 = 0$. Hence

$$\begin{aligned}
\alpha(x) &= \frac{1}{x}c_1x^2\\
\beta(x) &= -2c_1\\
\gamma(x) &= 0
\end{aligned}$$

Let $c_1 = 1$, hence

$$\begin{aligned}
\alpha(x) &= x\\
\beta(x) &= -2\\
\gamma(x) &= 0
\end{aligned}$$

Therefore, since $\xi = \alpha(x)$ and $\eta = \beta(x)\,y + \gamma(x)$ then $\xi = x, \eta = -2y$ which is the same as the earlier method. After working using this ansatz, I find using the polynomial ansatz better. First of all, I had to set constants above to values in order to obtain the same result as earlier. Setting these constants other values will give different result. For example, the following are another set of possible solutions obtained from Maple for this ODE

$$\left\{\alpha(x) = \frac{1}{x}, \beta(x) = 0, \gamma(x) = -\frac{2}{x^4}\right\}$$
$$\left\{\alpha(x) = -\frac{x}{2}, \beta(x) = 1, \gamma(x) = 0\right\}$$
$$\left\{\alpha(x) = -\frac{x^3}{4}, \beta(x) = x^2, \gamma(x) = \frac{1}{2}\right\}$$

Which gives

$$\left\{\xi = \frac{1}{x}, \eta = -\frac{2}{x^4}\right\}$$
$$\left\{\xi = -\frac{x}{2}, \eta = y\right\}$$
$$\left\{\xi = \frac{-x^3}{4}, \eta = x^2 y + \frac{1}{2}\right\}$$

1.12.5 Example $y' = \frac{y+1}{x} + \frac{y^2}{x^3}$

$$\begin{aligned} y' &= \frac{y+1}{x} + \frac{y^2}{x^3} \\ y' &= \omega(x,y) \end{aligned}$$

This can be written as

$$\begin{aligned} y' &= \frac{y}{x} + \frac{1}{x} + \frac{y^2}{x^3} \\ &= \frac{y}{x} + \frac{x^2+y^2}{x^3} \\ &= \frac{y}{x} + \frac{1}{x}\left(\frac{x^2+y^2}{x^2}\right) \\ &= \frac{y}{x} + \frac{1}{x}\left(1 + \left(\frac{y}{x}\right)^2\right) \end{aligned}$$

Hence this has the form $y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$ where $g(x) = \frac{1}{x}$ and $F = \left(1 + \left(\frac{y}{x}\right)^2\right)$. Therefore this is homogeneous class D. Lookup table gives

$$\begin{aligned} \xi &= x^2 \\ \eta &= xy \end{aligned}$$

Another way to find ξ, η is by solving the symmetry condition PDE and this is shown at the end of this problem. Hence

$$\begin{aligned} \bar{x} &= x + \xi\epsilon \\ &= x + x^2\epsilon \\ \bar{y} &= y + \eta\epsilon \\ &= y + xy\epsilon \end{aligned} \tag{2}$$

The integrating factor is therefore

$$\begin{aligned}
\mu(x,y) &= \frac{1}{\eta - \xi\omega} \\
&= \frac{1}{xy - x^2\left(\frac{y+1}{x} + \frac{y^2}{x^3}\right)} \\
&= -\frac{x}{x^2+y^2}
\end{aligned}$$

The ode is now verified that it remains invariant under (2) transformation.

$$\begin{aligned}
\frac{d\bar{y}}{d\bar{x}} &= \frac{\frac{d\bar{y}}{dx}}{\frac{d\bar{x}}{dx}} \\
&= \frac{\bar{y}_x + \bar{y}_y \frac{dy}{dx}}{\bar{x}_x + \bar{x}_y \frac{dy}{dx}}
\end{aligned}$$

But from (2) $\bar{y}_x = y\epsilon, \bar{y}_y = 1 + x\epsilon, \bar{x}_x = 1 + 2x\epsilon, \bar{x}_y = 0$ and the above becomes

$$\frac{d\bar{y}}{d\bar{x}} = \frac{1 + (1 + x\epsilon)\frac{dy}{dx}}{1 + 2x\epsilon}$$

Substituting $\bar{x}, \bar{y}, \frac{d\bar{y}}{d\bar{x}}$ in the original ode gives

$$\begin{aligned}
\frac{d\bar{y}}{d\bar{x}} &= \frac{\bar{y}+1}{\bar{x}} + \frac{\bar{y}^2}{\bar{x}^3} \\
\frac{1 + (1 + x\epsilon)\frac{dy}{dx}}{1 + 2x\epsilon} &= \frac{(y + xy\epsilon) + 1}{x + x^2\epsilon} + \frac{(y + xy\epsilon)^2}{(x + x^2\epsilon)^3}
\end{aligned}$$

Which as $\lim_{\epsilon \to 0}$ gives

$$\frac{dy}{dx} = \frac{y+1}{x} + \frac{y^2}{x^3}$$

The same original ode showing the transformation is valid symmetry.

```
Y:=y/(1-s*x):
X:=x/(1-s*x):
eq:=(diff(Y,x)+diff(Y,y)*Z)/(diff(X,x)+diff(X,y)*Z)=simplify((Y+1)/X+Y^2/X^3):
solve(simplify(eq),Z)
y/x + 1/x + y^2/x^3
```

Hence the transformation in (2) is invariant.

The next step is to determine what is called the canonical coordinates R, S. Where R is the independent variable and S is the dependent variable. So we are looking for $S(R)$ function. This is done by using the standard characteristic equation by writing

$$\begin{aligned}
\frac{dx}{\xi} &= \frac{dy}{\eta} = dS \\
\frac{dx}{x^2} &= \frac{dy}{xy} = dS \qquad (1)
\end{aligned}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)Y(x,y)=1$. Which is a first order PDE. We need to solve this for Y, which gives (1) using method of characteristic to solve first order PDE which is standard method. Starting with the first pair of ODE in (1) gives

$$\frac{dy}{dx}=\frac{xy}{x^2}=\frac{y}{x}$$

Integrating gives $\frac{y}{x}=c$ where c is constant of integration. In this method X is always c. Hence

$$X(x,y)=\frac{y}{x}$$

Now we find $Y(x,y)$ from the first equation in (1) and the last equation

$$\begin{aligned}dY&=\frac{dx}{\xi}\\Y&=\int\frac{dx}{x^2}\\Y&=\frac{-1}{x}\end{aligned}$$

Now that we found X and Y, we determine the ODE $\frac{dY}{dX}=f(X)$. The ODE comes out to be function of X only, so it is quadrature. This is the whole idea of this method. By solving for R we go back to x,y and solve for $y(x)$. How to find $\frac{dY}{dX}$? There is an equation to determine this given by

$$\frac{dY}{dX}=\frac{Y_x+\omega(x,y)\,Y_y}{X_x+\omega\,(x,y)\,X_y}$$

We know everything on the RHS. Substituting gives

$$\begin{aligned}\frac{dY}{dX}&=\frac{\frac{1}{x^2}+\left(\frac{y+1}{x}+\frac{y^2}{x^3}\right)(0)}{-\frac{y}{x^2}+\left(\frac{y+1}{x}+\frac{y^2}{x^3}\right)\frac{1}{x}}\\&=\frac{\frac{1}{x^2}}{-\frac{y}{x^2}+\left(\frac{y+1}{x}+\frac{y^2}{x^3}\right)\frac{1}{x}}\\&=\frac{x^2}{x^2+y^2}\\&=\frac{1}{1+\left(\frac{y}{x}\right)^2}\end{aligned}$$

But $X=\frac{y}{x}$, hence the above becomes

$$\frac{dY}{dX}=\frac{1}{1+X^2}$$

This is just quadrature. Integrating gives

$$Y=\arctan\left(X\right)+c_1$$

Now we go back to x,y. Since $Y=-\frac{1}{x}, X=\frac{y}{x}$, then the above becomes

$$
\begin{aligned}
-\frac{1}{x} &= \arctan\left(\frac{y}{x}\right)+c_1\\
\frac{-1}{x}+c_2 &= \arctan\left(\frac{y}{x}\right)\\
\frac{y}{x} &= \tan\left(\frac{-1}{x}+c_2\right)\\
y(x) &= x\tan\left(\frac{-1}{x}+c_2\right)
\end{aligned}
$$

And the above is the solution to original ODE.

Finding Lie symmetries for this example

The symmetry condition was derived earlier as

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{14}
$$

Let ansatz be

$$
\begin{aligned}
\xi &= c_1x+c_2y+c_3\\
\eta &= c_4x+c_5y+c_6
\end{aligned}
$$

Eq 14 becomes

$$
\begin{aligned}
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta &= 0\\
c_4+\omega(c_5-c_1)-\omega^2c_2-\omega_x(c_1x+c_2y+c_3)-\omega_y(c_4x+c_5y+c_6) &= 0
\end{aligned}
$$

But in this ODE $\omega=\frac{y+1}{x}+\frac{y^2}{x^3}$, hence $\omega_x=-\frac{y+1}{x^2}-3\frac{y^2}{x^4}$ and $\omega_y=\frac{1}{x}+\frac{2y}{x^3}$. The above becomes

$$
\begin{aligned}
c_4+\left(\frac{y+1}{x}+\frac{y^2}{x^3}\right)(c_5-c_1)-\left(\frac{y+1}{x}+\frac{y^2}{x^3}\right)^2c_2-\left(-\frac{y+1}{x^2}-3\frac{y^2}{x^4}\right)(c_1x+c_2y+c_3)-\left(\frac{1}{x}+\frac{2y}{x^3}\right)(c_4x\\
\frac{1}{x^2}c_3-\frac{1}{x^2}c_2+\frac{1}{x}c_5-\frac{1}{x}c_6+\frac{2}{x^3}y^2c_1-\frac{2}{x^4}y^2c_2+\frac{3}{x^4}y^2c_3+\frac{1}{x^4}y^3c_2-\frac{1}{x^3}y^2c_5-\frac{1}{x^6}y^4c_2-\frac{1}{x^2}yc_2+\frac{1}{x^2}yc_3-\frac{2}{x^2}\\
x^4c_3-x^4c_2+x^5c_5-x^5c_6+2x^3y^2c_1-2x^2y^2c_2+3x^2y^2c_3+x^2y^3c_2-x^3y^2c_5-y^4c_2-x^4yc_2+x^4yc_3-2x^4y\\
x^4(c_3-c_2)+x^5(c_5-c_6)+x^3y^2(2c_1-c_5)+x^2y^2(-2c_2+3c_3)+x^2y^3(c_2)+y^4(-c_2)+x^4y(-c_2+c_3-2c_4)-
\end{aligned}
$$

Each coefficient to each monomial must be zero. Hence

$$
\begin{aligned}
c_3-c_2 &= 0\\
c_5-c_6 &= 0\\
2c_1-c_5 &= 0\\
-2c_2+3c_3 &= 0\\
c_2 &= 0\\
-c_2+c_3-2c_4 &= 0\\
-2c_6 &= 0
\end{aligned}
$$

Which simplifies to (since $c_2 = 0, c_6 = 0$)

$$
\begin{aligned}
c_3 &= 0 \\
c_5 &= 0 \\
c_1 - c_5 &= 0 \\
3c_3 &= 0 \\
c_3 - 2c_4 &= 0
\end{aligned}
$$

Which simplifies to (since $c_3 = 0, c_5 = 0$)

$$
\begin{aligned}
c_5 &= 0 \\
c_1 - c_5 &= 0 \\
c_4 &= 0
\end{aligned}
$$

Hence $c_5 = 0, c_1 = 0, c_4 = 0$. We see that all $c_i = 0$, therefore there is no solution using this ansatz.

Trying ansatz

$$
\begin{aligned}
\xi &= a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 \\
\eta &= b_0 + b_1 x + b_2 y + b_3 xy + b_4 y^2
\end{aligned}
$$

Eq 9 becomes

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0
$$

Substituting the ansatz and simplifying gives

$$
-x^2y^3a_2 + y^4a_2 + x^4(-a_0 + a_2) + x^2y^2(-3a_0 + 2a_2) + xy^4a_3 + 2x^3yb_0 + x^4y(-a_0 + a_2 + 2b_1) + x^5(a_3 + b_0 - b_2) + x^3y^2(-
$$

Each coefficient to each monomial must be zero. Hence

$$
\begin{aligned}
a_2 &= 0 \\
-a_0 + a_2 &= 0 \\
-3a_0 + 2a_2 &= 0 \\
a_3 &= 0 \\
b_0 &= 0 \\
-a_0 + a_2 + 2b_1 &= 0 \\
a_3 + b_0 - b_2 &= 0 \\
-2a_1 + 2a_3 + b_2 &= 0 \\
a_4 - b_3 &= 0 \\
2a_3 - 2b_4 &= 0 \\
a_3 - b_4 &= 0
\end{aligned}
$$

Since $a_2 = a_3 = b_0 = 0$ the above simplifies to

$$\begin{aligned}
-a_0 &= 0 \\
-3a_0 &= 0 \\
-a_0 + 2b_1 &= 0 \\
-b_2 &= 0 \\
-2a_1 + b_2 &= 0 \\
a_4 - b_3 &= 0 \\
-2b_4 &= 0 \\
-b_4 &= 0
\end{aligned}$$

Since $a_0 = b_2 = a_4 = b_4 = 0$, The above now simplifies to

$$a_4 - b_3 = 0$$

Therefore, if we let $a_4 = 1$ then $b_3 = 1$ and the solution is

$$\begin{aligned}
\xi &= a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 \\
&= x^2 \\
\eta &= b_0 + b_1 x + b_2 y + b_3 xy + b_5 y^2 \\
&= xy
\end{aligned}$$

Which is what we used above to solve the ode.

