solving the spin 1 electron problem. Finding S_x, S_y, S_z

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Problem statement

Determine S_x, S_y, S_z angular momentum spin matrices for the electron using spin 1. The given is that experiments show that S_z has three possible values (eigenvalues). These are $1, 0, -1$.

Solution

Using eigenbasis for S_z as the following

$$\begin{aligned} |S_{z=1}\rangle &= |1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ |S_{z=0}\rangle &= |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ |S_{z=-1}\rangle &= |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned} \tag{1}$$

Then the following three equations result from writing $S_z|S_{z=\omega_i}\rangle = \omega_i|S_{z=\omega_i}\rangle$ where ω_i is the eigenvalue. These three equation are solved to determine S_z. Let $S_z = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & m \end{bmatrix}$.

Therefore

$$\begin{aligned} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & m \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & m \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &= 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & m \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &= -1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Or

$$\begin{bmatrix} a \\ d \\ g \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} b \\ e \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} c \\ f \\ m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Which gives

$$\begin{aligned} S_z &= \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & m \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned} \tag{1A}$$

Now that S_z is found, the goal is to determine S_x, S_y. Let $S_- = S_x - iS_y$ and $S_+ = S_x + iS_y$. We start with S_+ (starting with S_- will not work, as it will not be possible to determine S^2 that way. So we have to start with S_+). We always start with commutator $[S_z, S_+]$

$$\begin{aligned} [S_z, S_+] &= \left[S_z, S_x + iS_y\right] \\ &= [S_z, S_x] + i\left[S_z, S_y\right] \end{aligned}$$

But $[S_z, S_x] = i \sum_k \epsilon_{ijk} S_k$. Here $i = 3, j = 1$ since $z = 3, x = 1$. Then $[S_z, S_x] = i\epsilon_{312} S_2 = iS_y$ and similarly $\left[S_z S_y\right] = i \sum_k \epsilon_{ijk} S_k$. Here $i = 3, j = 2$ since $z = 3, y = 1$. Then $\left[S_z, S_y\right] = i\epsilon_{321} S_1 = -iS_x$. The above becomes

$$\begin{aligned} [S_z, S_+] &= iS_y + i(-iS_x) \\ &= iS_y + S_x \\ &= S_+ \end{aligned}$$

This implies

$$\begin{aligned} [S_z, S_+] &= S_+ \\ S_z S_+ - S_+ S_z &= S_+ \\ S_z S_+ &= S_+ S_z + S_+ \end{aligned} \tag{2}$$

Picking $|1\rangle$ to start with, as it lead to finding S^2, which we must find before making any progress. S^2 is proportional for the identity matrix I. From the above we obtain

$$\begin{aligned} S_z S_+ |1\rangle &= (S_+ S_z + S_+)|1\rangle \\ &= S_+ S_z |1\rangle + S_+ |1\rangle \end{aligned} \tag{3}$$

But $S_z|1\rangle = |1\rangle$ since the eigenvalue is 1 associated with $|1\rangle$ eigenvector. The above becomes

$$\begin{aligned} S_z S_+ |1\rangle &= S_+ |1\rangle + S_+ |1\rangle \\ &= 2S_+ |1\rangle \end{aligned}$$

The above shows that $S_+|1\rangle$ is eigenvector of S_z associated with the eigenvalue 2 which is not compatible with experiments. Therefore the only logical result is that

$$S_+|1\rangle = 0|1\rangle \tag{4}$$

Taking the adjoint gives

$$(S_+|1\rangle)^\dagger = 0|1\rangle^\dagger$$
$$\langle 1|S_+^\dagger = 0\langle 1|$$

Hence

$$\langle 1|S_+^\dagger S_+|1\rangle = 0\langle 1|1\rangle$$
$$\langle 1|S_+^\dagger S_+|1\rangle = 0 \tag{4A}$$

The above is used to find S^2. Since $S_+ = S_x + iS_y$ then the above becomes

$$\langle 1|\left(S_x^\dagger - iS_y^\dagger\right)\left(S_x + iS_y\right)|1\rangle = 0$$
$$\langle 1|S_x^\dagger S_x + iS_x^\dagger S_y - iS_y^\dagger S_x + S_y^\dagger S_y|1\rangle = 0$$

