Solving first and second order linear differential equations using Laplace transform method

Constant and time varying coefficients

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Contents

1 First order differential equation

1.1 Algorithm

<u>ode internal name</u> "first_order_laplace"

These are ode's solved using Laplace method. Currently only linear odes are supported. Both constant coefficients and time varying coefficients. For time varying only, only coefficients that are polynomial in t are allowed. For example the following ode can be solved using Laplace

$$\begin{aligned} ty' + y &= 0 \\ (1+t)\,y' + ty &= 0 \\ y' + 3t^2 y &= 0 \end{aligned}$$

But not

$$\sin(t)\,y' + y = 0$$

Initial conditions can be at zero or not at zero or not given. For time varying, the ode is transform to Laplace using the property

$$\mathcal{L}\left(t^n y(t)\right) = (-1)^n \frac{d^n}{ds^n} Y(s)$$

What this means, is that having t as coefficient will generate first order ode in $Y(s)$ which needs to be solved first to find $Y(s)$ before applying inverse Laplace transform to find the solution $y(t)$. A coefficient t^2 will generate second order ode in $Y(s)$ and t^3 will generate a third order ode in $Y(s)$ and so on. This means if we are to use Laplace transform to solve first order ode, we could end having to solving an ode in $Y(s)$ of much higher order and the generated solution $Y(s)$ might become too complicated to even inverse Laplace it.

So it is not really useful to use Laplace method to solve time varying first order ode of coefficient of polynomial of power t^n where $n > 1$.

When the initial condition of the original ode is not at zero, the original condition must be shifted so it is at zero. This is more critical to do for time varying than for constant coefficients ode when we use Laplace transform method. This means we have to do change of variables first. See examples below.

The following is the algorithm for solving using Laplace transform for time varying coefficients ode

```
-- Input is first ode in y(t) with possible IC in form y(t0)=y0
-- output is solution y(t) using Laplace transform.

-- The first step is convert the ODE in y(t) to ODE in Y(s) using
-- the relation L(t^n f(t) ) = -(1)^n d^n/ds^n( F(s) )
-- where F(s) is the Laplace of f(t). This is applied to each term in
-- the original ode in y(t).

-- Now we have an ODE in Y(s). This ode can be first order or higher
-- order depending on the power on t. For example if the input
-- is t^3*y(t)+y'(t)=0 then the ode in Y(s) will be 3rd order.

-- Next step is to solve the ode in Y(s). Let us say the solution
-- is Y(s)=.... This solution will have as many new constants as the
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```
-- order of the ode in Y(s)

IF no IC are given THEN
   Apply Laplace to the ODE and convert to ode in Y(s)
   solve this ode in Y(s)
   Apply inverse Laplace transform on solution Y(s). THis gives
   y(t)=.... which is the final solution.
ELSE -- IC is given as y(t0)=y0
    IF t0=0 THEN
        Apply Laplace to the ODE and convert to ode in Y(s)
        solve this ode in Y(s)

        LABEL L:

        Apply inverse Laplace transform on Y(s)
        now we have y(t)=.... with constants c_i in it     (*)
        these constants c_i come from solving the ode in Y(s)
        Apply IC to obtain equation   y0=.... with constants c_i in it.

        IF there is more than one unknown c_i in the RHS then solve
            for one of them and plug that into (*). This is final solution
        ELSE
            solve for c_1 from y0=.... c_1 .... and plugin into (*).
        END IF
    ELSE -- initial conditions not at zero, i.e. y(t0)=y0 and t0<>0
         -- This applies also even if y0=0 or not.

        Transform the original ode in y(t) such that IC is now
        shifted to zero.

        For example, if IC was y(1)=y0, then use transformation
        tau=t-1. This gives new ode in time, but with y(0)=y0.

        This is the one we will work with now. Not the orginal one.

        Apply Laplace to this new ODE and convert to ode in Y(s)
        solve this ode in Y(s)

        GOTO LABEL L to find solution y(tau)

        convert solution back to t, using tau=t-t0
    END IF
END IF
```

1.2 Examples solving using Laplace transform for constant coefficients ode

Local contents

1.2.1 Example 1 $y' - 2y = 6e^{5t}, y(0) = 3$

$$
\begin{aligned}
y' - 2y &= 6e^{5t} \\
y(0) &= 3
\end{aligned}
$$

Taking the Laplace transform gives

$$
\begin{aligned}
\mathcal{L}(y) &= Y(s) \\
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(6e^{5t}) &= \frac{6}{s-5}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
sY(s) - y(0) - 2Y(s) &= \frac{6}{s-5} \\
Y(s)(s-2) - y(0) &= \frac{6}{s-5} \\
Y(s)(s-2) &= \frac{6}{s-5} + y(0) \\
Y(s)(s-2) &= \frac{6}{s-5} + 3 \\
Y(s)(s-2) &= \frac{6+3(s-5)}{s-5} \\
Y(s)(s-2) &= \frac{3s-9}{s-5} \\
Y(s) &= \frac{3s-9}{(s-5)(s-2)} \\
&= \frac{2}{s-5} + \frac{1}{s-2}
\end{aligned}
$$

