Example solving non-linear first order ODE

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$$
\begin{aligned}
\frac{dy}{dt}+y^{\frac{3}{2}}(t)&=a^{\frac{3}{2}}\\
y(0)&=0
\end{aligned}
$$

Write as

$$
\begin{aligned}
\left(y^{\frac{3}{2}}-a^{\frac{3}{2}}\right)dt+dy&=0\\
M(t,y)\,dt+N(t,y)\,dy&=0 \qquad (1)
\end{aligned}
$$

Where

$$
\begin{aligned}
M&=y^{\frac{3}{2}}-a^{\frac{3}{2}}\\
N&=1
\end{aligned}
$$

Check if exact

$$
\begin{aligned}
\frac{\partial M(t,y)}{\partial y}&=\frac{3}{2}y^{\frac{1}{2}}\\
\frac{\partial N(t,y)}{\partial t}&=0
\end{aligned}
$$

Since $\frac{\partial M(t,y)}{\partial y}\neq\frac{\partial N(t,y)}{\partial t}$ then Not exact. Trying integrating factor $A=\frac{\frac{\partial N}{\partial t}-\frac{\partial M}{\partial y}}{M}=\frac{-\frac{3}{2}y^{\frac{1}{2}}}{y^{\frac{3}{2}}-a^{\frac{3}{2}}}$, Since it is a function of y alone, then it (1) can be made exact. The integrating factor is

$$
\begin{aligned}
\mu&=e^{\int A dy}\\
&=e^{\int\frac{-\frac{3}{2}y^{\frac{1}{2}}}{y^{\frac{3}{2}}-a^{\frac{3}{2}}}dy}\\
&=e^{-\ln\left(a^{\frac{3}{2}}-y^{\frac{3}{2}}\right)}\\
&=\frac{1}{a^{\frac{3}{2}}-y^{\frac{3}{2}}}
\end{aligned}
$$

Multiplying (1) by this integrating factor, now it becomes exact

$$
\mu M(t,y)\,dt+\mu N(t,y)\,dy=0
$$

Now we follow standard method for solving exact ODE. Let

$$
\begin{aligned}
\frac{dU}{dt}&=\mu M=\frac{y^{\frac{3}{2}}-a^{\frac{3}{2}}}{a^{\frac{3}{2}}-y^{\frac{3}{2}}}=-1 \qquad (2)\\
\frac{dU}{dy}&=\mu N=\frac{1}{a^{\frac{3}{2}}-y^{\frac{3}{2}}} \qquad (3)
\end{aligned}
$$

From (2)

$$
\begin{aligned}
U&=-\int dt\\
&=-t+f(y) \qquad (4)
\end{aligned}
$$

Substituting this into (3) to solve for $f(y)$

$$
\begin{aligned}
f'(y)&=\frac{1}{a^{\frac{3}{2}}-y^{\frac{3}{2}}}\\
f(y)&=\frac{-2\sqrt{3}}{3\sqrt{a}}\arctan\left(\frac{1+2\sqrt{\frac{y}{a}}}{\sqrt{3}}\right)-\frac{2}{3\sqrt{a}}\ln\left(\sqrt{a}-\sqrt{y}\right)+\frac{1}{3\sqrt{a}}\ln\left(a+\sqrt{ay}+y\right)+C
\end{aligned}
$$

Hence the solution from (4) is

$$U=-t+\frac{-2\sqrt{3}}{3\sqrt{a}}\arctan\left(\frac{1+2\sqrt{\frac{y}{a}}}{\sqrt{3}}\right)-\frac{2}{3\sqrt{a}}\ln\left(\sqrt{a}-\sqrt{y}\right)+\frac{1}{3\sqrt{a}}\ln\left(a+\sqrt{ay}+y\right)+C$$

But $\frac{dU}{dt}=0$, hence $U=C_1$. Therefore, collecting constants into one, the solution is (implicit form)

$$t+\frac{2\sqrt{3}}{3\sqrt{a}}\arctan\left(\frac{1+2\sqrt{\frac{y}{a}}}{\sqrt{3}}\right)+\frac{2}{3\sqrt{a}}\ln\left(\sqrt{a}-\sqrt{y}\right)-\frac{1}{3\sqrt{a}}\ln\left(a+\sqrt{ay}+y\right)=C$$

From initial conditions

$$\frac{2\sqrt{3}}{3\sqrt{a}}\arctan\left(\frac{1}{\sqrt{3}}\right)+\frac{2}{3\sqrt{a}}\ln\left(\sqrt{a}\right)-\frac{1}{3\sqrt{a}}\ln\left(a\right)=C$$

$$\begin{aligned}C&=\frac{2\sqrt{3}}{3\sqrt{a}}\frac{\pi}{6}+\frac{2}{3\sqrt{a}}\ln\left(\sqrt{a}\right)-\frac{1}{3\sqrt{a}}\ln\left(a\right)\\C&=\frac{2\sqrt{3}}{3\sqrt{a}}\frac{\pi}{6}\\C&=\frac{\pi\sqrt{3}}{9\sqrt{a}}\end{aligned}$$

Hence final solution for $y(t)$ in implicit form is

$$\begin{aligned}t+\frac{2\sqrt{3}}{3\sqrt{a}}\arctan\left(\frac{1+2\sqrt{\frac{y}{a}}}{\sqrt{3}}\right)+\frac{2}{3\sqrt{a}}\ln\left(\sqrt{a}-\sqrt{y}\right)-\frac{1}{3\sqrt{a}}\ln\left(a+\sqrt{ay}+y\right)&=\frac{\pi\sqrt{3}}{9\sqrt{a}}\\3t\sqrt{a}+2\sqrt{3}\arctan\left(\frac{\sqrt{a}+2\sqrt{y}}{\sqrt{3}\sqrt{a}}\right)+6\ln\left(\sqrt{a}-\sqrt{y}\right)-\ln\left(a+\sqrt{ay}+y\right)&=\frac{\pi\sqrt{3}}{3}\end{aligned}$$