But S_x, S_y are Hermitian. Therefore $S_x^\dagger = S_x, S_y^\dagger = S_y$ and the above reduces to

$$\langle 1|S_x^2 + iS_xS_y - iS_yS_x + S_y^2|1\rangle = 0$$
$$\langle 1|S_x^2 + i\left(S_xS_y - S_yS_x\right) + S_y^2|1\rangle = 0$$
$$\langle 1|S_x^2 + i\left[S_x, S_y\right] + S_y^2|1\rangle = 0$$

But $\left[S_x, S_y\right] = iS_z$, therefore the above becomes

$$\langle 1|S_x^2 - S_z + S_y^2|1\rangle = 0$$

And $S^2 = S_x^2 + S_y^2 + S_z^2$, therefore $S_x^2 + S_y^2 = S^2 - S_z^2$. Hence

$$S_+^\dagger S_+ = S^2 - S_z^2 - S_z \tag{4B}$$

Therefore (4A) becomes

$$\langle 1|S^2 - S_z^2 - S_z|1\rangle = 0$$

Expanding the above gives

$$\langle 1|S^2|1\rangle - \langle 1|S_z^2|1\rangle - \langle 1|S_z|1\rangle = 0$$
$$\langle 1|S^2|1\rangle = \langle 1|S_z^2|1\rangle + \langle 1|S_z|1\rangle$$

But $\langle 1|S_z|1\rangle = 1$ and $\langle 1|S_z^2|1\rangle = 1$, therefore the above becomes

$$\langle 1|S^2|1\rangle = 2 \tag{5}$$

It is not possible to use the above to solve for a general S^2 which is 3×3 matrix. But since S^2 must be proportional to the Identity matrix for all spin numbers, then it must diagonal matrix with same element on the diagonal, then let S be

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

Substituting this in (5) gives

$$\langle 1|\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}^2|1\rangle = 2$$
$$\langle 1|\begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix}|1\rangle = 2 =$$
$$a^2 = 2$$

Hence $a = \sqrt{2}$. Therefore

$$S = \sqrt{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

Now that S is found, the next step is to find $S_+|2\rangle$ and $S_+|3\rangle$. Starting with (2), but now applying it to $|2\rangle$ gives

$$S_z S_+|2\rangle = S_+ S_z|2\rangle + S_+|2\rangle$$

But $S_z|2\rangle = 0|2\rangle$ since $|2\rangle$ is the eigenvector associated with 0 eigenvalue, the above becomes

$$S_z S_+|2\rangle = S_+|2\rangle$$

Which means $S_+|2\rangle$ is eigenvector of S_z associated with eigenvalue +1 which is compatible with experiment. Hence

$$S_+|2\rangle = c|1\rangle$$

Where we used $|1\rangle$ since that is the eigenvector of S_z associated with +1 eigenvalue. Now we need to find c. Taking adjoint of both sides of the above gives

$$\begin{aligned} (S_+|2\rangle)^\dagger &= (c|1\rangle)^\dagger \\ \langle 2|S_+^\dagger &= c^*\langle 1| \end{aligned}$$

Therefore

$$\begin{aligned} \langle 2|S_+^\dagger S_+|2\rangle &= c^* c\langle 1|1\rangle \\ &= |c|^2 \end{aligned} \tag{7}$$

But $S_+^\dagger S_+ = S^2 - S_z^2 - S_z$ which was found earlier above in (4B). Therefore the above equation becomes

$$\langle 2|\left(S^2 - S_z^2 - S_z\right)|2\rangle = |c|^2$$

Using S^2 found in (6) the above becomes

$$\begin{aligned} \langle 2|\left(2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right)|2\rangle &= |c|^2 \\ \langle 2|\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}|2\rangle &= |c|^2 \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &= |c|^2 \\ 2 &= |c|^2 \end{aligned}$$

which implies $c = \sqrt{2}$. Therefore

$$S_+|2\rangle = \sqrt{2}\,|1\rangle \tag{8}$$

Finally, to find $S_+|3\rangle$, starting again with (2), but now applying it to $|3\rangle$ gives

$$S_z S_+|3\rangle = S_+ S_z|3\rangle + S_+|3\rangle$$

But $S_z|3\rangle = -|3\rangle$ since $|3\rangle$ is the eigenvector associated with -1 eigenvalue, the above becomes

$$\begin{aligned} S_z S_+|3\rangle &= -S_+|3\rangle + S_+|3\rangle \\ &= 0S_+|3\rangle \end{aligned}$$