1.12.6 Example $y' = \frac{y-4xy^2-16x^3}{y^3+4x^2y+x}$

Solve

$$\begin{aligned}
y' &= \frac{y - 4xy^2 - 16x^3}{y^3 + 4x^2 y + x} \\
y' &= \omega(x, y)
\end{aligned}$$

The first step is to find ξ and η. This is shown at the end of this problem below.

$$\begin{aligned}
\xi &= -y \\
\eta &= 4x
\end{aligned}$$

The integrating factor is therefore

$$\begin{aligned}
\mu(x, y) &= \frac{1}{\eta - \xi\omega} \\
&= \frac{1}{4x + y\left(\frac{y-4xy^2-16x^3}{y^3+4x^2y+x}\right)} \\
&= \frac{x^2 y + x + y^3}{4x^2 + y^2}
\end{aligned}$$

The next step is to determine what is called the canonical coordinates X, Y. Where X is the independent variable and Y is the dependent variable. This is done by using the standard characteristic equation by writing

$$
\begin{aligned}
\frac{dx}{\xi} &= \frac{dy}{\eta} = dY \\
\frac{dx}{-y} &= \frac{dy}{4x} = dY
\end{aligned} \tag{1}
$$

The first pair of ode's in (1) gives

$$
\frac{dy}{dx} = -\frac{4x}{y}
$$

Solving gives

$$
y = \sqrt{-4x^2 + c}
$$

Where c is constant of integration (For $y > 0$ only). In this method X is always c. Hence

$$
\begin{aligned}
y^2 &= -4x^2 + c \\
X &= y^2 + 4x^2
\end{aligned} \tag{2}
$$

The first equation in (1) and the last equation gives

$$
\begin{aligned}
dY &= \frac{dx}{\xi} \\
Y &= -\int \frac{dx}{y}
\end{aligned}
$$

But $y = \sqrt{-4x^2 + c}$. The above becomes

$$
\begin{aligned}
Y &= -\int \frac{dx}{\sqrt{-4x^2 + c}} \\
&= -\frac{1}{2} \arctan\left(\frac{2x}{\sqrt{-4x^2 + c}}\right) \\
&= -\frac{1}{2} \arctan\left(\frac{2x}{y}\right)
\end{aligned}
$$

For $y > 0$. Now that we found X and Y, we determine the ODE $\frac{dY}{dX} = f(X)$. The ODE comes out to be function of X only, so it is quadrature. This is the whole idea of this method. By solving for X we go back to x, y and solve for $y(x)$. How to find $\frac{dY}{dX}$? There is an equation to determine this given by

$$
\frac{dY}{dX} = \frac{Y_x + \omega(x, y)\, Y_y}{X_x + \omega\,(x, y)\, X_y}
$$

We know everything on the RHS. Substituting gives

$$\begin{aligned}
\frac{dY}{dX} &= \frac{\frac{d}{dx}\left(-\frac{1}{2}\arctan\left(\frac{2x}{y}\right)\right) + \left(\frac{y-4xy^2-16x^3}{y^3+4x^2y+x}\right)\frac{d}{dy}\left(-\frac{1}{2}\arctan\left(\frac{2x}{y}\right)\right)}{\frac{d}{dx}\sqrt{y^2+4x^2} + \left(\frac{y-4xy^2-16x^3}{y^3+4x^2y+x}\right)\frac{d}{dy}\sqrt{y^2+4x^2}} \\
&= \frac{\frac{-1}{y\left(\frac{4x^2}{y^2}+1\right)} + \left(\frac{y-4xy^2-16x^3}{y^3+4x^2y+x}\right)\frac{x}{y^2\left(\frac{4x^2}{y^2}+1\right)}}{\frac{4x}{\sqrt{y^2+4x^2}} + \left(\frac{y-4xy^2-16x^3}{y^3+4x^2y+x}\right)\frac{y}{\sqrt{y^2+4x^2}}} \\
&= -\sqrt{4x^2+y^2} \\
&= -X
\end{aligned}$$

Hence

$$\frac{dY}{dX} = -X$$

This is just quadrature. Integrating gives

$$Y = -\frac{X^2}{2} + c$$

Now we go back to x, y. Since $Y = -\frac{1}{2}\arctan\left(\frac{2x}{y}\right), X = \sqrt{y^2+4x^2}$, then the above becomes

$$\begin{aligned}
-\frac{1}{2}\arctan\left(\frac{2x}{y}\right) &= -\left(\frac{y^2+4x^2}{2}\right) + c \\
\frac{y^2}{2} - \frac{1}{2}\arctan\left(\frac{2x}{y}\right) + 2x^2 - c = 0 &\qquad y > 0
\end{aligned}$$

And the above is the solution to original ODE.

Finding Lie symmetries for this example

The symmetry condition was derived earlier as

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{14}$$

Let ansatz be

$$\begin{aligned}
\xi &= c_1x + c_2y + c_3 \\
\eta &= c_4x + c_5y + c_6
\end{aligned}$$

Eq 14 becomes

$$c_4 + \omega(c_5 - c_1) - \omega^2 c_2 - \omega_x(c_1x + c_2y + c_3) - \omega_y(c_4x + c_5y + c_6) = 0$$

But in this ODE $\omega = \frac{y-4xy^2-16x^3}{y^3+4x^2y+x}$, hence $\omega_x = \frac{-4y^5-32x^2y^3-8xy^2+(-64x^4-1)y-32x^3}{(4x^2y+y^3+x)^2}$ and $\omega_y = \frac{64x^5+32x^3y^2+4xy^4-8x^2y-2y^3+x}{(4x^2y+y^3+x)^2}$. Above becomes

$$c_4 + \left(\frac{y-4xy^2-16x^3}{y^3+4x^2y+x}\right)(c_5 - c_1) - \left(\frac{y-4xy^2-16x^3}{y^3+4x^2y+x}\right)^2 c_2 - \left(\frac{-4y^5-32x^2y^3-8xy^2+(-64x^4-1)y-32x^3}{(4x^2y+y^3+x)^2}\right.$$

Which expands to

$$\frac{8c_1xy^2}{4x^2y+y^3+x}+\frac{4c_5xy^2}{4x^2y+y^3+x}-\frac{256c_2x^4y^2}{(4x^2y+y^3+x)^2}-\frac{48c_2x^2y^4}{(4x^2y+y^3+x)^2}+\frac{16c_2x^3y}{(4x^2y+y^3+x)^2}+\frac{12c_2xy^3}{(4x^2y+y^3+x)^2}$$
$$+\frac{48x^2c_2y}{4x^2y+y^3+x}-\frac{128x^5yc_1}{(4x^2y+y^3+x)^2}-\frac{128x^4yc_3}{(4x^2y+y^3+x)^2}-\frac{32x^3y^3c_1}{(4x^2y+y^3+x)^2}-\frac{32x^2y^3c_3}{(4x^2y+y^3+x)^2}$$
$$+\frac{4x^2y^2c_1}{(4x^2y+y^3+x)^2}+\frac{4xy^2c_3}{(4x^2y+y^3+x)^2}+\frac{yc_1x}{(4x^2y+y^3+x)^2}+\frac{8x^2yc_4}{4x^2y+y^3+x}+\frac{8xyc_6}{4x^2y+y^3+x}-$$
$$\frac{64x^5c_5y}{(4x^2y+y^3+x)^2}-\frac{64x^4y^2c_4}{(4x^2y+y^3+x)^2}-\frac{64x^3y^3c_5}{(4x^2y+y^3+x)^2}-\frac{64x^3y^2c_6}{(4x^2y+y^3+x)^2}-\frac{12x^2y^4c_4}{(4x^2y+y^3+x)^2}-\frac{16c_5x^3}{4x^2y+y^3+}$$
$$-\frac{256c_2x^6}{(4x^2y+y^3+x)^2}+\frac{64c_1x^3}{4x^2y+y^3+x}-\frac{c_1y}{4x^2y+y^3+x}+\frac{48x^2c_3}{4x^2y+y^3+x}+\frac{4y^3c_2}{4x^2y+y^3+x}+\frac{4y^2c_3}{4x^2y+y^3+x}$$
$$-\frac{16x^4c_1}{(4x^2y+y^3+x)^2}-\frac{16x^3c_3}{(4x^2y+y^3+x)^2}+\frac{yc_3}{(4x^2y+y^3+x)^2}-\frac{c_4x}{4x^2y+y^3+x}-\frac{64x^6c_4}{(4x^2y+y^3+x)^2}-\frac{64x^5c_6}{(4x^2y+y^3+}$$
$$+\frac{3y^4c_5}{(4x^2y+y^3+x)^2}+\frac{3y^3c_6}{(4x^2y+y^3+x)^2}-\frac{c_6}{4x^2y+y^3+x}-\frac{12xy^5c_5}{(4x^2y+y^3+x)^2}-\frac{12xy^4c_6}{(4x^2y+y^3+x)^2}+\frac{4x^3yc_4}{(4x^2y+y^3+}$$
$$+\frac{4x^2y^2c_5}{(4x^2y+y^3+x)^2}+\frac{4x^2yc_6}{(4x^2y+y^3+x)^2}+\frac{3y^3c_4x}{(4x^2y+y^3+x)^2}+c_4=0$$

Multiplying each term by $(4x^2y+y^3+x)^2$ and expanding gives the multivariable polynomial

$$128x^5yc_1+64x^3y^3c_1+8c_1xy^5-256c_2x^6-64c_2x^4y^2+16c_2x^2y^4+4c_2y^6-64x^6c_4-16x^4y^2c_4+4x^2y^4c_4+c_4y^6$$
$$-128x^5c_5y-64x^3y^3c_5-8xy^5c_5+64x^4yc_3+32x^2y^3c_3+4c_3y^5-64x^5c_6-32x^3y^2c_6-4xy^4c_6+48x^4c_1+$$
$$8x^2y^2c_1-c_1y^4+64c_2x^3y+16c_2xy^3+16x^3yc_4+4y^3c_4x-16c_5x^4+8x^2y^2c_5+3y^4c_5+32x^3c_3+8xy^2c_3+8x^2yc_6+2y^3c_6$$

Each monomial coefficient must be zero. This gives the following equations to solve for c_i

$$-256c_2 - 64c_4 = 0$$
$$128c_1 - 128c_5 = 0$$
$$-64c_6 = 0$$
$$-64c_2 - 16c_4 = 0$$
$$64c_3 = 0$$
$$48c_1 - 16c_5 = 0$$
$$64c_1 - 64c_5 = 0$$
$$-32c_6 = 0$$
$$64c_2 + 16c_4 = 0$$
$$32c_3 = 0$$
$$16c_2 + 4c_4 = 0$$
$$32c_3 = 0$$
$$8c_1 + 8c_5 = 0$$
$$8c_6 = 0$$
$$8c_1 - 8c_5 = 0$$
$$-4c_6 = 0$$
$$16c_2 + 4c_4 = 0$$
$$8c_3 = 0$$
$$-c_6 = 0$$
$$4c_2 + c_4 = 0$$
$$4c_3 = 0$$
$$c_1 + 3c_5 = 0$$
$$2c_6 = 0$$
$$c_3 = 0$$

Hence we see that $c_6 = 0, c_3 = 0$. The above reduces to

$$
\begin{aligned}
256c_2 - 64c_4 &= 0 \\
128c_1 - 128c_5 &= 0 \\
64c_2 - 16c_4 &= 0 \\
48c_1 - 16c_5 &= 0 \\
64c_1 - 64c_5 &= 0 \\
64c_2 + 16c_4 &= 0 \\
8c_1 + 8c_5 &= 0 \\
8c_1 - 8c_5 &= 0 \\
16c_2 + 4c_4 &= 0 \\
4c_2 + c_4 &= 0 \\
-c_1 + 3c_5 &= 0
\end{aligned}
$$

Hence $Ac = b$ gives

$$
\begin{pmatrix}
0 & -256 & -64 & 0 \\
128 & 0 & 0 & -128 \\
0 & -64 & -16 & 0 \\
48 & 0 & 0 & -16 \\
64 & 0 & 0 & -64 \\
0 & 64 & 16 & 0 \\
0 & 16 & 4 & 0 \\
8 & 0 & 0 & -8 \\
0 & 16 & 4 & 0 \\
0 & 4 & 1 & 0 \\
-1 & 0 & 0 & 3
\end{pmatrix}
\begin{pmatrix} c_1 \\ c_2 \\ c_4 \\ c_5 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

The rank of A is 3 and the number of columns is 4. Hence non-trivial solution exist. Solving the above gives $c_4 = -4$ and $c_2 = 1$ and all other coefficients are zero. this means that , since

$$
\begin{aligned}
\xi &= c_1 x + c_2 y + c_3 \\
\eta &= c_4 x + c_5 y + c_6
\end{aligned}
$$

Then

$$
\begin{aligned}
\xi &= y \\
\eta &= -4x
\end{aligned}
$$

Which is what we wanted to show for this ODE.

