1 Introduction

A short review for solving the beam problem in 2D is given. The deflection curve, bending moment and shear force diagrams are calculated for a beam subject to bending moment and shear force using direct stiffness method and finite elements method. The problem is solved first by finding the stiffness matrix using the direct method and then using the virtual work method.

2 Direct method

Starting with only one element beam subject to bending and shear forces. There are 4 nodal degrees of freedom. Rotation at the left and right nodes of the beam and transverse displacements at the left and right nodes. The following diagram shows the sign convention used for external forces. Moments are always positive when anti-clockwise direction and vertical forces are positive when in the positive $y$ direction.

The two nodes are numbered 1 and 2 from left to right. $M_1$ is the moment at the left node (node 1), $M_2$ is the moment at the right node (node 2). $V_1$ is the vertical force at the left node and $V_2$ is the vertical force at the right node.
The above diagram shows the signs for the applied forces directions when acting in the positive sense. Since this is a one dimensional problem the displacement field (the unknown being solved for) will be function of one independent variable which is the $x$ coordinate. The displacement field is in the vertical direction called $v(x)$. This is the vertical displacement of a point $x$ on the beam from the $x-axis$. The following diagram shows the notation used for the coordinates

Angular displacement at $x$ of the beam is then found using $\theta(x) = \frac{dv(x)}{dx}$. At the left node the degrees of freedom, or the displacements, are called $v_1, \theta_1$ and at the right node called $v_2, \theta_2$. At an arbitrary location $x$ in the beam, the vertical displacement is $v(x)$ and the rotation is $\theta(x)$. The following diagram shows the displacement field $v(x)$
In the direct method of finding the stiffness matrix, the forces at the ends of the beam are found directly by the use of beam theory. In beam theory the signs are different from is shown in the first diagram above. Therefore, some of the moment and shear forces obtained using beam theory ($M_B$ and $V_B$ in the diagram below) will have different signs when compared to the external forces. The signs are then adjusted to reflect the convention as shown in the diagram above using $M$ and $V$.

For example, the external moment $M_1$ is opposite in sign to $M_{B1}$ and reaction $V_2$ is opposite to $V_{B2}$. To illustrate this more, a diagram with both sign conventions is shown below.

The goal now is to obtain expressions for external loads $M_i$ and $R_i$ in the above diagram as functions of the displacements at the nodes $\{d\} = \{v_1, \theta_1, v_2, \theta_2\}^T$.

In other words, the goal is to obtain an expression of the form $\{p\} = [K] \{d\}$ where $[K]$ is the stiffness matrix, $\{p\} = \{V_1, M_1, R_2, M_2\}^T$ is the nodal forces or load vector, $\{d\}$ is the nodal displacement vector. In this case $[K]$ will be a $4 \times 4$ matrix and $\{p\}$ is a $4 \times 1$ vector and $\{d\}$ is a $4 \times 1$ vector.

Starting with $V_1$, it is in the same direction as shear force $V_{B1}$. But $V_{B1} = \frac{dM_{B1}}{dx}$ hence

$$V_1 = \frac{dM_{B1}}{dx}$$

But $M_{B1} = -\sigma(x) \frac{L}{y}$, (from beam theory), hence

$$V_1 = -\frac{I}{y} \frac{d\sigma(x)}{dx}$$

But $\sigma(x) = E\varepsilon(x)$ and $\varepsilon(x) = \frac{-y}{\rho}$ where $\rho$ is radius of curvature, therefore

$$V_1 = EI \frac{d}{dx} \left(\frac{1}{\rho}\right)$$
But \( \frac{1}{\rho} = \frac{\left(\frac{d^2u}{dx^2}\right)}{\left(1+\left(\frac{du}{dx}\right)^2\right)^{3/2}} \) and for small angle of deflection \( \frac{du}{dx} \ll 1 \) hence \( \frac{1}{\rho} = \left(\frac{d^2u}{dx^2}\right) \), and the above becomes

\[
V_1 = EI \frac{d^3u(x)}{dx^3}
\]

Before continuing, the following diagram illustrates the above derivation. This comes from beam theory.

Now \( M_1 \) will be obtained. \( M_1 \) is in the opposite sense of bending moment \( M_{B1} \) hence a negative sign is added giving \( M_1 = -M_{B1} \). But \( M_{B1} = -\sigma(x) \frac{I}{y} \) hence

\[
M_1 = \sigma(x) \frac{I}{y} \\
= E\varepsilon(x) \frac{I}{y} \\
= E \left( -\frac{y}{\rho} \right) \frac{I}{y} \\
= -EI \left( \frac{1}{\rho} \right) \\
= -EI \frac{d^2w}{dx^2}
\]

Now \( V_2 \) is calculated. It is in the opposite sense of shear force \( V_{B2} \), hence a negative sign is added giving

\[
V_2 = -V_{B2} = -\frac{dM_{B2}}{dx}
\]
But $M_{B2} = -\sigma (x) \frac{I}{y}$, hence

$$V_2 = \frac{I}{y} \frac{d\sigma (x)}{dx}$$

But $\sigma (x) = E\varepsilon (x)$ and $\varepsilon (x) = \frac{y}{\rho}$ where $\rho$ is radius of curvature therefore

$$V_2 = -EI \frac{d}{dx} \left( \frac{1}{\rho} \right)$$

But $\frac{1}{\rho} = \left( \frac{\frac{d^2 y}{dx^2}}{1 + \left( \frac{dy}{dx} \right)^2} \right)^{3/2}$ and for small angle of deflection $\frac{dw}{dx} \ll 1$ hence $\frac{1}{\rho} = \left( \frac{d^2 u}{dx^2} \right)$ and the above becomes

$$V_2 = -EI \frac{d^3 u (x)}{dx^3}$$

Finally $M_2$ is in the same direction as $M_{B2}$ so no sign change is needed. $M_{B2} = -\sigma (x) \frac{I}{y}$ hence

$$M_2 = -\sigma (x) \frac{I}{y}$$

$$= -E\varepsilon (x) \frac{I}{y}$$

$$= -E \left( \frac{y}{\rho} \right) \frac{I}{y}$$

$$= EI \left( \frac{1}{\rho} \right)$$

$$= EI \frac{d^2 u}{dx^2}$$

The following is a summary of what was found so far. Notice that the above expressions are evaluates at $x = 0$ and $x = L$ accordingly to obtain the nodal end forces vector $\{p\}$

$$\{p\} = \begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} EI \frac{d^3 u (x)}{dx^3} \bigg|_{x=0} \\ -EI \frac{d^2 u}{dx^2} \bigg|_{x=0} \\ -EI \frac{d^2 u (x)}{dx^2} \bigg|_{x=L} \\ EI \frac{d^2 u}{dx^2} \bigg|_{x=L} \end{bmatrix} \tag{1}$$

