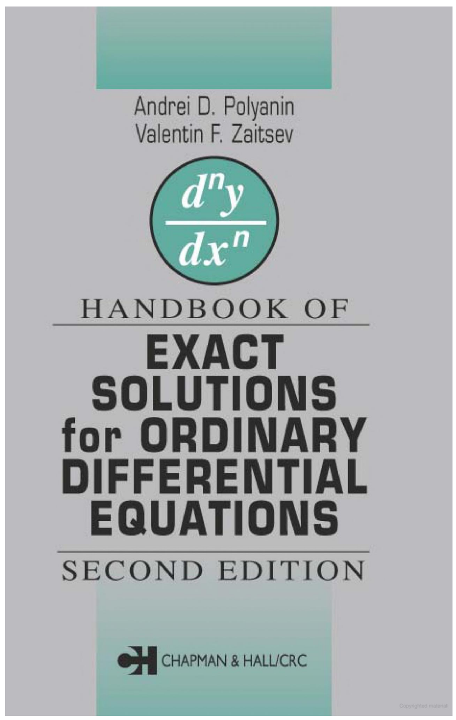
A Solution Manual For

Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition



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Contents

1 Chapter 1, First-Order differential equations

1.1 problem 1.1.1

Internal problem ID [10325]
Internal file name [`OUTPUT/9272_Monday_June_06_2022_01_45_19_PM_35985161/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, First-Order differential equations
Problem number: 1.1.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = f(x)}$$

1.1.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int f(x)\, \mathrm{d}x \\ &= \int f(x)\, dx + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \int f(x)\, dx + c_1 \tag{1}$$

Verification of solutions

$$y = \int f(x)\, dx + c_1$$

Verified OK.

1.1.2 Maple step by step solution

Let's solve

$y' = f(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int f(x) \, dx + c_1$

- Evaluate integral

  $y = \int f(x) \, dx + c_1$

- Solve for y

  $y = \int f(x) \, dx + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)=f(x),y(x), singsol=all)
```

$$y(x) = \int f(x) \, dx + c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 18

```
DSolve[y'[x]==f[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \int_1^x f(K[1]) dK[1] + c_1$$

1.2 problem 1.1.2

Internal problem ID [10326]
Internal file name [`OUTPUT/9273_Monday_June_06_2022_01_45_19_PM_3498373/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, First-Order differential equations
Problem number: 1.1.2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - f(y) = 0}$$

1.2.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{f(y)} dy = \int dx$$
$$\int^{y} \frac{1}{f(\_a)} d\_a = x + c_1$$

Summary
The solution(s) found are the following

$$\int^{y} \frac{1}{f(\_a)} d\_a = x + c_1 \tag{1}$$

Verification of solutions

$$\int^{y} \frac{1}{f(\_a)} d\_a = x + c_1$$

Verified OK.

1.2.2 Maple step by step solution

Let's solve

$y' - f(y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{f(y)} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{f(y)} dx = \int 1 dx + c_1$

- Cannot compute integral

  $\int \frac{y'}{f(y)} dx = x + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)=f(y(x)),y(x), singsol=all)
```

$$x - \left(\int^{y(x)} \frac{1}{f(\_a)} d\_a\right) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.442 (sec). Leaf size: 33

```
DSolve[y'[x]==f[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{1}{f(K[1])} dK[1]\&\right][x + c_1]$$

$$y(x) \to f^{(-1)}(0)$$

1.3 problem 1.1.3

Internal problem ID [10327]
Internal file name [`OUTPUT/9274_Monday_June_06_2022_01_45_19_PM_45060263/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, First-Order differential equations
Problem number: 1.1.3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - f(x)\, g(y) = 0}$$

1.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= f(x)\, g(y) \end{aligned}$$

Where $f(x) = f(x)$ and $g(y) = g(y)$. Integrating both sides gives

$$\frac{1}{g(y)}\, dy = f(x)\, dx$$

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

$$\int^{y} \frac{1}{g(\_a)} d\_a = \int f(x)\, dx + c_1$$

Which results in

$$\int^{y} \frac{1}{g(\_a)} d\_a = \int f(x)\, dx + c_1$$

The solution is

$$\int^{y} \frac{1}{g(\_a)} d\_a - \left(\int f(x)\, dx\right) - c_1 = 0$$

Summary

The solution(s) found are the following

$$\int^{y} \frac{1}{g(\_a)} d\_a - \left(\int f(x)\, dx\right) - c_1 = 0 \tag{1}$$

Verification of solutions

$$\int^{y} \frac{1}{g(\_a)} d\_a - \left(\int f(x)\, dx\right) - c_1 = 0$$

Verified OK.

1.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = f(x)\, g(y)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 3: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{f(x)} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{f(x)}} dx \end{aligned}$$

Which results in

$$S = \int f(x)\, dx$$

1.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{1}{g(y)}\right)\mathrm{d}y &= (f(x))\,\mathrm{d}x \\
(-f(x))\,\mathrm{d}x + \left(\frac{1}{g(y)}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= -f(x) \\
N(x,y) &= \frac{1}{g(y)}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-f(x)) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{g(y)}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -f(x) \, \mathrm{d}x$$

$$\phi = \int^{x} -f(\_a) \, d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{g(y)}$. Therefore equation (4) becomes

$$\frac{1}{g(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{g(y)}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{g(y)} \right) \mathrm{d}y$$

$$f(y) = \int_{0}^{y} \frac{1}{g(\_a)} d\_a + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^{x} -f(\_a) \, d\_a + \int_{0}^{y} \frac{1}{g(\_a)} d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^x -f(\_a)\, d\_a + \int_0^y \frac{1}{g(\_a)} d\_a$$

Summary

The solution(s) found are the following

$$\int^x -f(\_a)\, d\_a + \int_0^y \frac{1}{g(\_a)} d\_a = c_1 \tag{1}$$

Verification of solutions

$$\int^x -f(\_a)\, d\_a + \int_0^y \frac{1}{g(\_a)} d\_a = c_1$$

Verified OK.

1.3.4 Maple step by step solution

Let's solve

$y' - f(x)\, g(y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{g(y)} = f(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{g(y)} dx = \int f(x)\, dx + c_1$

- Cannot compute integral

  $\int \frac{y'}{g(y)} dx = \int f(x)\, dx + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x)=f(x)*g(y(x)),y(x), singsol=all)
```

$$\int f(x)\,dx - \left(\int^{y(x)} \frac{1}{g(\_a)} d\_a\right) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.452 (sec). Leaf size: 42

```
DSolve[y'[x]==f[x]*g[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{1}{g(K[1])} dK[1]\&\right]\left[\int_1^x f(K[2]) dK[2] + c_1\right]$$
$$y(x) \to g^{(-1)}(0)$$

1.4 problem 1.1.4

Internal problem ID [10328]
Internal file name [`OUTPUT/9275_Monday_June_06_2022_01_45_20_PM_51267741/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, First-Order differential equations
Problem number: 1.1.4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{g(x)\,y' - f_1(x)\,y = f_0(x)}$$

1.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{f_1(x)}{g\,(x)}$$
$$q(x) = \frac{f_0(x)}{g\,(x)}$$

Hence the ode is

$$y' - \frac{f_1(x)\,y}{g\,(x)} = \frac{f_0(x)}{g\,(x)}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{f_1(x)}{g(x)}dx}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{f_0(x)}{g(x)}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\int -\frac{f_1(x)}{g(x)}dx} y\right) = \left(\mathrm{e}^{\int -\frac{f_1(x)}{g(x)}dx}\right)\left(\frac{f_0(x)}{g(x)}\right)$$

$$\mathrm{d}\left(\mathrm{e}^{\int -\frac{f_1(x)}{g(x)}dx} y\right) = \left(\frac{f_0(x)\,\mathrm{e}^{-\left(\int \frac{f_1(x)}{g(x)}dx\right)}}{g(x)}\right)\mathrm{d}x$$

Integrating gives

$$\mathrm{e}^{\int -\frac{f_1(x)}{g(x)}dx} y = \int \frac{f_0(x)\,\mathrm{e}^{-\left(\int \frac{f_1(x)}{g(x)}dx\right)}}{g(x)}\,\mathrm{d}x$$

$$\mathrm{e}^{\int -\frac{f_1(x)}{g(x)}dx} y = \int \frac{f_0(x)\,\mathrm{e}^{-\left(\int \frac{f_1(x)}{g(x)}dx\right)}}{g(x)}dx + c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{f_1(x)}{g(x)}dx}$ results in

$$y = \mathrm{e}^{\int \frac{f_1(x)}{g(x)}dx}\left(\int \frac{f_0(x)\,\mathrm{e}^{-\left(\int \frac{f_1(x)}{g(x)}dx\right)}}{g(x)}dx\right) + c_1 \mathrm{e}^{\int \frac{f_1(x)}{g(x)}dx}$$

which simplifies to

$$y = \mathrm{e}^{\int \frac{f_1(x)}{g(x)}dx}\left(\int \frac{f_0(x)\,\mathrm{e}^{-\left(\int \frac{f_1(x)}{g(x)}dx\right)}}{g(x)}dx + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\int \frac{f_1(x)}{g(x)}dx}\left(\int \frac{f_0(x)\,\mathrm{e}^{-\left(\int \frac{f_1(x)}{g(x)}dx\right)}}{g(x)}dx + c_1\right) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{\int \frac{f_1(x)}{g(x)}dx}\left(\int \frac{f_0(x)\,\mathrm{e}^{-\left(\int \frac{f_1(x)}{g(x)}dx\right)}}{g(x)}dx + c_1\right)$$

Verified OK.

1.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{f_1(x)\, y + f_0(x)}{g\,(x)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 6: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \mathrm{e}^{\int \frac{f_1(x)}{g(x)}dx}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta}dy\\ &= \int \frac{1}{\mathrm{e}^{\int \frac{f_1(x)}{g(x)}dx}}dy\end{aligned}$$

1.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(g(x)) \, \mathrm{d}y &= (f_1(x) \, y + f_0(x)) \, \mathrm{d}x \\
(-f_1(x) \, y - f_0(x)) \, \mathrm{d}x + (g(x)) \, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -f_1(x) \, y - f_0(x) \\
N(x, y) &= g(x)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-f_1(x) \, y - f_0(x)) \\
&= -f_1(x)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(g(x)) \\
&= g'(x)
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{g(x)}\left((-f_1(x)) - (g'(x))\right) \\ &= \frac{-f_1(x) - g'(x)}{g(x)} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{-f_1(x) - g'(x)}{g(x)} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\int \frac{-f_1(x) - g'(x)}{g(x)} dx} \\ &= \mathrm{e}^{-\left(\int \frac{f_1(x) + g'(x)}{g(x)} dx\right)} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-\left(\int \frac{f_1(x) + g'(x)}{g(x)} dx\right)} (-f_1(x)\, y - f_0(x)) \\ &= -(f_1(x)\, y + f_0(x))\, \mathrm{e}^{-\left(\int \frac{f_1(x) + g'(x)}{g(x)} dx\right)} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-\left(\int \frac{f_1(x) + g'(x)}{g(x)} dx\right)} (g(x)) \\ &= g(x)\, \mathrm{e}^{-\left(\int \frac{f_1(x) + g'(x)}{g(x)} dx\right)} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-(f_1(x)\, y + f_0(x))\, \mathrm{e}^{-\left(\int \frac{f_1(x) + g'(x)}{g(x)} dx\right)}\right) + \left(g(x)\, \mathrm{e}^{-\left(\int \frac{f_1(x) + g'(x)}{g(x)} dx\right)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int \overline{M}\, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int -(f_1(x)\, y + f_0(x))\, \mathrm{e}^{-\left(\int \frac{f_1(x)+g'(x)}{g(x)} dx\right)}\, \mathrm{d}x$$

$$\phi = \int^x -(f_1(\_a)\, y + f_0(\_a))\, \mathrm{e}^{-\left(\int \frac{f_1(\_a)+\frac{d}{d\_a} g(\_a)}{g(\_a)} d\_a\right)} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\left(\int^x f_1(\_a)\, \mathrm{e}^{-\left(\int \frac{f_1(\_a)+\frac{d}{d\_a} g(\_a)}{g(\_a)} d\_a\right)} d\_a\right) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = g(x)\, \mathrm{e}^{-\left(\int \frac{f_1(x)+g'(x)}{g(x)} dx\right)}$. Therefore equation (4) becomes

$$g(x)\, \mathrm{e}^{-\left(\int \frac{f_1(x)+g'(x)}{g(x)} dx\right)} = -\left(\int^x f_1(\_a)\, \mathrm{e}^{-\left(\int \frac{f_1(\_a)+\frac{d}{d\_a} g(\_a)}{g(\_a)} d\_a\right)} d\_a\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = g(x)\, \mathrm{e}^{-\left(\int \frac{f_1(x)+g'(x)}{g(x)} dx\right)} + \int^x f_1(\_a)\, \mathrm{e}^{-\left(\int \frac{f_1(\_a)+\frac{d}{d\_a} g(\_a)}{g(\_a)} d\_a\right)} d\_a$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(g(x)\,\mathrm{e}^{-\left(\int \frac{f_1(x)+g'(x)}{g(x)}dx\right)} + \int^{x} f_1(\_a)\,\mathrm{e}^{-\left(\int \frac{f_1(\_a)+\frac{d}{d\_a}g(\_a)}{g(\_a)}d\_a\right)}d\_a\right)\mathrm{d}y$$

$$f(y) = \int_0^y \left(g(x)\,\mathrm{e}^{-\left(\int \frac{f_1(x)+g'(x)}{g(x)}dx\right)} + \int^{x} f_1(\_a)\,\mathrm{e}^{-\left(\int \frac{f_1(\_a)+\frac{d}{d\_a}g(\_a)}{g(\_a)}d\_a\right)}d\_a\right)d\_a + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^{x} -(f_1(\_a)\,y + f_0(\_a))\,\mathrm{e}^{-\left(\int \frac{f_1(\_a)+\frac{d}{d\_a}g(\_a)}{g(\_a)}d\_a\right)}d\_a + \int_0^y \left(g(x)\,\mathrm{e}^{-\left(\int \frac{f_1(x)+g'(x)}{g(x)}dx\right)} + \int^{x} f_1(\_a)\,\mathrm{e}^{-\left(\int \frac{f_1(\_a)+\frac{d}{d\_a}g(\_a)}{g(\_a)}d\_a\right)}d\_a\right)d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^{x} -(f_1(\_a)\,y + f_0(\_a))\,\mathrm{e}^{-\left(\int \frac{f_1(\_a)+\frac{d}{d\_a}g(\_a)}{g(\_a)}d\_a\right)}d\_a + \int_0^y \left(g(x)\,\mathrm{e}^{-\left(\int \frac{f_1(x)+g'(x)}{g(x)}dx\right)} + \int^{x} f_1(\_a)\,\mathrm{e}^{-\left(\int \frac{f_1(\_a)+\frac{d}{d\_a}g(\_a)}{g(\_a)}d\_a\right)}d\_a\right)d\_a$$

Summary

The solution(s) found are the following

$$\int^{x} -(f_1(\_a)\,y + f_0(\_a))\,\mathrm{e}^{-\left(\int \frac{f_1(\_a)+\frac{d}{d\_a}g(\_a)}{g(\_a)}d\_a\right)}d\_a + \int_0^y \left(g(x)\,\mathrm{e}^{-\left(\int \frac{f_1(x)+g'(x)}{g(x)}dx\right)} + \int^{x} f_1(\_a)\,\mathrm{e}^{-\left(\int \frac{f_1(\_a)+\frac{d}{d\_a}g(\_a)}{g(\_a)}d\_a\right)}d\_a\right)d\_a = c_1 \tag{1}$$

Verification of solutions

$$\int^{x} -(f_1(\_a)\, y+f_0(\_a))\, \mathrm{e}^{-\left(\int \frac{f_1(\_a)+\frac{d}{d\_a}g(\_a)}{g(\_a)}d\_a\right)}d\_a + \int_0^{y}\left(g(x)\, \mathrm{e}^{-\left(\int \frac{f_1(x)+g'(x)}{g(x)}dx\right)}+\int^{x} f_1(\_a)\, \mathrm{e}^{-\left(\int \frac{f_1(\_a)+\frac{d}{d\_a}g(\_a)}{g(\_a)}d\_a\right)}d\_a\right)d\_a = c_1$$

Verified OK.

1.4.4 Maple step by step solution

Let's solve

$g(x)\, y'-f_1(x)\, y=f_0(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=\frac{f_1(x)y}{g(x)}+\frac{f_0(x)}{g(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'-\frac{f_1(x)y}{g(x)}=\frac{f_0(x)}{g(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'-\frac{f_1(x)y}{g(x)}\right)=\frac{\mu(x)f_0(x)}{g(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y'-\frac{f_1(x)y}{g(x)}\right)=\mu'(x)\, y+\mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{\mu(x)f_1(x)}{g(x)}$

- Solve to find the integrating factor

  $\mu(x)=\mathrm{e}^{\int -\frac{f_1(x)}{g(x)}dx}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right)dx=\int \frac{\mu(x)f_0(x)}{g(x)}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y=\int \frac{\mu(x)f_0(x)}{g(x)}dx+c_1$

- Solve for y

$$y = \frac{\int \frac{\mu(x) f_0(x)}{g(x)}dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = e^{\int -\frac{f_1(x)}{g(x)}dx}$

$$y = \frac{\int \frac{f_0(x) e^{\int -\frac{f_1(x)}{g(x)}dx}}{g(x)}dx + c_1}{e^{\int -\frac{f_1(x)}{g(x)}dx}}$$

- Simplify

$$y = e^{\int \frac{f_1(x)}{g(x)}dx} \left(\int \frac{f_0(x) e^{-\left(\int \frac{f_1(x)}{g(x)}dx\right)}}{g(x)}dx + c_1 \right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(g(x)*diff(y(x),x)=f__1(x)*y(x)+f__0(x),y(x), singsol=all)
```

$$y(x) = \left(\int \frac{f_0(x)\, e^{-\left(\int \frac{f_1(x)}{g(x)}dx\right)}}{g(x)}dx + c_1 \right) e^{\int \frac{f_1(x)}{g(x)}dx}$$

✓ Solution by Mathematica

Time used: 0.135 (sec). Leaf size: 64

```
DSolve[g[x]*y'[x]==f1[x]*y[x]+f0[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \exp\left(\int_1^x \frac{\text{f1}(K[1])}{g(K[1])}dK[1]\right)\left(\int_1^x \frac{\exp\left(-\int_1^{K[2]}\frac{\text{f1}(K[1])}{g(K[1])}dK[1]\right)\text{f0}(K[2])}{g(K[2])}dK[2] + c_1\right)$$

1.5 problem 1.1.5

Internal problem ID [10329]
Internal file name [`OUTPUT/9276_Monday_June_06_2022_01_45_20_PM_6215492/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, First-Order differential equations
Problem number: 1.1.5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{g(x)\,y' - f_1(x)\,y - f_n(x)\,y^n = 0}$$

1.5.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{f_1(x)\,y + f_n(x)\,y^n}{g\,(x)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 9: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^n \mathrm{e}^{(n-1)\left(\int -\frac{f_1(x)}{g(x)} dx\right)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{y^n e^{(n-1)\left(\int -\frac{f_1(x)}{g(x)} dx\right)}} dy
\end{aligned}$$

1.5.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= \frac{f_1(x)\, y + f_n(x)\, y^n}{g\,(x)}
\end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{f_1(x)}{g\,(x)} y + \frac{f_n(x)}{g\,(x)} y^n \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x) y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= \frac{f_1(x)}{g\,(x)} \\
f_1(x) &= \frac{f_n(x)}{g\,(x)} \\
n &= n
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^n$ gives

$$y'y^{-n} = \frac{f_1(x)\, y^{1-n}}{g(x)} + \frac{f_n(x)}{g(x)} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^{1-n} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = (1-n)\, y^{-n} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{1-n} &= \frac{f_1(x)\, w(x)}{g(x)} + \frac{f_n(x)}{g(x)} \\ w' &= \frac{(1-n)\, f_1(x)\, w}{g(x)} + \frac{(1-n)\, f_n(x)}{g(x)} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{(n-1)\, f_1(x)}{g(x)} \\ q(x) &= -\frac{(n-1)\, f_n(x)}{g(x)} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{(n-1)\, f_1(x)\, w(x)}{g(x)} = -\frac{(n-1)\, f_n(x)}{g(x)}$$

The integrating factor μ is

$$\mu = e^{\int \frac{(n-1) f_1(x)}{g(x)} dx}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{(n-1)\, f_n(x)}{g(x)}\right) \\ \frac{d}{dx}\left(e^{\int \frac{(n-1) f_1(x)}{g(x)} dx} w\right) &= \left(e^{\int \frac{(n-1) f_1(x)}{g(x)} dx}\right)\left(-\frac{(n-1)\, f_n(x)}{g(x)}\right) \\ d\left(e^{\int \frac{(n-1) f_1(x)}{g(x)} dx} w\right) &= \left(-\frac{(n-1)\, f_n(x)\, e^{(n-1)\left(\int \frac{f_1(x)}{g(x)} dx\right)}}{g(x)}\right) dx \end{aligned}$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\int \frac{(n-1) f_1(x)}{g(x)} dx} w &= \int -\frac{(n-1)\, f_n(x)\, \mathrm{e}^{(n-1)\left(\int \frac{f_1(x)}{g(x)} dx\right)}}{g\,(x)}\, \mathrm{d}x \\
\mathrm{e}^{\int \frac{(n-1) f_1(x)}{g(x)} dx} w &= \int -\frac{(n-1)\, f_n(x)\, \mathrm{e}^{(n-1)\left(\int \frac{f_1(x)}{g(x)} dx\right)}}{g\,(x)} dx + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int \frac{(n-1) f_1(x)}{g(x)} dx}$ results in

$$
w(x) = \mathrm{e}^{-(n-1)\left(\int \frac{f_1(x)}{g(x)} dx\right)} \left(\int -\frac{(n-1)\, f_n(x)\, \mathrm{e}^{(n-1)\left(\int \frac{f_1(x)}{g(x)} dx\right)}}{g\,(x)} dx\right) + c_1 \mathrm{e}^{-(n-1)\left(\int \frac{f_1(x)}{g(x)} dx\right)}
$$

which simplifies to

$$
w(x) = -\mathrm{e}^{-(n-1)\left(\int \frac{f_1(x)}{g(x)} dx\right)} \left((n-1)\left(\int \frac{f_n(x)\, \mathrm{e}^{(n-1)\left(\int \frac{f_1(x)}{g(x)} dx\right)}}{g\,(x)} dx\right) - c_1\right)
$$

Replacing w in the above by y^{1-n} using equation (5) gives the final solution.

$$
y^{1-n} = -\mathrm{e}^{-(n-1)\left(\int \frac{f_1(x)}{g(x)} dx\right)} \left((n-1)\left(\int \frac{f_n(x)\, \mathrm{e}^{(n-1)\left(\int \frac{f_1(x)}{g(x)} dx\right)}}{g\,(x)} dx\right) - c_1\right)
$$

Summary

The solution(s) found are the following

$$
y^{1-n} = -\mathrm{e}^{-(n-1)\left(\int \frac{f_1(x)}{g(x)} dx\right)} \left((n-1)\left(\int \frac{f_n(x)\, \mathrm{e}^{(n-1)\left(\int \frac{f_1(x)}{g(x)} dx\right)}}{g\,(x)} dx\right) - c_1\right) \tag{1}
$$

Verification of solutions

$$
y^{1-n} = -\mathrm{e}^{-(n-1)\left(\int \frac{f_1(x)}{g(x)} dx\right)} \left((n-1)\left(\int \frac{f_n(x)\, \mathrm{e}^{(n-1)\left(\int \frac{f_1(x)}{g(x)} dx\right)}}{g\,(x)} dx\right) - c_1\right)
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple
Time used: 0.078 (sec). Leaf size: 74

```
dsolve(g(x)*diff(y(x),x)=f__1(x)*y(x)+f__n(x)*y(x)^n,y(x), singsol=all)
```

$$y(x) = e^{\int \frac{f_1(x)}{g(x)}dx} \left(-n\left(\int \frac{f_n(x)\, e^{(n-1)\left(\int \frac{f_1(x)}{g(x)}dx\right)}}{g(x)}dx\right) + c_1 + \int \frac{f_n(x)\, e^{(n-1)\left(\int \frac{f_1(x)}{g(x)}dx\right)}}{g(x)}dx\right)^{-\frac{1}{n-1}}$$

✓ Solution by Mathematica
Time used: 14.019 (sec). Leaf size: 84

```
DSolve[g[x]*y'[x]==f1[x]*y[x]+fn[x]*y[x]^n,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\exp\left(-\left((n-1)\int_1^x \frac{\mathrm{f1}(K[1])}{g(K[1])}dK[1]\right)\right)\left(-(n-1)\int_1^x \frac{\exp\left((n-1)\int_1^{K[2]} \frac{\mathrm{f1}(K[1])}{g(K[1])}dK[1]\right)\mathrm{fn}(K[2])}{g(K[2])}dK[2] + c_1\right)\right)^{\frac{1}{1-n}}$$

1.6 problem 1.1.6

Internal problem ID [10330]
Internal file name [`OUTPUT/9277_Monday_June_06_2022_01_45_23_PM_73837130/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, First-Order differential equations
Problem number: 1.1.6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y' - f\left(\frac{y}{x}\right) = 0}$$

1.6.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - f(u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{-u + f(u)}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -u + f(u)$. Integrating both sides gives

$$\frac{1}{-u + f(u)} \, du = \frac{1}{x} \, dx$$

$$\int \frac{1}{-u + f(u)} \, du = \int \frac{1}{x} \, dx$$

$$\int^{u} \frac{1}{-\_a + f(\_a)} d\_a = c_2 + \ln(x)$$

Which results in

$$\int^{u} \frac{1}{-\_a + f(\_a)} d\_a = c_2 + \ln(x)$$

The solution is

$$\int^{u(x)} \frac{1}{-\_a + f(\_a)} d\_a - c_2 - \ln(x) = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\int^{\frac{y}{x}} \frac{1}{-\_a + f(\_a)} d\_a - c_2 - \ln(x) = 0$$

$$\int^{\frac{y}{x}} \frac{1}{-\_a + f(\_a)} d\_a - c_2 - \ln(x) = 0$$

Summary

The solution(s) found are the following

$$\int^{\frac{y}{x}} \frac{1}{-\_a + f(\_a)} d\_a - c_2 - \ln(x) = 0 \tag{1}$$

Verification of solutions

$$\int^{\frac{y}{x}} \frac{1}{-\_a + f(\_a)} d\_a - c_2 - \ln(x) = 0$$

Verified OK.

1.6.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= f\left(\frac{y}{x}\right)\\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + f\left(\frac{y}{x}\right)(b_3 - a_2) - f\left(\frac{y}{x}\right)^2 a_3 + \frac{D(f)\left(\frac{y}{x}\right) y(x a_2 + y a_3 + a_1)}{x^2}\\
&- \frac{D(f)\left(\frac{y}{x}\right)(x b_2 + y b_3 + b_1)}{x} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{f\left(\frac{y}{x}\right)^2 a_3 x^2 + D(f)\left(\frac{y}{x}\right) x^2 b_2 - D(f)\left(\frac{y}{x}\right) x y a_2 + D(f)\left(\frac{y}{x}\right) x y b_3 - D(f)\left(\frac{y}{x}\right) y^2 a_3 + x^2 f\left(\frac{y}{x}\right) a_2 - x^2 f\left(\frac{y}{x}\right) b \cdots}{x^2}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-f\left(\frac{y}{x}\right)^2 a_3 x^2 - D(f)\left(\frac{y}{x}\right) x^2 b_2 + D(f)\left(\frac{y}{x}\right) x y a_2\\
&- D(f)\left(\frac{y}{x}\right) x y b_3 + D(f)\left(\frac{y}{x}\right) y^2 a_3 - x^2 f\left(\frac{y}{x}\right) a_2\\
&+ x^2 f\left(\frac{y}{x}\right) b_3 - D(f)\left(\frac{y}{x}\right) x b_1 + D(f)\left(\frac{y}{x}\right) y a_1 + b_2 x^2 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, f\left(\frac{y}{x}\right), D(f)\left(\frac{y}{x}\right)\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, f\left(\frac{y}{x}\right) = v_3, D(f)\left(\frac{y}{x}\right) = v_4\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-v_3^2 a_3 v_1^2 - v_1^2 v_3 a_2 + v_4 v_1 v_2 a_2 + v_4 v_2^2 a_3 - v_4 v_1^2 b_2 \\ &+ v_1^2 v_3 b_3 - v_4 v_1 v_2 b_3 + v_4 v_2 a_1 - v_4 v_1 b_1 + b_2 v_1^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned} &-v_3^2 a_3 v_1^2 + (b_3 - a_2) v_1^2 v_3 - v_4 v_1^2 b_2 + b_2 v_1^2 \\ &+ (-b_3 + a_2) v_1 v_2 v_4 - v_4 v_1 b_1 + v_4 v_2^2 a_3 + v_4 v_2 a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ a_3 &= 0 \\ b_2 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -b_2 &= 0 \\ -b_3 + a_2 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{y}{x} \\
&= \frac{y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x}
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{x}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= \ln(x)
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = f\left(\frac{y}{x}\right)
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{y}{x^2} \\
R_y &= \frac{1}{x} \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{x}{xf\left(\frac{y}{x}\right) - y} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{f(R) - R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{1}{f(R) - R} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = \int^{\frac{y}{x}} \frac{1}{f(\_a) - \_a} d\_a + c_1$$

Which simplifies to

$$\ln(x) = \int^{\frac{y}{x}} \frac{1}{f(\_a) - \_a} d\_a + c_1$$

Summary

The solution(s) found are the following

$$\ln(x) = \int^{\frac{y}{x}} \frac{1}{f(\_a) - \_a} d\_a + c_1 \tag{1}$$

Verification of solutions

$$\ln(x) = \int^{\frac{y}{x}} \frac{1}{f(\_a) - \_a} d\_a + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x)=f(y(x)/x),y(x), singsol=all)
```

$$y(x) = \mathrm{RootOf}\left(-\left(\int^{\_Z} -\frac{1}{-f(\_a)+\_a}d\_a\right)+\ln(x)+c_1\right)x$$

✓ Solution by Mathematica

Time used: 0.106 (sec). Leaf size: 33

```
DSolve[y'[x]==f[y[x]/x],y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\int_1^{\frac{y(x)}{x}} \frac{1}{K[1]-f(K[1])}dK[1] = -\log(x)+c_1, y(x)\right]$$

2 Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions

2.1 problem 1

Internal problem ID [10331]
Internal file name [`OUTPUT/9278_Monday_June_06_2022_01_45_23_PM_48991554/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - ay^2 = bx + c}$$

2.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= a\,y^2 + bx + c \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,y^2 + bx + c$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = bx + c$, $f_1(x) = 0$ and $f_2(x) = a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= a^2(bx+c) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$au''(x) + a^2(bx+c)\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{AiryAi}\left(-\frac{(ab)^{\frac{1}{3}}\,(bx+c)}{b}\right) + c_2 \operatorname{AiryBi}\left(-\frac{(ab)^{\frac{1}{3}}\,(bx+c)}{b}\right)$$

The above shows that

$$u'(x) = \left(-\operatorname{AiryBi}\left(1, -\frac{(ab)^{\frac{1}{3}}\,(bx+c)}{b}\right)c_2 - \operatorname{AiryAi}\left(1, -\frac{(ab)^{\frac{1}{3}}\,(bx+c)}{b}\right)c_1\right)(ab)^{\frac{1}{3}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-\operatorname{AiryBi}\left(1, -\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right)c_2 - \operatorname{AiryAi}\left(1, -\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right)c_1\right)(ab)^{\frac{1}{3}}}{a\left(c_1 \operatorname{AiryAi}\left(-\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right) + c_2 \operatorname{AiryBi}\left(-\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\operatorname{AiryAi}\left(1, -\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right)c_3 + \operatorname{AiryBi}\left(1, -\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right)\right)(ab)^{\frac{1}{3}}}{a\left(c_3 \operatorname{AiryAi}\left(-\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right) + \operatorname{AiryBi}\left(-\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right)\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\operatorname{AiryAi}\left(1, -\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right)c_3 + \operatorname{AiryBi}\left(1, -\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right)\right)(ab)^{\frac{1}{3}}}{a\left(c_3 \operatorname{AiryAi}\left(-\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right) + \operatorname{AiryBi}\left(-\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\operatorname{AiryAi}\left(1, -\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right)c_3 + \operatorname{AiryBi}\left(1, -\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right)\right)(ab)^{\frac{1}{3}}}{a\left(c_3 \operatorname{AiryAi}\left(-\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right) + \operatorname{AiryBi}\left(-\frac{(ab)^{\frac{1}{3}}(bx+c)}{b}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   <- Abel AIR successful: ODE belongs to the 0F1 0-parameter (Airy type) class`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 85

```
dsolve(diff(y(x),x)=a*y(x)^2+b*x+c,y(x), singsol=all)
```

$$y(x) = \frac{\left(\frac{b}{\sqrt{a}}\right)^{\frac{1}{3}} \left(\mathrm{AiryAi}\left(1, -\frac{bx+c}{\left(\frac{b}{\sqrt{a}}\right)^{\frac{2}{3}}}\right) c_1 + \mathrm{AiryBi}\left(1, -\frac{bx+c}{\left(\frac{b}{\sqrt{a}}\right)^{\frac{2}{3}}}\right)\right)}{\sqrt{a}\left(c_1 \,\mathrm{AiryAi}\left(-\frac{bx+c}{\left(\frac{b}{\sqrt{a}}\right)^{\frac{2}{3}}}\right) + \mathrm{AiryBi}\left(-\frac{bx+c}{\left(\frac{b}{\sqrt{a}}\right)^{\frac{2}{3}}}\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.325 (sec). Leaf size: 143

```
DSolve[y'[x]==a*y[x]^2+b*x+c,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{b\left(\mathrm{AiryBiPrime}\left(-\frac{a(c+bx)}{(-ab)^{2/3}}\right) + c_1 \,\mathrm{AiryAiPrime}\left(-\frac{a(c+bx)}{(-ab)^{2/3}}\right)\right)}{(-ab)^{2/3}\left(\mathrm{AiryBi}\left(-\frac{a(c+bx)}{(-ab)^{2/3}}\right) + c_1 \,\mathrm{AiryAi}\left(-\frac{a(c+bx)}{(-ab)^{2/3}}\right)\right)}$$

$$y(x) \to \frac{b \,\mathrm{AiryAiPrime}\left(-\frac{a(c+bx)}{(-ab)^{2/3}}\right)}{(-ab)^{2/3}\,\mathrm{AiryAi}\left(-\frac{a(c+bx)}{(-ab)^{2/3}}\right)}$$

2.2 problem 2

Internal problem ID [10332]
Internal file name [`OUTPUT/9279_Monday_June_06_2022_01_45_24_PM_3663866/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 = -a^2x^2 + 3a}$$

2.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -a^2x^2 + y^2 + 3a \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a^2x^2 + y^2 + 3a$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2x^2 + 3a$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= -a^2x^2 + 3a \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-a^2x^2 + 3a\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x\left(\sqrt{\pi}\ \mathrm{erf}\left(x\sqrt{-a}\right)\sqrt{-a}\,c_2 + c_1\right)\mathrm{e}^{-\frac{a\,x^2}{2}} + c_2\mathrm{e}^{\frac{a\,x^2}{2}}$$

The above shows that

$$u'(x) = \left(c_2\sqrt{\pi}\left(x^2(-a)^{\frac{3}{2}} + \sqrt{-a}\right)\mathrm{erf}\left(x\sqrt{-a}\right) - c_1a\,x^2 + c_1\right)\mathrm{e}^{-\frac{a\,x^2}{2}} - c_2xa\,\mathrm{e}^{\frac{a\,x^2}{2}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(c_2\sqrt{\pi}\left(x^2(-a)^{\frac{3}{2}} + \sqrt{-a}\right)\mathrm{erf}\left(x\sqrt{-a}\right) - c_1a\,x^2 + c_1\right)\mathrm{e}^{-\frac{a\,x^2}{2}} - c_2xa\,\mathrm{e}^{\frac{a\,x^2}{2}}}{x\left(\sqrt{\pi}\ \mathrm{erf}\left(x\sqrt{-a}\right)\sqrt{-a}\,c_2 + c_1\right)\mathrm{e}^{-\frac{a\,x^2}{2}} + c_2\mathrm{e}^{\frac{a\,x^2}{2}}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{xa\,\mathrm{e}^{a\,x^2} - \sqrt{\pi}\left(x^2(-a)^{\frac{3}{2}} + \sqrt{-a}\right)\mathrm{erf}\left(x\sqrt{-a}\right) + c_3(a\,x^2 - 1)}{\sqrt{\pi}\,\sqrt{-a}\ \mathrm{erf}\left(x\sqrt{-a}\right)x + \mathrm{e}^{a\,x^2} + c_3x}$$

Summary

The solution(s) found are the following

$$y = \frac{xa\,\mathrm{e}^{a\,x^2} - \sqrt{\pi}\left(x^2(-a)^{\frac{3}{2}} + \sqrt{-a}\right)\mathrm{erf}\left(x\sqrt{-a}\right) + c_3(a\,x^2 - 1)}{\sqrt{\pi}\,\sqrt{-a}\ \mathrm{erf}\left(x\sqrt{-a}\right)x + \mathrm{e}^{a\,x^2} + c_3x} \tag{1}$$

Verification of solutions

$$y = \frac{xa\,\mathrm{e}^{a\,x^2} - \sqrt{\pi}\left(x^2(-a)^{\frac{3}{2}} + \sqrt{-a}\right)\mathrm{erf}\left(x\sqrt{-a}\right) + c_3(a\,x^2 - 1)}{\sqrt{\pi}\,\sqrt{-a}\ \mathrm{erf}\left(x\sqrt{-a}\right)x + \mathrm{e}^{a\,x^2} + c_3 x}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a^2*x^2-3*a)*y(x), y(x)`
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Reducible group (found an exponential solution)
         Group is reducible, not completely reducible
      <- Kovacics algorithm successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 82

```
dsolve(diff(y(x),x)=y(x)^2-a^2*x^2+3*a,y(x), singsol=all)
```

$$y(x) = \frac{e^{a x^2} c_1 a x - c_1 \sqrt{\pi} \left((-a)^{\frac{3}{2}} x^2 + \sqrt{-a} \right) \operatorname{erf}\left(\sqrt{-a}\, x \right) + a x^2 - 1}{\sqrt{\pi} \sqrt{-a} \operatorname{erf}\left(\sqrt{-a}\, x \right) c_1 x + e^{a x^2} c_1 + x}$$

✓ Solution by Mathematica

Time used: 0.79 (sec). Leaf size: 192

```
DSolve[y'[x]==y[x]^2-a^2*x^2+3*a,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{ax \text{ ParabolicCylinderD}\left(-2, i\sqrt{2}\sqrt{a}x\right) + i\sqrt{2}\sqrt{a} \text{ ParabolicCylinderD}\left(-1, i\sqrt{2}\sqrt{a}x\right) - ac_1 x \text{ Parabolic}}{\text{ParabolicCylinderD}\left(-2, i\sqrt{2}\sqrt{a}x\right) + c_1 \text{ ParabolicCyli}}$$

$$y(x) \to \frac{\sqrt{2}\sqrt{a} \text{ ParabolicCylinderD}\left(2, \sqrt{2}\sqrt{a}x\right)}{\text{ParabolicCylinderD}\left(1, \sqrt{2}\sqrt{a}x\right)} - ax$$

2.3 problem 3

Internal problem ID [10333]
Internal file name [`OUTPUT/9280_Monday_June_06_2022_01_45_25_PM_48071204/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = a^2x^2 + bx + c}$$

2.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= a^2x^2 + bx + y^2 + c \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a^2x^2 + bx + y^2 + c$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a^2x^2 + bx + c$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= a^2x^2 + bx + c \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(a^2x^2 + bx + c\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = \mathrm{e}^{-\frac{ix\left(a^2x+b\right)}{2a}} \Bigg(&2x\, a^2 c_2 \,\mathrm{hypergeom}\left(\left[\frac{4ia^2c + 12a^3 - ib^2}{16a^3}\right], \left[\frac{3}{2}\right], \frac{i(2a^2x+b)^2}{4a^3}\right) \\ &+ bc_2 \,\mathrm{hypergeom}\left(\left[\frac{4ia^2c + 12a^3 - ib^2}{16a^3}\right], \left[\frac{3}{2}\right], \frac{i(2a^2x+b)^2}{4a^3}\right) \\ &+ \mathrm{hypergeom}\left(\left[\frac{4ia^2c + 4a^3 - ib^2}{16a^3}\right], \left[\frac{1}{2}\right], \frac{i(2a^2x+b)^2}{4a^3}\right) c_1\Bigg) \end{aligned}$$

The above shows that

$$u'(x) = \frac{2\left(\left(a^2x + \frac{b}{2}\right)^2 \left(ia^3 - \frac{1}{3}a^2c + \frac{1}{12}b^2\right) c_2 \,\mathrm{hypergeom}\left(\left[\frac{4ia^2c+28a^3-ib^2}{16a^3}\right], \left[\frac{5}{2}\right], \frac{i(2a^2x+b)^2}{4a^3}\right) + \frac{c_1\left(-a^2c+\frac{1}{4}b^2+ia^3\right)\left(a^2x\right.}{}\right.}{}$$

Using the above in (1) gives the solution

$$y = -\frac{2\left(\left(a^2x + \frac{b}{2}\right)^2 \left(ia^3 - \frac{1}{3}a^2c + \frac{1}{12}b^2\right) c_2 \,\mathrm{hypergeom}\left(\left[\frac{4ia^2c+28a^3-ib^2}{16a^3}\right], \left[\frac{5}{2}\right], \frac{i(2a^2x+b)^2}{4a^3}\right) + \frac{c_1\left(-a^2c+\frac{1}{4}b^2+ia^3\right)\left(\right.}{}\right.}{a^4\left(2x\, a^2 c_2 \,\mathrm{hypergeom}\left(\left[\frac{4ia^2c+12a^3-ib^2}{16a^3}\right]\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{4\left(-i a^3 + \frac{1}{3} a^2 c - \frac{1}{12} b^2\right) \left(a^2 x + \frac{b}{2}\right)^2 \text{hypergeom}\left(\left[\frac{4 i a^2 c + 28 a^3 - i b^2}{16 a^3}\right], \left[\frac{5}{2}\right], \frac{i \left(2 a^2 x + b\right)^2}{4 a^3}\right) + \left(4 i a^7 x^2 + 4 i a^5 b x - 4\right.}{4 a^4 \left(\left(a^2\right.\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{4\left(-i a^3 + \frac{1}{3} a^2 c - \frac{1}{12} b^2\right) \left(a^2 x + \frac{b}{2}\right)^2 \text{hypergeom}\left(\left[\frac{4 i a^2 c + 28 a^3 - i b^2}{16 a^3}\right], \left[\frac{5}{2}\right], \frac{i \left(2 a^2 x + b\right)^2}{4 a^3}\right) + \left(4 i a^7 x^2 + 4 i a^5 b x - 4\right.}{4 a^4 \left(\left(a^2\right.\right.} \tag{1}$$

Verification of solutions

$$y = \frac{4\left(-i a^3 + \frac{1}{3} a^2 c - \frac{1}{12} b^2\right) \left(a^2 x + \frac{b}{2}\right)^2 \text{hypergeom}\left(\left[\frac{4 i a^2 c + 28 a^3 - i b^2}{16 a^3}\right], \left[\frac{5}{2}\right], \frac{i \left(2 a^2 x + b\right)^2}{4 a^3}\right) + \left(4 i a^7 x^2 + 4 i a^5 b x - 4\right.}{4 a^4 \left(\left(a^2\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-a^2*x^2-b*x-c)*y(x), y(x)`
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: indirect Equivalence to 0F1 under \`\`^ @ Moebius\`\` is r
         <- hypergeometric successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 393

```
dsolve(diff(y(x),x)=y(x)^2+a^2*x^2+b*x+c,y(x), singsol=all)
```

$$y(x) = \frac{-48\left(ia^3 - \frac{1}{3}a^2c + \frac{1}{12}b^2\right)\left(a^2x + \frac{b}{2}\right)^2 c_1 \operatorname{hypergeom}\left(\left[\frac{4ia^2c+28a^3-ib^2}{16a^3}\right], \left[\frac{5}{2}\right], \frac{i(2a^2x+b)^2}{4a^3}\right) + 48c_1a^3\left(ia^4x^2 + ia\cdots\right.}{48\left(\left(a\cdots\right.\right.}$$

✓ Solution by Mathematica

Time used: 1.582 (sec). Leaf size: 664

```
DSolve[y'[x]==y[x]^2+a^2*x^2+b*x+c,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2i\sqrt{2}a^2x \operatorname{ParabolicCylinderD}\left(\frac{1}{8}\left(-\frac{ib^2}{a^3} + \frac{4ic}{a} - 4\right), -\frac{\left(\frac{1}{2}-\frac{i}{2}\right)(2xa^2+b)}{a^{3/2}}\right) + 4(-1)^{3/4}a^{3/2} \operatorname{ParabolicCylinderD}\cdots}{\cdots}$$

$$y(x) \to \frac{(1+i)\sqrt{a} \operatorname{ParabolicCylinderD}\left(\frac{1}{8}\left(\frac{ib^2}{a^3} - \frac{4ic}{a} + 4\right), \frac{\left(\frac{1}{2}+\frac{i}{2}\right)(2xa^2+b)}{a^{3/2}}\right)}{\operatorname{ParabolicCylinderD}\left(\frac{1}{8}\left(\frac{ib^2}{a^3} - \frac{4ic}{a} - 4\right), \frac{\left(\frac{1}{2}+\frac{i}{2}\right)(2xa^2+b)}{a^{3/2}}\right)} - \frac{i(2a^2x+b)}{2a}$$

$$y(x) \to \frac{(1+i)\sqrt{a} \operatorname{ParabolicCylinderD}\left(\frac{1}{8}\left(\frac{ib^2}{a^3} - \frac{4ic}{a} + 4\right), \frac{\left(\frac{1}{2}+\frac{i}{2}\right)(2xa^2+b)}{a^{3/2}}\right)}{\operatorname{ParabolicCylinderD}\left(\frac{1}{8}\left(\frac{ib^2}{a^3} - \frac{4ic}{a} - 4\right), \frac{\left(\frac{1}{2}+\frac{i}{2}\right)(2xa^2+b)}{a^{3/2}}\right)} - \frac{i(2a^2x+b)}{2a}$$

2.4 problem 4

Internal problem ID [10334]
Internal file name [`OUTPUT/9281_Monday_June_06_2022_01_45_26_PM_40145015/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[[_Riccati, _special]]`

$$\boxed{y' - a y^2 = b\,x^n}$$

2.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,y^2 + b\,x^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,y^2 + b\,x^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\,x^n$, $f_1(x) = 0$ and $f_2(x) = a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= b\,x^n a^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$a u''(x) + b\,x^n a^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\text{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\,x^{1+\frac{n}{2}}}{2+n}\right) c_1 + \text{BesselY}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\,x^{1+\frac{n}{2}}}{2+n}\right) c_2\right)\sqrt{x}$$

The above shows that

$$u'(x) = \frac{-\text{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{ab}\,x^{1+\frac{n}{2}}}{2+n}\right)\sqrt{ab}\,x^{1+\frac{n}{2}} c_1 - \text{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{ab}\,x^{1+\frac{n}{2}}}{2+n}\right)\sqrt{ab}\,x^{1+\frac{n}{2}} c_2 + \text{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\,x^{1+}}{2+n}\right.}{\sqrt{x}}$$

Using the above in (1) gives the solution

$$y = -\frac{-\text{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{ab}\,x^{1+\frac{n}{2}}}{2+n}\right)\sqrt{ab}\,x^{1+\frac{n}{2}} c_1 - \text{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{ab}\,x^{1+\frac{n}{2}}}{2+n}\right)\sqrt{ab}\,x^{1+\frac{n}{2}} c_2 + \text{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\,x}{2+}\right.}{xa\left(\text{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\,x^{1+\frac{n}{2}}}{2+n}\right) c_1 + \text{BesselY}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\,x^{1+\frac{n}{2}}}{2+n}\right) c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\text{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{ab}\,x^{1+\frac{n}{2}}}{2+n}\right) c_3 + \text{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{ab}\,x^{1+\frac{n}{2}}}{2+n}\right)\right)\sqrt{ab}\,x^{1+\frac{n}{2}} - \text{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\,x^{1+\frac{n}{2}}}{2+n}\right) c_3 - \text{Be}}{xa\left(\text{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\,x^{1+\frac{n}{2}}}{2+n}\right) c_3 + \text{BesselY}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\,x^{1+\frac{n}{2}}}{2+n}\right)\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\mathrm{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{ab}\, x^{1+\frac{n}{2}}}{2+n}\right) c_3 + \mathrm{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{ab}\, x^{1+\frac{n}{2}}}{2+n}\right)\right) \sqrt{ab}\, x^{1+\frac{n}{2}} - \mathrm{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\, x^{1+\frac{n}{2}}}{2+n}\right) c_3 - \mathrm{Be}}{xa \left(\mathrm{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\, x^{1+\frac{n}{2}}}{2+n}\right) c_3 + \mathrm{BesselY}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\, x^{1+\frac{n}{2}}}{2+n}\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\mathrm{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{ab}\, x^{1+\frac{n}{2}}}{2+n}\right) c_3 + \mathrm{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{ab}\, x^{1+\frac{n}{2}}}{2+n}\right)\right) \sqrt{ab}\, x^{1+\frac{n}{2}} - \mathrm{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\, x^{1+\frac{n}{2}}}{2+n}\right) c_3 - \mathrm{Be}}{xa \left(\mathrm{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\, x^{1+\frac{n}{2}}}{2+n}\right) c_3 + \mathrm{BesselY}\left(\frac{1}{2+n}, \frac{2\sqrt{ab}\, x^{1+\frac{n}{2}}}{2+n}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 207

```
dsolve(diff(y(x),x)=a*y(x)^2+b*x^n,y(x), singsol=all)
```

$$y(x) = \frac{\text{BesselJ}\left(\frac{3+n}{n+2}, \frac{2\sqrt{ab}\, x^{\frac{n}{2}+1}}{n+2}\right)\sqrt{ab}\, x^{\frac{n}{2}+1} c_1 + \text{BesselY}\left(\frac{3+n}{n+2}, \frac{2\sqrt{ab}\, x^{\frac{n}{2}+1}}{n+2}\right)\sqrt{ab}\, x^{\frac{n}{2}+1} - c_1 \text{BesselJ}\left(\frac{1}{n+2}, \frac{2\sqrt{ab}\, x^{\frac{n}{2}+1}}{n+2}\right)}{xa\left(c_1 \text{BesselJ}\left(\frac{1}{n+2}, \frac{2\sqrt{ab}\, x^{\frac{n}{2}+1}}{n+2}\right) + \text{BesselY}\left(\frac{1}{n+2}, \frac{2\sqrt{ab}\, x^{\frac{n}{2}+1}}{n+2}\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.696 (sec). Leaf size: 605

```
DSolve[y'[x]==a*y[x]^2+b*x^n,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{a}\sqrt{b}x^{\frac{n}{2}+1}\,\text{Gamma}\left(1+\frac{1}{n+2}\right)\text{BesselJ}\left(\frac{1}{n+2}-1, \frac{2\sqrt{a}\sqrt{b}x^{\frac{n}{2}+1}}{n+2}\right) - \sqrt{a}\sqrt{b}x^{\frac{n}{2}+1}\,\text{Gamma}\left(1+\frac{1}{n+2}\right)\text{BesselJ}}{}$$

$$y(x) \to \frac{\frac{\sqrt{a}\sqrt{b}x^{n/2}\left(\text{BesselJ}\left(\frac{n+1}{n+2}, \frac{2\sqrt{a}\sqrt{b}x^{\frac{n}{2}+1}}{n+2}\right) - \text{BesselJ}\left(-\frac{n+3}{n+2}, \frac{2\sqrt{a}\sqrt{b}x^{\frac{n}{2}+1}}{n+2}\right)\right)}{\text{BesselJ}\left(-\frac{1}{n+2}, \frac{2\sqrt{a}\sqrt{b}x^{\frac{n}{2}+1}}{n+2}\right)} - \frac{1}{x}}{2a}$$

2.5 problem 5

Internal problem ID [10335]
Internal file name [`OUTPUT/9282_Monday_June_06_2022_01_45_27_PM_95068594/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = an\,x^{n-1} - a^2x^{2n}}$$

2.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + an\,x^{n-1} - a^2x^{2n} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \frac{an\,x^n}{x} - a^2x^{2n}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = an\,x^{n-1} - a^2x^{2n}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= an\, x^{n-1} - a^2 x^{2n} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(an\, x^{n-1} - a^2 x^{2n}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = -\frac{c_2 x^{-1-\frac{3n}{2}} (2+n)^2 \operatorname{WhittakerM}\left(\frac{2+n}{2+2n}, \frac{3+2n}{2+2n}, -\frac{2x^{n+1}a}{n+1}\right)}{2} + c_2\left(\left(-\frac{n}{2} - 1\right) x^{-1-\frac{3n}{2}} + a\, x^{-\frac{n}{2}}\right) (n+1) \operatorname{WhittakerM}\left(-\frac{n}{2+2n}, \frac{3+2n}{2+2n}, -\frac{2x^{n+1}a}{n+1}\right) + c_1 \mathrm{e}^{-\frac{x^{n+1}a}{n+1}}$$

The above shows that

$$u'(x) = -\frac{\left(-\frac{3c_2\left(\left(\frac{1}{3}n^2+n+\frac{2}{3}\right)x^{-\frac{n}{2}}+ax\,x^{\frac{n}{2}}\left(n+\frac{4}{3}\right)\right)(2+n)\operatorname{WhittakerM}\left(\frac{2+n}{2+2n},\frac{3+2n}{2+2n},-\frac{2x^n xa}{n+1}\right)}{2} + c_2\left(\left(-\frac{1}{2}n^2 - \frac{3}{2}n - 1\right) x^{-\frac{n}{2}} + ax\left(\left(\cdots\right.\right.\right.\right.}{\cdots}$$

Using the above in (1) gives the solution

$$y = \frac{\left(-\frac{3c_2\left(\left(\frac{1}{3}n^2+n+\frac{2}{3}\right)x^{-\frac{n}{2}}+ax\,x^{\frac{n}{2}}\left(n+\frac{4}{3}\right)\right)(2+n)\operatorname{WhittakerM}\left(\frac{2+n}{2+2n},\frac{3+2n}{2+2n},-\frac{2x^n xa}{n+1}\right)}{2} + c_2\left(\left(-\frac{1}{2}n^2 - \frac{3}{2}n - 1\right) x^{-\frac{n}{2}} + ax\left(\left(-\frac{n}{2}\cdots\right.\right.\right.\right.}{x^2\left(-\frac{c_2 x^{-1-\frac{3n}{2}}(2+n)^2\operatorname{WhittakerM}\left(\frac{2+n}{2+2n},\frac{3+2n}{2+2n},-\frac{2x^{n+1}a}{n+1}\right)}{2} + c_2\left(\left(\cdots\right.\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-\frac{3\,\mathrm{e}^{\frac{x^{n} x a}{n+1}} \left(\left(\frac{1}{3}n^{2}+n+\frac{2}{3}\right)x^{-\frac{n}{2}}+a x\, x^{\frac{n}{2}} \left(n+\frac{4}{3}\right)\right)(2+n)\, \mathrm{WhittakerM}\left(\frac{2+n}{2+2n},\frac{3+2n}{2+2n},-\frac{2x^{n} x a}{n+1}\right)}{2}+\mathrm{e}^{\frac{x^{n} x a}{n+1}} \left(\left(-\frac{1}{2}n^{2}-\frac{3}{2}n-1\right) x^{-\frac{n}{2}}+a \cdots\right.\right.}{x \left(-\frac{x^{-\frac{3n}{2}}\, \mathrm{e}^{\frac{x^{n} x a}{n+1}} (2+n)^{2}\, \mathrm{WhittakerM}\left(\frac{2+n}{2+2n},\frac{3+2n}{2+2n},-\frac{2x^{n} x a}{n+1}\right)}{2}+\mathrm{e} \cdots\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-\frac{3\,\mathrm{e}^{\frac{x^{n} x a}{n+1}} \left(\left(\frac{1}{3}n^{2}+n+\frac{2}{3}\right)x^{-\frac{n}{2}}+a x\, x^{\frac{n}{2}} \left(n+\frac{4}{3}\right)\right)(2+n)\, \mathrm{WhittakerM}\left(\frac{2+n}{2+2n},\frac{3+2n}{2+2n},-\frac{2x^{n} x a}{n+1}\right)}{2}+\mathrm{e}^{\frac{x^{n} x a}{n+1}} \left(\left(-\frac{1}{2}n^{2}-\frac{3}{2}n-1\right) x^{-\frac{n}{2}}+a \cdots\right.\right.}{x \left(-\frac{x^{-\frac{3n}{2}}\, \mathrm{e}^{\frac{x^{n} x a}{n+1}} (2+n)^{2}\, \mathrm{WhittakerM}\left(\frac{2+n}{2+2n},\frac{3+2n}{2+2n},-\frac{2x^{n} x a}{n+1}\right)}{2}+\mathrm{e} \cdots\right.} \tag{1}$$

Verification of solutions

$$y = \frac{\left(-\frac{3\,\mathrm{e}^{\frac{x^{n} x a}{n+1}} \left(\left(\frac{1}{3}n^{2}+n+\frac{2}{3}\right)x^{-\frac{n}{2}}+a x\, x^{\frac{n}{2}} \left(n+\frac{4}{3}\right)\right)(2+n)\, \mathrm{WhittakerM}\left(\frac{2+n}{2+2n},\frac{3+2n}{2+2n},-\frac{2x^{n} x a}{n+1}\right)}{2}+\mathrm{e}^{\frac{x^{n} x a}{n+1}} \left(\left(-\frac{1}{2}n^{2}-\frac{3}{2}n-1\right) x^{-\frac{n}{2}}+a \cdots\right.\right.}{x \left(-\frac{x^{-\frac{3n}{2}}\, \mathrm{e}^{\frac{x^{n} x a}{n+1}} (2+n)^{2}\, \mathrm{WhittakerM}\left(\frac{2+n}{2+2n},\frac{3+2n}{2+2n},-\frac{2x^{n} x a}{n+1}\right)}{2}+\mathrm{e} \cdots\right.}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-a*n*x^(n-1)+a^2*x^(2*n))*y(x
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
         <- Kovacics algorithm successful
      <- Equivalence, under non-integer power transformations successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 388

