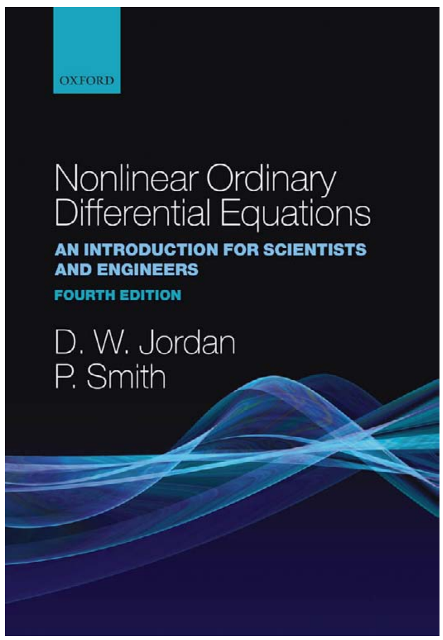
A Solution Manual For

Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY



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1.1 problem 2.1 (i)

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Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.1 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= x(t) - 5y(t) \\ y'(t) &= x(t) - y(t) \end{aligned}$$

1.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\sin(2t)}{2} + \cos(2t) & -\frac{5\sin(2t)}{2} \\ \frac{\sin(2t)}{2} & \cos(2t) - \frac{\sin(2t)}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{\sin(2t)}{2} + \cos(2t) & -\frac{5\sin(2t)}{2} \\ \frac{\sin(2t)}{2} & \cos(2t) - \frac{\sin(2t)}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{\sin(2t)}{2} + \cos(2t)\right) c_1 - \frac{5\sin(2t)c_2}{2} \\ \frac{\sin(2t)c_1}{2} + \left(\cos(2t) - \frac{\sin(2t)}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-5c_2)\sin(2t)}{2} + c_1\cos(2t) \\ \frac{(-c_2+c_1)\sin(2t)}{2} + c_2\cos(2t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2i$	1	complex eigenvalue
$-2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} - (-2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1+2i & -5 \\ 1 & -1+2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1+2i & -5 & 0 \\ 1 & -1+2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{5} + \frac{2i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 1+2i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+2i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1-2i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1-2\,\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1-2i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1-2\,\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1-2\,\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} - (2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1-2i & -5 & 0 \\ 1 & -1-2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{5} - \frac{2i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 1-2i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-2i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1+2i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1+2\ \mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1+2i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1+2\ \mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1+2\ \mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2i$	1	1	No	$\begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$
$-2i$	1	1	No	$\begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} (1+2i)\, e^{2it} \\ e^{2it} \end{bmatrix} + c_2 \begin{bmatrix} (1-2i)\, e^{-2it} \\ e^{-2it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (1+2i)\, c_1 e^{2it} + (1-2i)\, c_2 e^{-2it} \\ c_1 e^{2it} + c_2 e^{-2it} \end{bmatrix}$$

The following is the phase plot of the system.

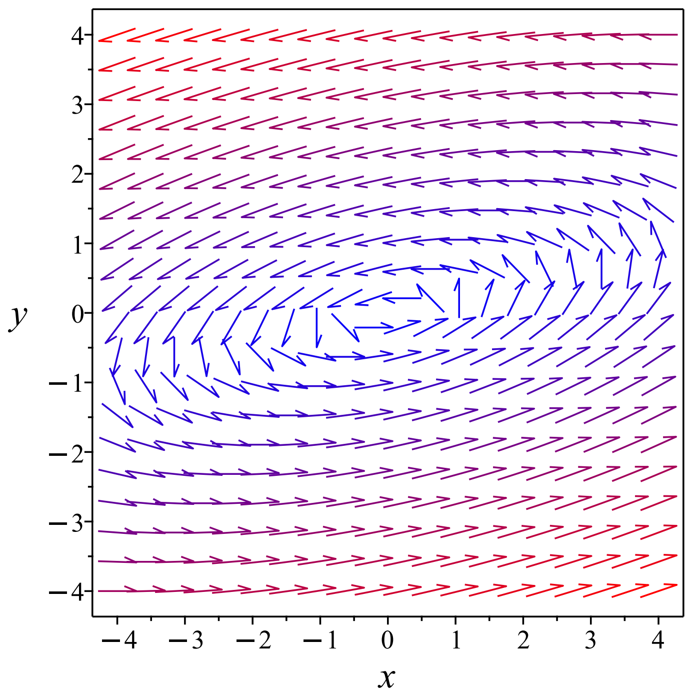

Figure 1: Phase plot

1.1.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) - 5y(t)\,, y'(t) = x(t) - y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2\,\mathrm{I}, \begin{bmatrix} 1-2\,\mathrm{I} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} 1+2\,\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} 1-2\,\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-2\,\mathrm{I}t} \cdot \begin{bmatrix} 1-2\,\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} 1-2\,\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} (1-2\,\mathrm{I})\,(\cos(2t)-\mathrm{I}\sin(2t)) \\ \cos(2t)-\mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t)=\begin{bmatrix} \cos(2t)-2\sin(2t) \\ \cos(2t) \end{bmatrix}, \vec{x}_2(t)=\begin{bmatrix} -2\cos(2t)-\sin(2t) \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x}=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x}=\begin{bmatrix} c_2(-2\cos(2t)-\sin(2t))+c_1(\cos(2t)-2\sin(2t)) \\ c_1\cos(2t)-c_2\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}=\begin{bmatrix} (c_1-2c_2)\cos(2t)-2\sin(2t)\left(\frac{c_2}{2}+c_1\right) \\ c_1\cos(2t)-c_2\sin(2t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t)=(c_1-2c_2)\cos(2t)-2\sin(2t)\left(\frac{c_2}{2}+c_1\right), y(t)=c_1\cos(2t)-c_2\sin(2t)\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 50

```
dsolve([diff(x(t),t)=x(t)-5*y(t),diff(y(t),t)=x(t)-y(t)],singsol=all)
```

$$x(t)=c_1\sin(2t)+c_2\cos(2t)$$
$$y(t)=-\frac{2c_1\cos(2t)}{5}+\frac{2c_2\sin(2t)}{5}+\frac{c_1\sin(2t)}{5}+\frac{c_2\cos(2t)}{5}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 48

```
DSolve[{x'[t]==x[t]-5*y[t],y'[t]==x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t)\to c_1\cos(2t)+(c_1-5c_2)\sin(t)\cos(t)$$
$$y(t)\to c_2\cos(2t)+(c_1-c_2)\sin(t)\cos(t)$$

1.2 problem 2.1 (ii)

Internal problem ID [12556]
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Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.1 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = x(t) + y(t)$$
$$y'(t) = x(t) - 2y(t)$$

1.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(3\sqrt{13}+13)\,e^{\frac{(-1+\sqrt{13})t}{2}}}{26} + \frac{(-3\sqrt{13}+13)\,e^{-\frac{(1+\sqrt{13})t}{2}}}{26} & -\frac{\left(-e^{\frac{(-1+\sqrt{13})t}{2}} + e^{-\frac{(1+\sqrt{13})t}{2}}\right)\sqrt{13}}{13} \\ -\frac{\left(-e^{\frac{(-1+\sqrt{13})t}{2}} + e^{-\frac{(1+\sqrt{13})t}{2}}\right)\sqrt{13}}{13} & \frac{(-3\sqrt{13}+13)\,e^{\frac{(-1+\sqrt{13})t}{2}}}{26} + \frac{e^{-\frac{(1+\sqrt{13})t}{2}}(3\sqrt{13}+13)}{26} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{\left(3\sqrt{13}+13\right)e^{\frac{\left(-1+\sqrt{13}\right)t}{2}}}{26} + \frac{\left(-3\sqrt{13}+13\right)e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}}{26} & -\frac{\left(-e^{\frac{\left(-1+\sqrt{13}\right)t}{2}} + e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}\right)\sqrt{13}}{13} \\ -\frac{\left(-e^{\frac{\left(-1+\sqrt{13}\right)t}{2}} + e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}\right)\sqrt{13}}{13} & \frac{\left(-3\sqrt{13}+13\right)e^{\frac{\left(-1+\sqrt{13}\right)t}{2}}}{26} + \frac{e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}\left(3\sqrt{13}+13\right)}{26} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\left(3\sqrt{13}+13\right)e^{\frac{\left(-1+\sqrt{13}\right)t}{2}}}{26} + \frac{\left(-3\sqrt{13}+13\right)e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}}{26}\right)c_1 - \frac{\left(-e^{\frac{\left(-1+\sqrt{13}\right)t}{2}} + e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}\right)\sqrt{13}\,c_2}{13} \\ -\frac{\left(-e^{\frac{\left(-1+\sqrt{13}\right)t}{2}} + e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}\right)\sqrt{13}\,c_1}{13} + \left(\frac{\left(-3\sqrt{13}+13\right)e^{\frac{\left(-1+\sqrt{13}\right)t}{2}}}{26} + \frac{e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}\left(3\sqrt{13}+13\right)}{26}\right)c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left((3c_1+2c_2)\sqrt{13}+13c_1\right)e^{\frac{\left(-1+\sqrt{13}\right)t}{2}}}{26} - \frac{3\,e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}\left(\left(c_1+\frac{2c_2}{3}\right)\sqrt{13}-\frac{13c_1}{3}\right)}{26} \\ \frac{\left((2c_1-3c_2)\sqrt{13}+13c_2\right)e^{\frac{\left(-1+\sqrt{13}\right)t}{2}}}{26} - \frac{e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}\left(\left(c_1-\frac{3c_2}{2}\right)\sqrt{13}-\frac{13c_2}{2}\right)}{13} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda - 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{2} + \frac{\sqrt{13}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{\sqrt{13}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{2} - \frac{\sqrt{13}}{2}$	1	real eigenvalue
$-\frac{1}{2} + \frac{\sqrt{13}}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{2} - \frac{\sqrt{13}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} - \left(-\frac{1}{2} - \frac{\sqrt{13}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} + \frac{\sqrt{13}}{2} & 1 \\ 1 & -\frac{3}{2} + \frac{\sqrt{13}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{3}{2} + \frac{\sqrt{13}}{2} & 1 & 0 \\ 1 & -\frac{3}{2} + \frac{\sqrt{13}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{\frac{3}{2}+\frac{\sqrt{13}}{2}} \implies \left[\begin{array}{cc|c} \frac{3}{2}+\frac{\sqrt{13}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{2}+\frac{\sqrt{13}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{3+\sqrt{13}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{3+\sqrt{13}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3+\sqrt{13}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{3+\sqrt{13}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3+\sqrt{13}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{3+\sqrt{13}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3+\sqrt{13}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{3+\sqrt{13}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3+\sqrt{13}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2} + \frac{\sqrt{13}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} - \left(-\frac{1}{2}+\frac{\sqrt{13}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2}-\frac{\sqrt{13}}{2} & 1 \\ 1 & -\frac{3}{2}-\frac{\sqrt{13}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{3}{2}-\frac{\sqrt{13}}{2} & 1 & 0 \\ 1 & -\frac{3}{2}-\frac{\sqrt{13}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{\frac{3}{2}-\frac{\sqrt{13}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{3}{2}-\frac{\sqrt{13}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{2}-\frac{\sqrt{13}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{-3+\sqrt{13}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{-3+\sqrt{13}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{-3+\sqrt{13}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{-3+\sqrt{13}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{-3+\sqrt{13}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{-3+\sqrt{13}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{-3+\sqrt{13}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{1}{2}+\frac{\sqrt{13}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{1}{-\frac{3}{2}+\frac{\sqrt{13}}{2}}\\ 1\end{array}\right]$
$-\frac{1}{2}-\frac{\sqrt{13}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{1}{-\frac{3}{2}-\frac{\sqrt{13}}{2}}\\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{1}{2}+\frac{\sqrt{13}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(-\frac{1}{2}+\frac{\sqrt{13}}{2}\right)t} \\
&= \left[\begin{array}{c}\frac{1}{-\frac{3}{2}+\frac{\sqrt{13}}{2}}\\ 1\end{array}\right] e^{\left(-\frac{1}{2}+\frac{\sqrt{13}}{2}\right)t}
\end{aligned}
$$

Since eigenvalue $-\frac{1}{2}-\frac{\sqrt{13}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(-\frac{1}{2}-\frac{\sqrt{13}}{2}\right)t} \\
&= \left[\begin{array}{c}\frac{1}{-\frac{3}{2}-\frac{\sqrt{13}}{2}}\\ 1\end{array}\right] e^{\left(-\frac{1}{2}-\frac{\sqrt{13}}{2}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = c_1 \left[\begin{array}{c}\frac{\mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{13}}{2}\right)t}}{-\frac{3}{2}+\frac{\sqrt{13}}{2}}\\ \mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{13}}{2}\right)t}\end{array}\right] + c_2 \left[\begin{array}{c}\frac{\mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{13}}{2}\right)t}}{-\frac{3}{2}-\frac{\sqrt{13}}{2}}\\ \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{13}}{2}\right)t}\end{array}\right]
$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1\left(3+\sqrt{13}\right)e^{\frac{\left(-1+\sqrt{13}\right)t}{2}}}{2} - \frac{e^{-\frac{\left(1+\sqrt{13}\right)t}{2}} c_2\left(-3+\sqrt{13}\right)}{2} \\ c_1 e^{\frac{\left(-1+\sqrt{13}\right)t}{2}} + c_2 e^{-\frac{\left(1+\sqrt{13}\right)t}{2}} \end{bmatrix}$$

The following is the phase plot of the system.

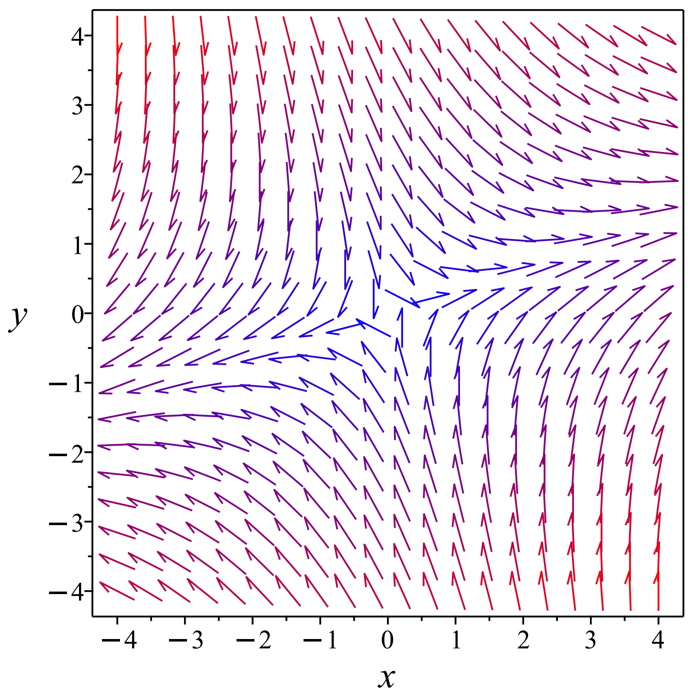

Figure 2: Phase plot

1.2.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + y(t)\,, y'(t) = x(t) - 2y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{2} - \frac{\sqrt{13}}{2}, \begin{bmatrix} \frac{1}{-\frac{3}{2} - \frac{\sqrt{13}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{\sqrt{13}}{2}, \begin{bmatrix} \frac{1}{-\frac{3}{2} + \frac{\sqrt{13}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{1}{2} - \frac{\sqrt{13}}{2}, \begin{bmatrix} \frac{1}{-\frac{3}{2} - \frac{\sqrt{13}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{\left(-\frac{1}{2} - \frac{\sqrt{13}}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{-\frac{3}{2} - \frac{\sqrt{13}}{2}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{1}{2} + \frac{\sqrt{13}}{2}, \begin{bmatrix} \frac{1}{-\frac{3}{2} + \frac{\sqrt{13}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{\left(-\frac{1}{2} + \frac{\sqrt{13}}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{-\frac{3}{2} + \frac{\sqrt{13}}{2}} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 e^{\left(-\frac{1}{2}-\frac{\sqrt{13}}{2}\right)t} \cdot \left[\begin{array}{c}\frac{1}{-\frac{3}{2}-\frac{\sqrt{13}}{2}} \\ 1\end{array}\right] + c_2 e^{\left(-\frac{1}{2}+\frac{\sqrt{13}}{2}\right)t} \cdot \left[\begin{array}{c}\frac{1}{-\frac{3}{2}+\frac{\sqrt{13}}{2}} \\ 1\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y(t)\end{array}\right] = \left[\begin{array}{c}\frac{c_2\left(3+\sqrt{13}\right)e^{\frac{\left(-1+\sqrt{13}\right)t}{2}}}{2} - \frac{e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}c_1\left(-3+\sqrt{13}\right)}{2} \\ c_1 e^{-\frac{\left(1+\sqrt{13}\right)t}{2}} + c_2 e^{\frac{\left(-1+\sqrt{13}\right)t}{2}}\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{c_2\left(3+\sqrt{13}\right)e^{\frac{\left(-1+\sqrt{13}\right)t}{2}}}{2} - \frac{e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}c_1\left(-3+\sqrt{13}\right)}{2}, y(t) = c_1 e^{-\frac{\left(1+\sqrt{13}\right)t}{2}} + c_2 e^{\frac{\left(-1+\sqrt{13}\right)t}{2}}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 86