1.12.7 Example $y' = \frac{-y^2}{e^x - y}$

Solve

$$
\begin{aligned}
y' &= \frac{-y^2}{e^x - y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The symmetry condition results in the PDE

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0
$$

End of the problem shows how this is solved for ξ, η which results in

$$
\begin{aligned}
\xi(x,y) &= 1 \\
\eta(x,y) &= y
\end{aligned}
$$

The integrating factor is therefore

$$
\begin{aligned}
\mu(x,y) &= \frac{1}{\eta - \xi\omega} \\
&= \frac{1}{y - \left(\frac{-y^2}{e^x - y}\right)} \\
&= \frac{1 - ye^{-x}}{y}
\end{aligned}
$$

The next step is to determine what is called the canonical coordinates X, Y. Where X is the independent variable and Y is the dependent variable. So we are looking for $Y(X)$ function. This is done by using the standard characteristic equation by writing

$$
\begin{aligned}
\frac{dx}{\xi} &= \frac{dy}{\eta} = dY \\
\frac{dx}{1} &= \frac{dy}{y} = dY \qquad (1)
\end{aligned}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) Y(x,y) = 1$. Which is a first order PDE. This is solved for Y, which gives (1) using the method of characteristic to solve first order PDE which is standard method. Starting with the first pair of ODE gives

$$
\frac{dy}{dx} = y
$$

Integrating gives $\ln|y| = x + c$ or $y = ce^x$ where c is constant of integration. In this method X is always c. Hence

$$
X(x,y) = ye^{-x}
$$

$Y(x,y)$ is now found from the first equation in (1) and the last equation which gives

$$
\begin{aligned}
dY &= \frac{dx}{\xi} \\
dY &= \frac{dx}{1} \\
dY &= dx \\
Y &= x
\end{aligned}
$$

Hence

$$
\begin{aligned}
X &= ye^{-x} \\
Y &= x
\end{aligned}
$$

Now that $X(x,y), Y(x,y)$ are found, the ODE $\frac{dY}{dX} = f(X)$ is setup. The ODE comes out to be function of X only, so it is quadrature. This is the main idea of this method. By solving for X we go back to x, y and solve for $y(x)$. How to find $\frac{dY}{dX}$? There is an equation to determine this given by

$$
\begin{aligned}
\frac{dY}{dX} &= \frac{\frac{dY}{dx} + \omega(x,y)\frac{dY}{dy}}{\frac{dX}{dx} + \omega(x,y)\frac{dX}{dy}} \\
&= \frac{Y_x + \omega(x,y)Y_y}{X_x + \omega(x,y)X_y}
\end{aligned}
$$

Everything on the RHS is known. $Y_x = 1, X_x = -ye^{-x}, Y_y = 0, X_y = e^{-x}$. Substituting gives

$$
\begin{aligned}
\frac{dY}{dX} &= \frac{1}{-ye^{-x} + \frac{-y^2}{e^x - y}e^{-x}} \\
&= \frac{ye^{-x} - 1}{ye^{-x}}
\end{aligned}
$$

But $X = ye^{-x}$, hence the above becomes

$$
\frac{dY}{dX} = \frac{X-1}{X}
$$

This is just quadrature. Integrating gives

$$
\begin{aligned}
Y &= \int \frac{X-1}{X} dX \\
&= X - \ln X + c_1
\end{aligned}
$$

This solution is converted back to x, y. Since $Y = x, X = ye^{-x}$, the above becomes

$$
x = ye^{-x} - \ln\left(ye^{-x}\right) + c_1
$$

Which is the solution to the original ODE.

Finding Lie symmetries for this example

The condition of symmetry is given above in equation (14) as

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{14}$$

Try

$$\begin{aligned}\xi &= c_1 x + c_2 y + c_3\\ \eta &= c_4 x + c_5 y + c_6\end{aligned}$$

Hence $\xi_x = c_1, \xi_y = c_2, \eta_x = c_4, \eta_y = c_5$ and (14) becomes

$$\begin{aligned}\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta &= 0\\ c_4 + \omega(c_5 - c_1) - \omega^2 c_2 - \omega_x(c_1 x + c_2 y + c_3) - \omega_y(c_4 x + c_5 y + c_6) &= 0\end{aligned}$$

But $\omega = \frac{-y^2}{e^x - y}, \omega_x = \frac{y^2 e^x}{(e^x-y)^2}, \omega_y = \left(-\frac{2y}{e^x-y} - \frac{y^2}{(e^x-y)^2}\right)$ and the above becomes

$$c_4 + \frac{-y^2}{e^x - y}(c_5 - c_1) - \left(\frac{-y^2}{e^x - y}\right)^2 c_2 - \frac{y^2 e^x}{(e^x - y)^2}(c_1 x + c_2 y + c_3) - \left(-\frac{2y}{e^x - y} - \frac{y^2}{(e^x - y)^2}\right)(c_4 x + c_5 y + c_6) =$$

Need to do this again. I should get $c_3 = 1, c_5 = 1$ and everything else zero.

$$\begin{aligned}\xi &= 1\\ \eta &= y\end{aligned}$$

1.12.8 Example $y' = \frac{x\sqrt{1+y}+\sqrt{1+y}+1+y}{1+x}$

Solve

$$\begin{aligned}y' &= \frac{x\sqrt{1+y}+\sqrt{1+y}+1+y}{1+x}\\ y' &= \omega(x,y)\end{aligned}$$

The symmetry condition results in the pde

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{1}$$

Let Ansatz be

$$\begin{aligned}\xi &= 0\\ \eta &= f(x)\,g(y)\end{aligned}$$

Hence (1) becomes

$$g(y)\frac{df}{dx} + \omega f(x)\frac{dg}{dy} - \omega_y f(x)\,g(y) = 0$$

But $\omega_x = \frac{d}{dx}\left(\frac{x\sqrt{1+y}+\sqrt{1+y}+1+y}{1+x}\right) = -\frac{(y+1)}{(x+1)^2}$ and $\omega_y = \frac{x+1+2\sqrt{1+y}}{\sqrt{1+y}(2+2x)}$. Hence the above becomes

$$g(y)\,\frac{df}{dx} + \left(\frac{x\sqrt{1+y}+\sqrt{1+y}+1+y}{1+x}\right) f(x)\,\frac{dg}{dy} - \frac{x+1+2\sqrt{1+y}}{\sqrt{1+y}\,(2+2x)}(f(x)\,g(y)) = 0 \quad (2)$$

The numerator of the normal form of the above is

$$2\frac{df}{dx}g\sqrt{1+y}x+2y\sqrt{1+y}f\frac{dg}{dy}+2f\frac{dg}{dy}xy+2\frac{df}{dx}g\sqrt{1+y}-2fg\sqrt{1+y}+2f\frac{dg}{dy}\sqrt{1+y}-fgx+2f\frac{dg}{dy}x+2fy\frac{dg}{dy}- \quad (3)$$

We can now either collect on y or x and try. Let us start with collecting on all terms with y. This gives

$$g\sqrt{1+y}\left(2x\frac{df}{dx}+2\frac{df}{dx}-2f\right)+y\sqrt{1+y}\frac{dg}{dy}(2f)+\frac{dg}{dy}\sqrt{1+y}(2f)+g(xf-f)+y\frac{dg}{dy}(2xf+2f)+\frac{dg}{dy}(2xf+2 \quad (3A)$$

The coefficients of all terms with $g(y)$ or y in them are from the above are the following, which each must be zero

$$\begin{aligned} 2f &= 0 \\ xf - f &= 0 \\ 2xf + 2f &= 0 \\ 2x\frac{df}{dx} + 2\frac{df}{dx} - 2f &= 0 \end{aligned}$$

Now we set each to zero and see if this produces $f(x)$ which can be used. We have 4 choices to try above. Starting from the most simple one. The first one above gives $2f = 0$ or $f = 0$. But this is not function of x. We try the next one $xf - f = 0$. This gives $f = 0$ or $x = 1$. Hence this does not give f as function of x. Next we try $2xf + 2f$. This also does not give f as function of x. The last one is $2x\frac{df}{dx} + 2\frac{df}{dx} - 2f = 0$ or $\frac{df}{dx} = \frac{2f}{2x+2}$. Solving this gives $f = c_1(x+1)$. This is successful since f is function of x. Hence

$$\begin{aligned} f(x) &= c_1(x+1) \\ \frac{df}{dx} &= c_1 \end{aligned}$$

Now we need to determine $g(y)$. Substituting the above into (3) gives

$$2c_1g(y)\sqrt{1+y}x+2\sqrt{1+y}c_1(x+1)\frac{dg}{dy}y+2c_1(x+1)\frac{dg}{dy}xy+2c_1g\sqrt{1+y}-2c_1(x+1)g\sqrt{1+y}+2c_1(x+1)\frac{dg}{dy}\sqrt{1}$$

Which simplifies to

$$2c_1\sqrt{1+y}\frac{dg}{dy}yx+2c_1\frac{dg}{dy}x^2y-c_1gx^2+2c_1\frac{dg}{dy}\sqrt{1+y}x+2\sqrt{1+y}c_1\frac{dg}{dy}y+2c_1\frac{dg}{dy}x^2+4c_1\frac{dg}{dy}xy-2c_1xg+2c_1\frac{dg}{dy}\sqrt{1} \quad (4)$$

Now factoring on all terms with x, and these are $\{x, x^2\}$ gives

$$-c_1x^2\left(-2\frac{dg}{dy}y+g-2\frac{dg}{dy}\right)-c_1x\left(-2\sqrt{1+y}\frac{dg}{dy}y-2\sqrt{1+y}\frac{dg}{dy}-2\frac{dg}{dy}y+g-2\frac{dg}{dy}\right)+T=0 \tag{4A}$$

Where T are terms that depends on y only. Each factor of x, x^2 must be zero. Hence the first above implies

$$\begin{aligned}-2\frac{dg}{dy}y+g-2\frac{dg}{dy}&=0\\ g'(y)&=\frac{g}{2(1+y)}\end{aligned}$$

Solving gives

$$g = c_2\sqrt{1+y} \tag{5}$$

Substituting (5) into (4) gives

$$c_1(1+x)\,c_2(1+y)=0$$

Which is not zero. Hence this term does not work. Now we try the second term in (4A) which means

$$\begin{aligned}-2\sqrt{1+y}\frac{dg}{dy}y-2\sqrt{1+y}\frac{dg}{dy}-2\frac{dg}{dy}y+g-2\frac{dg}{dy}&=0\\ \frac{dg}{dy}&=\frac{-g}{-2\sqrt{1+y}\,y-2\sqrt{1+y}-2y-2}\end{aligned}$$

Solving gives

$$g(y)=c_2\frac{\sqrt{1+y}}{1+\sqrt{1+y}}$$

Again, substituting the above back in (4) gives

$$c_1(1+x)\,c_2\frac{(1+y)\,x}{\left(1+\sqrt{1+y}\right)^2}=0$$

Which is not zero. Therefore starting with $f(x)=c_1(x+1)$ has failed to produce a valid $g(y)$ to satisfy the pde. This means we need to start all over again. Going back to (3) and now collecting on all terms with x instead. Here is (3) again

$$2\frac{df}{dx}g\sqrt{1+y}x+2y\sqrt{1+y}f\frac{dg}{dy}+2f\frac{dg}{dy}xy+2\frac{df}{dx}g\sqrt{1+y}-2fg\sqrt{1+y}+2f\frac{dg}{dy}\sqrt{1+y}-fgx+2f\frac{dg}{dy}x+2fy\frac{dg}{dy}- \tag{3}$$

Collecting on all terms that depend on x gives

$$x\frac{df}{dx}\left(2g\sqrt{1+y}\right)+f\left(2y\sqrt{1+y}\frac{dg}{dy}-2g\sqrt{1+y}+2\frac{dg}{dy}\sqrt{1+y}+2y\frac{dg}{dy}+2\frac{dg}{dy}-g\right)+xf\left(2\frac{dg}{dy}y-g+2\frac{dg}{dy}y\right. \tag{3B}$$

Each term must be zero, hence this gives these trials

$$
\begin{aligned}
2g\sqrt{1+y} &= 0 \\
2\frac{dg}{dy}y - g + 2\frac{dg}{dy}y &= 0 \\
2y\sqrt{1+y}\frac{dg}{dy} - 2g\sqrt{1+y} + 2\frac{dg}{dy}\sqrt{1+y} + 2y\frac{dg}{dy} + 2\frac{dg}{dy} - g &= 0
\end{aligned}
$$

Starting with the first one above $2g\sqrt{1+y} = 0$ which gives $g = 0$ which does not match the ansatz. Now we try the second one above, which gives

$$
\frac{dg}{dy} = \frac{g}{2+2y}
$$

Solving gives

$$
g = c_1\sqrt{1+y} \tag{6}
$$

Which meets the requirements of the ansatz. Now we need to use the above to generate $f(x)$. We do not need to try the third one above unless this fails. Substituting (6) into (3) gives

$$
\begin{aligned}
c_2\left(2\frac{df}{dx}xy + 2\frac{df}{dx}x + 2\frac{df}{dx}y - fy + 2\frac{df}{dx} - f\right) &= 0 \\
2\frac{df}{dx}xy + 2\frac{df}{dx}x + 2\frac{df}{dx}y - fy + 2\frac{df}{dx} - f &= 0 \qquad (7)
\end{aligned}
$$