```
dsolve(diff(y(x),x)=y(x)^2+a*n*x^(n-1)-a^2*x^(2*n),y(x), singsol=all)
```

$$y(x) = \frac{-3(n+2)\,c_1\left(\left(\frac{1}{3}n^2+n+\frac{2}{3}\right)x^{-\frac{3n}{2}}+ax\,x^{-\frac{n}{2}}\left(n+\frac{4}{3}\right)\right)\mathrm{e}^{\frac{ax\,x^n}{n+1}}\ \mathrm{WhittakerM}\left(\frac{n+2}{2n+2},\frac{2n+3}{2n+2},-\frac{2ax\,x^n}{n+1}\right)+2c_1\mathrm{e}^{\frac{ax\,x}{n+}}}{2\left(-\frac{\mathrm{e}^{\frac{ax\,x^n}{n+1}}x^{-\frac{3n}{2}}c_1(n+2)^2\,\mathrm{WhittakerM}\left(\frac{n+2}{2n+2},\frac{2n+3}{2n+2},\right.}{2}\right.}$$

✓ Solution by Mathematica

Time used: 1.61 (sec). Leaf size: 227

```
DSolve[y'[x]==y[x]^2+a*n*x^(n-1)-a^2*x^(2*n),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2^{\frac{1}{n+1}}(n+1)\left(-\frac{ax^{n+1}}{n+1}\right)^{\frac{1}{n+1}}\left(ax^n-c_1e^{\frac{2ax^{n+1}}{n+1}}\right)-ac_1x^{n+1}\Gamma\left(\frac{1}{n+1},-\frac{2ax^{n+1}}{n+1}\right)}{2^{\frac{1}{n+1}}(n+1)\left(-\frac{ax^{n+1}}{n+1}\right)^{\frac{1}{n+1}}-c_1x\Gamma\left(\frac{1}{n+1},-\frac{2ax^{n+1}}{n+1}\right)}$$

$$y(x) \to \frac{2^{\frac{1}{n+1}}(n+1)e^{\frac{2ax^{n+1}}{n+1}}\left(-\frac{ax^{n+1}}{n+1}\right)^{\frac{1}{n+1}}}{x\Gamma\left(\frac{1}{n+1},-\frac{2ax^{n+1}}{n+1}\right)}+ax^n$$

2.6 problem 6

Internal problem ID [10336]
Internal file name [`OUTPUT/9283_Monday_June_06_2022_01_46_32_PM_97394906/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - ay^2 = b\,x^{2n} + c\,x^{n-1}}$$

2.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,y^2 + b\,x^{2n} + c\,x^{n-1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,y^2 + b\,x^{2n} + \frac{c\,x^n}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\,x^{2n} + c\,x^{n-1}$, $f_1(x) = 0$ and $f_2(x) = a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= a^2 \left(b\, x^{2n} + c\, x^{n-1}\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$a u''(x) + a^2 \left(b\, x^{2n} + c\, x^{n-1}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{-\frac{n}{2}} \left(c_1 \,\text{WhittakerM}\left(-\frac{i\sqrt{a}\, c}{\sqrt{b}\,(2+2n)}, \frac{1}{2+2n}, \frac{2i\sqrt{a}\,\sqrt{b}\, x^n x}{n+1}\right) + c_2 \,\text{WhittakerW}\left(-\frac{i\sqrt{a}\, c}{\sqrt{b}\,(2+2n)}, \frac{1}{2+2n}, \frac{2i\sqrt{a}\,\sqrt{b}\, x^n x}{n+1}\right)\right)$$

The above shows that

$$u'(x) = \frac{\left(-\left(i\sqrt{a}\,\sqrt{b}\, c - b(2+n)\right) c_1 \,\text{WhittakerM}\left(-\frac{(-2n-2)\sqrt{b}+i\sqrt{a}\, c}{\sqrt{b}\,(2+2n)}, \frac{1}{2+2n}, \frac{2i\sqrt{a}\,\sqrt{b}\, x^n x}{n+1}\right) - 2c_2 b(n+1)\,\text{Whittaker}\right.}{}$$

Using the above in (1) gives the solution

$$y = -\frac{-\left(i\sqrt{a}\,\sqrt{b}\, c - b(2+n)\right) c_1 \,\text{WhittakerM}\left(-\frac{(-2n-2)\sqrt{b}+i\sqrt{a}\, c}{\sqrt{b}\,(2+2n)}, \frac{1}{2+2n}, \frac{2i\sqrt{a}\,\sqrt{b}\, x^n x}{n+1}\right) - 2c_2 b(n+1)\,\text{Whittake}}{2bxa\left(c_1 \,\text{Whitt}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(i\sqrt{a}\,\sqrt{b}\, c - b(2+n)\right) c_3 \,\text{WhittakerM}\left(-\frac{(-2n-2)\sqrt{b}+i\sqrt{a}\, c}{\sqrt{b}\,(2+2n)}, \frac{1}{2+2n}, \frac{2i\sqrt{a}\,\sqrt{b}\, x^n x}{n+1}\right) + 2b(n+1)\,\text{WhittakerW}\left(-\right.}{2bxa\left(c_3 \,\text{WhittakerI}\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(i\sqrt{a}\,\sqrt{b}\,c - b(2+n)\right) c_3 \,\mathrm{WhittakerM}\left(-\frac{(-2n-2)\sqrt{b}+i\sqrt{a}\,c}{\sqrt{b}\,(2+2n)}, \frac{1}{2+2n}, \frac{2i\sqrt{a}\,\sqrt{b}\,x^n x}{n+1}\right) + 2b(n+1)\,\mathrm{WhittakerW}\left(-\cdots\right.}{2bxa\left(c_3 \,\mathrm{WhittakerM}\cdots\right.} \tag{1}$$

Verification of solutions

$$y = \frac{\left(i\sqrt{a}\,\sqrt{b}\,c - b(2+n)\right) c_3 \,\mathrm{WhittakerM}\left(-\frac{(-2n-2)\sqrt{b}+i\sqrt{a}\,c}{\sqrt{b}\,(2+2n)}, \frac{1}{2+2n}, \frac{2i\sqrt{a}\,\sqrt{b}\,x^n x}{n+1}\right) + 2b(n+1)\,\mathrm{WhittakerW}\left(-\cdots\right.}{2bxa\left(c_3 \,\mathrm{WhittakerM}\cdots\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -a*(b*x^(2*n)+c*x^(n-1))*y(x),
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
         <- Whittaker successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 357

```
dsolve(diff(y(x),x)=a*y(x)^2+b*x^(2*n)+c*x^(n-1),y(x), singsol=all)
```

$$y(x) = -\frac{\left(\left(\frac{n}{2}+1\right)\sqrt{b}-\frac{ic\sqrt{a}}{2}\right)\mathrm{WhittakerM}\left(-\frac{(-2n-2)\sqrt{b}+ic\sqrt{a}}{\sqrt{b}\,(2n+2)},\frac{1}{2n+2},\frac{2i\sqrt{a}\,\sqrt{b}\,x\,x^{n}}{n+1}\right)-\sqrt{b}\,c_1(n+1)\,\mathrm{WhittakerW}\left(\right.}{\sqrt{b}\left(\mathrm{WhittakerW}\left(-\right.\right.}$$

✓ Solution by Mathematica

Time used: 1.818 (sec). Leaf size: 982

```
DSolve[y'[x]==a*y[x]^2+b*x^(2*n)+c*x^(n-1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^n\left(\sqrt{b}c_1(n+1)\sqrt{-(n+1)^2}\,\mathrm{HypergeometricU}\left(\frac{1}{2}\left(\frac{\sqrt{a}c}{\sqrt{b}\sqrt{-(n+1)^2}}+\frac{n}{n+1}\right),\frac{n}{n+1},\frac{2\sqrt{a}\sqrt{b}x^{n+1}}{\sqrt{-(n+1)^2}}\right)+c_1\left(\sqrt{a}c(n\right.\right.}{\sqrt{a}(n+1)^2}$$

$$y(x) \to \frac{x^n\left(-\frac{\left(\sqrt{a}c(n+1)+\sqrt{b}\sqrt{-(n+1)^2}n\right)\mathrm{HypergeometricU}\left(\frac{1}{2}\left(\frac{\sqrt{a}c}{\sqrt{b}\sqrt{-(n+1)^2}}+\frac{n}{n+1}+2\right),\frac{n}{n+1}+1,\frac{2\sqrt{a}\sqrt{b}x^{n+1}}{\sqrt{-(n+1)^2}}\right)}{\mathrm{HypergeometricU}\left(\frac{1}{2}\left(\frac{\sqrt{a}c}{\sqrt{b}\sqrt{-(n+1)^2}}+\frac{n}{n+1}\right),\frac{n}{n+1},\frac{2\sqrt{a}\sqrt{b}x^{n+1}}{\sqrt{-(n+1)^2}}\right)}-\sqrt{b}\sqrt{-(n+1)^2}(n\right.}{\sqrt{a}(n+1)^2}$$

$$y(x) \to \frac{x^n\left(-\frac{\left(\sqrt{a}c(n+1)+\sqrt{b}\sqrt{-(n+1)^2}n\right)\mathrm{HypergeometricU}\left(\frac{1}{2}\left(\frac{\sqrt{a}c}{\sqrt{b}\sqrt{-(n+1)^2}}+\frac{n}{n+1}+2\right),\frac{n}{n+1}+1,\frac{2\sqrt{a}\sqrt{b}x^{n+1}}{\sqrt{-(n+1)^2}}\right)}{\mathrm{HypergeometricU}\left(\frac{1}{2}\left(\frac{\sqrt{a}c}{\sqrt{b}\sqrt{-(n+1)^2}}+\frac{n}{n+1}\right),\frac{n}{n+1},\frac{2\sqrt{a}\sqrt{b}x^{n+1}}{\sqrt{-(n+1)^2}}\right)}-\sqrt{b}\sqrt{-(n+1)^2}(n\right.}{\sqrt{a}(n+1)^2}$$

2.7 problem 7

Internal problem ID [10337]
Internal file name [OUTPUT/9284_Monday_June_06_2022_01_46_37_PM_54916221/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _Riccati]
```

$$\boxed{y' - a\,x^n y^2 = b\,x^{-n-2}}$$

2.7.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= a\,x^n y^2 + b\,x^{-n-2} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \left(a\,x^n y^2 + b\,x^{-n-2}\right)(b_3 - a_2) - \left(a\,x^n y^2 + b\,x^{-n-2}\right)^2 a_3 - \left(\frac{a\,x^n n\,y^2}{x} + \frac{b\,x^{-n-2}(-n-2)}{x}\right)(xa_2 + ya_3 + a_1) - 2a\,x^n y(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^{2n}a^2x\,y^4a_3 + 2x^nx^{-n-2}abx\,y^2a_3 + x^nanx\,y^2a_2 + x^nan\,y^3a_3 + x^nan\,y^2a_1 + 2x^na\,x^2yb_2 + x^nax\,y^2a_2 + x^n}{}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}&-x^{2n}a^2x\,y^4a_3 - 2x^nx^{-n-2}abx\,y^2a_3 - x^nanx\,y^2a_2 - x^nan\,y^3a_3\\ &- x^nan\,y^2a_1 - 2x^na\,x^2yb_2 - x^nax\,y^2a_2 - x^nax\,y^2b_3 - x^{-4-2n}b^2xa_3\\ &- 2x^naxyb_1 + x^{-n-2}bnxa_2 + x^{-n-2}bnya_3 + x^{-n-2}bna_1\\ &+ x^{-n-2}bxa_2 + x^{-n-2}bxb_3 + 2x^{-n-2}bya_3 + 2x^{-n-2}ba_1 + xb_2 = 0\end{aligned} \tag{6E}$$

Simplifying the above gives

$$-\frac{(x^{4n}a^2x^4y^4a_3 + x^{3n}an\,x^4y^2a_2 + x^{3n}an\,y^3a_3x^3 + x^{3n}an\,y^2a_1x^3 + 2x^{3n}a\,x^5yb_2 + x^{3n}a\,x^4y^2a_2 + x^{3n}a\,x^4y^2b_3}{} \tag{6E}$$
$$= 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, x^n\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, x^n = v_3\}$$

The above PDE (6E) now becomes

$$-\frac{v_3^4 a^2 v_1^4 v_2^4 a_3 + v_3^3 a n v_1^4 v_2^2 a_2 + v_3^3 a n v_2^3 a_3 v_1^3 + v_3^3 a n v_2^2 a_1 v_1^3 + v_3^3 a v_1^4 v_2^2 a_2 + 2 v_3^3 a v_1^5 v_2 b_2 + v_3^3 a v_1^4 v_2^2 b_3 + 2 v_3^3 a v_1^4 v}{} \tag{7E}$$

$$= 0$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &-2ab_2v_2v_3v_1^2 - a^2a_3v_3^2v_1v_2^4 + (-ana_2 - aa_2 - ab_3)\,v_1v_2^2v_3 \\ &- 2ab_1v_3v_1v_2 + b_2v_1 - ana_3v_3v_2^3 - ana_1v_3v_2^2 - \frac{2aba_3v_2^2}{v_1} \\ &+ \frac{bna_2 + ba_2 + bb_3}{v_1v_3} + \frac{(bna_3 + 2ba_3)\,v_2}{v_1^2v_3} + \frac{bna_1 + 2ba_1}{v_1^2v_3} - \frac{b^2a_3}{v_3^2v_1^3} = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ -2ab_1 &= 0 \\ -2ab_2 &= 0 \\ -a^2a_3 &= 0 \\ -b^2a_3 &= 0 \\ -2aba_3 &= 0 \\ -ana_1 &= 0 \\ -ana_3 &= 0 \\ bna_1 + 2ba_1 &= 0 \\ bna_3 + 2ba_3 &= 0 \\ -ana_2 - aa_2 - ab_3 &= 0 \\ bna_2 + ba_2 + bb_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= a_2\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= -(n+1)\, a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x\\
\eta &= -y(n+1)
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi\\
&= -y(n+1) - \left(a\,x^n y^2 + b\,x^{-n-2}\right)(x)\\
&= -x^n a x\, y^2 - \frac{b\,x^{-n}}{x} - ny - y\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy\\
&= \int \frac{1}{-x^n a x\, y^2 - \frac{b\,x^{-n}}{x} - ny - y} dy
\end{aligned}
$$

Which results in

$$S = -\frac{2x^{n} x \arctan\left(\frac{2a\,x^{2+2n} y + x^{n+1} n + x^{n+1}}{\sqrt{-x^{2+2n} n^{2} + 4a\,x^{2+2n} b - 2x^{2+2n} n - x^{2+2n}}}\right)}{\sqrt{-x^{2+2n} n^{2} + 4a\,x^{2+2n} b - 2x^{2+2n} n - x^{2+2n}}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = a\,x^{n} y^{2} + b\,x^{-n-2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y\,x^{n}(n+1)}{a\,x^{2+2n} y^{2} + y\,(n+1)\,x^{n+1} + b} \\
S_y &= -\frac{x^{n+1}}{a\,x^{2+2n} y^{2} + y\,(n+1)\,x^{n+1} + b}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2\arctan\left(\frac{2a\,x^{n+1} y + n + 1}{\sqrt{4ab - n^{2} - 2n - 1}}\right)}{\sqrt{4ab - n^{2} - 2n - 1}} = -\ln(x) + c_1$$

Which simplifies to

$$-\frac{2\arctan\left(\frac{2a\,x^{n+1}y+n+1}{\sqrt{4ab-n^2-2n-1}}\right)}{\sqrt{4ab-n^2-2n-1}} = -\ln(x)+c_1$$

Which gives

$$y = -\frac{\left(\tan\left(-\frac{\ln(x)\sqrt{4ab-n^2-2n-1}}{2}+\frac{c_1\sqrt{4ab-n^2-2n-1}}{2}\right)\sqrt{4ab-n^2-2n-1}+n+1\right)x^{-n-1}}{2a}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\tan\left(-\frac{\ln(x)\sqrt{4ab-n^2-2n-1}}{2}+\frac{c_1\sqrt{4ab-n^2-2n-1}}{2}\right)\sqrt{4ab-n^2-2n-1}+n+1\right)x^{-n-1}}{2a} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\tan\left(-\frac{\ln(x)\sqrt{4ab-n^2-2n-1}}{2}+\frac{c_1\sqrt{4ab-n^2-2n-1}}{2}\right)\sqrt{4ab-n^2-2n-1}+n+1\right)x^{-n-1}}{2a}$$

Verified OK.

2.7.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,x^n y^2 + b\,x^{-n-2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,x^n y^2 + \frac{b\,x^{-n}}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\,x^{-n-2}$, $f_1(x) = 0$ and $f_2(x) = x^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^n a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{an\, x^n}{x} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= x^{2n} a^2 b\, x^{-n-2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^n a u''(x) - \frac{an\, x^n u'(x)}{x} + x^{2n} a^2 b\, x^{-n-2} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{\frac{n}{2}} \sqrt{x} \left(x^{\frac{\sqrt{-4ab+n^2+2n+1}}{2}} c_1 + x^{-\frac{\sqrt{-4ab+n^2+2n+1}}{2}} c_2 \right)$$

The above shows that

$$u'(x) = \frac{\Big(c_2 \big(n+1-\sqrt{-4ab+n^2+2n+1}\big)\, x^{-\frac{\sqrt{-4ab+n^2+2n+1}}{2}} + x^{\frac{\sqrt{-4ab+n^2+2n+1}}{2}} c_1 \big(n+1+\sqrt{-4ab+n^2+2n+1}\big)}{2\sqrt{x}}$$

Using the above in (1) gives the solution

$$y = -\frac{\Big(c_2 \big(n+1-\sqrt{-4ab+n^2+2n+1}\big)\, x^{-\frac{\sqrt{-4ab+n^2+2n+1}}{2}} + x^{\frac{\sqrt{-4ab+n^2+2n+1}}{2}} c_1 \big(n+1+\sqrt{-4ab+n^2+2n+}}{2xa \left(x^{\frac{\sqrt{-4ab+n^2+2n+1}}{2}} c_1 + x^{-\frac{\sqrt{-4ab+n^2+2n+1}}{2}} c_2 \right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\Big(\big(-n-1+\sqrt{-4ab+n^2+2n+1}\big)\, x^{-\frac{\sqrt{-4ab+n^2+2n+1}}{2}} - x^{\frac{\sqrt{-4ab+n^2+2n+1}}{2}} c_3 \big(n+1+\sqrt{-4ab+n^2+2n+1}\big)}{2xa \left(x^{\frac{\sqrt{-4ab+n^2+2n+1}}{2}} c_3 + x^{-\frac{\sqrt{-4ab+n^2+2n+1}}{2}} \right)}$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(\left(-n-1+\sqrt{-4ab+n^2+2n+1}\right)x^{-\frac{\sqrt{-4ab+n^2+2n+1}}{2}} - x^{\frac{\sqrt{-4ab+n^2+2n+1}}{2}} c_3 \left(n+1+\sqrt{-4ab+n^2+2n+1}\right)}{2xa\left(x^{\frac{\sqrt{-4ab+n^2+2n+1}}{2}} c_3 + x^{-\frac{\sqrt{-4ab+n^2+2n+1}}{2}}\right)} \tag{1}
$$

Verification of solutions

$$
y = \frac{\left(\left(-n-1+\sqrt{-4ab+n^2+2n+1}\right)x^{-\frac{\sqrt{-4ab+n^2+2n+1}}{2}} - x^{\frac{\sqrt{-4ab+n^2+2n+1}}{2}} c_3 \left(n+1+\sqrt{-4ab+n^2+2n+1}\right)}{2xa\left(x^{\frac{\sqrt{-4ab+n^2+2n+1}}{2}} c_3 + x^{-\frac{\sqrt{-4ab+n^2+2n+1}}{2}}\right)}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 62

```
dsolve(diff(y(x),x)=a*x^n*y(x)^2+b*x^(-n-2),y(x), singsol=all)
```

$$
y(x) = -\frac{x^{-n-1}\left(n+1-\tan\left(\frac{\sqrt{4ab-n^2-2n-1}\,(\ln(x)-c_1)}{2}\right)\sqrt{4ab-n^2-2n-1}\right)}{2a}
$$

✓ Solution by Mathematica

Time used: 0.778 (sec). Leaf size: 135

```
DSolve[y'[x]==a*x^n*y[x]^2+b*x^(-n-2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^{-n-1}\left(-\left(\sqrt{(n+1)^2-4ab}+n+1\right)x^{\sqrt{(n+1)^2-4ab}}+c_1\left(\sqrt{(n+1)^2-4ab}-n-1\right)\right)}{2a\left(x^{\sqrt{(n+1)^2-4ab}}+c_1\right)}$$

$$y(x) \to \frac{x^{-n-1}\left(\sqrt{(n+1)^2-4ab}-n-1\right)}{2a}$$

2.8 problem 8

Internal problem ID [10338]
Internal file name [`OUTPUT/9285_Monday_June_06_2022_01_46_38_PM_37397424/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - a\,x^n y^2 = b\,x^m}$$

2.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,x^n y^2 + b\,x^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,x^n y^2 + b\,x^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\,x^m$, $f_1(x) = 0$ and $f_2(x) = x^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^n a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{an\,x^n}{x} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= b\,x^m a^2 x^{2n} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^n a u''(x) - \frac{an\,x^n u'(x)}{x} + b\,x^m a^2 x^{2n} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\mathrm{BesselY}\left(\frac{-n-1}{m+n+2}, \frac{2\sqrt{ab}\,x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) c_2 + \mathrm{BesselJ}\left(\frac{-n-1}{m+n+2}, \frac{2\sqrt{ab}\,x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) c_1\right) x^{\frac{n}{2}+\frac{1}{2}}$$

The above shows that

$$u'(x) = x^{\frac{1}{2}+n+\frac{m}{2}}\sqrt{ab}\left(-\mathrm{BesselY}\left(\frac{m+1}{m+n+2}, \frac{2\sqrt{ab}\,x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) c_2 - \mathrm{BesselJ}\left(\frac{m+1}{m+n+2}, \frac{2\sqrt{ab}\,x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) c_1\right)$$

Using the above in (1) gives the solution

$$y = -\frac{x^{\frac{1}{2}+n+\frac{m}{2}}\sqrt{ab}\left(-\mathrm{BesselY}\left(\frac{m+1}{m+n+2}, \frac{2\sqrt{ab}\,x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) c_2 - \mathrm{BesselJ}\left(\frac{m+1}{m+n+2}, \frac{2\sqrt{ab}\,x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) c_1\right) x^{-n} x^{-\frac{n}{2}-\frac{1}{2}}}{a\left(\mathrm{BesselY}\left(\frac{-n-1}{m+n+2}, \frac{2\sqrt{ab}\,x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) c_2 + \mathrm{BesselJ}\left(\frac{-n-1}{m+n+2}, \frac{2\sqrt{ab}\,x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x^{\frac{m}{2}-\frac{n}{2}}\sqrt{ab}\left(\mathrm{BesselJ}\left(\frac{m+1}{m+n+2}, \frac{2\sqrt{ab}\, x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) c_3 + \mathrm{BesselY}\left(\frac{m+1}{m+n+2}, \frac{2\sqrt{ab}\, x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right)\right)}{a\left(\mathrm{BesselY}\left(\frac{-n-1}{m+n+2}, \frac{2\sqrt{ab}\, x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) + \mathrm{BesselJ}\left(\frac{-n-1}{m+n+2}, \frac{2\sqrt{ab}\, x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) c_3\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{x^{\frac{m}{2}-\frac{n}{2}}\sqrt{ab}\left(\mathrm{BesselJ}\left(\frac{m+1}{m+n+2}, \frac{2\sqrt{ab}\, x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) c_3 + \mathrm{BesselY}\left(\frac{m+1}{m+n+2}, \frac{2\sqrt{ab}\, x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right)\right)}{a\left(\mathrm{BesselY}\left(\frac{-n-1}{m+n+2}, \frac{2\sqrt{ab}\, x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) + \mathrm{BesselJ}\left(\frac{-n-1}{m+n+2}, \frac{2\sqrt{ab}\, x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{x^{\frac{m}{2}-\frac{n}{2}}\sqrt{ab}\left(\mathrm{BesselJ}\left(\frac{m+1}{m+n+2}, \frac{2\sqrt{ab}\, x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) c_3 + \mathrm{BesselY}\left(\frac{m+1}{m+n+2}, \frac{2\sqrt{ab}\, x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right)\right)}{a\left(\mathrm{BesselY}\left(\frac{-n-1}{m+n+2}, \frac{2\sqrt{ab}\, x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) + \mathrm{BesselJ}\left(\frac{-n-1}{m+n+2}, \frac{2\sqrt{ab}\, x^{\frac{m}{2}+\frac{n}{2}+1}}{m+n+2}\right) c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = n*(diff(y(x), x))/x-a*x^n*b*x^
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         <- Bessel successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 170

```
dsolve(diff(y(x),x)=a*x^n*y(x)^2+b*x^m,y(x), singsol=all)
```

$$y(x) = \frac{x^{-\frac{n}{2}+\frac{m}{2}}\sqrt{ab}\left(\text{BesselY}\left(\frac{1+m}{n+m+2}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}+\frac{m}{2}+1}}{n+m+2}\right)c_1 + \text{BesselJ}\left(\frac{1+m}{n+m+2}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}+\frac{m}{2}+1}}{n+m+2}\right)\right)}{a\left(\text{BesselY}\left(\frac{-n-1}{n+m+2}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}+\frac{m}{2}+1}}{n+m+2}\right)c_1 + \text{BesselJ}\left(\frac{-n-1}{n+m+2}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}+\frac{m}{2}+1}}{n+m+2}\right)\right)}$$

✓ Solution by Mathematica

Time used: 2.978 (sec). Leaf size: 1805

```
DSolve[y'[x]==a*x^n*y[x]^2+b*x^m,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{a^{-\frac{2m+3n+5}{2(m+n+2)}}b^{-\frac{n+1}{2(m+n+2)}}(m+n+1)^{\frac{n+1}{m+n+2}}\left((m+n+1)^2\right)^{\frac{n+1}{m+n+2}-\frac{1}{2}}x^{-n-1}\left(x^{m+n+1}\right)^{-\frac{n+1}{2(m+n+1)}}\left(a^{\frac{n+1}{2(m+n+2)}}b^{\frac{}{2(}}\right.}{}$$

$$y(x) \to \frac{x^{-n-1}\left(\sqrt{a}\sqrt{b}(m+n+1)\left(x^{m+n+1}\right)^{\frac{1}{2}\left(\frac{1}{m+n+1}+1\right)}\text{BesselJ}\left(\frac{m+1}{m+n+2}, \frac{2\sqrt{a}\sqrt{b}(m+n+1)\left(x^{m+n+1}\right)^{\frac{1}{2}\left(1+\frac{1}{m+n+1}\right)}}{\sqrt{(m+n+1)^2}(m+n+2)}\right) - \sqrt{}\right.}{2}$$

$$y(x) \to \frac{x^{-n-1}\left(\sqrt{a}\sqrt{b}(m+n+1)\left(x^{m+n+1}\right)^{\frac{1}{2}\left(\frac{1}{m+n+1}+1\right)}\text{BesselJ}\left(\frac{m+1}{m+n+2}, \frac{2\sqrt{a}\sqrt{b}(m+n+1)\left(x^{m+n+1}\right)^{\frac{1}{2}\left(1+\frac{1}{m+n+1}\right)}}{\sqrt{(m+n+1)^2}(m+n+2)}\right) - \sqrt{}\right.}{2}$$

2.9 problem 9

Internal problem ID [10339]
Internal file name [`OUTPUT/9286_Monday_June_06_2022_01_46_40_PM_23997105/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 = k(xa+b)^n (cx+d)^{-n-4}}$$

2.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + k(xa+b)^n (cx+d)^{-n-4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \frac{k(xa+b)^n (cx+d)^{-n}}{(cx+d)^4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = k(xa+b)^n (cx+d)^{-n-4}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= k(xa+b)^n\,(cx+d)^{-n-4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + k(xa+b)^n\,(cx+d)^{-n-4}\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) + k(xa+b)^n\,(cx+d)^{-n-4}\,\_Y(x)\right\}, \left\{\_Y(x)\right\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + k(xa+b)^n\,(cx+d)^{-n-4}\,\_Y(x)\right\}, \left\{\_Y(x)\right\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + k(xa+b)^n\,(cx+d)^{-n-4}\,\_Y(x)\right\}, \left\{\_Y(x)\right\}\right)}{\mathrm{DESol}\left(\left\{\_Y''(x) + k(xa+b)^n\,(cx+d)^{-n-4}\,\_Y(x)\right\}, \left\{\_Y(x)\right\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + k(xa+b)^n\,(cx+d)^{-n-4}\,\_Y(x)\right\}, \left\{\_Y(x)\right\}\right)}{\mathrm{DESol}\left(\left\{\_Y''(x) + k(xa+b)^n\,(cx+d)^{-n-4}\,\_Y(x)\right\}, \left\{\_Y(x)\right\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + k(xa+b)^n\,(cx+d)^{-n-4}\,\_Y(x)\right\}, \left\{\_Y(x)\right\}\right)}{\mathrm{DESol}\left(\left\{\_Y''(x) + k(xa+b)^n\,(cx+d)^{-n-4}\,\_Y(x)\right\}, \left\{\_Y(x)\right\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\_Y''(x) + k(xa+b)^n\,(cx+d)^{-n-4}\,\_Y(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\_Y''(x) + k\,(xa+b)^n\,(cx+d)^{-n-4}\,\_Y(x)\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -k*(a*x+b)^n*(c*x+d)^(-n-4)*y(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-((c^4*x^4/(c*x+d)^4+4*c^3*d*x^3/(c*x+d)^4
      Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2+k*(a*x+b)^n*(c*x+d)^(-n-4),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+k*(a*x+b)^n*(c*x+d)^(-n-4),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

2.10 problem 10

Internal problem ID [10340]
Internal file name [`OUTPUT/9287_Monday_June_06_2022_01_46_51_PM_95752007/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\,x^n y^2 = bm\,x^{m-1} - a\,b^2 x^{n+2m}}$$

2.10.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,x^n y^2 + bm\,x^{m-1} - a\,b^2 x^{n+2m} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,x^n y^2 + \frac{b\,x^m m}{x} - a\,b^2 x^n x^{2m}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = bm\,x^{m-1} - a\,b^2 x^{n+2m}$, $f_1(x) = 0$ and $f_2(x) = x^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^n a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{an\,x^n}{x} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= x^{2n} a^2 \left(bm\,x^{m-1} - a\,b^2 x^{n+2m}\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^n a u''(x) - \frac{an\,x^n u'(x)}{x} + x^{2n} a^2 \left(bm\,x^{m-1} - a\,b^2 x^{n+2m}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
u(x) = &-\frac{x^{-\frac{3m}{2}-1-n} c_2 (m+2n+2)^2 \operatorname{WhittakerM}\left(\frac{m+2n+2}{2+2m+2n}, \frac{2m+3n+3}{2+2m+2n}, -\frac{2ab\,x^{1+m+n}}{1+m+n}\right)}{2} \\
&+ (1+m+n)\,c_2 \left(\left(-\frac{m}{2} - n - 1\right) x^{-\frac{3m}{2}-1-n} \right. \\
&\quad \left. + x^{-\frac{m}{2}} ab\right) \operatorname{WhittakerM}\left(-\frac{m}{2+2m+2n}, \frac{2m+3n+3}{2+2m+2n}, -\frac{2ab\,x^{1+m+n}}{1+m+n}\right) \\
&+ c_1 e^{-\frac{ab\,x^{1+m+n}}{1+m+n}}
\end{aligned}$$

The above shows that

$$u'(x) = -x^{-m-2-n}\left(-\frac{3(m+2n+2)\left(ab\left(m+\frac{4n}{3}+\frac{4}{3}\right)x^{n+1+\frac{m}{2}}+\frac{x^{-\frac{m}{2}}(m+2n+2)(1+m+n)}{3}\right)c_2 \operatorname{WhittakerM}\left(\frac{m+2n}{2+2m+}\right.}{2}\right.$$

$$+(1+m+n)\left(a^2x^{\frac{3m}{2}+2n+2}b^2\right.$$

$$\left.-\frac{\left(ab\,x^{n+1+\frac{m}{2}}+x^{-\frac{m}{2}}(1+m+n)\right)(m+2n+2)}{2}\right)c_2 \operatorname{WhittakerM}\left(-\frac{m}{2+2m+2n}, \frac{2m+3n+3}{2+2m+2n},\right.$$

$$\left.-\frac{2ab\,x^{1+m+n}}{1+m+n}\right)$$

$$+\left(m+\frac{3n}{2}+\frac{3}{2}\right)(m+2n+2)^2 x^{-\frac{m}{2}} c_2 e^{\frac{ab\,x^{1+m+n}}{1+m+n}}\left(-\frac{2ab\,x^{1+m+n}}{1+m+n}\right)^{\frac{3m+4n+4}{2+2m+2n}}$$

$$\left.+abc_1x^{2+2m+2n}e^{-\frac{ab\,x^{1+m+n}}{1+m+n}}\right)$$

Using the above in (1) gives the solution

$$y = \frac{x^{-m-2-n}\left(-\frac{3(m+2n+2)\left(ab\left(m+\frac{4n}{3}+\frac{4}{3}\right)x^{n+1+\frac{m}{2}}+\frac{x^{-\frac{m}{2}}(m+2n+2)(1+m+n)}{3}\right)c_2 \operatorname{WhittakerM}\left(\frac{m+2n+2}{2+2m+2n}, \frac{2m+3n+3}{2+2m+2n}, -\frac{2ab\,x^{1+m+n}}{1+m+n}\right)}{2}+\right.}{a\left(-\frac{x^{-\frac{3m}{2}-1-n}c_2(m+2n+2)^2 \operatorname{WhittakerM}}{2}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x^{-1-m-n}\left(-\frac{3\,e^{\frac{ab\,x^{1+m+n}}{1+m+n}}(m+2n+2)\left(ab\left(m+\frac{4n}{3}+\frac{4}{3}\right)x^{n+1+\frac{m}{2}}+\frac{x^{-\frac{m}{2}}(m+2n+2)(1+m+n)}{3}\right)\operatorname{WhittakerM}\left(\frac{m+2n+2}{2+2m+2n}, \frac{2m+3n+3}{2+2m+2n}, -\frac{2ab\,x^{1+}}{1+m}\right.}{2}\right.}{\left(-\frac{x^{-\frac{3m}{2}}e^{\frac{ab\,x^{1+m+n}}{1+m+n}}(m+2n+2)^2 \operatorname{Whittak}}{}\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{x^{-1-m-n}\left(-\frac{3\,\mathrm{e}^{\frac{ab\,x^{1+m+n}}{1+m+n}}(m+2n+2)\left(ab\left(m+\frac{4n}{3}+\frac{4}{3}\right)x^{n+1+\frac{m}{2}}+\frac{x^{-\frac{m}{2}}(m+2n+2)(1+m+n)}{3}\right)\mathrm{WhittakerM}\left(\frac{m+2n+2}{2+2m+2n},\frac{2m+3n+3}{2+2m+2n},-\frac{2ab\,x^{1+\cdots}}{1+m\cdots}\right.\cdots}{2}\cdots\right.}{\left(-\frac{x^{-\frac{3m}{2}}\mathrm{e}^{\frac{ab\,x^{1+m+n}}{1+m+n}}(m+2n+2)^2\,\mathrm{Whittak}\cdots}{\cdots}\right.\cdots} \tag{1}$$

Verification of solutions

$$y = \frac{x^{-1-m-n}\left(-\frac{3\,\mathrm{e}^{\frac{ab\,x^{1+m+n}}{1+m+n}}(m+2n+2)\left(ab\left(m+\frac{4n}{3}+\frac{4}{3}\right)x^{n+1+\frac{m}{2}}+\frac{x^{-\frac{m}{2}}(m+2n+2)(1+m+n)}{3}\right)\mathrm{WhittakerM}\left(\frac{m+2n+2}{2+2m+2n},\frac{2m+3n+3}{2+2m+2n},-\frac{2ab\,x^{1+\cdots}}{1+m\cdots}\right.\cdots}{2}\cdots\right.}{\left(-\frac{x^{-\frac{3m}{2}}\mathrm{e}^{\frac{ab\,x^{1+m+n}}{1+m+n}}(m+2n+2)^2\,\mathrm{Whittak}\cdots}{\cdots}\right.\cdots}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = n*(diff(y(x), x))/x-a*x^n*b*(-
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
         <- Kovacics algorithm successful
      <- Equivalence, under non-integer power transformations successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 522

```
dsolve(diff(y(x),x)=a*x^n*y(x)^2+b*m*x^(m-1)-a*b^2*x^(n+2*m),y(x), singsol=all)
```

$$y(x) = \frac{x^{-n-1}\left(-\frac{3\left(ab\left(m+\frac{4n}{3}+\frac{4}{3}\right)x^{n+1-\frac{m}{2}}+\frac{x^{-\frac{3m}{2}}(m+2n+2)(1+m+n)}{3}\right)c_1(m+2n+2)\mathrm{e}^{\frac{ab\,x^{1+m+n}}{1+m+n}}\operatorname{WhittakerM}\left(\frac{m+2n+2}{2n+2m+2},\frac{2m+3n+3}{2n+2m+2},-\frac{2ab\,x^{1+m+n}}{1+m+\cdots}\right.}{2}\cdots\right)}{\left(-\frac{\mathrm{e}^{\frac{ab\,x^{1+m+n}}{1+m+n}}x^{-\frac{3m}{2}}c_1(m+2n+2)^2\operatorname{Whitt}\cdots}{\cdots}\right.}$$

✓ Solution by Mathematica

Time used: 2.322 (sec). Leaf size: 306

```
DSolve[y'[x]==a*x^n*y[x]^2+b*m*x^(m-1)-a*b^2*x^(n+2*m),y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{2^{\frac{n+1}{m+n+1}}(m+n+1)\left(-\frac{abx^{m+n+1}}{m+n+1}\right)^{\frac{n+1}{m+n+1}}\left(abx^m-c_1e^{\frac{2abx^{m+n+1}}{m+n+1}}\right)-abc_1x^{m+n+1}\Gamma\left(\frac{n+1}{m+n+1},-\frac{2abx^{m+n+1}}{m+n+1}\right)}{a\left(2^{\frac{n+1}{m+n+1}}(m+n+1)\left(-\frac{abx^{m+n+1}}{m+n+1}\right)^{\frac{n+1}{m+n+1}}-c_1x^{n+1}\Gamma\left(\frac{n+1}{m+n+1},-\frac{2abx^{m+n+1}}{m+n+1}\right)\right)}$$

$$y(x) \to bx^m-\frac{b2^{\frac{n+1}{m+n+1}}x^me^{\frac{2abx^{m+n+1}}{m+n+1}}\left(-\frac{abx^{m+n+1}}{m+n+1}\right)^{-\frac{m}{m+n+1}}}{\Gamma\left(\frac{n+1}{m+n+1},-\frac{2abx^{m+n+1}}{m+n+1}\right)}$$

2.11 problem 11

Internal problem ID [10341]
Internal file name [`OUTPUT/9288_Monday_June_06_2022_01_48_19_PM_41829524/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \left(a\,x^{2n} + b\,x^{n-1}\right) y^2 = c}$$

2.11.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x^{2n} a\,y^2 + x^{n-1} b\,y^2 + c \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^{2n} a\,y^2 + \frac{x^n b\,y^2}{x} + c$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = c$, $f_1(x) = 0$ and $f_2(x) = a\,x^{2n} + b\,x^{n-1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\left(a\,x^{2n} + b\,x^{n-1}\right) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2a\,x^{2n} n}{x} + \frac{b\,x^{n-1}(n-1)}{x} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \left(a\,x^{2n} + b\,x^{n-1}\right)^2 c \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\left(a\,x^{2n} + b\,x^{n-1}\right) u''(x) - \left(\frac{2a\,x^{2n} n}{x} + \frac{b\,x^{n-1}(n-1)}{x}\right) u'(x) + \left(a\,x^{2n} + b\,x^{n-1}\right)^2 c u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{-\mathrm{e}^{-\frac{i\sqrt{c}\,\sqrt{a}\,x^{n+1}}{n+1}} (2+n)\, c_1 \left(\left(2ia^{\frac{3}{2}}bn - 2a\sqrt{c}\,b^2\right) x^{1+2n} + \left(-a^2\sqrt{c}\,b + ia^{\frac{5}{2}}n\right) x^{2+3n} + x^n \left(-\sqrt{c}\,b + i\sqrt{a}\,n\right) b^2 \cdots}{\cdots}$$

The above shows that

$$u'(x) = \frac{\mathrm{e}^{\frac{i\left(-4\sqrt{c}\,\sqrt{a}\,x^{n+1} + \pi(2+n)\right)}{4n+4}} c c_2 (x^{5+5n} a^4 + 4a^3 b\, x^{4n+4} + 6a^2 x^{3n+3} b^2 + 4x^{2+2n} a\, b^3 + x^{n+1} b^4) \operatorname{hypergeom}\left(\left[\frac{(2+n)\sqrt{a}\cdots}{\sqrt{a}\,(2\cdots}\right.\right.\cdots}{\cdots}$$

Using the above in (1) gives the solution

$$y = -\frac{\cdots}{x\left(a\,x^{n+1} + b\right)\left(a\,x^{2n} + b\,x^{n-1}\right)\left(-\mathrm{e}^{-\frac{i\sqrt{c}\,\sqrt{a}\,x^{n+1}}{n+1}} (2+n)\, c_1 \left(\left(2ia^{\frac{3}{2}}bn - 2a\sqrt{c}\,b^2\right) x^{1+2n} + \left(-a^2\sqrt{c}\,b + i\cdots\right.\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y
$$

$$
= \frac{}{\left(\left(2\left(i a^{\frac{3}{2}} b n - a\sqrt{c}\, b^{2}\right) x^{1+2n} + \left(-a^{2}\sqrt{c}\, b + i a^{\frac{5}{2}} n\right) x^{2+3n} + x^{n}\left(-\sqrt{c}\, b + i\sqrt{a}\, n\right) b^{2}\right)(2+n)\, \mathrm{e}^{-\frac{i\sqrt{c}\,\sqrt{a}\, x^{n+1}}{n+1}}}
$$

Summary

The solution(s) found are the following

$$
y \tag{1}
$$

$$
= \frac{}{\left(\left(2\left(i a^{\frac{3}{2}} b n - a\sqrt{c}\, b^{2}\right) x^{1+2n} + \left(-a^{2}\sqrt{c}\, b + i a^{\frac{5}{2}} n\right) x^{2+3n} + x^{n}\left(-\sqrt{c}\, b + i\sqrt{a}\, n\right) b^{2}\right)(2+n)\, \mathrm{e}^{-\frac{i\sqrt{c}\,\sqrt{a}\, x^{n+1}}{n+1}}}
$$

Verification of solutions

$$
y
$$

$$
= \frac{}{\left(\left(2\left(i a^{\frac{3}{2}} b n - a\sqrt{c}\, b^{2}\right) x^{1+2n} + \left(-a^{2}\sqrt{c}\, b + i a^{\frac{5}{2}} n\right) x^{2+3n} + x^{n}\left(-\sqrt{c}\, b + i\sqrt{a}\, n\right) b^{2}\right)(2+n)\, \mathrm{e}^{-\frac{i\sqrt{c}\,\sqrt{a}\, x^{n+1}}{n+1}}}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (b*x^(n-1)*n+2*x^(2*n)*n*a-b*x
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
                  <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
         <- hypergeometric successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 1237

```
dsolve(diff(y(x),x)=(a*x^(2*n)+b*x^(n-1))*y(x)^2+c,y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 2.128 (sec). Leaf size: 1384

```
DSolve[y'[x]==(a*x^(2*n)+b*x^(n-1))*y[x]^2+c,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{c}(n+1)^2x^{-n}\Bigg(}{\sqrt{a}c_1(n+1)\sqrt{-(n+1)^2}\,\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{\sqrt{c}b}{\sqrt{a}\sqrt{-(n+1)^2}}+\frac{n}{n+1}\right),\frac{n}{n+1},\frac{2\sqrt{a}\sqrt{c}x^{n+1}}{\sqrt{-(n+1)^2}}\right)+c_1\left(\sqrt{a}\sqrt{-(n\right.}$$

$$y(x) \to \frac{\sqrt{c}(n+1)^2x^{-n}\,\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{\sqrt{c}b}{\sqrt{a}\sqrt{-(n}}\right.\right.}{\sqrt{a}(n+1)\sqrt{-(n+1)^2}\,\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{\sqrt{c}b}{\sqrt{a}\sqrt{-(n+1)^2}}+\frac{n}{n+1}\right),\frac{n}{n+1},\frac{2\sqrt{a}\sqrt{c}x^{n+1}}{\sqrt{-(n+1)^2}}\right)+\left(\sqrt{a}\sqrt{-(n+1)^2}\right.}$$

$$y(x) \to \frac{\sqrt{c}(n+1)^2x^{-n}\,\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{\sqrt{c}b}{\sqrt{a}\sqrt{-(n}}\right.\right.}{\sqrt{a}(n+1)\sqrt{-(n+1)^2}\,\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{\sqrt{c}b}{\sqrt{a}\sqrt{-(n+1)^2}}+\frac{n}{n+1}\right),\frac{n}{n+1},\frac{2\sqrt{a}\sqrt{c}x^{n+1}}{\sqrt{-(n+1)^2}}\right)+\left(\sqrt{a}\sqrt{-(n+1)^2}\right.}$$

$$y(x) \to \frac{\sqrt{c}(n+1)x^{-n}L_{-\frac{\sqrt{c}b}{2\sqrt{a}\sqrt{-(n+1)^2}}-\frac{n}{2(n+1)}}^{-\frac{1}{n+1}}\left(\frac{2\sqrt{a}\sqrt{c}x^{n+1}}{\sqrt{-(n+1)^2}}\right)}{\sqrt{a}\sqrt{-(n+1)^2}\left(2L_{-\frac{\sqrt{c}b}{2\sqrt{a}\sqrt{-(n+1)^2}}-\frac{n}{2(n+1)}-1}^{\frac{n}{n+1}}\left(\frac{2\sqrt{a}\sqrt{c}x^{n+1}}{\sqrt{-(n+1)^2}}\right)+L_{-\frac{\sqrt{c}b}{2\sqrt{a}\sqrt{-(n+1)^2}}-\frac{n}{2(n+1)}}^{-\frac{1}{n+1}}\left(\frac{2\sqrt{a}\sqrt{c}x^{n+1}}{\sqrt{-(n+1)^2}}\right)\right)}$$

2.12 problem 12

Internal problem ID [10342]
Internal file name [`OUTPUT/9289_Monday_June_06_2022_01_49_06_PM_3413175/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{(a_2x + b_2)\left(y' + \lambda y^2\right) = -a_0x - b_0}$$

2.12.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y^2a_2\lambda x + y^2b_2\lambda + a_0x + b_0}{a_2x + b_2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2a_2\lambda x}{a_2x + b_2} - \frac{y^2b_2\lambda}{a_2x + b_2} - \frac{a_0x}{a_2x + b_2} - \frac{b_0}{a_2x + b_2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{a_0x+b_0}{a_2x+b_2}$, $f_1(x) = 0$ and $f_2(x) = -\frac{\lambda a_2x+\lambda b_2}{a_2x+b_2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{-\frac{(\lambda a_2x+\lambda b_2)u}{a_2x+b_2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{\lambda a_2}{a_2x + b_2} + \frac{(\lambda a_2x + \lambda b_2)\,a_2}{(a_2x + b_2)^2}\\
f_1f_2 &= 0\\
f_2^2f_0 &= -\frac{(\lambda a_2x + \lambda b_2)^2\,(a_0x + b_0)}{(a_2x + b_2)^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{(\lambda a_2x + \lambda b_2)\,u''(x)}{a_2x + b_2} - \left(-\frac{\lambda a_2}{a_2x + b_2} + \frac{(\lambda a_2x + \lambda b_2)\,a_2}{(a_2x + b_2)^2}\right)u'(x) - \frac{(\lambda a_2x + \lambda b_2)^2\,(a_0x + b_0)\,u(x)}{(a_2x + b_2)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
u(x) = c_1 \operatorname{WhittakerM}&\left(\frac{i\sqrt{\lambda}\,(a_0b_2 - a_2b_0)}{2a_2^{\frac{3}{2}}\sqrt{a_0}}, \frac{1}{2}, \frac{2i\sqrt{a_0}\,\sqrt{\lambda}\,(a_2x + b_2)}{a_2^{\frac{3}{2}}}\right)\\
&+ c_2 \operatorname{WhittakerW}\left(\frac{i\sqrt{\lambda}\,(a_0b_2 - a_2b_0)}{2a_2^{\frac{3}{2}}\sqrt{a_0}}, \frac{1}{2}, \frac{2i\sqrt{a_0}\,\sqrt{\lambda}\,(a_2x + b_2)}{a_2^{\frac{3}{2}}}\right)
\end{aligned}$$

The above shows that

$$u'(x) = \frac{\frac{\left(2a_2^{\frac{3}{2}}a_0^{\frac{3}{2}} + i(a_0b_2 - a_2b_0)a_0\sqrt{\lambda}\right)c_1 \operatorname{WhittakerM}\left(\frac{i\sqrt{\lambda}\,(a_0b_2 - a_2b_0)}{2a_2^{\frac{3}{2}}\sqrt{a_0}} + 1, \frac{1}{2}, \frac{2i\sqrt{a_0}\,\sqrt{\lambda}\,(a_2x+b_2)}{a_2^{\frac{3}{2}}}\right)}{2} - a_2^{\frac{3}{2}}a_0^{\frac{3}{2}} \operatorname{WhittakerW}\left(\frac{i\sqrt{\lambda}\,(a_0b_2 - a_2b_0)}{2a_2^{\frac{3}{2}}\sqrt{a_0}}\right.}{}$$

Using the above in (1) gives the solution

$$y = \frac{\frac{\left(2a_2^{\frac{3}{2}}a_0^{\frac{3}{2}} + i(a_0b_2 - a_2b_0)a_0\sqrt{\lambda}\right)c_1 \operatorname{WhittakerM}\left(\frac{i\sqrt{\lambda}\,(a_0b_2 - a_2b_0)}{2a_2^{\frac{3}{2}}\sqrt{a_0}} + 1, \frac{1}{2}, \frac{2i\sqrt{a_0}\,\sqrt{\lambda}\,(a_2x+b_2)}{a_2^{\frac{3}{2}}}\right)}{2} - a_2^{\frac{3}{2}}a_0^{\frac{3}{2}} \operatorname{WhittakerW}\left(\frac{i\sqrt{\lambda}\,(a_0b_2 - a_2b_0)}{2a_2^{\frac{3}{2}}\sqrt{a_0}}\right.}{\sqrt{a_2}\,a_0^{\frac{3}{2}}\,(\lambda a_2x + \lambda b_2)\left(c_1 \operatorname{Whitta}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{\frac{c_3\left(2a_2^{\frac{3}{2}}\sqrt{a_0}+i\sqrt{\lambda}\,(a_0b_2-a_2b_0)\right)\operatorname{WhittakerM}\left(\frac{2a_2^{\frac{3}{2}}\sqrt{a_0}+i\sqrt{\lambda}\,(a_0b_2-a_2b_0)}{2a_2^{\frac{3}{2}}\sqrt{a_0}},\frac{1}{2},\frac{2i\sqrt{a_0}\,\sqrt{\lambda}\,(a_2x+b_2)}{a_2^{\frac{3}{2}}}\right)}{2} - a_2^{\frac{3}{2}}\operatorname{WhittakerW}\left(\frac{2a_2^{\frac{3}{2}}\sqrt{a_0}+i\sqrt{\lambda}\cdots}{2a_2^{\frac{3}{2}}\cdots}\right.\cdots}{\sqrt{a_0}\,\sqrt{a_2}\,\lambda\,(a_2x+b_2)\left(c_3\operatorname{Whi}\cdots\right.}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\frac{c_3\left(2a_2^{\frac{3}{2}}\sqrt{a_0}+i\sqrt{\lambda}\,(a_0b_2-a_2b_0)\right)\operatorname{WhittakerM}\left(\frac{2a_2^{\frac{3}{2}}\sqrt{a_0}+i\sqrt{\lambda}\,(a_0b_2-a_2b_0)}{2a_2^{\frac{3}{2}}\sqrt{a_0}},\frac{1}{2},\frac{2i\sqrt{a_0}\,\sqrt{\lambda}\,(a_2x+b_2)}{a_2^{\frac{3}{2}}}\right)}{2} - a_2^{\frac{3}{2}}\operatorname{WhittakerW}\left(\frac{2a_2^{\frac{3}{2}}\sqrt{a_0}+i\sqrt{\lambda}\cdots}{2a_2^{\frac{3}{2}}\cdots}\right.\cdots}{\sqrt{a_0}\,\sqrt{a_2}\,\lambda\,(a_2x+b_2)\left(c_3\operatorname{Whi}\cdots\right.} \tag{1}
$$

Verification of solutions

$$
y = \frac{\frac{c_3\left(2a_2^{\frac{3}{2}}\sqrt{a_0}+i\sqrt{\lambda}\,(a_0b_2-a_2b_0)\right)\operatorname{WhittakerM}\left(\frac{2a_2^{\frac{3}{2}}\sqrt{a_0}+i\sqrt{\lambda}\,(a_0b_2-a_2b_0)}{2a_2^{\frac{3}{2}}\sqrt{a_0}},\frac{1}{2},\frac{2i\sqrt{a_0}\,\sqrt{\lambda}\,(a_2x+b_2)}{a_2^{\frac{3}{2}}}\right)}{2} - a_2^{\frac{3}{2}}\operatorname{WhittakerW}\left(\frac{2a_2^{\frac{3}{2}}\sqrt{a_0}+i\sqrt{\lambda}\cdots}{2a_2^{\frac{3}{2}}\cdots}\right.\cdots}{\sqrt{a_0}\,\sqrt{a_2}\,\lambda\,(a_2x+b_2)\left(c_3\operatorname{Whi}\cdots\right.}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Abel AIR successful: ODE belongs to the 1F1 2-parameter class`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 461

```
dsolve((a__2*x+b__2)*(diff(y(x),x)+lambda*y(x)^2)+a__0*x+b__0=0,y(x), singsol=all)
```

$$y(x) = -\frac{\left(\frac{c_1 \lambda (a_0 b_2 - a_2 b_0) \operatorname{KummerU}\left(-\frac{\sqrt{-a_2 \lambda a_0}\, a_0 b_2 - \sqrt{-a_2 \lambda a_0}\, a_2 b_0 - 2 a_0 a_2^2}{2 a_0 a_2^2}, 1, \frac{2(a_2 x + b_2)\sqrt{-a_2 \lambda a_0}}{a_2^2}\right)}{2} + \sqrt{-a_2 \lambda a_0}\, a_2 \left(c_1 \operatorname{KummerU}\left(\right.\right.\right.}{\left(\frac{c_1 \sqrt{-a_2 \lambda a_0}\, (a_0 b_2 - a_2 b_0) \operatorname{KummerU}\left(-\frac{\sqrt{-a_2 \lambda a_0}\, a_0 b_2 - \sqrt{-a_2 \lambda a_0}\, a_2 b_0 - 2 a_0 a_2^2}{2 a_0 a_2^2}, 1, \frac{2(a_2 x + b_2)\sqrt{-a_2 \lambda a_0}}{a_2^2}\right)}{2} + a_0 a_2^2 \left(\operatorname{KummerM}\left(-\right.\right.\right.}$$

✓ Solution by Mathematica

Time used: 1.895 (sec). Leaf size: 690

```
DSolve[(a2*x+b2)*(y'[x]+\[Lambda]*y[x]^2)+a0*x+b0==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \rightarrow \frac{c_1\sqrt{\lambda}(\mathrm{a2b0}-\mathrm{a0b2})\,\mathrm{HypergeometricU}\left(\frac{i\sqrt{\lambda}(\mathrm{a2b0}-\mathrm{a0b2})}{2\sqrt{\mathrm{a0}}\mathrm{a2}^{3/2}}+1,1,\frac{2i\sqrt{\mathrm{a0}}(\mathrm{b2}+\mathrm{a2}x)\sqrt{\lambda}}{\mathrm{a2}^{3/2}}\right)-i\sqrt{\mathrm{a0}}\mathrm{a2}^{3/2}\left(c_1\,\mathrm{Hy}\right.}{\mathrm{a2}^2\sqrt{\lambda}\left(c_1\,\mathrm{HypergeometricU}\right.}$$

$$y(x) \rightarrow \frac{(\mathrm{a2b0}-\mathrm{a0b2})\,\mathrm{HypergeometricU}\left(\frac{i\sqrt{\lambda}(\mathrm{a2b0}-\mathrm{a0b2})}{2\sqrt{\mathrm{a0}}\mathrm{a2}^{3/2}}+1,1,\frac{2i\sqrt{\mathrm{a0}}(\mathrm{b2}+\mathrm{a2}x)\sqrt{\lambda}}{\mathrm{a2}^{3/2}}\right)}{\mathrm{a2}^2\,\mathrm{HypergeometricU}\left(\frac{i(\mathrm{a2b0}-\mathrm{a0b2})\sqrt{\lambda}}{2\sqrt{\mathrm{a0}}\mathrm{a2}^{3/2}},0,\frac{2i\sqrt{\mathrm{a0}}(\mathrm{b2}+\mathrm{a2}x)\sqrt{\lambda}}{\mathrm{a2}^{3/2}}\right)} - \frac{i\sqrt{\mathrm{a0}}}{\sqrt{\mathrm{a2}}\sqrt{\lambda}}$$

$$y(x) \rightarrow \frac{(\mathrm{a2b0}-\mathrm{a0b2})\,\mathrm{HypergeometricU}\left(\frac{i\sqrt{\lambda}(\mathrm{a2b0}-\mathrm{a0b2})}{2\sqrt{\mathrm{a0}}\mathrm{a2}^{3/2}}+1,1,\frac{2i\sqrt{\mathrm{a0}}(\mathrm{b2}+\mathrm{a2}x)\sqrt{\lambda}}{\mathrm{a2}^{3/2}}\right)}{\mathrm{a2}^2\,\mathrm{HypergeometricU}\left(\frac{i(\mathrm{a2b0}-\mathrm{a0b2})\sqrt{\lambda}}{2\sqrt{\mathrm{a0}}\mathrm{a2}^{3/2}},0,\frac{2i\sqrt{\mathrm{a0}}(\mathrm{b2}+\mathrm{a2}x)\sqrt{\lambda}}{\mathrm{a2}^{3/2}}\right)} - \frac{i\sqrt{\mathrm{a0}}}{\sqrt{\mathrm{a2}}\sqrt{\lambda}}$$