```
dsolve([diff(x(t),t)=x(t)+y(t),diff(y(t),t)=x(t)-2*y(t)],singsol=all)
```

$$x(t) = c_1 e^{\frac{\left(-1+\sqrt{13}\right)t}{2}} + c_2 e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}$$

$$y(t) = \frac{c_1 e^{\frac{\left(-1+\sqrt{13}\right)t}{2}}\sqrt{13}}{2} - \frac{c_2 e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}\sqrt{13}}{2} - \frac{3c_1 e^{\frac{\left(-1+\sqrt{13}\right)t}{2}}}{2} - \frac{3c_2 e^{-\frac{\left(1+\sqrt{13}\right)t}{2}}}{2}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 149

```
DSolve[{x'[t]==x[t]+y[t],y'[t]==x[t]-2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{26}e^{-\frac{1}{2}\left(1+\sqrt{13}\right)t}\left(c_1\left(\left(13+3\sqrt{13}\right)e^{\sqrt{13}t} + 13 - 3\sqrt{13}\right) + 2\sqrt{13}c_2\left(e^{\sqrt{13}t} - 1\right)\right)$$

$$y(t) \to \frac{1}{26}e^{-\frac{1}{2}\left(1+\sqrt{13}\right)t}\left(2\sqrt{13}c_1\left(e^{\sqrt{13}t} - 1\right) - c_2\left(\left(3\sqrt{13} - 13\right)e^{\sqrt{13}t} - 13 - 3\sqrt{13}\right)\right)$$

1.3 problem 2.1 (iii)

Internal problem ID [12557]
Internal file name [`OUTPUT/11209_Wednesday_October_18_2023_10_01_15_PM_78487314/index.tex`]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.1 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -4x(t) + 2y(t) \\ y'(t) &= 3x(t) - 2y(t) \end{aligned}$$

1.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(\sqrt{7}+7\right)e^{-\left(3+\sqrt{7}\right)t}}{14} - \frac{e^{\left(-3+\sqrt{7}\right)t}\left(\sqrt{7}-7\right)}{14} & -\frac{\left(-e^{\left(-3+\sqrt{7}\right)t}+e^{-\left(3+\sqrt{7}\right)t}\right)\sqrt{7}}{7} \\ -\frac{3\left(-e^{\left(-3+\sqrt{7}\right)t}+e^{-\left(3+\sqrt{7}\right)t}\right)\sqrt{7}}{14} & \frac{\left(-\sqrt{7}+7\right)e^{-\left(3+\sqrt{7}\right)t}}{14} + \frac{e^{\left(-3+\sqrt{7}\right)t}\left(\sqrt{7}+7\right)}{14} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{(\sqrt{7}+7)e^{-(3+\sqrt{7})t}}{14} - \frac{e^{(-3+\sqrt{7})t}(\sqrt{7}-7)}{14} & -\frac{\left(-e^{(-3+\sqrt{7})t}+e^{-(3+\sqrt{7})t}\right)\sqrt{7}}{7} \\ -\frac{3\left(-e^{(-3+\sqrt{7})t}+e^{-(3+\sqrt{7})t}\right)\sqrt{7}}{14} & \frac{(-\sqrt{7}+7)e^{-(3+\sqrt{7})t}}{14} + \frac{e^{(-3+\sqrt{7})t}(\sqrt{7}+7)}{14} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{(\sqrt{7}+7)e^{-(3+\sqrt{7})t}}{14} - \frac{e^{(-3+\sqrt{7})t}(\sqrt{7}-7)}{14}\right)c_1 - \frac{\left(-e^{(-3+\sqrt{7})t}+e^{-(3+\sqrt{7})t}\right)\sqrt{7}\,c_2}{7} \\ -\frac{3\left(-e^{(-3+\sqrt{7})t}+e^{-(3+\sqrt{7})t}\right)\sqrt{7}\,c_1}{14} + \left(\frac{(-\sqrt{7}+7)e^{-(3+\sqrt{7})t}}{14} + \frac{e^{(-3+\sqrt{7})t}(\sqrt{7}+7)}{14}\right)c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{((c_1-2c_2)\sqrt{7}+7c_1)e^{-(3+\sqrt{7})t}}{14} - \frac{e^{(-3+\sqrt{7})t}((c_1-2c_2)\sqrt{7}-7c_1)}{14} \\ \frac{((-3c_1-c_2)\sqrt{7}+7c_2)e^{-(3+\sqrt{7})t}}{14} + \frac{3e^{(-3+\sqrt{7})t}\left((c_1+\frac{c_2}{3})\sqrt{7}+\frac{7c_2}{3}\right)}{14} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -4 & 2 \\ 3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -4-\lambda & 2 \\ 3 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 6\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3 + \sqrt{7}$$
$$\lambda_2 = -3 - \sqrt{7}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-3-\sqrt{7}$	1	real eigenvalue
$-3+\sqrt{7}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3 - \sqrt{7}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 2 \\ 3 & -2 \end{bmatrix} - \left(-3-\sqrt{7}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1+\sqrt{7} & 2 \\ 3 & 1+\sqrt{7} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1+\sqrt{7} & 2 & 0 \\ 3 & 1+\sqrt{7} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{-1+\sqrt{7}} \Longrightarrow \left[\begin{array}{cc|c} -1+\sqrt{7} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+\sqrt{7} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{-1+\sqrt{7}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{-1+\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{-1+\sqrt{7}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{-1+\sqrt{7}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{-1+\sqrt{7}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{-1+\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{-1+\sqrt{7}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{-1+\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{-1+\sqrt{7}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -3 + \sqrt{7}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 2 \\ 3 & -2 \end{bmatrix} - \left(-3+\sqrt{7}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1-\sqrt{7} & 2 \\ 3 & 1-\sqrt{7} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1-\sqrt{7} & 2 & 0 \\ 3 & 1-\sqrt{7} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{-1-\sqrt{7}} \Longrightarrow \left[\begin{array}{cc|c} -1-\sqrt{7} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-\sqrt{7} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{1+\sqrt{7}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{1+\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{1+\sqrt{7}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{1+\sqrt{7}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{1+\sqrt{7}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{1+\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{1+\sqrt{7}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{1+\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{1+\sqrt{7}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-3+\sqrt{7}$	1	1	No	$\left[\begin{array}{c}\frac{2}{1+\sqrt{7}} \\ 1\end{array}\right]$
$-3-\sqrt{7}$	1	1	No	$\left[\begin{array}{c}\frac{2}{1-\sqrt{7}} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-3+\sqrt{7}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(-3+\sqrt{7}\right)t} \\
&= \left[\begin{array}{c}\frac{2}{1+\sqrt{7}} \\ 1\end{array}\right] e^{\left(-3+\sqrt{7}\right)t}
\end{aligned}
$$

Since eigenvalue $-3-\sqrt{7}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(-3-\sqrt{7}\right)t} \\
&= \left[\begin{array}{c}\frac{2}{1-\sqrt{7}} \\ 1\end{array}\right] e^{\left(-3-\sqrt{7}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c}x(t) \\ y(t)\end{array}\right] = c_1 \left[\begin{array}{c}\frac{2\,\mathrm{e}^{\left(-3+\sqrt{7}\right)t}}{1+\sqrt{7}} \\ \mathrm{e}^{\left(-3+\sqrt{7}\right)t}\end{array}\right] + c_2 \left[\begin{array}{c}\frac{2\,\mathrm{e}^{\left(-3-\sqrt{7}\right)t}}{1-\sqrt{7}} \\ \mathrm{e}^{\left(-3-\sqrt{7}\right)t}\end{array}\right]
$$

Which becomes

$$
\left[\begin{array}{c}x(t) \\ y(t)\end{array}\right] = \left[\begin{array}{c}-\frac{c_2\left(1+\sqrt{7}\right)\mathrm{e}^{-\left(3+\sqrt{7}\right)t}}{3} + \frac{\mathrm{e}^{\left(-3+\sqrt{7}\right)t} c_1 \left(-1+\sqrt{7}\right)}{3} \\ c_1 \mathrm{e}^{\left(-3+\sqrt{7}\right)t} + c_2 \mathrm{e}^{-\left(3+\sqrt{7}\right)t}\end{array}\right]
$$

The following is the phase plot of the system.

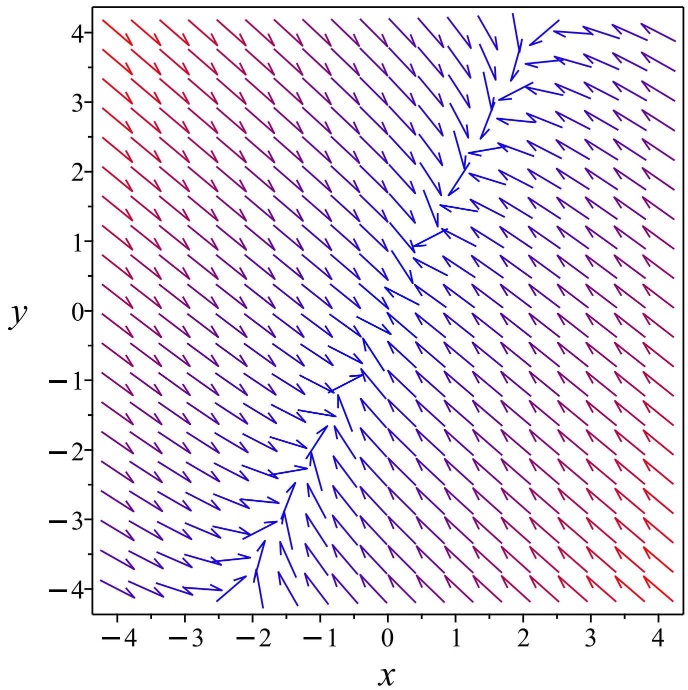

Figure 3: Phase plot

1.3.3 Maple step by step solution

Let's solve

$[x'(t) = -4x(t) + 2y(t), y'(t) = 3x(t) - 2y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -4 & 2 \\ 3 & -2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -4 & 2 \\ 3 & -2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -4 & 2 \\ 3 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3-\sqrt{7}, \begin{bmatrix} \frac{2}{1-\sqrt{7}} \\ 1 \end{bmatrix}\right], \left[-3+\sqrt{7}, \begin{bmatrix} \frac{2}{1+\sqrt{7}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3-\sqrt{7}, \begin{bmatrix} \frac{2}{1-\sqrt{7}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{\left(-3-\sqrt{7}\right)t} \cdot \begin{bmatrix} \frac{2}{1-\sqrt{7}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-3+\sqrt{7}, \begin{bmatrix} \frac{2}{1+\sqrt{7}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{\left(-3+\sqrt{7}\right)t} \cdot \begin{bmatrix} \frac{2}{1+\sqrt{7}} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{\left(-3-\sqrt{7}\right)t} \cdot \begin{bmatrix} \frac{2}{1-\sqrt{7}} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{\left(-3+\sqrt{7}\right)t} \cdot \begin{bmatrix} \frac{2}{1+\sqrt{7}} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1\left(1+\sqrt{7}\right)\mathrm{e}^{-\left(3+\sqrt{7}\right)t}}{3} + \frac{c_2\mathrm{e}^{\left(-3+\sqrt{7}\right)t}\left(-1+\sqrt{7}\right)}{3} \\ c_1\mathrm{e}^{-\left(3+\sqrt{7}\right)t} + c_2\mathrm{e}^{\left(-3+\sqrt{7}\right)t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{c_1\left(1+\sqrt{7}\right)\mathrm{e}^{-\left(3+\sqrt{7}\right)t}}{3} + \frac{c_2\mathrm{e}^{\left(-3+\sqrt{7}\right)t}\left(-1+\sqrt{7}\right)}{3}, y(t) = c_1\mathrm{e}^{-\left(3+\sqrt{7}\right)t} + c_2\mathrm{e}^{\left(-3+\sqrt{7}\right)t}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 83

```
dsolve([diff(x(t),t)=-4*x(t)+2*y(t),diff(y(t),t)=3*x(t)-2*y(t)],singsol=all)
```

$$x(t) = c_1\mathrm{e}^{\left(-3+\sqrt{7}\right)t} + c_2\mathrm{e}^{-\left(\sqrt{7}+3\right)t}$$

$$y(t) = \frac{c_1\mathrm{e}^{\left(-3+\sqrt{7}\right)t}\sqrt{7}}{2} - \frac{c_2\mathrm{e}^{-\left(\sqrt{7}+3\right)t}\sqrt{7}}{2} + \frac{c_1\mathrm{e}^{\left(-3+\sqrt{7}\right)t}}{2} + \frac{c_2\mathrm{e}^{-\left(\sqrt{7}+3\right)t}}{2}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 143

```
DSolve[{x'[t]==-4*x[t]+2*y[t],y'[t]==3*x[t]-2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -
```

$$x(t) \to \frac{1}{14}e^{-\left(\left(3+\sqrt{7}\right)t\right)}\left(c_1\left(-\left(\sqrt{7}-7\right)e^{2\sqrt{7}t} + 7 + \sqrt{7}\right) + 2\sqrt{7}c_2\left(e^{2\sqrt{7}t} - 1\right)\right)$$

$$y(t) \to \frac{1}{14}e^{-\left(\left(3+\sqrt{7}\right)t\right)}\left(3\sqrt{7}c_1\left(e^{2\sqrt{7}t} - 1\right) + c_2\left(\left(7+\sqrt{7}\right)e^{2\sqrt{7}t} + 7 - \sqrt{7}\right)\right)$$

1.4 problem 2.1 (iv)

Internal problem ID [12558]
Internal file name [OUTPUT/11210_Wednesday_October_18_2023_10_01_16_PM_79597876/index.tex]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.1 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = x(t) + 2y(t)$$
$$y'(t) = 2x(t) + 2y(t)$$