Collecting on y gives

$$
c_1(1+y)\left(2\frac{df}{dx}x + 2\frac{df}{dx} - f\right) = 0
$$

Hence $2\frac{df}{dx}x + 2\frac{df}{dx} - f$ must be zero. This gives as solution

$$
\begin{aligned}
f(x) &= c_2\sqrt{1+x} \\
\frac{df}{dx} &= c_2\frac{1}{2\sqrt{1+x}}
\end{aligned}
$$

Substituting the above into (7) to verify gives

$$
\begin{aligned}
2\left(c_2\frac{1}{2\sqrt{1+x}}\right)xy + 2\left(c_2\frac{1}{2\sqrt{1+x}}\right)x + 2\left(c_2\frac{1}{2\sqrt{1+x}}\right)y - \left(c_2\sqrt{1+x}\right)y + 2\left(c_2\frac{1}{2\sqrt{1+x}}\right) - c_2\sqrt{1+x} &= \\
c_2\frac{1}{\sqrt{1+x}}xy + c_2\frac{1}{\sqrt{1+x}}x + c_2\frac{1}{\sqrt{1+x}}y - c_2\sqrt{1+x}y + c_2\frac{1}{\sqrt{1+x}} - c_2\sqrt{1+x} &= \\
c_2\left(\frac{1}{\sqrt{1+x}}xy + \frac{1}{\sqrt{1+x}}x + \frac{1}{\sqrt{1+x}}y - \sqrt{1+x}y + \frac{1}{\sqrt{1+x}} - \sqrt{1+x}\right) &= \\
0 &=
\end{aligned}
$$

Verified, Hence we have found $f(x), g(y)$. Therefore

$$
\begin{aligned}
\xi &= 0 \\
\eta &= f(x)\,g(y) \\
&= \sqrt{1+x}\sqrt{1+y}
\end{aligned}
$$

Where we set $c_1 = c_2 = 1$. The integrating factor is therefore

$$
\begin{aligned}
\mu(x,y) &= \frac{1}{\eta - \xi\omega} \\
&= \frac{1}{\sqrt{1+x}\sqrt{1+y}}
\end{aligned}
$$

The next step is to determine the canonical coordinates X, Y. Where X is the independent variable and Y is the dependent variable. This is done by using the standard characteristic equation by writing

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dY
$$

For the special case $\xi = 0$ we have $X = x$. $Y(x,y)$ is now found from the last two pair of equations which gives

$$
\begin{aligned}
dY &= \frac{dy}{\eta} \\
dY &= \frac{dy}{\sqrt{1+x}\sqrt{1+y}} \\
Y &= 2\frac{\sqrt{1+y}}{\sqrt{1+x}}
\end{aligned}
$$

Hence (constant of integration is set to zero)

$$
\begin{aligned}
X &= x \\
Y &= 2\frac{\sqrt{1+y}}{\sqrt{1+x}}
\end{aligned} \tag{2}
$$

Now that $X(x,y), Y(x,y)$ are found, the ODE $\frac{dY}{dX} = f(X)$ is setup. The ODE comes out to be function of X only, so it is quadrature. This is the main idea of this method. By solving for X we go back to x, y and solve for $y(x)$. How to find $\frac{dY}{dX}$? There is an equation to determine this given by

$$
\begin{aligned}
\frac{dY}{dX} &= \frac{\frac{dY}{dx} + \omega(x,y)\frac{dY}{dy}}{\frac{dX}{dx} + \omega(x,y)\frac{dX}{dy}} \\
&= \frac{Y_x + \omega(x,y)Y_y}{X_x + \omega(x,y)X_y}
\end{aligned}
$$

Everything on the RHS is known. $Y_x = -\frac{\sqrt{1+y}}{(1+x)^{\frac{3}{2}}}, X_x = 1, Y_y = \frac{1}{\sqrt{1+x}\sqrt{1+y}}, X_y = 0$. Substituting into the above gives

$$
\begin{aligned}
\frac{dY}{dX} &= -\frac{\sqrt{1+y}}{(1+x)^{\frac{3}{2}}} + \omega(x,y)\frac{1}{\sqrt{1+x}\sqrt{1+y}} \\
&= -\frac{\sqrt{1+y}}{(1+x)^{\frac{3}{2}}} + \left(\frac{x\sqrt{1+y}+\sqrt{1+y}+1+y}{1+x}\right)\frac{1}{\sqrt{1+x}\sqrt{1+y}} \\
&= \frac{1}{\sqrt{x+1}} \\
&= \frac{1}{\sqrt{X+1}}
\end{aligned}
$$

Hence

$$
\frac{dY}{dX} = \frac{1}{\sqrt{X+1}}
$$

This is quadrature. Solving gives

$$
Y = 2\sqrt{X+1} + c_1
$$

Convecting back to x, y gives

$$
2\frac{\sqrt{1+y}}{\sqrt{1+x}} = 2\sqrt{x+1} + c_1
$$

1.12.9 Example $y' = \frac{-y}{2x - ye^y}$

Solve

$$
\begin{aligned}
y' &= \frac{-y}{2x - ye^y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The symmetry condition results in the pde

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{1}
$$

Let ansatz be

$$
\begin{aligned}
\xi &= g(y) \\
\eta &= 0
\end{aligned}
$$

Substituting this into (1) gives

$$
-\omega^2 \frac{dg}{dy} - \omega_x g = 0
$$

But $\omega^2 = \frac{y^2}{(2x-ye^y)^2}, \omega_x = \frac{d}{dx}\left(\frac{-y}{2x-ye^y}\right) = \frac{2y}{(2x-ye^y)^2}$. The above becomes

$$
\begin{aligned}
-\frac{y^2}{(2x-ye^y)^2}\frac{dg}{dy} - \frac{2y}{(2x-ye^y)^2}g &= 0 \\
-y^2\frac{dg}{dy} - 2yg &= 0 \\
\frac{dg}{dy} + \frac{2}{y}g &= 0
\end{aligned}
$$

This is linear ode. The solution is

$$
g = \frac{c_1}{y^2}
$$

Hence

$$
\begin{aligned}
\xi &= \frac{1}{y^2} \\
\eta &= 0
\end{aligned}
$$

But taking $c_1 = 1$. The integrating factor is therefore

$$
\begin{aligned}
\mu(x,y) &= \frac{1}{\eta - \xi\omega} \\
&= \frac{1}{-\frac{1}{y^2}\left(\frac{-y}{2x-ye^y}\right)} \\
&= y(2x-ye^y)
\end{aligned}
$$

The next step is to determine the canonical coordinates X, Y. Where X is the independent variable and Y is the dependent variable. This is done by using the standard characteristic equation by writing

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dY
$$

Since $\eta = 0$, then in this special case $X = c_1 = y$. To find Y we use $dY = \frac{dx}{\xi}$ or $dY = y^2 dx$. Hence $Y = c_1^2 x + c_2 = c_1^2 x$ by taking $c_2 = 0$. Therefore $S = y^2 x$ since $c_1 = y$.

$$
\begin{aligned}
X &= y \\
Y &= y^2 x
\end{aligned} \tag{2}
$$

Now that $X(x,y), Y(x,y)$ are found, the ODE $\frac{dY}{dX} = f(X)$ is setup. The ODE comes out to be function of X only, so it is quadrature. This is the main idea of this method. By solving for X we go back to x, y and solve for $y(x)$. How to find $\frac{dY}{dX}$? There is an equation to determine this given by

$$
\begin{aligned}
\frac{dY}{dX} &= \frac{\frac{dY}{dx} + \omega(x,y)\frac{dY}{dy}}{\frac{dX}{dx} + \omega(x,y)\frac{dX}{dy}} \\
&= \frac{Y_x + \omega(x,y)Y_y}{X_x + \omega(x,y)X_y}
\end{aligned}
$$

Everything on the RHS is known. $Y_x = y^2, X_x = 0, Y_y = 2yx, X_y = 1$. Substituting into the above gives

$$\begin{aligned}\frac{dY}{dX} &= \frac{y^2 + \omega(x,y)\,2yx}{\omega\,(x,y)}\\ &= \frac{y^2 + \left(\frac{-y}{2x-ye^y}\right)2yx}{\left(\frac{-y}{2x-ye^y}\right)}\\ &= y^2 e^y\end{aligned}$$

Now we need to express the RHS in terms of X, Y. From (2) we see that $y = X$, hence the above becomes

$$\frac{dY}{dX} = X^2 e^X$$

This is quadrature. Solving gives

$$Y = \left(X^2 - 2X + 2\right)e^X + c_1$$

Convecting back to x, y gives

$$y^2 x = \left(y^2 - 2y + 2\right)e^y + c_1$$

1.12.10 Example $y' = \frac{-1-2yx}{x^2+2y}$

Solve

$$\begin{aligned}y' &= \frac{-1-2yx}{x^2+2y}\\ y' &= \omega(x,y)\end{aligned}$$

The symmetry condition results in the pde

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{1}$$

Let ansatz be

$$\begin{aligned}\xi &= 0\\ \eta &= f(x)\,g(y)\end{aligned}$$

Substituting this into (1) gives

$$g\frac{df}{dx} + \omega f\frac{dg}{dy} - \omega_y f g = 0$$

But $\omega=\frac{-1-2yx}{x^2+2y}, \omega_y=\frac{d}{dy}\left(\frac{-1-2yx}{x^2+2y}\right)=\frac{2-2x^3}{(x^2+2y)^2}$. The above becomes

$$g\frac{df}{dx}+\left(\frac{-1-2yx}{x^2+2y}\right)f\frac{dg}{dy}-\left(\frac{2-2x^3}{(x^2+2y)^2}\right)fg=0$$

The numerator of the normal form is

$$\begin{aligned} g\frac{df}{dx}(x^2+2y)^2+(x^2+2y)(-1-2yx)f\frac{dg}{dy}-(2-2x^3)fg&=0 \\ g\frac{df}{dx}(x^4+4x^2y+4y^2)+(-2x^3y-x^2-4xy^2-2y)f\frac{dg}{dy}-(2-2x^3)fg&=0 \end{aligned} \tag{2}$$

To solve this for $f(x), g(y)$ we start by collecting on either x or y. Let us start by collecting on y. This gives

$$\left[4\frac{df}{dx}\right](gy^2)+\left[4\frac{df}{dx}x^2\right](yg)+\left[\frac{df}{dx}x^4-(-2x^3+2)f\right]g+\left[(-2x^3-4x-2)f\right]\left(\frac{dg}{dy}\right)-\left[x^2f\right]\frac{dg}{dy}=0 \tag{3}$$

The other option was to collect on x terms. This would give

$$\left[-2y\frac{dg}{dy}+2g\right](x^3f)-\left[x^2f\right]\left(\frac{dg}{dy}\right)-[4xf]\left(y\frac{dg}{dy}\right)+\left[-2\frac{dg}{dy}y-2g\right](f)+[g]\left(x^4\frac{df}{dx}\right)+[yg]\left(4\frac{df}{dx}x^2\right)+\left[y^2g\right] \tag{4}$$

We start from (3), and if this yields no solutions for $f(x), g(y)$ then we come back and try (4). In either form, the terms inside the $[\cdot]$ must all be zero to satisfy the ode. From (3) this gives

$$\begin{aligned} 4\frac{df}{dx}&=0 \\ 4\frac{df}{dx}x^2&=0 \\ \frac{df}{dx}x^4-(-2x^3+2)f&=0 \\ (-2x^3-4x-2)f&=0 \\ x^2f&=0 \end{aligned}$$

If one of these results in $f(x)$ which is function of x. Then we try it to solve for $g(y)$. If the solutions end up verifying the pde, then we are done. From the above, we start with the first one. This gives $f=c_1$. Which is not function of x. The second give same result. The this option which is $\frac{df}{dx}x^4-(-2x^3+2)f=0$ gives

$$f(x)=c_1\frac{e^{-\frac{2}{3x^3}}}{x^2}$$

Which is function of x. We now use this to find $g(y)$. It turns out this does not work. The whole ansatz will fail. So need to try different ansatz.