Now the RHS of the above is expressed as function of nodal displacements $v_1, \theta_1, v_2, \theta_2$. To do that, the field displacement $v (x)$ which is the transverse displacement of the beam is assumed to be a polynomial in $x$ of degree 3. Hence

$$v (x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\theta (x) = \frac{dv (x)}{dx} = a_1 + 2a_2 x + 3a_3 x^2 \tag{A}$$

Polynomial of degree 3 was used since there are 4 degrees of freedom, and a minimum of 4 free parameters are needed. Hence

$$v_1 = v (x) \big|_{x=0} = a_0 \tag{2}$$
and

\[ \theta_1 = \theta(x)|_{x=0} = a_1 \quad (3) \]

Assuming the length of the beam is \( L \), then

\[ v_2 = v(x)|_{x=L} = a_0 + a_1L + a_2L^2 + a_3L^3 \quad (4) \]

and

\[ \theta_2 = \theta(x)|_{x=L} = a_1 + 2a_2L + 3a_3L^2 \quad (5) \]

eqs (2-5) gives

\[
\{d\} = \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_0 + a_1L + a_2L^2 + a_3L^3 \\ a_1 + 2a_2L + 3a_3L^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}
\]

Solving for \( a_i \) gives

\[
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{3}{L^2} & \frac{1}{L^2} & 0 & 0 \\ \frac{3}{L^2} & \frac{1}{L^2} & -\frac{2}{L^3} & \frac{1}{L^3} \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}
= \begin{pmatrix} \frac{3}{L^2}v_2 - \frac{1}{L^2}\theta_2 - \frac{3}{L^3}v_1 - \frac{2}{L^3}\theta_1 \\ \frac{3}{L^2}\theta_1 + \frac{1}{L^2}\theta_2 + \frac{3}{L^3}v_1 - \frac{2}{L^3}v_2 \end{pmatrix}
\]

Hence \( v(x) \) the field displacement function from eq. (A) can now be written as function of the nodal displacements

\[
v(x) = a_0 + a_1x + a_2x^2 + a_3x^3
= v_1 + \theta_1x + \left( \frac{3}{L^2}v_2 - \frac{1}{L}\theta_2 - \frac{3}{L^3}v_1 - \frac{2}{L^3}\theta_1 \right) x^2 + \left( \frac{1}{L^2}\theta_1 + \frac{1}{L^2}\theta_2 + \frac{2}{L^3}v_1 - \frac{2}{L^3}v_2 \right) x^3
= v_1 + x\theta_1 - 2\frac{x^2}{L}\theta_1 + \frac{x^3}{L^2}\theta_1 - \frac{x^2}{L}\theta_2 + \frac{x^3}{L^2}\theta_2 - 3\frac{x^2}{L^2}v_1 + 2\frac{x^3}{L^3}v_1 + 3\frac{x^2}{L^2}v_2 - 2\frac{x^3}{L^3}v_2
\]

or in matrix form

\[
v(x) = \begin{pmatrix} 1 & -3\frac{x^2}{L^2} + 2\frac{x^3}{L^3} \\ -2\frac{x^2}{L} + \frac{x^3}{L^2} \\ 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3} \\ -\frac{x^2}{L} + \frac{x^3}{L^3} \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}
= \begin{pmatrix} \frac{1}{L^2}(L^3 - 3Lx^2 + 2x^3) \\ \frac{1}{L^2}(L^2x - 2Lx^2 + x^3) \\ \frac{1}{L^2}(3Lx^2 - 2x^3) \\ \frac{1}{L^2}(-Lx^2 + x^3) \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}
\]
The above can be written as

\[
v(x) = \left( N_1(x) \ N_2(x) \ N_3(x) \ N_4(x) \right) \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \tag{5A}
\]

\[
v(x) = [N] \{d\}
\]

Where the \(N_i\) are called the shape functions. The shape functions are

\[
\begin{pmatrix}
N_1(x) \\
N_2(x) \\
N_3(x) \\
N_4(x)
\end{pmatrix} = \begin{pmatrix}
\frac{1}{L_3} \left( L^3 - 3Lx^2 + 2x^3 \right) \\
\frac{1}{L_2} \left( L^2x - 2Lx^2 + x^3 \right) \\
\frac{1}{L_3} \left( 3Lx^2 - 2x^3 \right) \\
\frac{1}{L_2} \left( -Lx^2 + x^3 \right)
\end{pmatrix}
\]

Notice that

\(N_1(0) = 1\)

and

\(N_1(L) = 0\)

as expected. Also

\[
\left. \frac{dN_2(x)}{dx} \right|_{x=0} = \frac{1}{L^2} \left( L^2 - 4Lx + 3x^2 \right)_{x=0} = 1
\]

and

\[
\left. \frac{dN_2(x)}{dx} \right|_{x=L} = \frac{1}{L^2} \left( L^2 - 4Lx + 3x^2 \right)_{x=L} = 0
\]

and

\(N_3(0) = 0\)

and

\(N_3(L) = 1\)

and

\[
\left. \frac{dN_4(x)}{dx} \right|_{x=0} = \frac{1}{L^2} \left( -2Lx + 3x^2 \right)_{x=0} = 0
\]

and

\[
\left. \frac{dN_4(x)}{dx} \right|_{x=L} = \frac{1}{L^2} \left( -2Lx + 3x^2 \right)_{x=L} = 1
\]

The shape functions have thus been verified.

The stiffness matrix is now found by substituting eq (5A) into eq. (1), repeated below

\[
\{p\} = \begin{pmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{pmatrix} = \begin{pmatrix} EI \left. \frac{d^2v(x)}{dx^2} \right|_{x=0} \\ -EI \left. \frac{d^2v}{dx^2} \right|_{x=0} \\ -EI \left. \frac{d^3v(x)}{dx^3} \right|_{x=0} \\ EI \left. \frac{d^2v}{dx^2} \right|_{x=L} \end{pmatrix}
\]

Hence

\[
\begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}
\]
\[
\{ p \} = \begin{pmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{pmatrix} = \begin{pmatrix}
EI \frac{d^3}{dx^3} (N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2) \\
-EI \frac{d^2}{dx^2} (N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2) \\
-\frac{d^2}{dx^2} (N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2) \\
\frac{d}{dx} (N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2)
\end{pmatrix}
\]