1.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(\sqrt{17}+17\right)e^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}}{34} - \frac{e^{\frac{\left(3+\sqrt{17}\right)t}{2}}\left(\sqrt{17}-17\right)}{34} & -\frac{2\left(-e^{\frac{\left(3+\sqrt{17}\right)t}{2}}+e^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}\right)\sqrt{17}}{17} \\ -\frac{2\left(-e^{\frac{\left(3+\sqrt{17}\right)t}{2}}+e^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}\right)\sqrt{17}}{17} & \frac{\left(-\sqrt{17}+17\right)e^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}}{34} + \frac{e^{\frac{\left(3+\sqrt{17}\right)t}{2}}\left(\sqrt{17}+17\right)}{34} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{\left(\sqrt{17}+17\right)e^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}}{34} - \frac{e^{\frac{\left(3+\sqrt{17}\right)t}{2}}\left(\sqrt{17}-17\right)}{34} & -\frac{2\left(-e^{\frac{\left(3+\sqrt{17}\right)t}{2}}+e^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}\right)\sqrt{17}}{17} \\ -\frac{2\left(-e^{\frac{\left(3+\sqrt{17}\right)t}{2}}+e^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}\right)\sqrt{17}}{17} & \frac{\left(-\sqrt{17}+17\right)e^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}}{34} + \frac{e^{\frac{\left(3+\sqrt{17}\right)t}{2}}\left(\sqrt{17}+17\right)}{34} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\left(\sqrt{17}+17\right)e^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}}{34} - \frac{e^{\frac{\left(3+\sqrt{17}\right)t}{2}}\left(\sqrt{17}-17\right)}{34}\right)c_1 - \frac{2\left(-e^{\frac{\left(3+\sqrt{17}\right)t}{2}}+e^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}\right)\sqrt{17}\,c_2}{17} \\ -\frac{2\left(-e^{\frac{\left(3+\sqrt{17}\right)t}{2}}+e^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}\right)\sqrt{17}\,c_1}{17} + \left(\frac{\left(-\sqrt{17}+17\right)e^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}}{34} + \frac{e^{\frac{\left(3+\sqrt{17}\right)t}{2}}\left(\sqrt{17}+17\right)}{34}\right)c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left((c_1-4c_2)\sqrt{17}+17c_1\right)e^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}}{34} - \frac{e^{\frac{\left(3+\sqrt{17}\right)t}{2}}\left((c_1-4c_2)\sqrt{17}-17c_1\right)}{34} \\ \frac{\left((-4c_1-c_2)\sqrt{17}+17c_2\right)e^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}}{34} + \frac{2\,e^{\frac{\left(3+\sqrt{17}\right)t}{2}}\left(\left(c_1+\frac{c_2}{4}\right)\sqrt{17}+\frac{17c_2}{4}\right)}{17} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}1-\lambda & 2\\ 2 & 2-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda - 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{3}{2} + \frac{\sqrt{17}}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{\sqrt{17}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{3}{2} - \frac{\sqrt{17}}{2}$	1	real eigenvalue
$\frac{3}{2} + \frac{\sqrt{17}}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{3}{2} - \frac{\sqrt{17}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix} - \left(\frac{3}{2} - \frac{\sqrt{17}}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}-\frac{1}{2} + \frac{\sqrt{17}}{2} & 2\\ 2 & \frac{1}{2} + \frac{\sqrt{17}}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}-\frac{1}{2} + \frac{\sqrt{17}}{2} & 2 & 0\\ 2 & \frac{1}{2} + \frac{\sqrt{17}}{2} & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{-\frac{1}{2} + \frac{\sqrt{17}}{2}} \implies \left[\begin{array}{cc|c} -\frac{1}{2} + \frac{\sqrt{17}}{2} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{2} + \frac{\sqrt{17}}{2} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{4t}{-1+\sqrt{17}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{4t}{-1+\sqrt{17}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4t}{-1+\sqrt{17}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{4t}{-1+\sqrt{17}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{4}{-1+\sqrt{17}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{4t}{-1+\sqrt{17}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{-1+\sqrt{17}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{4t}{-1+\sqrt{17}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{-1+\sqrt{17}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{3}{2} + \frac{\sqrt{17}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} - \left(\frac{3}{2} + \frac{\sqrt{17}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{17}}{2} & 2 \\ 2 & \frac{1}{2} - \frac{\sqrt{17}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{1}{2}-\frac{\sqrt{17}}{2} & 2 & 0 \\ 2 & \frac{1}{2}-\frac{\sqrt{17}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{-\frac{1}{2}-\frac{\sqrt{17}}{2}} \Longrightarrow \left[\begin{array}{cc|c} -\frac{1}{2}-\frac{\sqrt{17}}{2} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{2}-\frac{\sqrt{17}}{2} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{4t}{1+\sqrt{17}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{4t}{1+\sqrt{17}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4t}{1+\sqrt{17}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{4t}{1+\sqrt{17}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{1+\sqrt{17}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{4t}{1+\sqrt{17}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{1+\sqrt{17}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{4t}{1+\sqrt{17}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{1+\sqrt{17}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{3}{2}+\frac{\sqrt{17}}{2}$	1	1	No	$\begin{bmatrix} \frac{2}{\frac{1}{2}+\frac{\sqrt{17}}{2}} \\ 1 \end{bmatrix}$
$\frac{3}{2}-\frac{\sqrt{17}}{2}$	1	1	No	$\begin{bmatrix} \frac{2}{\frac{1}{2}-\frac{\sqrt{17}}{2}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{3}{2}+\frac{\sqrt{17}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\left(\frac{3}{2}+\frac{\sqrt{17}}{2}\right)t} \\ &= \begin{bmatrix} \frac{2}{\frac{1}{2}+\frac{\sqrt{17}}{2}} \\ 1 \end{bmatrix} e^{\left(\frac{3}{2}+\frac{\sqrt{17}}{2}\right)t}\end{aligned}$$

Since eigenvalue $\frac{3}{2}-\frac{\sqrt{17}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{\left(\frac{3}{2}-\frac{\sqrt{17}}{2}\right)t} \\ &= \begin{bmatrix} \frac{2}{\frac{1}{2}-\frac{\sqrt{17}}{2}} \\ 1 \end{bmatrix} e^{\left(\frac{3}{2}-\frac{\sqrt{17}}{2}\right)t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{2\,e^{\left(\frac{3}{2}+\frac{\sqrt{17}}{2}\right)t}}{\frac{1}{2}+\frac{\sqrt{17}}{2}} \\ e^{\left(\frac{3}{2}+\frac{\sqrt{17}}{2}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{2\,e^{\left(\frac{3}{2}-\frac{\sqrt{17}}{2}\right)t}}{\frac{1}{2}-\frac{\sqrt{17}}{2}} \\ e^{\left(\frac{3}{2}-\frac{\sqrt{17}}{2}\right)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_2\left(1+\sqrt{17}\right)\mathrm{e}^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}}{4} + \frac{\mathrm{e}^{\frac{\left(3+\sqrt{17}\right)t}{2}} c_1\left(-1+\sqrt{17}\right)}{4} \\ c_1\mathrm{e}^{\frac{\left(3+\sqrt{17}\right)t}{2}} + c_2\mathrm{e}^{-\frac{\left(-3+\sqrt{17}\right)t}{2}} \end{bmatrix}$$

The following is the phase plot of the system.

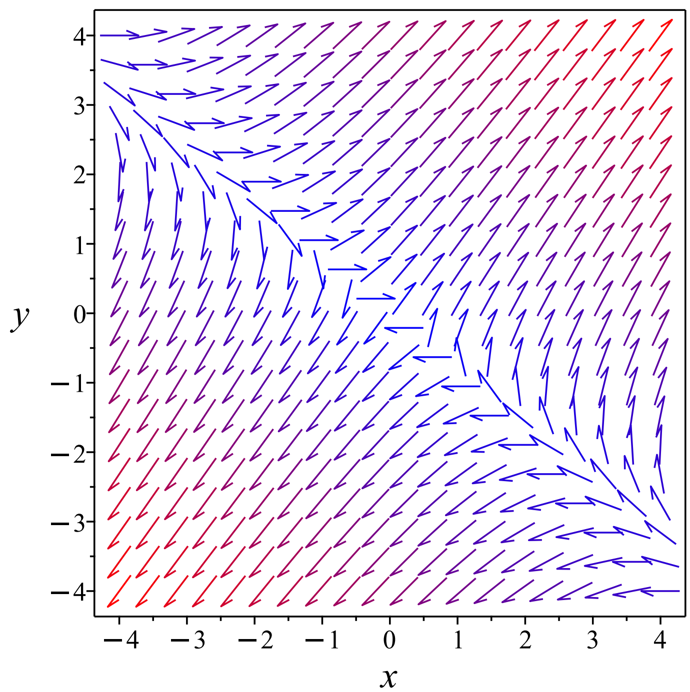

Figure 4: Phase plot

1.4.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + 2y(t)\,, y'(t) = 2x(t) + 2y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{3}{2} - \frac{\sqrt{17}}{2}, \begin{bmatrix} \frac{2}{\frac{1}{2} - \frac{\sqrt{17}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{3}{2} + \frac{\sqrt{17}}{2}, \begin{bmatrix} \frac{2}{\frac{1}{2} + \frac{\sqrt{17}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\frac{3}{2} - \frac{\sqrt{17}}{2}, \begin{bmatrix} \frac{2}{\frac{1}{2} - \frac{\sqrt{17}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{\left(\frac{3}{2} - \frac{\sqrt{17}}{2}\right)t} \cdot \begin{bmatrix} \frac{2}{\frac{1}{2} - \frac{\sqrt{17}}{2}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{3}{2} + \frac{\sqrt{17}}{2}, \begin{bmatrix} \frac{2}{\frac{1}{2} + \frac{\sqrt{17}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{\left(\frac{3}{2} + \frac{\sqrt{17}}{2}\right)t} \cdot \begin{bmatrix} \frac{2}{\frac{1}{2} + \frac{\sqrt{17}}{2}} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{\left(\frac{3}{2}-\frac{\sqrt{17}}{2}\right)t} \cdot \left[\begin{array}{c} \frac{2}{\frac{1}{2}-\frac{\sqrt{17}}{2}} \\ 1 \end{array}\right] + c_2 \mathrm{e}^{\left(\frac{3}{2}+\frac{\sqrt{17}}{2}\right)t} \cdot \left[\begin{array}{c} \frac{2}{\frac{1}{2}+\frac{\sqrt{17}}{2}} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} -\frac{c_1\left(1+\sqrt{17}\right)\mathrm{e}^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}}{4} + \frac{\mathrm{e}^{\frac{\left(3+\sqrt{17}\right)t}{2}} c_2\left(-1+\sqrt{17}\right)}{4} \\ c_1 \mathrm{e}^{-\frac{\left(-3+\sqrt{17}\right)t}{2}} + c_2 \mathrm{e}^{\frac{\left(3+\sqrt{17}\right)t}{2}} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{c_1\left(1+\sqrt{17}\right)\mathrm{e}^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}}{4} + \frac{\mathrm{e}^{\frac{\left(3+\sqrt{17}\right)t}{2}} c_2\left(-1+\sqrt{17}\right)}{4}, y(t) = c_1 \mathrm{e}^{-\frac{\left(-3+\sqrt{17}\right)t}{2}} + c_2 \mathrm{e}^{\frac{\left(3+\sqrt{17}\right)t}{2}}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 86

```
dsolve([diff(x(t),t)=x(t)+2*y(t),diff(y(t),t)=2*x(t)+2*y(t)],singsol=all)
```

$$x(t) = c_1 \mathrm{e}^{\frac{\left(3+\sqrt{17}\right)t}{2}} + c_2 \mathrm{e}^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}$$

$$y(t) = \frac{c_1 \mathrm{e}^{\frac{\left(3+\sqrt{17}\right)t}{2}} \sqrt{17}}{4} - \frac{c_2 \mathrm{e}^{-\frac{\left(-3+\sqrt{17}\right)t}{2}} \sqrt{17}}{4} + \frac{c_1 \mathrm{e}^{\frac{\left(3+\sqrt{17}\right)t}{2}}}{4} + \frac{c_2 \mathrm{e}^{-\frac{\left(-3+\sqrt{17}\right)t}{2}}}{4}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 143

```
DSolve[{x'[t]==x[t]+2*y[t],y'[t]==2*x[t]+2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to \frac{1}{34} e^{-\frac{1}{2}\left(\sqrt{17}-3\right)t}\left(c_1\left(-\left(\sqrt{17}-17\right)e^{\sqrt{17}t} + 17 + \sqrt{17}\right) + 4\sqrt{17}c_2\left(e^{\sqrt{17}t} - 1\right)\right)$$

$$y(t) \to \frac{1}{34} e^{-\frac{1}{2}\left(\sqrt{17}-3\right)t}\left(4\sqrt{17}c_1\left(e^{\sqrt{17}t} - 1\right) + c_2\left(\left(17 + \sqrt{17}\right)e^{\sqrt{17}t} + 17 - \sqrt{17}\right)\right)$$

1.5 problem 2.1 (v)

Internal problem ID [12559]
Internal file name [`OUTPUT/11211_Wednesday_October_18_2023_10_01_16_PM_97786729/index.tex`]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.1 (v).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 4x(t) - 2y(t) \\
y'(t) &= 3x(t) - y(t)
\end{aligned}
$$

1.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -2\,\mathrm{e}^{t} + 3\,\mathrm{e}^{2t} & -2\,\mathrm{e}^{2t} + 2\,\mathrm{e}^{t} \\ 3\,\mathrm{e}^{2t} - 3\,\mathrm{e}^{t} & 3\,\mathrm{e}^{t} - 2\,\mathrm{e}^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -2\,\mathrm{e}^{t}+3\,\mathrm{e}^{2t} & -2\,\mathrm{e}^{2t}+2\,\mathrm{e}^{t} \\ 3\,\mathrm{e}^{2t}-3\,\mathrm{e}^{t} & 3\,\mathrm{e}^{t}-2\,\mathrm{e}^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (-2\,\mathrm{e}^{t}+3\,\mathrm{e}^{2t})\,c_1+(-2\,\mathrm{e}^{2t}+2\,\mathrm{e}^{t})\,c_2 \\ (3\,\mathrm{e}^{2t}-3\,\mathrm{e}^{t})\,c_1+(3\,\mathrm{e}^{t}-2\,\mathrm{e}^{2t})\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (3c_1-2c_2)\,\mathrm{e}^{2t}-2\,\mathrm{e}^{t}(-c_2+c_1) \\ (3c_1-2c_2)\,\mathrm{e}^{2t}-3\,\mathrm{e}^{t}(-c_2+c_1) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & -2 \\ 3 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{2} \implies \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{t} \\
&= \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} e^{t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{2\,\mathrm{e}^{t}}{3} \\ \mathrm{e}^{t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{2t} + \frac{2 c_2 \mathrm{e}^{t}}{3} \\ c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{t} \end{bmatrix}
$$

The following is the phase plot of the system.

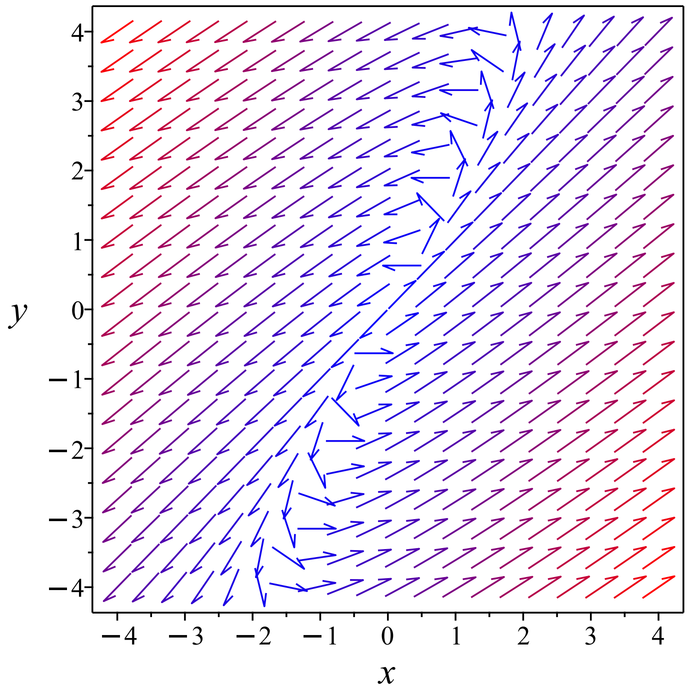

Figure 5: Phase plot

1.5.3 Maple step by step solution

Let's solve

$[x'(t) = 4x(t) - 2y(t), y'(t) = 3x(t) - y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1, \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[1, \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \mathrm{e}^t \cdot \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = c_1 \mathrm{e}^t \cdot \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{2t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{2c_1 e^t}{3} + c_2 e^{2t} \\ c_1 e^t + c_2 e^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{2c_1 e^t}{3} + c_2 e^{2t}, y(t) = c_1 e^t + c_2 e^{2t}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
dsolve([diff(x(t),t)=4*x(t)-2*y(t),diff(y(t),t)=3*x(t)-y(t)],singsol=all)
```