2.13 problem 13

Internal problem ID [10343]
Internal file name [`OUTPUT/9290_Monday_June_06_2022_01_49_09_PM_96814483/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Riccati, _special]]
```

$$\boxed{x^2y' - a\,x^2y^2 = b}$$

2.13.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{a\,x^2y^2 + b}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(a\,x^2y^2+b)\,(b_3-a_2)}{x^2} - \frac{(a\,x^2y^2+b)^2\,a_3}{x^4} \\ & - \left(\frac{2a\,y^2}{x} - \frac{2(a\,x^2y^2+b)}{x^3}\right)(xa_2+ya_3+a_1) - 2ya(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{a^2x^4y^4a_3 + 2a\,x^5yb_2 + a\,x^4y^2a_2 + a\,x^4y^2b_3 + 2ab\,x^2y^2a_3 + 2a\,x^4yb_1 - b_2x^4 - b\,x^2a_2 - b\,x^2b_3 - 2bxya_3 +}{x^4}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -a^2x^4y^4a_3 - 2a\,x^5yb_2 - a\,x^4y^2a_2 - a\,x^4y^2b_3 - 2ab\,x^2y^2a_3 \\ & - 2a\,x^4yb_1 + b_2x^4 + b\,x^2a_2 + b\,x^2b_3 + 2bxya_3 - b^2a_3 + 2bxa_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -a^2a_3v_1^4v_2^4 - aa_2v_1^4v_2^2 - 2ab_2v_1^5v_2 - ab_3v_1^4v_2^2 - 2aba_3v_1^2v_2^2 - 2ab_1v_1^4v_2 \\ & + b_2v_1^4 + ba_2v_1^2 + 2ba_3v_1v_2 + bb_3v_1^2 - b^2a_3 + 2ba_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-2ab_2v_1^5v_2 - a^2a_3v_1^4v_2^4 + (-aa_2 - ab_3)\,v_1^4v_2^2 - 2ab_1v_1^4v_2 + b_2v_1^4 \\
&- 2aba_3v_1^2v_2^2 + (ba_2 + bb_3)\,v_1^2 + 2ba_3v_1v_2 + 2ba_1v_1 - b^2a_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-2ab_1 &= 0 \\
-2ab_2 &= 0 \\
-a^2a_3 &= 0 \\
2ba_1 &= 0 \\
2ba_3 &= 0 \\
-b^2a_3 &= 0 \\
-2aba_3 &= 0 \\
-aa_2 - ab_3 &= 0 \\
ba_2 + bb_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{a\,x^2y^2 + b}{x^2}\right)(-x) \\
&= \frac{a\,x^2y^2 + yx + b}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{a\,x^2y^2+yx+b}{x}} dy \end{aligned}$$

Which results in

$$S = \frac{2x \arctan\left(\frac{2a\,x^2y+x}{\sqrt{4ab\,x^2-x^2}}\right)}{\sqrt{4ab\,x^2-x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{a\,x^2y^2+b}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{a\,x^2y^2+yx+b} \\ S_y &= \frac{x}{a\,x^2y^2+yx+b} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\arctan\left(\frac{2yax+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}} = \ln(x) + c_1$$

Which simplifies to

$$\frac{2\arctan\left(\frac{2yax+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}} = \ln(x) + c_1$$

Which gives

$$y = \frac{\tan\left(\frac{\ln(x)\sqrt{4ab-1}}{2} + \frac{c_1\sqrt{4ab-1}}{2}\right)\sqrt{4ab-1} - 1}{2xa}$$

Summary

The solution(s) found are the following

$$y = \frac{\tan\left(\frac{\ln(x)\sqrt{4ab-1}}{2} + \frac{c_1\sqrt{4ab-1}}{2}\right)\sqrt{4ab-1} - 1}{2xa} \tag{1}$$

Verification of solutions

$$y = \frac{\tan\left(\frac{\ln(x)\sqrt{4ab-1}}{2} + \frac{c_1\sqrt{4ab-1}}{2}\right)\sqrt{4ab-1} - 1}{2xa}$$

Verified OK.

2.13.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2\right)\mathrm{d}y&=\left(a\,x^2y^2+b\right)\mathrm{d}x\\ \left(-a\,x^2y^2-b\right)\mathrm{d}x+\left(x^2\right)\mathrm{d}y&=0 \qquad\qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-a\,x^2y^2-b\\ N(x,y)&=x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-a\,x^2y^2 - b\right)\\ &= -2a\,x^2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2\right)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^2}\left(\left(-2a\,x^2y\right) - (2x)\right)\\ &= \frac{-2axy - 2}{x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\frac{1}{a\,x^2y^2 + b}\left((2x) - \left(-2a\,x^2y\right)\right)\\ &= -\frac{2x(axy + 1)}{a\,x^2y^2 + b}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(2x) - (-2a\,x^2 y)}{x\,(-a\,x^2y^2 - b) - y\,(x^2)} \\
&= \frac{-2axy - 2}{a\,x^2y^2 + yx + b}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{-2at - 2}{a\,t^2 + b + t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(\frac{-2at-2}{a\,t^2+b+t}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\ln\left(a\,t^2+b+t\right) - \frac{2\arctan\left(\frac{2at+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}} \\
&= \frac{\mathrm{e}^{-\frac{2\arctan\left(\frac{2at+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}}{a\,t^2 + b + t}
\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{\mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}}{a\,x^2y^2 + yx + b}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}}{a\,x^2y^2 + yx + b}\left(-a\,x^2y^2 - b\right) \\
&= \frac{\left(-a\,x^2y^2 - b\right)\mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}}{a\,x^2y^2 + yx + b}
\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}}{a\,x^2y^2+yx+b}\left(x^2\right) \\
&= \frac{x^2\mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}}{a\,x^2y^2+yx+b}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x}=0
$$

$$
\left(\frac{\left(-a\,x^2y^2-b\right)\mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}}{a\,x^2y^2+yx+b}\right)+\left(\frac{x^2\mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}}{a\,x^2y^2+yx+b}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=0
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int\overline{M}\,\mathrm{d}x
$$

$$
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int\frac{\left(-a\,x^2y^2-b\right)\mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}}{a\,x^2y^2+yx+b}\,\mathrm{d}x
$$

$$
\phi=-\mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}x+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y}=\frac{4x^2a\,\mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}}{(4ab-1)\left(\frac{(2axy+1)^2}{4ab-1}+1\right)}+f'(y) \tag{4}
$$

$$= \frac{x^2 e^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}}{a\,x^2y^2 + yx + b} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2 e^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}}{a\,x^2y^2+yx+b}$. Therefore equation (4) becomes

$$\frac{x^2 e^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}}{a\,x^2y^2 + yx + b} = \frac{x^2 e^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}}}{a\,x^2y^2 + yx + b} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -e^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}} x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -e^{-\frac{2\arctan\left(\frac{2axy+1}{\sqrt{4ab-1}}\right)}{\sqrt{4ab-1}}} x$$

The solution becomes

$$y = -\frac{\tan\left(\frac{\ln\left(-\frac{c_1}{x}\right)\sqrt{4ab-1}}{2}\right)\sqrt{4ab-1} + 1}{2xa}$$

Summary

The solution(s) found are the following

$$y = -\frac{\tan\left(\frac{\ln\left(-\frac{c_1}{x}\right)\sqrt{4ab-1}}{2}\right)\sqrt{4ab-1} + 1}{2xa} \tag{1}$$

Verification of solutions

$$y = -\frac{\tan\left(\frac{\ln\left(-\frac{c_1}{x}\right)\sqrt{4ab-1}}{2}\right)\sqrt{4ab-1} + 1}{2xa}$$

Verified OK.

2.13.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,x^2y^2+b}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,y^2 + \frac{b}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{b}{x^2}$, $f_1(x) = 0$ and $f_2(x) = a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= \frac{a^2b}{x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$au''(x) + \frac{a^2bu(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sqrt{x}\left(c_1x^{\frac{\sqrt{-4ab+1}}{2}} + c_2x^{-\frac{\sqrt{-4ab+1}}{2}}\right)$$

The above shows that

$$u'(x) = \frac{-c_2\left(-1+\sqrt{-4ab+1}\right) x^{-\frac{\sqrt{-4ab+1}}{2}} + c_1 x^{\frac{\sqrt{-4ab+1}}{2}}\left(1+\sqrt{-4ab+1}\right)}{2\sqrt{x}}$$

Using the above in (1) gives the solution

$$y = -\frac{-c_2\left(-1+\sqrt{-4ab+1}\right) x^{-\frac{\sqrt{-4ab+1}}{2}} + c_1 x^{\frac{\sqrt{-4ab+1}}{2}}\left(1+\sqrt{-4ab+1}\right)}{2xa\left(c_1 x^{\frac{\sqrt{-4ab+1}}{2}} + c_2 x^{-\frac{\sqrt{-4ab+1}}{2}}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-1+\sqrt{-4ab+1}\right) x^{-\frac{\sqrt{-4ab+1}}{2}} - c_3 x^{\frac{\sqrt{-4ab+1}}{2}}\left(1+\sqrt{-4ab+1}\right)}{2xa\left(c_3 x^{\frac{\sqrt{-4ab+1}}{2}} + x^{-\frac{\sqrt{-4ab+1}}{2}}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-1+\sqrt{-4ab+1}\right) x^{-\frac{\sqrt{-4ab+1}}{2}} - c_3 x^{\frac{\sqrt{-4ab+1}}{2}}\left(1+\sqrt{-4ab+1}\right)}{2xa\left(c_3 x^{\frac{\sqrt{-4ab+1}}{2}} + x^{-\frac{\sqrt{-4ab+1}}{2}}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(-1+\sqrt{-4ab+1}\right) x^{-\frac{\sqrt{-4ab+1}}{2}} - c_3 x^{\frac{\sqrt{-4ab+1}}{2}}\left(1+\sqrt{-4ab+1}\right)}{2xa\left(c_3 x^{\frac{\sqrt{-4ab+1}}{2}} + x^{-\frac{\sqrt{-4ab+1}}{2}}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 40

```
dsolve(x^2*diff(y(x),x)=a*x^2*y(x)^2+b,y(x), singsol=all)
```

$$y(x) = \frac{-1 + \tan\left(\frac{\sqrt{4ab-1}\,(\ln(x)-c_1)}{2}\right)\sqrt{4ab-1}}{2ax}$$

✓ Solution by Mathematica
Time used: 0.292 (sec). Leaf size: 77

```
DSolve[x^2*y'[x]==a*x^2*y[x]^2+b,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-1 + \sqrt{1-4ab}\left(-1 + \frac{2c_1}{x^{\sqrt{1-4ab}}+c_1}\right)}{2ax}$$
$$y(x) \to \frac{\sqrt{1-4ab}-1}{2ax}$$

2.14 problem 14

Internal problem ID [10344]
Internal file name [`OUTPUT/9291_Monday_June_06_2022_01_49_10_PM_88481665/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{x^2y' - x^2y^2 = -a^2x^4 + a(1-2b)\,x^2 - b(b+1)}$$

2.14.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-a^2x^4 - 2ab\,x^2 + x^2y^2 + a\,x^2 - b^2 - b}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a^2x^2 - 2ab + y^2 + a - \frac{b^2}{x^2} - \frac{b}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{-a^2x^4 - 2ab\,x^2 + a\,x^2 - b^2 - b}{x^2}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x)-(f_2'+f_1f_2)\,u'(x)+f_2^2f_0u(x)=0 \tag{2}$$

But

$$\begin{aligned}f_2'&=0\\ f_1f_2&=0\\ f_2^2f_0&=\frac{-a^2x^4-2ab\,x^2+a\,x^2-b^2-b}{x^2}\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x)+\frac{(-a^2x^4-2ab\,x^2+a\,x^2-b^2-b)\,u(x)}{x^2}=0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x)=x^{-b}\mathrm{e}^{-\frac{a\,x^2}{2}}\left(c_2\Gamma\left(b+\frac{1}{2}\right)-c_2\Gamma\left(b+\frac{1}{2},-a\,x^2\right)+c_1\right)$$

The above shows that

$$u'(x)=-2\left(-\frac{\left(-c_2\Gamma\left(b+\frac{1}{2}\right)+c_2\Gamma\left(b+\frac{1}{2},-a\,x^2\right)-c_1\right)(a\,x^2+b)\,\mathrm{e}^{-\frac{a\,x^2}{2}}}{2}+\mathrm{e}^{\frac{a\,x^2}{2}}\left(-a\,x^2\right)^{b-\frac{1}{2}}c_2a\,x^2\right)x^{-1-b}$$

Using the above in (1) gives the solution

$$y=\frac{2\left(-\frac{\left(-c_2\Gamma\left(b+\frac{1}{2}\right)+c_2\Gamma\left(b+\frac{1}{2},-a\,x^2\right)-c_1\right)(a\,x^2+b)\mathrm{e}^{-\frac{a\,x^2}{2}}}{2}+\mathrm{e}^{\frac{a\,x^2}{2}}\left(-a\,x^2\right)^{b-\frac{1}{2}}c_2a\,x^2\right)x^{-1-b}x^b\mathrm{e}^{\frac{a\,x^2}{2}}}{c_2\Gamma\left(b+\frac{1}{2}\right)-c_2\Gamma\left(b+\frac{1}{2},-a\,x^2\right)+c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1}=c_3$ the following solution

$$y=\frac{-2x^2a(-a\,x^2)^{b-\frac{1}{2}}\,\mathrm{e}^{a\,x^2}+(a\,x^2+b)\left(-\Gamma\left(b+\frac{1}{2}\right)+\Gamma\left(b+\frac{1}{2},-a\,x^2\right)-c_3\right)}{x\left(-\Gamma\left(b+\frac{1}{2}\right)+\Gamma\left(b+\frac{1}{2},-a\,x^2\right)-c_3\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{-2x^2 a(-a\,x^2)^{b-\frac{1}{2}}\,\mathrm{e}^{a\,x^2} + (a\,x^2+b)\left(-\Gamma\left(b+\frac{1}{2}\right)+\Gamma\left(b+\frac{1}{2}, -a\,x^2\right)-c_3\right)}{x\left(-\Gamma\left(b+\frac{1}{2}\right)+\Gamma\left(b+\frac{1}{2}, -a\,x^2\right)-c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-2x^2 a(-a\,x^2)^{b-\frac{1}{2}}\,\mathrm{e}^{a\,x^2} + (a\,x^2+b)\left(-\Gamma\left(b+\frac{1}{2}\right)+\Gamma\left(b+\frac{1}{2}, -a\,x^2\right)-c_3\right)}{x\left(-\Gamma\left(b+\frac{1}{2}\right)+\Gamma\left(b+\frac{1}{2}, -a\,x^2\right)-c_3\right)}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a^2*x^4+2*a*b*x^2-a*x^2+b^2+b
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Reducible group (found an exponential solution)
         Group is reducible, not completely reducible
      <- Kovacics algorithm successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 84

```
dsolve(x^2*diff(y(x),x)=x^2*y(x)^2-a^2*x^4+a*(1-2*b)*x^2-b*(b+1),y(x), singsol=all)
```

$$y(x) = \frac{-2(-a\,x^2)^{b-\frac{1}{2}}\,\mathrm{e}^{a\,x^2}c_1 a\,x^2 + \left(-c_1\Gamma\left(b+\frac{1}{2}\right) + c_1\Gamma\left(b+\frac{1}{2}, -a\,x^2\right) - 1\right)\left(a\,x^2+b\right)}{x\left(-c_1\Gamma\left(b+\frac{1}{2}\right) + c_1\Gamma\left(b+\frac{1}{2}, -a\,x^2\right) - 1\right)}$$

✓ Solution by Mathematica

Time used: 1.225 (sec). Leaf size: 128

```
DSolve[x^2*y'[x]==x^2*y[x]^2-a^2*x^4+a*(1-2*b)*x^2-b*(b+1),y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{x^{2b+1}(ax^2+b)\,\Gamma\left(b+\frac{1}{2},-ax^2\right)-2(-ax^2)^{b+\frac{1}{2}}\left(-e^{ax^2}x^{2b+1}+c_1(ax^2+b)\right)}{x^{2b+2}\Gamma\left(b+\frac{1}{2},-ax^2\right)-2c_1x\left(-ax^2\right)^{b+\frac{1}{2}}}$$

$$y(x) \to ax+\frac{b}{x}$$

2.15 problem 15

Internal problem ID [10345]
Internal file name [`OUTPUT/9292_Monday_June_06_2022_01_49_11_PM_80921583/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{x^2y' - a\,x^2y^2 = b\,x^n + c}$$

2.15.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,x^2y^2 + b\,x^n + c}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,y^2 + \frac{b\,x^n}{x^2} + \frac{c}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{b\,x^n + c}{x^2}$, $f_1(x) = 0$ and $f_2(x) = a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= \frac{a^2(b\,x^n + c)}{x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$au''(x) + \frac{a^2(b\,x^n + c)\,u(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\text{BesselY}\left(\frac{\sqrt{-4ca+1}}{n}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)c_2 + \text{BesselJ}\left(\frac{\sqrt{-4ca+1}}{n}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)c_1\right)\sqrt{x}$$

The above shows that

$$u'(x) = \frac{-2\sqrt{ab}\left(\text{BesselJ}\left(\frac{\sqrt{-4ca+1}+n}{n}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)c_1 + \text{BesselY}\left(\frac{\sqrt{-4ca+1}+n}{n}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)c_2\right)x^{\frac{n}{2}} + \left(\text{BesselY}\left(\frac{\sqrt{-4ca+1}}{n},\right.\right.}{2\sqrt{x}}$$

Using the above in (1) gives the solution

$$y = -\frac{-2\sqrt{ab}\left(\text{BesselJ}\left(\frac{\sqrt{-4ca+1}+n}{n}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)c_1 + \text{BesselY}\left(\frac{\sqrt{-4ca+1}+n}{n}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)c_2\right)x^{\frac{n}{2}} + \left(\text{BesselY}\left(\frac{\sqrt{-4ca+}}{n}\right.\right.}{2xa\left(\text{BesselY}\left(\frac{\sqrt{-4ca+1}}{n}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)c_2 + \text{BesselJ}\left(\frac{\sqrt{-4c}}{n}\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2\sqrt{ab}\left(\text{BesselJ}\left(\frac{\sqrt{-4ca+1}}{n} + 1, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)c_3 + \text{BesselY}\left(\frac{\sqrt{-4ca+1}}{n} + 1, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)\right)x^{\frac{n}{2}} - \left(\text{BesselY}\left(\frac{\sqrt{-4ca+1}}{n},\right.\right.}{2xa\left(\text{BesselY}\left(\frac{\sqrt{-4ca+1}}{n}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right) + \text{BesselJ}\left(\frac{\sqrt{-4ca+1}}{n},\right.\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{2\sqrt{ab}\left(\text{BesselJ}\left(\frac{\sqrt{-4ca+1}}{n}+1, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)c_3 + \text{BesselY}\left(\frac{\sqrt{-4ca+1}}{n}+1, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)\right)x^{\frac{n}{2}} - \left(\text{BesselY}\left(\frac{\sqrt{-4ca+1}}{n}, \cdots\right.\right.}{2xa\left(\text{BesselY}\left(\frac{\sqrt{-4ca+1}}{n}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right) + \text{BesselJ}\left(\frac{\sqrt{-4ca+1}}{n}, \cdots\right.\right.} \tag{1}$$

Verification of solutions

$$y = \frac{2\sqrt{ab}\left(\text{BesselJ}\left(\frac{\sqrt{-4ca+1}}{n}+1, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)c_3 + \text{BesselY}\left(\frac{\sqrt{-4ca+1}}{n}+1, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)\right)x^{\frac{n}{2}} - \left(\text{BesselY}\left(\frac{\sqrt{-4ca+1}}{n}, \cdots\right.\right.}{2xa\left(\text{BesselY}\left(\frac{\sqrt{-4ca+1}}{n}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right) + \text{BesselJ}\left(\frac{\sqrt{-4ca+1}}{n}, \cdots\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -a*(x^(n-2)*b*x^2+c)*y(x)/x^2,
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         <- Bessel successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 220

```
dsolve(x^2*diff(y(x),x)=a*x^2*y(x)^2+b*x^n+c,y(x), singsol=all)
```

$$y(x) = \frac{2\sqrt{ab}\left(\mathrm{BesselY}\left(\frac{\sqrt{-4ac+1}}{n}+1, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)c_1 + \mathrm{BesselJ}\left(\frac{\sqrt{-4ac+1}}{n}+1, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)\right)x^{\frac{n}{2}} - \left(\sqrt{-4ac+1}+1\right)\left(\mathrm{B}\right.}{2xa\left(\mathrm{BesselY}\left(\frac{\sqrt{-4ac+1}}{n}, \frac{2\sqrt{ab}\,x^{\frac{n}{2}}}{n}\right)c_1 + \mathrm{BesselJ}\left(\frac{\sqrt{-4ac+1}}{n}\right.\right.}$$

✓ Solution by Mathematica

Time used: 1.898 (sec). Leaf size: 1779

```
DSolve[x^2*y'[x]==a*x^2*y[x]^2+b*x^n+c,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-a^{\frac{i\sqrt{4ac-1}}{n}+\frac{1}{2}}n^{\frac{2\sqrt{(1-4ac)n^2}}{n^2}+1}(x^n)^{\frac{i\sqrt{4ac-1}}{n}+1}\,\mathrm{BesselJ}\left(\frac{\sqrt{(1-4ac)n^2}}{n^2}-1, \frac{2\sqrt{a}\sqrt{b}\sqrt{x^n}}{n}\right)\mathrm{Gamma}\left(\frac{n+\sqrt{1-4ac}}{n}\right)b^{\frac{i\sqrt{4ac-1}}{n}\cdots}}{\cdots}$$

$$y(x) \to \frac{\frac{\sqrt{a}\sqrt{b}\sqrt{x^n}\left(\mathrm{BesselJ}\left(1-\frac{\sqrt{(1-4ac)n^2}}{n^2}, \frac{2\sqrt{a}\sqrt{b}\sqrt{x^n}}{n}\right)-\mathrm{BesselJ}\left(-\frac{\sqrt{(1-4ac)n^2}}{n^2}-1, \frac{2\sqrt{a}\sqrt{b}\sqrt{x^n}}{n}\right)\right)}{\mathrm{BesselJ}\left(-\frac{\sqrt{(1-4ac)n^2}}{n^2}, \frac{2\sqrt{a}\sqrt{b}\sqrt{x^n}}{n}\right)} - \frac{\sqrt{n^2(1-4ac)}}{n} + i\sqrt{4ac-1} - 1}{2ax}$$

2.16 problem 16

Internal problem ID [10346]
Internal file name [OUTPUT/9293_Monday_June_06_2022_01_49_12_PM_46893412/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{x^2y' - x^2y^2 = a\,x^{2m}(b\,x^m + c)^n - \frac{n^2}{4} + \frac{1}{4}}$$

2.16.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{4x^2y^2 + 4a\,x^{2m}(b\,x^m + c)^n - n^2 + 1}{4x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \frac{a\,x^{2m}(b\,x^m + c)^n}{x^2} - \frac{n^2}{4x^2} + \frac{1}{4x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{4a\,x^{2m}(b\,x^m+c)^n - n^2+1}{4x^2}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= \frac{4a\,x^{2m}(b\,x^m + c)^n - n^2 + 1}{4x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{(4a\,x^{2m}(b\,x^m + c)^n - n^2 + 1)\,u(x)}{4x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) + \frac{(4a\,x^{2m}(b\,x^m + c)^n - n^2 + 1)\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{(4a\,x^{2m}(b\,x^m + c)^n - n^2 + 1)\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{(4a\,x^{2m}(b\,x^m+c)^n - n^2+1)\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{(4a\,x^{2m}(b\,x^m+c)^n - n^2+1)\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{4x^{2m}(b\,x^m+c)^n\_Y(x)a + 4\_Y''(x)x^2 - \_Y(x)n^2 + \_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{4x^{2m}(b\,x^m+c)^n\_Y(x)a + 4\_Y''(x)x^2 - \_Y(x)n^2 + \_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{4x^{2m}(b\,x^m+c)^n\_\_Y(x)a+4\_\_Y''(x)x^2-\_\_Y(x)n^2+\_\_Y(x)}{4x^2}\right\},\{\_\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{4x^{2m}(b\,x^m+c)^n\_\_Y(x)a+4\_\_Y''(x)x^2-\_\_Y(x)n^2+\_\_Y(x)}{4x^2}\right\},\{\_\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{4x^{2m}(b\,x^m+c)^n\_\_Y(x)a+4\_\_Y''(x)x^2-\_\_Y(x)n^2+\_\_Y(x)}{4x^2}\right\},\{\_\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{4x^{2m}(b\,x^m+c)^n\_\_Y(x)a+4\_\_Y''(x)x^2-\_\_Y(x)n^2+\_\_Y(x)}{4x^2}\right\},\{\_\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(1/4)*(4*x^(2*m-2)*(b*x^m+c)^
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(y(x)^2+y(x)+x^2*(x^(2*m-2)*a*(b*x^m+c)^n
      Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(x^2*diff(y(x),x)=x^2*y(x)^2+a*x^(2*m)*(b*x^m+c)^n+1/4*(1-n^2),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*y'[x]==x^2*y[x]^2+a*x^(2*m)*(b*x^m+c)^n+1/4*(1-n^2),y[x],x,IncludeSingularSolutio
```

Not solved

2.17 problem 17

Internal problem ID [10347]
Internal file name [`OUTPUT/9294_Monday_June_06_2022_01_49_17_PM_87108479/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{(c_2x^2+b_2x+a_2)\left(y'+\lambda y^2\right)=-a_0}$$

2.17.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= -\frac{c_2\lambda\,x^2y^2+y^2b_2\lambda x+y^2a_2\lambda+a_0}{c_2x^2+b_2x+a_2}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2c_2\lambda\,x^2}{c_2x^2+b_2x+a_2}-\frac{y^2b_2\lambda x}{c_2x^2+b_2x+a_2}-\frac{y^2a_2\lambda}{c_2x^2+b_2x+a_2}-\frac{a_0}{c_2x^2+b_2x+a_2}$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=-\frac{a_0}{c_2x^2+b_2x+a_2}$, $f_1(x)=0$ and $f_2(x)=-\frac{c_2\lambda\,x^2+\lambda b_2x+\lambda a_2}{c_2x^2+b_2x+a_2}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2u}\\ &= \frac{-u'}{-\frac{(c_2\lambda\,x^2+\lambda b_2x+\lambda a_2)u}{c_2x^2+b_2x+a_2}}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = -\frac{2c_2\lambda x + \lambda b_2}{c_2 x^2 + b_2 x + a_2} + \frac{(c_2\lambda\, x^2 + \lambda b_2 x + \lambda a_2)\,(2c_2 x + b_2)}{(c_2 x^2 + b_2 x + a_2)^2}$$

$$f_1 f_2 = 0$$

$$f_2^2 f_0 = -\frac{(c_2\lambda\, x^2 + \lambda b_2 x + \lambda a_2)^2\, a_0}{(c_2 x^2 + b_2 x + a_2)^3}$$

Substituting the above terms back in equation (2) gives

$$-\frac{(c_2\lambda\, x^2 + \lambda b_2 x + \lambda a_2)\, u''(x)}{c_2 x^2 + b_2 x + a_2} - \left(-\frac{2c_2\lambda x + \lambda b_2}{c_2 x^2 + b_2 x + a_2} + \frac{(c_2\lambda\, x^2 + \lambda b_2 x + \lambda a_2)\,(2c_2 x + b_2)}{(c_2 x^2 + b_2 x + a_2)^2}\right) u'(x) - \frac{(c_2\lambda\, x}{}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = -2\left(c_2 x - \frac{\sqrt{-4a_2c_2 + b_2^2}}{2} + \frac{b_2}{2}\right)\left(c_3\left(2c_2 x + b_2\right.\right.$$
$$\left.+\sqrt{-4a_2c_2 + b_2^2}\right)^{\frac{-\sqrt{c_2}+\sqrt{-4a_0\lambda+c_2}}{2\sqrt{c_2}}} \text{hypergeom}\left(\left[-\frac{-\sqrt{c_2}+\sqrt{-4a_0\lambda+c_2}}{2\sqrt{c_2}}, -\frac{-3\sqrt{c_2}+\sqrt{-4a_0\lambda+c_2}}{2\sqrt{c_2}}\right], \left[\right.\right.$$
$$+ c_4\left(2c_2 x + b_2\right.$$
$$\left.+\sqrt{-4a_2c_2 + b_2^2}\right)^{-\frac{\sqrt{c_2}+\sqrt{-4a_0\lambda+c_2}}{2\sqrt{c_2}}} \text{hypergeom}\left(\left[\frac{\sqrt{c_2}+\sqrt{-4a_0\lambda+c_2}}{2\sqrt{c_2}}, \frac{3\sqrt{c_2}+\sqrt{-4a_0\lambda+c_2}}{2\sqrt{c_2}}\right], \left[\frac{\sqrt{c_2}+}{}\right.\right.$$

The above shows that

$$u'(x) =$$
$$-\frac{4\sqrt{c_2}\left(\left(2c_2 x + b_2 + \sqrt{-4a_2c_2 + b_2^2}\right)^{\frac{-\sqrt{c_2}+\sqrt{-4a_0\lambda+c_2}}{2\sqrt{c_2}}}\left(\left(\left(c_2 x + \frac{b_2}{2}\right)\sqrt{-4a_0\lambda + c_2} + c_2^{\frac{3}{2}} x + \frac{\sqrt{c_2}\, b_2}{2}\right)\sqrt{-4\right.\right.\right.}{}$$

Using the above in (1) gives the solution

Expression too large to display

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{4\left(\operatorname{hypergeom}\left(\left[\frac{1}{2}+\frac{\sqrt{-4a_0\lambda+1}}{2}, \frac{1}{2}+\frac{\sqrt{-4a_0\lambda+1}}{2}\right], \left[1+\sqrt{-4a_0\lambda+1}\right], \frac{2\sqrt{b_2^2-4a_2}}{2x+b_2+\sqrt{b_2^2-4a_2}}\right) c_4\left(-1+\sqrt{-4a_0\lambda\right.\right.}{\left(2x+b_2+\sqrt{b_2^2-4a_2}\right)^2\left(-\sqrt{b_2^2-4a_2}+2x+b_2\right)\lambda\left(c_3\left(2x+b_2+\sqrt{b_2^2-4a_2}\right)^{\sqrt{-4}}\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{4\left(\operatorname{hypergeom}\left(\left[\frac{1}{2}+\frac{\sqrt{-4a_0\lambda+1}}{2}, \frac{1}{2}+\frac{\sqrt{-4a_0\lambda+1}}{2}\right], \left[1+\sqrt{-4a_0\lambda+1}\right], \frac{2\sqrt{b_2^2-4a_2}}{2x+b_2+\sqrt{b_2^2-4a_2}}\right) c_4\left(-1+\sqrt{-4a_0\lambda\right.\right.}{\left(2x+b_2+\sqrt{b_2^2-4a_2}\right)^2\left(-\sqrt{b_2^2-4a_2}+2x+b_2\right)\lambda\left(c_3\left(2x+b_2+\sqrt{b_2^2-4a_2}\right)^{\sqrt{-4}}\right.} \quad (1)$$

Verification of solutions

$$y = -\frac{4\left(\operatorname{hypergeom}\left(\left[\frac{1}{2}+\frac{\sqrt{-4a_0\lambda+1}}{2}, \frac{1}{2}+\frac{\sqrt{-4a_0\lambda+1}}{2}\right], \left[1+\sqrt{-4a_0\lambda+1}\right], \frac{2\sqrt{b_2^2-4a_2}}{2x+b_2+\sqrt{b_2^2-4a_2}}\right) c_4\left(-1+\sqrt{-4a_0\lambda\right.\right.}{\left(2x+b_2+\sqrt{b_2^2-4a_2}\right)^2\left(-\sqrt{b_2^2-4a_2}+2x+b_2\right)\lambda\left(c_3\left(2x+b_2+\sqrt{b_2^2-4a_2}\right)^{\sqrt{-4}}\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -a__0*lambda*y(x)/(c__2*x^2+b_
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 1127

```
dsolve((c__2*x^2+b__2*x+a__2)*(diff(y(x),x)+lambda*y(x)^2)+a__0=0,y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 5.964 (sec). Leaf size: 1046

```
DSolve[(c2*x^2+b2*x+a2)*(y'[x]+\[Lambda]*y[x]^2)+a0==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{(\mathrm{b2}+2\mathrm{c2}x)\left(8\mathrm{c2}\left(\mathrm{b2}^2-4\mathrm{a2c2}\right)G_{2,2}^{2,0}\left(-\frac{4\mathrm{c2}(\mathrm{a2}+x(\mathrm{b2}+\mathrm{c2}x))}{\mathrm{b2}^2-4\mathrm{a2c2}}\middle| \begin{array}{c} \frac{1}{4}-\frac{\sqrt{\mathrm{c2}-4\mathrm{a0}\lambda}}{4\sqrt{\mathrm{c2}}},\frac{1}{4}\left(\frac{\sqrt{\mathrm{c2}-4\mathrm{a0}\lambda}}{\sqrt{\mathrm{c2}}}+1\right) \\ 0,0 \end{array}\right)+c_1\left(\right.}{2\lambda\left(\mathrm{b2}^2\right.}$$

$$y(x) \to \frac{(\mathrm{b2}+2\mathrm{c2}x)\left(2\left(\mathrm{b2}^2-4\mathrm{a2c2}\right)\mathrm{Hypergeometric2F1}\left(\frac{3\mathrm{c2}+\sqrt{\mathrm{c2}(\mathrm{c2}-4\mathrm{a0}\lambda)}}{4\mathrm{c2}},\frac{1}{4}\left(3-\frac{\sqrt{\mathrm{c2}(\mathrm{c2}-4\mathrm{a0}\lambda)}}{\mathrm{c2}}\right),2,-\frac{4\mathrm{c2}}{}\right.\right.}{2\lambda\left(\mathrm{b2}^2-4\mathrm{a2c2}\right)(\mathrm{a2}+x(\mathrm{b2}+\mathrm{c2}x))\,\mathrm{Hyper}}$$

$$y(x) \to \frac{(\mathrm{b2}+2\mathrm{c2}x)\left(2\left(\mathrm{b2}^2-4\mathrm{a2c2}\right)\mathrm{Hypergeometric2F1}\left(\frac{3\mathrm{c2}+\sqrt{\mathrm{c2}(\mathrm{c2}-4\mathrm{a0}\lambda)}}{4\mathrm{c2}},\frac{1}{4}\left(3-\frac{\sqrt{\mathrm{c2}(\mathrm{c2}-4\mathrm{a0}\lambda)}}{\mathrm{c2}}\right),2,-\frac{4\mathrm{c2}}{}\right.\right.}{2\lambda\left(\mathrm{b2}^2-4\mathrm{a2c2}\right)(\mathrm{a2}+x(\mathrm{b2}+\mathrm{c2}x))\,\mathrm{Hyper}}$$

2.18 problem 18

Internal problem ID [10348]
Internal file name [`OUTPUT/9295_Monday_June_06_2022_01_49_21_PM_53621118/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, [_Riccati, _special]]
```

$$\boxed{x^4y' + x^4y^2 = -a^2}$$

2.18.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x^4y^2 + a^2}{x^4}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2a_4 + yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = x^2b_4 + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &2xb_4 + yb_5 + b_2 - \frac{(x^4y^2 + a^2)(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{x^4} \\ &- \frac{(x^4y^2 + a^2)^2 (xa_5 + 2ya_6 + a_3)}{x^8} \\ &- \left(-\frac{4y^2}{x} + \frac{4x^4y^2 + 4a^2}{x^5}\right)(x^2a_4 + yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1) \\ &+ 2y\left(x^2b_4 + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1\right) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^9y^4a_5 + 2x^8y^5a_6 + x^8y^4a_3 - 2x^{10}yb_4 - 2x^9y^2a_4 - x^9y^2b_5 - x^8y^3a_5 - 2x^9yb_2 - x^8y^2a_2 - x^8y^2b_3 + 2a^2x^5}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^9y^4a_5 - 2x^8y^5a_6 - x^8y^4a_3 + 2x^{10}yb_4 + 2x^9y^2a_4 + x^9y^2b_5 + x^8y^3a_5 + 2x^9yb_2 \\ &+ x^8y^2a_2 + x^8y^2b_3 - 2a^2x^5y^2a_5 - 4a^2x^4y^3a_6 + 2x^9b_4 + 2x^8yb_1 + yb_5x^8 \\ &- 2a^2x^4y^2a_3 + b_2x^8 - 2a^2x^5a_4 - a^2x^5b_5 - 3a^2x^4ya_5 - 2a^2x^4yb_6 - 4a^2x^3y^2a_6 \\ &- 3a^2x^4a_2 - a^2x^4b_3 - 4a^2x^3ya_3 - a^4xa_5 - 2a^4ya_6 - 4a^2x^3a_1 - a^4a_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-a_5v_1^9v_2^4 - 2a_6v_1^8v_2^5 - a_3v_1^8v_2^4 + 2a_4v_1^9v_2^2 + a_5v_1^8v_2^3 + 2b_4v_1^{10}v_2 + b_5v_1^9v_2^2 + a_2v_1^8v_2^2 \\ &+ 2b_2v_1^9v_2 + b_3v_1^8v_2^2 - 2a^2a_5v_1^5v_2^2 - 4a^2a_6v_1^4v_2^3 + 2b_1v_1^8v_2 + 2b_4v_1^9 + b_5v_1^8v_2 \\ &- 2a^2a_3v_1^4v_2^2 + b_2v_1^8 - 2a^2a_4v_1^5 - 3a^2a_5v_1^4v_2 - 4a^2a_6v_1^3v_2^2 - a^2b_5v_1^5 - 2a^2b_6v_1^4v_2 \\ &- 3a^2a_2v_1^4 - 4a^2a_3v_1^3v_2 - a^2b_3v_1^4 - a^4a_5v_1 - 2a^4a_6v_2 - 4a^2a_1v_1^3 - a^4a_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &2b_4v_1^{10}v_2 - a_5v_1^9v_2^4 + (2a_4 + b_5)\,v_1^9v_2^2 + 2b_2v_1^9v_2 + 2b_4v_1^9 - 2a_6v_1^8v_2^5 \\ &- a_3v_1^8v_2^4 + a_5v_1^8v_2^3 + (a_2 + b_3)\,v_1^8v_2^2 + (2b_1 + b_5)\,v_1^8v_2 + b_2v_1^8 \\ &- 2a^2a_5v_1^5v_2^2 + \left(-2a^2a_4 - a^2b_5\right)v_1^5 - 4a^2a_6v_1^4v_2^3 - 2a^2a_3v_1^4v_2^2 \\ &+ \left(-3a^2a_5 - 2a^2b_6\right)v_1^4v_2 + \left(-3a^2a_2 - a^2b_3\right)v_1^4 - 4a^2a_6v_1^3v_2^2 \\ &- 4a^2a_3v_1^3v_2 - 4a^2a_1v_1^3 - a^4a_5v_1 - 2a^4a_6v_2 - a^4a_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_5 &= 0 \\ b_2 &= 0 \\ -a_3 &= 0 \\ -a_5 &= 0 \\ -2a_6 &= 0 \\ 2b_2 &= 0 \\ 2b_4 &= 0 \\ -4a^2a_1 &= 0 \\ -4a^2a_3 &= 0 \\ -2a^2a_3 &= 0 \\ -2a^2a_5 &= 0 \\ -4a^2a_6 &= 0 \\ -a^4a_3 &= 0 \\ -a^4a_5 &= 0 \\ -2a^4a_6 &= 0 \\ a_2 + b_3 &= 0 \\ 2a_4 + b_5 &= 0 \\ 2b_1 + b_5 &= 0 \\ -3a^2a_2 - a^2b_3 &= 0 \\ -2a^2a_4 - a^2b_5 &= 0 \\ -3a^2a_5 - 2a^2b_6 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= b_1 \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0 \\
b_4 &= 0 \\
b_5 &= -2b_1 \\
b_6 &= 0
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x^2 \\
\eta &= -2yx + 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -2yx + 1 - \left(-\frac{x^4y^2 + a^2}{x^4}\right)\left(x^2\right) \\
&= \frac{x^4y^2 - 2y\,x^3 + a^2 + x^2}{x^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^4y^2 - 2y\,x^3 + a^2 + x^2}{x^2}} dy \end{aligned}$$

Which results in

$$S = \frac{\arctan\left(\frac{2x^4y - 2x^3}{2a\,x^2}\right)}{a}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x^4y^2 + a^2}{x^4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2yx - 1}{x^4y^2 - 2y\,x^3 + a^2 + x^2} \\ S_y &= \frac{x^2}{x^4y^2 - 2y\,x^3 + a^2 + x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\arctan\left(\frac{x(xy-1)}{a}\right)}{a} = \frac{1}{x} + c_1$$

Which simplifies to

$$\frac{\arctan\left(\frac{x(xy-1)}{a}\right)}{a} = \frac{1}{x} + c_1$$

Which gives

$$y = \frac{\tan\left(\frac{a(c_1x+1)}{x}\right)a + x}{x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{\tan\left(\frac{a(c_1x+1)}{x}\right)a + x}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{\tan\left(\frac{a(c_1x+1)}{x}\right)a + x}{x^2}$$

Verified OK.

2.18.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{x^4y^2 + a^2}{x^4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y^2 - \frac{a^2}{x^4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{a^2}{x^4}$, $f_1(x) = 0$ and $f_2(x) = -1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -\frac{a^2}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) - \frac{a^2 u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x\left(c_1 \sin\left(\frac{a}{x}\right) + c_2 \cos\left(\frac{a}{x}\right)\right)$$

The above shows that

$$u'(x) = c_1 \sin\left(\frac{a}{x}\right) + c_2 \cos\left(\frac{a}{x}\right) + \frac{a\left(-c_1 \cos\left(\frac{a}{x}\right) + c_2 \sin\left(\frac{a}{x}\right)\right)}{x}$$

Using the above in (1) gives the solution

$$y = \frac{c_1 \sin\left(\frac{a}{x}\right) + c_2 \cos\left(\frac{a}{x}\right) + \frac{a\left(-c_1 \cos\left(\frac{a}{x}\right) + c_2 \sin\left(\frac{a}{x}\right)\right)}{x}}{x\left(c_1 \sin\left(\frac{a}{x}\right) + c_2 \cos\left(\frac{a}{x}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-a c_3 + x) \cos\left(\frac{a}{x}\right) + \sin\left(\frac{a}{x}\right)(c_3 x + a)}{x^2 \left(c_3 \sin\left(\frac{a}{x}\right) + \cos\left(\frac{a}{x}\right)\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{(-a c_3 + x) \cos\left(\frac{a}{x}\right) + \sin\left(\frac{a}{x}\right)(c_3 x + a)}{x^2 \left(c_3 \sin\left(\frac{a}{x}\right) + \cos\left(\frac{a}{x}\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{(-a c_3 + x) \cos\left(\frac{a}{x}\right) + \sin\left(\frac{a}{x}\right)(c_3 x + a)}{x^2 \left(c_3 \sin\left(\frac{a}{x}\right) + \cos\left(\frac{a}{x}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(x^4*diff(y(x),x)=-x^4*y(x)^2-a^2,y(x), singsol=all)
```

$$y(x) = \frac{-a\tan\left(\frac{a(c_1x-1)}{x}\right)+x}{x^2}$$

✓ Solution by Mathematica

Time used: 1.107 (sec). Leaf size: 87

```
DSolve[x^4*y'[x]==-x^4*y[x]^2-a^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-2ia^2c_1e^{\frac{2ia}{x}}+2ac_1xe^{\frac{2ia}{x}}+a-ix}{x^2\left(2ac_1e^{\frac{2ia}{x}}-i\right)}$$
$$y(x) \to \frac{x-ia}{x^2}$$

2.19 problem 19

Internal problem ID [10349]
Internal file name [`OUTPUT/9296_Monday_June_06_2022_01_49_22_PM_49385801/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{a\,x^2(x-1)^2\left(y'+\lambda y^2\right)=-b\,x^2-cx-s}$$

2.19.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y'&=F(x,y)\\&=-\frac{y^2a\lambda\,x^4-2y^2a\lambda\,x^3+y^2a\lambda\,x^2+b\,x^2+cx+s}{a\,x^2\,(x-1)^2}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=-\frac{x^2y^2\lambda}{(x-1)^2}+\frac{2x\,y^2\lambda}{(x-1)^2}-\frac{y^2\lambda}{(x-1)^2}-\frac{b}{a\,(x-1)^2}-\frac{c}{ax\,(x-1)^2}-\frac{s}{a\,x^2\,(x-1)^2}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x) = -\frac{b\,x^2+cx+s}{a\,x^2(x-1)^2}$, $f_1(x) = 0$ and $f_2(x) = -\frac{\lambda a\,x^4-2\lambda a\,x^3+\lambda a\,x^2}{a\,x^2(x-1)^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{(\lambda a\,x^4-2\lambda a\,x^3+\lambda a\,x^2)u}{a\,x^2(x-1)^2}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
f_2' = -\frac{4\lambda a\,x^3 - 6\lambda a\,x^2 + 2a\lambda x}{a\,x^2\,(x-1)^2} + \frac{2\lambda a\,x^4 - 4\lambda a\,x^3 + 2\lambda a\,x^2}{a\,x^3\,(x-1)^2} + \frac{2\lambda a\,x^4 - 4\lambda a\,x^3 + 2\lambda a\,x^2}{a\,x^2\,(x-1)^3}
$$

$$
f_1 f_2 = 0
$$

$$
f_2^2 f_0 = -\frac{(\lambda a\,x^4 - 2\lambda a\,x^3 + \lambda a\,x^2)^2\,(b\,x^2 + cx + s)}{a^3 x^6\,(x-1)^6}
$$

Substituting the above terms back in equation (2) gives

$$
-\frac{(\lambda a\,x^4 - 2\lambda a\,x^3 + \lambda a\,x^2)\,u''(x)}{a\,x^2\,(x-1)^2} - \left(-\frac{4\lambda a\,x^3 - 6\lambda a\,x^2 + 2a\lambda x}{a\,x^2\,(x-1)^2} + \frac{2\lambda a\,x^4 - 4\lambda a\,x^3 + 2\lambda a\,x^2}{a\,x^3\,(x-1)^2} + \frac{2\lambda a\,x^4 - \cdots}{a\,x^2 \cdots}\right.
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
\begin{aligned}
u(x) &= (x \\
&\quad -1)^{\frac{\sqrt{a}-\sqrt{(-4b-4c-4s)\lambda+a}}{2\sqrt{a}}} \left(c_2 x^{-\frac{-\sqrt{a}+\sqrt{-4\lambda s+a}}{2\sqrt{a}}} \operatorname{hypergeom}\left(\left[-\frac{\sqrt{(-4b-4c-4s)\lambda+a}-\sqrt{a}+\sqrt{-4\lambda s+a}\cdots}{2\sqrt{a}}\right.\right.\right. \\
&\quad +c_1 x^{\frac{\sqrt{-4\lambda s+a}+\sqrt{a}}{2\sqrt{a}}} \operatorname{hypergeom}\left(\left[\frac{-\sqrt{(-4b-4c-4s)\lambda+a}+\sqrt{a}+\sqrt{-4\lambda s+a}-\sqrt{-4b\lambda+a}}{2\sqrt{a}}, \frac{-\sqrt{(-4\cdots}}{}\right.\right.
\end{aligned}
$$

The above shows that

$$
\begin{aligned}
u'(x) &= \\
&\quad -\frac{x^{\frac{\sqrt{-4\lambda s+a}+\sqrt{a}}{2\sqrt{a}}}(x-1)^{\frac{\sqrt{a}-\sqrt{(-4b-4c-4s)\lambda+a}}{2\sqrt{a}}}\left(2s\sqrt{(-4b-4c-4s)\lambda+a}+(-c-2s)\sqrt{-4\lambda s+a}+\sqrt{a}\,c\right)c_1\cdots}{\cdots}
\end{aligned}
$$

Using the above in (1) gives the solution

Expression too large to display

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

Expression too large to display

Summary

The solution(s) found are the following

Expression too large to display (1)

Verification of solutions

Expression too large to display

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(b*x^2+c*x+s)*lambda*y(x)/(a*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            <- heuristic approach successful
         <- hypergeometric successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 1087

```
dsolve(a*x^2*(x-1)^2*(diff(y(x),x)+lambda*y(x)^2)+b*x^2+c*x+s=0,y(x), singsol=all)
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[a*x^2*(x-1)^2*(y'[x]+\[Lambda]*y[x]^2)+b*x^2+c*x+s==0,y[x],x,IncludeSingularSolutions
```

Timed out

2.20 problem 20

Internal problem ID [10350]
Internal file name [`OUTPUT/9297_Monday_June_06_2022_01_49_28_PM_64106840/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{\left(a\,x^{2}+bx+c\right)^{2}\left(y'+y^{2}\right)=-A}$$

2.20.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{y^{2}a^{2}x^{4}+2y^{2}ab\,x^{3}+2a\,x^{2}y^{2}c+y^{2}b^{2}x^{2}+2y^{2}bcx+c^{2}y^{2}+A}{\left(a\,x^{2}+bx+c\right)^{2}}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi=x^{2}a_4+yxa_5+y^{2}a_6+xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=x^{2}b_4+yxb_5+y^{2}b_6+xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&2xb_4 + yb_5 + b_2 \\
&- \frac{(y^2a^2x^4 + 2y^2ab\,x^3 + 2a\,x^2y^2c + y^2b^2x^2 + 2y^2bcx + c^2y^2 + A)\,(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{(a\,x^2 + bx + c)^2} \\
&- \frac{(y^2a^2x^4 + 2y^2ab\,x^3 + 2a\,x^2y^2c + y^2b^2x^2 + 2y^2bcx + c^2y^2 + A)^2\,(xa_5 + 2ya_6 + a_3)}{(a\,x^2 + bx + c)^4} \\
&- \left(-\frac{4a^2x^3y^2 + 6ab\,x^2y^2 + 4acx\,y^2 + 2b^2x\,y^2 + 2bc\,y^2}{(a\,x^2 + bx + c)^2}\right. \\
&\left.+ \frac{2(y^2a^2x^4 + 2y^2ab\,x^3 + 2a\,x^2y^2c + y^2b^2x^2 + 2y^2bcx + c^2y^2 + A)\,(2xa + b)}{(a\,x^2 + bx + c)^3}\right)(x^2a_4 \\
&+ yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1) \\
&+ \frac{(2a^2x^4y + 4ab\,x^3y + 4ac\,x^2y + 2b^2x^2y + 4bcxy + 2c^2y)\,(x^2b_4 + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1)}{(a\,x^2 + bx + c)^2} \\
&= 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

Expression too large to display (7E)

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\text{Expression too large to display} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$12$$
$$4$$
$$-8a$$
$$-24$$
$$-24$$
$$-8a$$

$$4$$
$$-12a^2c^2a_6 - 2$$
$$-4a\,c^3a_3 - 6$$
$$-4a^3ba_3 - 4$$
$$4a^3bb_2 + 8$$
$$12ab\,c^2a_5 + 4$$
$$-12ab\,c^2a_3 - 4a\,c^3a_5 - 4$$
$$-4a^3ca_3 - 6a^2b^2a_3 - 12$$
$$4a^3cb_2 + 6a^2b^2b_2 + 24$$
$$4a\,c^3a_5 + 6b^2c^2a_5 - 8$$
$$6a^2c^2a_5 + 12a\,b^2ca_5 +$$
$$a^4a_2 + a^4b_3 +$$
$$c^4a_2 + c^4b_3 - 2$$
$$-6a^2c^2a_3 - 12a\,b^2ca_3 - 12ab\,c^2a_5$$
$$-12a^2bca_3 - 6a^2c^2a_5 - 4a\,b^3a_3 -$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -\frac{cb_3}{b} \\
a_2 &= -b_3 \\
a_3 &= 0 \\
a_4 &= -\frac{ab_3}{b} \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= -\frac{ab_3}{b} \\
b_2 &= 0 \\
b_3 &= b_3 \\
b_4 &= 0 \\
b_5 &= \frac{2ab_3}{b} \\
b_6 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{a\,x^2 + bx + c}{b} \\
\eta &= \frac{2axy + by - a}{b}
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= \frac{2axy + by - a}{b} - \left(-\frac{y^2a^2x^4 + 2y^2ab\,x^3 + 2a\,x^2y^2c + y^2b^2x^2 + 2y^2bcx + c^2y^2 + A}{(a\,x^2 + bx + c)^2}\right)\left(-\frac{a\,x^2 + bx + c}{b}\right. \\
&= \frac{-y^2a^2x^4 - 2y^2ab\,x^3 + 2a^2x^3y - 2a\,x^2y^2c - y^2b^2x^2 + 3ab\,x^2y - 2y^2bcx - a^2x^2 + 2acxy + b^2xy - c^2y^2}{ab\,x^2 + b^2x + bc} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-y^2a^2x^4-2y^2ab\,x^3+2a^2x^3y-2a\,x^2y^2c-y^2b^2x^2+3ab\,x^2y-2y^2bcx-a^2x^2+2acxy+b^2xy-c^2y^2-abx+ybc-ca-A}{ab\,x^2+b^2x+bc}} dy \end{aligned}$$

Which results in

$$S = -\frac{2(ab\,x^2+b^2x+bc)\arctan\left(\frac{2y(a^2x^4+2b\,x^3a+2ac\,x^2+b^2x^2+2bxc+c^2)-}{\sqrt{4a^3c\,x^4-a^2b^2x^4+4A\,a^2x^4+8a^2bc\,x^3-2a\,b^3x^3+8Aab\,x^3+8a^2c^2x^2+2a\,b^2c\,x^2-b}}\right.}{\sqrt{4a^3c\,x^4-a^2b^2x^4+4A\,a^2x^4+8a^2bc\,x^3-2a\,b^3x^3+8Aab\,x^3+8a^2c^2x^2+2a\,b^2c\,x^2-b^4x^2+8Aac}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y^2a^2x^4 + 2y^2ab\,x^3 + 2a\,x^2y^2c + y^2b^2x^2 + 2y^2bcx + c^2y^2 + A}{(a\,x^2+bx+c)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2(4axy+2by-2a)\,b}{(4ca-b^2+4A)\left(\frac{(2a\,x^2y+2bxy-2xa+2yc-b)^2}{4ca-b^2+4A}+1\right)} \\ S_y &= -\frac{2(2a\,x^2+2bx+2c)\,b}{(4ca-b^2+4A)\left(\frac{(2a\,x^2y+2bxy-2xa+2yc-b)^2}{4ca-b^2+4A}+1\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{b}{a\,x^2 + bx + c} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{b}{R^2 a + Rb + c}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{2b \arctan\left(\frac{2Ra+b}{\sqrt{4ca-b^2}}\right)}{\sqrt{4ca-b^2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2\arctan\left(\frac{2a\,x^2y+2ybx-2xa+2yc-b}{\sqrt{4ca-b^2+4A}}\right)b}{\sqrt{4ca-b^2+4A}} = \frac{2b\arctan\left(\frac{2xa+b}{\sqrt{4ca-b^2}}\right)}{\sqrt{4ca-b^2}} + c_1$$

Which simplifies to

$$-\frac{2\arctan\left(\frac{2a\,x^2y+2ybx-2xa+2yc-b}{\sqrt{4ca-b^2+4A}}\right)b}{\sqrt{4ca-b^2+4A}} = \frac{2b\arctan\left(\frac{2xa+b}{\sqrt{4ca-b^2}}\right)}{\sqrt{4ca-b^2}} + c_1$$

Which gives

$$y = -\frac{\tan\left(\frac{\sqrt{4ca-b^2+4A}\left(\sqrt{4ca-b^2}\,c_1+2b\arctan\left(\frac{2xa+b}{\sqrt{4ca-b^2}}\right)\right)}{2\sqrt{4ca-b^2}\,b}\right)\sqrt{4ca-b^2+4A} - 2xa - b}{2\left(a\,x^2+bx+c\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\tan\left(\frac{\sqrt{4ca-b^2+4A}\left(\sqrt{4ca-b^2}\,c_1+2b\arctan\left(\frac{2xa+b}{\sqrt{4ca-b^2}}\right)\right)}{2\sqrt{4ca-b^2}\,b}\right)\sqrt{4ca-b^2+4A} - 2xa - b}{2\left(a\,x^2+bx+c\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\tan\left(\frac{\sqrt{4ca-b^2+4A}\left(\sqrt{4ca-b^2}\,c_1+2b\arctan\left(\frac{2xa+b}{\sqrt{4ca-b^2}}\right)\right)}{2\sqrt{4ca-b^2}\,b}\right)\sqrt{4ca-b^2+4A}-2xa-b}{2\left(a\,x^2+bx+c\right)}$$

Verified OK.