But
\[
\frac{d^3}{dx^3} N_1 (x) = \frac{1}{L^3} \frac{d^3}{dx^3} (L^3 - 3Lx^2 + 2x^3) = \frac{12}{L^3}
\]
and
\[
\frac{d^3}{dx^3} N_2 (x) = \frac{1}{L^3} \frac{d^3}{dx^3} (L^2 x - 2Lx^2 + x^3) = \frac{6}{L^2}
\]
and
\[
\frac{d^3}{dx^3} N_3 (x) = \frac{1}{L^3} \frac{d^3}{dx^3} (3Lx^2 - 2x^3) = \frac{-12}{L^3}
\]
and
\[
\frac{d^3}{dx^3} N_4 (x) = \frac{1}{L^3} \frac{d^3}{dx^3} (-Lx^2 + x^3) = \frac{6}{L^2}
\]
and now do the second derivatives
\[
\frac{d^2}{dx^2} N_1 (x) = \frac{1}{L^3} \frac{d^2}{dx^2} (L^3 - 3Lx^2 + 2x^3) = \frac{1}{L^3} (12x - 6L)
\]
and
\[
\frac{d^2}{dx^2} N_2 (x) = \frac{1}{L^2} \frac{d^2}{dx^2} (L^2 x - 2Lx^2 + x^3) = \frac{1}{L^2} (6x - 4L)
\]
and
\[
\frac{d^2}{dx^2} N_3 (x) = \frac{1}{L^3} \frac{d^2}{dx^2} (3Lx^2 - 2x^3) = \frac{1}{L^3} (6L - 12x)
\]
and

\[
\frac{d^2}{dx^2} N_4(x) = \frac{1}{L^2} \frac{d^2}{dx^2} \left( -Lx^2 + x^3 \right)
= \frac{1}{L^2} \left( 6x - 2L \right)
\]

Hence eq (6) becomes

\[
\{ p \} = \begin{pmatrix}
V_1 \\
M_1 \\
V_2 \\
M_2
\end{pmatrix} = \begin{pmatrix}
EI \frac{d^3}{dx^3} (N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2) \\
-\frac{EI}{L^2} (N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2) \\
-\frac{EI}{L^2} (N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2) \\
EI \frac{d^3}{dx^3} (N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2)
\end{pmatrix} \bigg|_{x=0}^{x=L}
\]

\[= \begin{pmatrix}
-\frac{6}{L^2} \left( 12x - 6L \right) v_1 + \frac{6}{L^2} \left( 6x - 4L \right) \theta_1 + \frac{1}{L^2} \left( 6L - 12x \right) v_2 + \frac{1}{L^2} \left( 6x - 2L \right) \theta_2 \\
-\frac{6}{L^2} \left( 12x - 6L \right) v_1 + \frac{6}{L^2} \left( 6x - 4L \right) \theta_1 + \frac{1}{L^2} \left( 6L - 12x \right) v_2 + \frac{1}{L^2} \left( 6x - 2L \right) \theta_2 \\
\frac{6}{L^2} \left( 12x - 6L \right) v_1 + \frac{6}{L^2} \left( 6x - 4L \right) \theta_1 + \frac{1}{L^2} \left( 6L - 12x \right) v_2 + \frac{1}{L^2} \left( 6x - 2L \right) \theta_2 \\
\frac{6}{L^2} \left( 12x - 6L \right) v_1 + \frac{6}{L^2} \left( 6x - 4L \right) \theta_1 + \frac{1}{L^2} \left( 6L - 12x \right) v_2 + \frac{1}{L^2} \left( 6x - 2L \right) \theta_2
\end{pmatrix}
\]

\[
= \frac{EI}{L^3} \begin{pmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
(12x - 6L) & L & (6L - 12x) & (6L - 2L)
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
\theta_1 \\
\theta_2
\end{pmatrix}
\]

or in matrix form, after evaluating the expressions above for \(x = L\) and \(x = 0\)

\[
\begin{pmatrix}
V_1 \\
M_1 \\
V_2 \\
M_2
\end{pmatrix} = \frac{EI}{L^3} \begin{pmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
\theta_1 \\
\theta_2
\end{pmatrix}
\]

The above now is in the form

\[
\{ p \} = [K] \{ d \}
\]

Hence the stiffness matrix is

\[
[K] = \frac{EI}{L^3} \begin{pmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{pmatrix}
\]

Knowing the stiffness matrix means knowing the nodal displacements \(\{d\}\) given the forces at the nodes. The power of the finite element method now comes after all the nodal displacements
\(v_1, \theta_1, v_2, \theta_2\) are calculated by solving \(\{p\} = [K] \{d\}\) because the polynomial \(v(x)\) is now completely determined and hence \(v(x)\) and \(\theta(x)\) can now be evaluated for any \(x\) along the beam and not just at its end nodes. Eq. 5A above can now be used to find the displacement \(v(x)\) and \(\theta(x)\) everywhere.

\[
v(x) = [N] \{d\} \\
v(x) = (N_1(x) \quad N_2(x) \quad N_3(x) \quad N_4(x)) \\
\begin{bmatrix}
v_1 \\
\theta_1 \\
v_2 \\
\theta_2
\end{bmatrix}
\]

Therefore, these are the steps to obtain \(v(x)\):

1. Find an expression for \([K]\)
2. Solve \(\{p\} = [K] \{d\}\) for \(\{d\}\)
3. Calculate \(v(x) = [N] \{d\}\) by assuming \(v(x)\) is a polynomial. This gives the displacement \(v(x)\) to use to evaluate the transverse displacement anywhere on the beam and not just at the end nodes.
4. Obtain \(\theta(x) = \frac{dv(x)}{dx} = \frac{d}{dx} [N] \{d\}\) to evaluate the rotation of the beam any where and not just at the end nodes.
5. Obtain strain \(\epsilon(x) = -y[B] \{d\}\) where \([B]\) is the gradient matrix \([B] = \frac{d^2}{dx^2} [N]\)
6. Obtain stress from \(\sigma = E\epsilon = -Ey[B] \{d\}\)
7. Obtain the bending moment diagram from \(M(x) = EI[B] \{d\}\)
8. Obtain shear force diagram from \(V(x) = \frac{d}{dx} M(x)\)

### 2.1 Examples using the direct beam stiffness matrix

Now the beam stiffness matrix is used to solve few beam problems. Starting with simple one span beam

#### 2.1.1 Example 1

A one span beam, a cantilever beam of length \(L\), with point load \(P\) at the free end.