Which means $S_+|3\rangle$ is eigenvector of S_z associated with eigenvalue 0 which is compatible with experiment. Hence

$$S_+|3\rangle = b|2\rangle$$

Where we used $|2\rangle$ since that is the eigenvector of S_z associated with 0 eigenvalue. Now we need to find b. Taking adjoint of both sides of the above gives

$$\begin{aligned}(S_+|3\rangle)^\dagger &= (b|2\rangle)^\dagger \\ \langle 3|S_+^\dagger &= b^*\langle 2|\end{aligned}$$

Therefore

$$\begin{aligned}\langle 3|S_+^\dagger S_+|3\rangle &= b^* b\langle 2|2\rangle \\ &= |b|^2\end{aligned} \tag{9}$$

But $S_+^\dagger S_+ = S^2 - S_z^2 - S_z$ which was found earlier in (4B). Therefore the above equation becomes

$$\langle 3|\left(S^2 - S_z^2 - S_z\right)|3\rangle = |b|^2$$

Using S^2 from eq (6) the above becomes

$$\begin{aligned}\langle 3|\left(2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix} - \begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}\right)|3\rangle &= |b|^2 \\ \langle 3|\begin{bmatrix}0&0&0\\0&2&0\\0&0&2\end{bmatrix}|3\rangle &= |b|^2 \\ \begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&2&0\\0&0&2\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} &= |b|^2 \\ 2 &= |b|^2\end{aligned}$$

which implies $b = \sqrt{2}$. Therefore

$$S_+|3\rangle = \sqrt{2}|2\rangle \tag{10}$$

Now that $S_+|1\rangle, S_+|2\rangle, S_+|3\rangle$ are found, S_+ can be calculated. From (4,6,8) the result is

$$\begin{aligned}S_+|1\rangle &= 0|1\rangle \\ S_+|2\rangle &= \sqrt{2}|1\rangle \\ S_+|3\rangle &= \sqrt{2}|2\rangle\end{aligned}$$

Hence

$$\begin{aligned}S_+ &= \begin{bmatrix}\langle 1|S_+|1\rangle & \langle 1|S_+|2\rangle & \langle 1|S_+|3\rangle \\ \langle 2|S_+|1\rangle & \langle 2|S_+|2\rangle & \langle 2|S_+|3\rangle \\ \langle 3|S_+|1\rangle & \langle 3|S_+|2\rangle & \langle 3|S_+|3\rangle\end{bmatrix} \\ &= \begin{bmatrix}0 & \langle 1|\sqrt{2}|1\rangle & \langle 1|\sqrt{2}|2\rangle \\ 0 & \langle 2|\sqrt{2}|1\rangle & \langle 2|\sqrt{2}|2\rangle \\ 0 & \langle 3|\sqrt{2}|1\rangle & \langle 3|\sqrt{2}|2\rangle\end{bmatrix} \\ &= \sqrt{2}\begin{bmatrix}0 & \langle 1|1\rangle & \langle 1|2\rangle \\ 0 & \langle 2|1\rangle & \langle 2|2\rangle \\ 0 & \langle 3|1\rangle & \langle 3|2\rangle\end{bmatrix}\end{aligned}$$

Therefore

$$S_+ = \begin{bmatrix}0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0\end{bmatrix} \tag{11}$$

Now that we know S_+ we turn our attention to finding S_-. Considering the commutator $[S_z, S_-]$

$$\begin{aligned}[S_z, S_-] &= \left[S_z, S_x - iS_y\right] \\ &= [S_z, S_x] - i\left[S_z, S_y\right]\end{aligned}$$

But $[S_z, S_x] = iS_y$ and $\left[S_z S_y\right] = -iS_x$. The above becomes

$$\begin{aligned}[S_z, S_-] &= iS_y - i(-iS_x) \\ &= iS_y - S_x \\ &= -S_-\end{aligned}$$

This implies

$$\begin{aligned}[S_z, S_-] &= -S_- \\ S_z S_- - S_- S_z &= -S_- \\ S_z S_- &= S_- S_z - S_- \end{aligned} \tag{12}$$