2.20.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y^2a^2x^4+2y^2ab\,x^3+2a\,x^2y^2c+y^2b^2x^2+2y^2bcx+c^2y^2+A}{\left(a\,x^2+bx+c\right)^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2a^2x^4}{\left(a\,x^2+bx+c\right)^2}-\frac{2y^2ab\,x^3}{\left(a\,x^2+bx+c\right)^2}-\frac{2a\,x^2y^2c}{\left(a\,x^2+bx+c\right)^2}-\frac{y^2b^2x^2}{\left(a\,x^2+bx+c\right)^2}-\frac{2y^2bcx}{\left(a\,x^2+bx+c\right)^2}-\frac{c^2}{\left(a\,x^2+\right.}$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x) = -\frac{A}{\left(a\,x^2+bx+c\right)^2}$, $f_1(x) = 0$ and $f_2(x) = -\frac{a^2x^4+2b\,x^3a+2ac\,x^2+b^2x^2+2bxc+c^2}{\left(a\,x^2+bx+c\right)^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{-\frac{\left(a^2x^4+2b\,x^3a+2ac\,x^2+b^2x^2+2bxc+c^2\right)u}{\left(a\,x^2+bx+c\right)^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x)-\left(f_2'+f_1f_2\right)u'(x)+f_2^2f_0u(x) = 0 \tag{2}$$

But

$$f_2' = -\frac{4a^2x^3+6ab\,x^2+4acx+2b^2x+2bc}{\left(a\,x^2+bx+c\right)^2}+\frac{2\left(a^2x^4+2b\,x^3a+2ac\,x^2+b^2x^2+2bxc+c^2\right)\left(2xa+b\right)}{\left(a\,x^2+bx+c\right)^3}$$

$$f_1f_2 = 0$$

$$f_2^2f_0 = -\frac{\left(a^2x^4+2b\,x^3a+2ac\,x^2+b^2x^2+2bxc+c^2\right)^2A}{\left(a\,x^2+bx+c\right)^6}$$

Substituting the above terms back in equation (2) gives

$$-\frac{(a^2x^4+2b\,x^3a+2ac\,x^2+b^2x^2+2bxc+c^2)\,u''(x)}{(a\,x^2+bx+c)^2}-\left(-\frac{4a^2x^3+6ab\,x^2+4acx+2b^2x+2bc}{(a\,x^2+bx+c)^2}+\frac{2(a^2x^4 \cdots}{}\right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x)=\left(\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{\frac{a\sqrt{\frac{-4ca+b^2-4A}{a^2}}}{2\sqrt{-4ca+b^2}}}c_1+\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{-4ca+b^2-4A}{a^2}}}{2\sqrt{-4ca+b^2}}}c_2\right)\sqrt{a\,x^2+bx+c}$$

The above shows that

$$u'(x)=\frac{2\left(c_2\left(i\sqrt{4ca-b^2}\,\sqrt{\frac{-4ca+b^2-4A}{a^2}}\,a-2\sqrt{-4ca+b^2}\,\left(xa+\frac{b}{2}\right)\right)\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{-4ca+b^2-4A}{a^2}}}{2\sqrt{-4ca+b^2}}}-\left(i\sqrt{4ca \cdots\right.\right.}{\sqrt{-4ca+b^2}\,\left(-b+i\sqrt{4ca-b^2}- \cdots\right.}$$

Using the above in (1) gives the solution

$$y=\frac{2\left(c_2\left(i\sqrt{4ca-b^2}\,\sqrt{\frac{-4ca+b^2-4A}{a^2}}\,a-2\sqrt{-4ca+b^2}\,\left(xa+\frac{b}{2}\right)\right)\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{-4ca+b^2-4A}{a^2}}}{2\sqrt{-4ca+b^2}}}-\left(i\sqrt{4ca \cdots\right.\right.}{\sqrt{-4ca+b^2}\,\left(-b+i\sqrt{4ca-b^2}-2xa\right)\left(b+i\sqrt{4ca-b^2}+2xa\right)\left(a^2x^4+2b\,x^3a+2ac\,x^2+b^2 \cdots\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1}=c_3$ the following solution

$$
y = \frac{2\left(\left(i\sqrt{4ca-b^2}\,\sqrt{\frac{-4ca+b^2-4A}{a^2}}\,a-2\sqrt{-4ca+b^2}\,\left(xa+\frac{b}{2}\right)\right)\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{-4ca+b^2-4A}{a^2}}}{2\sqrt{-4ca+b^2}}}-\left(i\sqrt{4ca-\cdots}\right.\right.}{\sqrt{-4ca+b^2}\,\left(\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{\frac{a\sqrt{\frac{-4ca+b^2-4A}{a^2}}}{2\sqrt{-4ca+b^2}}}c_3+\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{-4ca+b^{\cdots}}{a^2}}}{2\sqrt{-4ca\cdots}}}\cdots\right.}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2\left(\left(i\sqrt{4ca-b^2}\,\sqrt{\frac{-4ca+b^2-4A}{a^2}}\,a-2\sqrt{-4ca+b^2}\,\left(xa+\frac{b}{2}\right)\right)\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{-4ca+b^2-4A}{a^2}}}{2\sqrt{-4ca+b^2}}}-\left(i\sqrt{4ca-\cdots}\right.\right.}{\sqrt{-4ca+b^2}\,\left(\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{\frac{a\sqrt{\frac{-4ca+b^2-4A}{a^2}}}{2\sqrt{-4ca+b^2}}}c_3+\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{-4ca+b^{\cdots}}{a^2}}}{2\sqrt{-4ca\cdots}}}\cdots\right.} \quad (1)
$$

Verification of solutions

$$
y = \frac{2\left(\left(i\sqrt{4ca-b^2}\,\sqrt{\frac{-4ca+b^2-4A}{a^2}}\,a-2\sqrt{-4ca+b^2}\,\left(xa+\frac{b}{2}\right)\right)\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{-4ca+b^2-4A}{a^2}}}{2\sqrt{-4ca+b^2}}}-\left(i\sqrt{4ca-\cdots}\right.\right.}{\sqrt{-4ca+b^2}\,\left(\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{\frac{a\sqrt{\frac{-4ca+b^2-4A}{a^2}}}{2\sqrt{-4ca+b^2}}}c_3+\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{-4ca+b^{\cdots}}{a^2}}}{2\sqrt{-4ca\cdots}}}\cdots\right.}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -A*y(x)/(a^2*x^4+2*a*b*x^3+2*a
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Group is reducible or imprimitive
      <- Kovacics algorithm successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 493

```
dsolve((a*x^2+b*x+c)^2*(diff(y(x),x)+y(x)^2)+A=0,y(x), singsol=all)
```

$$y(x) = \frac{2\left(c_1\left(i\sqrt{\frac{-4ac+b^2-4A}{a^2}}\,a\sqrt{4ac-b^2}-2\sqrt{-4ac+b^2}\left(\frac{b}{2}+ax\right)\right)\left(\frac{-b+i\sqrt{4ac-b^2}-2ax}{i\sqrt{4ac-b^2}+2ax+b}\right)^{-\frac{a\sqrt{\frac{-4ac+b^2-4A}{a^2}}}{2\sqrt{-4ac+b^2}}}-\left(i\sqrt{\frac{-4a}{}}\right.\right.}{\sqrt{-4ac+b^2}\left(i\sqrt{4ac-b^2}+2ax+b\right)\left(-b+i\sqrt{4ac-b^2}-2ax\right)\left(c_1\left(\frac{-b+i\sqrt{4ac-}}{i\sqrt{4ac-b^2+}}\right.\right.}$$

✓ Solution by Mathematica

Time used: 5.579 (sec). Leaf size: 743

```
DSolve[(a*x^2+b*x+c)^2*(y'[x]+y[x]^2)+A==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{b^2c_1\left(-\exp\left(\frac{2\sqrt{4ac-b^2}\sqrt{1-\frac{4A}{b^2-4ac}}\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{b^2-4ac}}\right)\right)+bc_1\sqrt{b^2-4ac}\sqrt{1-\frac{4A}{b^2-4ac}}\exp\left(\frac{2\sqrt{4ac-b^2}\sqrt{1-\frac{4A}{b^2-4ac}}\,\mathrm{a}}{\sqrt{b^2-4a}}\right.}{}$$

$$y(x) \to \frac{2ax\sqrt{b^2-4ac}\sqrt{1-\frac{4A}{b^2-4ac}}+b\sqrt{b^2-4ac}\sqrt{1-\frac{4A}{b^2-4ac}}+4ac+4A-b^2}{2\sqrt{b^2-4ac}\sqrt{1-\frac{4A}{b^2-4ac}}(x(ax+b)+c)}$$

2.21 problem 21

Internal problem ID [10351]
Internal file name [`OUTPUT/9298_Monday_June_06_2022_01_49_29_PM_68667329/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{x^{n+1}y' - x^{2n}y^2 a = c\,x^m + d}$$

2.21.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \left(x^{2n} a\, y^2 + c\, x^m + d\right) x^{-n-1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x^n a\, y^2}{x} + \frac{x^{-n} c\, x^m}{x} + \frac{x^{-n} d}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = (c\, x^m + d)\, x^{-n-1}$, $f_1(x) = 0$ and $f_2(x) = x^{2n} a\, x^{-n-1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^{2n} a\, x^{-n-1} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = \frac{2x^{2n} na\, x^{-n-1}}{x} - \frac{x^{2n} a\, x^{-n-1}(n+1)}{x}$$
$$f_1 f_2 = 0$$
$$f_2^2 f_0 = x^{4n} a^2 x^{-3n-3}(c\, x^m + d)$$

Substituting the above terms back in equation (2) gives

$$x^{2n} a\, x^{-n-1} u''(x) - \left(\frac{2x^{2n} na\, x^{-n-1}}{x} - \frac{x^{2n} a\, x^{-n-1}(n+1)}{x}\right) u'(x) + x^{4n} a^2 x^{-3n-3}(c\, x^m + d)\, u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{\frac{n}{2}} \left(\text{BesselY}\left(\frac{\sqrt{-4ad+n^2}}{m}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}}}{m}\right) c_2 + \text{BesselJ}\left(\frac{\sqrt{-4ad+n^2}}{m}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}}}{m}\right) c_1\right)$$

The above shows that

$$u'(x) = \frac{x^{-1+\frac{n}{2}} \left(-2\left(\text{BesselJ}\left(\frac{\sqrt{-4ad+n^2}}{m} + 1, \frac{2\sqrt{ca}\, x^{\frac{m}{2}}}{m}\right) c_1 + \text{BesselY}\left(\frac{\sqrt{-4ad+n^2}}{m} + 1, \frac{2\sqrt{ca}\, x^{\frac{m}{2}}}{m}\right) c_2\right) \sqrt{ca}\, x^{\frac{m}{2}} + \left(\text{Bes}\right.\right.}{2}$$

Using the above in (1) gives the solution

$$y = -\frac{x^{-1+\frac{n}{2}} \left(-2\left(\text{BesselJ}\left(\frac{\sqrt{-4ad+n^2}}{m} + 1, \frac{2\sqrt{ca}\, x^{\frac{m}{2}}}{m}\right) c_1 + \text{BesselY}\left(\frac{\sqrt{-4ad+n^2}}{m} + 1, \frac{2\sqrt{ca}\, x^{\frac{m}{2}}}{m}\right) c_2\right) \sqrt{ca}\, x^{\frac{m}{2}} + \left(\text{B}\right.\right.}{2a\left(\text{BesselY}\left(\frac{\sqrt{-4ad+n^2}}{m}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}}}{m}\right) c_2\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(-2\left(\mathrm{BesselJ}\left(\frac{\sqrt{-4ad+n^2}}{m}+1, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right)c_3 + \mathrm{BesselY}\left(\frac{\sqrt{-4ad+n^2}}{m}+1, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right)\right)\sqrt{ca}\,x^{\frac{m}{2}} + \left(\mathrm{BesselY}\left(\right.\right.\right.}{2a\left(\mathrm{BesselY}\left(\frac{\sqrt{-4ad+n^2}}{m}, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right) + \mathrm{BesselJ}\left(\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(-2\left(\mathrm{BesselJ}\left(\frac{\sqrt{-4ad+n^2}}{m}+1, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right)c_3 + \mathrm{BesselY}\left(\frac{\sqrt{-4ad+n^2}}{m}+1, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right)\right)\sqrt{ca}\,x^{\frac{m}{2}} + \left(\mathrm{BesselY}\left(\right.\right.\right.}{2a\left(\mathrm{BesselY}\left(\frac{\sqrt{-4ad+n^2}}{m}, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right) + \mathrm{BesselJ}\left(\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(-2\left(\mathrm{BesselJ}\left(\frac{\sqrt{-4ad+n^2}}{m}+1, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right)c_3 + \mathrm{BesselY}\left(\frac{\sqrt{-4ad+n^2}}{m}+1, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right)\right)\sqrt{ca}\,x^{\frac{m}{2}} + \left(\mathrm{BesselY}\left(\right.\right.\right.}{2a\left(\mathrm{BesselY}\left(\frac{\sqrt{-4ad+n^2}}{m}, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right) + \mathrm{BesselJ}\left(\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (n-1)*(diff(y(x), x))/x-x^(n-1
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         <- Bessel successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 235

```
dsolve(x^(n+1)*diff(y(x),x)=a*x^(2*n)*y(x)^2+c*x^m+d,y(x), singsol=all)
```

$$y(x) = -\frac{x^{-n}\left(-2\sqrt{ac}\left(\text{BesselY}\left(\frac{\sqrt{-4ad+n^2}}{m}+1, \frac{2\sqrt{ac}\,x^{\frac{m}{2}}}{m}\right)c_1 + \text{BesselJ}\left(\frac{\sqrt{-4ad+n^2}}{m}+1, \frac{2\sqrt{ac}\,x^{\frac{m}{2}}}{m}\right)\right)x^{\frac{m}{2}} + \left(\text{Bessel}\right.\right.}{2a\left(\text{BesselY}\left(\frac{\sqrt{-4ad+n^2}}{m}, \frac{2\sqrt{ac}\,x^{\frac{m}{2}}}{m}\right)c_1 + \text{Bessel}\right.}$$

✓ Solution by Mathematica

Time used: 2.124 (sec). Leaf size: 1890

```
DSolve[x^(n+1)*y'[x]==a*x^(2*n)*y[x]^2+c*x^m+d,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^{-n}\left(a^{\frac{\sqrt{n^2-4ad}}{m}} m^{\frac{2\sqrt{m^2(n^2-4ad)}}{m^2}}\left(\sqrt{m^2(n^2-4ad)} - m\left(n+\sqrt{n^2-4ad}\right)\right)(x^m)^{\frac{\sqrt{n^2-4ad}}{m}+\frac{1}{2}}\text{BesselJ}\left(\frac{\sqrt{m^2(n^2-}}{m^2}\right.\right.}{}$$

$$y(x) \to \frac{x^{-n}\left(\frac{\sqrt{a}\sqrt{c}\sqrt{x^m}\left(\text{BesselJ}\left(1-\frac{\sqrt{m^2(n^2-4ad)}}{m^2}, \frac{2\sqrt{a}\sqrt{c}\sqrt{x^m}}{m}\right) - \text{BesselJ}\left(-\frac{\sqrt{m^2(n^2-4ad)}}{m^2}-1, \frac{2\sqrt{a}\sqrt{c}\sqrt{x^m}}{m}\right)\right)}{\text{BesselJ}\left(-\frac{\sqrt{m^2(n^2-4ad)}}{m^2}, \frac{2\sqrt{a}\sqrt{c}\sqrt{x^m}}{m}\right)} - \frac{\sqrt{m^2(n^2-4ad)}}{m} + \sqrt{n^2}\right.}{2a}$$

2.22 problem 22

Internal problem ID [10352]
Internal file name [`OUTPUT/9299_Monday_June_06_2022_01_49_31_PM_23841973/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{(x^n a + b)\, y' - b y^2 = a\, x^{-2+n}}$$

2.22.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2 b + a\, x^{-2+n}}{x^n a + b} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2 b}{x^n a + b} + \frac{a\, x^n}{(x^n a + b)\, x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = \frac{a\, x^{-2+n}}{x^n a+b}$, $f_1(x) = 0$ and $f_2(x) = \frac{b}{x^n a+b}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{bu}{x^n a+b}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = -\frac{ban\, x^n}{(x^n a + b)^2 x}$$
$$f_1 f_2 = 0$$
$$f_2^2 f_0 = \frac{b^2 a\, x^{-2+n}}{(x^n a + b)^3}$$

Substituting the above terms back in equation (2) gives

$$\frac{bu''(x)}{x^n a + b} + \frac{ban\, x^n u'(x)}{(x^n a + b)^2 x} + \frac{b^2 a\, x^{-2+n} u(x)}{(x^n a + b)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\left(\frac{x^n a + b}{b}\right)^{-\frac{2}{n}} c_1 x + \text{hypergeom}\left(\left[1, \frac{1}{n}\right], \left[1 - \frac{1}{n}\right], -\frac{a\, x^n}{b}\right) c_2\right) (x^n a + b)^{\frac{1}{n}}$$

The above shows that

$$u'(x) = \left(-\frac{2\left(\frac{x^n a+b}{b}\right)^{-\frac{2}{n}} a\, x^n c_1}{x^n a + b} + \left(\frac{x^n a + b}{b}\right)^{-\frac{2}{n}} c_1 - \frac{anc_2 x^{n-1}\, \text{hypergeom}\left(\left[2, 1 + \frac{1}{n}\right], \left[2 - \frac{1}{n}\right], -\frac{a\, x^n}{b}\right)}{b\,(n-1)}\right) (x^n a + b)^{\frac{1}{n}} + a(x^n a + b)^{-\frac{n-1}{n}} x^n \left(\left(\frac{x^n a + b}{b}\right)^{-\frac{2}{n}} c_1 + \frac{\text{hypergeom}\left(\left[1, \frac{1}{n}\right], \left[1 - \frac{1}{n}\right], -\frac{a\, x^n}{b}\right) c_2}{x}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\left(-\frac{2\left(\frac{x^n a+b}{b}\right)^{-\frac{2}{n}} a\, x^n c_1}{x^n a+b} + \left(\frac{x^n a+b}{b}\right)^{-\frac{2}{n}} c_1 - \frac{anc_2 x^{n-1}\, \text{hypergeom}\left(\left[2,1+\frac{1}{n}\right],\left[2-\frac{1}{n}\right],-\frac{a\, x^n}{b}\right)}{b(n-1)}\right) (x^n a + b)^{\frac{1}{n}} + a(x^n a + b}{b\left(\left(\frac{x^n a+b}{b}\right)^{-\frac{2}{n}} c_1 x + \text{hypergeom}\left(\left[1, \frac{1}{n}\right],\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{a\left(\frac{x^n a+b}{b}\right)^{\frac{2}{n}} n(a^2 x^{3n} + 2ab\, x^{2n} + x^n b^2) \operatorname{hypergeom}\left(\left[2, \frac{n+1}{n}\right], \left[\frac{2n-1}{n}\right], -\frac{a\, x^n}{b}\right) - b\Big(a\left(\frac{x^n a+b}{b}\right)^{\frac{2}{n}} (a\, x^{2n} + b\, x^n)}{b^2 (n-1)\, x \left(c_3 x + \operatorname{hypergeom}\left(\left[1, \frac{1}{n}\right], \left[\frac{n-1}{n}\right], -\frac{a\, x^n}{b}\right) \left(\frac{x^n}{}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{a\left(\frac{x^n a+b}{b}\right)^{\frac{2}{n}} n(a^2 x^{3n} + 2ab\, x^{2n} + x^n b^2) \operatorname{hypergeom}\left(\left[2, \frac{n+1}{n}\right], \left[\frac{2n-1}{n}\right], -\frac{a\, x^n}{b}\right) - b\Big(a\left(\frac{x^n a+b}{b}\right)^{\frac{2}{n}} (a\, x^{2n} + b\, x^n)}{b^2 (n-1)\, x \left(c_3 x + \operatorname{hypergeom}\left(\left[1, \frac{1}{n}\right], \left[\frac{n-1}{n}\right], -\frac{a\, x^n}{b}\right) \left(\frac{x^n}{}\right.\right.} \quad (1)$$

Verification of solutions

$$y = \frac{a\left(\frac{x^n a+b}{b}\right)^{\frac{2}{n}} n(a^2 x^{3n} + 2ab\, x^{2n} + x^n b^2) \operatorname{hypergeom}\left(\left[2, \frac{n+1}{n}\right], \left[\frac{2n-1}{n}\right], -\frac{a\, x^n}{b}\right) - b\Big(a\left(\frac{x^n a+b}{b}\right)^{\frac{2}{n}} (a\, x^{2n} + b\, x^n)}{b^2 (n-1)\, x \left(c_3 x + \operatorname{hypergeom}\left(\left[1, \frac{1}{n}\right], \left[\frac{n-1}{n}\right], -\frac{a\, x^n}{b}\right) \left(\frac{x^n}{}\right.\right.}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -a*n*x^n*(diff(y(x), x))/((x^n
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               -> Whittaker
                  -> hyper3: Equivalence to 1F1 under a power @ Moebius
               -> hypergeometric
                  -> heuristic approach
                  <- heuristic approach successful
               <- hypergeometric successful
            <- special function solution successful
               -> Trying to convert hypergeometric functions to elementary form...
               <- elementary form could result into a too large expression - returning speci
            <- Kovacics algorithm successful
      <- Equivalence, under non-integer power transformations successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 224

```
dsolve((a*x^n+b)*diff(y(x),x)=b*y(x)^2+a*x^(n-2),y(x), singsol=all)
```

$$y(x) = \frac{\left(\frac{a\,x^n+b}{b}\right)^{\frac{2}{n}} \left(anc_1(a^2x^{3n} + 2ab\,x^{2n} + b^2x^n) \operatorname{hypergeom}\left(\left[2, \frac{n+1}{n}\right], \left[\frac{2n-1}{n}\right], -\frac{a\,x^n}{b}\right) - (n-1)\,b\Big(ac_1(a\,x^{2n} + b\right.}{b^2\,(n-1)\,x\,(a\,x^n+b)\,\Big(x + \operatorname{hypergeom}\left(\left[1, \frac{1}{n}\right], \left[\frac{n-1}{n}\right],\right.}$$

✓ Solution by Mathematica

Time used: 1.899 (sec). Leaf size: 289

```
DSolve[(a*x^n+b)*y'[x]==b*y[x]^2+a*x^(n-2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-b^2(-1)^{\frac{1}{n}}(n-1)\left(-\frac{ax^n}{b}\right)^{\frac{1}{n}} - abc_1(n-1)x^n\left(\frac{ax^n}{b}+1\right)^{2/n}\text{Hypergeometric2F1}\left(1, \frac{1}{n}, \frac{n-1}{n}, -\frac{ax^n}{b}\right) + ac_1nx}{b^2(n-1)x\left((-1)^{\frac{1}{n}}\left(-\frac{ax^n}{b}\right)^{\frac{1}{n}} + c_1\left(\frac{ax^n}{b}+1\right)^{2/n}\text{Hypergeomet}\right.}$$

$$y(x) \to \frac{ax^{n-1}\left(\frac{n(ax^n+b)\,\text{Hypergeometric2F1}\left(2,1+\frac{1}{n},2-\frac{1}{n},-\frac{ax^n}{b}\right)}{\text{Hypergeometric2F1}\left(1,\frac{1}{n},\frac{n-1}{n},-\frac{ax^n}{b}\right)} + b(-n) + b\right)}{b^2(n-1)}$$

2.23 problem 23

Internal problem ID [10353]
Internal file name [`OUTPUT/9300_Monday_June_06_2022_01_49_33_PM_14490210/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{(x^n a + b\,x^m + c)\left(y' - y^2\right) = -an(n-1)\,x^{-2+n} - bm(m-1)\,x^{m-2}}$$

2.23.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,x^n y^2 + x^m y^2 b - a\,n^2 x^{-2+n} - b\,m^2 x^{m-2} + c\,y^2 + an\,x^{-2+n} + bm\,x^{m-2}}{x^n a + b\,x^m + c} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,x^n y^2}{x^n a + b\,x^m + c} + \frac{x^m y^2 b}{x^n a + b\,x^m + c} - \frac{a\,n^2 x^n}{(x^n a + b\,x^m + c)\,x^2} - \frac{b\,m^2 x^m}{(x^n a + b\,x^m + c)\,x^2} + \frac{c\,y^2}{x^n a + b\,x^m + c} + \frac{}{(x^n a}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{-a\,n^2 x^{-2+n} - b\,m^2 x^{m-2} + an\,x^{-2+n} + bm\,x^{m-2}}{x^n a + b\,x^m + c}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{-a\, n^2 x^{-2+n} - b\, m^2 x^{m-2} + an\, x^{-2+n} + bm\, x^{m-2}}{x^n a + b\, x^m + c} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{(-a\, n^2 x^{-2+n} - b\, m^2 x^{m-2} + an\, x^{-2+n} + bm\, x^{m-2})\, u(x)}{x^n a + b\, x^m + c} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\left(\int \frac{1}{(x^n a + b\, x^m + c)^2} dx\right) c_1 + c_2\right) (x^n a + b\, x^m + c)$$

The above shows that

$$u'(x) = \frac{c_1}{x^n a + b\, x^m + c} + \frac{\left(\left(\int \frac{1}{(x^n a + b\, x^m + c)^2} dx\right) c_1 + c_2\right) (x^n n a + b\, x^m m)}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{c_1}{x^n a + b\, x^m + c} + \frac{\left(\left(\int \frac{1}{(x^n a + b\, x^m + c)^2} dx\right) c_1 + c_2\right) (x^n n a + b\, x^m m)}{x}}{\left(\left(\int \frac{1}{(x^n a + b\, x^m + c)^2} dx\right) c_1 + c_2\right) (x^n a + b\, x^m + c)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\frac{c_3}{x^n a + b\, x^m + c} - \frac{\left(\left(\int \frac{1}{(x^n a + b\, x^m + c)^2} dx\right) c_3 + 1\right) (x^n n a + b\, x^m m)}{x}}{\left(\left(\int \frac{1}{(x^n a + b\, x^m + c)^2} dx\right) c_3 + 1\right) (x^n a + b\, x^m + c)}$$

Summary
The solution(s) found are the following

$$y = \frac{-\frac{c_3}{x^n a+b\,x^m+c}-\frac{\left(\left(\int \frac{1}{(x^n a+b\,x^m+c)^2}dx\right)c_3+1\right)(x^n n a+b\,x^m m)}{x}}{\left(\left(\int \frac{1}{(x^n a+b\,x^m+c)^2}dx\right)c_3+1\right)(x^n a+b\,x^m+c)} \tag{1}$$

Verification of solutions

$$y = \frac{-\frac{c_3}{x^n a+b\,x^m+c}-\frac{\left(\left(\int \frac{1}{(x^n a+b\,x^m+c)^2}dx\right)c_3+1\right)(x^n n a+b\,x^m m)}{x}}{\left(\left(\int \frac{1}{(x^n a+b\,x^m+c)^2}dx\right)c_3+1\right)(x^n a+b\,x^m+c)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (b*m^2*x^(-2+m)+a*n^2*x^(n-2)-
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      <- linear_1 successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 141

```
dsolve((a*x^n+b*x^m+c)*(diff(y(x),x)-y(x)^2)+a*n*(n-1)*x^(n-2)+b*m*(m-1)*x^(m-2)=0,y(x), sin
```

$$y(x) = \frac{-(an\,x^n + bm\,x^m)\,(a\,x^n + b\,x^m + c)\left(\int \frac{1}{(a\,x^n+b\,x^m+c)^2}dx\right) - x^{2m}c_1b^2m - b(a(n+m)\,x^n + cm)\,c_1x^m - x^2}{(a\,x^n + b\,x^m + c)^2\,x\left(c_1 + \int \frac{1}{(a\,x^n+b\,x^m+c)^2}dx\right)}$$

✓ Solution by Mathematica

Time used: 35.099 (sec). Leaf size: 201

```
DSolve[(a*x^n+b*x^m+c)*(y'[x]-y[x]^2)+a*n*(n-1)*x^(n-2)+b*m*(m-1)*x^(m-2)==0,y[x],x,IncludeS
```

$$y(x) \to -\frac{c_1\left(\frac{(anx^n+bmx^m)\int_1^x \frac{1}{(bK[1]^m+aK[1]^n+c)^2}dK[1]}{x} + \frac{1}{ax^n+bx^m+c}\right) + anx^{n-1} + bmx^{m-1}}{(ax^n+bx^m+c)\left(1 + c_1\int_1^x \frac{1}{(bK[1]^m+aK[1]^n+c)^2}dK[1]\right)}$$

$$y(x) \to -\frac{\frac{1}{\int_1^x \frac{1}{(bK[1]^m+aK[1]^n+c)^2}dK[1]} + \frac{(anx^n+bmx^m)(ax^n+bx^m+c)}{x}}{(ax^n+bx^m+c)^2}$$

2.24 problem 24

Internal problem ID [10354]
Internal file name [`OUTPUT/9301_Monday_June_06_2022_01_49_47_PM_70081015/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a y^2 - y b = c x + k}$$

2.24.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= a\, y^2 + b y + c x + k \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\, y^2 + b y + c x + k$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = c x + k$, $f_1(x) = b$ and $f_2(x) = a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= ab \\ f_2^2f_0 &= a^2(cx+k) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$au''(x) - abu'(x) + a^2(cx+k)\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = e^{\frac{bx}{2}}\left(\operatorname{AiryAi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)\,a - \frac{b^2}{4}\right)}{ac}\right)c_1 + \operatorname{AiryBi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)\,a - \frac{b^2}{4}\right)}{ac}\right)c_2\right)$$

The above shows that

$$u'(x) = \frac{e^{\frac{bx}{2}}\left(-2(ca)^{\frac{1}{3}}\operatorname{AiryAi}\left(1, -\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\frac{b^2}{4}\right)}{ac}\right)c_1 - 2(ca)^{\frac{1}{3}}\operatorname{AiryBi}\left(1, -\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\frac{b^2}{4}\right)}{ac}\right)c_2 + b\Big(\operatorname{AiryAi}\right.}{2}$$

Using the above in (1) gives the solution

$$y = -\frac{-2(ca)^{\frac{1}{3}}\operatorname{AiryAi}\left(1, -\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\frac{b^2}{4}\right)}{ac}\right)c_1 - 2(ca)^{\frac{1}{3}}\operatorname{AiryBi}\left(1, -\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\frac{b^2}{4}\right)}{ac}\right)c_2 + b\Big(\operatorname{AiryAi}\Big(}{2a\left(\operatorname{AiryAi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\frac{b^2}{4}\right)}{ac}\right)c_1 + \operatorname{AiryBi}\Big(-\frac{(ca)^{\frac{1}{3}}\left((c\right.}{}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\operatorname{AiryAi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\frac{b^2}{4}\right)}{ac}\right)c_3b - b\operatorname{AiryBi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\frac{b^2}{4}\right)}{ac}\right) + 2(ca)^{\frac{1}{3}}\left(\operatorname{AiryAi}\left(1, -\frac{(ca)^{\frac{1}{3}}\left((cx+k\right.}{ac}\right.\right.}{2a\left(\operatorname{AiryAi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\frac{b^2}{4}\right)}{ac}\right)c_3 + \operatorname{AiryBi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\right.}{ac}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{-\operatorname{AiryAi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\frac{b^2}{4}\right)}{ac}\right)c_3b - b\operatorname{AiryBi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\frac{b^2}{4}\right)}{ac}\right) + 2(ca)^{\frac{1}{3}}\left(\operatorname{AiryAi}\left(1, -\frac{(ca)^{\frac{1}{3}}\left((cx+k\right.}{ac}\right.\right.}{2a\left(\operatorname{AiryAi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\frac{b^2}{4}\right)}{ac}\right)c_3 + \operatorname{AiryBi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\right.}{ac}\right.\right.} \quad (1)$$

Verification of solutions

$$y = \frac{-\operatorname{AiryAi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\frac{b^2}{4}\right)}{ac}\right)c_3b - b\operatorname{AiryBi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\frac{b^2}{4}\right)}{ac}\right) + 2(ca)^{\frac{1}{3}}\left(\operatorname{AiryAi}\left(1, -\frac{(ca)^{\frac{1}{3}}\left((cx+k\right.}{ac}\right.\right.}{2a\left(\operatorname{AiryAi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\frac{b^2}{4}\right)}{ac}\right)c_3 + \operatorname{AiryBi}\left(-\frac{(ca)^{\frac{1}{3}}\left((cx+k)a-\right.}{ac}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Abel AIR successful: ODE belongs to the 0F1 0-parameter (Airy type) class`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 194

```
dsolve(diff(y(x),x)=a*y(x)^2+b*y(x)+c*x+k,y(x), singsol=all)
```

$$y(x) = \frac{2\sqrt{a}\left(\frac{c}{\sqrt{a}}\right)^{\frac{1}{3}}\left(\operatorname{AiryAi}\left(1, -\frac{a(cx+k)-\frac{b^2}{4}}{\left(\frac{c}{\sqrt{a}}\right)^{\frac{2}{3}}a}\right)c_1 + \operatorname{AiryBi}\left(1, -\frac{a(cx+k)-\frac{b^2}{4}}{\left(\frac{c}{\sqrt{a}}\right)^{\frac{2}{3}}a}\right)\right) - b\left(c_1 \operatorname{AiryAi}\left(-\frac{a(cx+k)-\frac{b^2}{4}}{\left(\frac{c}{\sqrt{a}}\right)^{\frac{2}{3}}a}\right)\right.}{2a\left(c_1 \operatorname{AiryAi}\left(-\frac{a(cx+k)-\frac{b^2}{4}}{\left(\frac{c}{\sqrt{a}}\right)^{\frac{2}{3}}a}\right) + \operatorname{AiryBi}\left(-\frac{a(cx+k)-\frac{b^2}{4}}{\left(\frac{c}{\sqrt{a}}\right)^{\frac{2}{3}}a}\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.51 (sec). Leaf size: 359

```
DSolve[y'[x]==a*y[x]^2+b*y[x]+c*x+k,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{c\left(-b(-ac)^{2/3}\operatorname{AiryBi}\left(\frac{b^2-4a(k+cx)}{4(-ac)^{2/3}}\right) + 2ac\operatorname{AiryBiPrime}\left(\frac{b^2-4a(k+cx)}{4(-ac)^{2/3}}\right) + c_1\left(2ac\operatorname{AiryAiPrime}\left(\frac{b^2-4a(k+}{4(-ac)^2}\right.\right.\right.}{2(-ac)^{5/3}\left(\operatorname{AiryBi}\left(\frac{b^2-4a(k+cx)}{4(-ac)^{2/3}}\right) + c_1\operatorname{AiryAi}\left(\frac{b^2-4a(k+cx)}{4(-ac)^{2/3}}\right)\right)}$$

$$y(x) \to -\frac{\frac{2\sqrt[3]{-ac}\operatorname{AiryAiPrime}\left(\frac{b^2-4a(k+cx)}{4(-ac)^{2/3}}\right)}{\operatorname{AiryAi}\left(\frac{b^2-4a(k+cx)}{4(-ac)^{2/3}}\right)} + b}{2a}$$

$$y(x) \to -\frac{\frac{2\sqrt[3]{-ac}\operatorname{AiryAiPrime}\left(\frac{b^2-4a(k+cx)}{4(-ac)^{2/3}}\right)}{\operatorname{AiryAi}\left(\frac{b^2-4a(k+cx)}{4(-ac)^{2/3}}\right)} + b}{2a}$$

2.25 problem 25

Internal problem ID [10355]
Internal file name [`OUTPUT/9302_Monday_June_06_2022_01_49_48_PM_63309599/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - a\,x^n y = a\,x^{n-1}}$$

2.25.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + a\,x^n y + a\,x^{n-1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\,x^n y + \frac{a\,x^n}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\,x^{n-1}$, $f_1(x) = x^n a$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= x^n a \\ f_2^2 f_0 &= a\, x^{n-1} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - x^n a u'(x) + a\, x^{n-1} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 \mathrm{e}^{\frac{a\, x^{n+1}}{2+2n}} (n+1) \left(a\, x^{-\frac{n}{2}} - n\, x^{-1-\frac{3n}{2}}\right) \mathrm{WhittakerM}\left(\frac{-n-2}{2+2n}, \frac{1+2n}{2+2n}, -\frac{x^{n+1} a}{n+1}\right) - x^{-1-\frac{3n}{2}} \mathrm{e}^{\frac{a\, x^{n+1}}{2+2n}} \mathrm{WhittakerM}\left(\frac{n}{2+2n}, \frac{1+2n}{2+2n}, -\frac{x^{n+1} a}{n+1}\right) c_2 n^2 + c_1 x$$

The above shows that

$$u'(x) = \frac{\left(\mathrm{e}^{\frac{x^n a x}{2+2n}} c_2 (n+1) \left(a x\, x^{\frac{n}{2}} + x^{-\frac{n}{2}} n^2\right) \mathrm{WhittakerM}\left(\frac{-n-2}{2+2n}, \frac{1+2n}{2+2n}, -\frac{x^n x a}{n+1}\right) + c_2 n \left(x^{-\frac{n}{2}} n^2 + a x\, x^{\frac{n}{2}} (n+1)\right) \mathrm{e}^{\frac{x^n a x}{2+2n}} \cdots}{x^2}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\mathrm{e}^{\frac{x^n a x}{2+2n}} c_2 (n+1) \left(a x\, x^{\frac{n}{2}} + x^{-\frac{n}{2}} n^2\right) \mathrm{WhittakerM}\left(\frac{-n-2}{2+2n}, \frac{1+2n}{2+2n}, -\frac{x^n x a}{n+1}\right) + c_2 n \left(x^{-\frac{n}{2}} n^2 + a x\, x^{\frac{n}{2}} (n+1)\right) \mathrm{e}^{\cdots}}{x^2 \left(c_2 \mathrm{e}^{\frac{a\, x^{n+1}}{2+2n}} (n+1) \left(a\, x^{-\frac{n}{2}} - n\, x^{-1-\frac{3n}{2}}\right) \mathrm{WhittakerM}\left(\frac{-n-2}{2+2n}, \frac{1+2n}{2+2n}, -\frac{x^{n+1} a}{n+1}\right) \cdots\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\mathrm{e}^{\frac{x^n a x}{2+2n}} (n+1) \left(a x\, x^{\frac{n}{2}} + x^{-\frac{n}{2}} n^2\right) \mathrm{WhittakerM}\left(\frac{-n-2}{2+2n}, \frac{1+2n}{2+2n}, -\frac{x^n x a}{n+1}\right) + n \left(x^{-\frac{n}{2}} n^2 + a x\, x^{\frac{n}{2}} (n+1)\right) \mathrm{e}^{\frac{x^n a x}{2+2n}} \cdots}{x \left(\mathrm{e}^{\frac{x^n a x}{2+2n}} (n+1) \left(x^{-\frac{n}{2}} a x - n\, x^{-\frac{3n}{2}}\right) \mathrm{WhittakerM}\left(\frac{-n-2}{2+2n}, \frac{1+2n}{2+2n}, -\frac{x^n x a}{n+1}\right) - \cdots\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(e^{\frac{x^n a x}{2+2n}}(n+1)\left(a x\, x^{\frac{n}{2}}+x^{-\frac{n}{2}} n^2\right) \mathrm{WhittakerM}\left(\frac{-n-2}{2+2n}, \frac{1+2n}{2+2n}, -\frac{x^n x a}{n+1}\right)+n\left(x^{-\frac{n}{2}} n^2+a x\, x^{\frac{n}{2}}(n+1)\right) e^{\frac{x^n a x}{2+2n}} \cdots\right.}{x\left(e^{\frac{x^n a x}{2+2n}}(n+1)\left(x^{-\frac{n}{2}} a x-n\, x^{-\frac{3n}{2}}\right) \mathrm{WhittakerM}\left(\frac{-n-2}{2+2n}, \frac{1+2n}{2+2n}, -\frac{x^n x a}{n+1}\right)- \cdots\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(e^{\frac{x^n a x}{2+2n}}(n+1)\left(a x\, x^{\frac{n}{2}}+x^{-\frac{n}{2}} n^2\right) \mathrm{WhittakerM}\left(\frac{-n-2}{2+2n}, \frac{1+2n}{2+2n}, -\frac{x^n x a}{n+1}\right)+n\left(x^{-\frac{n}{2}} n^2+a x\, x^{\frac{n}{2}}(n+1)\right) e^{\frac{x^n a x}{2+2n}} \cdots\right.}{x\left(e^{\frac{x^n a x}{2+2n}}(n+1)\left(x^{-\frac{n}{2}} a x-n\, x^{-\frac{3n}{2}}\right) \mathrm{WhittakerM}\left(\frac{-n-2}{2+2n}, \frac{1+2n}{2+2n}, -\frac{x^n x a}{n+1}\right)- \cdots\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 400

```
dsolve(diff(y(x),x)=y(x)^2+a*x^n*y(x)+a*x^(n-1),y(x), singsol=all)
```

$$y(x) = \frac{-\mathrm{e}^{\frac{x^n a x}{2n+2}}\left(-\frac{a x\, x^n}{n+1}\right)^{-\frac{n}{2n+2}}(n+1)^2\left(x^n a x-n\right)\mathrm{WhittakerM}\left(\frac{-n-2}{2n+2},\frac{2n+1}{2n+2},-\frac{a x\, x^n}{n+1}\right)-2n\left(-\frac{(n+1)n\,\mathrm{e}^{\frac{x^n a x}{2n+2}}\left(-\frac{a x\, x^n}{n+1}\right)}{\cdots}\right.}{x\left(\mathrm{e}^{\frac{x^n a x}{2n+2}}\left(-\frac{a x\, x^n}{n+1}\right)^{-\frac{n}{2n+2}}(n+1)^2\left(x^n a x-n\right)\mathrm{WhittakerM}\left(\frac{-n-2}{2n+2},\frac{2n+1}{2n+2},-\frac{a x\, x^n}{n+1}\right)+2n\left(-\frac{(n+1)n\,\mathrm{e}^{\frac{x^n a x}{2n+2}}\left(-\cdots\right.}{\cdots}\right.\right.}$$

✓ Solution by Mathematica

Time used: 2.82 (sec). Leaf size: 136

```
DSolve[y'[x]==y[x]^2+a*x^n*y[x]+a*x^(n-1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\left(-\frac{a x^{n+1}}{n+1}\right)^{\frac{1}{n+1}}\Gamma\left(-\frac{1}{n+1},-\frac{a x^{n+1}}{n+1}\right)-(n+1)\left(e^{\frac{a x^{n+1}}{n+1}}+c_1 x\right)}{x\left(-\left(-\frac{a x^{n+1}}{n+1}\right)^{\frac{1}{n+1}}\Gamma\left(-\frac{1}{n+1},-\frac{a x^{n+1}}{n+1}\right)+c_1(n+1)x\right)}$$

$$y(x) \to -\frac{1}{x}$$

2.26 problem 26

Internal problem ID [10356]
Internal file name [`OUTPUT/9303_Monday_June_06_2022_01_50_21_PM_5228272/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - a\,x^n y = b\,x^{n-1}}$$

2.26.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + a\,x^n y + b\,x^{n-1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\,x^n y + \frac{b\,x^n}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\,x^{n-1}$, $f_1(x) = x^n a$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= x^n a \\ f_2^2 f_0 &= b\, x^{n-1} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - x^n a u'(x) + b\, x^{n-1} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x\left(\text{KummerM}\left(\frac{a-b}{a\,(n+1)}, \frac{2+n}{n+1}, \frac{x^{n+1}a}{n+1}\right) c_1 + \text{KummerU}\left(\frac{a-b}{a\,(n+1)}, \frac{2+n}{n+1}, \frac{x^{n+1}a}{n+1}\right) c_2\right)$$

The above shows that

$$u'(x) = \frac{ac_1(n+1)\,(a-b)\,\text{KummerM}\left(-\frac{an+b}{a(n+1)} + 2, \frac{2+n}{n+1}, \frac{x^{n+1}a}{n+1}\right) - b\Big(c_2(a-b)\,\text{KummerU}\left(\frac{(2+n)a-b}{a(n+1)}, \frac{2+n}{n+1}, \frac{x^{n+1}a}{n+1}\right)}{a^2\,(n+1)}$$

Using the above in (1) gives the solution

$$y = -\frac{ac_1(n+1)\,(a-b)\,\text{KummerM}\left(-\frac{an+b}{a(n+1)} + 2, \frac{2+n}{n+1}, \frac{x^{n+1}a}{n+1}\right) - b\Big(c_2(a-b)\,\text{KummerU}\left(\frac{(2+n)a-b}{a(n+1)}, \frac{2+n}{n+1}, \frac{x^{n+1}}{n+1}\right.}{a^2\,(n+1)\,x\left(\text{KummerM}\left(\frac{a-b}{a(n+1)}, \frac{2+n}{n+1}, \frac{x^{n+1}a}{n+1}\right) c_1 +\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-ac_3(n+1)\,(a-b)\,\text{KummerM}\left(\frac{2+n-\frac{b}{a}}{n+1}, \frac{2+n}{n+1}, \frac{x^n xa}{n+1}\right) + b\Big((a-b)\,\text{KummerU}\left(\frac{2+n-\frac{b}{a}}{n+1}, \frac{2+n}{n+1}, \frac{x^n xa}{n+1}\right) - a\Big(\text{Ku}}{a^2\,(n+1)\,x\left(\text{KummerM}\left(\frac{a-b}{a(n+1)}, \frac{2+n}{n+1}, \frac{x^n xa}{n+1}\right) c_3 + \text{Kumm}\right.}$$

Summary
The solution(s) found are the following

$$y = \frac{-a c_3 (n+1)(a-b)\,\mathrm{KummerM}\left(\frac{2+n-\frac{b}{a}}{n+1}, \frac{2+n}{n+1}, \frac{x^n x a}{n+1}\right) + b\Big((a-b)\,\mathrm{KummerU}\left(\frac{2+n-\frac{b}{a}}{n+1}, \frac{2+n}{n+1}, \frac{x^n x a}{n+1}\right) - a\Big(\mathrm{Ku}}{a^2 (n+1)\, x \left(\mathrm{KummerM}\left(\frac{a-b}{a(n+1)}, \frac{2+n}{n+1}, \frac{x^n x a}{n+1}\right) c_3 + \mathrm{Kumm}\right.} \tag{1}$$

Verification of solutions

$$y = \frac{-a c_3 (n+1)(a-b)\,\mathrm{KummerM}\left(\frac{2+n-\frac{b}{a}}{n+1}, \frac{2+n}{n+1}, \frac{x^n x a}{n+1}\right) + b\Big((a-b)\,\mathrm{KummerU}\left(\frac{2+n-\frac{b}{a}}{n+1}, \frac{2+n}{n+1}, \frac{x^n x a}{n+1}\right) - a\Big(\mathrm{Ku}}{a^2 (n+1)\, x \left(\mathrm{KummerM}\left(\frac{a-b}{a(n+1)}, \frac{2+n}{n+1}, \frac{x^n x a}{n+1}\right) c_3 + \mathrm{Kumm}\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (diff(y(x), x))*x^n*a-b*x^(n-1
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
         <- Kummer successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 267

```
dsolve(diff(y(x),x)=y(x)^2+a*x^n*y(x)+b*x^(n-1),y(x), singsol=all)
```

$$y(x) = \frac{-a(n+1)(a-b)\,\mathrm{KummerM}\left(\frac{a(n+2)-b}{a(n+1)}, \frac{n+2}{n+1}, \frac{a x\, x^n}{n+1}\right) + \left((a-b)\,c_1\,\mathrm{KummerU}\left(\frac{2+n-\frac{b}{a}}{n+1}, \frac{n+2}{n+1}, \frac{a x\, x^n}{n+1}\right) - a\left(\mathrm{K}\right.\right.}{a^2(n+1)\,x\left(\mathrm{KummerU}\left(\frac{a-b}{a(n+1)}, \frac{n+2}{n+1}, \frac{a x\, x^n}{n+1}\right) c_1 + \mathrm{Kum}\right.}$$

✓ Solution by Mathematica

Time used: 0.956 (sec). Leaf size: 453

```
DSolve[y'[x]==y[x]^2+a*x^n*y[x]+b*x^(n-1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(x^n)^{\frac{1}{n}}\left(-(-1)^{\frac{1}{n+1}}n(n+2)a^{\frac{1}{n+1}}\,\mathrm{Hypergeometric1F1}\left(\frac{a-b}{na+a}, \frac{n+2}{n+1}, \frac{a(x^n)^{1+\frac{1}{n}}}{n+1}\right) + x^n\left(-(-1)^{\frac{1}{n+1}}n(a-b)a^{\frac{}{n}}\right.\right.}{n(n+2)x\left((-1)^{\frac{1}{n+1}}a^{\frac{1}{n+1}}(x^n)^{\frac{1}{n}}\,\mathrm{Hypergeometric1F1}\left(\frac{a}{na}\right.\right.}$$

$$y(x) \to \frac{bx^{n-1}(x^n)^{\frac{1}{n}}\,\mathrm{Hypergeometric1F1}\left(\frac{na+a-b}{na+a}, \frac{2n+1}{n+1}, \frac{a(x^n)^{1+\frac{1}{n}}}{n+1}\right)}{n\,\mathrm{Hypergeometric1F1}\left(-\frac{b}{na+a}, \frac{n}{n+1}, \frac{a(x^n)^{1+\frac{1}{n}}}{n+1}\right)}$$

2.27 problem 27

Internal problem ID [10357]
Internal file name [`OUTPUT/9304_Monday_June_06_2022_01_50_23_PM_9360412/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - (\alpha x + \beta)\, y = a\, x^2 + bx + c}$$

2.27.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= a\, x^2 + \alpha xy + bx + \beta y + y^2 + c \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\, x^2 + \alpha xy + bx + \beta y + y^2 + c$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\, x^2 + bx + c$, $f_1(x) = \alpha x + \beta$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= \alpha x + \beta \\ f_2^2 f_0 &= a\,x^2 + bx + c \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - (\alpha x + \beta)\,u'(x) + \left(a\,x^2 + bx + c\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = 4\left(c_2\left(-\frac{1}{4}\alpha^2 x + xa - \frac{1}{4}\alpha\beta + \frac{1}{2}b\right)\text{hypergeom}\left(\left[\frac{\left(-\alpha^3 - 2\alpha^2 c + (2\beta b + 4a)\,\alpha + (-2\beta^2 + 8c)\,a - 2b^2\right)\sqrt{\alpha^2 - 4a} + 48\left(-\frac{\alpha^2}{4} + a\right)^2}{4\left(-\alpha^2 + 4a\right)^2}\right. \right.\right.$$

$$+ \frac{\text{hypergeom}\left(\left[\frac{\left(-\alpha^3 - 2\alpha^2c + (2\beta b + 4a)\alpha + (-2\beta^2 + 8c)a - 2b^2\right)\sqrt{\alpha^2 - 4a} + 16\left(-\frac{\alpha^2}{4} + a\right)^2}{4(-\alpha^2 + 4a)^2}\right], \left[\frac{1}{2}\right], \frac{\left(-\alpha^2 x + 4xa - \alpha\beta + 2b\right)^2}{2(\alpha^2 - 4a)^{\frac{3}{2}}}\right) c_1}{4}\Bigg) \mathrm{e}$$

The above shows that

Expression too large to display

Using the above in (1) gives the solution

Expression too large to display

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

Expression too large to display

Summary

The solution(s) found are the following

Expression too large to display (1)

Verification of solutions

Expression too large to display

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (alpha*x+beta)*(diff(y(x), x))
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: indirect Equivalence to 0F1 under \`\`^ @ Moebius\`\` is r
         <- hypergeometric successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 974

```
dsolve(diff(y(x),x)=y(x)^2+(alpha*x+beta)*y(x)+a*x^2+b*x+c,y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 4.264 (sec). Leaf size: 1291

```
DSolve[y'[x]==y[x]^2+(\[Alpha]*x+\[Beta])*y[x]+a*x^2+b*x+c,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to -\frac{2\left(2b+4ax+\left(\sqrt{\alpha^2-4a}-\alpha\right)(x\alpha+\beta)\right)\text{Hypergeometric1F1}\left(-\frac{2b^2-2\alpha\beta b+\alpha^2\left(2c+\alpha-\sqrt{\alpha^2-4a}\right)+2a\left(\beta^2-4c-\right.}{4(\alpha^2-4a)^{3/2}}\right.}{}$$

$$y(x) \to \frac{(4a-\alpha^2)\left(\left(\sqrt{\alpha^2-4a}-\alpha\right)(\beta+\alpha x)+4ax+2b\right)-\frac{\sqrt{2}\sqrt[4]{\alpha^2-4a}\left(2a\left(2\sqrt{\alpha^2-4a}-2\alpha+\beta^2-4c\right)+\alpha^2\left(-\sqrt{\alpha^2-4a}+\alpha+\right.\right.}{\text{HermiteH}\left(-\frac{-2b^2+2\alpha\beta b+\alpha}{}\right.}}{2\left(\alpha^2-4a\right)^{3/2}}$$

2.28 problem 28

Internal problem ID [10358]
Internal file name [`OUTPUT/9305_Monday_June_06_2022_01_50_51_PM_42556060/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - a\,x^n y = -a\,x^n b - b^2}$$

2.28.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + a\,x^n y - a\,x^n b - b^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\,x^n y - a\,x^n b - b^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a\,x^n b - b^2$, $f_1(x) = x^n a$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= x^n a \\ f_2^2 f_0 &= -a\, x^n b - b^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - x^n a u'(x) + \left(-a\, x^n b - b^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\int \frac{-\left(\int \mathrm{e}^{\frac{a\, x^{n+1}+2bx(n+1)}{n+1}} dx\right) b + c_1 b + \mathrm{e}^{\frac{a\, x^{n+1}+2bx(n+1)}{n+1}}}{\int \mathrm{e}^{\frac{x(x^n a+2b(n+1))}{n+1}} dx - c_1} dx} c_2$$

The above shows that

$$u'(x) = - \frac{c_2 \left(-\left(\int \mathrm{e}^{\frac{x(x^n a+2b(n+1))}{n+1}} dx\right) b + c_1 b + \mathrm{e}^{\frac{x(x^n a+2b(n+1))}{n+1}}\right) \mathrm{e}^{\int \frac{-\left(\int \mathrm{e}^{\frac{x(x^n a+2b(n+1))}{n+1}} dx\right) b + c_1 b + \mathrm{e}^{\frac{x(x^n a+2b(n+1))}{n+1}}}{\int \mathrm{e}^{\frac{x(x^n a+2b(n+1))}{n+1}} dx - c_1} dx}}{-\left(\int \mathrm{e}^{\frac{x(x^n a+2b(n+1))}{n+1}} dx\right) + c_1}$$

Using the above in (1) gives the solution

$$y = \frac{\left(-\left(\int \mathrm{e}^{\frac{x(x^n a+2b(n+1))}{n+1}} dx\right) b + c_1 b + \mathrm{e}^{\frac{x(x^n a+2b(n+1))}{n+1}}\right) \mathrm{e}^{\int \frac{-\left(\int \mathrm{e}^{\frac{x(x^n a+2b(n+1))}{n+1}} dx\right) b + c_1 b + \mathrm{e}^{\frac{x(x^n a+2b(n+1))}{n+1}}}{\int \mathrm{e}^{\frac{x(x^n a+2b(n+1))}{n+1}} dx - c_1} dx}\ \mathrm{e}^{\int - \frac{\int -b\, \mathrm{e}^{\frac{a\, x^{n+ \cdots}}{}}}{}}}{-\left(\int \mathrm{e}^{\frac{x(x^n a+2b(n+1))}{n+1}} dx\right) + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\int e^{\frac{a\,x^{n+1}+2bx(n+1)}{n+1}}dx - c_3\right)b - e^{\frac{a\,x^{n+1}+2bx(n+1)}{n+1}}}{\int e^{\frac{x(x^n a+2b(n+1))}{n+1}}dx - c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\int e^{\frac{a\,x^{n+1}+2bx(n+1)}{n+1}}dx - c_3\right)b - e^{\frac{a\,x^{n+1}+2bx(n+1)}{n+1}}}{\int e^{\frac{x(x^n a+2b(n+1))}{n+1}}dx - c_3} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\int e^{\frac{a\,x^{n+1}+2bx(n+1)}{n+1}}dx - c_3\right)b - e^{\frac{a\,x^{n+1}+2bx(n+1)}{n+1}}}{\int e^{\frac{x(x^n a+2b(n+1))}{n+1}}dx - c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (diff(y(x), x))*x^n*a+(b*x^n*a
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 74

```
dsolve(diff(y(x),x)=y(x)^2+a*x^n*y(x)-a*b*x^n-b^2,y(x), singsol=all)
```

$$\frac{(b-y(x))\left(\int^{x} \mathrm{e}^{\frac{(\_a^{n} a+2b(n+1))\_a}{n+1}} d\_a\right)+c_1 b-c_1 y(x)-\mathrm{e}^{\frac{(a\,x^{n}+2b(n+1))x}{n+1}}}{b-y\left(x\right)}=0$$