The first step is to make the free body diagram and show all moments and forces at the nodes.
$P$ is the given force. $M_2 = 0$ since there is no external moment at the right end. Hence \( \{p\} = [K] \{d\} \) for this system is

\[
\begin{pmatrix} R \\ M_1 \\ -P \\ 0 \end{pmatrix} = \frac{EI}{L^3} \begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}
\]

Now is an important step. The known end displacements from boundary conditions is substituted into \( \{d\} \), and the corresponding row and columns from the above system of equations are removed. Boundary conditions indicates that there is no rotation on the left end (since fixed). Hence \( v_1 = 0 \) and \( \theta_1 = 0 \). Hence the only unknowns are \( v_2 \) and \( \theta_2 \). Therefore the first and the second rows and columns are removed, giving

\[
\begin{pmatrix} -P \\ 0 \end{pmatrix} = \frac{EI}{L^3} \begin{pmatrix} 12 & -6L \\ -6L & 4L^2 \end{pmatrix} \begin{pmatrix} v_2 \\ \theta_2 \end{pmatrix}
\]

Now the above is solved for \( \begin{pmatrix} v_2 \\ \theta_2 \end{pmatrix} \). Let \( E = 30 \times 10^6 \text{ psi} \) (steel), \( I = 57 \text{ in}^4 \), \( L = 144 \text{ in} \), and \( P = 400 \text{ lb} \), hence

\[
P=400; \\
L=144; \\
E=30*10^6; \\
I=57.1; \\
A=(E*I/L^3)*[12 6*L -12 6*L; \\
6*L 4*L^2 -6*L 2*L^2; \\
-12 -6*L 12 -6*L; \\
6*L 2*L^2 -6*L 4*L^2];
\]

\[
\text{load}=[P;P*L;-P;0];
\]

\[
x=A(3:end,3:end)\text{load}(3:end)
\]

which gives

\[
1\text{Instead of removing rows/columns for known boundary conditions, we can also just put a 1 on the diagonal of the stiffness matrix for that boundary conditions. I will do this example again using this method.}
\]
Therefore the vertical displacement at the right end is \( v_2 = 0.2324 \) inches (downwards) and \( \theta_2 = -0.0024 \) radians. Now that all nodal displacements are found, the field displacement function is completely determined.

\[
\{d\} = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.2324 \\ -0.0024 \end{bmatrix}
\]

Hence from eq. 5A

\[
v(x) = [N] \{d\}
\]

\[
= \begin{pmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.2324 \\ -0.0024 \end{bmatrix}
\]

\[
= \left( \frac{1}{L^3} (L^3 - 3Lx^2 + 2x^3) \right) \left( \frac{1}{L^2} (L^2x - 2Lx^2 + x^3) \right) \left( \frac{1}{L^2} (3Lx^2 - 2x^3) \right) \left( \frac{1}{L^2} (-Lx^2 + x^3) \right) \begin{bmatrix} 0 \\ -0.2324 \\ -0.0024 \end{bmatrix}
\]

But \( L = 144 \) inches, hence

\[
v(x) = 3.9920 \times 10^{-8} x^3 - 1.6956 \times 10^{-5} x^2
\]

To verify, let \( x = 144 \) in the above

\[
v(x = 144) = 3.9920 \times 10^{-8} 144^3 - 1.6956 \times 10^{-5} 144^2
\]

\[
= -0.23240
\]

Here is a plot of the deflection curve for the beam

\[
v=@(x) 3.992*10^{-8}*x.^3-1.6956*10^{-5}*x.\ ^2
x=0:0.1:144;
pplot(x,v(x),'r-','LineWidth',2);
ylim([-0.8 0.3]);
ttitle('beam deflection curve');
xlabel('x inch'); ylabel('deflection inch');
ggrid
\]
Now instead of removing rows/columns for the known boundary conditions, a 1 is put on the diagonal. Starting again with
\[
\begin{pmatrix}
R \\
M_1 \\
-P \\
0
\end{pmatrix} = \frac{EI}{L^3} \begin{pmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{pmatrix} \begin{pmatrix}
v_1 \\
v_2 \\
\theta_1 \\
\theta_2
\end{pmatrix}
\]

since \(v_1 = 0\) and \(\theta_1 = 0\), then
\[
\begin{pmatrix}
0 \\
0 \\
-P \\
0
\end{pmatrix} = \frac{EI}{L^3} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 12 & -6L \\
0 & 0 & -6L & 4L^2
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
v_2 \\
\theta_2
\end{pmatrix}
\]

And now the above system is solved as before. \(E = 30 \times 10^6\) psi, \(I = 57\) in\(^4\), \(L = 144\) in,

\[P = 400 lb\]

\begin{verbatim}
P=400; L=144; E=30*10^6; I=57.1; A=(E*I/L^3)*[12 6*L -12 6*L; 6*L 4*L^2 -6*L 2*L^2; -12 -6*L 12 -6*L; 6*L 2*L^2 -6*L 4*L^2]; load=[0;0;-P;0]; %put zeros for known B.C. A(:,1)=0; A(1,:)=0; A(1,1)=1; %put 1 on diagonal A(:,2)=0; A(2,:)=0; A(2,2)=1; %put 1 on diagonal A
\end{verbatim}
The same solution is obtained as before, but without the need to remove rows/column from the stiffness matrix. This method might be easier for programming than the first method of removing rows/columns.

The rest now is the same as was done earlier and will not be repeated.

2.1.2 Example 2

The same example as above, but the vertical load $P$ now is placed in the middle of the beam.

In using stiffness method, all loads must be on the nodes. The vector $\{p\}$ is nodal forces vector. Hence equivalent nodal loads are found for the load in the middle of the beam. The equivalent loading is the following.
Therefore, the problem is as if it was the following problem

Now that equivalent loading is in place, we continue as before. Make a free body diagram showing all loads (including reaction forces)

Now the stiffness equation is written as

\[
\{p\} = [K] \{d\}
\]

\[
\begin{bmatrix}
R - P/2 \\
M_1 - PL/8 \\
-P/2 \\
PL/8
\end{bmatrix} = \begin{bmatrix}
EI \\
-12 \\
6L \\
-12
\end{bmatrix} \begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{bmatrix} \begin{bmatrix}
v_1 \\
v_2 \\
\theta_1 \\
\theta_2
\end{bmatrix}
\]

There is no need to determine \(R\) and \(M_1\) at this point since these rows will be removed due to boundary conditions \(v_1 = 0\) and \(\theta_1 = 0\) and hence those quantities are not needed to solve the equations. Remember that rows and columns are removed for the known boundary displacements before solving \(\{p\} = [K] \{d\}\). Hence, after removing the first two rows and columns, the system becomes
\[
\begin{bmatrix}
-P/2 \\
PL/8
\end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} \begin{bmatrix} v_2 \\ \theta_2 \end{bmatrix}
\]

Now the above is solved for \[\{v_2, \theta_2\}\] using the same numerical values for \[P, E, I, L\] as in the first example.