1.12.11 Example $y' = 3\sqrt{yx}$

Solve

$$
\begin{aligned}
y' &= 3\sqrt{yx} \\
y' &= \omega(x, y)
\end{aligned}
$$

The symmetry condition results in the pde

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{1}$$

Trying polynomial ansatz

$$
\begin{aligned}
\xi &= a_0 + a_1 x \\
\eta &= b_0 + b_1 y
\end{aligned}
$$

And substituting these into (1) and simplifying gives

$$(-9a_1 + 3b_1)\, yx - 3xb_0 - 3ya_0 = 0$$

Setting all coefficients to zero gives

$$
\begin{aligned}
-9a_1 + 3b_1 &= 0 \\
b_0 &= 0 \\
a_0 &= 0
\end{aligned}
$$

Hence $a_1 = \frac{1}{3} b_1$. Letting $b_1 = 1$ then $a_1 = \frac{1}{3}$ and the infinitesimals are

$$
\begin{aligned}
\xi &= \frac{1}{3} x \\
\eta &= y
\end{aligned}
$$

The integrating factor is therefore

$$
\begin{aligned}
\mu(x, y) &= \frac{1}{\eta - \xi \omega} \\
&= \frac{1}{y - \frac{1}{3} x \left(3\sqrt{yx}\right)} \\
&= -\frac{y + x\sqrt{xy}}{x^3 y - y^2}
\end{aligned}
$$

The next step is to determine the canonical coordinates X, Y. This is done by using the standard characteristic equation by writing

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dY$$

The first pair of equations gives

$$\frac{dy}{dx} = \frac{\eta}{\xi} = \frac{3y}{x}$$

Solving gives

$$y = c_1 x^3$$

Hence

$$X = c_1 = \frac{y}{x^3} \tag{2}$$

And Y is found from

$$dY = \frac{dx}{\xi} = 3\frac{dx}{x}$$

Integrating gives

$$\begin{aligned} Y &= 3\ln x + c_1 \\ &= 3\ln x \end{aligned}$$

By choosing $c_1 = 0$. Now that $X(x,y), Y(x,y)$ are found, the ODE $\frac{dY}{dX} = f(X)$ is determined. This is determined from

$$\begin{aligned} \frac{dY}{dX} &= \frac{\frac{dY}{dx} + \omega(x,y)\frac{dY}{dy}}{\frac{dX}{dx} + \omega(x,y)\frac{dX}{dy}} \\ &= \frac{Y_x + \omega(x,y) Y_y}{X_x + \omega(x,y) X_y} \end{aligned}$$

But $Y_x = \frac{3}{x}, X_x = -3\frac{y}{x^4}, Y_y = 0, X_y = \frac{1}{x^3}$. Substituting these into the above gives

$$\begin{aligned} \frac{dY}{dX} &= \frac{\frac{3}{x}}{-3\frac{y}{x^4} + \omega(x,y)\frac{1}{x^3}} \\ &= \frac{3x^3}{-3y + x\omega(x,y)} \end{aligned}$$

But $\omega(x,y) = 3\sqrt{yx}$. The above becomes

$$\begin{aligned} \frac{dY}{dX} &= \frac{3x^3}{-3y + 3x\sqrt{yx}} \\ &= \frac{x^3}{x\sqrt{yx} - y} \\ &= \frac{-1}{\sqrt{\frac{y}{x^3}} - \frac{y}{x^3}} \end{aligned} \tag{3}$$

But $X = \frac{y}{x^3}$ and the above becomes

$$\frac{dY}{dX} = \frac{-1}{X - \sqrt{X}}$$

Which is a quadrature. Solving gives

$$\int dY = \int \frac{-1}{X - \sqrt{X}} dX$$
$$Y = -2\ln\left(\sqrt{X} - 1\right) + c_1$$

Converting back to x, y gives

$$3\ln x = -2\ln\left(\sqrt{\frac{y}{x^3}} - 1\right) + c_1$$
$$\ln x^3 + \ln\left(\sqrt{\frac{y}{x^3}} - 1\right)^2 = c_1$$
$$\ln\left(x^3\left(\sqrt{\frac{y}{x^3}} - 1\right)^2\right) = c_1$$
$$x^3\left(\sqrt{\frac{y}{x^3}} - 1\right)^2 = c_2$$

Or

$$y_1(x) = 2x\left(x^2 + x\sqrt{xc_1}\right) - x^3 + c_1$$
$$y_2(x) = -2x\left(-x^2 + x\sqrt{xc_1}\right) - x^3 + c_1$$

1.12.12 Example $y' = 4(yx)^{\frac{1}{3}}$

Solve

$$y' = 4(yx)^{\frac{1}{3}}$$
$$y' = \omega(x, y)$$

The symmetry condition results in the pde

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{1}$$

Trying polynomial ansatz

$$\xi = a_0 + a_1 x$$
$$\eta = b_0 + b_1 y$$

And substituting these into (1) and simplifying gives

$$(-16a_1 + 8b_1)\, yx - 4xb_0 - 4ya_0 = 0$$

Setting all coefficients to zero gives

$$-16a_1 + 8b_1 = 0$$
$$b_0 = 0$$
$$a_0 = 0$$

Hence $a_1 = \frac{1}{2}b_1$. Letting $b_1 = 1$ then $a_1 = \frac{1}{2}$ and the infinitesimals are

$$\xi = \frac{1}{2}x$$
$$\eta = y$$

The integrating factor is therefore

$$\begin{aligned}\mu(x,y) &= \frac{1}{\eta - \xi\omega}\\ &= \frac{1}{y - \frac{1}{2}x\left(4\,(yx)^{\frac{1}{3}}\right)}\\ &= \frac{1}{y - 2x\,(xy)^{\frac{1}{3}}}\end{aligned}$$

The next step is to determine the canonical coordinates X, Y. This is done by using the standard characteristic equation by writing

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dY$$

The first pair of equations gives

$$\frac{dy}{dx} = \frac{\eta}{\xi} = \frac{2y}{x}$$

Solving gives

$$y = c_1 x^2$$

Hence

$$X = c_1 = \frac{y}{x^2} \tag{2}$$

And Y is found from

$$dY = \frac{dx}{\xi} = 2\frac{dx}{x}$$

Integrating gives

$$\begin{aligned}Y &= 2\ln x + c_1\\ &= 2\ln x\end{aligned}$$

By choosing $c_1 = 0$. Now the ODE $\frac{dY}{dX} = f(X)$ is found from

$$\begin{aligned}\frac{dY}{dX} &= \frac{\frac{dY}{dx} + \omega(x,y)\,\frac{dY}{dy}}{\frac{dX}{dx} + \omega\,(x,y)\,\frac{dX}{dy}}\\ &= \frac{Y_x + \omega(x,y)\,Y_y}{X_x + \omega\,(x,y)\,X_y}\end{aligned}$$

But $S_x = \frac{2}{x}, R_x = -2\frac{y}{x^3}, S_y = 0, R_y = \frac{2}{x^2}$. Substituting these into the above and simplifying gives

$$\begin{aligned}\frac{dY}{dX} &= \frac{x^2}{2x\,(yx)^{\frac{1}{3}} - y} \\ &= \frac{1}{2\frac{1}{x}\,(yx)^{\frac{1}{3}} - \frac{y}{x^2}} \\ &= \frac{1}{2y^{\frac{1}{3}}x^{-\frac{2}{3}} - \frac{y}{x^2}} \\ &= \frac{1}{2\left(\frac{y}{x^2}\right)^{\frac{1}{3}} - \frac{y}{x^2}} \\ &= \frac{1}{2\,(X)^{\frac{1}{3}} - X}\end{aligned}$$

Hence

$$\frac{dY}{dX} = \frac{1}{2X^{\frac{1}{3}} - X}$$

Which is a quadrature. Solving gives

$$\begin{aligned}\int dY &= \int \frac{1}{2X^{\frac{1}{3}} - X} dX \\ Y &= -\frac{3}{2}\ln\left(-2 + X^{\frac{2}{3}}\right) + c_1\end{aligned}$$

Converting back to x, y gives

$$2\ln x = -\frac{3}{2}\ln\left(-2 + \left(\frac{y}{x^2}\right)^{\frac{2}{3}}\right) + c_1$$

The above can be simplified more if needed to solve for $y(x)$ explicitly.

1.12.13 Example $y' = 2y + 3e^{2x}$

Solve

$$\begin{aligned}y' &= 2y + 3e^{2x} \\ y' &= \omega(x, y)\end{aligned}$$

From the lookup table, since this is linear ode $y' = f(x)\,y + g(x)$ then

$$\begin{aligned}\xi &= 0 \\ \eta &= e^{\int f dx} \\ &= e^{\int 2 dx} \\ &= e^{2x}.\end{aligned}$$

If we were to use the integrating factor method, then

$$\begin{aligned}\mu(x,y) &= \frac{1}{\eta - \xi\omega}\\ &= \frac{1}{e^{2x}}\\ &= e^{-2x}\end{aligned}$$

Then the general solution is

$$\begin{aligned}\int \mu(x,y)\,(dy - \omega dx) &= c_1\\ \int e^{-2x}\left(dy - \left(2y + 3e^{2x}\right)dx\right) &= c_1\\ \int e^{-2x}dy - \left(2ye^{-2x} + 3\right)dx &= c_1\\ \int e^{-2x}dy - 2ye^{-2x}dx &= \int 3dx + c_1\\ \int d\left(e^{-2x}y\right) &= \int 3dx + c_1\end{aligned}$$

Hence

$$\begin{aligned}e^{-2x}y &= 3x + c_1\\ y &= e^{2x}(3x + c_1)\end{aligned}$$

But if we were to use the basic Lie symmetry method, then the next step is to determine the canonical coordinates X, Y. This is done by using the standard characteristic equation by writing

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dY$$

Since $\xi = 0$ then this is the special case where $X = x$. And Y is found from

$$dY = \frac{dy}{\eta} = e^{-2x}dy$$

Integrating gives

$$\begin{aligned}Y &= e^{-2x}y + c_1\\ &= e^{-2x}y\end{aligned}$$

By choosing $c_1 = 0$. Now the ODE $\frac{dY}{dX} = f(X)$ is found from

$$\begin{aligned}\frac{dY}{dX} &= \frac{\frac{dY}{dx} + \omega(x,y)\frac{dY}{dy}}{\frac{dX}{dx} + \omega(x,y)\frac{dX}{dy}}\\ &= \frac{Y_x + \omega(x,y)\,Y_y}{X_x + \omega(x,y)\,X_y}\end{aligned}$$

But $Y_x = -2e^{-2x}y, X_x = 1, Y_y = e^{-2x}, X_y = 0$. Substituting these into the above and simplifying gives

$$\begin{aligned}\frac{dY}{dX} &= -2e^{-2x}y + \left(2y + 3e^{2x}\right)e^{-2x} \\ &= -2e^{-2x}y + 2ye^{-2x} + 3 \\ &= 3\end{aligned}$$

Which is a quadrature. Solving gives

$$\begin{aligned}\int dY &= \int 3dX \\ Y &= 3X + c_1\end{aligned}$$

Converting back to x, y gives

$$\begin{aligned}e^{-2x}y &= 3x + c_1 \\ y &= (3x + c_1)e^{2x}\end{aligned}$$

Of course, this ode is first order linear and can be solved much easier using integrating factor method. But this is just to illustrate the Lie symmetry method.

1.12.14 Example $y' = \frac{1}{3}\frac{2y+y^3-x^2}{x}$

Solve

$$\begin{aligned}y' &= \frac{1}{3}\frac{2y + y^3 - x^2}{x} \\ y' &= \omega(x, y)\end{aligned}$$

Using Maple the infinitesimals are

$$\begin{aligned}\xi &= \frac{3}{2x^{\frac{1}{3}}} \\ \eta &= \frac{y}{x^{\frac{4}{3}}}\end{aligned}$$

(Will need to show how to obtain these). Lets solve this using the integration factor method first. The integrating factor is given by

$$\begin{aligned}\mu(x, y) &= \frac{1}{\eta - \xi\omega} \\ &= \frac{1}{\frac{y}{x^{\frac{4}{3}}} - \frac{3}{2x^{\frac{1}{3}}}\left(\frac{1}{3}\frac{2y+y^3-x^2}{x}\right)} \\ &= 2\frac{x^{\frac{4}{3}}}{x^2 - y^3}\end{aligned}$$

Then the general solution is

$$
\begin{aligned}
\int \mu(x,y)\,(dy-\omega dx) &= c_1\\
\int 2\frac{x^{\frac{4}{3}}}{x^2-y^3}\left(dy-\left(\frac{1}{3}\frac{2y+y^3-x^2}{x}\right)dx\right) &= c_1\\
\int \left(2\frac{x^{\frac{4}{3}}}{x^2-y^3}dy-\left(2\frac{x^{\frac{4}{3}}}{x^2-y^3}\right)\left(\frac{1}{3}\frac{2y+y^3-x^2}{x}\right)dx\right) &= c_1\\
\int \left(2\frac{x^{\frac{4}{3}}}{x^2-y^3}dy-\left(\frac{2}{3}\frac{x^{\frac{1}{3}}}{x^2-y^3}\right)\left(2y+y^3-x^2\right)dx\right) &= c_1
\end{aligned}
$$

Hence we need to find $F(x,y)$ s.t. $dF=\left(2\frac{x^{\frac{4}{3}}}{x^2-y^3}dy-\left(\frac{2}{3}\frac{x^{\frac{1}{3}}}{x^2-y^3}\right)\left(2y+y^3-x^2\right)dx\right)$ which will make the solution $F=c$. Therefore

$$
\begin{aligned}
dF &= \frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy\\
&= 2\frac{x^{\frac{4}{3}}}{x^2-y^3}dy-\left(\frac{2}{3}\frac{x^{\frac{1}{3}}}{x^2-y^3}\right)\left(2y+y^3-x^2\right)dx
\end{aligned}
$$