✓ Solution by Mathematica

Time used: 1.948 (sec). Leaf size: 195

```
DSolve[y'[x]==y[x]^2+a*x^n*y[x]-a*b*x^n-b^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned}\text{Solve}\Bigg[&\int_1^{y(x)}\left(\frac{e^{\frac{ax^{n+1}}{n+1}+2bx}}{an(K[2]-b)^2}\right.\\&-\int_1^x\left(\frac{e^{\frac{aK[1]^{n+1}}{n+1}+2bK[1]}(aK[1]^n+b+K[2])}{an(b-K[2])^2}+\frac{e^{\frac{aK[1]^{n+1}}{n+1}+2bK[1]}}{an(b-K[2])}\right)dK[1]\Bigg)dK[2]\\&+\int_1^x\frac{e^{\frac{aK[1]^{n+1}}{n+1}+2bK[1]}(aK[1]^n+b+y(x))}{an(b-y(x))}dK[1]=c_1,y(x)\Bigg]\end{aligned}$$

2.29 problem 29

Internal problem ID [10359]
Internal file name [`OUTPUT/9306_Monday_June_06_2022_01_50_54_PM_18768528/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + (n+1)\, x^n y^2 = a\, x^{1+m+n} - a\, x^m}$$

2.29.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -x^n n\, y^2 - x^n y^2 + a\, x^{1+m+n} - a\, x^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -x^n n\, y^2 - x^n y^2 + ax\, x^m x^n - a\, x^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\, x^{1+m+n} - a\, x^m$, $f_1(x) = 0$ and $f_2(x) = -n\, x^n - x^n$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(-n\, x^n - x^n)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{n^2x^n}{x} - \frac{x^n n}{x} \\ f_1f_2 &= 0 \\ f_2^2f_0 &= (-n\,x^n - x^n)^2\left(a\,x^{1+m+n} - a\,x^m\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(-n\,x^n - x^n)\,u''(x) - \left(-\frac{n^2x^n}{x} - \frac{x^n n}{x}\right)u'(x) + (-n\,x^n - x^n)^2\left(a\,x^{1+m+n} - a\,x^m\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{-a\_Y(x)\,(n+1)\,x^{m+2n+2} + a\_Y(x)\,(n+1)\,x^{1+m+n} - n\_Y'(x) + \_Y''(x)\,x}{x}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-a\_Y(x)\,(n+1)\,x^{m+2n+2} + a\_Y(x)\,(n+1)\,x^{1+m+n} - n\_Y'(x) + \_Y''(x)\,x}{x}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-a\_Y(x)(n+1)x^{m+2n+2}+a\_Y(x)(n+1)x^{1+m+n}-n\_Y'(x)+\_Y''(x)x}{x}\right\}, \{\_Y(x)\}\right)}{(-n\,x^n - x^n)\,\mathrm{DESol}\left(\left\{\frac{-a\_Y(x)(n+1)x^{m+2n+2}+a\_Y(x)(n+1)x^{1+m+n}-n\_Y'(x)+\_Y''(x)x}{x}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-a\_Y(x)(n+1)x^{m+2n+2}+a\_Y(x)(n+1)x^{1+m+n}-n\_Y'(x)+\_Y''(x)x}{x}\right\}, \{\_Y(x)\}\right)\right)x^{-n}}{(n+1)\,\mathrm{DESol}\left(\left\{\frac{-a\_Y(x)(n+1)x^{m+2n+2}+a\_Y(x)(n+1)x^{1+m+n}-n\_Y'(x)+\_Y''(x)x}{x}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{-a\_\_Y(x)(n+1)x^{m+2n+2}+a\_\_Y(x)(n+1)x^{1+m+n}-n\_\_Y'(x)+\_\_Y''(x)x}{x}\right\},\{\_\_Y(x)\}\right)\right)x^{-n}}{(n+1)\,\mathrm{DESol}\left(\left\{\frac{-a\_\_Y(x)(n+1)x^{m+2n+2}+a\_\_Y(x)(n+1)x^{1+m+n}-n\_\_Y'(x)+\_\_Y''(x)x}{x}\right\},\{\_\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{-a\_\_Y(x)(n+1)x^{m+2n+2}+a\_\_Y(x)(n+1)x^{1+m+n}-n\_\_Y'(x)+\_\_Y''(x)x}{x}\right\},\{\_\_Y(x)\}\right)\right)x^{-n}}{(n+1)\,\mathrm{DESol}\left(\left\{\frac{-a\_\_Y(x)(n+1)x^{m+2n+2}+a\_\_Y(x)(n+1)x^{1+m+n}-n\_\_Y'(x)+\_\_Y''(x)x}{x}\right\},\{\_\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = n*(diff(y(x), x))/x+x^n*(n+1)*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=-(n+1)*x^n*y(x)^2+a*x^(n+m+1)-a*x^m,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==-(n+1)*x^n*y[x]^2+a*x^(n+m+1)-a*x^m,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

2.30 problem 30

Internal problem ID [10360]
Internal file name [`OUTPUT/9307_Monday_June_06_2022_01_50_57_PM_28947109/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\,x^n y^2 - b\,x^m y = bc\,x^m - a\,c^2 x^n}$$

2.30.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,x^n y^2 + x^m b y + bc\,x^m - a\,c^2 x^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,x^n y^2 + x^m b y + bc\,x^m - a\,c^2 x^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = bc\,x^m - a\,c^2 x^n$, $f_1(x) = b\,x^m$ and $f_2(x) = x^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^n a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{an\, x^n}{x} \\ f_1 f_2 &= b\, x^m x^n a \\ f_2^2 f_0 &= x^{2n} a^2 \left(bc\, x^m - a\, c^2 x^n\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^n a u''(x) - \left(\frac{an\, x^n}{x} + b\, x^m x^n a\right) u'(x) + x^{2n} a^2 \left(bc\, x^m - a\, c^2 x^n\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \text{DESol}\left(\left\{\_Y''(x) - \_Y'(x)\left(\frac{n}{x} + b\, x^m\right) + a\_Y(x)\left(bc\, x^{m+n} - a\, c^2 x^{2n}\right)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$\begin{aligned} u'(x) = \frac{\partial}{\partial x} \text{DESol}\bigg(\bigg\{&\_Y''(x) - \_Y'(x)\left(\frac{n}{x} + b\, x^m\right) \\ &+ a\_Y(x)\left(bc\, x^{m+n} - a\, c^2 x^{2n}\right)\bigg\}, \{\_Y(x)\}\bigg) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \text{DESol}\left(\left\{\_Y''(x) - \_Y'(x)\left(\frac{n}{x} + b\, x^m\right) + a\_Y(x)\left(bc\, x^{m+n} - a\, c^2 x^{2n}\right)\right\}, \{\_Y(x)\}\right)\right) x^{-n}}{a\, \text{DESol}\left(\left\{\_Y''(x) - \_Y'(x)\left(\frac{n}{x} + b\, x^m\right) + a\_Y(x)\left(bc\, x^{m+n} - a\, c^2 x^{2n}\right)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \text{DESol}\left(\left\{\frac{-x^{2n}\_Y(x) a^2 c^2 x + x^{m+n}\_Y(x) abcx - x^{m+1}\_Y'(x) b + \_Y''(x) x - n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)\right) x^{-n}}{a\, \text{DESol}\left(\left\{\frac{-x^{1+2n} a^2 c^2\_Y(x) + abc\, x^{1+m+n}\_Y(x) + \_Y''(x) x - \_Y'(x)(x^{m+1} b + n)}{x}\right\}, \{\_Y(x)\}\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-x^{2n}\_Y(x)a^2c^2x+x^{m+n}\_Y(x)abcx-x^{m+1}\_Y'(x)b+\_Y''(x)x-n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)\right)x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{-x^{1+2n}a^2c^2\_Y(x)+abc\,x^{1+m+n}\_Y(x)+\_Y''(x)x-\_Y'(x)(x^{m+1}b+n)}{x}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-x^{2n}\_Y(x)a^2c^2x+x^{m+n}\_Y(x)abcx-x^{m+1}\_Y'(x)b+\_Y''(x)x-n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)\right)x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{-x^{1+2n}a^2c^2\_Y(x)+abc\,x^{1+m+n}\_Y(x)+\_Y''(x)x-\_Y'(x)(x^{m+1}b+n)}{x}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x*x^m*b+n)*(diff(y(x), x))/x+
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 98

```
dsolve(diff(y(x),x)=a*x^n*y(x)^2+b*x^m*y(x)+b*c*x^m-a*c^2*x^n,y(x), singsol=all)
```

$$\frac{a(c+y(x))\left(\int^{x} \mathrm{e}^{-\frac{2\left(-\frac{b(n+1)\_a^{m}}{2}+a\_a^{n}c(1+m)\right)\_a}{(1+m)(n+1)}}\_a^{n}d\_a\right)+c_1y(x)+c_1c+\mathrm{e}^{-\frac{2\left(-\frac{b(n+1)x^{m}}{2}+a\,x^{n}c(1+m)\right)x}{(1+m)(n+1)}}}{c+y\left(x\right)}$$
$$=0$$

✓ Solution by Mathematica

Time used: 3.436 (sec). Leaf size: 286

```
DSolve[y'[x]==a*x^n*y[x]^2+b*x^m*y[x]+b*c*x^m-a*c^2*x^n,y[x],x,IncludeSingularSolutions -> T
```

$$\text{Solve}\left[\int_1^{y(x)}\left(\frac{e^{\frac{bx^{m+1}}{m+1}-\frac{2acx^{n+1}}{n+1}}}{ab(m-n)(c+K[2])^2}\right.\right.$$
$$-\int_1^x\left(-\frac{\exp\left(\frac{bK[1]^{m+1}}{m+1}-\frac{2acK[1]^{n+1}}{n+1}\right)K[1]^n}{b(m-n)(c+K[2])}-\frac{\exp\left(\frac{bK[1]^{m+1}}{m+1}-\frac{2acK[1]^{n+1}}{n+1}\right)(-bK[1]^m+acK[1]^n-aK[2]K[}{ab(m-n)(c+K[2])^2}\right.$$
$$\left.+\int_1^x\frac{\exp\left(\frac{bK[1]^{m+1}}{m+1}-\frac{2acK[1]^{n+1}}{n+1}\right)(-bK[1]^m+acK[1]^n-ay(x)K[1]^n)}{ab(m-n)(c+y(x))}dK[1]=c_1,y(x)\right]$$

2.31 problem 31

Internal problem ID [10361]
Internal file name [`OUTPUT/9308_Monday_June_06_2022_01_51_00_PM_6355186/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - a\,x^n y^2 + a\,x^n (b\,x^m + c)\,y = bm\,x^{m-1}}$$

2.31.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -x^n x^m aby - x^n acy + a\,x^n y^2 + bm\,x^{m-1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -x^n x^m aby - x^n acy + a\,x^n y^2 + \frac{b\,x^m m}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = bm\,x^{m-1}$, $f_1(x) = -b\,x^m x^n a - x^n ac$ and $f_2(x) = x^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^n au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{an\, x^n}{x} \\ f_1f_2 &= (-b\, x^m x^n a - x^n ac)\, x^n a \\ f_2^2 f_0 &= x^{2n} a^2 bm\, x^{m-1} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^n au''(x) - \left((-b\, x^m x^n a - x^n ac)\, x^n a + \frac{an\, x^n}{x}\right) u'(x) + x^{2n} a^2 bm\, x^{m-1} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{\_Y'(x)\, x^{1+m+n} ab + \_Y''(x)\, x + abm\, x^{m+n}\_Y(x) + \_Y'(x)\,(a\, x^{n+1} c - n)}{x}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\_Y'(x)\, x^{1+m+n} ab + \_Y''(x)\, x + abm\, x^{m+n}\_Y(x) + \_Y'(x)\,(a\, x^{n+1} c - n)}{x}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\_Y'(x) x^{1+m+n} ab + \_Y''(x) x + abm\, x^{m+n}\_Y(x) + \_Y'(x)(a\, x^{n+1} c - n)}{x}\right\}, \{\_Y(x)\}\right)\right) x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{\_Y'(x) x^{1+m+n} ab + \_Y''(x) x + abm\, x^{m+n}\_Y(x) + \_Y'(x)(a\, x^{n+1} c - n)}{x}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\_Y'(x) x^{1+m+n} ab + \_Y''(x) x + abm\, x^{m+n}\_Y(x) + \_Y'(x)(a\, x^{n+1} c - n)}{x}\right\}, \{\_Y(x)\}\right)\right) x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{\_Y'(x) x^{1+m+n} ab + \_Y''(x) x + abm\, x^{m+n}\_Y(x) + \_Y'(x)(a\, x^{n+1} c - n)}{x}\right\}, \{\_Y(x)\}\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\_Y'(x)x^{1+m+n}ab+\_Y''(x)x+abm\,x^{m+n}\_Y(x)+\_Y'(x)(a\,x^{n+1}c-n)}{x}\right\},\{\_Y(x)\}\right)\right)x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{\_Y'(x)x^{1+m+n}ab+\_Y''(x)x+abm\,x^{m+n}\_Y(x)+\_Y'(x)(a\,x^{n+1}c-n)}{x}\right\},\{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\_Y'(x)x^{1+m+n}ab+\_Y''(x)x+abm\,x^{m+n}\_Y(x)+\_Y'(x)(a\,x^{n+1}c-n)}{x}\right\},\{\_Y(x)\}\right)\right)x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{\_Y'(x)x^{1+m+n}ab+\_Y''(x)x+abm\,x^{m+n}\_Y(x)+\_Y'(x)(a\,x^{n+1}c-n)}{x}\right\},\{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(a*x^n*c*x+x^(n+m)*a*b*x-n)*(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=a*x^n*y(x)^2-a*x^n*(b*x^m+c)*y(x)+b*m*x^(m-1),y(x), singsol=all)
```

No solution found

✓ Solution by Mathematica

Time used: 59.342 (sec). Leaf size: 353

```
DSolve[y'[x]==a*x^n*y[x]^2-a*x^n*(b*x^m+c)*y[x]+b*m*x^(m-1),y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{bm(bx^m+c)^2\left(1+c_1\int_1^x \frac{\exp\left(aK[1]^{n+1}\left(\frac{bK[1]^m}{m+n+1}+\frac{c}{n+1}\right)\right)K[1]^{m-1}}{(bK[1]^m+c)^2}dK[1]\right)}{bc_1m\left(bx^m+c\right)\int_1^x \frac{\exp\left(aK[1]^{n+1}\left(\frac{bK[1]^m}{m+n+1}+\frac{c}{n+1}\right)\right)K[1]^{m-1}}{(bK[1]^m+c)^2}dK[1]+c_1e^{ax^{n+1}\left(\frac{bx^m}{m+n+1}+\frac{c}{n+1}\right)}+b^2mx^m+bcm}$$

$$y(x) \to \frac{bm(bx^m+c)^2\int_1^x \frac{\exp\left(aK[1]^{n+1}\left(\frac{bK[1]^m}{m+n+1}+\frac{c}{n+1}\right)\right)K[1]^{m-1}}{(bK[1]^m+c)^2}dK[1]}{bm\left(bx^m+c\right)\int_1^x \frac{\exp\left(aK[1]^{n+1}\left(\frac{bK[1]^m}{m+n+1}+\frac{c}{n+1}\right)\right)K[1]^{m-1}}{(bK[1]^m+c)^2}dK[1]+e^{ax^{n+1}\left(\frac{bx^m}{m+n+1}+\frac{c}{n+1}\right)}}$$

2.32 problem 32

Internal problem ID [10362]
Internal file name [`OUTPUT/9309_Monday_June_06_2022_01_51_03_PM_81658841/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + an\,x^{n-1}y^2 - c\,x^m(x^n a + b)\,y = -c\,x^m}$$

2.32.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -x^{n-1}an\,y^2 + x^m x^n acy + x^m bcy - c\,x^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{x^n an\,y^2}{x} + x^m x^n acy + x^m bcy - c\,x^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -c\,x^m$, $f_1(x) = ac\,x^m x^n + bc\,x^m$ and $f_2(x) = -an\,x^{n-1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-an\,x^{n-1}u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{an\,x^{n-1}(n-1)}{x} \\
f_1 f_2 &= -(ac\,x^m x^n + bc\,x^m)\,an\,x^{n-1} \\
f_2^2 f_0 &= -a^2 n^2 x^{2n-2} c\,x^m
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-an\,x^{n-1}u''(x) - \left(-\frac{an\,x^{n-1}(n-1)}{x} - (ac\,x^m x^n + bc\,x^m)\,an\,x^{n-1}\right)u'(x) - a^2 n^2 x^{2n-2} c\,x^m u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\left(\int \frac{x^{n-1} e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}}}{(x^n a + b)^2} dx\right) c_2 + c_1\right)(x^n a + b)$$

The above shows that

$$u'(x) = \frac{\left(anc_2(x^n a + b)\left(\int \frac{x^n e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}}}{x(x^n a+b)^2} dx\right) + e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}} c_2 + anc_1(x^n a + b)\right) x^n}{x\,(x^n a + b)}$$

Using the above in (1) gives the solution

$$y = \frac{\left(anc_2(x^n a + b)\left(\int \frac{x^n e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}}}{x(x^n a+b)^2} dx\right) + e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}} c_2 + anc_1(x^n a + b)\right) x^n x^{1-n}}{x\,(x^n a + b)^2 an\left(\left(\int \frac{x^{n-1} e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}}}{(x^n a+b)^2} dx\right) c_2 + c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{an(x^n a + b)\left(\int \frac{x^n e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}}}{x(x^n a+b)^2}dx\right) + a^2 n c_3 x^n + a n c_3 b + e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}}}{an\,(x^n a + b)^2\left(\int \frac{x^n e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}}}{x(x^n a+b)^2}dx + c_3\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{an(x^n a + b)\left(\int \frac{x^n e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}}}{x(x^n a+b)^2}dx\right) + a^2 n c_3 x^n + a n c_3 b + e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}}}{an\,(x^n a + b)^2\left(\int \frac{x^n e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}}}{x(x^n a+b)^2}dx + c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{an(x^n a + b)\left(\int \frac{x^n e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}}}{x(x^n a+b)^2}dx\right) + a^2 n c_3 x^n + a n c_3 b + e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}}}{an\,(x^n a + b)^2\left(\int \frac{x^n e^{\frac{(a(m+1)x^n + b(1+m+n))c\,x^m x}{(m+1)(1+m+n)}}}{x(x^n a+b)^2}dx + c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x*x^m*b*c+x^(n+m)*a*c*x+n-1)*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 199

```
dsolve(diff(y(x),x)=-a*n*x^(n-1)*y(x)^2+c*x^m*(a*x^n+b)*y(x)-c*x^m,y(x), singsol=all)
```

$$y(x) = \frac{an(a\,x^n+b)\left(\int \frac{x^{n-1}\mathrm{e}^{\frac{c\left(a(1+m)x^{1+m+n}+b\,x^{1+m}(1+m+n)\right)}{(1+m)(1+m+n)}}}{(a\,x^n+b)^2}dx\right)-x^n c_1 a-c_1 b+\mathrm{e}^{\frac{c\left(a(1+m)x^{1+m+n}+b\,x^{1+m}(1+m+n)\right)}{(1+m)(1+m+n)}}}{\left(a\left(\int \frac{x^{n-1}\mathrm{e}^{\frac{x(a(1+m)x^n+b(1+m+n))c\,x^m}{(1+m)(1+m+n)}}}{(a\,x^n+b)^2}dx\right)n-c_1\right)(a^2x^{2n}+2x^n ab+b^2)}$$

✓ Solution by Mathematica

Time used: 8.659 (sec). Leaf size: 304

```
DSolve[y'[x]==-a*n*x^(n-1)*y[x]^2+c*x^m*(a*x^n+b)*y[x]-c*x^m,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{ac_1n(ax^n+b)\int_1^x \frac{\exp\left(cK[1]^{m+1}\left(\frac{aK[1]^n}{m+n+1}+\frac{b}{m+1}\right)\right)K[1]^{n-1}}{(aK[1]^n+b)^2}dK[1]+a^2nx^n+c_1e^{cx^{m+1}\left(\frac{ax^n}{m+n+1}+\frac{b}{m+1}\right)}+abn}{an\,(ax^n+b)^2\left(1+c_1\int_1^x \frac{\exp\left(cK[1]^{m+1}\left(\frac{aK[1]^n}{m+n+1}+\frac{b}{m+1}\right)\right)K[1]^{n-1}}{(aK[1]^n+b)^2}dK[1]\right)}$$

$$y(x) \to \frac{\frac{e^{cx^{m+1}\left(\frac{ax^n}{m+n+1}+\frac{b}{m+1}\right)}}{an\int_1^x \frac{\exp\left(cK[1]^{m+1}\left(\frac{aK[1]^n}{m+n+1}+\frac{b}{m+1}\right)\right)K[1]^{n-1}}{(aK[1]^n+b)^2}dK[1]}+ax^n+b}{(ax^n+b)^2}$$

2.33 problem 33

Internal problem ID [10363]
Internal file name [OUTPUT/9310_Monday_June_06_2022_01_51_07_PM_78619454/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - a\,x^n y^2 - b\,x^m y = ck\,x^{k-1} - bc\,x^{k+m} - a\,c^2 x^{n+2k}}$$

2.33.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,x^n y^2 + x^m b y + ck\,x^{k-1} - bc\,x^{k+m} - a\,c^2 x^{n+2k} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,x^n y^2 + x^m b y + \frac{ck\,x^k}{x} - bc\,x^k x^m - a\,c^2 x^n x^{2k}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = ck\,x^{k-1} - bc\,x^{k+m} - a\,c^2 x^{n+2k}$, $f_1(x) = b\,x^m$ and $f_2(x) = x^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^n a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{an\,x^n}{x} \\
f_1f_2 &= b\,x^m x^n a \\
f_2^2f_0 &= x^{2n}a^2\left(ck\,x^{k-1} - bc\,x^{k+m} - a\,c^2x^{n+2k}\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^n a u''(x) - \left(\frac{an\,x^n}{x} + b\,x^m x^n a\right)u'(x) + x^{2n}a^2\left(ck\,x^{k-1} - bc\,x^{k+m} - a\,c^2x^{n+2k}\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \_Y'(x)\left(\frac{n}{x} + b\,x^m\right) + x^n a\left(ck\,x^{k-1} - bc\,x^{k+m} - a\,c^2x^{n+2k}\right)\_Y(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \_Y'(x)\left(\frac{n}{x} + b\,x^m\right) + x^n a\left(ck\,x^{k-1} - bc\,x^{k+m} - a\,c^2x^{n+2k}\right)\_Y(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \_Y'(x)\left(\frac{n}{x} + b\,x^m\right) + x^n a\left(ck\,x^{k-1} - bc\,x^{k+m} - a\,c^2x^{n+2k}\right)\_Y(x)\right\}, \{\_Y(x)\}\right)\right.}{a\,\mathrm{DESol}\left(\left\{\_Y''(x) - \_Y'(x)\left(\frac{n}{x} + b\,x^m\right) + x^n a\left(ck\,x^{k-1} - bc\,x^{k+m} - a\,c^2x^{n+2k}\right)\_Y(x)\right\}, \{\_Y(x)\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-x^{n+2k}x^{n+1}\_Y(x)a^2c^2 - ac\_Y(x)(-k\,x^{k-1} + x^{k+m}b)x^{n+1} + \_Y''(x)x - \_Y'(x)(x^{m+1}b + n)}{x}\right\}, \{\_Y(x)\}\right)\right.}{a\,\mathrm{DESol}\left(\left\{\frac{-x^{2n+2k+1}\_Y(x)a^2c^2 - x^{k+1+m+n}\_Y(x)abc + \_Y''(x)x + x^{k+n}\_Y(x)ack - \_Y'(x)(x^{m+1}b+n)}{x}\right\}, \{\_Y(x\right.}$$

Summary
The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-x^{n+2k}x^{n+1}\_\_Y(x)a^2c^2 - ac\_\_Y(x)(-k\,x^{k-1}+x^{k+m}b)x^{n+1} + \_\_Y''(x)x - \_\_Y'(x)(x^{m+1}b+n)}{x}\right\}, \{\_\_Y(x)\}\right)}{a\,\mathrm{DESol}\left(\left\{\frac{-x^{2n+2k+1}\_\_Y(x)a^2c^2 - x^{k+1+m+n}\_\_Y(x)abc + \_\_Y''(x)x + x^{k+n}\_\_Y(x)ack - \_\_Y'(x)(x^{m+1}b+n)}{x}\right\}, \{\_\_Y(x)}\tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-x^{n+2k}x^{n+1}\_\_Y(x)a^2c^2 - ac\_\_Y(x)(-k\,x^{k-1}+x^{k+m}b)x^{n+1} + \_\_Y''(x)x - \_\_Y'(x)(x^{m+1}b+n)}{x}\right\}, \{\_\_Y(x)\}\right)}{a\,\mathrm{DESol}\left(\left\{\frac{-x^{2n+2k+1}\_\_Y(x)a^2c^2 - x^{k+1+m+n}\_\_Y(x)abc + \_\_Y''(x)x + x^{k+n}\_\_Y(x)ack - \_\_Y'(x)(x^{m+1}b+n)}{x}\right\}, \{\_\_Y(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x*x^m*b+n)*(diff(y(x), x))/x-
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=a*x^n*y(x)^2+b*x^m*y(x)+c*k*x^(k-1)-b*c*x^(m+k)-a*c^2*x^(n+2*k),y(x), si
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==a*x^n*y[x]^2+b*x^m*y[x]+c*k*x^(k-1)-b*c*x^(m+k)-a*c^2*x^(n+2*k),y[x],x,Include
```

Not solved

2.34 problem 34

Internal problem ID [10364]
Internal file name [`OUTPUT/9311_Monday_June_06_2022_01_51_13_PM_28351703/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{y'x - a y^2 - yb = c\,x^{2b}}$$

2.34.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,y^2 + by + c\,x^{2b}}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,y^2}{x} + \frac{c\,x^{2b}}{x} + \frac{by}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{c\,x^{2b}}{x}$, $f_1(x) = \frac{b}{x}$ and $f_2(x) = \frac{a}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{au}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{a}{x^2} \\ f_1 f_2 &= \frac{ab}{x^2} \\ f_2^2 f_0 &= \frac{a^2 c\, x^{2b}}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{a u''(x)}{x} - \left(-\frac{a}{x^2} + \frac{ab}{x^2}\right) u'(x) + \frac{a^2 c\, x^{2b} u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(\frac{x^b \sqrt{ca}}{b}\right) + c_2 \cos\left(\frac{x^b \sqrt{ca}}{b}\right)$$

The above shows that

$$u'(x) = \frac{x^b \sqrt{ca}\left(c_1 \cos\left(\frac{x^b \sqrt{ca}}{b}\right) - c_2 \sin\left(\frac{x^b \sqrt{ca}}{b}\right)\right)}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{x^b \sqrt{ca}\left(c_1 \cos\left(\frac{x^b \sqrt{ca}}{b}\right) - c_2 \sin\left(\frac{x^b \sqrt{ca}}{b}\right)\right)}{a\left(c_1 \sin\left(\frac{x^b \sqrt{ca}}{b}\right) + c_2 \cos\left(\frac{x^b \sqrt{ca}}{b}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-c_3 \cos\left(\frac{x^b \sqrt{ca}}{b}\right) + \sin\left(\frac{x^b \sqrt{ca}}{b}\right)\right) x^b \sqrt{ca}}{\left(c_3 \sin\left(\frac{x^b \sqrt{ca}}{b}\right) + \cos\left(\frac{x^b \sqrt{ca}}{b}\right)\right) a}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(-c_3 \cos\left(\frac{x^b \sqrt{ca}}{b}\right) + \sin\left(\frac{x^b \sqrt{ca}}{b}\right)\right) x^b \sqrt{ca}}{\left(c_3 \sin\left(\frac{x^b \sqrt{ca}}{b}\right) + \cos\left(\frac{x^b \sqrt{ca}}{b}\right)\right) a} \tag{1}$$

Verification of solutions

$$y = \frac{\left(-c_3 \cos\left(\frac{x^b \sqrt{ca}}{b}\right) + \sin\left(\frac{x^b \sqrt{ca}}{b}\right)\right) x^b \sqrt{ca}}{\left(c_3 \sin\left(\frac{x^b \sqrt{ca}}{b}\right) + \cos\left(\frac{x^b \sqrt{ca}}{b}\right)\right) a}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve(x*diff(y(x),x)=a*y(x)^2+b*y(x)+c*x^(2*b),y(x), singsol=all)
```

$$y(x) = \frac{\tan\left(\frac{x^b \sqrt{a}\, \sqrt{c} - c_1 b}{b}\right) \sqrt{c}\, x^b}{\sqrt{a}}$$

✓ Solution by Mathematica

Time used: 0.544 (sec). Leaf size: 139

```
DSolve[x*y'[x]==a*y[x]^2+b*y[x]+c*x^(2*b),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{c}x^b\left(-\cos\left(\frac{\sqrt{a}\sqrt{c}x^b}{b}\right)+c_1\sin\left(\frac{\sqrt{a}\sqrt{c}x^b}{b}\right)\right)}{\sqrt{a}\left(\sin\left(\frac{\sqrt{a}\sqrt{c}x^b}{b}\right)+c_1\cos\left(\frac{\sqrt{a}\sqrt{c}x^b}{b}\right)\right)}$$

$$y(x) \to \frac{\sqrt{c}x^b\tan\left(\frac{\sqrt{a}\sqrt{c}x^b}{b}\right)}{\sqrt{a}}$$

2.35 problem 35

Internal problem ID [10365]
Internal file name [`OUTPUT/9312_Monday_June_06_2022_01_51_13_PM_77067681/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{y'x - ay^2 - yb = c\,x^n}$$

2.35.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,y^2 + by + c\,x^n}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,y^2}{x} + \frac{c\,x^n}{x} + \frac{by}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{c\,x^n}{x}$, $f_1(x) = \frac{b}{x}$ and $f_2(x) = \frac{a}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{au}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{a}{x^2} \\ f_1f_2 &= \frac{ab}{x^2} \\ f_2^2 f_0 &= \frac{a^2 c\, x^n}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{au''(x)}{x} - \left(-\frac{a}{x^2} + \frac{ab}{x^2}\right) u'(x) + \frac{a^2 c\, x^n u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\text{BesselJ}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_1 + \text{BesselY}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_2\right) x^{\frac{b}{2}}$$

The above shows that

$$\begin{aligned} u'(x) = -x^{-1+\frac{b}{2}}\Bigg(\sqrt{ca}\, x^{\frac{n}{2}} \,\text{BesselJ}\left(\frac{b+n}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_1 \\ + \text{BesselY}\left(\frac{b+n}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) \sqrt{ca}\, x^{\frac{n}{2}} c_2 \\ - b\left(\text{BesselJ}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_1 + \text{BesselY}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_2\right)\Bigg) \end{aligned}$$

Using the above in (1) gives the solution

$$y = \frac{x^{-1+\frac{b}{2}}\left(\sqrt{ca}\, x^{\frac{n}{2}} \,\text{BesselJ}\left(\frac{b+n}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_1 + \text{BesselY}\left(\frac{b+n}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) \sqrt{ca}\, x^{\frac{n}{2}} c_2 - b\left(\text{BesselJ}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c\right.\right.}{a\left(\text{BesselJ}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_1 + \text{BesselY}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\mathrm{BesselJ}\left(\frac{b+n}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 + \mathrm{BesselY}\left(\frac{b+n}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right)\right)\sqrt{ca}\, x^{\frac{n}{2}} - b\left(\mathrm{BesselJ}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 + \mathrm{BesselY}\left(\cdots\right.\right.}{a\left(\mathrm{BesselJ}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 + \mathrm{BesselY}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right)\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\mathrm{BesselJ}\left(\frac{b+n}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 + \mathrm{BesselY}\left(\frac{b+n}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right)\right)\sqrt{ca}\, x^{\frac{n}{2}} - b\left(\mathrm{BesselJ}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 + \mathrm{BesselY}\left(\cdots\right.\right.}{a\left(\mathrm{BesselJ}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 + \mathrm{BesselY}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\mathrm{BesselJ}\left(\frac{b+n}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 + \mathrm{BesselY}\left(\frac{b+n}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right)\right)\sqrt{ca}\, x^{\frac{n}{2}} - b\left(\mathrm{BesselJ}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 + \mathrm{BesselY}\left(\cdots\right.\right.}{a\left(\mathrm{BesselJ}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 + \mathrm{BesselY}\left(\frac{b}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (b-1)*(diff(y(x), x))/x-a*c*x^
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         <- Bessel successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 164

```
dsolve(x*diff(y(x),x)=a*y(x)^2+b*y(x)+c*x^n,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{ac}\left(\operatorname{BesselY}\left(\frac{b+n}{n}, \frac{2\sqrt{ac}\,x^{\frac{n}{2}}}{n}\right)c_1 + \operatorname{BesselJ}\left(\frac{b+n}{n}, \frac{2\sqrt{ac}\,x^{\frac{n}{2}}}{n}\right)\right)x^{\frac{n}{2}} - b\left(\operatorname{BesselY}\left(\frac{b}{n}, \frac{2\sqrt{ac}\,x^{\frac{n}{2}}}{n}\right)c_1 + \operatorname{BesselJ}\left(\right.\right.}{a\left(\operatorname{BesselY}\left(\frac{b}{n}, \frac{2\sqrt{ac}\,x^{\frac{n}{2}}}{n}\right)c_1 + \operatorname{BesselJ}\left(\frac{b}{n}, \frac{2\sqrt{ac}\,x^{\frac{n}{2}}}{n}\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.513 (sec). Leaf size: 402

```
DSolve[x*y'[x]==a*y[x]^2+b*y[x]+c*x^n,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{a}\sqrt{c}x^{n/2}\left(-2\operatorname{BesselJ}\left(\frac{b}{n}-1, \frac{2\sqrt{a}\sqrt{c}x^{n/2}}{n}\right) + c_1\left(\operatorname{BesselJ}\left(1-\frac{b}{n}, \frac{2\sqrt{a}\sqrt{c}x^{n/2}}{n}\right) - \operatorname{BesselJ}\left(-\frac{b+n}{n}, \frac{2\sqrt{a}\sqrt{c}x^{n/2}}{n}\right.\right.\right.}{2a\left(\operatorname{BesselJ}\left(\frac{b}{n}, \frac{2\sqrt{a}\sqrt{c}x^{n/2}}{n}\right) + c_1\operatorname{BesselJ}\left(-\frac{b}{n}, \frac{2\sqrt{a}\sqrt{c}x^{n/2}}{n}\right)\right)}$$

$$y(x) \to -\frac{-\sqrt{a}\sqrt{c}x^{n/2}\operatorname{BesselJ}\left(1-\frac{b}{n}, \frac{2\sqrt{a}\sqrt{c}x^{n/2}}{n}\right) + \sqrt{a}\sqrt{c}x^{n/2}\operatorname{BesselJ}\left(-\frac{b+n}{n}, \frac{2\sqrt{a}\sqrt{c}x^{n/2}}{n}\right) + b\operatorname{BesselJ}\left(-\frac{b}{n}, \frac{2\sqrt{}}{}\right.}{2a\operatorname{BesselJ}\left(-\frac{b}{n}, \frac{2\sqrt{a}\sqrt{c}x^{n/2}}{n}\right)}$$

2.36 problem 36

Internal problem ID [10366]
Internal file name [`OUTPUT/9313_Monday_June_06_2022_01_51_14_PM_27493872/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{y'x - ay^2 - (n + b\,x^n)\,y = c\,x^{2n}}$$

2.36.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^n by + a\,y^2 + c\,x^{2n} + ny}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x^n by}{x} + \frac{a\,y^2}{x} + \frac{c\,x^{2n}}{x} + \frac{ny}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{c\,x^{2n}}{x}$, $f_1(x) = \frac{n+b\,x^n}{x}$ and $f_2(x) = \frac{a}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{au}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{a}{x^2} \\ f_1 f_2 &= \frac{(n + b\,x^n)\,a}{x^2} \\ f_2^2 f_0 &= \frac{a^2 c\,x^{2n}}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{a u''(x)}{x} - \left(-\frac{a}{x^2} + \frac{(n + b\,x^n)\,a}{x^2}\right) u'(x) + \frac{a^2 c\,x^{2n} u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\frac{x^n b}{2n}} \left(c_1 \sinh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) + c_2 \cosh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right)\right)$$

The above shows that

$$u'(x) = \frac{\left(\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n c_1 + c_2 b\right) \cosh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) + \sinh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) \left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n c_2 + c_1 b\right)\right) x^{n-1} \mathrm{e}^{\frac{x^n b}{2n}}}{2}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n c_1 + c_2 b\right) \cosh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) + \sinh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) \left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n c_2 + c_1 b\right)\right) x^{n-1} x}{2a \left(c_1 \sinh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) + c_2 \cosh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x^n \left(\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n c_3 + b \right) \cosh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2} \right) + \sinh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2} \right) \left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n + b c_3 \right) \right)}{2a \left(c_3 \sinh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2} \right) + \cosh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2} \right) \right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^n \left(\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n c_3 + b \right) \cosh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2} \right) + \sinh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2} \right) \left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n + b c_3 \right) \right)}{2a \left(c_3 \sinh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2} \right) + \cosh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2} \right) \right)} \tag{1}$$

Verification of solutions

$$y = -\frac{x^n \left(\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n c_3 + b \right) \cosh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2} \right) + \sinh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2} \right) \left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n + b c_3 \right) \right)}{2a \left(c_3 \sinh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2} \right) + \cosh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2} \right) \right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 69

```
dsolve(x*diff(y(x),x)=a*y(x)^2+(n+b*x^n)*y(x)+c*x^(2*n),y(x), singsol=all)
```

$$y(x) = -\frac{x^n\left(b^2-\sqrt{4a\,b^2c-b^4}\,\tan\left(\frac{\sqrt{4a\,b^2c-b^4}\,(b\,x^n+c_1n)}{2b^2n}\right)\right)}{2ab}$$

✓ Solution by Mathematica

Time used: 1.07 (sec). Leaf size: 114

```
DSolve[x*y'[x]==a*y[x]^2+(n+b*x^n)*y[x]+c*x^(2*n),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^n\left(-b+\frac{\sqrt{b^2-4ac}\left(-e^{\frac{x^n\sqrt{b^2-4ac}}{n}}+c_1\right)}{e^{\frac{x^n\sqrt{b^2-4ac}}{n}}+c_1}\right)}{2a}$$
$$y(x) \to \frac{x^n\left(\sqrt{b^2-4ac}-b\right)}{2a}$$

2.37 problem 37

Internal problem ID [10367]
Internal file name [`OUTPUT/9314_Monday_June_06_2022_01_51_15_PM_48702340/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{y'x - y^2x - ay = b\,x^n}$$

2.37.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x\,y^2 + ya + b\,x^n}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \frac{b\,x^n}{x} + \frac{ya}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{b\,x^n}{x}$, $f_1(x) = \frac{a}{x}$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \frac{a}{x} \\ f_2^2 f_0 &= \frac{b\, x^n}{x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \frac{a u'(x)}{x} + \frac{b\, x^n u(x)}{x} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\text{BesselY}\left(\frac{-1-a}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_2 + \text{BesselJ}\left(\frac{-1-a}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_1\right) x^{\frac{1}{2}+\frac{a}{2}}$$

The above shows that

$$\begin{aligned} u'(x) = x^{\frac{a}{2}+\frac{n}{2}} \sqrt{b} \Bigg(&- \text{BesselJ}\left(\frac{-a+n}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_1 \\ &- \text{BesselY}\left(\frac{-a+n}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_2 \Bigg) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{x^{\frac{a}{2}+\frac{n}{2}} \sqrt{b} \left(-\text{BesselJ}\left(\frac{-a+n}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_1 - \text{BesselY}\left(\frac{-a+n}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_2\right) x^{-\frac{1}{2}-\frac{a}{2}}}{\text{BesselY}\left(\frac{-1-a}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_2 + \text{BesselJ}\left(\frac{-1-a}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\sqrt{b}\, x^{-\frac{1}{2}+\frac{n}{2}} \left(\text{BesselJ}\left(\frac{-a+n}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_3 + \text{BesselY}\left(\frac{-a+n}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right)\right)}{\text{BesselY}\left(\frac{-1-a}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) + \text{BesselJ}\left(\frac{-1-a}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_3}$$

Summary

The solution(s) found are the following

$$
y = \frac{\sqrt{b}\, x^{-\frac{1}{2}+\frac{n}{2}} \left(\operatorname{BesselJ}\left(\frac{-a+n}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_3 + \operatorname{BesselY}\left(\frac{-a+n}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right)\right)}{\operatorname{BesselY}\left(\frac{-1-a}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) + \operatorname{BesselJ}\left(\frac{-1-a}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_3} \tag{1}
$$

Verification of solutions

$$
y = \frac{\sqrt{b}\, x^{-\frac{1}{2}+\frac{n}{2}} \left(\operatorname{BesselJ}\left(\frac{-a+n}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_3 + \operatorname{BesselY}\left(\frac{-a+n}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right)\right)}{\operatorname{BesselY}\left(\frac{-1-a}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) + \operatorname{BesselJ}\left(\frac{-1-a}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_3}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (diff(y(x), x))*a/x-b*x^(n-1)*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         <- Bessel successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 139

```
dsolve(x*diff(y(x),x)=x*y(x)^2+a*y(x)+b*x^n,y(x), singsol=all)
```

$$y(x) = \frac{x^{\frac{n}{2}-\frac{1}{2}}\sqrt{b}\left(\text{BesselY}\left(\frac{-a+n}{n+1}, \frac{2\sqrt{b}\,x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right)c_1 + \text{BesselJ}\left(\frac{-a+n}{n+1}, \frac{2\sqrt{b}\,x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right)\right)}{\text{BesselY}\left(\frac{-a-1}{n+1}, \frac{2\sqrt{b}\,x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right)c_1 + \text{BesselJ}\left(\frac{-a-1}{n+1}, \frac{2\sqrt{b}\,x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right)}$$

✓ Solution by Mathematica

Time used: 1.381 (sec). Leaf size: 855

```
DSolve[x*y'[x]==x*y[x]^2+a*y[x]+b*x^n,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to$$

$$-\frac{\sqrt{b}(x^n)^{\frac{n+1}{2n}}\,\text{Gamma}\left(\frac{a+n+2}{n+1}\right)\text{BesselJ}\left(\frac{a-n}{n+1}, \frac{2\sqrt{b}(x^n)^{\frac{n+1}{2n}}}{n+1}\right) - \sqrt{b}(x^n)^{\frac{n+1}{2n}}\,\text{Gamma}\left(\frac{a+n+2}{n+1}\right)\text{BesselJ}\left(\frac{a+n+2}{n+1}\right.}{}$$

$$y(x) \to -\frac{\frac{\sqrt{b}(x^n)^{\frac{n+1}{2n}}\left(\text{BesselJ}\left(-\frac{a+n+2}{n+1}, \frac{2\sqrt{b}(x^n)^{\frac{n+1}{2n}}}{n+1}\right) - \text{BesselJ}\left(\frac{n-a}{n+1}, \frac{2\sqrt{b}(x^n)^{\frac{n+1}{2n}}}{n+1}\right)\right)}{\text{BesselJ}\left(-\frac{a+1}{n+1}, \frac{2\sqrt{b}(x^n)^{\frac{n+1}{2n}}}{n+1}\right)} + a + 1}{2x}$$

$$y(x) \to -\frac{\frac{\sqrt{b}(x^n)^{\frac{n+1}{2n}}\left(\text{BesselJ}\left(-\frac{a+n+2}{n+1}, \frac{2\sqrt{b}(x^n)^{\frac{n+1}{2n}}}{n+1}\right) - \text{BesselJ}\left(\frac{n-a}{n+1}, \frac{2\sqrt{b}(x^n)^{\frac{n+1}{2n}}}{n+1}\right)\right)}{\text{BesselJ}\left(-\frac{a+1}{n+1}, \frac{2\sqrt{b}(x^n)^{\frac{n+1}{2n}}}{n+1}\right)} + a + 1}{2x}$$

2.38 problem 38

Internal problem ID [10368]
Internal file name [`OUTPUT/9315_Monday_June_06_2022_01_51_16_PM_89246891/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{y'x + a_3 x y^2 + a_2 y = -a_1 x - a_0}$$

2.38.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{a_3 x\, y^2 + a_1 x + a_2 y + a_0}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a_3 y^2 - a_1 - \frac{a_2 y}{x} - \frac{a_0}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{a_1 x + a_0}{x}$, $f_1(x) = -\frac{a_2}{x}$ and $f_2(x) = -a_3$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-a_3 u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \frac{a_2 a_3}{x} \\ f_2^2 f_0 &= -\frac{a_3^2 (a_1 x + a_0)}{x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-a_3 u''(x) - \frac{a_2 a_3 u'(x)}{x} - \frac{a_3^2 (a_1 x + a_0) u(x)}{x} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = e^{-i\sqrt{a_1}\,\sqrt{a_3}\,x} \left(\text{KummerM}\left(\frac{i a_0 \sqrt{a_3} + a_2 \sqrt{a_1}}{2\sqrt{a_1}}, a_2, 2i\sqrt{a_1}\,\sqrt{a_3}\,x\right) c_1 + \text{KummerU}\left(\frac{i a_0 \sqrt{a_3} + a_2 \sqrt{a_1}}{2\sqrt{a_1}}, a_2, 2i\sqrt{a_1}\,\sqrt{a_3}\,x\right) c_2\right)$$

The above shows that

$$u'(x) = -\frac{\left(-\frac{\left(\left(-\frac{1}{2}a_2^2 + a_2\right) a_1^{\frac{3}{2}} + a_0 \left(i\sqrt{a_3}\, a_1 - \frac{a_0 a_3 \sqrt{a_1}}{2}\right)\right) c_2 \,\text{KummerU}\left(\frac{(a_2+2)\sqrt{a_1} + i a_0 \sqrt{a_3}}{2\sqrt{a_1}}, a_2, 2i\sqrt{a_1}\,\sqrt{a_3}\,x\right)}{2} - \frac{c_1 \left(i a_1 \sqrt{a_3}\, a_0 + a_1^{\frac{3}{2}} a_2\right) \text{Kumme}}{}\right.}{}$$

Using the above in (1) gives the solution

$$y = -\frac{-\frac{\left(\left(-\frac{1}{2}a_2^2 + a_2\right) a_1^{\frac{3}{2}} + a_0 \left(i\sqrt{a_3}\, a_1 - \frac{a_0 a_3 \sqrt{a_1}}{2}\right)\right) c_2 \,\text{KummerU}\left(\frac{(a_2+2)\sqrt{a_1} + i a_0 \sqrt{a_3}}{2\sqrt{a_1}}, a_2, 2i\sqrt{a_1}\,\sqrt{a_3}\,x\right)}{2} - \frac{c_1 \left(i a_1 \sqrt{a_3}\, a_0 + a_1^{\frac{3}{2}} a_2\right) \text{KummerM}}{}}{a_1^{\frac{3}{2}} x a_3 \left(\text{KummerM}\left(\frac{ia_0}{}\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\frac{\left(-\left(\frac{1}{2}a_2^2-a_2\right)a_1^{\frac{3}{2}}+ia_1\sqrt{a_3}\,a_0-\frac{a_0^2a_3\sqrt{a_1}}{2}\right)\mathrm{KummerU}\left(\frac{(a_2+2)\sqrt{a_1}+ia_0\sqrt{a_3}}{2\sqrt{a_1}},a_2,2i\sqrt{a_1}\,\sqrt{a_3}\,x\right)}{2}+\frac{c_3\left(ia_1\sqrt{a_3}\,a_0+a_1^{\frac{3}{2}}a_2\right)\mathrm{KummerM}\left(\frac{(a_2+2)\sqrt{a}}{2\sqrt{}}\right.}{2}}{a_1^{\frac{3}{2}}xa_3\left(\mathrm{KummerM}\left(\frac{ia_0\sqrt{a_3}+}{2\sqrt{a}}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\frac{\left(-\left(\frac{1}{2}a_2^2-a_2\right)a_1^{\frac{3}{2}}+ia_1\sqrt{a_3}\,a_0-\frac{a_0^2a_3\sqrt{a_1}}{2}\right)\mathrm{KummerU}\left(\frac{(a_2+2)\sqrt{a_1}+ia_0\sqrt{a_3}}{2\sqrt{a_1}},a_2,2i\sqrt{a_1}\,\sqrt{a_3}\,x\right)}{2}+\frac{c_3\left(ia_1\sqrt{a_3}\,a_0+a_1^{\frac{3}{2}}a_2\right)\mathrm{KummerM}\left(\frac{(a_2+2)\sqrt{a}}{2\sqrt{}}\right.}{2}}{a_1^{\frac{3}{2}}xa_3\left(\mathrm{KummerM}\left(\frac{ia_0\sqrt{a_3}+}{2\sqrt{a}}\right.\right.} \quad (1)$$

Verification of solutions

$$y = \frac{\frac{\left(-\left(\frac{1}{2}a_2^2-a_2\right)a_1^{\frac{3}{2}}+ia_1\sqrt{a_3}\,a_0-\frac{a_0^2a_3\sqrt{a_1}}{2}\right)\mathrm{KummerU}\left(\frac{(a_2+2)\sqrt{a_1}+ia_0\sqrt{a_3}}{2\sqrt{a_1}},a_2,2i\sqrt{a_1}\,\sqrt{a_3}\,x\right)}{2}+\frac{c_3\left(ia_1\sqrt{a_3}\,a_0+a_1^{\frac{3}{2}}a_2\right)\mathrm{KummerM}\left(\frac{(a_2+2)\sqrt{a}}{2\sqrt{}}\right.}{2}}{a_1^{\frac{3}{2}}xa_3\left(\mathrm{KummerM}\left(\frac{ia_0\sqrt{a_3}+}{2\sqrt{a}}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Abel AIR successful: ODE belongs to the 1F1 2-parameter class`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 403

```
dsolve(x*diff(y(x),x)+a__3*x*y(x)^2+a__2*y(x)+a__1*x+a__0=0,y(x), singsol=all)
```

$$y(x) = -\frac{4a_1\left(a_1^3a_3\left(a_3a_0 - a_2\sqrt{-a_1a_3}\right)\operatorname{KummerM}\left(\frac{\sqrt{-a_1a_3}\,a_0+a_1(a_2+2)}{2a_1}, a_2+1, 2x\sqrt{-a_1a_3}\right.\right.}{4a_1^3a_3^2\left(\sqrt{-a_1a_3}\,a_0 + a_1a_2\right)\operatorname{KummerM}\left(\frac{\sqrt{-a_1a_3}\,a_0+a_1(a_2+2)}{2a_1}, a_2+1, 2x\sqrt{-a_1a_3}\right) - c_1\sqrt{-a_1a_3}\,\left(a_0^2a_3 + a\right.}$$

✓ Solution by Mathematica

Time used: 0.748 (sec). Leaf size: 541

```
DSolve[x*y'[x]+a3*x*y[x]^2+a2*y[x]+a1*x+a0==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{i\left(\sqrt{\text{a1}}c_1\,\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{i\sqrt{\text{a3}}\text{a0}}{\sqrt{\text{a1}}}+\text{a2}\right),\text{a2},2i\sqrt{\text{a1}}\sqrt{\text{a3}}x\right)+c_1\left(\sqrt{\text{a1}}\text{a2}+i\text{a0}\sqrt{\text{a3}}\right)\text{Hypergeom}\right.}{\sqrt{\text{a3}}\left(c_1\,\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{i\sqrt{\text{a3}}}{\sqrt{\text{a}}}\right.\right.\right.}$$

$$y(x) \to \frac{\frac{\left(\text{a0}\sqrt{\text{a3}}-i\sqrt{\text{a1}}\text{a2}\right)\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{i\sqrt{\text{a3}}\text{a0}}{\sqrt{\text{a1}}}+\text{a2}+2\right),\text{a2}+1,2i\sqrt{\text{a1}}\sqrt{\text{a3}}x\right)}{\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{i\sqrt{\text{a3}}\text{a0}}{\sqrt{\text{a1}}}+\text{a2}\right),\text{a2},2i\sqrt{\text{a1}}\sqrt{\text{a3}}x\right)}-i\sqrt{\text{a1}}}{\sqrt{\text{a3}}}$$

$$y(x) \to \frac{\frac{\left(\text{a0}\sqrt{\text{a3}}-i\sqrt{\text{a1}}\text{a2}\right)\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{i\sqrt{\text{a3}}\text{a0}}{\sqrt{\text{a1}}}+\text{a2}+2\right),\text{a2}+1,2i\sqrt{\text{a1}}\sqrt{\text{a3}}x\right)}{\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{i\sqrt{\text{a3}}\text{a0}}{\sqrt{\text{a1}}}+\text{a2}\right),\text{a2},2i\sqrt{\text{a1}}\sqrt{\text{a3}}x\right)}-i\sqrt{\text{a1}}}{\sqrt{\text{a3}}}$$

2.39 problem 39

Internal problem ID [10369]
Internal file name [`OUTPUT/9316_Monday_June_06_2022_01_51_19_PM_59962069/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Riccati]
```

$$\boxed{y'x - a\,x^n y^2 - yb = c\,x^{-n}}$$

2.39.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{a\,x^n y^2 + by + c\,x^{-n}}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2 + \frac{(a\,x^n y^2 + by + c\,x^{-n})\,(b_3 - a_2)}{x} - \frac{(a\,x^n y^2 + by + c\,x^{-n})^2\, a_3}{x^2} \\ &- \left(\frac{\frac{x^n a n\, y^2}{x} - \frac{c\,x^{-n} n}{x}}{x} - \frac{a\,x^n y^2 + by + c\,x^{-n}}{x^2}\right)(x a_2 + y a_3 + a_1) \\ &- \frac{(2a\,x^n y + b)\,(x b_2 + y b_3 + b_1)}{x} = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^n x^{-n} ac\,y^2 a_3 + x^n anx\,y^2 a_2 + x^{2n} a^2 y^4 a_3 - x^n a\,y^3 a_3 - x^n a\,y^2 a_1 - x^{-n} cna_1 - x^{-n} cxb_3 - x^{-n} cya_3 - b_2 x}{}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}&-2x^n x^{-n} ac\,y^2 a_3 - x^n anx\,y^2 a_2 - x^{2n} a^2 y^4 a_3 + x^n a\,y^3 a_3 + x^n a\,y^2 a_1 \\ &+ x^{-n} cna_1 + x^{-n} cxb_3 + x^{-n} cya_3 + b_2 x^2 - 2x^n ab\,y^3 a_3 - x^n an\,y^3 a_3 \\ &- x^n an\,y^2 a_1 - 2x^n a\,x^2 y b_2 - x^n ax\,y^2 b_3 - 2x^n axyb_1 - 2x^{-n} bcya_3 + x^{-n} cnxa_2 \\ &+ x^{-n} cnya_3 - b^2 y^2 a_3 - b\,x^2 b_2 + b\,y^2 a_3 - x^{-2n} c^2 a_3 - bxb_1 + bya_1 + x^{-n} ca_1 = 0\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}&-\big(x^{3n} anx\,y^2 a_2 - b_2 x^2 x^{2n} - cna_1 x^n - cxb_3 x^n - cya_3 x^n + x^{4n} a^2 y^4 a_3 \\ &- x^{3n} a\,y^3 a_3 - x^{3n} a\,y^2 a_1 + b^2 y^2 a_3 x^{2n} + b\,x^2 b_2 x^{2n} - b\,y^2 a_3 x^{2n} \\ &+ bxb_1 x^{2n} - bya_1 x^{2n} + 2bcya_3 x^n - cnxa_2 x^n - cnya_3 x^n \\ &+ 2x^{3n} ab\,y^3 a_3 + x^{3n} an\,y^3 a_3 + x^{3n} an\,y^2 a_1 + 2x^{3n} a\,x^2 y b_2 \\ &+ x^{3n} ax\,y^2 b_3 + 2x^{3n} axyb_1 + 2ac\,y^2 a_3 x^{2n} + c^2 a_3 - ca_1 x^n\big)\, x^{-2n} = 0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, x^n\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, x^n = v_3\}$$

The above PDE (6E) now becomes

$$-\frac{v_3^4 a^2 v_2^4 a_3 + 2v_3^3 abv_2^3 a_3 + v_3^3 anv_1 v_2^2 a_2 + v_3^3 anv_2^3 a_3 + v_3^3 anv_2^2 a_1 - v_3^3 av_2^3 a_3 + 2v_3^3 av_1^2 v_2 b_2 + v_3^3 av_1 v_2^2 b_3 + 2ac}{} \tag{7E}$$

$$= 0$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &-2ab_2 v_3 v_1^2 v_2 + (-bb_2 + b_2) v_1^2 + (-ana_2 - ab_3) v_1 v_2^2 v_3 - 2ab_1 v_3 v_1 v_2 \\ &- bb_1 v_1 + \frac{(cna_2 + cb_3) v_1}{v_3} - a^2 a_3 v_3^2 v_2^4 + (-2aba_3 - ana_3 + aa_3) v_2^3 v_3 \\ &+ (-ana_1 + aa_1) v_2^2 v_3 + \left(-2aca_3 - b^2 a_3 + ba_3\right) v_2^2 + ba_1 v_2 \\ &+ \frac{(-2bca_3 + cna_3 + ca_3) v_2}{v_3} + \frac{cna_1 + ca_1}{v_3} - \frac{c^2 a_3}{v_3^2} = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} ba_1 &= 0 \\ -2ab_1 &= 0 \\ -2ab_2 &= 0 \\ -a^2 a_3 &= 0 \\ -bb_1 &= 0 \\ -c^2 a_3 &= 0 \\ -bb_2 + b_2 &= 0 \\ -ana_1 + aa_1 &= 0 \\ cna_1 + ca_1 &= 0 \\ -ana_2 - ab_3 &= 0 \\ cna_2 + cb_3 &= 0 \\ -2aca_3 - b^2 a_3 + ba_3 &= 0 \\ -2aba_3 - ana_3 + aa_3 &= 0 \\ -2bca_3 + cna_3 + ca_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -n a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -ny
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -ny - \left(\frac{a\,x^n y^2 + by + c\,x^{-n}}{x}\right)(x) \\
&= -a\,x^n y^2 - by - c\,x^{-n} - ny \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-a\,x^n y^2 - by - c\,x^{-n} - ny} dy
\end{aligned}
$$

Which results in

$$S = -\frac{2x^n \arctan\left(\frac{2x^{2n}ay+b\,x^n+n\,x^n}{\sqrt{-x^{2n}b^2-2x^{2n}bn-x^{2n}n^2+4ca\,x^{2n}}}\right)}{\sqrt{-x^{2n}b^2-2x^{2n}bn-x^{2n}n^2+4ca\,x^{2n}}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{a\,x^n y^2 + by + c\,x^{-n}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{ny\,x^{n-1}}{x^{2n}a\,y^2 + y\,(b+n)\,x^n + c} \\
S_y &= -\frac{x^n}{x^{2n}a\,y^2 + y\,(b+n)\,x^n + c}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2\arctan\left(\frac{2a\,x^n y+b+n}{\sqrt{4ca-b^2-2bn-n^2}}\right)}{\sqrt{4ca-b^2-2bn-n^2}} = -\ln(x)+c_1$$

Which simplifies to

$$-\frac{2\arctan\left(\frac{2a\,x^n y+b+n}{\sqrt{4ca-b^2-2bn-n^2}}\right)}{\sqrt{4ca-b^2-2bn-n^2}} = -\ln(x)+c_1$$

Which gives

$$y = -\frac{\left(\tan\left(-\frac{\ln(x)\sqrt{4ca-b^2-2bn-n^2}}{2}+\frac{c_1\sqrt{4ca-b^2-2bn-n^2}}{2}\right)\sqrt{4ca-b^2-2bn-n^2}+b+n\right)x^{-n}}{2a}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\tan\left(-\frac{\ln(x)\sqrt{4ca-b^2-2bn-n^2}}{2}+\frac{c_1\sqrt{4ca-b^2-2bn-n^2}}{2}\right)\sqrt{4ca-b^2-2bn-n^2}+b+n\right)x^{-n}}{2a} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\tan\left(-\frac{\ln(x)\sqrt{4ca-b^2-2bn-n^2}}{2}+\frac{c_1\sqrt{4ca-b^2-2bn-n^2}}{2}\right)\sqrt{4ca-b^2-2bn-n^2}+b+n\right)x^{-n}}{2a}$$

Verified OK.

2.39.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,x^n y^2+by+c\,x^{-n}}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x^n a\,y^2}{x}+\frac{c\,x^{-n}}{x}+\frac{by}{x}$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x) = \frac{c\,x^{-n}}{x}$, $f_1(x) = \frac{b}{x}$ and $f_2(x) = \frac{a\,x^n}{x}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{a\,x^n u}{x}}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= \frac{an\,x^n}{x^2} - \frac{a\,x^n}{x^2} \\
f_1 f_2 &= \frac{ba\,x^n}{x^2} \\
f_2^2 f_0 &= \frac{a^2 x^{2n} c\,x^{-n}}{x^3}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{a\,x^n u''(x)}{x} - \left(\frac{an\,x^n}{x^2} - \frac{a\,x^n}{x^2} + \frac{ba\,x^n}{x^2}\right) u'(x) + \frac{a^2 x^{2n} c\,x^{-n} u(x)}{x^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = x^{\frac{b}{2}} x^{\frac{n}{2}} \left(x^{\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}} c_1 + x^{-\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}} c_2\right)
$$

The above shows that

$$
u'(x) = -\frac{x^{\frac{n}{2}} \left(c_2 \left(-b-n+\sqrt{-4ca+b^2+2bn+n^2}\right) x^{-\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}} - x^{\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}} c_1 \left(b+n+\sqrt{-4ca+b\right.\right.}{2x}
$$

Using the above in (1) gives the solution

$$
y = \frac{\left(c_2 \left(-b-n+\sqrt{-4ca+b^2+2bn+n^2}\right) x^{-\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}} - x^{\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}} c_1 \left(b+n+\sqrt{-4ca+b^2+2\right.\right.}{2a \left(x^{\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}} c_1 + x^{-\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}} c_2\right)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\left(-b-n+\sqrt{-4ca+b^2+2bn+n^2}\right)x^{-\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}} - x^{\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}}c_3\left(b+n+\sqrt{-4ca+b^2+2bn\right.\right.}{2a\left(x^{\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}}c_3+x^{-\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}}\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\left(-b-n+\sqrt{-4ca+b^2+2bn+n^2}\right)x^{-\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}} - x^{\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}}c_3\left(b+n+\sqrt{-4ca+b^2+2bn\right.\right.}{2a\left(x^{\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}}c_3+x^{-\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\left(-b-n+\sqrt{-4ca+b^2+2bn+n^2}\right)x^{-\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}} - x^{\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}}c_3\left(b+n+\sqrt{-4ca+b^2+2bn\right.\right.}{2a\left(x^{\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}}c_3+x^{-\frac{\sqrt{-4ca+b^2+2bn+n^2}}{2}}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 73