\[
P=400;
L=144;
E=30*10^{-6};
I=57.1;
\]

\[
\]

\[
load=[-P/2;P*L/8];
x=A(3:end,3:end)\load
\]

\[
x =
\begin{bmatrix}
-0.072630472854641 \\
-0.000605253940455
\end{bmatrix}
\]

Therefore

\[
\{d\} = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.072630472854641 \\ -0.000605253940455 \end{bmatrix}
\]

This is enough to obtain \(v(x)\) as before. Now the reactions \(R\) and \(M_1\) can be determined if needed. Going back to the full \(\{p\} = [K]\{d\}\), results in

\[
\begin{bmatrix}
R - P/2 \\
M_1 - PL/8 \\
-P/2 \\
PL/8
\end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.072630472854641 \\ -0.000605253940455 \end{bmatrix}
\]

\[
= \frac{EI}{L^3} \begin{bmatrix} 0.87157 - 3.6315 \times 10^{-3}L \\ 0.43578L - 1.2105 \times 10^{-3}L^2 \\ 3.6315 \times 10^{-3}L - 0.87157 \\ 0.43578L - 2.421 \times 10^{-3}L^2 \end{bmatrix}
\]

Hence the first equation becomes
\[ R - \frac{P}{2} = \frac{EI}{L^3} \left( 0.87157 - 3.6315 \times 10^{-3}L \right) \]

and since \( E = 30 \times 10^6 \text{psi} \) (steel) and \( I = 57 \text{ in}^4 \) and \( L = 144 \text{ in} \) and \( P = 400 \text{lb} \), then
\[ R = \frac{(30 \times 10^6)^{57}}{144^3} \left( 0.87157 - 3.6315 \times 10^{-3} \times 144 \right) + 400/2 \text{ hence} \]
\[ R = 400 \text{ lb} \]

and \( M_1 - PL/8 = \frac{EI}{L^3} \left( 0.43578L - 1.2105 \times 10^{-3}L^2 \right) + PL/8 \), hence
\[ M_1 = 28762 \text{ lbft} \]

Now all nodal reactions are found and the displacement field is found. The deflection curve can be plotted.

\[
v(x) = [N] \{d\}
\]
\[
v(x) = (N_1(x) \quad N_2(x) \quad N_3(x) \quad N_4(x)) \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = (N_1(x) \quad N_2(x) \quad N_3(x) \quad N_4(x)) \begin{bmatrix} 0 \\ 0 \\ -0.07263047285461 \\ -0.000605253940455 \end{bmatrix}
\]
\[
= \frac{1}{L^3} (3Lx^2 - 2x^3) \quad \frac{1}{L^3} (-Lx^2 + x^3) \begin{bmatrix} -0.07263047285461 \\ -0.000605253940455 \end{bmatrix}
\]
\[
= \frac{6.0525 \times 10^{-4}}{L^2} (Lx^2 - x^3) + \frac{0.07263}{L^3} (2x^3 - 3Lx^2)
\]

Since \( L = 144 \text{ inches} \), then
\[
v(x) = \frac{6.0525 \times 10^{-4}}{144^2} ((144)x^2 - x^3) + \frac{0.07263}{144^3} (2x^3 - 3(144)x^2)
\]
\[
= 1.9459 \times 10^{-8}x^3 - 6.3047 \times 10^{-6}x^2
\]

The following is the plot

```matlab
clear all; close all;
v=@(x) 1.9459*10^-8*x.^3-6.3047*10^-6*x.^2
x=0:0.1:144;
plot(x,v(x), 'r-', 'LineWidth',2);
ylim([-0.8 0.3]);
title('beam deflection curve');
xlabel('x inch'); ylabel('deflection inch');
grid
```
2.1.3 Example 3

Now assume the beam is fixed on the left end as above, but simply supported on the right end, and the vertical load $P$ now at distance $a$ from the left end and distance $b$ from the right end, and a uniform distributed load of density $m$ lb/in on the beam.

Using the following values: $a = 0.625L, b = 0.375L, E = 30 \times 10^6$ psi (steel), $I = 57$ in$^4$, $L = 144$ in, $P = 1000$ lb, $m = 200$ lb/in.

In the above, the left end reaction forces are shown as $R_1$ and moment reaction as $M_1$ and the reaction at the right end as $R_2$. Starting by finding equivalent loads for the point load $P$ and equivalent loads for the uniform distributed load $m$. All external loads must be transferred to the nodes for the stiffness method to work. Equivalent load for the above point load is
And equivalent load for the uniform distributed loading is

Hence, using free body diagram, with all the loads on it gives the following diagram (In this diagram $M$ is the reaction moment and $R_1, R_2$ are the reaction forces)

Now that all loads are on the nodes, the stiffness equation is applied
\[ \{p\} = [K] \{d\} \]

\[
\begin{align*}
R_1 - \frac{Pb^2(L+2a)}{L^2} & - \frac{mL^2}{2} = M = \frac{Pab^2}{L^2} - \frac{mL^2}{12} \\
R_2 - \frac{Pa^2(L+2b)}{L^2} & + \frac{mL^2}{12} \end{align*}
\]

\[
= \begin{bmatrix} \frac{12}{E} & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}
\]

Now boundary conditions are applied. \( v_1 = 0, \theta_1 = 0, v_2 = 0 \) hence the first, second and third rows/columns are removed giving

\[
\frac{Pa^2b}{L^2} + \frac{mL^2}{12} = \frac{EI}{L^3} 4L^2 \theta_2
\]

Hence

\[
\theta_2 = \left( \frac{Pa^2b}{L^2} + \frac{mL^2}{12} \right) \left( \frac{L}{4EI} \right)
\]

Substituting numerical values for the above as given at the top of the problem results in

\[
\theta_2 = \left( \frac{1000}{144} \right) \left( \frac{0.625 (144)^2 (0.375 (144))}{(144)^2} + \frac{200}{12} \right) \left( \frac{144}{4(30 \times 10^6)(57)} \right)
\]

\[ = 7.7199 \times 10^{-3} \text{rad} \]

Hence the field displacement \( u(x) \) is now found

\[
v(x) = [N] \{d\}
\]

\[
v(x) = (N_1(x) \quad N_2(x) \quad N_3(x) \quad N_4(x)) \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}
\]

\[
= (N_1(x) \quad N_2(x) \quad N_3(x) \quad N_4(x)) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.7199 \times 10^{-3} \end{bmatrix}
\]

\[
= \frac{1}{L^2} (-Lx^2 + x^3) (7.7199 \times 10^{-3})
\]

\[
= 3.7229 \times 10^{-7}x^3 - 5.361 \times 10^{-5}x^2
\]

And a plot of the deflection curve is
clear all; close all;
v=@(x) 3.7229*10^-7*x.^3-5.361*10^-5*x.^2
x=0:0.1:144;
plot(x,v(x), 'r-','LineWidth',2);
ylim([-0.5 0.3]);
title('beam deflection curve');
xlabel('x inch'); ylabel('deflection inch');
grid

2.2 Adding more elements

Finite elements generated displacements are smaller in value than the actual analytical values. To improve the accuracy, more elements are added. To add more element, the beam is divided into 2, 3, 4 and more beam elements. To show how this work, example 3 above is solved again using 2 elements. It will be found that displacement field \( v(x) \) becomes more accurate (It was compared this with an exact solution based on using the beam 4th order differential equation and found to be almost the same with only 2 elements)

2.2.1 Example 3 redone with 2 elements

The first step is to divide the beam into 2 element and number the degrees of freedom and global nodes as follows
There will be 6 total degrees of freedom. 2 at each node. Hence the stiffness matrix for the whole beam (including both elements) will be 6 by 6. For each element however, the same stiffness matrix will be used as above and that will remain 4 by 4.