$$x(t) = c_1 e^t + c_2 e^{2t}$$
$$y(t) = \frac{3c_1 e^t}{2} + c_2 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 56

```
DSolve[{x'[t]==4*x[t]-2*y[t],y'[t]==3*x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to e^t\left(c_1\left(3e^t - 2\right) - 2c_2\left(e^t - 1\right)\right)$$
$$y(t) \to e^t\left(3c_1\left(e^t - 1\right) + c_2\left(3 - 2e^t\right)\right)$$

1.6 problem 2.1 (vi)

Internal problem ID [12560]
Internal file name [`OUTPUT/11212_Wednesday_October_18_2023_10_01_17_PM_76338301/index.tex`]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.1 (vi).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) + y(t) \\
y'(t) &= -x(t) + y(t)
\end{aligned}
$$

1.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{\frac{3t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{\sqrt{3}\,e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} & \frac{2\sqrt{3}\,e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} \\ -\frac{2\sqrt{3}\,e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} & e^{\frac{3t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) - \frac{\sqrt{3}\,e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{\frac{3t}{2}}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{2}\right)+3\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)}{3} & \frac{2\sqrt{3}\,e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} \\ -\frac{2\sqrt{3}\,e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} & -\frac{e^{\frac{3t}{2}}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{2}\right)-3\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{\frac{3t}{2}}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{2}\right)+3\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)}{3} & \frac{2\sqrt{3}\,e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} \\ -\frac{2\sqrt{3}\,e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)}{3} & -\frac{e^{\frac{3t}{2}}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{2}\right)-3\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{\frac{3t}{2}}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{2}\right)+3\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)c_1}{3} + \frac{2\sqrt{3}\,e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2}{3} \\ -\frac{2\sqrt{3}\,e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)c_1}{3} - \frac{e^{\frac{3t}{2}}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{2}\right)-3\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)c_2}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{\frac{3t}{2}}\left(\sqrt{3}\,(c_1+2c_2)\sin\left(\frac{\sqrt{3}\,t}{2}\right)+3\cos\left(\frac{\sqrt{3}\,t}{2}\right)c_1\right)}{3} \\ -\frac{2\left(\sqrt{3}\left(\frac{c_2}{2}+c_1\right)\sin\left(\frac{\sqrt{3}\,t}{2}\right)-\frac{3\cos\left(\frac{\sqrt{3}\,t}{2}\right)c_2}{2}\right)e^{\frac{3t}{2}}}{3} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda + 3 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= \frac{3}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{3}{2} + \frac{i\sqrt{3}}{2}$	1	complex eigenvalue
$\frac{3}{2} - \frac{i\sqrt{3}}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{3}{2} - \frac{i\sqrt{3}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} - \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} + \frac{i\sqrt{3}}{2} & 1 \\ -1 & -\frac{1}{2} + \frac{i\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2}+\frac{i\sqrt{3}}{2} & 1 & 0 \\ -1 & -\frac{1}{2}+\frac{i\sqrt{3}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{\frac{1}{2}+\frac{i\sqrt{3}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2}+\frac{i\sqrt{3}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}+\frac{i\sqrt{3}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{1+i\sqrt{3}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{1+i\sqrt{3}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{3}{2} + \frac{i\sqrt{3}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} - \left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} - \frac{i\sqrt{3}}{2} & 1 \\ -1 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2} - \frac{i\sqrt{3}}{2} & 1 & 0 \\ -1 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2} - \frac{i\sqrt{3}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} - \frac{i\sqrt{3}}{2} & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{i\sqrt{3}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{i\sqrt{3}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = t\begin{bmatrix} \frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity: algebraic m	multiplicity: geometric k	defective?	eigenvectors
$\frac{3}{2}+\frac{i\sqrt{3}}{2}$	1	1	No	$\begin{bmatrix} \frac{1}{-\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\ 1 \end{bmatrix}$
$\frac{3}{2}-\frac{i\sqrt{3}}{2}$	1	1	No	$\begin{bmatrix} \frac{1}{-\frac{1}{2}-\frac{i\sqrt{3}}{2}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{\left(\frac{3}{2}+\frac{i\sqrt{3}}{2}\right)t}}{-\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\ e^{\left(\frac{3}{2}+\frac{i\sqrt{3}}{2}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{\left(\frac{3}{2}-\frac{i\sqrt{3}}{2}\right)t}}{-\frac{1}{2}-\frac{i\sqrt{3}}{2}} \\ e^{\left(\frac{3}{2}-\frac{i\sqrt{3}}{2}\right)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{i\left(\sqrt{3}+i\right)c_2 e^{-\frac{\left(i\sqrt{3}-3\right)t}{2}}}{2} + \frac{i e^{\frac{\left(i\sqrt{3}+3\right)t}{2}} c_1 \left(i-\sqrt{3}\right)}{2} \\ c_1 e^{\frac{\left(i\sqrt{3}+3\right)t}{2}} + c_2 e^{-\frac{\left(i\sqrt{3}-3\right)t}{2}} \end{bmatrix}$$

The following is the phase plot of the system.

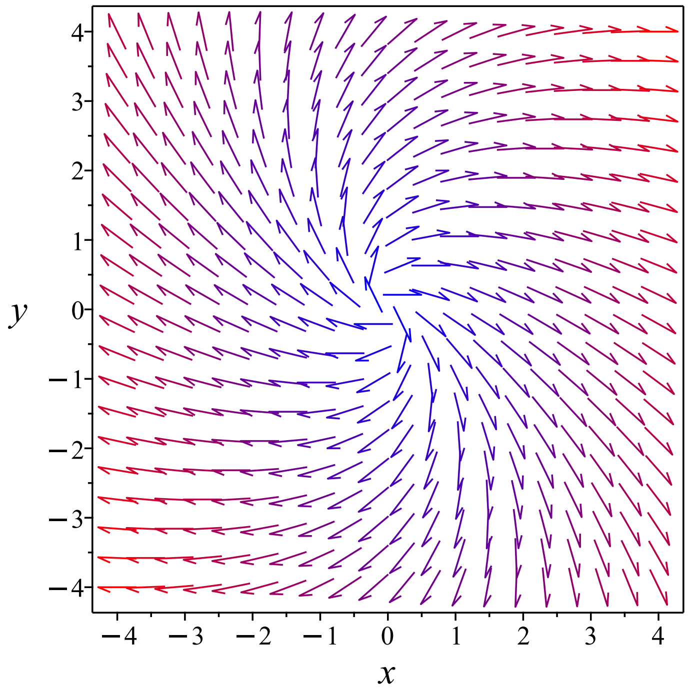

Figure 6: Phase plot

1.6.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) + y(t), y'(t) = -x(t) + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{3}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{3}{2} + \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{-\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{3}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(\frac{3}{2} - \frac{\mathrm{I}\sqrt{3}}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{\frac{3t}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,t}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right) \cdot \begin{bmatrix} \frac{1}{-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{\frac{3t}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{3}\,t}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\,t}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,t}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \mathrm{e}^{\frac{3t}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} + \frac{\sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \\ \cos\left(\frac{\sqrt{3}\,t}{2}\right) \end{bmatrix}, \vec{x}_2(t) = \mathrm{e}^{\frac{3t}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{3}\,t}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,t}{2}\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{\frac{3t}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,t}{2}\right)}{2} + \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \\ \cos\left(\frac{\sqrt{3}\,t}{2}\right) \end{bmatrix} + c_2 \mathrm{e}^{\frac{3t}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{3}\,t}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,t}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,t}{2}\right) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{\left(\left(c_2\sqrt{3}-c_1\right)\cos\left(\frac{\sqrt{3}\,t}{2}\right)+\sin\left(\frac{\sqrt{3}\,t}{2}\right)\left(\sqrt{3}\,c_1+c_2\right)\right)\mathrm{e}^{\frac{3t}{2}}}{2} \\ \mathrm{e}^{\frac{3t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,t}{2}\right) - c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{\left(\left(c_2\sqrt{3}-c_1\right)\cos\left(\frac{\sqrt{3}\,t}{2}\right)+\sin\left(\frac{\sqrt{3}\,t}{2}\right)\left(\sqrt{3}\,c_1+c_2\right)\right)\mathrm{e}^{\frac{3t}{2}}}{2}, y(t) = \mathrm{e}^{\frac{3t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,t}{2}\right) - c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 82

```
dsolve([diff(x(t),t)=2*x(t)+y(t),diff(y(t),t)=-x(t)+y(t)],singsol=all)
```

$$x(t) = \mathrm{e}^{\frac{3t}{2}}\left(\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_1 + \cos\left(\frac{\sqrt{3}\,t}{2}\right)c_2\right)$$

$$y(t) = -\frac{\mathrm{e}^{\frac{3t}{2}}\left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{2}\right)c_2 - \sqrt{3}\,\cos\left(\frac{\sqrt{3}\,t}{2}\right)c_1 + \sin\left(\frac{\sqrt{3}\,t}{2}\right)c_1 + \cos\left(\frac{\sqrt{3}\,t}{2}\right)c_2\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 111

```
DSolve[{x'[t]==2*x[t]+y[t],y'[t]==-x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True
```

$$x(t) \to \frac{1}{3}e^{3t/2}\left(3c_1\cos\left(\frac{\sqrt{3}t}{2}\right) + \sqrt{3}(c_1+2c_2)\sin\left(\frac{\sqrt{3}t}{2}\right)\right)$$

$$y(t) \to \frac{1}{3}e^{3t/2}\left(3c_2\cos\left(\frac{\sqrt{3}t}{2}\right) - \sqrt{3}(2c_1+c_2)\sin\left(\frac{\sqrt{3}t}{2}\right)\right)$$

1.7 problem 2.2 (i)

Internal problem ID [12561]
Internal file name [`OUTPUT/11213_Wednesday_October_18_2023_10_01_17_PM_26773818/index.tex`]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.2 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 3x(t) - y(t)$$
$$y'(t) = x(t) + y(t)$$

1.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{2t}(1+t) & -\mathrm{e}^{2t}t \\ \mathrm{e}^{2t}t & \mathrm{e}^{2t}(1-t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t}(1+t) & -e^{2t}t \\ e^{2t}t & e^{2t}(1-t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(1+t)\,c_1 - e^{2t}tc_2 \\ e^{2t}tc_1 + e^{2t}(1-t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(tc_1 - c_2t + c_1) \\ e^{2t}(tc_1 - c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

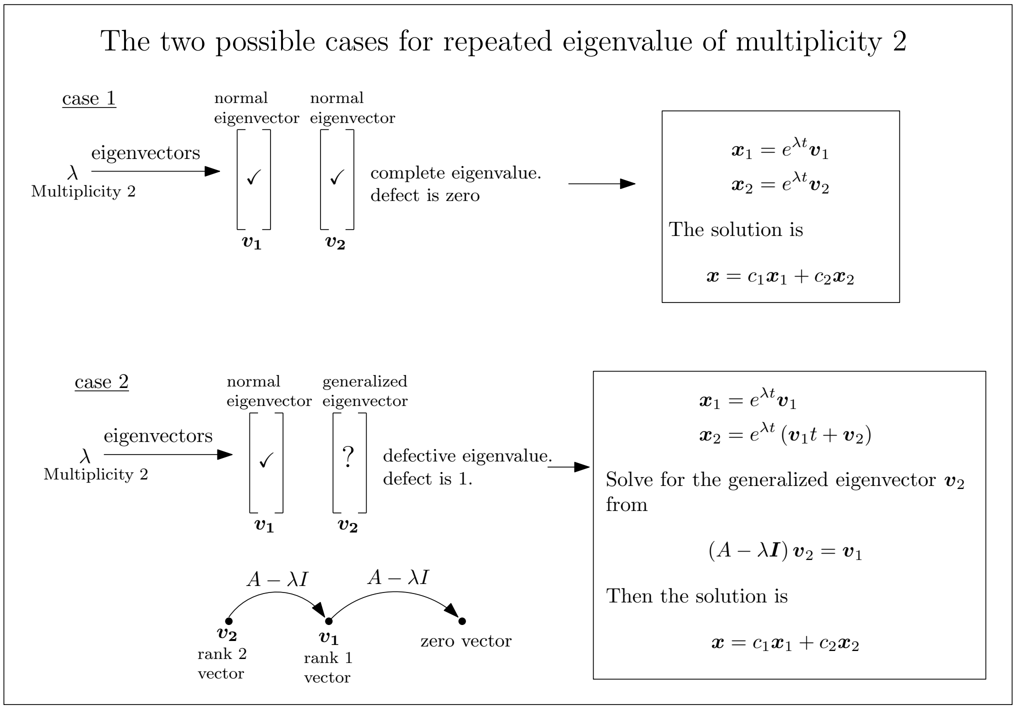

Figure 7: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\begin{aligned}\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}\end{aligned}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} \\ &= \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) e^{2t} \\ &= \begin{bmatrix} e^{2t}(t+2) \\ e^{2t}(1+t) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{2t}(t+2) \\ e^{2t}(1+t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} ((t+2)\, c_2 + c_1)\, e^{2t} \\ e^{2t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

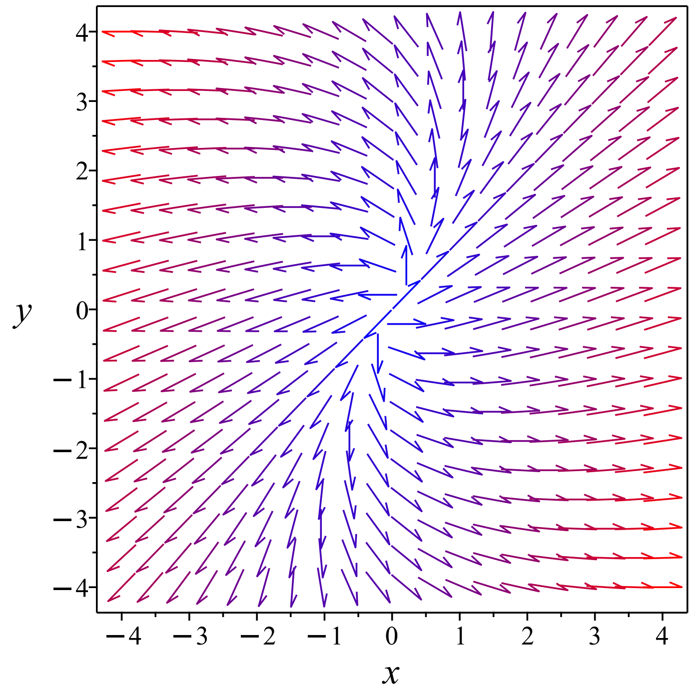

Figure 8: Phase plot

1.7.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t) - y(t), y'(t) = x(t) + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

  $$\overrightarrow{x}_1(t) = e^{2t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, and

  $\overrightarrow{x}_2(t) = e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = \left(e^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = e^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$

- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

$$\vec{x}_2(t) = e^{2t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{2t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{2t}(c_2 t + c_1 + c_2) \\ e^{2t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = e^{2t}(c_2 t + c_1 + c_2), y(t) = e^{2t}(c_2 t + c_1)\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve([diff(x(t),t)=3*x(t)-y(t),diff(y(t),t)=x(t)+y(t)],singsol=all)
```