Applying inverse Laplace transform and using $\mathcal{L}^{-1}\left(\frac{2}{s-5}\right) = 2e^{5t}, \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$ then the above gives

$$
y(t) = 2e^{5t} + e^{2t}
$$

1.2.2 Example 2 $y' - 6y = 0, y(-1) = 4$

$$
\begin{aligned}
y' - 6y &= 0 \\
y(-1) &= 4
\end{aligned}
$$

There are two ways to solve an ode using Laplace transform when IC are not at zero. Either we do change of variables to shift the IC to zero, or solve as is. Both methods are shown below.

method 1 (no change of variable)

Taking the Laplace transform of the ode gives

$$
\begin{aligned}
\mathcal{L}(y) &= Y(s) \\
\mathcal{L}(y') &= sY(s) - y(0)
\end{aligned}
$$

The ode becomes

$$sY - y(0) - 6Y = 0$$

Solving for Y gives

$$\begin{aligned} Y(s-6) - y(0) &= 0 \\ Y &= \frac{y(0)}{s-6} \end{aligned}$$

Taking inverse Laplace transform gives

$$y(t) = y(0)\, e^{6t} \tag{1}$$

Now we need to find $y(0)$, for this, we use the given IC $y(-1) = 4$. The above becomes

$$\begin{aligned} 4 &= y(0)\, e^{-6} \\ y(0) &= 4e^{6} \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} y(t) &= 4e^{6} e^{6t} \\ &= 4e^{6t+6} \end{aligned}$$

<u>method 2 (change of variable)</u>

Let

$$\tau = t + 1$$

The ode $y' - 6y = 0$ becomes

$$\begin{aligned} y'(\tau) - 6y(\tau) &= 0 \\ y(0) &= 4 \end{aligned}$$

Taking Laplace transform gives

$$\begin{aligned} sY - y(0) - 6Y &= 0 \\ sY - 4 - 6Y &= 0 \\ Y &= \frac{4}{s-6} \end{aligned}$$

The inverse Laplace transform is

$$y(\tau) = 4e^{6\tau}$$

Changing back to t the above becomes

$$y(t) = 4e^{6(t+1)}$$

Which is the same answer as before. The change of variable method seems to be more common.

1.2.3 Example 3 $y' + y = \sin(t), y(1) = y_0$

$$
\begin{aligned}
y' + y &= \sin(t) \\
y(1) &= y_0
\end{aligned}
$$

There are two ways to solve an ode using Laplace transform when IC are not at zero. Either we do change of variables to shift the IC to zero, or solve as is. Both methods are shown below.

method 1 (no change of variable)

Taking the Laplace transform of the ode gives

$$
\begin{aligned}
\mathcal{L}(y) &= Y(s) \\
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(\sin t) &= \frac{1}{1+s^2}
\end{aligned}
$$

The ode becomes

$$
sY - y(0) + Y = \frac{1}{1+s^2}
$$

Solving for Y gives

$$
\begin{aligned}
Y(s+1) - y(0) &= \frac{1}{1+s^2} \\
Y &= \frac{\frac{1}{1+s^2} + y(0)}{s+1} \\
&= \frac{1}{(1+s^2)(s+1)} + \frac{y(0)}{s+1}
\end{aligned}
$$

Taking inverse Laplace transform gives

$$
y(t) = \frac{e^{-t}}{2}(2y(0)+1) - \frac{1}{2}\cos t + \frac{1}{2}\sin t \tag{1}
$$

Now we need to find $y(0)$, for this, we use the original given IC $y(1) = y_0$. The above becomes

$$
\begin{aligned}
y_0 &= \frac{e^{-1}}{2}(2y(0)+1) - \frac{1}{2}\cos 1 + \frac{1}{2}\sin 1 \\
y_0 + \frac{1}{2}\cos 1 - \frac{1}{2}\sin 1 &= \frac{e^{-1}}{2}(2y(0)+1) \\
2e\left(y_0 + \frac{1}{2}\cos 1 - \frac{1}{2}\sin 1\right) &= (2y(0)+1) \\
y(0) &= e\left(y_0 + \frac{1}{2}\cos 1 - \frac{1}{2}\sin 1\right) - \frac{1}{2}
\end{aligned}
$$