```
dsolve(x*diff(y(x),x)=a*x^n*y(x)^2+b*y(x)+c*x^(-n),y(x), singsol=all)
```

$$y(x) = \frac{x^{-n}\left(\tan\left(\frac{\sqrt{4ac-b^2-2bn-n^2}\,(\ln(x)-c_1)}{2}\right)\sqrt{4ac-b^2-2bn-n^2}-b-n\right)}{2a}$$

✓ Solution by Mathematica

Time used: 0.978 (sec). Leaf size: 138

```
DSolve[x*y'[x]==a*x^n*y[x]^2+b*y[x]+c*x^(-n),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^{-n}\left(\frac{\sqrt{-4ac+b^2+2bn+n^2}\left(-x^{\sqrt{-4ac+b^2+2bn+n^2}}+c_1\right)}{x^{\sqrt{-4ac+b^2+2bn+n^2}}+c_1}-b-n\right)}{2a}$$

$$y(x) \to \frac{x^{-n}\left(\sqrt{-4ac+b^2+2bn+n^2}-b-n\right)}{2a}$$

2.40 problem 40

Internal problem ID [10370]
Internal file name [`OUTPUT/9317_Monday_June_06_2022_01_51_20_PM_83489069/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{y'x - a\,x^n y^2 - ym = -a\,b^2 x^{n+2m}}$$

2.40.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,x^n y^2 + ym - a\,b^2 x^{n+2m}}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x^n a\,y^2}{x} + \frac{ym}{x} - \frac{a\,b^2 x^n x^{2m}}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{a\,b^2 x^{n+2m}}{x}$, $f_1(x) = \frac{m}{x}$ and $f_2(x) = \frac{a\,x^n}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{a\,x^n u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{an\,x^n}{x^2} - \frac{a\,x^n}{x^2}\\
f_1f_2 &= \frac{ma\,x^n}{x^2}\\
f_2^2f_0 &= -\frac{a^3x^{2n}b^2x^{n+2m}}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{a\,x^n u''(x)}{x} - \left(\frac{an\,x^n}{x^2} - \frac{a\,x^n}{x^2} + \frac{ma\,x^n}{x^2}\right)u'(x) - \frac{a^3x^{2n}b^2x^{n+2m}u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1\sinh\left(\frac{ab\,x^{m+n}}{m+n}\right) + c_2\cosh\left(\frac{ab\,x^{m+n}}{m+n}\right)$$

The above shows that

$$u'(x) = ab\,x^{m+n-1}\left(c_1\cosh\left(\frac{ab\,x^{m+n}}{m+n}\right) + c_2\sinh\left(\frac{ab\,x^{m+n}}{m+n}\right)\right)$$

Using the above in (1) gives the solution

$$y = -\frac{b\,x^{m+n-1}\left(c_1\cosh\left(\frac{ab\,x^{m+n}}{m+n}\right) + c_2\sinh\left(\frac{ab\,x^{m+n}}{m+n}\right)\right)x^{-n}x}{c_1\sinh\left(\frac{ab\,x^{m+n}}{m+n}\right) + c_2\cosh\left(\frac{ab\,x^{m+n}}{m+n}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{b\,x^m\left(c_3\cosh\left(\frac{ab\,x^{m+n}}{m+n}\right) + \sinh\left(\frac{ab\,x^{m+n}}{m+n}\right)\right)}{c_3\sinh\left(\frac{ab\,x^{m+n}}{m+n}\right) + \cosh\left(\frac{ab\,x^{m+n}}{m+n}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{b\,x^{m}\left(c_{3} \cosh\left(\frac{a b\,x^{m+n}}{m+n}\right)+\sinh\left(\frac{a b\,x^{m+n}}{m+n}\right)\right)}{c_{3} \sinh\left(\frac{a b\,x^{m+n}}{m+n}\right)+\cosh\left(\frac{a b\,x^{m+n}}{m+n}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{b\,x^{m}\left(c_{3} \cosh\left(\frac{a b\,x^{m+n}}{m+n}\right)+\sinh\left(\frac{a b\,x^{m+n}}{m+n}\right)\right)}{c_{3} \sinh\left(\frac{a b\,x^{m+n}}{m+n}\right)+\cosh\left(\frac{a b\,x^{m+n}}{m+n}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve(x*diff(y(x),x)=a*x^n*y(x)^2+m*y(x)-a*b^2*x^(n+2*m),y(x), singsol=all)
```

$$y(x) = i \tan\left(\frac{c_{1}(n+m)+i a b\,x^{n+m}}{n+m}\right) b\,x^{m}$$

✓ Solution by Mathematica

Time used: 1.736 (sec). Leaf size: 43

```
DSolve[x*y'[x]==a*x^n*y[x]^2+m*y[x]-a*b^2*x^(n+2*m),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{-b^2}x^m \tan\left(\frac{a\sqrt{-b^2}x^{m+n}}{m+n} + c_1\right)$$

2.41 problem 41

Internal problem ID [10371]
Internal file name [`OUTPUT/9318_Monday_June_06_2022_01_51_21_PM_22191330/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{y'x - x^{2n}y^2 - (m-n)\,y = x^{2m}}$$

2.41.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^{2n}y^2 + ym - ny + x^{2m}}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x^{2n}y^2}{x} + \frac{ym}{x} - \frac{ny}{x} + \frac{x^{2m}}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{x^{2m}}{x}$, $f_1(x) = \frac{m-n}{x}$ and $f_2(x) = \frac{x^{2n}}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{x^{2n}u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2x^{2n}n}{x^2} - \frac{x^{2n}}{x^2} \\ f_1f_2 &= \frac{(m-n)\,x^{2n}}{x^2} \\ f_2^2f_0 &= \frac{x^{4n}x^{2m}}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{x^{2n}u''(x)}{x} - \left(\frac{2x^{2n}n}{x^2} - \frac{x^{2n}}{x^2} + \frac{(m-n)\,x^{2n}}{x^2}\right)u'(x) + \frac{x^{4n}x^{2m}u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(\frac{x^{m+n}}{m+n}\right) + c_2 \cos\left(\frac{x^{m+n}}{m+n}\right)$$

The above shows that

$$u'(x) = x^{m+n-1}\left(c_1 \cos\left(\frac{x^{m+n}}{m+n}\right) - c_2 \sin\left(\frac{x^{m+n}}{m+n}\right)\right)$$

Using the above in (1) gives the solution

$$y = -\frac{x^{m+n-1}\left(c_1 \cos\left(\frac{x^{m+n}}{m+n}\right) - c_2 \sin\left(\frac{x^{m+n}}{m+n}\right)\right)x^{-2n}x}{c_1 \sin\left(\frac{x^{m+n}}{m+n}\right) + c_2 \cos\left(\frac{x^{m+n}}{m+n}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x^{m-n}\left(-c_3 \cos\left(\frac{x^{m+n}}{m+n}\right) + \sin\left(\frac{x^{m+n}}{m+n}\right)\right)}{c_3 \sin\left(\frac{x^{m+n}}{m+n}\right) + \cos\left(\frac{x^{m+n}}{m+n}\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{x^{m-n}\left(-c_3 \cos\left(\frac{x^{m+n}}{m+n}\right) + \sin\left(\frac{x^{m+n}}{m+n}\right)\right)}{c_3 \sin\left(\frac{x^{m+n}}{m+n}\right) + \cos\left(\frac{x^{m+n}}{m+n}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{x^{m-n}\left(-c_3 \cos\left(\frac{x^{m+n}}{m+n}\right) + \sin\left(\frac{x^{m+n}}{m+n}\right)\right)}{c_3 \sin\left(\frac{x^{m+n}}{m+n}\right) + \cos\left(\frac{x^{m+n}}{m+n}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 34

```
dsolve(x*diff(y(x),x)=x^(2*n)*y(x)^2+(m-n)*y(x)+x^(2*m),y(x), singsol=all)
```

$$y(x) = \tan\left(\frac{x^{n+m} + (-n-m)\,c_1}{n+m}\right) x^{-n+m}$$

✓ Solution by Mathematica

Time used: 0.727 (sec). Leaf size: 28

```
DSolve[x*y'[x]==x^(2*n)*y[x]^2+(m-n)*y[x]+x^(2*m),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^{m-n} \tan \left(\frac{x^{m+n}}{m+n} + c_1 \right)$$

2.42 problem 42

Internal problem ID [10372]
Internal file name [OUTPUT/9319_Monday_June_06_2022_01_51_22_PM_29239198/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{y'x - a\,x^n y^2 - yb = c\,x^m}$$

2.42.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,x^n y^2 + by + c\,x^m}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x^n a\,y^2}{x} + \frac{c\,x^m}{x} + \frac{by}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{c\,x^m}{x}$, $f_1(x) = \frac{b}{x}$ and $f_2(x) = \frac{a\,x^n}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{a\,x^n u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{an\, x^n}{x^2} - \frac{a\, x^n}{x^2} \\
f_1 f_2 &= \frac{ba\, x^n}{x^2} \\
f_2^2 f_0 &= \frac{a^2 x^{2n} c\, x^m}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{a\, x^n u''(x)}{x} - \left(\frac{an\, x^n}{x^2} - \frac{a\, x^n}{x^2} + \frac{ba\, x^n}{x^2}\right) u'(x) + \frac{a^2 x^{2n} c\, x^m u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\text{BesselJ}\left(\frac{-b-n}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right) c_1 + \text{BesselY}\left(\frac{-b-n}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right) c_2\right) x^{\frac{b}{2}+\frac{n}{2}}$$

The above shows that

$$\begin{aligned}
u'(x) = x^{-1+\frac{b}{2}+n+\frac{m}{2}} \sqrt{ca} \Bigg(&-\text{BesselY}\left(\frac{-b+m}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right) c_2 \\
&- \text{BesselJ}\left(\frac{-b+m}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right) c_1\Bigg)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{x^{-1+\frac{b}{2}+n+\frac{m}{2}} \sqrt{ca} \left(-\text{BesselY}\left(\frac{-b+m}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right) c_2 - \text{BesselJ}\left(\frac{-b+m}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right) c_1\right) x^{-n} x\, x^{-\frac{b}{2}-\frac{n}{2}}}{a \left(\text{BesselJ}\left(\frac{-b-n}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right) c_1 + \text{BesselY}\left(\frac{-b-n}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right) c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x^{\frac{m}{2}-\frac{n}{2}} \sqrt{ca} \left(\text{BesselJ}\left(\frac{-b+m}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right) c_3 + \text{BesselY}\left(\frac{-b+m}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right)\right)}{a \left(\text{BesselJ}\left(\frac{-b-n}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right) c_3 + \text{BesselY}\left(\frac{-b-n}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right)\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{x^{\frac{m}{2}-\frac{n}{2}} \sqrt{ca}\left(\mathrm{BesselJ}\left(\frac{-b+m}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right) c_3 + \mathrm{BesselY}\left(\frac{-b+m}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right)\right)}{a\left(\mathrm{BesselJ}\left(\frac{-b-n}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right) c_3 + \mathrm{BesselY}\left(\frac{-b-n}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{x^{\frac{m}{2}-\frac{n}{2}} \sqrt{ca}\left(\mathrm{BesselJ}\left(\frac{-b+m}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right) c_3 + \mathrm{BesselY}\left(\frac{-b+m}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right)\right)}{a\left(\mathrm{BesselJ}\left(\frac{-b-n}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right) c_3 + \mathrm{BesselY}\left(\frac{-b-n}{m+n}, \frac{2\sqrt{ca}\, x^{\frac{m}{2}+\frac{n}{2}}}{m+n}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (b+n-1)*(diff(y(x), x))/x-x^(n
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         <- Bessel successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 166

```
dsolve(x*diff(y(x),x)=a*x^(n)*y(x)^2+b*y(x)+c*x^(m),y(x), singsol=all)
```

$$y(x) = \frac{x^{-\frac{n}{2}+\frac{m}{2}}\sqrt{ac}\left(\mathrm{BesselY}\left(\frac{-b+m}{n+m}, \frac{2\sqrt{ac}\, x^{\frac{m}{2}+\frac{n}{2}}}{n+m}\right)c_1 + \mathrm{BesselJ}\left(\frac{-b+m}{n+m}, \frac{2\sqrt{ac}\, x^{\frac{m}{2}+\frac{n}{2}}}{n+m}\right)\right)}{a\left(\mathrm{BesselY}\left(\frac{-b-n}{n+m}, \frac{2\sqrt{ac}\, x^{\frac{m}{2}+\frac{n}{2}}}{n+m}\right)c_1 + \mathrm{BesselJ}\left(\frac{-b-n}{n+m}, \frac{2\sqrt{ac}\, x^{\frac{m}{2}+\frac{n}{2}}}{n+m}\right)\right)}$$

✓ Solution by Mathematica

Time used: 1.49 (sec). Leaf size: 1321

```
DSolve[x*y'[x]==a*x^(n)*y[x]^2+b*y[x]+c*x^(m),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x^{-n}\left(\sqrt{a}\sqrt{c}(m+n)x^{m+n}\,\mathrm{BesselJ}\left(\frac{m-b}{m+n}, \frac{2\sqrt{a}\sqrt{c}\sqrt{x^{m+n}}}{\sqrt{(m+n)^2}}\right)c_1\,\mathrm{Gamma}\left(\frac{m-b}{m+n}\right)\left((m+n)^2\right)^{\frac{b+n}{m+n}} - \sqrt{a}\sqrt{c}mx^{m+n}\right.}{}$$

$$y(x) \rightarrow \frac{x^{-n}\left(\sqrt{a}\sqrt{c}(m+n)\sqrt{x^{m+n}}\,\mathrm{BesselJ}\left(\frac{m-b}{m+n}, \frac{2\sqrt{a}\sqrt{c}\sqrt{x^{m+n}}}{\sqrt{(m+n)^2}}\right) - (b+n)\sqrt{(m+n)^2}\,\mathrm{BesselJ}\left(-\frac{b+n}{m+n}, \frac{2\sqrt{a}\sqrt{c}\sqrt{x^{m}}}{\sqrt{(m+n)}}\right.\right.}{2a\sqrt{(m+n)^2}\,\mathrm{BesselJ}\left(-\frac{b+n}{m+n}, \frac{2\sqrt{a}\sqrt{c}\sqrt{x^{m+n}}}{\sqrt{(m+n)^2}}\right.}$$

2.43 problem 43

Internal problem ID [10373]
Internal file name [`OUTPUT/9320_Monday_June_06_2022_01_51_23_PM_60768990/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{y'x - x^{2n}y^2a - (b\,x^n - n)\,y = c}$$

2.43.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^{2n}a\,y^2 + x^n by - ny + c}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x^{2n}a\,y^2}{x} + \frac{x^n by}{x} - \frac{ny}{x} + \frac{c}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{c}{x}$, $f_1(x) = \frac{b\,x^n - n}{x}$ and $f_2(x) = \frac{x^{2n}a}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{x^{2n}au}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{2x^{2n}na}{x^2} - \frac{a\,x^{2n}}{x^2}\\
f_1 f_2 &= \frac{(b\,x^n - n)\,x^{2n}a}{x^2}\\
f_2^2 f_0 &= \frac{x^{4n}a^2 c}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{x^{2n}au''(x)}{x} - \left(\frac{2x^{2n}na}{x^2} - \frac{a\,x^{2n}}{x^2} + \frac{(b\,x^n - n)\,x^{2n}a}{x^2}\right) u'(x) + \frac{x^{4n}a^2cu(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\frac{x^n b}{2n}} \left(c_1 \sinh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) + c_2 \cosh\left(\frac{x^n \sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right)\right)$$

The above shows that

$$u'(x) = \frac{x^{n-1}\mathrm{e}^{\frac{x^n b}{2n}}\left(\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, nc_1 + c_2 b\right)\cosh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) + \sinh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right)\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, nc_2 + c_1 b\right)\right)}{2}$$

Using the above in (1) gives the solution

$$y = -\frac{x^{n-1}\left(\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, nc_1 + c_2 b\right)\cosh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) + \sinh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right)\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, nc_2 + c_1 b\right)\right) x^{-2n}}{2a\left(c_1 \sinh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) + c_2\cosh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x^{-n}\left(\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n c_3 + b\right)\cosh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) + \sinh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right)\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n + b c_3\right)\right)}{2a\left(c_3\sinh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) + \cosh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right)\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^{-n}\left(\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n c_3 + b\right)\cosh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) + \sinh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right)\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n + b c_3\right)\right)}{2a\left(c_3\sinh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) + \cosh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right)\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{x^{-n}\left(\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n c_3 + b\right)\cosh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) + \sinh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right)\left(\sqrt{\frac{-4ca+b^2}{n^2}}\, n + b c_3\right)\right)}{2a\left(c_3\sinh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right) + \cosh\left(\frac{x^n\sqrt{\frac{-4ca+b^2}{n^2}}}{2}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 72

```
dsolve(x*diff(y(x),x)=a*x^(2*n)*y(x)^2+(b*x^n-n)*y(x)+c,y(x), singsol=all)
```

$$y(x) = \frac{\left(\sqrt{4a\,b^{2}c-b^{4}}\,\tan\left(\frac{\sqrt{4a\,b^{2}c-b^{4}}\,(b\,x^{n}+c_{1}n)}{2b^{2}n}\right)-b^{2}\right)x^{-n}}{2ba}$$

✓ Solution by Mathematica

Time used: 1.071 (sec). Leaf size: 118

```
DSolve[x*y'[x]==a*x^(2*n)*y[x]^2+(b*x^n-n)*y[x]+c,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^{-n}\left(-b+\frac{\sqrt{b^{2}-4ac}\left(-e^{\frac{x^{n}\sqrt{b^{2}-4ac}}{n}}+c_{1}\right)}{e^{\frac{x^{n}\sqrt{b^{2}-4ac}}{n}}+c_{1}}\right)}{2a}$$

$$y(x) \to \frac{x^{-n}\left(\sqrt{b^{2}-4ac}-b\right)}{2a}$$

2.44 problem 44

Internal problem ID [10374]
Internal file name [`OUTPUT/9321_Monday_June_06_2022_01_51_24_PM_98601438/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{y'x - a\,x^{m+2n}y^2 - \left(b\,x^{m+n} - n\right)y = c\,x^m}$$

2.44.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{a\,x^{m+2n}y^2 + x^{m+n}by + c\,x^m - ny}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,x^m x^{2n}y^2}{x} + \frac{x^m x^n by}{x} + \frac{c\,x^m}{x} - \frac{ny}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{c\,x^m}{x}$, $f_1(x) = \frac{b\,x^{m+n}-n}{x}$ and $f_2(x) = \frac{x^{m+2n}a}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{x^{m+2n}au}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{x^{m+2n}(m+2n)\, a}{x^2} - \frac{a\, x^{m+2n}}{x^2} \\ f_1 f_2 &= \frac{(b\, x^{m+n} - n)\, x^{m+2n} a}{x^2} \\ f_2^2 f_0 &= \frac{x^{2m+4n} a^2 c\, x^m}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{x^{m+2n} a u''(x)}{x} - \left(\frac{x^{m+2n}(m+2n)\, a}{x^2} - \frac{a\, x^{m+2n}}{x^2} + \frac{(b\, x^{m+n} - n)\, x^{m+2n} a}{x^2}\right) u'(x) + \frac{x^{2m+4n} a^2 c\, x^m u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = e^{\frac{b\, x^{m+n}}{2m+2n}} \left(c_1 \sinh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right) + c_2 \cosh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right)\right)$$

The above shows that

$$u'(x) = \frac{e^{\frac{b\, x^{m+n}}{2m+2n}} x^{m+n-1} \left(\left(c_1 (m+n)\sqrt{\frac{-4ca+b^2}{(m+n)^2}} + c_2 b\right) \cosh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right) + \left(c_2 (m+n)\sqrt{\frac{-4ca+b^2}{(m+n)^2}} + c_1 b\right) \sinh \cdots\right.}{2}$$

Using the above in (1) gives the solution

$$y = -\frac{x^{m+n-1}\left(\left(c_1 (m+n)\sqrt{\frac{-4ca+b^2}{(m+n)^2}} + c_2 b\right) \cosh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right) + \left(c_2 (m+n)\sqrt{\frac{-4ca+b^2}{(m+n)^2}} + c_1 b\right) \sinh\left(\cdots\right.\right.}{2a\left(c_1 \sinh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right) + c_2 \cosh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\left(c_3(m+n)\sqrt{\frac{-4ca+b^2}{(m+n)^2}}+b\right)\cosh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right)+\left((m+n)\sqrt{\frac{-4ca+b^2}{(m+n)^2}}+bc_3\right)\sinh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right)\right)}{2a\left(c_3\sinh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right)+\cosh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right)\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\left(c_3(m+n)\sqrt{\frac{-4ca+b^2}{(m+n)^2}}+b\right)\cosh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right)+\left((m+n)\sqrt{\frac{-4ca+b^2}{(m+n)^2}}+bc_3\right)\sinh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right)\right)}{2a\left(c_3\sinh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right)+\cosh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right)\right)} \quad (1)$$

Verification of solutions

$$y = -\frac{\left(\left(c_3(m+n)\sqrt{\frac{-4ca+b^2}{(m+n)^2}}+b\right)\cosh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right)+\left((m+n)\sqrt{\frac{-4ca+b^2}{(m+n)^2}}+bc_3\right)\sinh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right)\right)}{2a\left(c_3\sinh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right)+\cosh\left(\frac{x^{m+n}\sqrt{\frac{-4ca+b^2}{(m+n)^2}}}{2}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 78

```
dsolve(x*diff(y(x),x)=a*x^(2*n+m)*y(x)^2+(b*x^(n+m)-n)*y(x)+c*x^m,y(x), singsol=all)
```

$$y(x) = \frac{x^{-n}\left(\sqrt{4a\,b^2c-b^4}\,\tan\left(\frac{(x^{n+m}b+c_1(n+m))\sqrt{4a\,b^2c-b^4}}{2b^2(n+m)}\right)-b^2\right)}{2ab}$$

✓ Solution by Mathematica

Time used: 1.566 (sec). Leaf size: 126

```
DSolve[x*y'[x]==a*x^(2*n+m)*y[x]^2+(b*x^(n+m)-n)*y[x]+c*x^m,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{x^{-n}\left(-b+\frac{\sqrt{b^2-4ac}\left(-e^{\frac{\sqrt{b^2-4ac}x^{m+n}}{m+n}}+c_1\right)}{e^{\frac{\sqrt{b^2-4ac}x^{m+n}}{m+n}}+c_1}\right)}{2a}$$

$$y(x) \to \frac{x^{-n}\left(\sqrt{b^2-4ac}-b\right)}{2a}$$

2.45 problem 45

Internal problem ID [10375]
Internal file name [OUTPUT/9322_Monday_June_06_2022_01_51_26_PM_43318158/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{(a_2x+b_2)\left(y'+\lambda y^2\right)+(a_1x+b_1)\,y=-a_0x-b_0}$$

2.45.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y^2a_2\lambda x+y^2b_2\lambda+ya_1x+a_0x+yb_1+b_0}{a_2x+b_2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=-\frac{y^2a_2\lambda x}{a_2x+b_2}-\frac{y^2b_2\lambda}{a_2x+b_2}-\frac{ya_1x}{a_2x+b_2}-\frac{a_0x}{a_2x+b_2}-\frac{yb_1}{a_2x+b_2}-\frac{b_0}{a_2x+b_2}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=-\frac{a_0x+b_0}{a_2x+b_2}$, $f_1(x)=-\frac{a_1x+b_1}{a_2x+b_2}$ and $f_2(x)=-\frac{\lambda a_2x+\lambda b_2}{a_2x+b_2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{-\frac{(\lambda a_2x+\lambda b_2)u}{a_2x+b_2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{\lambda a_2}{a_2 x + b_2} + \frac{(\lambda a_2 x + \lambda b_2) a_2}{(a_2 x + b_2)^2} \\ f_1 f_2 &= \frac{(a_1 x + b_1)(\lambda a_2 x + \lambda b_2)}{(a_2 x + b_2)^2} \\ f_2^2 f_0 &= -\frac{(\lambda a_2 x + \lambda b_2)^2 (a_0 x + b_0)}{(a_2 x + b_2)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{(\lambda a_2 x + \lambda b_2) u''(x)}{a_2 x + b_2} - \left(-\frac{\lambda a_2}{a_2 x + b_2} + \frac{(\lambda a_2 x + \lambda b_2) a_2}{(a_2 x + b_2)^2} + \frac{(a_1 x + b_1)(\lambda a_2 x + \lambda b_2)}{(a_2 x + b_2)^2}\right) u'(x) - \frac{(\lambda a_2 x + \lambda b_2)}{(a_2}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{-\frac{\left(\sqrt{-4a_0 a_2 \lambda + a_1^2} + a_1\right) x}{2a_2}} (a_2 x + b_2)^{\frac{b_2 a_1 + a_2^2 - a_2 b_1}{a_2^2}} \left(\mathrm{KummerM}\left(\frac{(b_2 a_1 + 2a_2^2 - a_2 b_1)\sqrt{-4a_0 a_2 \lambda + a_1^2} - 2a_2^2 b_0 \lambda + (2a_0 \lambda b_2 + a_1 b_1) a_2 - a_1^2}{2\sqrt{-4a_0 a_2 \lambda + a_1^2}\, a_2^2}\right.\right.$$
$$+\mathrm{KummerU}\left(\frac{(b_2 a_1 + 2a_2^2 - a_2 b_1)\sqrt{-4a_0 a_2 \lambda + a_1^2} - 2a_2^2 b_0 \lambda + (2a_0 \lambda b_2 + a_1 b_1) a_2 - a_1^2 b_2}{2\sqrt{-4a_0 a_2 \lambda + a_1^2}\, a_2^2}, \frac{b_2 a_1 + 2a_2^2 -}{a_2^2}\right.$$

The above shows that

$$u'(x) = -\frac{2\left(\left(\frac{(a_1 x + b_1)\sqrt{-4a_0 a_2 \lambda + a_1^2}}{4} + \lambda\left(a_0 x + \frac{b_0}{2}\right) a_2 - \frac{x a_1^2}{4} + \frac{a_0 \lambda b_2}{2} - \frac{a_1 b_1}{4}\right) c_1 a_2 \,\mathrm{KummerM}\left(\frac{(b_2 a_1 + 2a_2^2 - a_2 b_1)\sqrt{-4a}}{2}\right.\right.}{}$$

Using the above in (1) gives the solution

Expression too large to display

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

Expression too large to display

Summary

The solution(s) found are the following

Expression too large to display (1)

Verification of solutions

Expression too large to display

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(a__1*x+b__1)*(diff(y(x), x))
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
         <- Kummer successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 827

```
dsolve((a__2*x+b__2)*(diff(y(x),x)+lambda*y(x)^2)+(a__1*x+b__1)*y(x)+a__0*x+b__0=0,y(x), sin
```

Expression too large to display

✓ Solution by Mathematica

Time used: 3.165 (sec). Leaf size: 1418

```
DSolve[(a2*x+b2)*(y'[x]+\[Lambda]*y[x]^2)+(a1*x+b1)*y[x]+a0*x+b0==0,y[x],x,IncludeSingularSo
```

Too large to display

2.46 problem 46

Internal problem ID [10376]
Internal file name [`OUTPUT/9323_Monday_June_06_2022_01_51_33_PM_76540793/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational, _Riccati]
```

$$\boxed{(xa+c)\,y'-\alpha(ay+bx)^2-\beta(ay+bx)=-bx+\gamma}$$

2.46.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{a^2\alpha\,y^2+2a\alpha bxy+\alpha\,b^2x^2+a\beta y+b\beta x-bx+\gamma}{xa+c}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
& b_2+\frac{\left(a^2 \alpha y^2+2 a \alpha b x y+\alpha b^2 x^2+a \beta y+b \beta x-b x+\gamma\right)\left(b_3-a_2\right)}{x a+c} \\
& -\frac{\left(a^2 \alpha y^2+2 a \alpha b x y+\alpha b^2 x^2+a \beta y+b \beta x-b x+\gamma\right)^2 a_3}{(x a+c)^2} \\
& -\left(\frac{2 a \alpha b y+2 \alpha b^2 x+\beta b-b}{x a+c}\right. \\
& \left.-\frac{\left(a^2 \alpha y^2+2 a \alpha b x y+\alpha b^2 x^2+a \beta y+b \beta x-b x+\gamma\right) a}{(x a+c)^2}\right)\left(x a_2+y a_3+a_1\right) \\
& -\frac{\left(2 a^2 \alpha y+2 \alpha a x b+\beta a\right)\left(x b_2+y b_3+b_1\right)}{x a+c}=0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{a^4\alpha^2y^4a_3+4a^3\alpha^2bx\,y^3a_3+6a^2\alpha^2b^2x^2y^2a_3+4a\,\alpha^2b^3x^3ya_3+\alpha^2b^4x^4a_3+2a^3\alpha\beta\,y^3a_3+6a^2\alpha b\beta x\,y^2a_3+6}{}$$

$$=0$$

Setting the numerator to zero gives

$$\begin{aligned}
& -a^4 \alpha^2 y^4 a_3-4 a^3 \alpha^2 b x\, y^3 a_3-6 a^2 \alpha^2 b^2 x^2 y^2 a_3-4 a\, \alpha^2 b^3 x^3 y a_3 \\
& -\alpha^2 b^4 x^4 a_3-2 a^3 \alpha \beta\, y^3 a_3-6 a^2 \alpha b \beta x\, y^2 a_3-6 a \alpha\, b^2 \beta\, x^2 y a_3 \\
& -2 \alpha\, b^3 \beta\, x^3 a_3-2 a^3 \alpha\, x^2 y b_2-a^3 \alpha x\, y^2 b_3+a^3 \alpha\, y^3 a_3-2 a^2 \alpha b\, x^3 b_2 \\
& -2 a^2 \alpha b\, x^2 y a_2+2 a^2 \alpha b x\, y^2 a_3-2 a \alpha\, b^2 x^3 a_2+a \alpha\, b^2 x^3 b_3 \\
& +3 a \alpha\, b^2 x^2 y a_3+2 \alpha\, b^3 x^3 a_3-2 a^3 \alpha x y b_1+a^3 \alpha\, y^2 a_1-2 a^2 \alpha b\, x^2 b_1 \\
& -2 a^2 \alpha c x y b_2-a^2 \alpha c\, y^2 a_2-a^2 \alpha c\, y^2 b_3-2 a^2 \alpha \gamma\, y^2 a_3-a^2 \beta^2 y^2 a_3 \\
& -a \alpha\, b^2 x^2 a_1-2 a \alpha b c\, x^2 b_2-4 a \alpha b c x y a_2-2 a \alpha b c\, y^2 a_3 \\
& -4 a \alpha b \gamma x y a_3-2 a b\, \beta^2 x y a_3-3 \alpha\, b^2 c\, x^2 a_2+\alpha\, b^2 c\, x^2 b_3 \\
& -2 \alpha\, b^2 c x y a_3-2 \alpha\, b^2 \gamma\, x^2 a_3-b^2 \beta^2 x^2 a_3-2 a^2 \alpha c y b_1-a^2 \beta\, x^2 b_2 \\
& +a^2 \beta\, y^2 a_3-2 a \alpha b c x b_1-2 a \alpha b c y a_1-a b \beta\, x^2 a_2+a b \beta\, x^2 b_3 \\
& +2 a b \beta x y a_3-2 \alpha\, b^2 c x a_1+2 b^2 \beta\, x^2 a_3-a^2 \beta x b_1+a^2 \beta y a_1 \\
& +a^2 x^2 b_2+a b\, x^2 a_2-a b\, x^2 b_3-a \beta c x b_2-a \beta c y a_2-2 a \beta \gamma y a_3 \\
& -b^2 x^2 a_3-2 b \beta c x a_2+b \beta c x b_3-b \beta c y a_3-2 b \beta \gamma x a_3-a \beta c b_1 \\
& +2 a c x b_2+a \gamma x b_3+a \gamma y a_3-b \beta c a_1+2 b c x a_2-b c x b_3+b c y a_3 \\
& +2 b \gamma x a_3+a \gamma a_1+b c a_1+c^2 b_2-c \gamma a_2+c \gamma b_3-\gamma^2 a_3=0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a^4\alpha^2a_3v_2^4 - 4a^3\alpha^2ba_3v_1v_2^3 - 6a^2\alpha^2b^2a_3v_1^2v_2^2 - 4a\,\alpha^2b^3a_3v_1^3v_2 \\
&- \alpha^2b^4a_3v_1^4 - 2a^3\alpha\beta a_3v_2^3 - 6a^2\alpha b\beta a_3v_1v_2^2 - 6a\alpha\,b^2\beta a_3v_1^2v_2 \\
&- 2\alpha\,b^3\beta a_3v_1^3 + a^3\alpha a_3v_2^3 - 2a^3\alpha b_2v_1^2v_2 - a^3\alpha b_3v_1v_2^2 \\
&- 2a^2\alpha ba_2v_1^2v_2 + 2a^2\alpha ba_3v_1v_2^2 - 2a^2\alpha bb_2v_1^3 - 2a\alpha\,b^2a_2v_1^3 \\
&+ 3a\alpha\,b^2a_3v_1^2v_2 + a\alpha\,b^2b_3v_1^3 + 2\alpha\,b^3a_3v_1^3 + a^3\alpha a_1v_2^2 - 2a^3\alpha b_1v_1v_2 \\
&- 2a^2\alpha bb_1v_1^2 - a^2\alpha ca_2v_2^2 - 2a^2\alpha cb_2v_1v_2 - a^2\alpha cb_3v_2^2 - 2a^2\alpha\gamma a_3v_2^2 \\
&- a^2\beta^2a_3v_2^2 - a\alpha\,b^2a_1v_1^2 - 4a\alpha bca_2v_1v_2 - 2a\alpha bca_3v_2^2 - 2a\alpha bcb_2v_1^2 \\
&- 4a\alpha b\gamma a_3v_1v_2 - 2ab\,\beta^2a_3v_1v_2 - 3\alpha\,b^2ca_2v_1^2 - 2\alpha\,b^2ca_3v_1v_2 \\
&+ \alpha\,b^2cb_3v_1^2 - 2\alpha\,b^2\gamma a_3v_1^2 - b^2\beta^2a_3v_1^2 - 2a^2\alpha cb_1v_2 + a^2\beta a_3v_2^2 \\
&- a^2\beta b_2v_1^2 - 2a\alpha bca_1v_2 - 2a\alpha bcb_1v_1 - ab\beta a_2v_1^2 + 2ab\beta a_3v_1v_2 \\
&+ ab\beta b_3v_1^2 - 2\alpha\,b^2ca_1v_1 + 2b^2\beta a_3v_1^2 + a^2\beta a_1v_2 - a^2\beta b_1v_1 \\
&+ a^2b_2v_1^2 + aba_2v_1^2 - abb_3v_1^2 - a\beta ca_2v_2 - a\beta cb_2v_1 - 2a\beta\gamma a_3v_2 \\
&- b^2a_3v_1^2 - 2b\beta ca_2v_1 - b\beta ca_3v_2 + b\beta cb_3v_1 - 2b\beta\gamma a_3v_1 - a\beta cb_1 \\
&+ 2acb_2v_1 + a\gamma a_3v_2 + a\gamma b_3v_1 - b\beta ca_1 + 2bca_2v_1 + bca_3v_2 - bcb_3v_1 \\
&+ 2b\gamma a_3v_1 + a\gamma a_1 + bca_1 + c^2b_2 - c\gamma a_2 + c\gamma b_3 - \gamma^2a_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-\alpha^2 b^4 a_3 v_1^4 - 4a\,\alpha^2 b^3 a_3 v_1^3 v_2 \\
&+ \left(-2\alpha\, b^3 \beta a_3 - 2a^2\alpha b b_2 - 2a\alpha\, b^2 a_2 + a\alpha\, b^2 b_3 + 2\alpha\, b^3 a_3\right) v_1^3 \\
&- 6a^2\alpha^2 b^2 a_3 v_1^2 v_2^2 \\
&+ \left(-6a\alpha\, b^2 \beta a_3 - 2a^3\alpha b_2 - 2a^2\alpha b a_2 + 3a\alpha\, b^2 a_3\right) v_1^2 v_2 + \left(-2a^2\alpha b b_1\right. \\
&- a\alpha\, b^2 a_1 - 2a\alpha b c b_2 - 3\alpha\, b^2 c a_2 + \alpha\, b^2 c b_3 - 2\alpha\, b^2 \gamma a_3 - b^2\beta^2 a_3 \\
&\left. - a^2\beta b_2 - ab\beta a_2 + ab\beta b_3 + 2b^2\beta a_3 + a^2 b_2 + aba_2 - abb_3 - b^2 a_3\right) v_1^2 \\
&- 4a^3\alpha^2 b a_3 v_1 v_2^3 + \left(-6a^2\alpha b\beta a_3 - a^3\alpha b_3 + 2a^2\alpha b a_3\right) v_1 v_2^2 \\
&+ \left(-2a^3\alpha b_1 - 2a^2\alpha c b_2 - 4a\alpha b c a_2 - 4a\alpha b\gamma a_3 - 2ab\,\beta^2 a_3 - 2\alpha\, b^2 c a_3\right. \\
&\left. + 2ab\beta a_3\right) v_1 v_2 + \left(-2a\alpha b c b_1 - 2\alpha\, b^2 c a_1 - a^2\beta b_1 - a\beta c b_2 - 2b\beta c a_2\right. \\
&\left. + b\beta c b_3 - 2b\beta\gamma a_3 + 2acb_2 + a\gamma b_3 + 2bca_2 - bcb_3 + 2b\gamma a_3\right) v_1 \\
&- a^4\alpha^2 a_3 v_2^4 + \left(-2a^3\alpha\beta a_3 + a^3\alpha a_3\right) v_2^3 + \left(a^3\alpha a_1 - a^2\alpha c a_2\right. \\
&\left. - a^2\alpha c b_3 - 2a^2\alpha\gamma a_3 - a^2\beta^2 a_3 - 2a\alpha b c a_3 + a^2\beta a_3\right) v_2^2 + \left(-2a^2\alpha c b_1\right. \\
&\left. - 2a\alpha b c a_1 + a^2\beta a_1 - a\beta c a_2 - 2a\beta\gamma a_3 - b\beta c a_3 + a\gamma a_3 + bca_3\right) v_2 \\
&- a\beta c b_1 - b\beta c a_1 + a\gamma a_1 + bca_1 + c^2 b_2 - c\gamma a_2 + c\gamma b_3 - \gamma^2 a_3 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-6a\alpha\, b^2\beta a_3 +$$

$$-2\alpha\, b^3\beta a_3 - 2a^2\alpha b b_2$$

$$a^3\alpha a_1 - a^2\alpha c a_2 - a^2\alpha c b_3 - 2a^2\alpha\gamma a$$

$$-2a^3\alpha b_1 - 2a^2\alpha c b_2 - 4a\alpha b c a_2 - 4a\alpha b\gamma a_3 -$$

$$-2a^2\alpha c b_1 - 2a\alpha b c a_1 + a^2\beta a_1 - a\beta c a_2 -$$

$$-a\beta c b_1 - b\beta c a_1 + a\gamma a_1 + bc$$

$$-2a\alpha b c b_1 - 2\alpha\, b^2 c a_1 - a^2\beta b_1 - a\beta c b_2 - 2b\beta c a_2 + b\beta c b_3 - 2b\beta\gamma a_3 + 2acb_2$$

$$-2a^2\alpha b b_1 - a\alpha\, b^2 a_1 - 2a\alpha b c b_2 - 3\alpha\, b^2 c a_2 + \alpha\, b^2 c b_3 - 2\alpha\, b^2\gamma a_3 - b^2\beta^2 a_3 - a^2\beta b_2 - ab\beta a_2 + ab\beta b_3 + 2b^2\beta$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -\frac{b_1 a}{b} \\
a_2 &= -\frac{b_1 a^2}{cb} \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= \frac{b_1 a}{c} \\
b_3 &= 0
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -\frac{a(xa+c)}{bc} \\
\eta &= \frac{xa+c}{c}
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= \frac{xa+c}{c} - \left(\frac{a^2\alpha\, y^2 + 2a\alpha bxy + \alpha\, b^2x^2 + a\beta y + b\beta x - bx + \gamma}{xa+c}\right)\left(-\frac{a(xa+c)}{bc}\right) \\
&= \frac{a^3\alpha\, y^2 + 2a^2\alpha bxy + a\alpha\, b^2x^2 + a^2\beta y + ab\beta x + a\gamma + bc}{bc} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{a^3\alpha\, y^2+2a^2\alpha bxy+a\alpha\, b^2x^2+a^2\beta y+ab\beta x+a\gamma+bc}{bc}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{2bc \arctan\left(\frac{2a^3\alpha y+2\alpha\, a^2xb+a^2\beta}{\sqrt{4a^4\alpha\gamma-a^4\beta^2+4a^3\alpha bc}}\right)}{\sqrt{4a^4\alpha\gamma - a^4\beta^2 + 4a^3\alpha bc}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{a^2\alpha\, y^2 + 2a\alpha bxy + \alpha\, b^2x^2 + a\beta y + b\beta x - bx + \gamma}{xa + c}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{b^2c}{a\left(a^3\alpha\, y^2 + 2y\left(bx\alpha + \frac{\beta}{2}\right)a^2 + \left(\alpha\, b^2x^2 + b\beta x + \gamma\right)a + bc\right)} \\
S_y &= \frac{bc}{a^3\alpha\, y^2 + 2y\left(bx\alpha + \frac{\beta}{2}\right)a^2 + \left(\alpha\, b^2x^2 + b\beta x + \gamma\right)a + bc}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{bc}{a\,(xa + c)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{bc}{a\,(Ra + c)}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{bc \ln(Ra + c)}{a^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2bc \arctan\left(\frac{\sqrt{a}\,(2a\alpha y+2bx\alpha+\beta)}{\sqrt{(4\alpha\gamma-\beta^2)a+4bc\alpha}}\right)}{a^{\frac{3}{2}}\sqrt{(4\alpha\gamma-\beta^2)\,a+4bc\alpha}} = \frac{bc \ln(xa+c)}{a^2} + c_1$$

Which simplifies to

$$\frac{2bc \arctan\left(\frac{\sqrt{a}\,(2a\alpha y+2bx\alpha+\beta)}{\sqrt{(4\alpha\gamma-\beta^2)a+4bc\alpha}}\right)}{a^{\frac{3}{2}}\sqrt{(4\alpha\gamma-\beta^2)\,a+4bc\alpha}} = \frac{bc \ln(xa+c)}{a^2} + c_1$$

Which gives

$$y = \frac{-2\sqrt{a}\,\alpha b x + \tan\left(\frac{\sqrt{4\alpha\gamma a-a\,\beta^2+4bc\alpha}\,(c_1 a^2+bc\ln(xa+c))}{2\sqrt{a}\,bc}\right)\sqrt{4\alpha\gamma a-a\,\beta^2+4bc\alpha}-\sqrt{a}\,\beta}{2a^{\frac{3}{2}}\alpha}$$

Summary

The solution(s) found are the following

$$y = \frac{-2\sqrt{a}\,\alpha b x + \tan\left(\frac{\sqrt{4\alpha\gamma a-a\,\beta^2+4bc\alpha}\,(c_1 a^2+bc\ln(xa+c))}{2\sqrt{a}\,bc}\right)\sqrt{4\alpha\gamma a-a\,\beta^2+4bc\alpha}-\sqrt{a}\,\beta}{2a^{\frac{3}{2}}\alpha} \tag{1}$$

Verification of solutions

$$y = \frac{-2\sqrt{a}\,\alpha b x + \tan\left(\frac{\sqrt{4\alpha\gamma a-a\,\beta^2+4bc\alpha}\,(c_1 a^2+bc\ln(xa+c))}{2\sqrt{a}\,bc}\right)\sqrt{4\alpha\gamma a-a\,\beta^2+4bc\alpha}-\sqrt{a}\,\beta}{2a^{\frac{3}{2}}\alpha}$$

Verified OK.

2.46.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a^2\alpha\,y^2+2a\alpha bxy+\alpha\,b^2x^2+a\beta y+b\beta x-bx+\gamma}{xa+c} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a^2\alpha\, y^2}{xa+c} + \frac{2a\alpha bxy}{xa+c} + \frac{\alpha\, b^2x^2}{xa+c} + \frac{a\beta y}{xa+c} + \frac{b\beta x}{xa+c} - \frac{bx}{xa+c} + \frac{\gamma}{xa+c}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\alpha\, b^2x^2+b\beta x-bx+\gamma}{xa+c}$, $f_1(x) = \frac{2\alpha axb+\beta a}{xa+c}$ and $f_2(x) = \frac{\alpha\, a^2}{xa+c}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{\frac{\alpha\, a^2u}{xa+c}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{\alpha\, a^3}{(xa+c)^2} \\ f_1f_2 &= \frac{(2\alpha axb+\beta a)\,\alpha\, a^2}{(xa+c)^2} \\ f_2^2f_0 &= \frac{\alpha^2a^4(\alpha\, b^2x^2+b\beta x-bx+\gamma)}{(xa+c)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\alpha\, a^2u''(x)}{xa+c} - \left(-\frac{\alpha\, a^3}{(xa+c)^2} + \frac{(2\alpha axb+\beta a)\,\alpha\, a^2}{(xa+c)^2}\right)u'(x) + \frac{\alpha^2a^4(\alpha\, b^2x^2+b\beta x-bx+\gamma)\, u(x)}{(xa+c)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{bx\alpha}\left((xa+c)^{\frac{-2bc\alpha+\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}\,a+\beta a}{2a}}c_1 + (xa+c)^{-\frac{2bc\alpha+\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}\,a-\beta a}{2a}}c_2\right)$$

The above shows that

$$u'(x) = \frac{a\,\mathrm{e}^{bx\alpha}\left(-c_2\left(-2bx\alpha+\sqrt{\frac{(-4\alpha\gamma+\beta^2)a-4bc\alpha}{a}}-\beta\right)(xa+c)^{-\frac{2bc\alpha+\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}\,a-\beta a}{2a}} + (xa+c)^{\frac{-2bc\alpha+\sqrt{\frac{\left(-4\alpha\gamma\right.}{}}}{}}\right.}{2xa+2c}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-c_2\left(-2bx\alpha+\sqrt{\frac{(-4\alpha\gamma+\beta^2)a-4bc\alpha}{a}}-\beta\right)(xa+c)^{-\frac{2bc\alpha+\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}\,a-\beta a}{2a}}+(xa+c)^{\frac{-2bc\alpha+\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)}{a}}}{2a}}\right.}{a\,(2xa+2c)\,\alpha\left((xa+c)^{\frac{-2bc\alpha+\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}\,a+\beta a}{2a}}c_1+(xa+c)^{-\frac{2bc\alpha+}{}}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1}=c_3$ the following solution

$$y = \frac{\left(-2bx\alpha+\sqrt{\frac{(-4\alpha\gamma+\beta^2)a-4bc\alpha}{a}}-\beta\right)(xa+c)^{-\frac{\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}}{2}}-2c_3\left(bx\alpha+\frac{\beta}{2}+\frac{\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}}{2}\right)(xa-}{2\alpha a\left((xa+c)^{\frac{\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}}{2}}c_3+(xa+c)^{-\frac{\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}}{2}}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-2bx\alpha+\sqrt{\frac{(-4\alpha\gamma+\beta^2)a-4bc\alpha}{a}}-\beta\right)(xa+c)^{-\frac{\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}}{2}}-2c_3\left(bx\alpha+\frac{\beta}{2}+\frac{\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}}{2}\right)(xa-}{2\alpha a\left((xa+c)^{\frac{\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}}{2}}c_3+(xa+c)^{-\frac{\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}}{2}}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(-2bx\alpha+\sqrt{\frac{(-4\alpha\gamma+\beta^2)a-4bc\alpha}{a}}-\beta\right)(xa+c)^{-\frac{\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}}{2}}-2c_3\left(bx\alpha+\frac{\beta}{2}+\frac{\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}}{2}\right)(xa-}{2\alpha a\left((xa+c)^{\frac{\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}}{2}}c_3+(xa+c)^{-\frac{\sqrt{\frac{\left(-4\alpha\gamma+\beta^2\right)a-4bc\alpha}{a}}}{2}}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = (-a*x-c)*b/(a*(a*x+c)), y(x)`      ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 94

```
dsolve((a*x+c)*diff(y(x),x)=alpha*(a*y(x)+b*x)^2+beta*(a*y(x)+b*x)-b*x+gamma,y(x), singsol=a
```

$$y(x) = \frac{-2a^2\alpha x b - a^2\beta + \sqrt{-\left(\left(-4\gamma\alpha+\beta^2\right)a - 4\alpha b c\right)a^3}\,\tan\left(\frac{-2c_1a^2+\ln(ax+c)\sqrt{-\left(\left(-4\gamma\alpha+\beta^2\right)a-4\alpha bc\right)a^3}}{2a^2}\right)}{2a^3\alpha}$$

✓ Solution by Mathematica

Time used: 60.527 (sec). Leaf size: 98

```
DSolve[(a*x+c)*y'[x]==\[Alpha]*(a*y[x]+b*x)^2+\[Beta]*(a*y[x]+b*x)-b*x+\[Gamma],y[x],x,Inclu
```

$$y(x) \to -\frac{-a\alpha\sqrt{\frac{4a\alpha\gamma-a\beta^2+4\alpha bc}{a^3\alpha^2}}\tan\left(\frac{1}{2}a\alpha\log(ax+c)\sqrt{\frac{4a\alpha\gamma-a\beta^2+4\alpha bc}{a^3\alpha^2}}+c_1\right)+2\alpha bx+\beta}{2a\alpha}$$

2.47 problem 47

Internal problem ID [10377]
Internal file name [`OUTPUT/9324_Monday_June_06_2022_01_51_35_PM_72746468/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 47.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{2x^2y' - 2y^2 - xy = -2a^2x}$$

2.47.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-2a^2x + yx + 2y^2}{2x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{a^2}{x} + \frac{y}{2x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{a^2}{x}$, $f_1(x) = \frac{1}{2x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1 f_2 &= \frac{1}{2x^3} \\ f_2^2 f_0 &= -\frac{a^2}{x^5} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{3u'(x)}{2x^3} - \frac{a^2 u(x)}{x^5} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sinh\left(\frac{2a}{\sqrt{x}}\right) + c_2 \cosh\left(\frac{2a}{\sqrt{x}}\right)$$

The above shows that

$$u'(x) = \frac{a\left(-c_1 \cosh\left(\frac{2a}{\sqrt{x}}\right) - c_2 \sinh\left(\frac{2a}{\sqrt{x}}\right)\right)}{x^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$y = -\frac{a\sqrt{x}\left(-c_1 \cosh\left(\frac{2a}{\sqrt{x}}\right) - c_2 \sinh\left(\frac{2a}{\sqrt{x}}\right)\right)}{c_1 \sinh\left(\frac{2a}{\sqrt{x}}\right) + c_2 \cosh\left(\frac{2a}{\sqrt{x}}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(c_3 \cosh\left(\frac{2a}{\sqrt{x}}\right) + \sinh\left(\frac{2a}{\sqrt{x}}\right)\right) a\sqrt{x}}{c_3 \sinh\left(\frac{2a}{\sqrt{x}}\right) + \cosh\left(\frac{2a}{\sqrt{x}}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_3 \cosh\left(\frac{2a}{\sqrt{x}}\right) + \sinh\left(\frac{2a}{\sqrt{x}}\right)\right) a\sqrt{x}}{c_3 \sinh\left(\frac{2a}{\sqrt{x}}\right) + \cosh\left(\frac{2a}{\sqrt{x}}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(c_3 \cosh\left(\frac{2a}{\sqrt{x}}\right) + \sinh\left(\frac{2a}{\sqrt{x}}\right)\right) a\sqrt{x}}{c_3 \sinh\left(\frac{2a}{\sqrt{x}}\right) + \cosh\left(\frac{2a}{\sqrt{x}}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(2*x^2*diff(y(x),x)=2*y(x)^2+x*y(x)-2*a^2*x,y(x), singsol=all)
```

$$y(x) = \tanh\left(\frac{ic_1\sqrt{x} + 2a}{\sqrt{x}}\right)\sqrt{x}\, a$$

✓ Solution by Mathematica

Time used: 0.619 (sec). Leaf size: 43

```
DSolve[2*x^2*y'[x]==2*y[x]^2+x*y[x]-2*a^2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-a^2}\sqrt{x}\tan\left(\frac{2\sqrt{-a^2}}{\sqrt{x}} - c_1\right)$$

2.48 problem 48

Internal problem ID [10378]
Internal file name [OUTPUT/9325_Monday_June_06_2022_01_51_35_PM_24216642/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 48.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{2x^2y' - 2y^2 - 3xy = -2a^2x}$$

2.48.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-2a^2x + 3yx + 2y^2}{2x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{a^2}{x} + \frac{3y}{2x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{a^2}{x}$, $f_1(x) = \frac{3}{2x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1f_2 &= \frac{3}{2x^3} \\ f_2^2f_0 &= -\frac{a^2}{x^5} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{u'(x)}{2x^3} - \frac{a^2u(x)}{x^5} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sqrt{x}\left(c_1 \sinh\left(\frac{2a}{\sqrt{x}}\right) + c_2\cosh\left(\frac{2a}{\sqrt{x}}\right)\right)$$

The above shows that

$$u'(x) = \frac{\left(-2\sqrt{x}\, c_1a + c_2x\right)\cosh\left(\frac{2a}{\sqrt{x}}\right) + \left(-2\sqrt{x}\, c_2a + c_1x\right)\sinh\left(\frac{2a}{\sqrt{x}}\right)}{2x^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-2\sqrt{x}\, c_1a + c_2x\right)\cosh\left(\frac{2a}{\sqrt{x}}\right) + \left(-2\sqrt{x}\, c_2a + c_1x\right)\sinh\left(\frac{2a}{\sqrt{x}}\right)}{2\left(c_1\sinh\left(\frac{2a}{\sqrt{x}}\right) + c_2\cosh\left(\frac{2a}{\sqrt{x}}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\sinh\left(\frac{2a}{\sqrt{x}}\right)\left(2a\sqrt{x} - c_3x\right) + 2\left(\sqrt{x}\, c_3a - \frac{x}{2}\right)\cosh\left(\frac{2a}{\sqrt{x}}\right)}{2c_3\sinh\left(\frac{2a}{\sqrt{x}}\right) + 2\cosh\left(\frac{2a}{\sqrt{x}}\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{\sinh\left(\frac{2a}{\sqrt{x}}\right)\left(2a\sqrt{x}-c_3x\right)+2\left(\sqrt{x}\,c_3a-\frac{x}{2}\right)\cosh\left(\frac{2a}{\sqrt{x}}\right)}{2c_3\sinh\left(\frac{2a}{\sqrt{x}}\right)+2\cosh\left(\frac{2a}{\sqrt{x}}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\sinh\left(\frac{2a}{\sqrt{x}}\right)\left(2a\sqrt{x}-c_3x\right)+2\left(\sqrt{x}\,c_3a-\frac{x}{2}\right)\cosh\left(\frac{2a}{\sqrt{x}}\right)}{2c_3\sinh\left(\frac{2a}{\sqrt{x}}\right)+2\cosh\left(\frac{2a}{\sqrt{x}}\right)}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Group is reducible or imprimitive
      <- Kovacics algorithm successful
   <- Abel AIR successful: ODE belongs to the 0F1 1-parameter (Bessel type) class`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 102

```
dsolve(2*x^2*diff(y(x),x)=2*y(x)^2+3*x*y(x)-2*a^2*x,y(x), singsol=all)
```

$$y(x) = \frac{\left(-2xc_1\sqrt{-\frac{a^2}{x}} - x\right)\sin\left(2\sqrt{-\frac{a^2}{x}}\right) - x\left(c_1 - 2\sqrt{-\frac{a^2}{x}}\right)\cos\left(2\sqrt{-\frac{a^2}{x}}\right)}{2\cos\left(2\sqrt{-\frac{a^2}{x}}\right)c_1 + 2\sin\left(2\sqrt{-\frac{a^2}{x}}\right)}$$

✓ Solution by Mathematica

Time used: 0.457 (sec). Leaf size: 94