$$x(t) = e^{2t}(c_2 t + c_1)$$
$$y(t) = e^{2t}(c_2 t + c_1 - c_2)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 44

```
DSolve[{x'[t]==3*x[t]-y[t],y'[t]==x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{2t}(c_1(t+1) - c_2 t)$$
$$y(t) \to e^{2t}((c_1 - c_2)t + c_2)$$

1.8 problem 2.2 (ii)

Internal problem ID [12562]
Internal file name [`OUTPUT/11214_Wednesday_October_18_2023_10_01_17_PM_4787677/index.tex`]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.2 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = x(t) - y(t)$$
$$y'(t) = 2x(t) - 2y(t)$$

1.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\mathrm{e}^{-t} + 2 & \mathrm{e}^{-t} - 1 \\ 2 - 2\,\mathrm{e}^{-t} & 2\,\mathrm{e}^{-t} - 1 \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\mathrm{e}^{-t}+2 & \mathrm{e}^{-t}-1 \\ 2-2\,\mathrm{e}^{-t} & 2\,\mathrm{e}^{-t}-1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (-\mathrm{e}^{-t}+2)\,c_1+(\mathrm{e}^{-t}-1)\,c_2 \\ (2-2\,\mathrm{e}^{-t})\,c_1+(2\,\mathrm{e}^{-t}-1)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_2-c_1)\,\mathrm{e}^{-t}+2c_1-c_2 \\ (-2c_1+2c_2)\,\mathrm{e}^{-t}+2c_1-c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 \\ 2 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{0} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{0}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{-t}}{2} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{-t}}{2} + c_2 \\ c_1 e^{-t} + c_2 \end{bmatrix}$$

The following is the phase plot of the system.

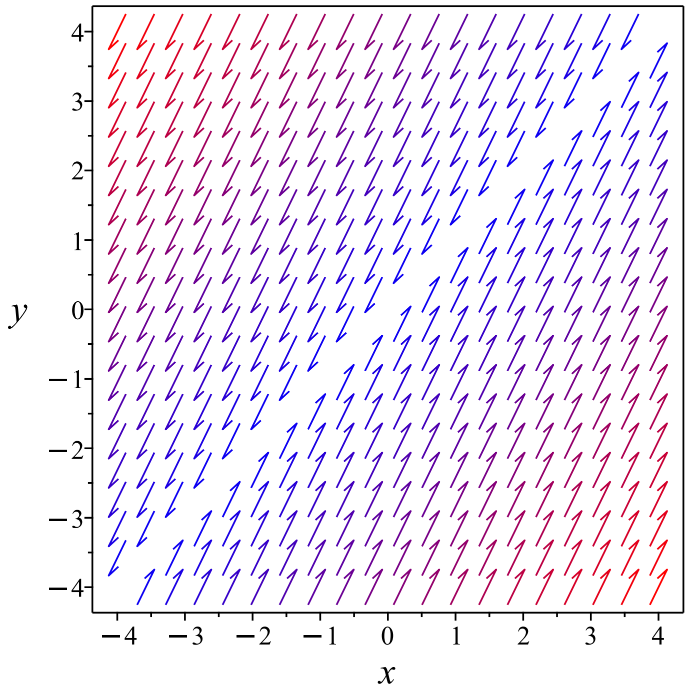

Figure 9: Phase plot

1.8.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) - y(t), y'(t) = 2x(t) - 2y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ c_2 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 \mathrm{e}^{-t}}{2} + c_2 \\ c_1 \mathrm{e}^{-t} + c_2 \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x(t) = \frac{c_1 \mathrm{e}^{-t}}{2} + c_2, y(t) = c_1 \mathrm{e}^{-t} + c_2\right\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve([diff(x(t),t)=x(t)-y(t),diff(y(t),t)=2*x(t)-2*y(t)],singsol=all)
```

$$x(t) = c_1 + c_2 \mathrm{e}^{-t}$$
$$y(t) = 2c_2 \mathrm{e}^{-t} + c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 59

```
DSolve[{x'[t]==x[t]-y[t],y'[t]==2*x[t]-2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tru
```

$$x(t) \to e^{-t}\left(c_1\left(2e^t - 1\right) - c_2\left(e^t - 1\right)\right)$$
$$y(t) \to e^{-t}\left(2c_1\left(e^t - 1\right) - c_2\left(e^t - 2\right)\right)$$

1.9 problem 2.2 (iii)

Internal problem ID [12563]
Internal file name [`OUTPUT/11215_Wednesday_October_18_2023_10_01_18_PM_39063778/index.tex`]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.2 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = x(t)$$
$$y'(t) = 2x(t) - 3y(t)$$

1.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{t} & 0 \\ \frac{(\mathrm{e}^{4t}-1)\mathrm{e}^{-3t}}{2} & \mathrm{e}^{-3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{t} & 0 \\ \frac{(e^{4t}-1)e^{-3t}}{2} & e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{t}c_1 \\ \frac{(e^{4t}-1)e^{-3t}c_1}{2} + e^{-3t}c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{t}c_1 \\ \frac{(e^{4t}c_1-c_1+2c_2)e^{-3t}}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 \\ 2 & -3-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1-\lambda)(-3-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 4 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{t} \\
&= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{t}
\end{aligned}
$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-3t} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-3t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{-3t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2c_1 \mathrm{e}^{t} \\ (c_1 \mathrm{e}^{4t} + c_2)\,\mathrm{e}^{-3t} \end{bmatrix}
$$

The following is the phase plot of the system.

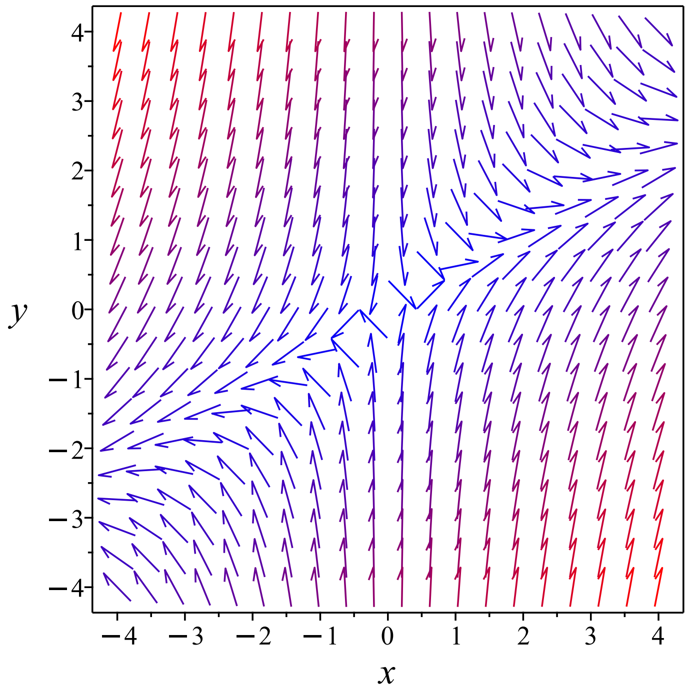

Figure 10: Phase plot

1.9.3 Maple step by step solution

Let's solve

$[x'(t) = x(t), y'(t) = 2x(t) - 3y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-3, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]\right]$$
- Consider eigenpair

  $$\left[-3, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$
- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \mathrm{e}^{-3t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
- Consider eigenpair

  $$\left[1, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$
- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \mathrm{e}^{t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$
- Substitute solutions into the general solution

  $$\overrightarrow{x} = \mathrm{e}^{-3t} c_1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2c_2 e^t \\ (c_2 e^{4t} + c_1) e^{-3t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = 2c_2 e^t, y(t) = (c_2 e^{4t} + c_1) e^{-3t}\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve([diff(x(t),t)=x(t),diff(y(t),t)=2*x(t)-3*y(t)],singsol=all)
```

$$x(t) = c_2 e^t$$
$$y(t) = \frac{c_2 e^t}{2} + c_1 e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 38

```
DSolve[{x'[t]==x[t],y'[t]==2*x[t]-3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^t$$
$$y(t) \to \frac{1}{2} e^{-3t} \left(c_1 \left(e^{4t} - 1\right) + 2c_2\right)$$

1.10 problem 2.2 (iv)

Internal problem ID [12564]
Internal file name [`OUTPUT/11216_Wednesday_October_18_2023_10_01_18_PM_75237406/index.tex`]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.2 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) \\
y'(t) &= x(t) + 3y(t)
\end{aligned}
$$

1.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{t} & 0 \\ \frac{e^{3t}}{2} - \frac{e^{t}}{2} & e^{3t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t & 0 \\ \frac{e^{3t}}{2} - \frac{e^t}{2} & e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t c_1 \\ \left(\frac{e^{3t}}{2} - \frac{e^t}{2}\right) c_1 + e^{3t} c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t c_1 \\ \frac{(c_1+2c_2)e^{3t}}{2} - \frac{e^t c_1}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & 0 \\ 1 & 3-\lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1-\lambda)(3-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 2 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \implies \left[\begin{array}{cc|c} -2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^t = \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^t$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{3t} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -2\,\mathrm{e}^t \\ \mathrm{e}^t \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -2c_1 \mathrm{e}^t \\ c_1 \mathrm{e}^t + c_2 \mathrm{e}^{3t} \end{bmatrix}$$

The following is the phase plot of the system.

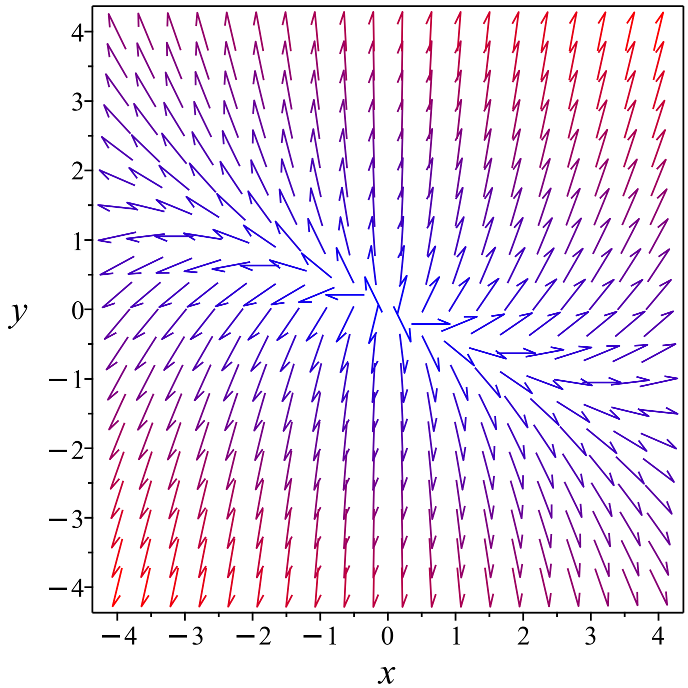

Figure 11: Phase plot

1.10.3 Maple step by step solution

Let's solve

$$[x'(t) = x(t), y'(t) = x(t) + 3y(t)]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[1, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_1 = e^t \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[3, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_2 = e^{3t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

- Substitute solutions into the general solution

  $$\vec{x} = c_1 e^t \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -2c_1 e^t \\ c_1 e^t + c_2 e^{3t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = -2c_1 e^t, y(t) = c_1 e^t + c_2 e^{3t}\}$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 24

```
dsolve([diff(x(t),t)=x(t),diff(y(t),t)=x(t)+3*y(t)],singsol=all)
```

$$x(t) = c_2 e^t$$
$$y(t) = -\frac{c_2 e^t}{2} + c_1 e^{3t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 39

```
DSolve[{x'[t]==x[t],y'[t]==x[t]+3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^t$$
$$y(t) \to \left(\frac{c_1}{2} + c_2\right) e^{3t} - \frac{c_1 e^t}{2}$$

1.11 problem 2.2 (v)

Internal problem ID [12565]
Internal file name [`OUTPUT/11217_Wednesday_October_18_2023_10_01_18_PM_19494184/index.tex`]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.2 (v).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -y(t) \\
y'(t) &= 2x(t) - 4y(t)
\end{aligned}
$$

1.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(-\sqrt{2}+1\right)\mathrm{e}^{-\left(2+\sqrt{2}\right)t}}{2} + \frac{\mathrm{e}^{\left(\sqrt{2}-2\right)t}\left(1+\sqrt{2}\right)}{2} & \frac{\left(-\mathrm{e}^{\left(\sqrt{2}-2\right)t}+\mathrm{e}^{-\left(2+\sqrt{2}\right)t}\right)\sqrt{2}}{4} \\ -\frac{\left(-\mathrm{e}^{\left(\sqrt{2}-2\right)t}+\mathrm{e}^{-\left(2+\sqrt{2}\right)t}\right)\sqrt{2}}{2} & \frac{\left(1+\sqrt{2}\right)\mathrm{e}^{-\left(2+\sqrt{2}\right)t}}{2} - \frac{\mathrm{e}^{\left(\sqrt{2}-2\right)t}\left(\sqrt{2}-1\right)}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{\left(-\sqrt{2}+1\right)e^{-\left(2+\sqrt{2}\right)t}}{2} + \frac{e^{\left(\sqrt{2}-2\right)t}\left(1+\sqrt{2}\right)}{2} & \frac{\left(-e^{\left(\sqrt{2}-2\right)t}+e^{-\left(2+\sqrt{2}\right)t}\right)\sqrt{2}}{4} \\ -\frac{\left(-e^{\left(\sqrt{2}-2\right)t}+e^{-\left(2+\sqrt{2}\right)t}\right)\sqrt{2}}{2} & \frac{\left(1+\sqrt{2}\right)e^{-\left(2+\sqrt{2}\right)t}}{2} - \frac{e^{\left(\sqrt{2}-2\right)t}\left(\sqrt{2}-1\right)}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\left(-\sqrt{2}+1\right)e^{-\left(2+\sqrt{2}\right)t}}{2} + \frac{e^{\left(\sqrt{2}-2\right)t}\left(1+\sqrt{2}\right)}{2}\right)c_1 + \frac{\left(-e^{\left(\sqrt{2}-2\right)t}+e^{-\left(2+\sqrt{2}\right)t}\right)\sqrt{2}\,c_2}{4} \\ -\frac{\left(-e^{\left(\sqrt{2}-2\right)t}+e^{-\left(2+\sqrt{2}\right)t}\right)\sqrt{2}\,c_1}{2} + \left(\frac{\left(1+\sqrt{2}\right)e^{-\left(2+\sqrt{2}\right)t}}{2} - \frac{e^{\left(\sqrt{2}-2\right)t}\left(\sqrt{2}-1\right)}{2}\right)c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left((-2c_1+c_2)\sqrt{2}+2c_1\right)e^{-\left(2+\sqrt{2}\right)t}}{4} + \frac{\left(\left(c_1-\frac{c_2}{2}\right)\sqrt{2}+c_1\right)e^{\left(\sqrt{2}-2\right)t}}{2} \\ \frac{\left((c_2-c_1)\sqrt{2}+c_2\right)e^{-\left(2+\sqrt{2}\right)t}}{2} + \frac{\left(\sqrt{2}\,(-c_2+c_1)+c_2\right)e^{\left(\sqrt{2}-2\right)t}}{2} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -1 \\ 2 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 2 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \sqrt{2} - 2$$
$$\lambda_2 = -2 - \sqrt{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\sqrt{2} - 2$	1	real eigenvalue
$-2 - \sqrt{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2 - \sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 2 & -4 \end{bmatrix} - \left(-2-\sqrt{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+\sqrt{2} & -1 \\ 2 & \sqrt{2}-2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+\sqrt{2} & -1 & 0 \\ 2 & \sqrt{2}-2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{2+\sqrt{2}} \Longrightarrow \left[\begin{array}{cc|c} 2+\sqrt{2} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+\sqrt{2} & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2+\sqrt{2}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2+\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2+\sqrt{2}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2+\sqrt{2}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2+\sqrt{2}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2+\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2+\sqrt{2}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2+\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2+\sqrt{2}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \sqrt{2} - 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 2 & -4 \end{bmatrix} - \left(\sqrt{2}-2\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-\sqrt{2} & -1 \\ 2 & -2-\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2-\sqrt{2} & -1 & 0 \\ 2 & -2-\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{2-\sqrt{2}} \implies \left[\begin{array}{cc|c} 2-\sqrt{2} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-\sqrt{2} & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{\sqrt{2}-2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{\sqrt{2}-2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{\sqrt{2}-2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{\sqrt{2}-2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{\sqrt{2}-2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{\sqrt{2}-2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}-2} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity: algebraic m	multiplicity: geometric k	defective?	eigenvectors
$\sqrt{2}-2$	1	1	No	$\left[\begin{array}{c} -\frac{1}{\sqrt{2}-2} \\ 1 \end{array}\right]$
$-2-\sqrt{2}$	1	1	No	$\left[\begin{array}{c} -\frac{1}{-2-\sqrt{2}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\sqrt{2}-2$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(\sqrt{2}-2\right)t} \\
&= \left[\begin{array}{c} -\frac{1}{\sqrt{2}-2} \\ 1 \end{array}\right] e^{\left(\sqrt{2}-2\right)t}
\end{aligned}
$$

Since eigenvalue $-2-\sqrt{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(-2-\sqrt{2}\right)t} \\
&= \left[\begin{array}{c} -\frac{1}{-2-\sqrt{2}} \\ 1 \end{array}\right] e^{\left(-2-\sqrt{2}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = c_1 \left[\begin{array}{c} -\frac{e^{\left(\sqrt{2}-2\right)t}}{\sqrt{2}-2} \\ e^{\left(\sqrt{2}-2\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{e^{\left(-2-\sqrt{2}\right)t}}{-2-\sqrt{2}} \\ e^{\left(-2-\sqrt{2}\right)t} \end{array}\right]
$$

Which becomes

$$
\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} -\frac{c_2\left(\sqrt{2}-2\right)e^{-\left(2+\sqrt{2}\right)t}}{2} + \frac{e^{\left(\sqrt{2}-2\right)t} c_1 \left(2+\sqrt{2}\right)}{2} \\ c_1 e^{\left(\sqrt{2}-2\right)t} + c_2 e^{-\left(2+\sqrt{2}\right)t} \end{array}\right]
$$

The following is the phase plot of the system.

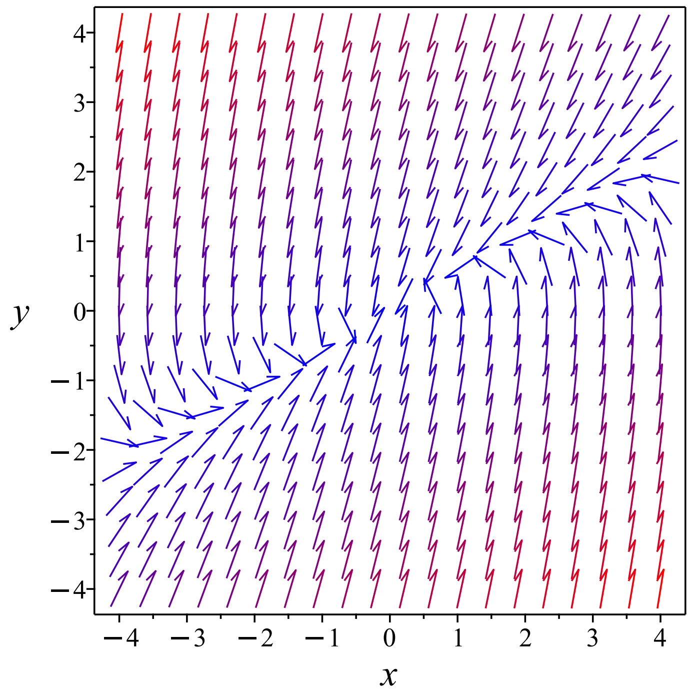

Figure 12: Phase plot

1.11.3 Maple step by step solution

Let's solve

$[x'(t) = -y(t), y'(t) = 2x(t) - 4y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -1 \\ 2 & -4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -1 \\ 2 & -4 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -1 \\ 2 & -4 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2-\sqrt{2}, \begin{bmatrix} -\frac{1}{-2-\sqrt{2}} \\ 1 \end{bmatrix}\right], \left[\sqrt{2}-2, \begin{bmatrix} -\frac{1}{\sqrt{2}-2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2-\sqrt{2}, \begin{bmatrix} -\frac{1}{-2-\sqrt{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{\left(-2-\sqrt{2}\right)t} \cdot \begin{bmatrix} -\frac{1}{-2-\sqrt{2}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\sqrt{2}-2, \begin{bmatrix} -\frac{1}{\sqrt{2}-2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{\left(\sqrt{2}-2\right)t} \cdot \begin{bmatrix} -\frac{1}{\sqrt{2}-2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{\left(-2-\sqrt{2}\right)t} \cdot \begin{bmatrix} -\frac{1}{-2-\sqrt{2}} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{\left(\sqrt{2}-2\right)t} \cdot \begin{bmatrix} -\frac{1}{\sqrt{2}-2} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1(\sqrt{2}-2)e^{-(2+\sqrt{2})t}}{2} + \frac{c_2 e^{(\sqrt{2}-2)t}(2+\sqrt{2})}{2} \\ c_1 e^{-(2+\sqrt{2})t} + c_2 e^{(\sqrt{2}-2)t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{c_1(\sqrt{2}-2)e^{-(2+\sqrt{2})t}}{2} + \frac{c_2 e^{(\sqrt{2}-2)t}(2+\sqrt{2})}{2}, y(t) = c_1 e^{-(2+\sqrt{2})t} + c_2 e^{(\sqrt{2}-2)t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 64