Hence

$$
\frac{\partial F}{\partial x} = -\frac{2}{3}\frac{x^{\frac{1}{3}}\left(2y+y^3-x^2\right)}{x^2-y^3} \tag{1}
$$

$$
\frac{\partial F}{\partial y} = 2\frac{x^{\frac{4}{3}}}{x^2-y^3} \tag{2}
$$

Integrating (1) gives

$$
\begin{aligned}
F &= \left(\int -\frac{2}{3}\frac{x^{\frac{1}{3}}\left(2y+y^3-x^2\right)}{x^2-y^3}dx\right)+g(y)\\
&= \frac{1}{2}x^{\frac{4}{3}}+\frac{1}{3}\ln\left(x^{\frac{4}{3}}+x^{\frac{2}{3}}y+y^2\right)-\frac{2}{3}\sqrt{3}\arctan\left(\frac{1}{3}\frac{\left(2x^{\frac{2}{3}}+y\right)\sqrt{3}}{y}\right)-\frac{2}{3}\ln\left(x^{\frac{2}{3}}-y\right)+g(y)
\end{aligned} \tag{3}
$$

Where $g(y)$ acts as the integration constant but F depends on x,y it becomes an arbitrary function. Taking derivative of the above w.r.t. y gives

$$
\frac{\partial F}{\partial y} = 2\frac{x^{\frac{4}{3}}}{x^2-y^3}+g'(y) \tag{4}
$$

Equating (4,2) gives

$$
\begin{aligned}
2\frac{x^{\frac{4}{3}}}{x^2-y^3} &= 2\frac{x^{\frac{4}{3}}}{x^2-y^3}+g'(y)\\
0 &= g'(y)\\
g(y) &= c_1
\end{aligned}
$$

Hence (3) becomes

$$F=\frac{1}{2}x^{\frac{4}{3}}+\frac{1}{3}\ln\left(x^{\frac{4}{3}}+x^{\frac{2}{3}}y+y^{2}\right)-\frac{2}{3}\sqrt{3}\arctan\left(\frac{1}{3}\frac{\left(2x^{\frac{2}{3}}+y\right)\sqrt{3}}{y}\right)-\frac{2}{3}\ln\left(x^{\frac{2}{3}}-y\right)+c_1$$

Therefore the solution is

$$F=c$$

$$\frac{1}{2}x^{\frac{4}{3}}+\frac{1}{3}\ln\left(x^{\frac{4}{3}}+x^{\frac{2}{3}}y+y^{2}\right)-\frac{2}{3}\sqrt{3}\arctan\left(\frac{1}{3}\frac{\left(2x^{\frac{2}{3}}+y\right)\sqrt{3}}{y}\right)-\frac{2}{3}\ln\left(x^{\frac{2}{3}}-y\right)=c_2$$

Where constants c_1, c were combined into c_2. Now this ode will be solved using direct symmetry by converting to canonical coordinates. This is done by using the standard characteristic equation by writing

$$\frac{dx}{\xi}=\frac{dy}{\eta}=dY$$
$$\frac{dx}{\frac{3}{2x^{\frac{1}{3}}}}=\frac{dy}{\frac{y}{x^{\frac{4}{3}}}}=dY$$

First pair of ode's give

$$\frac{dy}{dx}=\frac{\frac{y}{x^{\frac{4}{3}}}}{\frac{3}{2x^{\frac{1}{3}}}}=\frac{2}{3x}y$$

Hence

$$y=c_1x^{\frac{2}{3}}$$

Therefore

$$X=yx^{-\frac{2}{3}}$$

And

$$dY=\frac{dx}{\xi}=\frac{2}{3}x^{\frac{1}{3}}dx$$

Integrating gives

$$\begin{aligned}Y&=\int\frac{2}{3}x^{\frac{1}{3}}dx\\&=\frac{1}{2}x^{\frac{4}{3}}+c_1\\&=\frac{1}{2}x^{\frac{4}{3}}\end{aligned}$$

By choosing $c_1=0$. Now the ODE $\frac{dY}{dX}=f(X)$ is found from

$$
\begin{aligned}
\frac{dY}{dX} &= \frac{\frac{dY}{dx} + \omega(x,y)\frac{dY}{dy}}{\frac{dX}{dx} + \omega(x,y)\frac{dX}{dy}} \\
&= \frac{Y_x + \omega(x,y)\,Y_y}{X_x + \omega(x,y)\,X_y}
\end{aligned}
$$

But $Y_x = \frac{2}{3}x^{\frac{1}{3}}, X_x = -\frac{2}{3}yx^{-\frac{5}{3}}, Y_y = 0, X_y = x^{-\frac{2}{3}}$. Substituting these into the above and simplifying gives

$$
\begin{aligned}
\frac{dY}{dX} &= \frac{\frac{2}{3}x^{\frac{1}{3}}}{-\frac{2}{3}yx^{-\frac{5}{3}} + \omega(x,y)\,x^{-\frac{2}{3}}} \\
&= \frac{\frac{2}{3}x^{\frac{1}{3}}}{-\frac{2}{3}yx^{-\frac{5}{3}} + \left(\frac{1}{3}\frac{2y+y^3-x^2}{x}\right)x^{-\frac{2}{3}}} \\
&= -2\frac{x^2}{x^2-y^3}
\end{aligned}
$$

But $X = yx^{-\frac{2}{3}}$ or $y = Xx^{\frac{2}{3}}$. The above becomes

$$
\begin{aligned}
\frac{dY}{dX} &= -2\frac{x^2}{x^2 - X^3x^2} \\
&= \frac{-2}{1-X^3}
\end{aligned}
$$

Which is a quadrature. Solving gives

$$
\begin{aligned}
\int dY &= \int \frac{-2}{1-X^3}dX \\
Y &= -\frac{1}{3}\ln\left(X^2+x+1\right) - \frac{2}{3}\sqrt{3}\arctan\left(\frac{1}{3}(1+2X)\sqrt{3}\right) + \frac{2}{3}\ln(X-1) + c_1
\end{aligned}
$$

Converting back to x, y gives

$$
\begin{aligned}
\frac{1}{2}x^{\frac{4}{3}} &= -\frac{1}{3}\ln\left(\left(yx^{-\frac{2}{3}}\right)^2 + x + 1\right) - \frac{2}{3}\sqrt{3}\arctan\left(\frac{1}{3}\left(1+2\left(yx^{-\frac{2}{3}}\right)\right)\sqrt{3}\right) + \frac{2}{3}\ln\left(\left(yx^{-\frac{2}{3}}\right)-1\right) + c_1 \\
\frac{1}{2}x^{\frac{4}{3}} &= -\frac{1}{3}\ln\left(y^2x^{-\frac{4}{3}} + x + 1\right) - \frac{2}{3}\sqrt{3}\arctan\left(\frac{1}{3}\left(1+2yx^{-\frac{2}{3}}\right)\sqrt{3}\right) + \frac{2}{3}\ln\left(yx^{-\frac{2}{3}}-1\right) + c_1
\end{aligned}
$$

1.12.15 Example $y' = 3 - 2\frac{y}{x}$

This is homogeneous ODE of Class A of form $y' = F\left(\frac{y}{x}\right)$, hence from the lookup table

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

The first step is to verify that $\bar{x} = \epsilon x, \bar{y} = \epsilon y$ leaves the ode invariant.

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\bar{y}_x + \bar{y}_y y'}{\bar{x}_x + \bar{x}_y y'} = \frac{\epsilon y'}{\epsilon} = y'$$

Hence the ode becomes

$$\begin{aligned}
\frac{d\bar{y}}{d\bar{x}} &= 3 - 2\frac{\bar{y}}{\bar{x}} \\
y' &= 3 - 2\frac{\epsilon y}{\epsilon x} \\
&= 3 - 2\frac{y}{x}
\end{aligned}$$

Verified. Now the ode is solved. The tangent curves are computed directly from the Lie group symmetry given above

$$\begin{aligned}
\xi &= \frac{\partial \bar{x}}{\partial \epsilon}\bigg|_{\epsilon=0} = x \\
\eta &= \frac{\partial \bar{y}}{\partial \epsilon}\bigg|_{\epsilon=0} = y
\end{aligned}$$

The canonical coordinates (X, Y) are now found. Using

$$\begin{aligned}
\frac{dx}{\xi} &= \frac{dy}{\eta} = dY \\
\frac{dx}{x} &= \frac{dy}{y} = dY \qquad (1)
\end{aligned}$$

The first pair gives

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y}{x} \\
\ln y &= \ln x + c_1 \\
y &= cx
\end{aligned}$$

Hence

$$\begin{aligned}
X &= c \\
&= \frac{y}{x}
\end{aligned}$$

Now we find Y from the last pair of equations

$$\begin{aligned}
\frac{dy}{y} &= dY \\
Y &= \ln y
\end{aligned}$$

What is left is to find $\frac{dY}{dX}$. This is given by

$$\frac{dY}{dX} = f(X)$$

To find $f(X)$, we use $dY = Y_x dx + Y_y dy = \frac{1}{y} dy$ and $dX = X_x dx + X_y dy = -\frac{y}{x^2} dx + \frac{1}{x} dy$. Hence

$$\begin{aligned} \frac{dY}{dX} &= \frac{\frac{1}{y} dy}{-\frac{y}{x^2} dx + \frac{1}{x} dy} \\ &= \frac{\frac{dy}{dx}}{-\frac{y^2}{x^2} + \frac{y}{x}\frac{dy}{dx}} \\ &= \frac{\frac{dy}{dx}}{-X^2 + X\frac{dy}{dx}} \end{aligned}$$

But $\frac{dy}{dx} = 3 - 2\frac{y}{x} = 3 - 2X$, hence

$$\begin{aligned} \frac{dY}{dX} &= \frac{3 - 2X}{-X^2 + X(3 - 2X)} \\ &= \frac{3 - 2X}{3(X - X^2)} \end{aligned}$$

Which is a quadrature. In Lie method, for first order ode, we always obtain $\frac{dY}{dX} = f(X)$. Integrating the above gives

$$\begin{aligned} \int dY &= \int \frac{3 - 2X}{3(X - X^2)} dX \\ Y &= \ln X - \frac{1}{3} \ln (X - 1) + c_1 \end{aligned}$$

Final step is to replace X, Y back with x, y which gives

$$\begin{aligned} \ln y &= \ln \frac{y}{x} - \frac{1}{3} \ln \left(\frac{y}{x} - 1\right) + c_1 \\ y &= c_1 \frac{\frac{y}{x}}{\left(\frac{y}{x} - 1\right)^{\frac{1}{3}}} \\ \left(\frac{y}{x} - 1\right)^{\frac{1}{3}} &= c_1 \frac{1}{x} \\ \frac{y}{x} - 1 &= c_2 \frac{1}{x^3} \\ y &= \left(c_2 \frac{1}{x^3} + 1\right) x \end{aligned}$$

1.12.16 Example $y' = \frac{-3+\frac{y}{x}}{-1-\frac{y}{x}}$

This is homogeneous ODE of Class A of form $y' = F\left(\frac{y}{x}\right)$, hence from the lookup table

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Canonical coordinates (X, Y) are found similar to the above which gives

$$
\begin{aligned}
X &= \frac{y}{x} \\
Y &= \ln y
\end{aligned}
$$

What is left is to find $\frac{dY}{dX}$. This is given by

$$
\frac{dY}{dX} = f(X)
$$

Which is the same as above

$$
\frac{dS}{dR} = \frac{\frac{dy}{dx}}{-R^2 + R\frac{dy}{dx}}
$$

But in this problem, the only difference is that $\frac{dy}{dx} = \frac{-3+\frac{y}{x}}{-1-\frac{y}{x}} = \frac{-3+X}{-1-X}$, hence

$$
\begin{aligned}
\frac{dY}{dX} &= \frac{\frac{-3+X}{-1-X}}{-X^2 + X\left(\frac{-3+X}{-1-X}\right)} \\
&= \frac{1}{X}\frac{X-3}{X^2+2X-3}
\end{aligned}
$$

Which is a quadrature. In Lie method, for first order ode, we always obtain $\frac{dY}{dX} = f(X)$. Integrating the above gives

$$
\begin{aligned}
\int dY &= \int \frac{1}{X}\left(\frac{X-3}{X^2+2X-3}\right) dX \\
Y &= \ln(X) - \frac{1}{2}\ln(X+3) - \frac{1}{2}\ln(X-1) + c_1
\end{aligned}
$$

Final step is to replace X, Y back with x, y which gives

$$
\ln y = \ln\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(\frac{y}{x}+3\right) - \frac{1}{2}\ln\left(\frac{y}{x}-1\right) + c_1
$$

This can be solved for y if an explicit solution is needed.