Hence (1) becomes

$$
\begin{aligned}
y(t) &= \frac{e^{-t}}{2}\left(2\left(e\left(y_0 + \frac{1}{2}\cos 1 - \frac{1}{2}\sin 1\right) - \frac{1}{2}\right) + 1\right) - \frac{1}{2}\cos t + \frac{1}{2}\sin t \\
&= e^{1-t}\left(y_0 + \frac{1}{2}\cos 1 - \frac{1}{2}\sin 1\right) - \frac{1}{2}\cos t + \frac{1}{2}\sin t \\
&= \frac{1}{2}e^{1-t}(2y_0 + \cos 1 - \sin 1) - \frac{1}{2}\cos t + \frac{1}{2}\sin t
\end{aligned}
$$

method 2 (change of variable)

Let

$$\tau = t - 1$$

The ode $y' + y = \sin(t)$ becomes

$$\begin{aligned} y'(\tau) + y(\tau) &= \sin(\tau + 1) \\ y(0) &= y_0 \end{aligned}$$

Taking Laplace transform gives

$$\begin{aligned} sY - y(0) + Y &= \frac{\sin(1)s + \cos(1)}{1 + s^2} \\ Y(1+s) &= \frac{\sin(1)s + \cos(1)}{1 + s^2} + y_0 \\ Y &= \frac{\sin(1)s + \cos(1)}{(1+s^2)(1+s)} + \frac{y_0}{1+s} \end{aligned}$$

The inverse Laplace transform is

$$y(\tau) = \frac{1}{2}e^{-\tau}(2y_0 + \cos 1 - \sin 1) + \frac{\cos 1}{2}(\sin\tau - \cos\tau) + \frac{\sin 1}{2}(\sin\tau + \cos\tau)$$

Finally, changing back to t the above becomes

$$y(t) = \frac{1}{2}e^{1-t}(2y_0 + \cos 1 - \sin 1) + \frac{\cos 1}{2}(\sin(t-1) - \cos(t-1)) + \frac{\sin 1}{2}(\sin(t-1) + \cos(t-1))$$

Which simplifies to

$$y(t) = \frac{1}{2}e^{1-t}(2y_0 + \cos 1 - \sin 1) - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

Which is the same answer as before.

1.3 Examples with time varying coefficients

Local contents

1.3.1 Example 1 $y' - ty = 0, y(0) = 0$

$$\begin{aligned} y' - ty &= 0 \\ y(0) &= 0 \end{aligned}$$

For this we will use relation $\mathcal{L}(tf(t)) = -\frac{d}{ds}F(s)$. Hence taking the Laplace transform gives

$$\begin{aligned} \mathcal{L}(ty) &= -\frac{d}{ds}\mathcal{L}(y) \\ &= -\frac{d}{ds}Y(s) \\ \mathcal{L}(y') &= sY(s) - y(0) \end{aligned}$$

The ode becomes

$$\begin{aligned} sY(s) - y(0) + \frac{d}{ds}Y(s) &= 0 \\ sY(s) + \frac{d}{ds}Y(s) &= 0 \end{aligned}$$

Replacing initial conditions $y(0) = 0$ the above becomes

$$sY(s) + \frac{d}{ds}Y(s) = 0$$

This is linear ode in $Y(s)$. The integrating factor is $e^{\int s ds} = e^{\frac{s^2}{2}}$. Hence the above becomes

$$\frac{d}{ds}\left(Ye^{\frac{s^2}{2}}\right) = 0$$

Integrating gives

$$\begin{aligned} Ye^{\frac{s^2}{2}} &= c_1 \\ Y &= c_1 e^{\frac{-s^2}{2}} \end{aligned} \tag{1}$$

Taking the inverse Laplace gives

$$y(t) = c_1 \mathcal{L}^{-1}\left(e^{\frac{-s^2}{2}}\right) \tag{2}$$

And now apply IC which gives

$$0 = c_1 \mathcal{L}^{-1}\left(e^{\frac{-s^2}{2}}\right)$$

Hence $c_1 = 0$. Therefore (2) becomes

$$y(t) = 0$$

1.3.2 Example 2 $ty' + y = 0, y(0) = 0$

$$\begin{aligned} ty' + y &= 0 \\ y(0) &= 0 \end{aligned}$$

We will use the property

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}F(s)$$

Hence taking Laplace transform of each term of the ode gives

$$\begin{aligned} \mathcal{L}(ty') &= -\frac{d}{ds}(\mathcal{L}(y')) \\ &= -\frac{d}{ds}(sY - y(0)) \\ &= -\left(Y + s\frac{dY}{ds}\right) \\ &= -s\frac{dY}{ds} - Y \end{aligned}$$