```
dsolve([diff(x(t),t)=-y(t),diff(y(t),t)=2*x(t)-4*y(t)],singsol=all)
```

$$x(t) = c_1 e^{(-2+\sqrt{2})t} + c_2 e^{-(2+\sqrt{2})t}$$
$$y(t) = \left(2+\sqrt{2}\right) c_2 e^{-(2+\sqrt{2})t} + \left(2-\sqrt{2}\right) c_1 e^{(-2+\sqrt{2})t}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 143

```
DSolve[{x'[t]==-y[t],y'[t]==2*x[t]-4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{4}e^{-\left(\left(2+\sqrt{2}\right)t\right)}\left(2c_1\left(\left(1+\sqrt{2}\right)e^{2\sqrt{2}t} + 1 - \sqrt{2}\right) - \sqrt{2}c_2\left(e^{2\sqrt{2}t} - 1\right)\right)$$
$$y(t) \to \frac{1}{2}e^{-\left(\left(2+\sqrt{2}\right)t\right)}\left(\sqrt{2}c_1\left(e^{2\sqrt{2}t} - 1\right) + c_2\left(-\left(\sqrt{2}-1\right)e^{2\sqrt{2}t} + 1 + \sqrt{2}\right)\right)$$

1.12 problem 2.2 (vi)

Internal problem ID [12566]
Internal file name [`OUTPUT/11218_Wednesday_October_18_2023_10_01_18_PM_3236521/index.tex`]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.2 (vi).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = x(t)$$
$$y'(t) = y(t)$$

1.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{c}\\ &= \begin{bmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\\ &= \begin{bmatrix} \mathrm{e}^t c_1 \\ \mathrm{e}^t c_2 \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1-\lambda)(1-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1, v_2\}$ and there are no leading variables. Let $v_1 = t$. Let $v_2 = s$. Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ s \end{bmatrix}$$
$$= t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	2	2	No	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

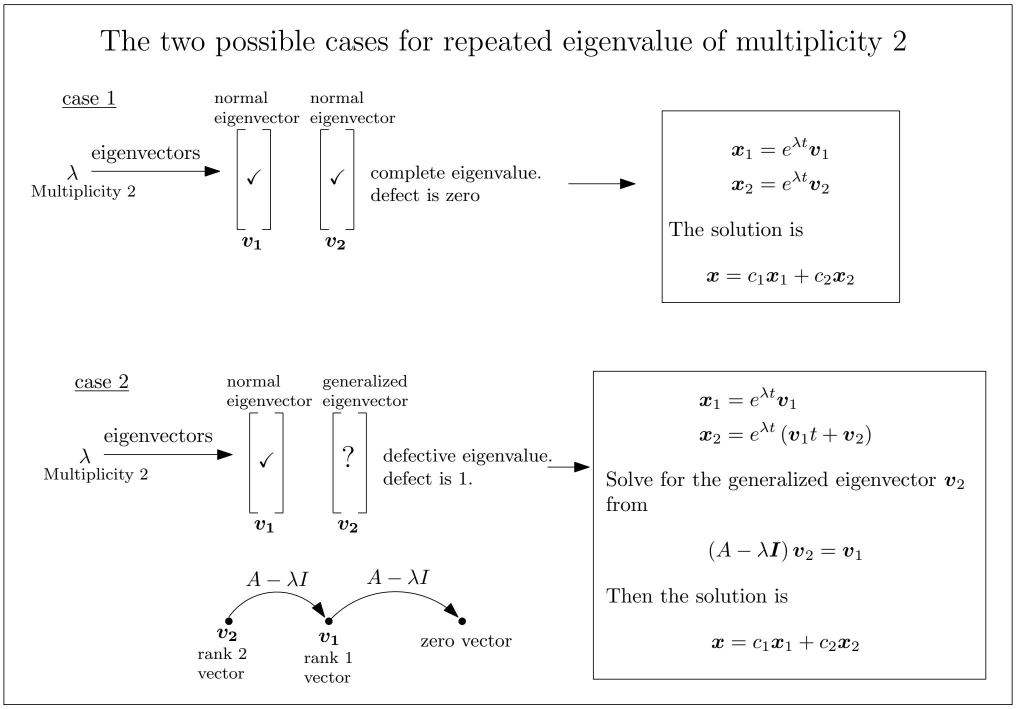

Figure 13: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{t} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{t} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^{t} \end{bmatrix} + c_2 \begin{bmatrix} e^{t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2 e^{t} \\ c_1 e^{t} \end{bmatrix}$$

The following is the phase plot of the system.

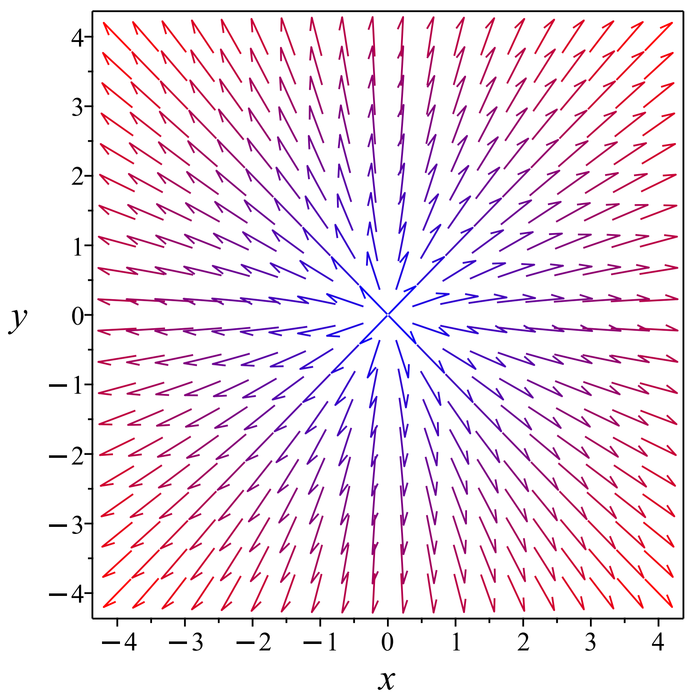

Figure 14: Phase plot

1.12.3 Maple step by step solution

Let's solve

$[x'(t) = x(t), y'(t) = y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

$$\vec{x}_1(t) = e^t \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, and

$$\vec{x}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_2(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda t}\vec{v}=\left(\mathrm{e}^{\lambda t}A\right)\cdot\left(t\vec{v}+\vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda t}\vec{v}=\mathrm{e}^{\lambda t}\left(\lambda t\vec{v}+A\cdot\vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-1\cdot\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}0\\ 1\end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{x}_2(t)=\mathrm{e}^{t}\cdot\left(t\cdot\begin{bmatrix}0\\ 1\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{x}=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x}=c_1\mathrm{e}^{t}\cdot\begin{bmatrix}0\\ 1\end{bmatrix}+c_2\mathrm{e}^{t}\cdot\left(t\cdot\begin{bmatrix}0\\ 1\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=\begin{bmatrix}0\\ \mathrm{e}^{t}(c_2t+c_1)\end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = 0, y(t) = e^t(c_2t + c_1)\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(x(t),t)=x(t),diff(y(t),t)=y(t)],singsol=all)
```

$$x(t) = c_2e^t$$
$$y(t) = c_1e^t$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 57

```
DSolve[{x'[t]==x[t],y'[t]==y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1e^t$$
$$y(t) \to c_2e^t$$
$$x(t) \to c_1e^t$$
$$y(t) \to 0$$
$$x(t) \to 0$$
$$y(t) \to c_2e^t$$
$$x(t) \to 0$$
$$y(t) \to 0$$

1.13 problem 2.2 (vii)

Internal problem ID [12567]
Internal file name [`OUTPUT/11219_Wednesday_October_18_2023_10_01_19_PM_79343540/index.tex`]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.2 (vii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 0$$
$$y'(t) = x(t)$$

1.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} c_1 \\ tc_1 + c_2 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 0 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-\lambda)(-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	2	1	Yes	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

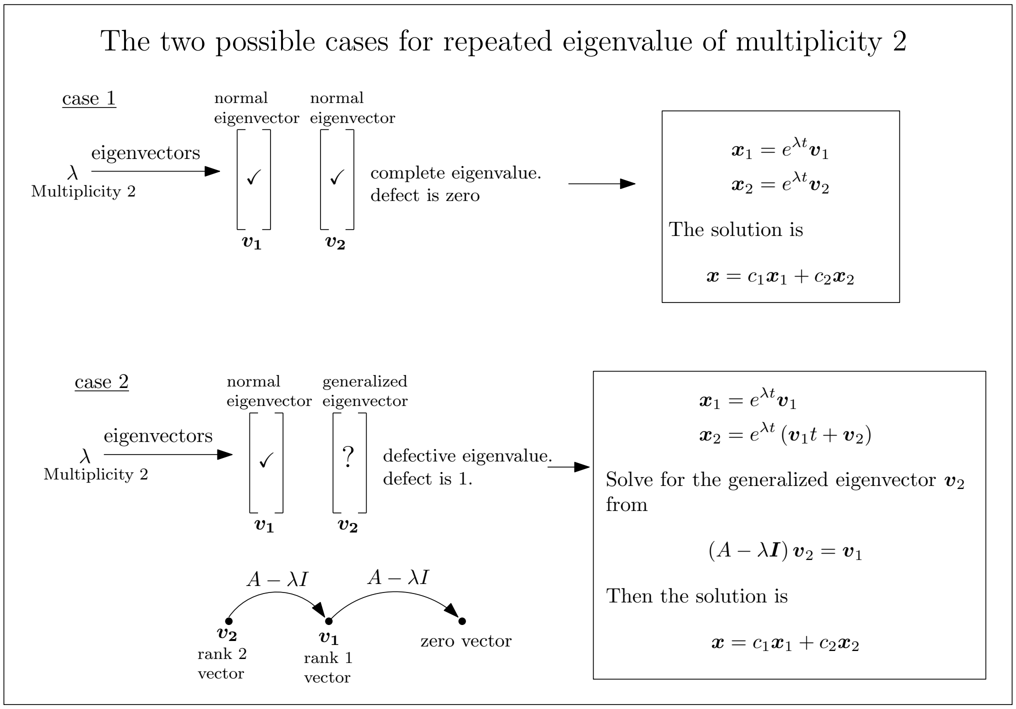

Figure 15: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) 1 \\
&= \begin{bmatrix} 1 \\ 1+t \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1+t \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2 \\ c_2 t + c_1 + c_2 \end{bmatrix}
$$

The following is the phase plot of the system.