Picking $|1\rangle$ to start with, gives

$$\begin{aligned}S_z S_- |1\rangle &= (S_- S_z - S_-)|1\rangle \\ &= S_- S_z |1\rangle - S_- |1\rangle\end{aligned} \tag{13}$$

But $S_z|1\rangle = |1\rangle$ since the eigenvalue is 1 associated with $|1\rangle$ eigenvector from (1). The above now becomes

$$\begin{aligned}S_z S_- |1\rangle &= S_- |1\rangle - S_- |1\rangle \\ &= 0 S_- |1\rangle\end{aligned}$$

The above shows that $S_+|1\rangle$ is eigenvector of S_z associated with the eigenvalue 0 which is compatible with experiments. This implies

$$S_- |1\rangle = c|2\rangle \tag{14}$$

Where $|2\rangle$ was used above, since that is the eigenvector with 0 eigenvalue. c is constant to be found. Taking adjoint of both sides of the above gives

$$\begin{aligned}(S_- |1\rangle)^\dagger &= (c|2\rangle)^\dagger \\ \langle 1| S_-^\dagger &= c^* \langle 2|\end{aligned}$$

Therefore

$$\begin{aligned}\langle 1| S_-^\dagger S_- |1\rangle &= c^* c \langle 2|2\rangle \\ &= |c|^2\end{aligned} \tag{15}$$

But

$$\begin{aligned}S_-^\dagger S_- &= \left(S_x - iS_y\right)^\dagger \left(S_x - iS_y\right) \\ &= \left(S_x^\dagger + iS_y^\dagger\right)\left(S_x - iS_y\right)\end{aligned}$$

Since S_x, S_y are Hermitian, then $S_x^\dagger = S_x$ and $S_y^\dagger = S_y$. The above becomes

$$\begin{aligned}S_-^\dagger S_- &= \left(S_x + iS_y\right)\left(S_x - iS_y\right) \\ &= S_x^2 - iS_x S_y + iS_y S_x + S_y^2 \\ &= S_x^2 + i\left(S_y S_x - S_x S_y\right) + S_y^2 \\ &= S_x^2 + i\left[S_y, S_x\right] + S_y^2\end{aligned}$$

But $\left[S_y, S_x\right] = -iS_z$. The above becomes

$$
\begin{aligned}
S_-^\dagger S_- &= S_x^2 + i(-iS_z) + S_y^2 \\
&= S_x^2 + S_y^2 + S_z
\end{aligned} \tag{16}
$$

Since $S^2 = S_x^2 + S_y^2 + S_z^2$, then $S_x^2 + S_y^2 = S^2 - S_z^2$. This implies

$$
S_-^\dagger S_- = S^2 - S_z^2 + S_z \tag{16A}
$$

Substituting (16A) in (15) gives

$$
\langle 1|\left(S^2 - S_z^2 + S_z\right)|1\rangle = |c|^2 \tag{17}
$$

But

$$
S_z^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{18}
$$

And S^2 was found in (6). Substituting (6,18) back in (17) gives an equation to solve for c

$$
\begin{aligned}
\langle 1|\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}|1\rangle &= |c|^2 \\
\langle 1|\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}|1\rangle &= |c|^2 \\
\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= |c|^2 \\
2 &= |c|^2
\end{aligned}
$$

Hence $c = \sqrt{2}$. From (14) this implies

$$
S_-|1\rangle = \sqrt{2}\,|2\rangle \tag{19}
$$

Now we pick $|2\rangle$ and using (12) gives

$$
\begin{aligned}
S_z S_-|2\rangle &= (S_- S_z - S_-)|2\rangle \\
&= S_- S_z|2\rangle - S_-|2\rangle
\end{aligned} \tag{20}
$$