And

$$\mathcal{L}(y) = Y$$

Hence the ode becomes in Laplace domain as

$$\begin{aligned} -s\frac{dY}{ds} - Y + Y &= 0 \\ -s\frac{dY}{ds} &= 0 \\ \frac{dY}{ds} &= 0 \end{aligned}$$

Solving this ode for $Y(s)$ gives

$$Y = c_1 \tag{1}$$

Taking the inverse Laplace transform gives

$$y(t) = c_1 \delta(t) \tag{2}$$

Applying initial conditions

$$0 = c_1 \delta(0)$$

Hence $c_1 = 0$ and the solution (2) becomes

$$y(t) = 0$$

1.3.3 Example 3 $ty' + y = 0, y(0) = y_0$

$$\begin{aligned} ty' + y &= 0 \\ y(0) &= y_0 \end{aligned}$$

The following property is used

$$\mathcal{L}(tf(t)) = -\frac{d}{ds} F(s)$$

Taking Laplace transform of each term of the ode gives

$$\begin{aligned} \mathcal{L}(ty') &= -\frac{d}{ds}(\mathcal{L}(y')) \\ &= -\frac{d}{ds}(sY - y(0)) \\ &= -\left(Y + s\frac{dY}{ds}\right) \\ &= -s\frac{dY}{ds} - Y \end{aligned}$$

And

$$\mathcal{L}(y) = Y$$

TThe ode becomes in Laplace domain becomes

$$\begin{aligned} -s\frac{dY}{ds} - Y + Y &= 0 \\ -s\frac{dY}{ds} &= 0 \\ \frac{dY}{ds} &= 0 \end{aligned}$$

Solving this ode for $Y(s)$ gives

$$Y = c_1 \tag{1}$$

Taking inverse Laplace gives

$$y(t) = \delta(t) c_1 \tag{2}$$

Applying initial conditions gives

$$\begin{aligned} y_0 &= \delta(0) c_1 \\ c_1 &= \frac{y_0}{\delta(0)} \end{aligned}$$

The solution (2) becomes

$$y(t) = y_0 \frac{\delta(t)}{\delta(0)}$$

1.3.4 Example 4 $ty' + y = 0, y(x_0) = y_0$

$$\begin{aligned} ty' + y &= 0 \\ y(x_0) &= y_0 \end{aligned}$$

Since IC given is not at zero, change of variables must be made so that the IC at zero. Let $\tau = t - x_0$ then the ode becomes

$$\begin{aligned} (x_0 + \tau) y'(\tau) + y(\tau) &= 0 \\ x_0 y'(\tau) + \tau y'(\tau) + y(\tau) &= 0 \\ y(0) &= y_0 \end{aligned}$$

Converting the above new ode to Laplace domain using

$$\mathcal{L}(tf(\tau)) = -\frac{d}{ds} F(s)$$

Gives (using $Y(s)$ as the Laplace of $y(\tau)$) and simplifying using $y(0) = y_0$

$$\begin{aligned} x_0(sY - y(0)) + (-1)\frac{d}{ds}(sY - y(0)) + Y &= 0 \\ x_0(sY - y_0) - \left(Y + s\frac{dY}{ds}\right) + Y &= 0 \\ x_0 sY - x_0 y_0 - Y - s\frac{dY}{ds} + Y &= 0 \\ x_0 sY - s\frac{dY}{ds} &= x_0 y_0 \\ \frac{dY}{ds} - x_0 Y &= -\frac{x_0 y_0}{s} \end{aligned}$$

The solution is

$$Y = c_1 e^{sx_0} + (x_0 y_0 \operatorname{Ei}(sx_0)) e^{sx_0}$$

Taking inverse Laplace gives

$$y(\tau) = \frac{x_0 y_0}{\tau + x_0} + c_1 \mathcal{L}^{-1}(e^{sx_0}) \tag{1}$$

Applying initial conditions gives $y(0) = y_0$ gives

$$\begin{aligned} y_0 &= \frac{x_0 y_0}{x_0} + c_1 \mathcal{L}^{-1}(e^{sx_0}) \\ y_0 &= y_0 + c_1 \mathcal{L}^{-1}(e^{sx_0}) \\ 0 &= c_1 \mathcal{L}^{-1}(e^{sx_0}) \\ c_1 &= 0 \end{aligned}$$

Hence the solution (1) becomes

$$y(\tau) = \frac{x_0 y_0}{\tau + x_0}$$

Converting back to t using $\tau = t - x_0$ the above becomes

$$y(\tau) = \frac{x_0 y_0}{t}$$

1.3.5 Example 5 $ty' + y = 0$

$$ty' + y = 0$$

We will use the property

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}F(s)$$

Hence taking Laplace transform of each term of the ode gives

$$\begin{aligned}
\mathcal{L}(ty') &= -\frac{d}{ds}(\mathcal{L}(y')) \\
&= -\frac{d}{ds}(sY - y(0)) \\
&= -\left(Y + s\frac{dY}{ds}\right) \\
&= -s\frac{dY}{ds} - Y
\end{aligned}$$