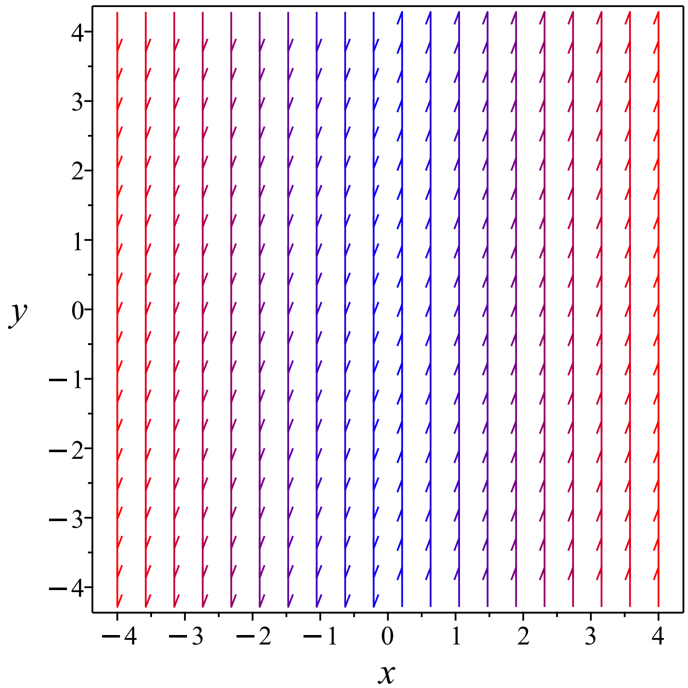

Figure 16: Phase plot

1.13.3 Maple step by step solution

Let's solve

$[x'(t) = 0, y'(t) = x(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

- Substitute solutions into the general solution

$$\vec{x} = \begin{bmatrix} 0 \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ c_1 \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = 0, y(t) = c_1\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(x(t),t)=0,diff(y(t),t)=x(t)],singsol=all)
```

$$x(t) = c_2$$
$$y(t) = c_2 t + c_1$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 17

```
DSolve[{x'[t]==0,y'[t]==x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1$$
$$y(t) \to c_1 t + c_2$$

1.14 problem 2.4 (i)

Internal problem ID [12568]
Internal file name [OUTPUT/11220_Wednesday_October_18_2023_10_01_19_PM_60338816/index.tex]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.4 (i).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Duffing, [_2nd_order, _reducible,
   _mu_x_y1]]
```

$$\boxed{x'' + x - x^3 = 0}$$

1.14.1 Solving as second order ode can be made integrable ode

Multiplying the ode by x' gives

$$x'x'' + (1 - x^2)\, x'x = 0$$

Integrating the above w.r.t t gives

$$\int (x'x'' + (1 - x^2)\, x'x)\, dt = 0$$

$$\frac{x'^2}{2} - \frac{(1 - x^2)^2}{4} = c_2$$

Which is now solved for x. Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$x' = \frac{\sqrt{2 + 2x^4 - 4x^2 + 8c_1}}{2} \tag{1}$$

$$x' = -\frac{\sqrt{2 + 2x^4 - 4x^2 + 8c_1}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2}{\sqrt{2x^4 - 4x^2 + 8c_1 + 2}}dx = \int dt$$
$$\int^x \frac{2}{\sqrt{2\_a^4 - 4\_a^2 + 8c_1 + 2}}d\_a = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2}{\sqrt{2x^4 - 4x^2 + 8c_1 + 2}}dx = \int dt$$
$$\int^x -\frac{2}{\sqrt{2\_a^4 - 4\_a^2 + 8c_1 + 2}}d\_a = t + c_3$$

Summary

The solution(s) found are the following

$$\int^x \frac{2}{\sqrt{2\_a^4 - 4\_a^2 + 8c_1 + 2}}d\_a = t + c_2 \tag{1}$$
$$\int^x -\frac{2}{\sqrt{2\_a^4 - 4\_a^2 + 8c_1 + 2}}d\_a = t + c_3 \tag{2}$$

Verification of solutions

$$\int^x \frac{2}{\sqrt{2\_a^4 - 4\_a^2 + 8c_1 + 2}}d\_a = t + c_2$$

Verified OK.

$$\int^x -\frac{2}{\sqrt{2\_a^4 - 4\_a^2 + 8c_1 + 2}}d\_a = t + c_3$$

Verified OK.

1.14.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable x an independent variable. Using

$$x' = p(x)$$

Then

$$\begin{aligned} x'' &= \frac{dp}{dt} \\ &= \frac{dx}{dt}\frac{dp}{dx} \\ &= p\frac{dp}{dx} \end{aligned}$$

Hence the ode becomes

$$p(x)\left(\frac{d}{dx}p(x)\right) + \left(-x^2+1\right)x = 0$$

Which is now solved as first order ode for $p(x)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= \frac{(x^2-1)\,x}{p} \end{aligned}$$

Where $f(x) = (x^2-1)\,x$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\frac{1}{\frac{1}{p}}\,dp = \left(x^2-1\right)x\,dx$$

$$\int \frac{1}{\frac{1}{p}}\,dp = \int \left(x^2-1\right)x\,dx$$

$$\frac{p^2}{2} = \frac{\left(x^2-1\right)^2}{4} + c_1$$

The solution is

$$\frac{p(x)^2}{2} - \frac{\left(x^2-1\right)^2}{4} - c_1 = 0$$

For solution (1) found earlier, since $p = x'$ then we now have a new first order ode to solve which is

$$\frac{x'^2}{2} - \frac{(x^2-1)^2}{4} - c_1 = 0$$

Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$x' = \frac{\sqrt{2+2x^4-4x^2+8c_1}}{2} \tag{1}$$

$$x' = -\frac{\sqrt{2+2x^4-4x^2+8c_1}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2}{\sqrt{2x^4-4x^2+8c_1+2}}dx = \int dt$$

$$\int^x \frac{2}{\sqrt{2\_a^4-4\_a^2+8c_1+2}}d\_a = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2}{\sqrt{2x^4-4x^2+8c_1+2}}dx = \int dt$$

$$\int^x -\frac{2}{\sqrt{2\_a^4-4\_a^2+8c_1+2}}d\_a = t + c_3$$

Summary
The solution(s) found are the following

$$\int^x \frac{2}{\sqrt{2\_a^4-4\_a^2+8c_1+2}}d\_a = t + c_2 \tag{1}$$

$$\int^x -\frac{2}{\sqrt{2\_a^4-4\_a^2+8c_1+2}}d\_a = t + c_3 \tag{2}$$

Verification of solutions

$$\int^{x} \frac{2}{\sqrt{2\_a^{4}-4\_a^{2}+8c_{1}+2}}d\_a = t+c_{2}$$

Verified OK.

$$\int^{x} -\frac{2}{\sqrt{2\_a^{4}-4\_a^{2}+8c_{1}+2}}d\_a = t+c_{3}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
<- 2nd_order JacobiSN successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 43

```
dsolve(diff(x(t),t$2)+x(t)-x(t)^3=0,x(t), singsol=all)
```

$$x(t) = c_{2}\sqrt{2}\,\sqrt{\frac{1}{c_{2}^{2}+1}}\;\mathrm{JacobiSN}\left(\frac{\left(\sqrt{2}\,t+2c_{1}\right)\sqrt{2}\,\sqrt{\frac{1}{c_{2}^{2}+1}}}{2}, c_{2}\right)$$

✓ Solution by Mathematica

Time used: 60.266 (sec). Leaf size: 171

```
DSolve[x''[t]+x[t]-x[t]^3==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{i\,\mathrm{sn}\left(\frac{\sqrt{\left(\sqrt{1-2c_1}+1\right)(t+c_2)^2}}{\sqrt{2}}\middle|\frac{1-\sqrt{1-2c_1}}{\sqrt{1-2c_1}+1}\right)}{\sqrt{\frac{1}{-1+\sqrt{1-2c_1}}}}$$

$$x(t) \to \frac{i\,\mathrm{sn}\left(\frac{\sqrt{\left(\sqrt{1-2c_1}+1\right)(t+c_2)^2}}{\sqrt{2}}\middle|\frac{1-\sqrt{1-2c_1}}{\sqrt{1-2c_1}+1}\right)}{\sqrt{\frac{1}{-1+\sqrt{1-2c_1}}}}$$

1.15 problem 2.4 (ii)

Internal problem ID [12569]
Internal file name [`OUTPUT/11221_Wednesday_October_18_2023_10_01_20_PM_50415031/index.tex`]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.4 (ii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Duffing, [_2nd_order, _reducible,
   _mu_x_y1]]
```

$$\boxed{x''+x+x^3=0}$$

1.15.1 Solving as second order ode can be made integrable ode

Multiplying the ode by x' gives

$$x'x''+\left(1+x^2\right)x'x=0$$

Integrating the above w.r.t t gives

$$\int \left(x'x''+\left(1+x^2\right)x'x\right)dt=0$$

$$\frac{x'^2}{2}+\frac{\left(1+x^2\right)^2}{4}=c_2$$

Which is now solved for x. Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$x'=\frac{\sqrt{-2-2x^4-4x^2+8c_1}}{2} \tag{1}$$

$$x'=-\frac{\sqrt{-2-2x^4-4x^2+8c_1}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2}{\sqrt{-2x^4 - 4x^2 + 8c_1 - 2}} dx = \int dt$$
$$\int^x \frac{2}{\sqrt{-2\_a^4 - 4\_a^2 + 8c_1 - 2}} d\_a = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2}{\sqrt{-2x^4 - 4x^2 + 8c_1 - 2}} dx = \int dt$$
$$\int^x -\frac{2}{\sqrt{-2\_a^4 - 4\_a^2 + 8c_1 - 2}} d\_a = t + c_3$$

Summary

The solution(s) found are the following

$$\int^x \frac{2}{\sqrt{-2\_a^4 - 4\_a^2 + 8c_1 - 2}} d\_a = t + c_2 \tag{1}$$
$$\int^x -\frac{2}{\sqrt{-2\_a^4 - 4\_a^2 + 8c_1 - 2}} d\_a = t + c_3 \tag{2}$$

Verification of solutions

$$\int^x \frac{2}{\sqrt{-2\_a^4 - 4\_a^2 + 8c_1 - 2}} d\_a = t + c_2$$

Verified OK.

$$\int^x -\frac{2}{\sqrt{-2\_a^4 - 4\_a^2 + 8c_1 - 2}} d\_a = t + c_3$$

Verified OK.

1.15.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable x an independent variable. Using

$$x' = p(x)$$

Then

$$\begin{aligned} x'' &= \frac{dp}{dt} \\ &= \frac{dx}{dt}\frac{dp}{dx} \\ &= p\frac{dp}{dx} \end{aligned}$$

Hence the ode becomes

$$p(x)\left(\frac{d}{dx}p(x)\right) + x\left(x^2+1\right) = 0$$

Which is now solved as first order ode for $p(x)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= -\frac{x(x^2+1)}{p} \end{aligned}$$

Where $f(x) = -x(x^2+1)$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\frac{1}{\frac{1}{p}}\, dp = -x\left(x^2+1\right)\, dx$$

$$\int \frac{1}{\frac{1}{p}}\, dp = \int -x\left(x^2+1\right)\, dx$$

$$\frac{p^2}{2} = -\frac{\left(x^2+1\right)^2}{4} + c_1$$

The solution is

$$\frac{p(x)^2}{2} + \frac{\left(x^2+1\right)^2}{4} - c_1 = 0$$

For solution (1) found earlier, since $p = x'$ then we now have a new first order ode to solve which is

$$\frac{x'^2}{2} + \frac{(1 + x^2)^2}{4} - c_1 = 0$$

Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$x' = \frac{\sqrt{-2 - 2x^4 - 4x^2 + 8c_1}}{2} \tag{1}$$

$$x' = -\frac{\sqrt{-2 - 2x^4 - 4x^2 + 8c_1}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2}{\sqrt{-2x^4 - 4x^2 + 8c_1 - 2}} dx = \int dt$$

$$\int^x \frac{2}{\sqrt{-2\_a^4 - 4\_a^2 + 8c_1 - 2}} d\_a = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2}{\sqrt{-2x^4 - 4x^2 + 8c_1 - 2}} dx = \int dt$$

$$\int^x -\frac{2}{\sqrt{-2\_a^4 - 4\_a^2 + 8c_1 - 2}} d\_a = t + c_3$$

Summary

The solution(s) found are the following

$$\int^x \frac{2}{\sqrt{-2\_a^4 - 4\_a^2 + 8c_1 - 2}} d\_a = t + c_2 \tag{1}$$

$$\int^x -\frac{2}{\sqrt{-2\_a^4 - 4\_a^2 + 8c_1 - 2}} d\_a = t + c_3 \tag{2}$$

Verification of solutions

$$\int^{x} \frac{2}{\sqrt{-2\_a^{4}-4\_a^{2}+8c_{1}-2}}d\_a = t+c_{2}$$

Verified OK.

$$\int^{x} -\frac{2}{\sqrt{-2\_a^{4}-4\_a^{2}+8c_{1}-2}}d\_a = t+c_{3}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
<- 2nd_order JacobiSN successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 56

```
dsolve(diff(x(t),t$2)+x(t)+x(t)^3=0,x(t), singsol=all)
```

$$x(t) = c_{2} \operatorname{JacobiSN}\left(\frac{\left(\sqrt{3}\,\sqrt{2}\,t+2c_{1}\right)\sqrt{2}\,\sqrt{-\frac{1}{c_{2}^{2}-3}}}{2}, \frac{ic_{2}\sqrt{3}}{3}\right)\sqrt{2}\,\sqrt{-\frac{1}{c_{2}^{2}-3}}$$

✓ Solution by Mathematica

Time used: 60.261 (sec). Leaf size: 169

```
DSolve[x''[t]+x[t]+x[t]^3==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -i\sqrt{1+\sqrt{1+2c_{1}}}\operatorname{sn}\left(\frac{\sqrt{-\left(\left(\sqrt{2c_{1}+1}-1\right)(t+c_{2})^{2}\right)}}{\sqrt{2}}\Big|\frac{\sqrt{2c_{1}+1}+1}{1-\sqrt{2c_{1}+1}}\right)$$

$$x(t) \to i\sqrt{1+\sqrt{1+2c_{1}}}\operatorname{sn}\left(\frac{\sqrt{-\left(\left(\sqrt{2c_{1}+1}-1\right)(t+c_{2})^{2}\right)}}{\sqrt{2}}\Big|\frac{\sqrt{2c_{1}+1}+1}{1-\sqrt{2c_{1}+1}}\right)$$

1.16 problem 2.4 (iii)

Internal problem ID [12570]
Internal file name [OUTPUT/11222_Wednesday_October_18_2023_10_01_21_PM_93984434/index.tex]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.4 (iii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

Unable to solve or complete the solution.

$$\boxed{x'' + x' + x - x^3 = 0}$$

1.16.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable x an independent variable. Using

$$x' = p(x)$$

Then

$$\begin{aligned} x'' &= \frac{dp}{dt} \\ &= \frac{dx}{dt}\frac{dp}{dx} \\ &= p\frac{dp}{dx} \end{aligned}$$

Hence the ode becomes

$$p(x)\left(\frac{d}{dx}p(x)\right) + p(x) + \left(-x^2 + 1\right)x = 0$$

Which is now solved as first order ode for $p(x)$. Unable to determine ODE type.

Unable to solve. Terminating

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)+_a-_a^3 = 0, _b(_a)`   **
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   trying inverse linear
   trying homogeneous types:
   trying Chini
   differential order: 1; looking for linear symmetries
   trying exact
   trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
   Looking for potential symmetries
   trying inverse_Riccati
   trying an equivalence to an Abel ODE
   differential order: 1; trying a linearization to 2nd order
   --- trying a change of variables {x -> y(x), y(x) -> x}
   differential order: 1; trying a linearization to 2nd order
   trying 1st order ODE linearizable_by_differentiation
   --- Trying Lie symmetry methods, 1st order ---
   `, `-> Computing symmetries using: way = 3
   `, `-> Computing symmetries using: way = 4
   `, `-> Computing symmetries using: way = 2
   trying symmetry patterns for 1st order ODEs
   -> trying a symmetry pattern of the form [F(x)*G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)*G(y)]
   -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
   -> trying a symmetry pattern of the form [F(x),G(x)]
   -> trying a symmetry pattern of the form [F(y),G(y)]
   -> trying a symmetry pattern of the form [F(x)+G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)+G(y)]
```

✗ Solution by Maple

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)-x(t)^3=0,x(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x''[t]+x'[t]+x[t]-x[t]^3==0,x[t],t,IncludeSingularSolutions -> True]
```

Not solved

1.17 problem 2.4 (iv)

Internal problem ID [12571]
Internal file name [`OUTPUT/11223_Wednesday_October_18_2023_10_01_21_PM_54312799/index.tex`]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.4 (iv).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

Unable to solve or complete the solution.

$$\boxed{x'' + x' + x + x^3 = 0}$$

1.17.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable x an independent variable. Using

$$x' = p(x)$$

Then

$$\begin{aligned} x'' &= \frac{dp}{dt} \\ &= \frac{dx}{dt}\frac{dp}{dx} \\ &= p\frac{dp}{dx} \end{aligned}$$

Hence the ode becomes

$$p(x)\left(\frac{d}{dx}p(x)\right) + p(x) + x(x^2 + 1) = 0$$

Which is now solved as first order ode for $p(x)$. Unable to determine ODE type.