1.12.17 Example $y' = \frac{1+3\left(\frac{y}{x}\right)^2}{2\frac{y}{x}}$

This is homogeneous ODE of Class A of form $y' = F\left(\frac{y}{x}\right)$, hence from the lookup table

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

The canonical ode is

$$
\frac{dY}{dX} = \frac{\frac{dy}{dx}}{-X^2 + X\frac{dy}{dx}}
$$

The above is the same ode in canonical coordinates for any ode of the form $y' = F\left(\frac{y}{x}\right)$. We just need to express y' as function of X. In this case the above becomes

$$
\begin{aligned}
\frac{dY}{dX} &= \frac{\frac{1+3X^2}{2X}}{-X^2 + X\left(\frac{1+3X^2}{2X}\right)} \\
&= \frac{3X^2+1}{X^3+X}
\end{aligned}
$$

Integrating gives

$$
Y = \ln\left(X\left(X^2+1\right)\right) + c_1
$$

Final step is to replace X, Y back with x, y which gives

$$
\begin{aligned}
\ln y &= \ln\left(\frac{y}{x}\left(\left(\frac{y}{x}\right)^2 + 1\right)\right) + c_1 \\
y &= c_2\frac{y}{x}\left(\left(\frac{y}{x}\right)^2 + 1\right) \\
1 &= \frac{c_2}{x}\left(\left(\frac{y}{x}\right)^2 + 1\right) \\
\frac{y^2}{x^2} &= c_3 x - 1 \\
y^2 &= c_3 x^3 - x^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= \pm\sqrt{c_3 x^3 - x^2} \\
&= \pm x\sqrt{c_3 x - 1}
\end{aligned}
$$

Finding ξ, η from symmetry condition for the above ode This shows how to find ξ, η directly also. The condition of symmetry is given above in equation (14) as

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{14}
$$

Try Ansatz

$$
\begin{aligned}
\xi &= c_0 + c_1 x \\
\eta &= c_2 + c_3 y
\end{aligned}
$$

And given

$$\omega = \frac{1}{2}\frac{x^2+3y^2}{xy}$$
$$\omega^2 = \frac{1}{4}\frac{(x^2+3y^2)^2}{x^2y^2}$$
$$\omega_x = \frac{1}{2}\frac{x^2-3y^2}{yx^2}$$
$$\omega_y = \frac{1}{2}\frac{3y^2-x^2}{xy^2}$$

Hence (14) becomes

$$\eta_x + \frac{1}{2}\frac{x^2+3y^2}{xy}\eta_y - \frac{1}{2}\frac{x^2-3y^2}{yx^2}\xi - \frac{1}{2}\frac{3y^2-x^2}{xy^2}\eta = 0$$

Therefore the above becomes

$$\frac{1}{2}\frac{x^2+3y^2}{xy}c_3 - \frac{1}{2}\frac{x^2-3y^2}{yx^2}(c_0+c_1x) - \frac{1}{2}\frac{3y^2-x^2}{xy^2}(c_2+c_3y) = 0$$

Using the computer the above simplifies to

$$\frac{x}{y}(c_3-c_1) + \frac{1}{2}c_2\frac{x}{y^2} - \frac{1}{y}\left(\frac{1}{2}c_0\right) - \frac{1}{x}\frac{3}{2}c_2 + \frac{3}{2}c_0\frac{y}{x^2} = 0$$

Hence

$$c_3 - c_1 = 0$$
$$\frac{1}{2}c_2 = 0$$
$$-\frac{1}{2}c_0 = 0$$
$$-\frac{3}{2}c_2 = 0$$
$$\frac{3}{2}c_0 = 0$$

Solving gives $c_0 = 0, c_2 = 0$ and $c_3 = c_1$. Hence the solution is

$$\xi = c_1x$$
$$\eta = c_3y$$

Let $c_1 = 1$, therefore $c_3 = 1$ and we obtain

$$\xi = x$$
$$\eta = y$$

Which is the result we used in solving the above problem. Notice that any scaler will also work. Hence

$$
\begin{aligned}
\xi &= 5x \\
\eta &= 5y
\end{aligned}
$$

And

$$
\begin{aligned}
\xi &= 10x \\
\eta &= 10y
\end{aligned}
$$

This will also give same solution.

1.12.18 Example $y' = \frac{y}{x} + \frac{1}{x} F\left(\frac{y}{x}\right)$

This is homogeneous class D $y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$. Hence from lookup table

$$
\begin{aligned}
\xi &= x^2 \\
\eta &= xy
\end{aligned}
$$

Now we just need to find canonical coordinates (X, Y) since ξ, η are known. Using

$$
\begin{aligned}
\frac{dx}{\xi} &= \frac{dy}{\eta} = dY \\
\frac{dx}{x^2} &= \frac{dy}{xy} = dY
\end{aligned} \tag{1}
$$

The first pair gives

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{y}{x} \\
\ln y &= \ln x + c_1 \\
y &= cx
\end{aligned}
$$

Hence

$$
\begin{aligned}
X &= c \\
&= \frac{y}{x}
\end{aligned}
$$

Now we find Y from the last pair of equations (we could also use the first and last equations in (1)).

$$
\begin{aligned}
\frac{dy}{xy} &= dY \\
Y &= \frac{1}{x} \ln y
\end{aligned}
$$

What is left is to find $\frac{dY}{dX}$. This is given by

$$\begin{aligned}\frac{dY}{dX} &= f(X)\\ &= \frac{Y_x + Y_y y'}{X_x + X_y y'}\end{aligned}$$

To find $f(X)$, we use $Y_x = \frac{-1}{x^2}\ln y, Y_y = \frac{1}{xy}$ and $X_x = -\frac{y}{x^2}, X_y = \frac{1}{x}$. Hence

$$\begin{aligned}\frac{dY}{dX} &= \frac{\frac{-1}{x^2}\ln y + \frac{1}{xy}y'}{-\frac{y}{x^2} + \frac{1}{x}y'}\\ &= \frac{-\ln y - \frac{x}{y}y'}{y + xy'}\\ &= \frac{-\ln y - \frac{1}{R}y'}{y + xy'}\end{aligned}$$

But $y' = \frac{y}{x} + \frac{1}{x}f\left(\frac{y}{x}\right) = X + \frac{1}{x}f(X)$. The above becomes

$$\begin{aligned}\frac{dY}{dX} &= \frac{-\ln y - \frac{1}{X}\left(X + \frac{1}{x}f(X)\right)}{y + x\left(X + \frac{1}{x}f\left(X\right)\right)}\\ &= \frac{-\ln y - 1 - \frac{1}{xX}f(X)}{y + xX + f\left(X\right)}\\ &= \frac{-\ln y - 1 - \frac{1}{x\frac{y}{x}}f(X)}{y + x\frac{y}{x} + f\left(X\right)}\\ &= \frac{-\ln y - 1 - \frac{1}{y}f(X)}{2y + f\left(X\right)}\end{aligned}$$

Something is wrong. $\frac{dY}{dX}$ should only be a function of X. Need to find out why. Let me try the other pair of equations from (1) to solve for S and see what happens.

$$\begin{aligned}\frac{dx}{x^2} &= dY\\ Y &= -\frac{1}{x}\end{aligned}$$

What is left is to find $\frac{dY}{dX}$. This is given by

$$\begin{aligned}\frac{dY}{dX} &= G(X)\\ &= \frac{Y_x + Y_y y'}{X_x + X_y y'}\end{aligned}$$

To find $G(X)$, we use $Y_x = \frac{1}{x^2}, Y_y = 0$ and $X_x = -\frac{y}{x^2}, X_y = \frac{1}{x}$. Hence

$$\begin{aligned}\frac{dS}{dR} &= \frac{\frac{1}{x^2}}{-\frac{y}{x^2} + \frac{1}{x}y'}\\ &= \frac{1}{-y + xy'}\end{aligned}$$

But $y' = \frac{y}{x} + \frac{1}{x}F\left(\frac{y}{x}\right) = R + \frac{1}{x}F(R)$. The above becomes

$$\begin{aligned}\frac{dY}{dX} &= \frac{1}{-y + x\left(X + \frac{1}{x}f\left(X\right)\right)}\\ &= \frac{1}{-y + xX + f\left(X\right)}\\ &= \frac{1}{-y + x\frac{y}{x} + f\left(X\right)}\\ &= \frac{1}{f\left(X\right)}\end{aligned}$$

This worked. But why the first choice did not work? OK, let me continue now. Integrating the above gives

$$Y = \int \frac{1}{f\left(X\right)} dX + c$$

But $Y = -\frac{1}{x}$, hence

$$\begin{aligned}-\frac{1}{x} &= \int^{\frac{y}{x}} \frac{1}{f\left(r\right)} dr + c\\ 0 &= \int^{\frac{y}{x}} \frac{1}{f\left(r\right)} dr + c + \frac{1}{x}\end{aligned}$$

This example shows that when solving for Y from

$$\frac{dx}{x^2} = \frac{dy}{xy} = dY$$

There are two choice. One is $dY = \frac{dy}{xy}$ and the other $dY = \frac{dx}{x^2}$. Using the first choice did not work here (unless I made a mistake, but do not see it)., Only the second choice worked because we must end up with $\frac{dY}{dX} = G(X)$ where RHS is function of R only. I need to look more into this. In theory, any choice should have worked.

1.12.19 Example $y' = \frac{y}{x} + \frac{1}{x}e^{-\frac{y}{x}}$

This is homogeneous class D $y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$. Hence from lookup table

$$\begin{aligned}\xi &= x^2\\ \eta &= xy\end{aligned}$$

From above we found the solution to be

$$Y = \int \frac{1}{F\left(X\right)} dX + c$$

In this case $F(X) = e^{-X}$. Hence

$$\begin{aligned}Y &= \int e^X dX + c\\ Y &= e^X + c\end{aligned}$$

Now we just need to find canonical coordinates (X, Y) since ξ, η are known. From above

$$
\begin{aligned}
X &= \frac{y}{x} \\
Y &= -\frac{1}{x}
\end{aligned}
$$

Hence the solution becomes

$$
\begin{aligned}
-\frac{1}{x} &= e^{\frac{y}{x}} + c \\
e^{\frac{y}{x}} &= c_2 - \frac{1}{x} \\
\frac{y}{x} &= \ln\left(c_2 - \frac{1}{x}\right) \\
y &= x \ln\left(c_2 - \frac{1}{x}\right)
\end{aligned}
$$

The nice thing about this method is that once we solve for one pattern of an ode, then the same solution in canonical coordinates is used, the only change need is to plug-in in the RHS of the original ode in the solution and integrate.

1.12.20 Example $y' = \frac{1-y^2+x^2}{1+y^2-x^2}$

$$
\begin{aligned}
y' &= \frac{1-y^2+x^2}{1+y^2-x^2} \\
&= \omega(x, y)
\end{aligned}
$$

Using ansatz it is found that

$$
\begin{aligned}
\xi &= x - y \\
\eta &= y - x
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{dx}{\xi} &= \frac{dy}{\eta} = dS \\
\frac{dx}{x-y} &= \frac{dy}{y-x} = dS \qquad (1)
\end{aligned}
$$

The first two give

$$
\frac{dy}{dx} = \frac{\eta}{\xi} = \frac{y-x}{x-y} = -1
$$

Hence

$$
y = -x + c_1 \qquad (2)
$$

Therefore

$$
\begin{aligned}
X &= c_1 \\
&= y + x
\end{aligned}
$$

To find Y, since both ξ, η depend on both x, y, then $\frac{dy}{\eta} = dS$ or $\frac{dx}{\xi} = dS$ can be used. Lets try both to show same answer results.

$$
\begin{aligned}
\frac{dy}{\eta} &= dY \\
dY &= \frac{dy}{y - x}
\end{aligned}
$$

But from (2), $x = c_1 - y$. The above becomes

$$
\begin{aligned}
dY &= \frac{dy}{y - (c_1 - y)} \\
&= \frac{dy}{2y - c_1}
\end{aligned}
$$

Hence

$$
Y = \frac{1}{2} \ln (2y - c_1)
$$

But $c_1 = y + x$. So the above becomes

$$
\begin{aligned}
Y &= \frac{1}{2} \ln (2y - (y + x)) \\
&= \frac{1}{2} \ln (y - x) \qquad (3)
\end{aligned}
$$

Let us now try the other ode

$$
\begin{aligned}
\frac{dx}{\xi} &= dS \\
dS &= \frac{dx}{x - y}
\end{aligned}
$$

But from (2) $y = -x + c_1$. The above becomes

$$
\begin{aligned}
dY &= \frac{dx}{x - (-x + c_1)} \\
&= \frac{dx}{2x - c_1}
\end{aligned}
$$

Therefore

$$
Y = \frac{1}{2} \ln (2x - c_1)
$$

But $c_1 = y + x$. Therefore

$$
\begin{aligned}
Y &= \frac{1}{2}\ln\left(2x - (y+x)\right) \\
&= \frac{1}{2}\ln\left(x - y\right) \qquad (4)
\end{aligned}
$$

The constant of integration is set to zero when finding Y. What is left is to find $\frac{dY}{dX}$. This is given by

$$
\frac{dY}{dX} = \frac{Y_x + Y_y\omega}{X_x + X_y\omega} \qquad (5)
$$

But, and using (4) for Y we have

$$
\begin{aligned}
X_x &= 1 \\
X_y &= 1 \\
Y_x &= \frac{-1}{y-x} \\
Y_y &= \frac{1}{y-x}
\end{aligned}
$$

Hence (2) becomes

$$
\begin{aligned}
\frac{dY}{dX} &= \frac{\frac{-1}{y-x} + \frac{1}{y-x}\omega}{1+\omega} \\
&= \frac{-\frac{\omega-1}{x-y}}{1+\omega} \\
&= \frac{1-\omega}{(1+\omega)(x-y)} \\
&= \frac{1-\left(\frac{1-y^2+x^2}{1+y^2-x^2}\right)}{\left(1+\left(\frac{1-y^2+x^2}{1+y^2-x^2}\right)\right)(x-y)} \\
&= -x-y \\
&= -(x+y) \\
&= -X
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{dY}{dX} &= -X \\
Y &= -\frac{X^2}{2}
\end{aligned}
$$