And

$$\mathcal{L}(y) = Y$$

Hence the ode becomes in Laplace domain as

$$\begin{aligned}
-s\frac{dY}{ds} - Y + Y &= 0 \\
-s\frac{dY}{ds} &= 0 \\
\frac{dY}{ds} &= 0
\end{aligned}$$

Solving this ode for $Y(s)$ gives

$$Y = c_1 \tag{1}$$

Taking inverse Laplace gives

$$y(t) = \delta(t)\, c_1$$

Since no initial conditions are given, then the above is the final solution. Notice that $y(0)$ do not have to be known, since it cancels out in the above. What is left is the c_1 which is generated from solve the ode in $Y(s)$.

1.3.6 Example 6 $ty' + y = 0, y(1) = 5$

$$\begin{aligned}
ty' + y &= 0 \\
y(1) &= 5
\end{aligned}$$

<u>method 1</u>

Since IC given is not at zero, change of variables must be made so that the IC at zero. Let $\tau = t - 1$ then the ode becomes

$$\begin{aligned}
(1+\tau)\, y'(\tau) + y(\tau) &= 0 \\
y'(\tau) + \tau y'(\tau) + y(\tau) &= 0 \\
y(0) &= 5
\end{aligned}$$

Converting the above new ode to Laplace domain using

$$\mathcal{L}(tf(\tau)) = -\frac{d}{ds}F(s)$$

Gives (using $Y(s)$ as the Laplace of $y(\tau)$)

$$\begin{aligned}
sY - y(0) + (-1)\frac{d}{ds}(sY - y(0)) + Y &= 0 \\
sY - y(0) - \left(Y + s\frac{d}{ds}Y\right) + Y &= 0 \\
sY - 5 - Y - s\frac{d}{ds}Y + Y &= 0 \\
sY - s\frac{d}{ds}Y &= 5 \\
\frac{d}{ds}Y - Y &= -\frac{5}{s}
\end{aligned}$$

The solution is

$$Y = c_1 e^s + (5\,\mathrm{Ei}(s))\, e^s$$

Taking inverse Laplace transform gives

$$\begin{aligned}
y(\tau) &= c_1 \mathcal{L}^{-1}(e^s, s, \tau) + \mathcal{L}^{-1}((5\,\mathrm{Ei}(s))\, e^s) \\
&= c_1 \mathcal{L}^{-1}(e^s, s, \tau) + \frac{5}{1+\tau}
\end{aligned} \tag{1}$$

Applying IC $y(0) = 5$ the above becomes

$$\begin{aligned}
5 &= c_1 \mathcal{L}^{-1}(e^s, s, 0) + 5 \\
0 &= c_1 \mathcal{L}^{-1}(e^s, s, 0)
\end{aligned}$$

Hence

$$c_1 = 0$$

Therefore the solution (1) becomes

$$y(\tau) = \frac{5}{1+\tau} \tag{2}$$

Converting back to t the above becomes

$$y(t) = \frac{5}{t}$$

Note that this ode can be solved much more easily but not using Laplace transform. Let see how. The given ode is

$$y' + \frac{y}{t} = 0 \qquad t \neq 0$$

This is linear ode, its solution can be easily found as

$$y = \frac{1}{t} c_1$$

Applying IC

$$\begin{aligned}
5 &= \frac{1}{1} c_1 \\
c_1 &= 5
\end{aligned}$$

Hence the solution is

$$y = \frac{5}{t}$$

method 2

This method shows what happens in the case of time varying ode whose IC is not at zero, and if we do not do change of variables as was done above.

Taking Laplace transform of original ode $ty' + y = 0$ gives

$$\begin{aligned}
-\frac{d}{ds}\left(sY - y\left(0\right)\right) + Y &= 0 \\
-\left(Y + s\frac{dY}{ds}\right) + Y &= 0 \\
-s\frac{dY}{ds} &= 0 \\
\frac{dY}{ds} &= 0
\end{aligned}$$

Hence

$$Y = c_1$$

Taking inverse Laplace transform gives

$$y\left(t\right) = c_1\delta\left(t\right) \tag{1}$$

Applying IC $y\left(1\right) = 5$ to the above

$$\begin{aligned}
5 &= c_1\delta\left(1\right) \\
c_1 &= \frac{5}{\delta\left(1\right)}
\end{aligned}$$

Which is off course is not valid, since $\delta\left(1\right) = 0$. This shows that time varying ode, using Laplace transform, we *must* apply change of variables (as done in method 1) first. Notice that for constant coefficients, both methods work OK. See example above under constant coefficient for problem where IC was not at zero.