Unable to solve. Terminating

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, _a^3+(diff(_b(_a), _a))*_b(_a)+_b(_a)+_a = 0, _b(_a)`   **
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   trying inverse linear
   trying homogeneous types:
   trying Chini
   differential order: 1; looking for linear symmetries
   trying exact
   trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
   Looking for potential symmetries
   trying inverse_Riccati
   trying an equivalence to an Abel ODE
   differential order: 1; trying a linearization to 2nd order
   --- trying a change of variables {x -> y(x), y(x) -> x}
   differential order: 1; trying a linearization to 2nd order
   trying 1st order ODE linearizable_by_differentiation
   --- Trying Lie symmetry methods, 1st order ---
   `, `-> Computing symmetries using: way = 3
   `, `-> Computing symmetries using: way = 4
   `, `-> Computing symmetries using: way = 2
   trying symmetry patterns for 1st order ODEs
   -> trying a symmetry pattern of the form [F(x)*G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)*G(y)]
   -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
   -> trying a symmetry pattern of the form [F(x),G(x)]
   -> trying a symmetry pattern of the form [F(y),G(y)]
   -> trying a symmetry pattern of the form [F(x)+G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)+G(y)]
```

✗ Solution by Maple

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)+x(t)^3=0,x(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x''[t]+x'[t]+x[t]+x[t]^3==0,x[t],t,IncludeSingularSolutions -> True]
```

Not solved

1.18 problem 2.4 (v)

Internal problem ID [12572]
Internal file name [`OUTPUT/11224_Wednesday_October_18_2023_10_01_22_PM_50942243/index.tex`]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.4 (v).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{x'' - (2\cos(x) - 1)\sin(x) = 0}$$

1.18.1 Solving as second order ode can be made integrable ode

Multiplying the ode by x' gives

$$x'x'' - x'(\sin(2x) - \sin(x)) = 0$$

Integrating the above w.r.t t gives

$$\int (x'x'' - x'(\sin(2x) - \sin(x)))\,dt = 0$$
$$\frac{x'^2}{2} - \cos(x) + \frac{\cos(2x)}{2} = c_2$$

Which is now solved for x. Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$x' = \sqrt{1 - 2\cos(x)^2 + 2\cos(x) + 2c_1} \tag{1}$$
$$x' = -\sqrt{1 - 2\cos(x)^2 + 2\cos(x) + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{1-2\cos(x)^2+2\cos(x)+2c_1}}dx = \int dt$$

$$\int^x \frac{1}{\sqrt{1-2\cos(\_a)^2+2\cos(\_a)+2c_1}}d\_a = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{1-2\cos(x)^2+2\cos(x)+2c_1}}dx = \int dt$$

$$\int^x -\frac{1}{\sqrt{1-2\cos(\_a)^2+2\cos(\_a)+2c_1}}d\_a = t + c_3$$

Summary

The solution(s) found are the following

$$\int^x \frac{1}{\sqrt{1-2\cos(\_a)^2+2\cos(\_a)+2c_1}}d\_a = t + c_2 \tag{1}$$

$$\int^x -\frac{1}{\sqrt{1-2\cos(\_a)^2+2\cos(\_a)+2c_1}}d\_a = t + c_3 \tag{2}$$

Verification of solutions

$$\int^x \frac{1}{\sqrt{1-2\cos(\_a)^2+2\cos(\_a)+2c_1}}d\_a = t + c_2$$

Verified OK.

$$\int^x -\frac{1}{\sqrt{1-2\cos(\_a)^2+2\cos(\_a)+2c_1}}d\_a = t + c_3$$

Verified OK.

1.18.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable x an independent variable. Using

$$x' = p(x)$$

Then

$$\begin{aligned} x'' &= \frac{dp}{dt} \\ &= \frac{dx}{dt}\frac{dp}{dx} \\ &= p\frac{dp}{dx} \end{aligned}$$

Hence the ode becomes

$$p(x)\left(\frac{d}{dx}p(x)\right) = \sin(2x) - \sin(x)$$

Which is now solved as first order ode for $p(x)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= \frac{\sin(2x) - \sin(x)}{p} \end{aligned}$$

Where $f(x) = \sin(2x) - \sin(x)$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{1}{p}}\,dp &= \sin(2x) - \sin(x)\,dx \\ \int \frac{1}{\frac{1}{p}}\,dp &= \int \sin(2x) - \sin(x)\,dx \\ \frac{p^2}{2} &= \cos(x) - \frac{\cos(2x)}{2} + c_1 \end{aligned}$$

The solution is

$$\frac{p(x)^2}{2} - \cos(x) + \frac{\cos(2x)}{2} - c_1 = 0$$

For solution (1) found earlier, since $p = x'$ then we now have a new first order ode to solve which is

$$\frac{x'^2}{2} - \cos(x) + \frac{\cos(2x)}{2} - c_1 = 0$$

Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$x' = \sqrt{1 - 2\cos(x)^2 + 2\cos(x) + 2c_1} \tag{1}$$

$$x' = -\sqrt{1 - 2\cos(x)^2 + 2\cos(x) + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{1 - 2\cos(x)^2 + 2\cos(x) + 2c_1}} dx = \int dt$$

$$\int^x \frac{1}{\sqrt{1 - 2\cos(\_a)^2 + 2\cos(\_a) + 2c_1}} d\_a = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{1 - 2\cos(x)^2 + 2\cos(x) + 2c_1}} dx = \int dt$$

$$\int^x -\frac{1}{\sqrt{1 - 2\cos(\_a)^2 + 2\cos(\_a) + 2c_1}} d\_a = t + c_3$$

Summary

The solution(s) found are the following

$$\int^x \frac{1}{\sqrt{1 - 2\cos(\_a)^2 + 2\cos(\_a) + 2c_1}} d\_a = t + c_2 \tag{1}$$

$$\int^x -\frac{1}{\sqrt{1 - 2\cos(\_a)^2 + 2\cos(\_a) + 2c_1}} d\_a = t + c_3 \tag{2}$$

Verification of solutions

$$\int^{x} \frac{1}{\sqrt{1-2\cos\left(\_a\right)^{2}+2\cos\left(\_a\right)+2c_{1}}}d\_a = t+c_{2}$$

Verified OK.

$$\int^{x} -\frac{1}{\sqrt{1-2\cos\left(\_a\right)^{2}+2\cos\left(\_a\right)+2c_{1}}}d\_a = t+c_{3}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-sin(2*_a)+sin(_a) = 0, _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 59

```
dsolve(diff(x(t),t$2)=(2*cos(x(t))-1)*sin(x(t)),x(t), singsol=all)
```

$$\int^{x(t)} \frac{1}{\sqrt{2 \sin \left(\_a\right)^{2}+2 \cos \left(\_a\right)+c_{1}}}d\_a - t - c_{2} = 0$$

$$-\left(\int^{x(t)} \frac{1}{\sqrt{2 \sin \left(\_a\right)^{2}+2 \cos \left(\_a\right)+c_{1}}}d\_a\right) - t - c_{2} = 0$$

✓ Solution by Mathematica

Time used: 61.831 (sec). Leaf size: 437

```
DSolve[x''[t]==(2*Cos[x[t]]-1)*Sin[x[t]],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -2\arccos\left(-\frac{1}{2}\sqrt{3-\sqrt{3+2c_1}}\right)$$
$$x(t) \to 2\arccos\left(-\frac{1}{2}\sqrt{3-\sqrt{3+2c_1}}\right)$$
$$x(t) \to -2\arccos\left(\frac{1}{2}\sqrt{3-\sqrt{3+2c_1}}\right)$$
$$x(t) \to 2\arccos\left(\frac{1}{2}\sqrt{3-\sqrt{3+2c_1}}\right)$$
$$x(t) \to -2\arccos\left(-\frac{1}{2}\sqrt{3+\sqrt{3+2c_1}}\right)$$
$$x(t) \to 2\arccos\left(-\frac{1}{2}\sqrt{3+\sqrt{3+2c_1}}\right)$$
$$x(t) \to -2\arccos\left(\frac{1}{2}\sqrt{3+\sqrt{3+2c_1}}\right)$$
$$x(t) \to 2\arccos\left(\frac{1}{2}\sqrt{3+\sqrt{3+2c_1}}\right)$$
$$x(t) \to -2i\operatorname{arctanh}\left(\frac{\operatorname{sn}\left(\frac{1}{2}\sqrt{\left(-c_1+2\sqrt{2c_1+3}-3\right)(t+c_2)^2}\middle|\frac{c_1+2\sqrt{2c_1+3}+3}{c_1-2\sqrt{2c_1+3}+3}\right)}{\sqrt{\frac{-3+c_1}{3+c_1+2\sqrt{3+2c_1}}}}\right)$$
$$x(t) \to 2i\operatorname{arctanh}\left(\frac{\operatorname{sn}\left(\frac{1}{2}\sqrt{\left(-c_1+2\sqrt{2c_1+3}-3\right)(t+c_2)^2}\middle|\frac{c_1+2\sqrt{2c_1+3}+3}{c_1-2\sqrt{2c_1+3}+3}\right)}{\sqrt{\frac{-3+c_1}{3+c_1+2\sqrt{3+2c_1}}}}\right)$$

1.19 problem 2.5

Internal problem ID [12573]
Internal file name [`OUTPUT/11225_Wednesday_October_18_2023_10_03_50_PM_88559711/index.tex`]

Book: Nonlinear Ordinary Differential Equations by D.W.Jordna and P.Smith. 4th edition 1999. Oxford Univ. Press. NY
Section: Chapter 2. Plane autonomous systems and linearization. Problems page 79
Problem number: 2.5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= x - 5y(t) \\
y'(t) &= x - y(t)
\end{aligned}
$$

1.19.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{\sin(2t)}{2} + \cos(2t) & -\frac{5\sin(2t)}{2} \\ \frac{\sin(2t)}{2} & \cos(2t) - \frac{\sin(2t)}{2} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{\sin(2t)}{2} + \cos(2t) & -\frac{5\sin(2t)}{2} \\ \frac{\sin(2t)}{2} & \cos(2t) - \frac{\sin(2t)}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{\sin(2t)}{2} + \cos(2t)\right) c_1 - \frac{5\sin(2t)c_2}{2} \\ \frac{\sin(2t)c_1}{2} + \left(\cos(2t) - \frac{\sin(2t)}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-5c_2)\sin(2t)}{2} + c_1\cos(2t) \\ \frac{(c_1-c_2)\sin(2t)}{2} + c_2\cos(2t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.19.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2i$	1	complex eigenvalue
$-2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} - (-2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1+2i & -5 \\ 1 & -1+2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1+2i & -5 & 0 \\ 1 & -1+2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{5} + \frac{2i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 1+2i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+2i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1-2i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1-2\,\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1-2i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1-2\,\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1-2\,\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} - (2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1-2i & -5 & 0 \\ 1 & -1-2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{5} - \frac{2i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 1-2i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-2i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1+2i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1+2\,\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1+2i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1+2\,\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1+2\,\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2i$	1	1	No	$\begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$
$-2i$	1	1	No	$\begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} (1+2i)\,\mathrm{e}^{2it} \\ \mathrm{e}^{2it} \end{bmatrix} + c_2 \begin{bmatrix} (1-2i)\,\mathrm{e}^{-2it} \\ \mathrm{e}^{-2it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} (1+2i)\,c_1\mathrm{e}^{2it} + (1-2i)\,c_2\mathrm{e}^{-2it} \\ c_1\mathrm{e}^{2it} + c_2\mathrm{e}^{-2it} \end{bmatrix}$$

The following is the phase plot of the system.

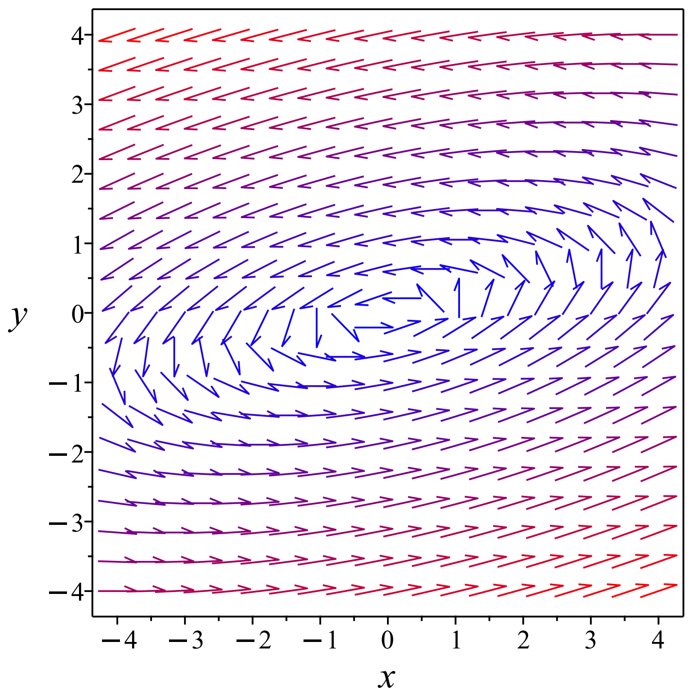

Figure 17: Phase plot

1.19.3 Maple step by step solution

Let's solve

$[x' = x - 5y(t)\,, y'(t) = x - y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2\,\mathrm{I}, \begin{bmatrix} 1-2\,\mathrm{I} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} 1+2\,\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} 1-2\,\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-2\,\mathrm{I}t} \cdot \begin{bmatrix} 1-2\,\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} 1-2\,\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\left[\begin{array}{c}(1-2\,\mathrm{I})\left(\cos\left(2t\right)-\mathrm{I}\sin\left(2t\right)\right)\\ \cos\left(2t\right)-\mathrm{I}\sin\left(2t\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t)=\left[\begin{array}{c}\cos\left(2t\right)-2\sin\left(2t\right)\\ \cos\left(2t\right)\end{array}\right],\vec{x}_2(t)=\left[\begin{array}{c}-2\cos\left(2t\right)-\sin\left(2t\right)\\ -\sin\left(2t\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{x}=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x}=\left[\begin{array}{c}c_2(-2\cos\left(2t\right)-\sin\left(2t\right))+c_1(\cos\left(2t\right)-2\sin\left(2t\right))\\ c_1\cos\left(2t\right)-c_2\sin\left(2t\right)\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x\\ y(t)\end{array}\right]=\left[\begin{array}{c}(c_1-2c_2)\cos\left(2t\right)-2\sin\left(2t\right)\left(\frac{c_2}{2}+c_1\right)\\ c_1\cos\left(2t\right)-c_2\sin\left(2t\right)\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x=(c_1-2c_2)\cos\left(2t\right)-2\sin\left(2t\right)\left(\frac{c_2}{2}+c_1\right),y(t)=c_1\cos\left(2t\right)-c_2\sin\left(2t\right)\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve([diff(x(t),t)=x(t)-5*y(t),diff(y(t),t)=x(t)-y(t)],singsol=all)
```

$$x(t)=c_1\sin\left(2t\right)+c_2\cos\left(2t\right)$$
$$y(t)=-\frac{2c_1\cos\left(2t\right)}{5}+\frac{2c_2\sin\left(2t\right)}{5}+\frac{c_1\sin\left(2t\right)}{5}+\frac{c_2\cos\left(2t\right)}{5}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 48

```
DSolve[{x'[t]==x[t]-5*y[t],y'[t]==x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t)\to c_1\cos(2t)+(c_1-5c_2)\sin(t)\cos(t)$$
$$y(t)\to c_2\cos(2t)+(c_1-c_2)\sin(t)\cos(t)$$