```
DSolve[2*x^2*y'[x]==2*y[x]^2+3*x*y[x]-2*a^2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4a^2c_1\sqrt{x} + 2a\sqrt{x}e^{\frac{4a}{\sqrt{x}}} - xe^{\frac{4a}{\sqrt{x}}} + 2ac_1x}{2e^{\frac{4a}{\sqrt{x}}} - 4ac_1}$$

$$y(x) \to a(-\sqrt{x}) - \frac{x}{2}$$

2.49 problem 49

Internal problem ID [10379]
Internal file name [`OUTPUT/9326_Monday_June_06_2022_01_51_36_PM_57238639/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 49.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Riccati]
```

$$\boxed{x^2y' - a\,x^2y^2 - ybx = c}$$

2.49.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{a\,x^2y^2 + bxy + c}{x^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{(a\,x^2y^2 + bxy + c)\,(b_3 - a_2)}{x^2} - \frac{(a\,x^2y^2 + bxy + c)^2\,a_3}{x^4} \\ &- \left(\frac{2ax\,y^2 + by}{x^2} - \frac{2(a\,x^2y^2 + bxy + c)}{x^3}\right)(xa_2 + ya_3 + a_1) \\ &- \frac{(2a\,x^2y + bx)\,(xb_2 + yb_3 + b_1)}{x^2} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{a^2x^4y^4a_3 + 2ab\,x^3y^3a_3 + 2a\,x^5yb_2 + a\,x^4y^2a_2 + a\,x^4y^2b_3 + 2ac\,x^2y^2a_3 + 2a\,x^4yb_1 + b^2x^2y^2a_3 + b\,x^4b_2 - b}{x^4} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-a^2x^4y^4a_3 - 2ab\,x^3y^3a_3 - 2a\,x^5yb_2 - a\,x^4y^2a_2 - a\,x^4y^2b_3 - 2ac\,x^2y^2a_3 \\ &- 2a\,x^4yb_1 - b^2x^2y^2a_3 - b\,x^4b_2 + b\,x^2y^2a_3 - 2bcxya_3 - b\,x^3b_1 \\ &+ b\,x^2ya_1 + b_2x^4 + c\,x^2a_2 + c\,x^2b_3 + 2cxya_3 - c^2a_3 + 2cxa_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-a^2a_3v_1^4v_2^4 - 2aba_3v_1^3v_2^3 - aa_2v_1^4v_2^2 - 2ab_2v_1^5v_2 - ab_3v_1^4v_2^2 - 2aca_3v_1^2v_2^2 \\ &- 2ab_1v_1^4v_2 - b^2a_3v_1^2v_2^2 + ba_3v_1^2v_2^2 - bb_2v_1^4 - 2bca_3v_1v_2 + ba_1v_1^2v_2 \\ &- bb_1v_1^3 + b_2v_1^4 + ca_2v_1^2 + 2ca_3v_1v_2 + cb_3v_1^2 - c^2a_3 + 2ca_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-2ab_2v_1^5v_2 - a^2a_3v_1^4v_2^4 + (-aa_2 - ab_3)\, v_1^4v_2^2 - 2ab_1v_1^4v_2 + (-bb_2 + b_2)\, v_1^4 \\
&- 2aba_3v_1^3v_2^3 - bb_1v_1^3 + \left(-2aca_3 - b^2a_3 + ba_3\right) v_1^2v_2^2 + ba_1v_1^2v_2 \\
&+ (ca_2 + cb_3)\, v_1^2 + (-2bca_3 + 2ca_3)\, v_1v_2 + 2ca_1v_1 - c^2a_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
ba_1 &= 0 \\
-2ab_1 &= 0 \\
-2ab_2 &= 0 \\
-a^2a_3 &= 0 \\
-bb_1 &= 0 \\
2ca_1 &= 0 \\
-c^2a_3 &= 0 \\
-2aba_3 &= 0 \\
-bb_2 + b_2 &= 0 \\
-aa_2 - ab_3 &= 0 \\
-2bca_3 + 2ca_3 &= 0 \\
ca_2 + cb_3 &= 0 \\
-2aca_3 - b^2a_3 + ba_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{a\,x^2y^2 + bxy + c}{x^2}\right)(-x) \\
&= \frac{a\,x^2y^2 + bxy + yx + c}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{a\,x^2y^2+bxy+yx+c}{x}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{2x \arctan\left(\frac{2a\,x^2y+bx+x}{\sqrt{4ac\,x^2-b^2x^2-2b\,x^2-x^2}}\right)}{\sqrt{4ac\,x^2 - b^2x^2 - 2b\,x^2 - x^2}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{a\,x^2y^2 + bxy + c}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{y}{a\,x^2y^2 + y\,(b+1)\,x + c}\\
S_y &= \frac{x}{a\,x^2y^2 + y\,(b+1)\,x + c}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\arctan\left(\frac{2yax+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}} = \ln(x) + c_1$$

Which simplifies to

$$\frac{2\arctan\left(\frac{2yax+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}} = \ln(x) + c_1$$

Which gives

$$y = \frac{\tan\left(\frac{\ln(x)\sqrt{4ca-b^2-2b-1}}{2} + \frac{c_1\sqrt{4ca-b^2-2b-1}}{2}\right)\sqrt{4ca-b^2-2b-1} - b - 1}{2xa}$$

Summary

The solution(s) found are the following

$$y = \frac{\tan\left(\frac{\ln(x)\sqrt{4ca-b^2-2b-1}}{2} + \frac{c_1\sqrt{4ca-b^2-2b-1}}{2}\right)\sqrt{4ca-b^2-2b-1} - b - 1}{2xa} \tag{1}$$

Verification of solutions

$$y = \frac{\tan\left(\frac{\ln(x)\sqrt{4ca-b^2-2b-1}}{2} + \frac{c_1\sqrt{4ca-b^2-2b-1}}{2}\right)\sqrt{4ca-b^2-2b-1} - b - 1}{2xa}$$

Verified OK.

2.49.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^{2}\right)\mathrm{d}y&=\left(a\,x^{2}y^{2}+bxy+c\right)\mathrm{d}x\\ \left(-a\,x^{2}y^{2}-bxy-c\right)\mathrm{d}x+\left(x^{2}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-a\,x^{2}y^{2}-bxy-c\\ N(x,y)&=x^{2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-a\,x^{2}y^{2}-bxy-c\right)\\ &=-2a\,x^{2}y-bx\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(x^{2}\right)\\ &=2x\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=\frac{1}{x^{2}}\left(\left(-2a\,x^{2}y-bx\right)-(2x)\right)\\ &=\frac{-2axy-b-2}{x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{a\,x^{2}y^{2}+bxy+c}\left((2x)-\left(-2a\,x^{2}y-bx\right)\right) \\
&= -\frac{x(2axy+b+2)}{a\,x^{2}y^{2}+bxy+c}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R=\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{xM-yN}
$$

R is now checked to see if it is a function of only $t=xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{xM-yN} \\
&= \frac{(2x)-\left(-2a\,x^{2}y-bx\right)}{x\left(-a\,x^{2}y^{2}-bxy-c\right)-y\left(x^{2}\right)} \\
&= \frac{-2axy-b-2}{a\,x^{2}y^{2}+y\left(b+1\right)x+c}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R=\frac{-2at-b-2}{a\,t^{2}+\left(b+1\right)t+c}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R\,\mathrm{d}t} \\
&= e^{\int \left(\frac{-2at-b-2}{a\,t^{2}+(b+1)t+c}\right)\mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln\left(a\,t^{2}+tb+c+t\right)-\frac{2\arctan\left(\frac{2at+b+1}{\sqrt{4ca-b^{2}-2b-1}}\right)}{\sqrt{4ca-b^{2}-2b-1}}} \\
&= \frac{\mathrm{e}^{-\frac{2\arctan\left(\frac{2at+b+1}{\sqrt{4ca-b^{2}-2b-1}}\right)}{\sqrt{4ca-b^{2}-2b-1}}}}{a\,t^{2}+tb+c+t}
\end{aligned}
$$

Now t is replaced back with xy giving

$$\mu = \frac{e^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}}}{a\,x^2y^2+bxy+yx+c}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{e^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}}}{a\,x^2y^2+bxy+yx+c}(-a\,x^2y^2-bxy-c) \\ &= -\frac{(a\,x^2y^2+bxy+c)\,e^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}}}{a\,x^2y^2+y\,(b+1)\,x+c}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{e^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}}}{a\,x^2y^2+bxy+yx+c}(x^2) \\ &= \frac{x^2 e^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}}}{a\,x^2y^2+y\,(b+1)\,x+c}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(-\frac{(a\,x^2y^2+bxy+c)\,e^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}}}{a\,x^2y^2+y\,(b+1)\,x+c}\right) + \left(\frac{x^2 e^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}}}{a\,x^2y^2+y\,(b+1)\,x+c}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{\left(a\,x^2y^2+bxy+c\right) \mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}}}{a\,x^2y^2+y\,(b+1)\,x+c} \mathrm{d}x$$

$$\phi = -\mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}} x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= \frac{4x^2 a\,\mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}}}{\left(4ca-b^2-2b-1\right)\left(\frac{(2axy+b+1)^2}{4ca-b^2-2b-1}+1\right)} + f'(y) \\ &= \frac{x^2 \mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}}}{a\,x^2y^2+y\,(b+1)\,x+c} + f'(y) \end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2 \mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}}}{a\,x^2y^2+y(b+1)x+c}$. Therefore equation (4) becomes

$$\frac{x^2 \mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}}}{a\,x^2y^2+y\,(b+1)\,x+c} = \frac{x^2 \mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}}}{a\,x^2y^2+y\,(b+1)\,x+c} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}} x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\mathrm{e}^{-\frac{2\arctan\left(\frac{2axy+b+1}{\sqrt{4ca-b^2-2b-1}}\right)}{\sqrt{4ca-b^2-2b-1}}} x$$

The solution becomes

$$y = -\frac{\tan\left(\frac{\ln\left(-\frac{c_1}{x}\right)\sqrt{4ca-b^2-2b-1}}{2}\right)\sqrt{4ca-b^2-2b-1}+b+1}{2xa}$$

Summary

The solution(s) found are the following

$$y = -\frac{\tan\left(\frac{\ln\left(-\frac{c_1}{x}\right)\sqrt{4ca-b^2-2b-1}}{2}\right)\sqrt{4ca-b^2-2b-1}+b+1}{2xa} \tag{1}$$

Verification of solutions

$$y = -\frac{\tan\left(\frac{\ln\left(-\frac{c_1}{x}\right)\sqrt{4ca-b^2-2b-1}}{2}\right)\sqrt{4ca-b^2-2b-1}+b+1}{2xa}$$

Verified OK.

2.49.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,x^2y^2+bxy+c}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,y^2 + \frac{by}{x} + \frac{c}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{c}{x^2}$, $f_1(x) = \frac{b}{x}$ and $f_2(x) = a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= \frac{ab}{x} \\ f_2^2f_0 &= \frac{a^2c}{x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$au''(x) - \frac{abu'(x)}{x} + \frac{a^2cu(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{\frac{b}{2}}\sqrt{x}\left(x^{\frac{\sqrt{-4ca+b^2+2b+1}}{2}}c_1 + x^{-\frac{\sqrt{-4ca+b^2+2b+1}}{2}}c_2\right)$$

The above shows that

$$u'(x) = \frac{\left(c_2\left(1+b-\sqrt{-4ca+b^2+2b+1}\right)x^{-\frac{\sqrt{-4ca+b^2+2b+1}}{2}} + x^{\frac{\sqrt{-4ca+b^2+2b+1}}{2}}c_1\left(1+b+\sqrt{-4ca+b^2+2b+1}\right)\right)x}{2\sqrt{x}}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2\left(1+b-\sqrt{-4ca+b^2+2b+1}\right)x^{-\frac{\sqrt{-4ca+b^2+2b+1}}{2}} + x^{\frac{\sqrt{-4ca+b^2+2b+1}}{2}}c_1\left(1+b+\sqrt{-4ca+b^2+2b+1}\right)}{2xa\left(x^{\frac{\sqrt{-4ca+b^2+2b+1}}{2}}c_1 + x^{-\frac{\sqrt{-4ca+b^2+2b+1}}{2}}c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-1-b+\sqrt{-4ca+b^2+2b+1}\right)x^{-\frac{\sqrt{-4ca+b^2+2b+1}}{2}} - x^{\frac{\sqrt{-4ca+b^2+2b+1}}{2}}c_3\left(1+b+\sqrt{-4ca+b^2+2b+1}\right)}{2xa\left(x^{\frac{\sqrt{-4ca+b^2+2b+1}}{2}}c_3 + x^{-\frac{\sqrt{-4ca+b^2+2b+1}}{2}}\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(-1-b+\sqrt{-4ca+b^2+2b+1}\right) x^{-\frac{\sqrt{-4ca+b^2+2b+1}}{2}} - x^{\frac{\sqrt{-4ca+b^2+2b+1}}{2}} c_3 \left(1+b+\sqrt{-4ca+b^2+2b+1}\right)}{2xa \left(x^{\frac{\sqrt{-4ca+b^2+2b+1}}{2}} c_3 + x^{-\frac{\sqrt{-4ca+b^2+2b+1}}{2}}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(-1-b+\sqrt{-4ca+b^2+2b+1}\right) x^{-\frac{\sqrt{-4ca+b^2+2b+1}}{2}} - x^{\frac{\sqrt{-4ca+b^2+2b+1}}{2}} c_3 \left(1+b+\sqrt{-4ca+b^2+2b+1}\right)}{2xa \left(x^{\frac{\sqrt{-4ca+b^2+2b+1}}{2}} c_3 + x^{-\frac{\sqrt{-4ca+b^2+2b+1}}{2}}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 59

```
dsolve(x^2*diff(y(x),x)=a*x^2*y(x)^2+b*x*y(x)+c,y(x), singsol=all)
```

$$y(x) = \frac{-1-b+\tan\left(\frac{\sqrt{4ac-b^2-2b-1}\,(\ln(x)-c_1)}{2}\right)\sqrt{4ac-b^2-2b-1}}{2ax}$$

✓ Solution by Mathematica

Time used: 0.43 (sec). Leaf size: 99

```
DSolve[x^2*y'[x]==a*x^2*y[x]^2+b*x*y[x]+c,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-4ac+b^2+2b+1}\left(1-\frac{2c_1}{x^{\sqrt{-4ac+b^2+2b+1}}+c_1}\right)+b+1}{2ax}$$
$$y(x) \to -\frac{-\sqrt{-4ac+b^2+2b+1}+b+1}{2ax}$$

2.50 problem 50

Internal problem ID [10380]
Internal file name [`OUTPUT/9327_Monday_June_06_2022_01_51_39_PM_55641103/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 50.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{x^2y' - y^2c\,x^2 - \left(a\,x^2 + bx\right)y = \alpha\,x^2 + \beta x + \gamma}$$

2.50.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2c\,x^2 + a\,x^2y + \alpha\,x^2 + bxy + \beta x + \gamma}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = c\,y^2 + ya + \alpha + \frac{by}{x} + \frac{\beta}{x} + \frac{\gamma}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\alpha\,x^2+\beta x+\gamma}{x^2}$, $f_1(x) = \frac{a\,x^2+bx}{x^2}$ and $f_2(x) = c$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{cu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \frac{(a\,x^2 + bx)\,c}{x^2} \\ f_2^2 f_0 &= \frac{c^2(\alpha\,x^2 + \beta x + \gamma)}{x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$c u''(x) - \frac{(a\,x^2 + bx)\,c u'(x)}{x^2} + \frac{c^2(\alpha\,x^2 + \beta x + \gamma)\,u(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = x^{\frac{b}{2}} \mathrm{e}^{\frac{xa}{2}} \Bigg(&c_1 \operatorname{WhittakerM}\left(-\frac{ab - 2\beta c}{2\sqrt{a^2 - 4\alpha c}}, \frac{\sqrt{b^2 - 4c\gamma + 2b + 1}}{2}, \sqrt{a^2 - 4\alpha c}\,x\right) \\ &+ c_2 \operatorname{WhittakerW}\left(-\frac{ab - 2\beta c}{2\sqrt{a^2 - 4\alpha c}}, \frac{\sqrt{b^2 - 4c\gamma + 2b + 1}}{2}, \sqrt{a^2 - 4\alpha c}\,x\right)\Bigg) \end{aligned}$$

The above shows that

$$u'(x) = \frac{\left(-c_1\left(ab - 2\beta c - \sqrt{b^2 - 4c\gamma + 2b + 1}\,\sqrt{a^2 - 4\alpha c} - \sqrt{a^2 - 4\alpha c}\right) \operatorname{WhittakerM}\left(-\frac{ab - 2\beta c - 2\sqrt{a^2 - 4\alpha c}}{2\sqrt{a^2 - 4\alpha c}}, \frac{\sqrt{b^2 - 4c}}{}\right.\right.}{}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-c_1\left(ab - 2\beta c - \sqrt{b^2 - 4c\gamma + 2b + 1}\,\sqrt{a^2 - 4\alpha c} - \sqrt{a^2 - 4\alpha c}\right) \operatorname{WhittakerM}\left(-\frac{ab - 2\beta c - 2\sqrt{a^2 - 4\alpha c}}{2\sqrt{a^2 - 4\alpha c}}, \frac{\sqrt{b^2}}{}\right.\right.}{}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{-c_3\left(ab - 2\beta c - \sqrt{b^2 - 4c\gamma + 2b + 1}\,\sqrt{a^2 - 4\alpha c} - \sqrt{a^2 - 4\alpha c}\right)\mathrm{WhittakerM}\left(-\frac{ab-2\beta c-2\sqrt{a^2-4\alpha c}}{2\sqrt{a^2-4\alpha c}}, \frac{\sqrt{b^2-\cdots}}{}\right.}{}$$

Summary

The solution(s) found are the following

$$y = -\frac{-c_3\left(ab - 2\beta c - \sqrt{b^2 - 4c\gamma + 2b + 1}\,\sqrt{a^2 - 4\alpha c} - \sqrt{a^2 - 4\alpha c}\right)\mathrm{WhittakerM}\left(-\frac{ab-2\beta c-2\sqrt{a^2-4\alpha c}}{2\sqrt{a^2-4\alpha c}}, \frac{\sqrt{b^2-\cdots}}{}\right.}{} \tag{1}$$

Verification of solutions

$$y = -\frac{-c_3\left(ab - 2\beta c - \sqrt{b^2 - 4c\gamma + 2b + 1}\,\sqrt{a^2 - 4\alpha c} - \sqrt{a^2 - 4\alpha c}\right)\mathrm{WhittakerM}\left(-\frac{ab-2\beta c-2\sqrt{a^2-4\alpha c}}{2\sqrt{a^2-4\alpha c}}, \frac{\sqrt{b^2-\cdots}}{}\right.}{}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a*x+b)*(diff(y(x), x))/x-c*(a
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
         <- Whittaker successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 443

```
dsolve(x^2*diff(y(x),x)=c*x^2*y(x)^2+(a*x^2+b*x)*y(x)+alpha*x^2+beta*x+gamma,y(x), singsol=a
```

$$y(x) = -\frac{\left(\sqrt{b^2-4c\gamma+2b+1}\,\sqrt{a^2-4\alpha c}-ab+2\beta c+\sqrt{a^2-4\alpha c}\right)\mathrm{WhittakerM}\left(-\frac{ab-2\beta c-2\sqrt{a^2-4\alpha c}}{2\sqrt{a^2-4\alpha c}},\frac{\sqrt{b^2-4c\gamma+2}}{2}\right.}{}$$

✓ Solution by Mathematica

Time used: 1.712 (sec). Leaf size: 1584

```
DSolve[x^2*y'[x]==c*x^2*y[x]^2+(a*x^2+b*x)*y[x]+\[Alpha]*x^2+\[Beta]*x+\[Gamma],y[x],x,Inclu
```

$$y(x) \to -\frac{\left(b+ax-x\sqrt{a^2-4c\alpha}+\sqrt{b^2+2b-4c\gamma+1}+1\right)c_1\,\mathrm{HypergeometricU}\left(\frac{ab-2c\beta+\sqrt{a^2-4c\alpha}\left(\sqrt{b^2+2b-4c\gamma+1}\right.}{2\sqrt{a^2-4c\alpha}}\right.}{}$$

$$y(x) \to \frac{\frac{\left(\sqrt{a^2-4\alpha c}\left(\sqrt{b^2+2b-4c\gamma+1}+1\right)+ab-2\beta c\right)\mathrm{HypergeometricU}\left(\frac{ab-2c\beta+\sqrt{a^2-4c\alpha}\left(\sqrt{b^2+2b-4c\gamma+1}+3\right)}{2\sqrt{a^2-4c\alpha}},\sqrt{b^2+2b-4c\gamma+1}+2,x\sqrt{a^2-4c\alpha}\right)}{\mathrm{HypergeometricU}\left(\frac{ab-2c\beta+\sqrt{a^2-4c\alpha}\left(\sqrt{b^2+2b-4c\gamma+1}+1\right)}{2\sqrt{a^2-4c\alpha}},\sqrt{b^2+2b-4c\gamma+1}+1,x\sqrt{a^2-4c\alpha}\right)} -}{2c}$$

$$y(x) \to \frac{\frac{\left(\sqrt{a^2-4\alpha c}\left(\sqrt{b^2+2b-4c\gamma+1}+1\right)+ab-2\beta c\right)\mathrm{HypergeometricU}\left(\frac{ab-2c\beta+\sqrt{a^2-4c\alpha}\left(\sqrt{b^2+2b-4c\gamma+1}+3\right)}{2\sqrt{a^2-4c\alpha}},\sqrt{b^2+2b-4c\gamma+1}+2,x\sqrt{a^2-4c\alpha}\right)}{\mathrm{HypergeometricU}\left(\frac{ab-2c\beta+\sqrt{a^2-4c\alpha}\left(\sqrt{b^2+2b-4c\gamma+1}+1\right)}{2\sqrt{a^2-4c\alpha}},\sqrt{b^2+2b-4c\gamma+1}+1,x\sqrt{a^2-4c\alpha}\right)} -}{2c}$$

2.51 problem 51

Internal problem ID [10381]
Internal file name [OUTPUT/9328_Monday_June_06_2022_01_51_52_PM_78719202/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 51.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{x^2y' - a\,x^2y^2 - ybx = c\,x^n + s}$$

2.51.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,x^2y^2 + bxy + c\,x^n + s}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,y^2 + \frac{by}{x} + \frac{c\,x^n}{x^2} + \frac{s}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{c\,x^n+s}{x^2}$, $f_1(x) = \frac{b}{x}$ and $f_2(x) = a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \frac{ab}{x} \\ f_2^2 f_0 &= \frac{a^2(c\,x^n + s)}{x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$a u''(x) - \frac{abu'(x)}{x} + \frac{a^2(c\,x^n + s)\,u(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{\frac{b}{2}}\sqrt{x}\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}, \frac{2\sqrt{ca}\,x^{\frac{n}{2}}}{n}\right)c_1 + \operatorname{BesselY}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}, \frac{2\sqrt{ca}\,x^{\frac{n}{2}}}{n}\right)c_2\right)$$

The above shows that

$$u'(x) = \frac{\left(-2\sqrt{ca}\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}+1, \frac{2\sqrt{ca}\,x^{\frac{n}{2}}}{n}\right)c_1 + \operatorname{BesselY}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}+1, \frac{2\sqrt{ca}\,x^{\frac{n}{2}}}{n}\right)c_2\right)x^{\frac{n}{2}} + \left(\mathrm{Be}\right.\right.}{}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-2\sqrt{ca}\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}+1, \frac{2\sqrt{ca}\,x^{\frac{n}{2}}}{n}\right)c_1 + \operatorname{BesselY}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}+1, \frac{2\sqrt{ca}\,x^{\frac{n}{2}}}{n}\right)c_2\right)x^{\frac{n}{2}} + \left(\right.\right.}{2a\sqrt{x}\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}, \right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2\sqrt{ca}\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}+1, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 + \operatorname{BesselY}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}+1, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right)\right) x^{\frac{n}{2}} - \left(\operatorname{BesselJ}\right.}{2ax\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 +\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{2\sqrt{ca}\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}+1, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 + \operatorname{BesselY}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}+1, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right)\right) x^{\frac{n}{2}} - \left(\operatorname{BesselJ}\right.}{2ax\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 +\right.} \tag{1}$$

Verification of solutions

$$y = \frac{2\sqrt{ca}\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}+1, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 + \operatorname{BesselY}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}+1, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right)\right) x^{\frac{n}{2}} - \left(\operatorname{BesselJ}\right.}{2ax\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}, \frac{2\sqrt{ca}\, x^{\frac{n}{2}}}{n}\right) c_3 +\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = b*(diff(y(x), x))/x-a*(x^(n-2)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         <- Bessel successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 263

```
dsolve(x^2*diff(y(x),x)=a*x^2*y(x)^2+b*x*y(x)+c*x^n+s,y(x), singsol=all)
```

$$y(x) = \frac{2\left(\text{BesselY}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}+1, \frac{2\sqrt{ac}\,x^{\frac{n}{2}}}{n}\right)c_1 + \text{BesselJ}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}+1, \frac{2\sqrt{ac}\,x^{\frac{n}{2}}}{n}\right)\right)\sqrt{ac}\,x^{\frac{n}{2}} - \left(\text{BesselY}\right.}{2xa\left(\text{BesselY}\left(\frac{\sqrt{-4as+b^2+2b+1}}{n}, \frac{2\sqrt{ac}\,x^{\frac{n}{2}}}{n}\right)c_1 + \right.}$$

✓ Solution by Mathematica

Time used: 2.637 (sec). Leaf size: 2281

```
DSolve[x^2*y'[x]==a*x^2*y[x]^2+b*x*y[x]+c*x^n+s,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

2.52 problem 52

Internal problem ID [10382]
Internal file name [`OUTPUT/9329_Monday_June_06_2022_01_51_53_PM_16676066/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 52.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{x^2y' - a\,x^2y^2 - ybx = c\,x^{2n} + s\,x^n}$$

2.52.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,x^2y^2 + bxy + c\,x^{2n} + s\,x^n}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,y^2 + \frac{by}{x} + \frac{c\,x^{2n}}{x^2} + \frac{s\,x^n}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{c\,x^{2n}+s\,x^n}{x^2}$, $f_1(x) = \frac{b}{x}$ and $f_2(x) = a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= \frac{ab}{x} \\
f_2^2 f_0 &= \frac{a^2(c\,x^{2n} + s\,x^n)}{x^2}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$au''(x) - \frac{abu'(x)}{x} + \frac{a^2(c\,x^{2n} + s\,x^n)\,u(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
u(x) = \Bigg(&\mathrm{WhittakerW}\left(-\frac{i\sqrt{a}\,s}{2n\sqrt{c}}, \frac{b+1}{2n}, \frac{2i\sqrt{c}\,\sqrt{a}\,x^n}{n}\right) c_2 \\
&+ \mathrm{WhittakerM}\left(-\frac{i\sqrt{a}\,s}{2n\sqrt{c}}, \frac{b+1}{2n}, \frac{2i\sqrt{c}\,\sqrt{a}\,x^n}{n}\right) c_1\Bigg)\, x^{\frac{b}{2}-\frac{n}{2}+\frac{1}{2}}
\end{aligned}$$

The above shows that

$$u'(x) = \frac{\left(-\left(i\sqrt{a}\,\sqrt{c}\,s - c(1+b+n)\right) c_1\,\mathrm{WhittakerM}\left(-\frac{i\sqrt{a}\,s-2n\sqrt{c}}{2n\sqrt{c}}, \frac{b+1}{2n}, \frac{2i\sqrt{c}\,\sqrt{a}\,x^n}{n}\right) - 2cnc_2\,\mathrm{WhittakerW}\left(-\frac{i\sqrt{a}\cdots}{\cdots}\right.\right.}{\cdots}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-\left(i\sqrt{a}\,\sqrt{c}\,s - c(1+b+n)\right) c_1\,\mathrm{WhittakerM}\left(-\frac{i\sqrt{a}\,s-2n\sqrt{c}}{2n\sqrt{c}}, \frac{b+1}{2n}, \frac{2i\sqrt{c}\,\sqrt{a}\,x^n}{n}\right) - 2cnc_2\,\mathrm{WhittakerW}\left(-\cdots\right.\right.}{2ca\left(\mathrm{Wh}\cdots\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(i\sqrt{a}\,\sqrt{c}\,s - c(1+b+n)\right) c_3 \,\mathrm{WhittakerM}\left(-\frac{i\sqrt{a}\,s}{2n\sqrt{c}}+1, \frac{b+1}{2n}, \frac{2i\sqrt{c}\,\sqrt{a}\,x^n}{n}\right) + 2\,\mathrm{WhittakerW}\left(-\frac{i\sqrt{a}\,s}{2n\sqrt{c}}+1, \cdots\right.}{2acx\left(\mathrm{WhittakerW}\left(-\cdots\right.\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(i\sqrt{a}\,\sqrt{c}\,s - c(1+b+n)\right) c_3 \,\mathrm{WhittakerM}\left(-\frac{i\sqrt{a}\,s}{2n\sqrt{c}}+1, \frac{b+1}{2n}, \frac{2i\sqrt{c}\,\sqrt{a}\,x^n}{n}\right) + 2\,\mathrm{WhittakerW}\left(-\frac{i\sqrt{a}\,s}{2n\sqrt{c}}+1, \cdots\right.}{2acx\left(\mathrm{WhittakerW}\left(-\cdots\right.\right.} \tag{1}$$

Verification of solutions

$$y = \frac{\left(i\sqrt{a}\,\sqrt{c}\,s - c(1+b+n)\right) c_3 \,\mathrm{WhittakerM}\left(-\frac{i\sqrt{a}\,s}{2n\sqrt{c}}+1, \frac{b+1}{2n}, \frac{2i\sqrt{c}\,\sqrt{a}\,x^n}{n}\right) + 2\,\mathrm{WhittakerW}\left(-\frac{i\sqrt{a}\,s}{2n\sqrt{c}}+1, \cdots\right.}{2acx\left(\mathrm{WhittakerW}\left(-\cdots\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = b*(diff(y(x), x))/x-a*(x^(2*n-
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
         <- Kummer successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 373

```
dsolve(x^2*diff(y(x),x)=a*x^2*y(x)^2+b*x*y(x)+c*x^(2*n)+s*x^n,y(x), singsol=all)
```

$$y(x) = \frac{\operatorname{KummerM}\left(\frac{(b-n+1)\sqrt{c}+i\sqrt{a}\,s}{2\sqrt{c}\,n}, \frac{b+n+1}{n}, \frac{2i\sqrt{a}\,\sqrt{c}\,x^n}{n}\right)\left((-b-n-1)\sqrt{c}+i\sqrt{a}\,s\right)+2\sqrt{c}\operatorname{KummerU}\left(\frac{(b-n+1)\sqrt{c}\cdots}{2\sqrt{c}\,n}\right.}{2\sqrt{c}\,xa\left(\operatorname{KummerU}\left(\frac{(b+\cdots}{}\right.\right.}$$

✓ Solution by Mathematica

Time used: 1.839 (sec). Leaf size: 819

```
DSolve[x^2*y'[x]==a*x^2*y[x]^2+b*x*y[x]+c*x^(2*n)+s*x^n,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\frac{i\sqrt{a}c_1x^n\left(\sqrt{c}(b+n+1)-i\sqrt{a}s\right)\text{HypergeometricU}\left(\frac{b+3n-\frac{i\sqrt{a}s}{\sqrt{c}}+1}{2n}, \frac{b+2n+1}{n}, -\frac{2i\sqrt{a}\sqrt{c}x^n}{n}\right)+c_1n\left(i\sqrt{a}\sqrt{c}\cdots\right.}{anx\left(c_1\,\text{Hyperge}\cdots\right.}$$

$$y(x) \to -\frac{\frac{\sqrt{a}x^n\left(\sqrt{a}s+i\sqrt{c}(b+n+1)\right)\text{HypergeometricU}\left(\frac{b+3n-\frac{i\sqrt{a}s}{\sqrt{c}}+1}{2n}, \frac{b+2n+1}{n}, -\frac{2i\sqrt{a}\sqrt{c}x^n}{n}\right)}{n\,\text{HypergeometricU}\left(\frac{b+n-\frac{i\sqrt{a}s}{\sqrt{c}}+1}{2n}, \frac{b+n+1}{n}, -\frac{2i\sqrt{a}\sqrt{c}x^n}{n}\right)}+i\sqrt{a}\sqrt{c}x^n+b+1}{ax}$$

$$y(x) \to -\frac{\frac{\sqrt{a}x^n\left(\sqrt{a}s+i\sqrt{c}(b+n+1)\right)\text{HypergeometricU}\left(\frac{b+3n-\frac{i\sqrt{a}s}{\sqrt{c}}+1}{2n}, \frac{b+2n+1}{n}, -\frac{2i\sqrt{a}\sqrt{c}x^n}{n}\right)}{n\,\text{HypergeometricU}\left(\frac{b+n-\frac{i\sqrt{a}s}{\sqrt{c}}+1}{2n}, \frac{b+n+1}{n}, -\frac{2i\sqrt{a}\sqrt{c}x^n}{n}\right)}+i\sqrt{a}\sqrt{c}x^n+b+1}{ax}$$

2.53 problem 53

Internal problem ID [10383]
Internal file name [`OUTPUT/9330_Monday_June_06_2022_01_51_58_PM_89123123/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 53.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{x^2y' - y^2c\,x^2 - (x^n a + b)\,xy = \alpha\,x^{2n} + \beta\,x^n + \gamma}$$

2.53.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2c\,x^2 + a\,x^nxy + bxy + \beta\,x^n + \alpha\,x^{2n} + \gamma}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = c\,y^2 + \frac{x^nay}{x} + \frac{by}{x} + \frac{\beta\,x^n}{x^2} + \frac{\alpha\,x^{2n}}{x^2} + \frac{\gamma}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\alpha\,x^{2n}+\beta\,x^n+\gamma}{x^2}$, $f_1(x) = \frac{x^nax+bx}{x^2}$ and $f_2(x) = c$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{cu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= 0\\
f_1f_2 &= \frac{(x^n a x + b x)\,c}{x^2}\\
f_2^2 f_0 &= \frac{c^2(\alpha\,x^{2n} + \beta\,x^n + \gamma)}{x^2}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$cu''(x) - \frac{(x^n a x + b x)\,c u'(x)}{x^2} + \frac{c^2(\alpha\,x^{2n} + \beta\,x^n + \gamma)\,u(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
&u(x)\\
&= x^{\frac{b}{2}-\frac{n}{2}+\frac{1}{2}} \mathrm{e}^{\frac{x^n a}{2n}} \left(c_1 \,\mathrm{WhittakerM}\left(-\frac{(b-n+1)\,a - 2\beta c}{2\sqrt{a^2-4\alpha c}\,n}, \frac{\sqrt{b^2-4c\gamma+2b+1}}{2n}, \frac{\sqrt{a^2-4\alpha c}\,x^n}{n}\right)\right.\\
&\quad \left. + c_2 \,\mathrm{WhittakerW}\left(-\frac{(b-n+1)\,a - 2\beta c}{2\sqrt{a^2-4\alpha c}\,n}, \frac{\sqrt{b^2-4c\gamma+2b+1}}{2n}, \frac{\sqrt{a^2-4\alpha c}\,x^n}{n}\right)\right)
\end{aligned}$$

The above shows that

$$\begin{aligned}
&u'(x)\\
&= \frac{\left(-x^{\frac{n}{2}}\left(-\sqrt{b^2-4c\gamma+2b+1}\,\sqrt{a^2-4\alpha c} - \sqrt{a^2-4\alpha c}\,n + (b-n+1)\,a - 2\beta c\right) c_1 \,\mathrm{WhittakerM}\left(-\frac{-2\sqrt{a^2}}{}\right.\right.}{}
\end{aligned}$$

Using the above in (1) gives the solution

$$\begin{aligned}
&y =\\
&-\frac{\left(-x^{\frac{n}{2}}\left(-\sqrt{b^2-4c\gamma+2b+1}\,\sqrt{a^2-4\alpha c} - \sqrt{a^2-4\alpha c}\,n + (b-n+1)\,a - 2\beta c\right) c_1 \,\mathrm{WhittakerM}\left(-\frac{-2}{}\right.\right.}{}
\end{aligned}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$y =$

$$-\frac{\left(-x^{\frac{n}{2}}\left(-\sqrt{b^2-4c\gamma+2b+1}\,\sqrt{a^2-4\alpha c}-\sqrt{a^2-4\alpha c}\,n+(b-n+1)\,a-2\beta c\right)c_3\,\text{WhittakerM}\left(-\frac{-2\cdots}{}\right.\right.}{}$$

Summary

The solution(s) found are the following

$y =$ (1)

$$-\frac{\left(-x^{\frac{n}{2}}\left(-\sqrt{b^2-4c\gamma+2b+1}\,\sqrt{a^2-4\alpha c}-\sqrt{a^2-4\alpha c}\,n+(b-n+1)\,a-2\beta c\right)c_3\,\text{WhittakerM}\left(-\frac{-2\cdots}{}\right.\right.}{}$$

Verification of solutions

$y =$

$$-\frac{\left(-x^{\frac{n}{2}}\left(-\sqrt{b^2-4c\gamma+2b+1}\,\sqrt{a^2-4\alpha c}-\sqrt{a^2-4\alpha c}\,n+(b-n+1)\,a-2\beta c\right)c_3\,\text{WhittakerM}\left(-\frac{-2\cdots}{}\right.\right.}{}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x^(n-1)*a*x+b)*(diff(y(x), x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
         <- Whittaker successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 560

```
dsolve(x^2*diff(y(x),x)=c*x^2*y(x)^2+(a*x^n+b)*x*y(x)+alpha*x^(2*n)+beta*x^n+gamma,y(x), sin
```

$y(x) =$

$$-\frac{\left(\sqrt{b^2-4c\gamma+2b+1}\,\sqrt{a^2-4\alpha c}+\sqrt{a^2-4\alpha c}\,n+(n-b-1)\,a+2\beta c\right)\mathrm{WhittakerM}\left(-\frac{-2\sqrt{a^2-4\alpha c}\,n+a}{2\sqrt{a^2-4}}\right.}{}$$

✓ Solution by Mathematica

Time used: 3.672 (sec). Leaf size: 1837

```
DSolve[x^2*y'[x]==c*x^2*y[x]^2+(a*x^n+b)*x*y[x]+\[Alpha]*x^(2*n)+\[Beta]*x^n+\[Gamma],y[x],x
```

$y(x)$

$$\rightarrow \frac{-\left(\left(-\left(\left(n^2+\sqrt{n^2\left(b^2+2b-4c\gamma+1\right)}\right)a^2\right)+n(-b+n-1)\sqrt{a^2-4c\alpha}a+2c\left(2\alpha n^2+\sqrt{a^2-4c\alpha}\beta n\right.\right.\right.}{}$$

$y(x)$

$$\rightarrow -\frac{x^n\left(2c\left(\beta n\sqrt{a^2-4\alpha c}+2\alpha\sqrt{n^2(b^2+2b-4c\gamma+1)}+2\alpha n^2\right)-\left(a^2\left(\sqrt{n^2(b^2+2b-4c\gamma+1)}+n^2\right)\right)+an(-b+n-1)\sqrt{a^2-4\alpha c}\right)\mathrm{HypergeometricU}\left(\left(\right.\right.}{\mathrm{HypergeometricU}\left(\frac{\left(n^2+\sqrt{n^2\left(b^2+2b-4c\gamma+1\right)}\right)a^2+(b-n+1)n\sqrt{a^2-4c\alpha}a-2c\left(2\alpha n^2\right.}{2n^2\left(a^2-4c\alpha\right)}\right.}$$

2.54 problem 54

Internal problem ID [10384]
Internal file name [`OUTPUT/9331_Monday_June_06_2022_01_52_31_PM_88664424/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 54.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{x^2y' - \left(\alpha\, x^{2n} + \beta\, x^n + \gamma\right) y^2 - (x^n a + b)\, xy = c\, x^2}$$

2.54.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^{2n}\alpha\, y^2 + a\, x^n xy + x^n \beta\, y^2 + bxy + c\, x^2 + \gamma\, y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x^{2n}\alpha\, y^2}{x^2} + \frac{x^n ay}{x} + \frac{x^n \beta\, y^2}{x^2} + \frac{by}{x} + c + \frac{\gamma\, y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = c$, $f_1(x) = \frac{x^n ax + bx}{x^2}$ and $f_2(x) = \frac{\alpha\, x^{2n} + \beta\, x^n + \gamma}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{(\alpha\, x^{2n} + \beta\, x^n + \gamma)u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{\frac{2\alpha\, x^{2n} n}{x} + \frac{\beta\, x^n n}{x}}{x^2} - \frac{2(\alpha\, x^{2n} + \beta\, x^n + \gamma)}{x^3} \\
f_1 f_2 &= \frac{(x^n a x + b x)\,(\alpha\, x^{2n} + \beta\, x^n + \gamma)}{x^4} \\
f_2^2 f_0 &= \frac{(\alpha\, x^{2n} + \beta\, x^n + \gamma)^2\, c}{x^4}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{(\alpha\, x^{2n} + \beta\, x^n + \gamma)\, u''(x)}{x^2} - \left(\frac{\frac{2\alpha\, x^{2n} n}{x} + \frac{\beta\, x^n n}{x}}{x^2} - \frac{2(\alpha\, x^{2n} + \beta\, x^n + \gamma)}{x^3} + \frac{(x^n a x + b x)\,(\alpha\, x^{2n} + \beta\, x^n + \gamma)}{x^4}\right) u$$

Solving the above ODE (this ode solved using Maple, not this program), gives

Expression too large to display

The above shows that

$$u'(x) = \frac{c\, x^{-\frac{3}{2}+\frac{b}{2}} \left(\left(-n + \sqrt{b^2 - 4c\gamma - 2b + 1}\right) \left((3\gamma^2\alpha + 3\beta^2\gamma)\, x^{2n - \frac{\sqrt{b^2-4c\gamma-2b+1}}{2}} + (6\beta\gamma\alpha + \beta^3)\, x^{3n - \frac{\sqrt{b^2-4c\gamma-2b+1}}{2}}\right.\right.}{}$$

Using the above in (1) gives the solution

Expression too large to display

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

Expression too large to display

Summary

The solution(s) found are the following

Expression too large to display (1)

Verification of solutions

Expression too large to display

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x^(n-1)*x^(2*n-2)*a*alpha*x^3
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
                  <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
         <- hypergeometric successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 215200

```
dsolve(x^2*diff(y(x),x)=(alpha*x^(2*n)+beta*x^n+gamma)*y(x)^2+(a*x^n+b)*x*y(x)+c*x^2,y(x), s
```

Expression too large to display

✓ Solution by Mathematica

Time used: 4.676 (sec). Leaf size: 2649

```
DSolve[x^2*y'[x]==(\[Alpha]*x^(2*n)+\[Beta]*x^n+\[Gamma])*y[x]^2+(a*x^n+b)*x*y[x]+c*x^2,y[x]
```

Too large to display

2.55 problem 55

Internal problem ID [10385]
Internal file name [`OUTPUT/9332_Monday_June_06_2022_01_56_49_PM_58358154/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 55.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{(x^2-1)\,y'+\lambda(y^2-2xy+1)=0}$$

2.55.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= -\frac{\lambda(-2yx+y^2+1)}{x^2-1}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=\frac{2\lambda yx}{x^2-1}-\frac{\lambda\,y^2}{x^2-1}-\frac{\lambda}{x^2-1}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=-\frac{\lambda}{x^2-1}$, $f_1(x)=\frac{2\lambda x}{x^2-1}$ and $f_2(x)=-\frac{\lambda}{x^2-1}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2u}\\ &= \frac{-u'}{-\frac{\lambda u}{x^2-1}}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2\lambda x}{(x^2-1)^2} \\ f_1 f_2 &= -\frac{2\lambda^2 x}{(x^2-1)^2} \\ f_2^2 f_0 &= -\frac{\lambda^3}{(x^2-1)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{\lambda u''(x)}{x^2-1} - \left(-\frac{2\lambda^2 x}{(x^2-1)^2} + \frac{2\lambda x}{(x^2-1)^2}\right) u'(x) - \frac{\lambda^3 u(x)}{(x^2-1)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = (\mathrm{LegendreP}\,(\lambda-1, x)\, c_1 + \mathrm{LegendreQ}\,(\lambda-1, x)\, c_2)\, (x^2-1)^{\frac{\lambda}{2}}$$

The above shows that

$$u'(x) = \lambda (x^2-1)^{-1+\frac{\lambda}{2}}\, (\mathrm{LegendreP}\,(\lambda, x)\, c_1 + \mathrm{LegendreQ}\,(\lambda, x)\, c_2)$$

Using the above in (1) gives the solution

$$y = \frac{(x^2-1)^{-1+\frac{\lambda}{2}}\, (\mathrm{LegendreP}\,(\lambda, x)\, c_1 + \mathrm{LegendreQ}\,(\lambda, x)\, c_2)\, (x^2-1)\, (x^2-1)^{-\frac{\lambda}{2}}}{\mathrm{LegendreP}\,(\lambda-1, x)\, c_1 + \mathrm{LegendreQ}\,(\lambda-1, x)\, c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\mathrm{LegendreP}\,(\lambda, x)\, c_3 + \mathrm{LegendreQ}\,(\lambda, x)}{\mathrm{LegendreP}\,(\lambda-1, x)\, c_3 + \mathrm{LegendreQ}\,(\lambda-1, x)}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{LegendreP}\,(\lambda, x)\, c_3 + \mathrm{LegendreQ}\,(\lambda, x)}{\mathrm{LegendreP}\,(\lambda-1, x)\, c_3 + \mathrm{LegendreQ}\,(\lambda-1, x)} \tag{1}$$

Verification of solutions

$$y = \frac{\text{LegendreP}(\lambda, x)\, c_3 + \text{LegendreQ}(\lambda, x)}{\text{LegendreP}(\lambda - 1, x)\, c_3 + \text{LegendreQ}(\lambda - 1, x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Abel AIR successful: ODE belongs to the 2F1 3-parameter class`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 280

```
dsolve((x^2-1)*diff(y(x),x)+lambda*(y(x)^2-2*x*y(x)+1)=0,y(x), singsol=all)
```

$$y(x) = -\frac{2\left(\left(-\frac{x}{2} - \frac{1}{2}\right)^{-2\lambda} (1 + x) (-1 + x)^2 \,\text{HeunCPrime}\left(0, 2\lambda - 1, 0, 0, \lambda^2 - \lambda + \frac{1}{2}, \frac{2}{1+x}\right) + 8(-1 + x)^2 \,\text{Heun}\right.}{\left(8c_1 \,\text{hyp}\right.}$$

✓ Solution by Mathematica

Time used: 0.643 (sec). Leaf size: 47

```
DSolve[(x^2-1)*y'[x]+\[Lambda]*(y[x]^2-2*x*y[x]+1)==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{\text{LegendreQ}(\lambda, x) + c_1 \text{LegendreP}(\lambda, x)}{\text{LegendreQ}(\lambda - 1, x) + c_1 \text{LegendreP}(\lambda - 1, x)}$$

$$y(x) \to \frac{\text{LegendreP}(\lambda, x)}{\text{LegendreP}(\lambda - 1, x)}$$

2.56 problem 56

Internal problem ID [10386]
Internal file name [`OUTPUT/9333_Monday_June_06_2022_01_56_50_PM_34885414/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 56.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{\left(a\,x^2+b\right)y'+\alpha y^2+\beta x y=-\frac{b(a+\beta)}{\alpha}}$$

2.56.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{\alpha^2 y^2+\beta x y \alpha+ab+\beta b}{\left(a\,x^2+b\right)\alpha} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=-\frac{\alpha\,y^2}{a\,x^2+b}-\frac{\beta x y}{a\,x^2+b}-\frac{ba}{\left(a\,x^2+b\right)\alpha}-\frac{b\beta}{\left(a\,x^2+b\right)\alpha}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x) = -\frac{ab+\beta b}{(a\,x^2+b)\alpha}$, $f_1(x) = -\frac{\beta x}{a\,x^2+b}$ and $f_2(x) = -\frac{\alpha}{a\,x^2+b}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{\alpha u}{a\,x^2+b}}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= \frac{2\alpha x a}{\left(a\,x^2+b\right)^2} \\
f_1 f_2 &= \frac{\beta x \alpha}{\left(a\,x^2+b\right)^2} \\
f_2^2 f_0 &= -\frac{\alpha(ab+\beta b)}{\left(a\,x^2+b\right)^3}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-\frac{\alpha u''(x)}{a\,x^2+b} - \left(\frac{\beta x \alpha}{\left(a\,x^2+b\right)^2} + \frac{2\alpha x a}{\left(a\,x^2+b\right)^2}\right) u'(x) - \frac{\alpha(ab+\beta b)\,u(x)}{\left(a\,x^2+b\right)^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
\begin{aligned}
u(x) = \left(a\,x^2+b\right)^{-\frac{\beta}{4a}} \Bigg(&\mathrm{LegendreQ}\left(\frac{\beta}{2a}, \frac{2a+\beta}{2a}, \frac{ax}{\sqrt{-ab}}\right) c_2 \\
&+ \mathrm{LegendreP}\left(\frac{\beta}{2a}, \frac{2a+\beta}{2a}, \frac{ax}{\sqrt{-ab}}\right) c_1\Bigg)
\end{aligned}
$$

The above shows that

$$
\begin{aligned}
u'(x) = x(a+\beta)\left(a\,x^2+b\right)^{-\frac{4a+\beta}{4a}} \Bigg(&-\mathrm{LegendreQ}\left(\frac{\beta}{2a}, \frac{2a+\beta}{2a}, \frac{ax}{\sqrt{-ab}}\right) c_2 \\
&- \mathrm{LegendreP}\left(\frac{\beta}{2a}, \frac{2a+\beta}{2a}, \frac{ax}{\sqrt{-ab}}\right) c_1\Bigg)
\end{aligned}
$$

Using the above in (1) gives the solution

$$
y = \frac{x(a+\beta)\left(a\,x^2+b\right)^{-\frac{4a+\beta}{4a}} \left(-\mathrm{LegendreQ}\left(\frac{\beta}{2a}, \frac{2a+\beta}{2a}, \frac{ax}{\sqrt{-ab}}\right) c_2 - \mathrm{LegendreP}\left(\frac{\beta}{2a}, \frac{2a+\beta}{2a}, \frac{ax}{\sqrt{-ab}}\right) c_1\right) \left(a\,x^2+b\right.}{\alpha\left(\mathrm{LegendreQ}\left(\frac{\beta}{2a}, \frac{2a+\beta}{2a}, \frac{ax}{\sqrt{-ab}}\right) c_2 + \mathrm{LegendreP}\left(\frac{\beta}{2a}, \frac{2a+\beta}{2a}, \frac{ax}{\sqrt{-ab}}\right) c_1\right)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{(a+\beta)\,x}{\alpha}$$

Summary

The solution(s) found are the following

$$y = -\frac{(a+\beta)\,x}{\alpha} \tag{1}$$

Verification of solutions

$$y = -\frac{(a+\beta)\,x}{\alpha}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Abel AIR successful: ODE belongs to the 2F1 3-parameter class`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 517

```
dsolve((a*x^2+b)*diff(y(x),x)+alpha*y(x)^2+beta*x*y(x)+b/alpha*(a+beta)=0,y(x), singsol=all)
```

$$y(x) = -\frac{b\,a^2\left(-\frac{\left(-\frac{-ax+\sqrt{-ab}}{2\sqrt{-ab}}\right)^{\frac{\beta}{a}}(a\,x^2+b)\left(a\,x^2+2\sqrt{-ab}\,x-b\right)\mathrm{HeunCPrime}\left(0,-1-\frac{\beta}{a},1+\frac{\beta}{2a},0,\frac{1}{2}+\frac{\beta}{2a}+\frac{\beta^2}{4a^2},\frac{2\sqrt{-ab}}{-ax+\sqrt{-ab}}\right)}{2} - 2c_1 b\left(\left(3a\,x^2 -\right.\right.\right.}{}$$

✓ Solution by Mathematica

Time used: 1.111 (sec). Leaf size: 27

```
DSolve[(a*x^2+b)*y'[x]+\[Alpha]*y[x]^2+\[Beta]*x*y[x]+b/\[Alpha]*(a+\[Beta])==0,y[x],x,Inclu
```

$$y(x) \to -\frac{x(a+\beta)}{\alpha}$$
$$y(x) \to -\frac{x(a+\beta)}{\alpha}$$

2.57 problem 57

Internal problem ID [10387]
Internal file name [`OUTPUT/9334_Monday_June_06_2022_01_56_51_PM_67885841/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 57.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{\left(a\,x^2+b\right)y'+\alpha y^2+\beta xy=-\gamma}$$

2.57.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{\alpha\,y^2+\beta xy+\gamma}{a\,x^2+b} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=-\frac{\alpha\,y^2}{a\,x^2+b}-\frac{\beta xy}{a\,x^2+b}-\frac{\gamma}{a\,x^2+b}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=-\frac{\gamma}{a\,x^2+b}$, $f_1(x)=-\frac{\beta x}{a\,x^2+b}$ and $f_2(x)=-\frac{\alpha}{a\,x^2+b}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{-\frac{\alpha u}{a\,x^2+b}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2\alpha x a}{(a\,x^2+b)^2} \\ f_1f_2 &= \frac{\beta x \alpha}{(a\,x^2+b)^2} \\ f_2^2 f_0 &= -\frac{\alpha^2\gamma}{(a\,x^2+b)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{\alpha u''(x)}{a\,x^2+b} - \left(\frac{\beta x\alpha}{(a\,x^2+b)^2} + \frac{2\alpha x a}{(a\,x^2+b)^2}\right) u'(x) - \frac{\alpha^2\gamma u(x)}{(a\,x^2+b)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = (a\,x^2+b)^{-\frac{\beta}{4a}} \left(\mathrm{LegendreP}\left(\frac{\beta}{2a}, \frac{\sqrt{4\alpha\gamma a + b\,\beta^2}}{2a\sqrt{b}}, \frac{ax}{\sqrt{-ab}}\right) c_1 + \mathrm{LegendreQ}\left(\frac{\beta}{2a}, \frac{\sqrt{4\alpha\gamma a + b\,\beta^2}}{2a\sqrt{b}}, \frac{ax}{\sqrt{-ab}}\right) c_2\right)$$

The above shows that

$$u'(x) = \frac{(a\,x^2+b)^{-\frac{4a+\beta}{4a}} \left(\left(\left(-\frac{\sqrt{4\alpha\gamma a+b\,\beta^2}\, b}{2} + b^{\frac{3}{2}}\left(a+\frac{\beta}{2}\right)\right) c_1\sqrt{-ab}\, \mathrm{LegendreP}\left(\frac{2a+\beta}{2a}, \frac{\sqrt{4\alpha\gamma a+b\,\beta^2}}{2a\sqrt{b}}, \frac{ax}{\sqrt{-ab}}\right) + \left(-\frac{\sqrt{4\alpha\gamma a}}{2}\right.\right.\right.}{}$$

Using the above in (1) gives the solution

$$y = \frac{(a\,x^2+b)^{-\frac{4a+\beta}{4a}} \left(\left(\left(-\frac{\sqrt{4\alpha\gamma a+b\,\beta^2}\, b}{2} + b^{\frac{3}{2}}\left(a+\frac{\beta}{2}\right)\right) c_1\sqrt{-ab}\, \mathrm{LegendreP}\left(\frac{2a+\beta}{2a}, \frac{\sqrt{4\alpha\gamma a+b\,\beta^2}}{2a\sqrt{b}}, \frac{ax}{\sqrt{-ab}}\right) + \left(-\frac{\sqrt{4\alpha\gamma a}}{2}\right.\right.\right.}{a}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\left(-2a-\beta\right)\sqrt{b}+\sqrt{4\alpha\gamma a+b\,\beta^{2}}\right)\left(\mathrm{LegendreP}\left(\frac{2a+\beta}{2a},\frac{\sqrt{4\alpha\gamma a+b\,\beta^{2}}}{2a\sqrt{b}},\frac{ax}{\sqrt{-ab}}\right)c_{3}+\mathrm{LegendreQ}\left(\frac{2a+\beta}{2a},\frac{\sqrt{4\alpha\gamma}}{2a}\right.\right.}{2\sqrt{b}\,a\alpha\left(\mathrm{LegendreP}\left(\frac{\beta}{2a},\frac{\sqrt{4\alpha\gamma a+b\,\beta^{2}}}{2a\sqrt{b}}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\left(-2a-\beta\right)\sqrt{b}+\sqrt{4\alpha\gamma a+b\,\beta^{2}}\right)\left(\mathrm{LegendreP}\left(\frac{2a+\beta}{2a},\frac{\sqrt{4\alpha\gamma a+b\,\beta^{2}}}{2a\sqrt{b}},\frac{ax}{\sqrt{-ab}}\right)c_{3}+\mathrm{LegendreQ}\left(\frac{2a+\beta}{2a},\frac{\sqrt{4\alpha\gamma}}{2a}\right.\right.}{2\sqrt{b}\,a\alpha\left(\mathrm{LegendreP}\left(\frac{\beta}{2a},\frac{\sqrt{4\alpha\gamma a+b\,\beta^{2}}}{2a\sqrt{b}}\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\left(-2a-\beta\right)\sqrt{b}+\sqrt{4\alpha\gamma a+b\,\beta^{2}}\right)\left(\mathrm{LegendreP}\left(\frac{2a+\beta}{2a},\frac{\sqrt{4\alpha\gamma a+b\,\beta^{2}}}{2a\sqrt{b}},\frac{ax}{\sqrt{-ab}}\right)c_{3}+\mathrm{LegendreQ}\left(\frac{2a+\beta}{2a},\frac{\sqrt{4\alpha\gamma}}{2a}\right.\right.}{2\sqrt{b}\,a\alpha\left(\mathrm{LegendreP}\left(\frac{\beta}{2a},\frac{\sqrt{4\alpha\gamma a+b\,\beta^{2}}}{2a\sqrt{b}}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Abel AIR successful: ODE belongs to the 2F1 3-parameter class`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 858

```
dsolve((a*x^2+b)*diff(y(x),x)+alpha*y(x)^2+beta*x*y(x)+gamma=0,y(x), singsol=all)
```

$$y(x) =$$

$$-\frac{}{2\left(-ab\left(-\sqrt{-ab}\,x+b\right)\left(a\,x^2+b\right)\text{HeunCPrime}\left(0,\frac{-a+\beta}{a},-\frac{\sqrt{4\gamma\alpha ab+\beta^2b^2}}{2ab},0,\frac{2a^2-2\beta a+\beta^2}{4a^2},-\frac{2\sqrt{-ab}}{ax-\sqrt{-ab}}\right)+c\right.}$$

✓ Solution by Mathematica

Time used: 1.098 (sec). Leaf size: 598