So to be consistent, it seems better to stick to one method which works for both time varying and constant coefficients, which is to do change of variables if the IC is given and it is not at zero.

1.3.7 Example 7 $ty' + y = \sin\left(t\right), y\left(1\right) = 0$

$$\begin{aligned}
ty' + y &= \sin\left(t\right) \\
y\left(1\right) &= 0
\end{aligned}$$

Change of variables is made to make the IC at zero. Let $\tau = t - 1$. The ode becomes

$$\begin{aligned}
\left(1+\tau\right)y'\left(\tau\right) + y\left(\tau\right) &= \sin\left(1+\tau\right) \\
y'\left(\tau\right) + \tau y'\left(\tau\right) + y\left(\tau\right) &= \sin\left(1+\tau\right) \\
y\left(0\right) &= 0
\end{aligned}$$

Converting the above new ode to Laplace domain using

$$\mathcal{L}\left(tf\left(\tau\right)\right) = -\frac{d}{ds}F\left(s\right)$$

Gives (using $Y(s)$ as the Laplace of $y(\tau)$)

$$\begin{aligned}(sY - y(0)) + (-1)\frac{d}{ds}(sY - y(0)) + Y &= \frac{\sin(1)s + \cos(1)}{1+s^2}\\ sY - y(0) - \left(Y + s\frac{dY}{ds}\right) + Y &= \frac{\sin(1)s + \cos(1)}{1+s^2}\\ sY - 0 - Y - s\frac{dY}{ds} + Y &= \frac{\sin(1)s + \cos(1)}{1+s^2}\\ sY - s\frac{d}{ds}Y &= \frac{\sin(1)s + \cos(1)}{1+s^2}\\ \frac{d}{ds}Y - Y &= -\frac{\sin(1)s + \cos(1)}{s(1+s^2)}\end{aligned}$$

The above is linear ode. Solving it gives

$$\begin{aligned}Y &= \frac{e^s}{2}\left(2\operatorname{Ei}(1,s)\cos(1) - \operatorname{Ei}(1,s+i) - \operatorname{Ei}(1,s-i) + 2c_1\right)\\ &= e^s\operatorname{Ei}(1,s)\cos(1) - \frac{e^s}{2}\operatorname{Ei}(1,s+i) - \frac{e^s}{2}\operatorname{Ei}(1,s-i) + c_1e^s \qquad (1)\end{aligned}$$

Taking inverse Laplace transform gives

$$y(\tau) = \frac{\cos 1}{\tau+1} - \frac{\cos(\tau+1)}{\tau+1} + c_1\mathcal{L}^{-1}(e^s) \qquad (4)$$

Applying IC $y(0) = 0$

$$\begin{aligned}0 &= \cos(1) - \cos(1) + c_1\mathcal{L}^{-1}(e^s)\\ 0 &= c_1\mathcal{L}^{-1}(e^s)\end{aligned}$$

Hence $c_1 = 0$. Therefore (3) becomes

$$y(\tau) = \frac{\cos 1}{\tau+1} - \frac{\cos(\tau+1)}{\tau+1}$$

Going back to t using $\tau = t - 1$ the above becomes

$$y(t) = \frac{\cos 1}{t} - \frac{\cos(t)}{t}$$

1.3.8 Example 8 $ty' + y = t, y(1) = 0$

$$\begin{aligned}ty' + y &= t\\ y(1) &= 0\end{aligned}$$

Applying change of variables to make the IC at zero. Let $\tau = t - 1$ the ode becomes

$$\begin{aligned}(\tau+1)y'(\tau) + y(\tau) &= \tau + 1\\ y'(\tau) + \tau y'(\tau) + y(\tau) &= \tau + 1\\ y(0) &= 0\end{aligned}$$

Converting the above new ode to Laplace domain using

$$\mathcal{L}(tf(\tau)) = -\frac{d}{ds}F(s)$$

Gives (using $Y(s)$ as the Laplace of $y(\tau)$)

$$\begin{aligned}
(sY - y(0)) + (-1)\frac{d}{ds}(sY - y(0)) + Y &= \frac{s+1}{s^2} \\
sY - y(0) - \left(Y + s\frac{dY}{ds}\right) + Y &= \frac{s+1}{s^2} \\
sY - 0 - Y - s\frac{dY}{ds} + Y &= \frac{s+1}{s^2} \\
sY - s\frac{d}{ds}Y &= \frac{s+1}{s^2} \\
\frac{d}{ds}Y - Y &= -\frac{s+1}{s^3}
\end{aligned}$$