The stiffness matrix for each element is found then the global stiffness matrix is constructed, then \( \{p_{\text{global}}\} = [K_{\text{global}}] \{d_{\text{global}}\} \) is solved as before. The first step is to move all loads to the nodes as before. This is done for each element. The formulas for equivalent loads remain the same, but \( L \) becomes \( L/2 \). The following diagram show the equivalent loading for \( P \).
The equivalent loading for distributed load \( m \) is

\[
\begin{align*}
\frac{m(L/2)}{2} & \quad \frac{m(L/2)}{2} \\
\frac{m(L/2)^2}{12} & \quad \frac{m(L/2)^2}{12} \\
\frac{m(L/2)^2}{12} & \quad \frac{m(L/2)^2}{12}
\end{align*}
\]

Now the above 2 diagrams are put together to show all equivalent loads with the original reaction forces to obtain the following diagram
Now \( \{p\} = [K] \{d\} \) for each element is constructed. Starting with the first element

\[
\begin{pmatrix}
R_1 - \frac{m(\frac{L}{2})}{2} \\
M_1 - \frac{m(\frac{L}{2})^2}{12} \\
\frac{Pb^2(\frac{L}{2}+2a)}{(L/2)^3} - \frac{m(\frac{L}{2})}{2} \\
\frac{m(\frac{L}{2})^2}{12} - \frac{Pab^2}{(L/2)^2} - \frac{m(\frac{L}{2})}{12}
\end{pmatrix}
= EI \frac{(L/2)^3}{(L/2)}
\begin{pmatrix}
12 & 6(\frac{L}{2})^2 & -12 & 6(\frac{L}{2})^2 \\
6(\frac{L}{2}) & -12 & -6(\frac{L}{2}) & 12 & -6(\frac{L}{2}) \\
6(\frac{L}{2}) & 2(\frac{L}{2})^2 & -6(\frac{L}{2}) & 4(\frac{L}{2})^2
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
\theta_1 \\
\theta_2
\end{pmatrix}
\]

And for the second element
As was mentioned above, another method is to remove the rows/columns which results in

\[
\begin{pmatrix}
-2 P b^2 \left( \frac{L}{2} + 2a \right) - 2 m \left( \frac{L}{2} \right)^2

= \frac{EI}{(\frac{L}{2})^3} \begin{pmatrix}
24 & 0 & 6 \left( \frac{L}{2} \right)^2 \\
0 & 8 \left( \frac{L}{2} \right)^2 & 2 \left( \frac{L}{2} \right)^2 \\
6 \left( \frac{L}{2} \right) & 2 \left( \frac{L}{2} \right)^2 & 4 \left( \frac{L}{2} \right)^2
\end{pmatrix}
\begin{pmatrix}
v_2 \\
\theta_2 \\
\theta_3
\end{pmatrix}
\]
giving the same solution.

Hence, there are 3 unknowns to solve for. Once there are solved, $v(x)$ for the first element and for the second element are determined. The following code displays the deflection curve for the above beam.

```matlab
clear all; close all;
P=400;
L=144;
E=30*10^6;
I=57.1;
m=200;
a=0.125*L;
b=0.375*L;

A=E*I/(L/2)^3*[1 0 0 0 0 0;
0 1 0 0 0 0;
0 0 24 0 0 6*L/2;
0 0 0 8*(L/2)^2 0 2*(L/2)^2;
0 0 0 0 1 0;
0 0 6*L/2 2*(L/2)^2 0 4*(L/2)^2];

load = [0;
0;
-(m*L/2) - 2*P*b^2*(L/2+2*a)/(L/2)^3;
-2*P*a*b^2/(L/2)^2;
0;
P*a^2*b/(L/2)^2+(m*(L/2)^2)/12]

sol=A\load
```
\[A =
\begin{bmatrix}
1.0e+08 * \\
0.0000 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.0000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0011 & 0 & 0 & 0.0198 \\
0 & 0 & 0 & 1.9033 & 0 & 0.4758 \\
0 & 0 & 0 & 0 & 0.0000 & 0 \\
0 & 0 & 0.0198 & 0.4758 & 0 & 0.9517
\end{bmatrix}
\]

\[\text{load} =
\begin{bmatrix}
0 \\
0 \\
-15075 \\
-8100 \\
0 \\
87750
\end{bmatrix}
\]

\[\text{sol} =
\begin{bmatrix}
0 \\
0 \\
-0.2735 \\
-0.0019 \\
0 \\
0.0076
\end{bmatrix}
\]

The above solution gives \(v_2 = -0.2735\) inch (downwards displacement) and \(\theta_2 = -0.0019\) radians and \(\theta_3 = 0.0076\) radians. Now the \(v(x)\) polynomial is found for each element

\[
v_{\text{elem}} (x) = \begin{pmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{pmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{8} (3 \left( \frac{L}{2} \right) x^2 - 2 x^3) \\
\frac{1}{8} \left( - \left( \frac{L}{2} \right)^2 x^2 + x^3 \right)
\end{pmatrix}
\begin{bmatrix}
-0.2735 \\
-0.0019
\end{bmatrix}
\]

\[
= \frac{2.188}{L^3} \left( 2 x^3 - \frac{3}{2} L x^2 \right) - \frac{0.0076}{L^2} \left( x^3 - \frac{1}{2} L x^2 \right)
\]