But $S_z|2\rangle = 0|2\rangle$ since the eigenvalue is 0 associated with $|2\rangle$ eigenvector. The above now becomes

$$
S_z S_-|2\rangle = -S_-|2\rangle
$$

The above shows that $S_-|2\rangle$ is eigenvector of S_z associated with the eigenvalue -1 which is compatible with experiments. This implies

$$
S_-|2\rangle = b|3\rangle \tag{21}
$$

Where $|3\rangle$ was used above, since that is the eigenvector with -1 eigenvalue. b is constant to be found. Taking adjoint of both sides of the above gives

$$
\begin{aligned}
(S_-|2\rangle)^\dagger &= (b|3\rangle)^\dagger \\
\langle 2|S_-^\dagger &= b^*\langle 3|
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\langle 2|S_-^\dagger S_-|2\rangle &= b^* b\langle 3|3\rangle \\
&= |b|^2
\end{aligned}
$$

But $S_-^\dagger S_- = S^2 - S_z^2 + S_z$ as calculated earlier in (16A). Hence the above becomes

$$\langle 2|\left(S^2 - S_z^2 + S_z\right)|2\rangle = |b|^2$$

Using S_z^2, S^2 calculated earlier in the above gives

$$\langle 2|\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}|2\rangle = |b|^2$$

$$\langle 2|\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}|2\rangle = |b|^2$$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = |b|^2$$

$$2 = |b|^2$$

Hence $b = \sqrt{2}$. From (21) this implies

$$S_-|2\rangle = \sqrt{2}\,|3\rangle \tag{22}$$

And finally using $|3\rangle$ in (12) results in

$$\begin{aligned} S_z S_-|3\rangle &= (S_- S_z - S_-)|3\rangle \\ &= S_- S_z|3\rangle - S_-|3\rangle \end{aligned}$$

But $S_z|3\rangle = -|3\rangle$ since the eigenvalue is -1 associated with $|3\rangle$ eigenvector. The above now becomes

$$S_z S_-|3\rangle = -2S_-|3\rangle$$

The above shows that $S_-|3\rangle$ is eigenvector of S_z associated with the eigenvalue -2 which is not compatible with experiments. This implies

$$S_-|3\rangle = 0|1\rangle \tag{23}$$

Now that $S_-|1\rangle, S_-|2\rangle, S_-|3\rangle$ are all known, we are ready to determine S_-. From (19,22,23)

$$\begin{aligned} S_-|1\rangle &= \sqrt{2}\,|2\rangle \\ S_-|2\rangle &= \sqrt{2}\,|3\rangle \\ S_-|3\rangle &= 0|1\rangle \end{aligned}$$

Therefore

$$\begin{aligned} S_- &= \begin{bmatrix} \langle 1|S_-|1\rangle & \langle 1|S_-|2\rangle & \langle 1|S_-|3\rangle \\ \langle 2|S_-|1\rangle & \langle 2|S_-|2\rangle & \langle 2|S_-|3\rangle \\ \langle 3|S_-|1\rangle & \langle 3|S_-|2\rangle & \langle 3|S_-|3\rangle \end{bmatrix} \\ &= \begin{bmatrix} \langle 1|\sqrt{2}\,|2\rangle & \langle 1|\sqrt{2}\,|3\rangle & \langle 1|0|1\rangle \\ \langle 2|\sqrt{2}\,|2\rangle & \langle 2|\sqrt{2}\,|3\rangle & \langle 2|0|1\rangle \\ \langle 3|\sqrt{2}\,|2\rangle & \langle 3|\sqrt{2}\,|3\rangle & \langle 3|0|1\rangle \end{bmatrix} \\ &= \sqrt{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

Therefore

$$S_- = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \tag{24}$$

Now that S_+, S_- are known, S_x, S_y can be found. Using

$$S_- = S_x - iS_y \tag{25}$$
$$S_+ = S_x + iS_y \tag{26}$$

Adding the above two equations gives, and using (11,24)

$$\begin{aligned}
S_- + S_+ &= 2S_x \\
\begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} &= 2S_x \\
S_x &= \frac{1}{2}\begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} \\
&= \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned}$$

Hence

$$S_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{27}$$

And subtracting (25,26) gives

$$\begin{aligned}
S_- - S_x &= -2iS_y \\
\begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} &= -2iS_y \\
\begin{bmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} &= -2iS_y \\
S_y &= \frac{-1}{2i}\begin{bmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} \\
&= \frac{i}{2}\begin{bmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} \\
&= \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}
\end{aligned}$$

Hence

$$S_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \tag{28}$$

This completes the solution. We have found S_x, S_y, starting from just knowing the eigenvalues of S_z.