The above is linear ode. Solving it gives

$$Y = \frac{1}{2s^2} + \frac{1}{2s} - \frac{e^s \operatorname{Ei}(1,s)}{2} + c_1 e^s$$

Taking the inverse Laplace transform gives

$$y(\tau) = \frac{\tau}{2} + \frac{1}{2} - \frac{1}{2(1+\tau)} + c_1 \mathcal{L}^{-1}(e^s) \tag{1}$$

Applying $y(0) = 0$

$$\begin{aligned}
0 &= \frac{1}{2} - \frac{1}{2} + c_1 \mathcal{L}^{-1}(e^s) \\
0 &= c_1 \mathcal{L}^{-1}(e^s) \qquad (2)
\end{aligned}$$

Hence $c_1 = 0$. Therefore (1) becomes

$$y(\tau) = \frac{\tau}{2} + \frac{1}{2} - \frac{1}{2(1+\tau)}$$

Going back to t using $\tau = t - 1$ the above becomes

$$\begin{aligned}
y(t) &= \frac{t-1}{2} + \frac{1}{2} - \frac{1}{2t} \\
&= \frac{t}{2} - \frac{1}{2t}
\end{aligned}$$

We see in the above, we did not have to use initial value theorem to find c_1. This is because the IC was $y(0) = 0$. But if the IC was $y(0) = y_0$, where $y_0 \neq 0$ then (2) would becomes

$$y_0 = c_1 \mathcal{L}^{-1}(e^s)$$

And then we can not solve for c_1. So the above method works for homogeneous IC. The following example solve this same problem but with IC $y(1) = 1$ to show how to handle these cases.

1.3.9 Example 9 $ty' + y = t, y(1) = 1$

This is the same example as above, but with $y(1) = 1$ instead of homogeneous IC $y(1) = 0$.

$$\begin{aligned}
ty' + y &= t \\
y(1) &= 1
\end{aligned}$$

Applying change of variables to make the IC at zero. Let $\tau = t - 1$ the ode becomes

$$\begin{aligned}
(\tau+1)\,y'(\tau) + y(\tau) &= \tau + 1 \\
y'(\tau) + \tau y'(\tau) + y(\tau) &= \tau + 1 \\
y(0) &= 1
\end{aligned}$$

Converting the above new ode to Laplace domain using

$$\mathcal{L}\left(tf\left(\tau\right)\right) = -\frac{d}{ds}F\left(s\right)$$

Gives (using $Y\left(s\right)$ as the Laplace of $y\left(\tau\right)$)

$$\begin{aligned}
\left(sY - y\left(0\right)\right) + (-1)\frac{d}{ds}\left(sY - y\left(0\right)\right) + Y &= \frac{s+1}{s^2} \\
sY - y\left(0\right) - \left(Y + s\frac{dY}{ds}\right) + Y &= \frac{s+1}{s^2} \\
sY - 1 - Y - s\frac{dY}{ds} + Y &= \frac{s+1}{s^2} \\
sY - s\frac{d}{ds}Y &= \frac{s+1}{s^2} + 1 \\
\frac{d}{ds}Y - Y &= -\frac{s+1}{s^3} - \frac{1}{s}
\end{aligned}$$

The above is linear ode. Solving it gives

$$Y = \frac{1}{2s^2} + \frac{1}{2s} + \frac{e^s \operatorname{Ei}\left(1, s\right)}{2} + c_1 e^s \tag{1}$$

Taking the inverse Laplace gives

$$y\left(\tau\right) = \frac{\tau}{2} + \frac{1}{2} + \frac{1}{2\left(1+\tau\right)} + c_1\mathcal{L}^{-1}\left(e^s\right) \tag{2}$$

Applying IC $y\left(0\right) = 1$ gives

$$\begin{aligned}
1 &= \frac{1}{2} + \frac{1}{2} + c_1\mathcal{L}^{-1}\left(e^s\right) \\
0 &= c_1\mathcal{L}^{-1}\left(e^s\right) \\
c_1 &= 0
\end{aligned}$$

Hence (2) becomes

$$y\left(\tau\right) = \frac{\tau}{2} + \frac{1}{2} + \frac{1}{2\left(1+\tau\right)}$$

Going back to t using $\tau = t - 1$ the above becomes

$$\begin{aligned}
y\left(t\right) &= \frac{t-1}{2} + \frac{1}{2} + \frac{1}{2t} \\
&= \frac{t}{2} + \frac{1}{2t}
\end{aligned}$$

1.3.10 Example 10 $y' + t^2 y = 0, y(0) = 0$

$$\begin{aligned}
y' + t^2 y &= 0 \\
y\left(0\right) &= 0
\end{aligned}$$

Using the property

$$\mathcal{L}\left(t^n f\left(t\right)\right) = (-1)^n \frac{d^n}{ds^n}F\left(s\right)$$