The above polynomial is the transverse deflection of the beam for the region \(0 \leq x \leq L/2\).
\[ v(x) \text{ for the second element is found in similar way} \]

\[
v_{\text{elem}2}(x) = \begin{pmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{pmatrix} \begin{bmatrix} v_2 \\ \theta_2 \\ 0 \\ \theta_3 \end{bmatrix}
\]

\[
= \begin{pmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{pmatrix} \begin{bmatrix} -0.2735 \\ -0.0019 \\ 0 \\ 0.0076 \end{bmatrix}
\]

\[
= \begin{pmatrix} \frac{1}{(\frac{L}{2})^4} \left( \left(\frac{L}{2}\right)^3 - 3 \left(\frac{L}{2}\right) x^2 + 2x^3 \right) - \frac{1}{(\frac{L}{2})^4} \left( \left(\frac{L}{2}\right)^2 x - 2 \left(\frac{L}{2}\right) x^2 + x^3 \right) - \frac{1}{(\frac{L}{2})^4} \left( - \left(\frac{L}{2}\right) x^2 + x^3 \right) \end{pmatrix}
\]

Hence

\[
v_{\text{elem}2}(x) = \frac{0.0304}{L^2} \left( x^3 - \frac{1}{2} L x^2 \right) - \frac{0.0076}{L^2} \left( \frac{1}{4} L^2 x - L x^2 + x^3 \right) - \frac{2.188}{L^3} \left( \frac{1}{8} L^3 - \frac{3}{2} L x^2 + x^3 \right)
\]

Which is valid for \( L/2 \leq x \leq L \). The following is a plot of the deflection curve using the above 2 equations.

When the above plot is compared to the case with one element, It can be seen the deflection is larger now. Comparing the above to the analytical solution shows that now the deflection is almost exactly the same as the analytical solution. Hence by using 2 elements instead of one element, the solution now agrees with the analytical solution.

The following diagram shows the deflection curve of problem 3 above when using one element and two elements on the same plot to help illustrate the difference in the result more clearly.
The analytical deflection for the beam in problem 3 above (fixed on the left and simply supported at the right end) when there is uniformly loaded with \( w \) lbs per unit length is given by

\[
v(x) = -\frac{wx^2}{48EI} (3L^2 - 5Lx + 2x^2),
\]

while the analytical deflection for the same beam but when there is a point load \( P \) at distance \( a \) from the left end is given by

\[
v(x) = -P(\langle L - a \rangle^3 (3L - x)x^2 + L^2(3(L - a)(L - x)x^2 + 2L(x - a)^3)),
\]

therefore, the analytical expression for deflection is given by the sum of the above expressions, giving

\[
v(x) = -\frac{wx^2}{48EI} (3L^2 - 5Lx + 2x^2) - P(\langle L - a \rangle^3 (3L - x)x^2 + L^2(3(L - a)(L - x)x^2 + 2L(x - a)^3))
\]

Where \( \langle x - a \rangle \) means it is zero when \( x-a \) is negative. In other words \( \langle x - a \rangle = (x - a) \text{UnitStep}(x - a) \)

The following diagram shows a plot of the analytical deflection with the 2 elements deflection calculated using Finite elements above.

![Deflection Curve Diagram](image-url)
In the above, the blue dashed curve is the analytical solution, and the red curve is the finite elements solution using 2 elements. It can be seen that the finite element solution for the deflection is now in a very good agreement with the analytical solution.

3 Generating shear and bending moments diagrams

After solving the problem using finite elements and obtaining the field displacement function \( v(x) \) as was shown in the above examples, the shear force and bending moments along the beam can be calculated. Since the bending moment is given by \( M(x) = -EI \frac{d^2v(x)}{dx^2} \) and shear force is given by \( V(x) = \frac{dM(x)}{dx} = -EI \frac{d^3v(x)}{dx^3} \) then these diagrams can now be readily plotted as shown below for example 3 above using the result from the finite elements with 2 elements. Recall from above that

\[
v(x) = \frac{0.0304}{L^2} (x^3 - \frac{1}{2} Lx^2) - \frac{0.0076}{L^2} (\frac{1}{4} L^2 x - Lx^2 + x^3) - \frac{2.188}{L^3} (\frac{1}{8} L^3 - \frac{3}{2} Lx^2 + 2x^3) \quad \{ \begin{array}{ll}
0 \leq x \leq L/2 \\
L/2 \leq x \leq L
\end{array} \}
\]

Hence

\[
M(x) = -EI \left\{ \begin{array}{ll}
-0.0076(6x-L) & 0 \leq x \leq L/2 \\
-0.0076(6x-2L) + 2.188(12x-3L) & L/2 \leq x \leq L
\end{array} \right. \\
L^4 L^2 L^3
\left( \begin{array}{ll}
\frac{1}{2} L^2 & L^2 & L^3
\end{array} \right)
\]

using \( E = 30 \times 10^6 \text{psi} \) and \( I = 57 \text{ inch}^4 \) and \( L = 144 \) the bending moment diagram plot is

The bending moment diagram clearly does not agree with the bending moment diagram that can be generated from the analytical solution show below (generated using my other program which solves this problem analytically)
The reason for this is because the solution \( v(x) \) obtained using the finite elements method is a 3\(^{rd} \) degree polynomial and after differentiating twice to obtain the bending moment \( (M(x) = -EI\frac{d^2v(x)}{dx^2}) \) the result is a linear function in \( x \) while in the analytical solution case, when the load is distributed, then solution \( v(x) \) is a 4\(^{th} \) degree polynomial. Hence the bending moment will be quadratic function in \( x \) in the analytical case.

Therefore, in order to obtain good approximation for the bending moment and shear force diagrams using finite elements, more elements will be needed.

4  Finding the stiffness matrix using methods other than direct method

There are 3 main methods to obtain the stiffness matrix

1. Variational method (minimize a functional). This functional is the potential energy of the structure and loads.

2. Weighted residual. requires the differential equation as starting point. Approximated in weighted average. Galerkin weighted residual method is the most common for implementation.

3. Virtual work method. Make the virtual work zero for arbitrary allowed displacement.

4.1 Virtual work method for derivation of the stiffness matrix

In virtual work method, a small displacement is assumed to occur. Looking at small volume element, the amount of work done by external loads to cause the small displacement is equal to amount of increased internal strain energy. Assuming the field of displacement is given by \( u = \{u, v, w\} \) and assuming the external loads are given by \( \{p\} \) acting on the nodes, hence these point loads will do work given by \( \{δd\}^T \{p\} \) on that unit volume where \( \{d\} \) is the nodal displacements. In all these derivation, only loads acting directly on the nodes are considered for now. In other words, body forces and traction forces are not considered in order to simplify the derivations.

