

Example solving non-linear first order ODE

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$$\begin{aligned}\frac{dy}{dt} + y^{\frac{3}{2}}(t) &= a^{\frac{3}{2}} \\ y(0) &= 0\end{aligned}$$

Write as

$$\begin{aligned}(y^{\frac{3}{2}} - a^{\frac{3}{2}}) dt + dy &= 0 \\ M(t, y) dt + N(t, y) dy &= 0\end{aligned}\tag{1}$$

Where

$$\begin{aligned}M &= y^{\frac{3}{2}} - a^{\frac{3}{2}} \\ N &= 1\end{aligned}$$

Check if exact

$$\begin{aligned}\frac{\partial M(t, y)}{\partial y} &= \frac{3}{2}y^{\frac{1}{2}} \\ \frac{\partial N(t, y)}{\partial t} &= 0\end{aligned}$$

Since $\frac{\partial M(t, y)}{\partial y} \neq \frac{\partial N(t, y)}{\partial t}$ then Not exact. Trying integrating factor $A = \frac{\frac{\partial N}{\partial t} - \frac{\partial M}{\partial y}}{M} = \frac{-\frac{3}{2}y^{\frac{1}{2}}}{y^{\frac{3}{2}} - a^{\frac{3}{2}}}$, Since it is a function of y alone, then it (1) can be made exact. The integrating factor is

$$\begin{aligned}\mu &= e^{\int A dy} \\ &= e^{\int \frac{-\frac{3}{2}y^{\frac{1}{2}}}{y^{\frac{3}{2}} - a^{\frac{3}{2}}} dy} \\ &= e^{-\ln(a^{\frac{3}{2}} - y^{\frac{3}{2}})} \\ &= \frac{1}{a^{\frac{3}{2}} - y^{\frac{3}{2}}}\end{aligned}$$

Multiplying (1) by this integrating factor, now it becomes exact

$$\mu M(t, y) dt + \mu N(t, y) dy = 0$$

Now we follow standard method for solving exact ODE. Let

$$\frac{dU}{dt} = \mu M = \frac{y^{\frac{3}{2}} - a^{\frac{3}{2}}}{a^{\frac{3}{2}} - y^{\frac{3}{2}}} = -1 \quad (2)$$

$$\frac{dU}{dy} = \mu N = \frac{1}{a^{\frac{3}{2}} - y^{\frac{3}{2}}} \quad (3)$$

From (2)

$$\begin{aligned} U &= - \int dt \\ &= -t + f(y) \end{aligned} \quad (4)$$

Substituting this into (3) to solve for $f(y)$

$$\begin{aligned} f'(y) &= \frac{1}{a^{\frac{3}{2}} - y^{\frac{3}{2}}} \\ f(y) &= \frac{-2\sqrt{3}}{3\sqrt{a}} \arctan\left(\frac{1 + 2\sqrt{\frac{y}{a}}}{\sqrt{3}}\right) - \frac{2}{3\sqrt{a}} \ln(\sqrt{a} - \sqrt{y}) + \frac{1}{3\sqrt{a}} \ln(a + \sqrt{ay} + y) + C \end{aligned}$$

Hence the solution from (4) is

$$U = -t + \frac{-2\sqrt{3}}{3\sqrt{a}} \arctan\left(\frac{1 + 2\sqrt{\frac{y}{a}}}{\sqrt{3}}\right) - \frac{2}{3\sqrt{a}} \ln(\sqrt{a} - \sqrt{y}) + \frac{1}{3\sqrt{a}} \ln(a + \sqrt{ay} + y) + C$$

But $\frac{dU}{dt} = 0$, hence $U = C_1$. Therefore, collecting constants into one, the solution is (implicit form)

$$t + \frac{2\sqrt{3}}{3\sqrt{a}} \arctan\left(\frac{1 + 2\sqrt{\frac{y}{a}}}{\sqrt{3}}\right) + \frac{2}{3\sqrt{a}} \ln(\sqrt{a} - \sqrt{y}) - \frac{1}{3\sqrt{a}} \ln(a + \sqrt{ay} + y) = C$$

From initial conditions

$$\begin{aligned} \frac{2\sqrt{3}}{3\sqrt{a}} \arctan\left(\frac{1}{\sqrt{3}}\right) + \frac{2}{3\sqrt{a}} \ln(\sqrt{a}) - \frac{1}{3\sqrt{a}} \ln(a) &= C \\ C &= \frac{2\sqrt{3}}{3\sqrt{a}} \frac{\pi}{6} + \frac{2}{3\sqrt{a}} \ln(\sqrt{a}) - \frac{1}{3\sqrt{a}} \ln(a) \\ C &= \frac{2\sqrt{3}}{3\sqrt{a}} \frac{\pi}{6} \\ C &= \frac{\pi\sqrt{3}}{9\sqrt{a}} \end{aligned}$$

Hence final solution for $y(t)$ in implicit form is

$$t + \frac{2\sqrt{3}}{3\sqrt{a}} \arctan\left(\frac{1 + 2\sqrt{\frac{y}{a}}}{\sqrt{3}}\right) + \frac{2}{3\sqrt{a}} \ln(\sqrt{a} - \sqrt{y}) - \frac{1}{3\sqrt{a}} \ln(a + \sqrt{ay} + y) = \frac{\pi\sqrt{3}}{9\sqrt{a}}$$

$$3t\sqrt{a} + 2\sqrt{3} \arctan\left(\frac{\sqrt{a} + 2\sqrt{y}}{\sqrt{3}\sqrt{a}}\right) + 6 \ln(\sqrt{a} - \sqrt{y}) - \ln(a + \sqrt{ay} + y) = \frac{\pi\sqrt{3}}{3}$$