Converting back to x, y gives

$$
\ln\left(y-x\right) = -\frac{(y+x)^2}{2}
$$

1.12.21 Example $y' = -\frac{1}{4}xe^{-2y} + \frac{1}{4}\sqrt{\left(e^{-2y}\right)^2 x^2 + 4e^{-2y}}$

$$
\begin{aligned}
y' &= -\frac{1}{4}xe^{-2y} + \frac{1}{4}\sqrt{\left(e^{-2y}\right)^2 x^2 + 4e^{-2y}} \\
&= \omega(x, y)
\end{aligned}
$$

Using ansatz it is found that

$$
\begin{aligned}
\xi &= x \\
\eta &= 1
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{dx}{\xi} &= \frac{dy}{\eta} = dS \\
\frac{dx}{x} &= dy = dS \qquad (1)
\end{aligned}
$$

The first two give

$$
\frac{dy}{dx} = \frac{1}{x}
$$

Hence

$$
y = \ln x + c_1
$$

Therefore

$$
\begin{aligned}
X &= c_1 \\
&= y - \ln x
\end{aligned}
$$

And Y is found from either $\frac{dy}{\eta} = dY$ or $\frac{dx}{\xi} = dY$. Since $\eta = 1$, it is simpler to use $\frac{dy}{\eta} = dY$ instead.

$$
\begin{aligned}
\frac{dy}{\eta} &= dY \\
dy &= dY \\
Y &= y
\end{aligned}
$$

Where constant of integration is set to zero. What is left is to find $\frac{dY}{dX}$. This is given by

$$
\frac{dY}{dX} = \frac{Y_x + Y_y\omega}{X_x + X_y\omega} \qquad (2)
$$

But

$$
\begin{aligned}
X_x &= -\frac{1}{x} \\
X_y &= 1 \\
Y_x &= 0 \\
Y_y &= 1
\end{aligned}
$$

Hence (2) becomes

$$
\begin{aligned}
\frac{dY}{dX} &= \frac{\omega}{-\frac{1}{x} + \omega} = \frac{1}{-\frac{1}{x\omega} + 1} \\
&= \frac{1}{1 - \frac{1}{x\left(-\frac{1}{4}xe^{-2y} + \frac{1}{4}\sqrt{(e^{-2y})^2 x^2 + 4e^{-2y}}\right)}}
\end{aligned}
$$

But $y = X + \ln x$. The above becomes

$$
\begin{aligned}
\frac{dY}{dX} &= \frac{1}{1 - \frac{1}{x\left(-\frac{1}{4}xe^{-2(R+\ln x)} + \frac{1}{4}\sqrt{\left(e^{-2(R+\ln x)}\right)^2 x^2 + 4e^{-2R+\ln x}}\right)}} \\
&= \frac{1}{1 - \frac{1}{x\left(-\frac{1}{4}\frac{xe^{-2R}}{x^2} + \frac{1}{4}\frac{1}{x}\sqrt{e^{-4R} + 4e^{-2R}}\right)}} \\
&= \frac{1}{1 - \frac{1}{\left(-\frac{1}{4}e^{-2R} + \frac{1}{4}\sqrt{e^{-4R} + 4e^{-2R}}\right)}}
\end{aligned}
$$

Integrating gives

$$
Y = \frac{\sqrt{\frac{1+4e^{2X}}{e^{4X}}}\, e^{2X} \operatorname{arctanh}\left(\frac{1}{\sqrt{1+4e^{X}}}\right)}{\sqrt{1 + 4e^{2X}}}
$$

Converting back to x, y gives

$$
y = \frac{\sqrt{\frac{1+4e^{2(y-\ln x)}}{e^{4(y-\ln x)}}}\, e^{2(y-\ln x)} \operatorname{arctanh}\left(\frac{1}{\sqrt{1+4e^{2(y-\ln x)}}}\right)}{\sqrt{1 + 4e^{2(y-\ln x)}}}
$$

1.12.22 Example $y' = \frac{y - xf(x^2+ay^2)}{x+ayf(x^2+ay^2)}$

$$
\begin{aligned}
y' &= \frac{y - xf(x^2 + ay^2)}{x + ayf\left(x^2 + ay^2\right)} \\
&= \omega(x, y)
\end{aligned}
$$

Using ansatz it is found that

$$
\begin{aligned}
\xi &= -ay \\
\eta &= x
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{dx}{\xi} &= \frac{dy}{\eta} = dS \\
\frac{dx}{-ay} &= \frac{dy}{x} = dS \qquad (1)
\end{aligned}
$$

The first two give

$$\frac{dy}{dx} = \frac{x}{-ay}$$

This is separable. Solving gives (taking one root)

$$y = \frac{\sqrt{a\left(a c_1 - x^2\right)}}{a}$$

Solving for c_1 gives

$$c_1 = \frac{x^2 + a y^2}{a}$$

Hence

$$X = \frac{x^2 + a y^2}{a}$$

Y is found from either $\frac{dy}{\eta} = dY$ or $\frac{dx}{\xi} = dY$. Using $\frac{dx}{-ay} = dY$ then

$$\frac{dx}{-ay} = dY$$

But $y = \frac{\sqrt{a(ac_1 - x^2)}}{a}$. Hence

$$\begin{aligned}
\frac{dx}{-a\frac{\sqrt{a(ac_1-x^2)}}{a}} &= dY \\
\frac{dx}{-\sqrt{a\left(a c_1 - x^2\right)}} &= dY \\
-\frac{1}{\sqrt{a}} \arctan\left(\frac{\sqrt{a}x}{\sqrt{c_1 a^2 - x^2 a}}\right) &= Y \\
-\frac{1}{\sqrt{a}} \arctan\left(\frac{\sqrt{a}x}{ay}\right) &= Y
\end{aligned}$$

Where constant of integration is set to zero. What is left is to find $\frac{dY}{dX}$. This is given by

$$\frac{dY}{dX} = \frac{Y_x + Y_y \omega}{X_x + X_y \omega} \tag{2}$$

But

$$\begin{aligned}
X_x &= \frac{2x}{a} \\
X_y &= 2y \\
Y_x &= -\frac{y}{x^2 y^2 + a} \\
Y_y &= -\frac{x}{a\left(1 + \frac{x^2 y^2}{a}\right)}
\end{aligned}$$

Hence (2) becomes

$$\frac{dY}{dX} = \frac{-\frac{y}{x^2y^2+a} + \left(-\frac{x}{a\left(1+\frac{x^2y^2}{a}\right)}\right)\omega}{\frac{2x}{a} + 2y\omega}$$

But $X = \frac{x^2+ay^2}{a}$. The above becomes

$$\frac{dY}{dX} = \frac{-\frac{y}{aX} + \left(-\frac{x}{a\left(1+\frac{x^2y^2}{a}\right)}\right)\omega}{\frac{2x}{a} + 2y\omega}$$

To finish. Another hard part of this Lie method is to convert back $\frac{dS}{dR} = \frac{S_x+S_y\omega}{R_x+R_y\omega}$ so that the RHS is only a function of R. Need to find a robust way to do this. This is now a weak point in my program as I have few ode's that it can't do it

1.13 Alternative form for the similarity condition PDE

This section shows how to obtain eq. (8) in paper "Computer Algebra Solving of First Order ODEs Using Symmetry Methods" 1996 by Durate, Terrab, Mota. Which is an alternative equation to solve instead of the main Lie condition for symmetry we were looking at above.

Starting with the main linearized symmetry pde

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{14}$$

Assuming ansatz

$$\eta = \xi\omega + \chi \tag{A}$$

Hence

$$\begin{aligned}\eta_x &= \xi_x\omega + \xi\omega_x + \chi_x \\ \eta_y &= \xi_y\omega + \xi\omega_y + \chi_y\end{aligned}$$

Then (14) becomes

$$\begin{aligned}(\xi_x\omega + \xi\omega_x + \chi_x) + \omega((\xi_y\omega + \xi\omega_y + \chi_y) - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y(\xi\omega + \chi) &= 0 \\ \xi_x\omega + \xi\omega_x + \chi_x + \xi_y\omega^2 + \xi\omega_y\omega + \chi_y\omega - \omega\xi_x - \omega^2\xi_y - \omega_x\xi - \xi\omega\omega_y - \omega_y\chi &= 0 \\ \xi_x\omega + \chi_x + \xi_y\omega^2 + \xi\omega_y\omega + \chi_y\omega - \omega\xi_x - \omega^2\xi_y - \xi\omega\omega_y - \omega_y\chi &= 0 \\ \chi_x + \xi_y\omega^2 + \xi\omega_y\omega + \chi_y\omega - \omega^2\xi_y - \xi\omega\omega_y - \omega_y\chi &= 0 \\ \chi_x + \xi\omega_y\omega + \chi_y\omega - \xi\omega\omega_y - \omega_y\chi &= 0\end{aligned}$$

Or

$$\chi_x + \chi_y\omega - \omega_y\chi = 0 \tag{1}$$

And hence (1) is now solved for $\chi(x,y)$. If we are able to find χ then we can use the ansatz $\eta = \xi\omega + \chi$. This leaves only one unknown ξ. The paper does not explain how to solve for this, ξ, which I assume is by using (14) again. The paper only said

The knowledge of χ, in turn, allows one to set ξ and η as desired using (A)

Which is not too clear how in practice this is done. I need to work an example showing this. The paper says that (1) is solved for $\chi(x,y)$ by using bivariate polynomial ansatz. The degree can be set by a user, or Maple internally determines this.

2 Second order ODE

Local contents

2.1 Linearized PDE of the similarity condition

Obtaining the linearized PDE of the similarity condition for second order ode, which is used to solve for ξ, η follows similar method as given earlier for the first order ode. The difference is that instead of $y' = \omega(x,y)$ the ode now $y'' = \omega(x,y,y')$.

$$\begin{aligned} \frac{d^2y}{dx^2} &= \omega\left(x, y, \frac{dy}{dx}\right) \\ y'' &= \omega(x,y,y') \end{aligned} \tag{A}$$

The linearized similarity condition for second order ode when $\omega = 0$ is

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})\, y' + (\eta_{yy} - 2\xi_{xy})\,(y')^2 - \xi_{yy}(y')^3 = 0$$

Which is polynomial in y' hence all the coefficients must be zero giving

$$\begin{aligned} 2\eta_{xy} - \xi_{xx} &= 0 \\ \eta_{yy} - 2\xi_{xy} &= 0 \\ \xi_{yy} &= 0 \\ \eta_{xx} &= 0 \end{aligned}$$

And for general $\omega(x,y,y')$, the linearized similarity condition is

$$-\eta\omega_y + (-3y'\xi_y - 2\xi_x + \eta_y)\,\omega - \xi\omega_x + \left(-y'\eta_y + (y')^2\,\xi_y + y'\xi_x - \eta_x\right)\omega_{y'} + \eta_{xx} - \xi_{yy}(y')^3 + (\eta_{yy} - 2\xi_{yx})\,(y')^2 + (2\eta$$

To continue

3 Analysis of Maple's symgen methods for finding symmetries

This section gives an overview of Maple's methods of solving for Lie symmetries. There are 16 total algorithms.

4 Notes, things to find out

1. Given an ODE $y'(x) = \omega(x, y)$ then we want to find nontrivial Lie symmetry. The condition for this is that
$$\eta(x, y) \neq \xi(x, y)\,\omega(x, y)$$
so any values for η, ξ must satisfies the above.

2. Can we always find ξ, η for non-trivial symmetry for first order ODE? When I tried some in Maple, it could not find symmetries for some first order ODE's. How does one check if nontrivial symmetry exist before trying to find one? For example $y' + y^3 + xy^2 = 0$ which is Abel ode type, Maple found no symmetry using all methods.

5 References

1. Symmetry Methods for Differential Equations. By PETER E. HYDON. CAMBRIDGE UNIVERSITY PRESS. 2000.

2. Symmetry and integration methods for differential equations. By Bluman and Anco. Springer publishing.

3. Introduction to symmetry analysis. By Brian J. Cantwell. See page 158 for table of known ξ, η per first ode type.

4. Using Lie Symmetry to solve first and second order Linear Differential Equation. Int. J. Adv. Appl. Math. andMech. 7(4) (2020) 91 – 99 (ISSN: 2347-2529).

5. Sym Mathematica software package. By Stilianos Dimas and Dimitris Tsoubelis.

6. SymmetryAnalysis Mathematica package. By Brian J. Cantwell.

7. Maple symgen and related commands in the ODEtools package.

8. Computer Algebra Solving of First Order ODEs Using Symmetry Methods. 1996 by Durate, Terrab, Mota. IF-UERJ-27/96

9. The Truth About Lie Symmetries: Solving Differential Equations With Symmetry Methods by Ruth A. Steinhour, 2013.

10. A practical course in differential equations and Mathematical modeling. By Nail H. Ibragimov. 2010. Chapter 6. Nonlinear ordinary differential equations.

11. Symmetry Analysis of Differential Equations by Daniel J. Arrigo.

12. Many other references, listed in my links page under computer algebra section.