The increase of strain energy is \( \{δε\}^T \{σ\} \) in that same unit volume.
Hence, for a unit volume
\[ \{\delta d\}^T\{p\} = \{\delta \varepsilon\}^T\{\sigma\} \]

And for the whole volume
\[ \{\delta d\}^T\{p\} = \int_V \{\delta \varepsilon\}^T\{\sigma\} \, dV \quad (1) \]

Assuming that displacement can be written as a function of the nodal displacements of the element results in
\[ u = [N]\{d\} \]

Therefore
\[ \delta u = [N]\{\delta d\} \]
\[ \{\delta u\}^T = \{\delta d\}^T[N]^T \quad (2) \]

Since \( \{\varepsilon\} = \partial\{u\} \) then \( \{\varepsilon\} = \partial [N]\{d\} = [B]\{d\} \) where \( B \) is the strain displacement matrix \( [B] = \partial [N] \), hence
\[ \{\varepsilon\} = [B]\{d\} \]
\[ \{\delta \varepsilon\} = [B]\{\delta d\} \]
\[ \{\delta \varepsilon\}^T = \{\delta d\}^T[B]^T \quad (3) \]

Now from the stress-strain relation \( \{\sigma\} = [E]\{\varepsilon\} \), hence
\[ \{\sigma\} = [E][B]\{d\} \quad (4) \]

Substituting Eqs. (2,3,4) into (1) results in
\[ \{\delta d\}^T\{p\} - \int_V \{\delta d\}^T[B]^T[E][B]\{d\} \, dV = 0 \]

Since \( \{\delta d\} \) and \( \{d\} \) do not depend on the integration variables they can be moved outside the integral, giving
\[ \{\delta d\}^T\left(\{p\} - \{d\} \int_V [B]^T[E][B] \, dV\right) = 0 \]

Since the above is true for any admissible \( \delta d \) then the only condition is that
\[ \{p\} = \{d\} \int_V [B]^T[E][B] \, dV \]

Hence this is in the form \( P = K\Delta \), therefore
\[ [K] = \int_V [B]^T[E][B] \, dV \]

knowing \( [B] \) allows finding \([k]\) by integrating over the volume. For the beam element though, \( u = v(x) \) the transverse displacement. This means \( [B] = \frac{d^2}{dx^2} [N] \). Recall from the above that for the beam element,
\[ [N] = \left( \frac{1}{L^3} \left( L^3 - 3Lx^2 + 2x^3 \right) \right) \left( \frac{1}{L^2} \left( L^2x - 2Lx^2 + x^3 \right) \right) \left( \frac{1}{L^2} \left( 3Lx^2 - 2x^3 \right) \right) \left( \frac{1}{L^2} \left( -Lx^2 + x^3 \right) \right) \]

Hence

\[ [B] = \frac{d^2}{dx^2} [N] = \left( \frac{1}{L^2} \left( -6L + 12x \right) \right) \left( \frac{1}{L^2} \left( -4L + 6x \right) \right) \left( \frac{1}{L^2} \left( 6L - 12x \right) \right) \left( \frac{1}{L^2} \left( -2L + 6x \right) \right) \]

Hence

\[ [K] = \int_V [B]^T [E] [B] \, dV \]

\[ = \int_V \left\{ \frac{1}{L^2} \left( -6L + 12x \right) \right\} \left\{ \frac{1}{L^2} \left( -4L + 6x \right) \right\} \left\{ \frac{1}{L^2} \left( 6L - 12x \right) \right\} \left\{ \frac{1}{L^2} \left( -2L + 6x \right) \right\} \, dV \]

\[ = EI \int_0^L \left\{ \frac{1}{L^2} \left( -6L + 12x \right) \right\} \left\{ \frac{1}{L^2} \left( -4L + 6x \right) \right\} \left\{ \frac{1}{L^2} \left( 6L - 12x \right) \right\} \left\{ \frac{1}{L^2} \left( -2L + 6x \right) \right\} \, dx \]

\[ = EI \int_0^L \left\{ \frac{1}{L^2} \left( 6L - 12x \right)^2 \right\} \left\{ \frac{1}{L^2} \left( 4L - 6x \right) (6L - 12x) \right\} \left\{ \frac{1}{L^2} \left( 4L - 6x \right)^2 \right\} \left\{ \frac{1}{L^2} \left( 2L - 6x \right) (6L - 12x) \right\} \, dx \]

\[ = EI \left( \frac{12}{L^3} \frac{6}{L^3} \frac{-12}{L^3} \frac{6}{L^3} \right) \]

\[ = EI \left( \frac{12}{L^3} \frac{6}{L^3} \frac{-12}{L^3} \frac{6}{L^3} \right) \]

\[ = EI \left( \begin{array}{cccc}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2 \\
\end{array} \right) \]

Which is the stiffness matrix found earlier.

### 4.2 Potential energy (minimize a functional) method to derive the stiffness matrix

This method is very similar to the first method actually. It all comes down to finding a functional, which is the potential energy of the system, and minimizing this with respect to the nodal displacements. The result gives the stiffness matrix.

Let the system total potential energy by called \( \Pi \) and let the total internal energy in the system be \( U \) and let the work done by external loads acting on the nodes be \( \Omega \), then

\[ \Pi = U - \Omega \]

Work done by external loads have a negative sign since they are an external agent to the system and work is being done onto the system.
The internal strain energy is given by $\frac{1}{2} \int_V \{\sigma\}^T \{\varepsilon\} dV$ and the work done by external loads is $\{d\}^T \{p\}$, hence

$$\Pi = \frac{1}{2} \int_V \{\sigma\}^T \{\varepsilon\} dV - \{d\}^T \{p\}$$

(1)

Now the rest follows as before. Assuming that displacement can be written as a function of the nodal displacements $\{d\}$, hence

$$\mathbf{u} = [N] \{d\}$$

Since $\{\varepsilon\} = \partial \{\mathbf{u}\}$ then $\{\varepsilon\} = \partial [N] \{d\} = [B] \{d\}$ where $B$ is the strain displacement matrix $[B] = \partial [N]$, hence

$$\{\varepsilon\} = [B] \{d\}$$

Now from the stress-strain relation $\{\sigma\} = [E] \{\varepsilon\}$, hence

$$\{\sigma\}^T = \{d\}^T [B]^T [E]$$

(4)

Substituting Eqs. (2,3,4) into (1) results in

$$\Pi = \frac{1}{2} \int_V \{d\}^T [B]^T [E] [B] \{d\} dV - \{d\}^T \{p\}$$

Hence $\frac{\partial \Pi}{\partial \{d\}^T} = 0$ gives

$$0 = \{d\} \int_V [B]^T [E] [B] dV - \{p\}$$

Which is on the form $P = [K] D$ which means that

$$[K] = \int_V [B]^T [E] [B] dV$$

As was found by the virtual work method.

5. References

1. A first course in Finite element method, 3rd edition, by Daryl L. Logan

2. Matrix analysis of framed structures, 2nd edition, by William Weaver, James Gere