Taking Laplace transform of each term of the ode gives

$$\mathcal{L}\left(y'\right) = sY - y\left(0\right)$$

And

$$\begin{aligned}\mathcal{L}\left(t^{2}y\right) &= (-1)^{2}\frac{d^{2}}{ds^{2}}L\left(y\right)\\ &= \frac{d^{2}}{ds^{2}}Y\end{aligned}$$

Hence the ode becomes in Laplace domain as

$$\begin{aligned}sY - y\left(0\right) + \frac{d^{2}}{ds^{2}}Y &= 0\\ \frac{d^{2}}{ds^{2}}Y + sY &= y\left(0\right)\end{aligned}$$

Replacing $y\left(0\right)$ from initial conditions

$$\frac{d^{2}}{ds^{2}}Y + sY = 0$$

This is Airy ode. The solution is

$$Y = c_1 \operatorname{AiryAi}\left(-s\right) + c_2 \operatorname{AiryBi}\left(-s\right) \tag{1}$$

Taking inverse Laplace transform gives

$$y = c_1\mathcal{L}^{-1}\operatorname{AiryAi}\left(-s\right) + c_2\mathcal{L}^{-1}\operatorname{AiryBi}\left(-s\right) \tag{2}$$

Since $y_0 = 0$ at $t = 0$, the above becomes

$$0 = c_1\mathcal{L}^{-1}\operatorname{AiryAi}\left(-s\right) + c_2\mathcal{L}^{-1}\operatorname{AiryBi}\left(-s\right)$$

if we take $c_1 = 0, c_2 = 0$, this will make the LHS equal to RHS. Hence (2) becomes

$$y\left(t\right) = 0$$

I need to double check I could do the above or not. If not, then this is not possible to solve using Laplace, since there is no inverse Laplace transform for Airy functions.

1.3.11 Example 11 $(1 + at)\,y' + y = t, y(1) = 0$

$$\begin{aligned}(1 + at)\,y' + y &= t\\ y\left(1\right) &= 0\end{aligned}$$

Applying change of variables to make the IC at zero. Let $\tau = t - 1$ the ode becomes

$$\begin{aligned}\left(1 + a\left(\tau + 1\right)\right)y' + y &= \tau + 1\\ y' + a\left(\tau + 1\right)y' + y &= \tau + 1\\ y' + a\tau y' + ay' + y &= \tau + 1\\ \left(1 + a\right)y' + a\tau y' + y &= \tau + 1\\ y\left(0\right) &= 0\end{aligned}$$

Converting the above new ode to Laplace domain using

$$\mathcal{L}\left(tf\left(\tau\right)\right) = -\frac{d}{ds}F\left(s\right)$$

Gives (using $Y(s)$ as the Laplace of $y(\tau)$)

$$
\begin{aligned}
(1+a)(sY - y(0)) + a(-1)\frac{d}{ds}(sY - y(0)) + Y &= \frac{s+1}{s^2} \\
(1+a)sY - a\frac{d}{ds}(sY) + Y &= \frac{s+1}{s^2} \\
sY + asY - a\left(Y + s\frac{dY}{ds}\right) + Y &= \frac{s+1}{s^2} \\
sY + asY - aY - as\frac{dY}{ds} + Y &= \frac{s+1}{s^2} \\
-as\frac{dY}{ds} + Y(1+s+as-a) &= \frac{s+1}{s^2} \\
\frac{dY}{ds} - Y\frac{(1+s+as-a)}{as} &= -\frac{s+1}{as^3}
\end{aligned}
$$

This is linear in $Y(s)$. Solving gives

$$
Y(s) = \frac{1}{s^2(a+1)} + c_1 \frac{s^{\frac{a+1}{a}} e^{s\frac{(a+1)}{a}}}{s^2}
$$

Taking inverse Laplace gives

$$
y(\tau) = \frac{\tau}{a+1} + c_1 \mathcal{L}^{-1}\left(\frac{s^{\frac{a+1}{a}} e^{s\frac{(a+1)}{a}}}{s^2}\right)
$$

Applying IC $y(0) = 0$ the above becomes

$$
0 = c_1 \mathcal{L}^{-1}\left(\frac{s^{\frac{a+1}{a}} e^{s\frac{(a+1)}{a}}}{s^2}\right)
$$

Hence $c_1 = 0$ and the solution (1) becomes

$$
y(\tau) = \frac{\tau}{a+1}
$$

Going back to t using $\tau = t - 1$ gives

$$
y(t) = \frac{t-1}{a+1}
$$