```
DSolve[(a*x^2+b)*y'[x]+\[Alpha]*y[x]^2+\[Beta]*x*y[x]+\[Gamma]==0,y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{i\left(c_1\left(\sqrt{4a\alpha\gamma+b\beta^2}-2a\sqrt{b}-\sqrt{b}\beta\right)P_{\frac{\beta}{2a}+1}^{\frac{\sqrt{b\beta^2+4a\alpha\gamma}}{2a\sqrt{b}}}\left(\frac{i\sqrt{a}x}{\sqrt{b}}\right)+2i\sqrt{a}x(a+\beta)Q_{\frac{\beta}{2a}}^{\frac{\sqrt{b\beta^2+4a\alpha\gamma}}{2a\sqrt{b}}}\left(\frac{i\sqrt{a}x}{\sqrt{b}}\right)+\left(\sqrt{4a\alpha\gamma\right.\right.}{2\sqrt{a}\alpha\left(c_1P_{\frac{\beta}{2a}}^{\frac{\sqrt{b\beta^2+4a\alpha\gamma}}{2a\sqrt{b}}}\left(\frac{i\sqrt{a}x}{\sqrt{b}}\right)+Q_{\frac{\beta}{2a}}^{\frac{\sqrt{b\beta^2+}}{2a}}\right.}$$

$$y(x) \to \frac{-2x(a+\beta)+\frac{i\left(\sqrt{4a\alpha\gamma+b\beta^2}-2a\sqrt{b}-\sqrt{b}\beta\right)P_{\frac{\beta}{2a}+1}^{\frac{\sqrt{b\beta^2+4a\alpha\gamma}}{2a\sqrt{b}}}\left(\frac{i\sqrt{a}x}{\sqrt{b}}\right)}{\sqrt{a}P_{\frac{\beta}{2a}}^{\frac{\sqrt{b\beta^2+4a\alpha\gamma}}{2a\sqrt{b}}}\left(\frac{i\sqrt{a}x}{\sqrt{b}}\right)}}{2\alpha}$$

2.58 problem 58

Internal problem ID [10388]
Internal file name [OUTPUT/9335_Monday_June_06_2022_01_56_55_PM_79509054/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 58.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x),G(x)]`],
   _Riccati]
```

$$\boxed{\left(a\,x^2+b\right)y'+y^2-2xy=-(1-a)\,x^2+b}$$

2.58.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= -\frac{-a\,x^2+x^2-2yx+y^2-b}{a\,x^2+b}\\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2a_4+yxa_5+y^2a_6+xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = x^2b_4+yxb_5+y^2b_6+xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&2xb_4 + yb_5 + b_2 \\
&- \frac{(-a\,x^2 + x^2 - 2yx + y^2 - b)\,(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{a\,x^2 + b} \\
&- \frac{(-a\,x^2 + x^2 - 2yx + y^2 - b)^2\,(xa_5 + 2ya_6 + a_3)}{(a\,x^2 + b)^2} - \left(-\frac{-2xa + 2x - 2y}{a\,x^2 + b}\right. \\
&\left. + \frac{2(-a\,x^2 + x^2 - 2yx + y^2 - b)\,xa}{(a\,x^2 + b)^2}\right)\left(x^2a_4 + yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1\right) \\
&+ \frac{(-2x + 2y)\,(x^2b_4 + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1)}{a\,x^2 + b} = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2a^2x^5a_4 + a^2x^5a_5 - 2a^2x^5b_4 - a^2x^5b_5 + a^2x^4ya_5 + 2a^2x^4ya_6 - a^2x^4yb_5 - 2a^2x^4yb_6 + a^2x^4a_2 + a^2x^4a_3 -}{}$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-2a^2x^5a_4 - a^2x^5a_5 + 2a^2x^5b_4 + a^2x^5b_5 - a^2x^4ya_5 \\
&- 2a^2x^4ya_6 + a^2x^4yb_5 + 2a^2x^4yb_6 - a^2x^4a_2 - a^2x^4a_3 \\
&+ a^2x^4b_2 + a^2x^4b_3 + 2a\,x^5a_4 + 2a\,x^5a_5 - 2a\,x^5b_4 - a\,x^5b_5 \\
&- 2a\,x^4ya_4 - 3a\,x^4ya_5 + 4a\,x^4ya_6 + 2a\,x^4yb_4 - 2a\,x^4yb_6 \\
&+ 2a\,x^3y^2a_5 - 8a\,x^3y^2a_6 + a\,x^3y^2b_5 + 2a\,x^3y^2b_6 - a\,x^2y^3a_5 \\
&+ 6a\,x^2y^3a_6 - 2ax\,y^4a_6 - 4ab\,x^3a_4 - 2ab\,x^3a_5 + 4ab\,x^3b_4 \\
&+ 2ab\,x^3b_5 - 2ab\,x^2ya_5 - 4ab\,x^2ya_6 + 2ab\,x^2yb_5 + 4ab\,x^2yb_6 \\
&+ a\,x^4a_2 + 2a\,x^4a_3 - 2a\,x^4b_2 - a\,x^4b_3 - 4a\,x^3ya_3 + 2a\,x^3yb_2 \\
&- a\,x^2y^2a_2 + 4a\,x^2y^2a_3 + a\,x^2y^2b_3 - 2ax\,y^3a_3 - x^5a_5 \\
&+ 4x^4ya_5 - 2x^4ya_6 - 6x^3y^2a_5 + 8x^3y^2a_6 + 4x^2y^3a_5 \\
&- 12x^2y^3a_6 - x\,y^4a_5 + 8x\,y^4a_6 - 2y^5a_6 - 2ab\,x^2a_2 \\
&- 2ab\,x^2a_3 + 2ab\,x^2b_2 + 2ab\,x^2b_3 - 2a\,x^3b_1 + 2a\,x^2ya_1 \\
&+ 2a\,x^2yb_1 - 2ax\,y^2a_1 + 4b\,x^3a_4 + 2b\,x^3a_5 - 2b\,x^3b_4 - b\,x^3b_5 \\
&- 6b\,x^2ya_4 - b\,x^2ya_5 + 4b\,x^2ya_6 + 2b\,x^2yb_4 - 2b\,x^2yb_6 \\
&+ 2bx\,y^2a_4 - 2bx\,y^2a_5 - 6bx\,y^2a_6 + bx\,y^2b_5 + 2bx\,y^2b_6 \\
&+ b\,y^3a_5 + 2b\,y^3a_6 - x^4a_3 + 4x^3ya_3 - 6x^2y^2a_3 + 4x\,y^3a_3 \\
&- y^4a_3 - 2b^2xa_4 - b^2xa_5 + 2b^2xb_4 + b^2xb_5 - b^2ya_5 - 2b^2ya_6 \\
&+ b^2yb_5 + 2b^2yb_6 + 3b\,x^2a_2 + 2b\,x^2a_3 - 2b\,x^2b_2 - b\,x^2b_3 \\
&- 4bxya_2 - 2bxya_3 + 2bxyb_2 + b\,y^2a_2 + b\,y^2b_3 - b^2a_2 \\
&- b^2a_3 + b^2b_2 + b^2b_3 + 2bxa_1 - 2bxb_1 - 2bya_1 + 2byb_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2a^2a_4v_1^5 - a^2a_5v_1^5 - a^2a_5v_1^4v_2 - 2a^2a_6v_1^4v_2 + 2a^2b_4v_1^5 + a^2b_5v_1^5 + a^2b_5v_1^4v_2 + 2a^2b_6v_1^4v_2 \\
&- a^2a_2v_1^4 - a^2a_3v_1^4 + a^2b_2v_1^4 + a^2b_3v_1^4 + 2aa_4v_1^5 - 2aa_4v_1^4v_2 + 2aa_5v_1^5 - 3aa_5v_1^4v_2 \\
&+ 2aa_5v_1^3v_2^2 - aa_5v_1^2v_2^3 + 4aa_6v_1^4v_2 - 8aa_6v_1^3v_2^2 + 6aa_6v_1^2v_2^3 - 2aa_6v_1v_2^4 - 2ab_4v_1^5 + 2ab_4v_1^4v_2 \\
&- ab_5v_1^5 + ab_5v_1^3v_2^2 - 2ab_6v_1^4v_2 + 2ab_6v_1^3v_2^2 - 4aba_4v_1^3 - 2aba_5v_1^3 - 2aba_5v_1^2v_2 - 4aba_6v_1^2v_2 \\
&+ 4abb_4v_1^3 + 2abb_5v_1^3 + 2abb_5v_1^2v_2 + 4abb_6v_1^2v_2 + aa_2v_1^4 - aa_2v_1^2v_2^2 + 2aa_3v_1^4 - 4aa_3v_1^3v_2 \\
&+ 4aa_3v_1^2v_2^2 - 2aa_3v_1v_2^3 - 2ab_2v_1^4 + 2ab_2v_1^3v_2 - ab_3v_1^4 + ab_3v_1^2v_2^2 - a_5v_1^5 + 4a_5v_1^4v_2 - 6a_5v_1^3v_2^2 \\
&+ 4a_5v_1^2v_2^3 - a_5v_1v_2^4 - 2a_6v_1^4v_2 + 8a_6v_1^3v_2^2 - 12a_6v_1^2v_2^3 + 8a_6v_1v_2^4 - 2a_6v_2^5 - 2aba_2v_1^2 \\
&- 2aba_3v_1^2 + 2abb_2v_1^2 + 2abb_3v_1^2 + 2aa_1v_1^2v_2 - 2aa_1v_1v_2^2 - 2ab_1v_1^3 + 2ab_1v_1^2v_2 + 4ba_4v_1^3 \\
&- 6ba_4v_1^2v_2 + 2ba_4v_1v_2^2 + 2ba_5v_1^3 - ba_5v_1^2v_2 - 2ba_5v_1v_2^2 + ba_5v_2^3 + 4ba_6v_1^2v_2 - 6ba_6v_1v_2^2 \\
&+ 2ba_6v_2^3 - 2bb_4v_1^3 + 2bb_4v_1^2v_2 - bb_5v_1^3 + bb_5v_1v_2^2 - 2bb_6v_1^2v_2 + 2bb_6v_1v_2^2 - a_3v_1^4 + 4a_3v_1^3v_2 \\
&- 6a_3v_1^2v_2^2 + 4a_3v_1v_2^3 - a_3v_2^4 - 2b^2a_4v_1 - b^2a_5v_1 - b^2a_5v_2 - 2b^2a_6v_2 + 2b^2b_4v_1 + b^2b_5v_1 \\
&+ b^2b_5v_2 + 2b^2b_6v_2 + 3ba_2v_1^2 - 4ba_2v_1v_2 + ba_2v_2^2 + 2ba_3v_1^2 - 2ba_3v_1v_2 - 2bb_2v_1^2 + 2bb_2v_1v_2 \\
&- bb_3v_1^2 + bb_3v_2^2 - b^2a_2 - b^2a_3 + b^2b_2 + b^2b_3 + 2ba_1v_1 - 2ba_1v_2 - 2bb_1v_1 + 2bb_1v_2 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$\begin{aligned}
&\left(-2a^2a_4 - a^2a_5 + 2a^2b_4 + a^2b_5 + 2aa_4 + 2aa_5 - 2ab_4 - ab_5 - a_5\right) v_1^5 \\
&+ \left(-a^2a_5 - 2a^2a_6 + a^2b_5 + 2a^2b_6 - 2aa_4\right. \\
&\left.- 3aa_5 + 4aa_6 + 2ab_4 - 2ab_6 + 4a_5 - 2a_6\right) v_1^4 v_2 \\
&+ \left(-a^2a_2 - a^2a_3 + a^2b_2 + a^2b_3 + aa_2 + 2aa_3 - 2ab_2 - ab_3 - a_3\right) v_1^4 \\
&+ (2aa_5 - 8aa_6 + ab_5 + 2ab_6 - 6a_5 + 8a_6)\, v_1^3 v_2^2 \\
&+ (-4aa_3 + 2ab_2 + 4a_3)\, v_1^3 v_2 + (-4aba_4 - 2aba_5 \\
&+ 4abb_4 + 2abb_5 - 2ab_1 + 4ba_4 + 2ba_5 - 2bb_4 - bb_5)\, v_1^3 \\
&+ (-aa_5 + 6aa_6 + 4a_5 - 12a_6)\, v_1^2 v_2^3 \\
&+ (-aa_2 + 4aa_3 + ab_3 - 6a_3)\, v_1^2 v_2^2 + (-2aba_5 - 4aba_6 + 2abb_5 \\
&+ 4abb_6 + 2aa_1 + 2ab_1 - 6ba_4 - ba_5 + 4ba_6 + 2bb_4 - 2bb_6)\, v_1^2 v_2 \\
&+ (-2aba_2 - 2aba_3 + 2abb_2 + 2abb_3 + 3ba_2 + 2ba_3 - 2bb_2 - bb_3)\, v_1^2 \\
&+ (-2aa_6 - a_5 + 8a_6)\, v_1 v_2^4 + (-2aa_3 + 4a_3)\, v_1 v_2^3 \\
&+ (-2aa_1 + 2ba_4 - 2ba_5 - 6ba_6 + bb_5 + 2bb_6)\, v_1 v_2^2 \\
&+ (-4ba_2 - 2ba_3 + 2bb_2)\, v_1 v_2 \\
&+ \left(-2b^2a_4 - b^2a_5 + 2b^2b_4 + b^2b_5 + 2ba_1 - 2bb_1\right) v_1 \\
&- 2a_6 v_2^5 - a_3 v_2^4 + (ba_5 + 2ba_6)\, v_2^3 + (ba_2 + bb_3)\, v_2^2 \\
&+ \left(-b^2a_5 - 2b^2a_6 + b^2b_5 + 2b^2b_6 - 2ba_1 + 2bb_1\right) v_2 \\
&- b^2a_2 - b^2a_3 + b^2b_2 + b^2b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-a_3 &= 0\\
-2a_6 &= 0\\
-2aa_3 + 4a_3 &= 0\\
ba_2 + bb_3 &= 0\\
ba_5 + 2ba_6 &= 0\\
-4aa_3 + 2ab_2 + 4a_3 &= 0\\
-2aa_6 - a_5 + 8a_6 &= 0\\
-4ba_2 - 2ba_3 + 2bb_2 &= 0\\
-aa_5 + 6aa_6 + 4a_5 - 12a_6 &= 0\\
-aa_2 + 4aa_3 + ab_3 - 6a_3 &= 0\\
-b^2a_2 - b^2a_3 + b^2b_2 + b^2b_3 &= 0\\
2aa_5 - 8aa_6 + ab_5 + 2ab_6 - 6a_5 + 8a_6 &= 0\\
-2b^2a_4 - b^2a_5 + 2b^2b_4 + b^2b_5 + 2ba_1 - 2bb_1 &= 0\\
-b^2a_5 - 2b^2a_6 + b^2b_5 + 2b^2b_6 - 2ba_1 + 2bb_1 &= 0\\
-2aa_1 + 2ba_4 - 2ba_5 - 6ba_6 + bb_5 + 2bb_6 &= 0\\
-2aba_2 - 2aba_3 + 2abb_2 + 2abb_3 + 3ba_2 + 2ba_3 - 2bb_2 - bb_3 &= 0\\
-a^2a_2 - a^2a_3 + a^2b_2 + a^2b_3 + aa_2 + 2aa_3 - 2ab_2 - ab_3 - a_3 &= 0\\
-2a^2a_4 - a^2a_5 + 2a^2b_4 + a^2b_5 + 2aa_4 + 2aa_5 - 2ab_4 - ab_5 - a_5 &= 0\\
-4aba_4 - 2aba_5 + 4abb_4 + 2abb_5 - 2ab_1 + 4ba_4 + 2ba_5 - 2bb_4 - bb_5 &= 0\\
-a^2a_5 - 2a^2a_6 + a^2b_5 + 2a^2b_6 - 2aa_4 - 3aa_5 + 4aa_6 + 2ab_4 - 2ab_6 + 4a_5 - 2a_6 &= 0\\
-2aba_5 - 4aba_6 + 2abb_5 + 4abb_6 + 2aa_1 + 2ab_1 - 6ba_4 - ba_5 + 4ba_6 + 2bb_4 - 2bb_6 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= b_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= \frac{ab_1}{b} \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0 \\
b_4 &= \frac{ab_1 + bb_6}{b} \\
b_5 &= -2b_6 \\
b_6 &= b_6
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= x^2 - 2yx + y^2
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{x^2 - 2yx + y^2} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{1}{-x + y}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{-a\, x^2 + x^2 - 2yx + y^2 - b}{a\, x^2 + b}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{(x - y)^2} \\
S_y &= \frac{1}{(x - y)^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{a\, x^2 + b} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R^2 a + b}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\arctan\left(\frac{aR}{\sqrt{ab}}\right)}{\sqrt{ab}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{x-y} = -\frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}} + c_1$$

Which simplifies to

$$\frac{1}{x-y} = -\frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}} + c_1$$

Which gives

$$y = \frac{c_1\sqrt{ab}\,x - \arctan\left(\frac{ax}{\sqrt{ab}}\right)x - \sqrt{ab}}{c_1\sqrt{ab} - \arctan\left(\frac{ax}{\sqrt{ab}}\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1\sqrt{ab}\,x - \arctan\left(\frac{ax}{\sqrt{ab}}\right)x - \sqrt{ab}}{c_1\sqrt{ab} - \arctan\left(\frac{ax}{\sqrt{ab}}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{ab}\,x - \arctan\left(\frac{ax}{\sqrt{ab}}\right)x - \sqrt{ab}}{c_1\sqrt{ab} - \arctan\left(\frac{ax}{\sqrt{ab}}\right)}$$

Verified OK.

2.58.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{-a\,x^2+x^2-2yx+y^2-b}{a\,x^2+b} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,x^2}{a\,x^2+b} - \frac{x^2}{a\,x^2+b} + \frac{2yx}{a\,x^2+b} - \frac{y^2}{a\,x^2+b} + \frac{b}{a\,x^2+b}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{-a\,x^2+x^2-b}{a\,x^2+b}$, $f_1(x) = \frac{2x}{a\,x^2+b}$ and $f_2(x) = -\frac{1}{a\,x^2+b}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{a\,x^2+b}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2xa}{(a\,x^2+b)^2} \\ f_1 f_2 &= -\frac{2x}{(a\,x^2+b)^2} \\ f_2^2 f_0 &= -\frac{-a\,x^2+x^2-b}{(a\,x^2+b)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{a\,x^2+b} - \left(\frac{2xa}{(a\,x^2+b)^2} - \frac{2x}{(a\,x^2+b)^2}\right)u'(x) - \frac{(-a\,x^2+x^2-b)\,u(x)}{(a\,x^2+b)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\arctan\left(\frac{\sqrt{ab}\, x}{b}\right) c_2 + c_1\right) \left(a\, x^2 + b\right)^{\frac{1}{2a}}$$

The above shows that

$$u'(x) = \left(a\, x^2 + b\right)^{-\frac{2a-1}{2a}} \left(\arctan\left(\frac{\sqrt{ab}\, x}{b}\right) c_2 x + \sqrt{ab}\, c_2 + c_1 x\right)$$

Using the above in (1) gives the solution

$$y = \frac{\left(a\, x^2 + b\right)^{-\frac{2a-1}{2a}} \left(\arctan\left(\frac{\sqrt{ab}\, x}{b}\right) c_2 x + \sqrt{ab}\, c_2 + c_1 x\right) \left(a\, x^2 + b\right) \left(a\, x^2 + b\right)^{-\frac{1}{2a}}}{\arctan\left(\frac{\sqrt{ab}\, x}{b}\right) c_2 + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\arctan\left(\frac{\sqrt{ab}\, x}{b}\right) x + \sqrt{ab} + c_3 x}{\arctan\left(\frac{\sqrt{ab}\, x}{b}\right) + c_3}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\arctan\left(\frac{\sqrt{ab}\, x}{b}\right) x + \sqrt{ab} + c_3 x}{\arctan\left(\frac{\sqrt{ab}\, x}{b}\right) + c_3} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{\arctan\left(\frac{\sqrt{ab}\, x}{b}\right) x + \sqrt{ab} + c_3 x}{\arctan\left(\frac{\sqrt{ab}\, x}{b}\right) + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (d) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve((a*x^2+b)*diff(y(x),x)+y(x)^2-2*x*y(x)+(1-a)*x^2-b=0,y(x), singsol=all)
```

$$y(x) = x + \frac{\sqrt{ab}}{c_1\sqrt{ab} + \arctan\left(\frac{ax}{\sqrt{ab}}\right)}$$

✓ Solution by Mathematica

Time used: 0.562 (sec). Leaf size: 41

```
DSolve[(a*x^2+b)*y'[x]+y[x]^2-2*x*y[x]+(1-a)*x^2-b==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to x + \frac{1}{\frac{\arctan\left(\frac{\sqrt{a}x}{\sqrt{b}}\right)}{\sqrt{a}\sqrt{b}} + c_1}$$

$$y(x) \to x$$

2.59 problem 59

Internal problem ID [10389]
Internal file name [`OUTPUT/9336_Monday_June_06_2022_01_56_57_PM_94594974/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 59.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x),G(x)]`],
   _Riccati]
```

$$\boxed{\left(a\,x^2+bx+c\right)y'-y^2-(2\lambda x+b)\,y=\lambda(\lambda-a)\,x^2+\mu}$$

2.59.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{-\lambda\,x^2a+\lambda^2x^2+2\lambda xy+by+y^2+\mu}{a\,x^2+bx+c}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi=x^2a_4+yxa_5+y^2a_6+xa_2+ya_3+a_1 \tag{1E}$$

$$\eta=x^2b_4+yxb_5+y^2b_6+xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&2xb_4 + yb_5 + b_2 \\
&+ \frac{(-\lambda\, x^2 a + \lambda^2 x^2 + 2\lambda xy + by + y^2 + \mu)\,(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{a\,x^2 + bx + c} \\
&- \frac{(-\lambda\, x^2 a + \lambda^2 x^2 + 2\lambda xy + by + y^2 + \mu)^2\,(xa_5 + 2ya_6 + a_3)}{(a\,x^2 + bx + c)^2} \\
&- \left(\frac{-2a\lambda x + 2\lambda^2 x + 2\lambda y}{a\,x^2 + bx + c} - \frac{(-\lambda\, x^2 a + \lambda^2 x^2 + 2\lambda xy + by + y^2 + \mu)\,(2xa + b)}{(a\,x^2 + bx + c)^2}\right)(x^2 a_4 \\
&+ yxa_5 + y^2 a_6 + xa_2 + ya_3 + a_1) - \frac{(2\lambda x + b + 2y)\,(x^2 b_4 + yxb_5 + y^2 b_6 + xb_2 + yb_3 + b_1)}{a\,x^2 + bx + c} \\
&= 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

Expression too large to display (7E)

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\text{Expression too large to display} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-a^2\lambda^2$$

$$4ab\lambda a_5 - 2a$$

$$-2a^2\lambda^2 a_6 + 4a\lambda^3 a_6 - 2\lambda$$

$$2ac\lambda a_1 - 2c$$

$$5ab\lambda a_6 - 5b\lambda^2 a$$

$$2ac\lambda a_3 - 2c\lambda^2 a_3 + 2aba_1$$

$$-a^2\lambda^2 a_3 + 2a\lambda^3 a_3 - \lambda^4 a_3 + a^2\lambda a_2 - a^2\lambda b_3 + 3ab\lambda a_4 -$$

$$2ac\lambda a_6 - 2c\lambda^2 a_6 + 2aba_3 + 2a\mu a_6 - b^2 a_5 +$$

$$2ab\lambda a_2 - ab\lambda b_3 + 4ac\lambda a_4 - ac\lambda b_5 + 2a\lambda\mu a_5 - 2b\lambda^2 a_2 + b\lambda^2 b_3 - 4c\lambda^2 a_4$$

$$ab\lambda a_1 + 3ac\lambda a_2 - ac\lambda b_3 + 2a\lambda\mu a_3 - b\lambda^2 a_1 - 3c\lambda^2 a_2 + c\lambda^2 b_3 - 2\lambda^2$$

$$3ab\lambda a_3 + 3ac\lambda a_5 - 2ac\lambda b_6 + 4a\lambda\mu a_6 - 3b\lambda^2 a_3 - 3c\lambda^2 a_5 + 2c\lambda^2 b_6 - 4\lambda^2\mu a_6 + aba_2 + 2acb_5 + 2a\lambda a_1 + a$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= \frac{ca_2}{b} \\
a_2 &= a_2 \\
a_3 &= 0 \\
a_4 &= \frac{aa_2}{b} \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= \frac{bc\lambda b_6 + b\mu b_6 - c\lambda a_2}{b} \\
b_2 &= \lambda(bb_6 - a_2) \\
b_3 &= bb_6 \\
b_4 &= -\frac{\lambda(-b\lambda b_6 + aa_2)}{b} \\
b_5 &= 2\lambda b_6 \\
b_6 &= b_6
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= \lambda^2 x^2 + b\lambda x + 2\lambda xy + by + c\lambda + y^2 + \mu
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\lambda^2 x^2 + b\lambda x + 2\lambda xy + by + c\lambda + y^2 + \mu} dy
\end{aligned}
$$

Which results in

$$
S = \frac{2 \arctan\left(\frac{2\lambda x + b + 2y}{\sqrt{-b^2 + 4c\lambda + 4\mu}}\right)}{\sqrt{-b^2 + 4c\lambda + 4\mu}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{-\lambda\, x^2 a + \lambda^2 x^2 + 2\lambda xy + by + y^2 + \mu}{a\, x^2 + bx + c}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{\lambda}{\lambda^2 x^2 + ((b + 2y)\, x + c)\, \lambda + by + y^2 + \mu} \\
S_y &= \frac{1}{\lambda^2 x^2 + ((b + 2y)\, x + c)\, \lambda + by + y^2 + \mu}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{a\, x^2 + bx + c} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2 a + Rb + c}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{2\arctan\left(\frac{2Ra+b}{\sqrt{4ca-b^2}}\right)}{\sqrt{4ca-b^2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\arctan\left(\frac{2\lambda x+b+2y}{\sqrt{-b^2+4c\lambda+4\mu}}\right)}{\sqrt{-b^2+4c\lambda+4\mu}} = \frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ca-b^2}}\right)}{\sqrt{4ca-b^2}} + c_1$$

Which simplifies to

$$\frac{2\arctan\left(\frac{2\lambda x+b+2y}{\sqrt{-b^2+4c\lambda+4\mu}}\right)}{\sqrt{-b^2+4c\lambda+4\mu}} = \frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ca-b^2}}\right)}{\sqrt{4ca-b^2}} + c_1$$

Which gives

$$y = \frac{\tan\left(\frac{\sqrt{-b^2+4c\lambda+4\mu}\left(\sqrt{4ca-b^2}\,c_1+2\arctan\left(\frac{2xa+b}{\sqrt{4ca-b^2}}\right)\right)}{2\sqrt{4ca-b^2}}\right)\sqrt{-b^2+4c\lambda+4\mu}}{2} - \lambda x - \frac{b}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\tan\left(\frac{\sqrt{-b^2+4c\lambda+4\mu}\left(\sqrt{4ca-b^2}\,c_1+2\arctan\left(\frac{2xa+b}{\sqrt{4ca-b^2}}\right)\right)}{2\sqrt{4ca-b^2}}\right)\sqrt{-b^2+4c\lambda+4\mu}}{2} - \lambda x - \frac{b}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\tan\left(\frac{\sqrt{-b^2+4c\lambda+4\mu}\left(\sqrt{4ca-b^2}\,c_1+2\arctan\left(\frac{2xa+b}{\sqrt{4ca-b^2}}\right)\right)}{2\sqrt{4ca-b^2}}\right)\sqrt{-b^2+4c\lambda+4\mu}}{2} - \lambda x - \frac{b}{2}$$

Verified OK.

2.59.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-\lambda\, x^2 a+\lambda^2 x^2+2\lambda x y+b y+y^2+\mu}{a\, x^2+b x+c} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{\lambda\, x^2 a}{a\, x^2+b x+c}+\frac{\lambda^2 x^2}{a\, x^2+b x+c}+\frac{2\lambda x y}{a\, x^2+b x+c}+\frac{b y}{a\, x^2+b x+c}+\frac{y^2}{a\, x^2+b x+c}+\frac{\mu}{a\, x^2+b x+c}$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=\frac{-\lambda\, x^2 a+\lambda^2 x^2+\mu}{a\, x^2+b x+c}$, $f_1(x)=\frac{2\lambda x+b}{a\, x^2+b x+c}$ and $f_2(x)=\frac{1}{a\, x^2+b x+c}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{a\, x^2+b x+c}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x)-\left(f_2'+f_1 f_2\right)u'(x)+f_2^2 f_0 u(x)=0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2 x a+b}{\left(a\, x^2+b x+c\right)^2} \\ f_1 f_2 &= \frac{2\lambda x+b}{\left(a\, x^2+b x+c\right)^2} \\ f_2^2 f_0 &= \frac{-\lambda\, x^2 a+\lambda^2 x^2+\mu}{\left(a\, x^2+b x+c\right)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{a\, x^2+b x+c}-\left(-\frac{2 x a+b}{\left(a\, x^2+b x+c\right)^2}+\frac{2\lambda x+b}{\left(a\, x^2+b x+c\right)^2}\right)u'(x)+\frac{\left(-\lambda\, x^2 a+\lambda^2 x^2+\mu\right)u(x)}{\left(a\, x^2+b x+c\right)^3}=0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\frac{2xa+b+\sqrt{-4ca+b^2}}{-2xa-b+\sqrt{-4ca+b^2}}\right)^{-\frac{b}{2\sqrt{-4ca+b^2}}} \left(\frac{-2xa-b+\sqrt{-4ca+b^2}}{2xa+b+\sqrt{-4ca+b^2}}\right)^{-\frac{\lambda b}{2a\sqrt{-4ca+b^2}}} \left(a\,x^2 + bx + c\right)^{\frac{\lambda}{2a}} \left(c_1 \left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{\frac{a\sqrt{\frac{b^2-4c\lambda-4\mu}{a^2}}}{2\sqrt{-4ca+b^2}}} + c_2 \left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{b^2-4c\lambda-4\mu}{a^2}}}{2\sqrt{-4ca+b^2}}}\right)$$

The above shows that

$$u'(x) = \frac{8\left(\left(ia\sqrt{4ca-b^2}\,\sqrt{\frac{b^2-4c\lambda-4\mu}{a^2}} - \sqrt{-4ca+b^2}\,(2\lambda x+b)\right) c_2 \left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{b^2-4c\lambda-4\mu}{a^2}}}{2\sqrt{-4ca+b^2}}} - \left(\frac{-b+i\sqrt{4ca-b}}{b+i\sqrt{4ca-b^2}}\right.\right.}{\sqrt{-4ca+b^2}\,\left(2xa+b-\right.}$$

Using the above in (1) gives the solution

$$y = -\frac{8\left(\left(ia\sqrt{4ca-b^2}\,\sqrt{\frac{b^2-4c\lambda-4\mu}{a^2}} - \sqrt{-4ca+b^2}\,(2\lambda x+b)\right) c_2 \left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{b^2-4c\lambda-4\mu}{a^2}}}{2\sqrt{-4ca+b^2}}} - \left(\frac{-b+i\sqrt{4c}}{b+i\sqrt{4ca}}\right.\right.}{\sqrt{-4ca+b^2}\,\left(2xa+b-\sqrt{-4ca+b^2}\right)\left(2xa+b+\sqrt{-4ca+b^2}\right)\left(b+i\sqrt{4ca-b^2}+2xa\right)\left(-b+\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{8\left(\left(ia\sqrt{4ca-b^2}\,\sqrt{\frac{b^2-4c\lambda-4\mu}{a^2}} - \sqrt{-4ca+b^2}\,(2\lambda x+b)\right)\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{b^2-4c\lambda-4\mu}{a^2}}}{2\sqrt{-4ca+b^2}}} - \left(\frac{-b+i\sqrt{4ca-}}{b+i\sqrt{4ca-b}}\right.}{\sqrt{-4ca+b^2}\,\left(2xa+b-\sqrt{-4ca+b^2}\right)\left(2xa+b+\sqrt{-4ca+b^2}\right)\left(b+i\sqrt{4ca-b^2}+2xa\right)\left(-b+\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{8\left(\left(ia\sqrt{4ca-b^2}\,\sqrt{\frac{b^2-4c\lambda-4\mu}{a^2}} - \sqrt{-4ca+b^2}\,(2\lambda x+b)\right)\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{b^2-4c\lambda-4\mu}{a^2}}}{2\sqrt{-4ca+b^2}}} - \left(\frac{-b+i\sqrt{4ca-}}{b+i\sqrt{4ca-b}}\right.}{\sqrt{-4ca+b^2}\,\left(2xa+b-\sqrt{-4ca+b^2}\right)\left(2xa+b+\sqrt{-4ca+b^2}\right)\left(b+i\sqrt{4ca-b^2}+2xa\right)\left(-b+\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{8\left(\left(ia\sqrt{4ca-b^2}\,\sqrt{\frac{b^2-4c\lambda-4\mu}{a^2}} - \sqrt{-4ca+b^2}\,(2\lambda x+b)\right)\left(\frac{-b+i\sqrt{4ca-b^2}-2xa}{b+i\sqrt{4ca-b^2}+2xa}\right)^{-\frac{a\sqrt{\frac{b^2-4c\lambda-4\mu}{a^2}}}{2\sqrt{-4ca+b^2}}} - \left(\frac{-b+i\sqrt{4ca-}}{b+i\sqrt{4ca-b}}\right.}{\sqrt{-4ca+b^2}\,\left(2xa+b-\sqrt{-4ca+b^2}\right)\left(2xa+b+\sqrt{-4ca+b^2}\right)\left(b+i\sqrt{4ca-b^2}+2xa\right)\left(-b+\right.}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -2*x*(a-lambda)*(diff(y(x), x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Group is reducible or imprimitive
      <- Kovacics algorithm successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 542

```
dsolve((a*x^2+b*x+c)*diff(y(x),x)=y(x)^2+(2*lambda*x+b)*y(x)+lambda*(lambda-a)*x^2+mu,y(x),
```

$$y(x) = -\frac{8\left(\left(ia\sqrt{4ac-b^2}\,\sqrt{\frac{b^2-4c\lambda-4\mu}{a^2}}-\sqrt{-4ac+b^2}\,(2x\lambda+b)\right)c_1\left(\frac{-b+i\sqrt{4ac-b^2}-2ax}{i\sqrt{4ac-b^2}+2ax+b}\right)^{-\frac{a\sqrt{\frac{b^2-4c\lambda-4\mu}{a^2}}}{2\sqrt{-4ac+b^2}}}-\left(ia\sqrt{4ac\right.}{\sqrt{-4ac+b^2}\,\left(2ax-\sqrt{-4ac+b^2}+b\right)\left(2ax+\sqrt{-4ac+b^2}+b\right)\left(i\sqrt{4ac-b^2}+2ax+b\right)(-b+}$$

✓ Solution by Mathematica

Time used: 17.168 (sec). Leaf size: 93

```
DSolve[(a*x^2+b*x+c)*y'[x]==y[x]^2+(2*\[Lambda]*x+b)*y[x]+\[Lambda]*(\[Lambda]-a)*x^2+\[Mu],
```

$$y(x)\to \frac{1}{2}\left(\sqrt{4(c\lambda+\mu)-b^2}\tan\left(\frac{\sqrt{-b^2+4c\lambda+4\mu}\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}+c_1\right)-b-2\lambda x\right)$$

2.60 problem 60

Internal problem ID [10390]
Internal file name [`OUTPUT/9337_Monday_June_06_2022_01_57_00_PM_79642548/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 60.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{\left(a\,x^{2}+bx+c\right)y'-y^{2}-\left(xa+\mu\right)y=-\lambda^{2}x^{2}+\lambda(b-\mu)\,x+c\lambda}$$

2.60.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= \frac{-\lambda^{2}x^{2}+axy+b\lambda x-\lambda x\mu+c\lambda+\mu y+y^{2}}{a\,x^{2}+bx+c}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{\lambda^{2}x^{2}}{a\,x^{2}+bx+c}+\frac{axy}{a\,x^{2}+bx+c}+\frac{b\lambda x}{a\,x^{2}+bx+c}-\frac{\lambda x\mu}{a\,x^{2}+bx+c}+\frac{c\lambda}{a\,x^{2}+bx+c}+\frac{\mu y}{a\,x^{2}+bx+c}+\frac{y}{a\,x^{2}+}$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=\frac{-\lambda^2x^2+b\lambda x-\lambda x\mu+c\lambda}{a\,x^2+bx+c}$, $f_1(x)=\frac{xa+\mu}{a\,x^2+bx+c}$ and $f_2(x)=\frac{1}{a\,x^2+bx+c}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2u}\\ &= \frac{-u'}{\frac{u}{a\,x^2+bx+c}}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$f_2' = -\frac{2xa + b}{(a\,x^2 + bx + c)^2}$$
$$f_1f_2 = \frac{xa + \mu}{(a\,x^2 + bx + c)^2}$$
$$f_2^2f_0 = \frac{-\lambda^2x^2 + b\lambda x - \lambda x\mu + c\lambda}{(a\,x^2 + bx + c)^3}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{a\,x^2 + bx + c} - \left(-\frac{2xa + b}{(a\,x^2 + bx + c)^2} + \frac{xa + \mu}{(a\,x^2 + bx + c)^2}\right)u'(x) + \frac{(-\lambda^2x^2 + b\lambda x - \lambda x\mu + c\lambda)\,u(x)}{(a\,x^2 + bx + c)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

Expression too large to display

The above shows that

Expression too large to display

Using the above in (1) gives the solution

Expression too large to display

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

Expression too large to display

Summary

The solution(s) found are the following

Expression too large to display (1)

Verification of solutions

Expression too large to display

Warning, solution could not be verified

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(a*x+b-mu)*(diff(y(x), x))/(a
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Reducible group (found an exponential solution)
         Group is reducible, not completely reducible
         Solution has integrals. Trying a special function solution free of integrals...
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
               <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
            <- hypergeometric successful
         <- special function solution successful
            -> Trying to convert hypergeometric functions to elementary form...
            <- elementary form is not straightforward to achieve - returning special functio
         <- Kovacics algorithm successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 6204

```
dsolve((a*x^2+b*x+c)*diff(y(x),x)=y(x)^2+(a*x+mu)*y(x)-lambda^2*x^2+lambda*(b-mu)*x+lambda*c
```

Expression too large to display

✓ Solution by Mathematica

Time used: 23.352 (sec). Leaf size: 433

```
DSolve[(a*x^2+b*x+c)*y'[x]==y[x]^2+(a*x+\[Mu])*y[x]-\[Lambda]^2*x^2+\[Lambda]*(b-\[Mu])*x+\[
```

$$y(x) \to \frac{(x(ax+b)+c)^{\frac{\lambda}{a}-\frac{1}{2}} \exp\left(-\frac{(a(b-2\mu)+2b\lambda)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}\right) \left(\lambda x(x(ax+b)+c)^{\frac{1}{2}-\frac{\lambda}{a}} \exp\left(\frac{(a(b-2\mu)+2b\lambda)\arctan}{a\sqrt{4ac-b^2}}\right.\right.}{\int_1^x \exp}$$

$$y(x) \to \lambda x$$

2.61 problem 61

Internal problem ID [10391]
Internal file name [`OUTPUT/9338_Monday_June_06_2022_02_01_30_PM_6267541/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\left(a_2x^2+b_2x+c_2\right)y'-y^2-\left(a_1x+b_1\right)y=-\lambda(\lambda+a_1-a_2)\,x^2+\lambda(b_2-b_1)\,x+\lambda c_2$$

2.61.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= \frac{-\lambda\,x^2a_1+\lambda\,x^2a_2-\lambda^2x^2+a_1xy-\lambda xb_1+\lambda xb_2+b_1y+c_2\lambda+y^2}{a_2x^2+b_2x+c_2}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=-\frac{\lambda\,x^2a_1}{a_2x^2+b_2x+c_2}+\frac{\lambda\,x^2a_2}{a_2x^2+b_2x+c_2}-\frac{\lambda^2x^2}{a_2x^2+b_2x+c_2}+\frac{a_1xy}{a_2x^2+b_2x+c_2}-\frac{\lambda xb_1}{a_2x^2+b_2x+c_2}+\frac{\lambda xb_2}{a_2x^2+b_2}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x) = \frac{-\lambda x^2 a_1 + \lambda x^2 a_2 - \lambda^2 x^2 - \lambda x b_1 + \lambda x b_2 + c_2 \lambda}{a_2 x^2 + b_2 x + c_2}$, $f_1(x) = \frac{a_1 x + b_1}{a_2 x^2 + b_2 x + c_2}$ and $f_2(x) = \frac{1}{a_2 x^2 + b_2 x + c_2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{a_2 x^2 + b_2 x + c_2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - \left(f_2' + f_1 f_2\right) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2a_2 x + b_2}{\left(a_2 x^2 + b_2 x + c_2\right)^2} \\ f_1 f_2 &= \frac{a_1 x + b_1}{\left(a_2 x^2 + b_2 x + c_2\right)^2} \\ f_2^2 f_0 &= \frac{-\lambda x^2 a_1 + \lambda x^2 a_2 - \lambda^2 x^2 - \lambda x b_1 + \lambda x b_2 + c_2 \lambda}{\left(a_2 x^2 + b_2 x + c_2\right)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{a_2 x^2 + b_2 x + c_2} - \left(-\frac{2a_2 x + b_2}{\left(a_2 x^2 + b_2 x + c_2\right)^2} + \frac{a_1 x + b_1}{\left(a_2 x^2 + b_2 x + c_2\right)^2}\right) u'(x) + \frac{\left(-\lambda x^2 a_1 + \lambda x^2 a_2 - \lambda^2 x^2 - \lambda x b\right.}{\left(a_2 x^2 + b_2 x + c\right.}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

Expression too large to display

The above shows that

Expression too large to display

Using the above in (1) gives the solution

Expression too large to display

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

Expression too large to display

Summary

The solution(s) found are the following

$$\text{Expression too large to display} \tag{1}$$

Verification of solutions

$$\text{Expression too large to display}$$

Warning, solution could not be verified

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a__1*x-2*a__2*x+b__1-b__2)*(d
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Reducible group (found an exponential solution)
         Group is reducible, not completely reducible
         Solution has integrals. Trying a special function solution free of integrals...
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
               <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
            <- hypergeometric successful
         <- special function solution successful
            -> Trying to convert hypergeometric functions to elementary form...
            <- elementary form is not straightforward to achieve - returning special functio
         <- Kovacics algorithm successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 5771

```
dsolve((a__2*x^2+b__2*x+c__2)*diff(y(x),x)=y(x)^2+(a__1*x+b__1)*y(x)-lambda*(lambda+a__1-a__
```

Expression too large to display

✓ Solution by Mathematica

Time used: 34.34 (sec). Leaf size: 284

```
DSolve[(a2*x^2+b2*x+c2)*y'[x]==y[x]^2+(a1*x+b1)*y[x]-\[Lambda]*(\[Lambda]+a1-a2)*x^2+\[Lambd
```

$$y(x) \to \frac{\lambda x \int_1^x \exp\left(\frac{(\text{a1}-2\text{a2}+2\lambda)\log(\text{c2}+K[1](\text{b2}+\text{a2}K[1]))-\frac{2(\text{b2}(\text{a1}+2\lambda)-2\text{a2}\text{b1})\arctan\left(\frac{\text{b2}+2\text{a2}K[1]}{\sqrt{4\text{a2}\text{c2}-\text{b2}^2}}\right)}{\sqrt{4\text{a2}\text{c2}-\text{b2}^2}}}{2\text{a2}}\right)dK[1]+(x(\text{a2}x -}{\int_1^x \exp\left(\frac{(\text{a1}-2\text{a2}+2\lambda)\log(\text{c2}+K[1](\text{b2}+\text{a2}K[1]))-\frac{2(\text{b2}(\text{a1}+2\lambda)-2\text{a2}\text{b1}}{\sqrt{4\text{a}}}}{2\text{a2}}\right.}$$

$$y(x) \to \lambda x$$

2.62 problem 62

Internal problem ID [10392]
Internal file name [`OUTPUT/9339_Monday_June_06_2022_02_08_40_PM_75216398/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 62.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{(a_2x^2+b_2x+c_2)\,y'-y^2-(a_1x+b_1)\,y=a_0x^2+b_0x+c_0}$$

2.62.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= \frac{a_0x^2+a_1xy+b_0x+b_1y+y^2+c_0}{a_2x^2+b_2x+c_2}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=\frac{a_0x^2}{a_2x^2+b_2x+c_2}+\frac{a_1xy}{a_2x^2+b_2x+c_2}+\frac{b_0x}{a_2x^2+b_2x+c_2}+\frac{b_1y}{a_2x^2+b_2x+c_2}+\frac{y^2}{a_2x^2+b_2x+c_2}+\frac{c_0}{a_2x^2+b_2x}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=\frac{a_0x^2+b_0x+c_0}{a_2x^2+b_2x+c_2}$, $f_1(x)=\frac{a_1x+b_1}{a_2x^2+b_2x+c_2}$ and $f_2(x)=\frac{1}{a_2x^2+b_2x+c_2}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2u}\\ &= \frac{-u'}{\frac{u}{a_2x^2+b_2x+c_2}}\end{aligned}\tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = -\frac{2a_2 x + b_2}{(a_2 x^2 + b_2 x + c_2)^2}$$
$$f_1 f_2 = \frac{a_1 x + b_1}{(a_2 x^2 + b_2 x + c_2)^2}$$
$$f_2^2 f_0 = \frac{a_0 x^2 + b_0 x + c_0}{(a_2 x^2 + b_2 x + c_2)^3}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{a_2 x^2 + b_2 x + c_2} - \left(-\frac{2a_2 x + b_2}{(a_2 x^2 + b_2 x + c_2)^2} + \frac{a_1 x + b_1}{(a_2 x^2 + b_2 x + c_2)^2}\right) u'(x) + \frac{(a_0 x^2 + b_0 x + c_0) u(x)}{(a_2 x^2 + b_2 x + c_2)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

Expression too large to display

The above shows that

Expression too large to display

Using the above in (1) gives the solution

Expression too large to display

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

Expression too large to display

Summary

The solution(s) found are the following

Expression too large to display (1)

Verification of solutions

Expression too large to display

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a__1*x-2*a__2*x+b__1-b__2)*(d
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 6462

```
dsolve((a__2*x^2+b__2*x+c__2)*diff(y(x),x)=y(x)^2+(a__1*x+b__1)*y(x)+a__0*x^2+b__0*x+c__0,y(
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(a2*x^2+b2*x+c2)*y'[x]==y[x]^2+(a1*x+b1)*y[x]+a0*x^2+b0*x+c0,y[x],x,IncludeSingularSo
```

Timed out

2.63 problem 63

Internal problem ID [10393]
Internal file name [OUTPUT/9340_Monday_June_06_2022_02_12_56_PM_17631241/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 63.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x),G(x)]`],
  _Riccati]
```

$$\boxed{(x-a)(x-b)y'+y^2+k(y+x-a)(y+x-b)=0}$$

2.63.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= -\frac{abk-akx-aky-bkx-bky+k\,x^2+2kxy+k\,y^2+y^2}{(a-x)(b-x)}\\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2a_4+yxa_5+y^2a_6+xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = x^2b_4+yxb_5+y^2b_6+xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&2xb_4 + yb_5 + b_2 \\
&- \frac{(abk - akx - aky - bkx - bky + k\,x^2 + 2kxy + k\,y^2 + y^2)\,(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{(a-x)\,(b-x)} \\
&- \frac{(abk - akx - aky - bkx - bky + k\,x^2 + 2kxy + k\,y^2 + y^2)^2\,(xa_5 + 2ya_6 + a_3)}{(a-x)^2\,(b-x)^2} \\
&- \left(-\frac{-ak - bk + 2kx + 2ky}{(a-x)\,(b-x)} \right.\\
&- \frac{abk - akx - aky - bkx - bky + k\,x^2 + 2kxy + k\,y^2 + y^2}{(a-x)^2\,(b-x)} \\
&\left. - \frac{abk - akx - aky - bkx - bky + k\,x^2 + 2kxy + k\,y^2 + y^2}{(a-x)\,(b-x)^2}\right)(x^2a_4 \\
&+ yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1) \\
&+ \frac{(-ak - bk + 2kx + 2ky + 2y)\,(x^2b_4 + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1)}{(a-x)\,(b-x)} = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\text{Expression too large to display} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-a^2b^2k$$

$$4a^2b\,k^2a$$

$$-a^2k^2a_5 - 4a^2k^2a_6 - 4ab\,k^2a_5 - 16ab\,k^2a_6$$
$$2a^2b\,k^2a_5 + 4a^2b\,k^2a_6 + 2a\,b^2k^2a_5 + 4a\,b^2k^2a_6 - 2a^2bka_4 - 2a^2bka_5 + 4$$
$$-a^2k^2a_5 - 4ab\,k^2a_5 - b^2k^2a_5 + 2a^2ka_4 + a^2kb_4 - a^2kb_5 + 8a$$
$$2a^2b\,k^2a_5 + 2a\,b^2k^2a_5 - 4a^2bka_4 - a^2bkb_4 + 2a^2bkb_5 - a^2k^2a_3 - 4a\,b^2ka_4 - a\,b^2kb_4$$
$$-2a^2k^2a_5 - 2a^2k^2a_6 - 8ab\,k^2a_5 - 8ab\,k^2a_6 - 2b^2k^2a_5 - 2b^2k^2a_6 + a^2ka_4 + a^2ka_5 - 2a^2kb_6 + 8abka_4 + 4a$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= ab a_4 \\
a_2 &= -(a+b)\, a_4 \\
a_3 &= 0 \\
a_4 &= a_4 \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= ab b_4 \\
b_2 &= -(a+b)\, b_4 \\
b_3 &= -\frac{aka_4 + akb_4 + bka_4 + bkb_4 + ab_4 + bb_4}{k} \\
b_4 &= b_4 \\
b_5 &= \frac{2ka_4 + 2kb_4 + 2b_4}{k} \\
b_6 &= \frac{(k+1)\,(ka_4 + kb_4 + b_4)}{k^2}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= \frac{ab\,k^2 - a\,k^2 x - a\,k^2 y - b\,k^2 x - b\,k^2 y + x^2 k^2 + 2yx\,k^2 + y^2 k^2 - aky - bky + 2kxy + 2k\,y^2 + y^2}{k^2}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{ab\,k^2 - a\,k^2 x - a\,k^2 y - b\,k^2 x - b\,k^2 y + x^2 k^2 + 2yx\,k^2 + y^2 k^2 - aky - bky + 2kxy + 2k\,y^2 + y^2}{k^2}} dy
\end{aligned}$$

Which results in

$$S = \frac{k \ln(-ak + kx + ky + y)}{(a-b)(k+1)} - \frac{k \ln(-bk + kx + ky + y)}{(a-b)(k+1)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{abk - akx - aky - bkx - bky + k\,x^2 + 2kxy + k\,y^2 + y^2}{(a-x)(b-x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{k^3}{(k+1)((a-x-y)k - y)(k(b-x-y) - y)} \\
S_y &= \frac{k^2}{((a-x-y)k - y)(k(b-x-y) - y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{k^2}{(a-x)(b-x)(k+1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{k^2}{(-R+a)(-R+b)(k+1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{k^2\left(-\frac{\ln(R-b)}{a-b} + \frac{\ln(R-a)}{a-b}\right)}{k+1} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{k(\ln(k(y+x-a)+y) - \ln(k(y+x-b)+y))}{(a-b)(k+1)} = -\frac{k^2\left(-\frac{\ln(x-b)}{a-b} + \frac{\ln(x-a)}{a-b}\right)}{k+1} + c_1$$

Which simplifies to

$$\frac{k\ln(k(y+x-a)+y) - k\ln(k(y+x-b)+y) + k^2\ln(x-a) - k^2\ln(x-b) - c_1(a-b)(k+1)}{(a-b)(k+1)} = 0$$

Summary

The solution(s) found are the following

$$\frac{k\ln(k(y+x-a)+y) - k\ln(k(y+x-b)+y) + k^2\ln(x-a) - k^2\ln(x-b) - c_1(a-b)(k+1)}{(a-b)(k+1)} = 0 \tag{1}$$

Verification of solutions

$$\frac{k\ln(k(y+x-a)+y) - k\ln(k(y+x-b)+y) + k^2\ln(x-a) - k^2\ln(x-b) - c_1(a-b)(k+1)}{(a-b)(k+1)} = 0$$

Verified OK.

2.63.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{abk - akx - aky - bkx - bky + k\,x^2 + 2kxy + k\,y^2 + y^2}{(a-x)(b-x)} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{abk}{(a-x)(b-x)} + \frac{akx}{(a-x)(b-x)} + \frac{aky}{(a-x)(b-x)} + \frac{bkx}{(a-x)(b-x)} + \frac{bky}{(a-x)(b-x)} - \frac{k\,x^2}{(a-x)(b-}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{abk-akx-bkx+k\,x^2}{(a-x)(b-x)}$, $f_1(x) = -\frac{-ak-bk+2kx}{(a-x)(b-x)}$ and $f_2(x) = -\frac{k+1}{(a-x)(b-x)}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{(k+1)u}{(a-x)(b-x)}}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{k+1}{(a-x)^2\,(b-x)} - \frac{k+1}{(a-x)\,(b-x)^2} \\
f_1 f_2 &= \frac{(-ak-bk+2kx)\,(k+1)}{(a-x)^2\,(b-x)^2} \\
f_2^2 f_0 &= -\frac{(k+1)^2\,(abk-akx-bkx+k\,x^2)}{(a-x)^3\,(b-x)^3}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-\frac{(k+1)\,u''(x)}{(a-x)\,(b-x)} - \left(-\frac{k+1}{(a-x)^2\,(b-x)} - \frac{k+1}{(a-x)\,(b-x)^2} + \frac{(-ak-bk+2kx)\,(k+1)}{(a-x)^2\,(b-x)^2}\right)u'(x) - \frac{(k+1)
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1(b-x)^{-k} + c_2(a-x)^{-k}
$$

The above shows that

$$
u'(x) = k\Big(c_1(b-x)^{-k-1} + c_2(a-x)^{-k-1}\Big)
$$

Using the above in (1) gives the solution

$$
y = \frac{k\Big(c_1(b-x)^{-k-1} + c_2(a-x)^{-k-1}\Big)\,(a-x)\,(b-x)}{(k+1)\,\Big(c_1\,(b-x)^{-k} + c_2\,(a-x)^{-k}\Big)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{k(a-x)^{k+1} (b-x)^{k+1} \left(c_3(b-x)^{-k-1} + (a-x)^{-k-1}\right)}{(k+1)\left(c_3 (a-x)^k + (b-x)^k\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{k(a-x)^{k+1} (b-x)^{k+1} \left(c_3(b-x)^{-k-1} + (a-x)^{-k-1}\right)}{(k+1)\left(c_3 (a-x)^k + (b-x)^k\right)} \tag{1}$$

Verification of solutions

$$y = \frac{k(a-x)^{k+1} (b-x)^{k+1} \left(c_3(b-x)^{-k-1} + (a-x)^{-k-1}\right)}{(k+1)\left(c_3 (a-x)^k + (b-x)^k\right)}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a*k+b*k-2*k*x+a+b-2*x)*(diff(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Reducible group (found an exponential solution)
         Reducible group (found another exponential solution)
      <- Kovacics algorithm successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
dsolve((x-a)*(x-b)*diff(y(x),x)+y(x)^2+k*(y(x)+x-a)*(y(x)+x-b)=0,y(x), singsol=all)
```

$$y(x) = \frac{k\left((b-x)^{1+k} + c_1(a-x)^k (a-x)\right)}{(1+k)\left(c_1 (a-x)^k + (b-x)^k\right)}$$

✓ Solution by Mathematica

Time used: 60.572 (sec). Leaf size: 99

```
DSolve[(x-a)*(x-b)*y'[x]+y[x]^2+k*(y[x]+x-a)*(y[x]+x-b)==0,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{1}{2}\left(\frac{k(a+b-2x)}{k+1} + \sqrt{-\frac{k^2(a-b)^2}{(k+1)^2}}\tan\left(\frac{(k+1)\sqrt{-\frac{k^2(a-b)^2}{(k+1)^2}}(\log(x-b)-\log(x-a))}{2(a-b)}+c_1\right)\right)$$

2.64 problem 64

Internal problem ID [10394]
Internal file name [`OUTPUT/9341_Monday_June_06_2022_02_12_57_PM_55541898/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 64.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{(c_2x^2 + b_2x + a_2)(y' + \lambda y^2) + (b_1x + a_1)y = -a_0}$$

2.64.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{c_2\lambda x^2y^2 + y^2b_2\lambda x + y^2a_2\lambda + yb_1x + ya_1 + a_0}{c_2x^2 + b_2x + a_2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2c_2\lambda x^2}{c_2x^2 + b_2x + a_2} - \frac{y^2b_2\lambda x}{c_2x^2 + b_2x + a_2} - \frac{y^2a_2\lambda}{c_2x^2 + b_2x + a_2} - \frac{yb_1x}{c_2x^2 + b_2x + a_2} - \frac{ya_1}{c_2x^2 + b_2x + a_2} - \frac{a_0}{c_2x^2 + b_2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{a_0}{c_2 x^2+b_2 x+a_2}$, $f_1(x) = -\frac{b_1 x+a_1}{c_2 x^2+b_2 x+a_2}$ and $f_2(x) = -\frac{c_2\lambda\, x^2+\lambda x b_2+\lambda a_2}{c_2 x^2+b_2 x+a_2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{(c_2\lambda\, x^2+\lambda x b_2+\lambda a_2)u}{c_2 x^2+b_2 x+a_2}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2c_2\lambda x + \lambda b_2}{c_2 x^2 + b_2 x + a_2} + \frac{(c_2\lambda\, x^2 + \lambda x b_2 + \lambda a_2)(2c_2 x + b_2)}{(c_2 x^2 + b_2 x + a_2)^2} \\
f_1 f_2 &= \frac{(b_1 x + a_1)(c_2\lambda\, x^2 + \lambda x b_2 + \lambda a_2)}{(c_2 x^2 + b_2 x + a_2)^2} \\
f_2^2 f_0 &= -\frac{(c_2\lambda\, x^2 + \lambda x b_2 + \lambda a_2)^2 a_0}{(c_2 x^2 + b_2 x + a_2)^3}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-\frac{(c_2\lambda\, x^2 + \lambda x b_2 + \lambda a_2)\, u''(x)}{c_2 x^2 + b_2 x + a_2} - \left(-\frac{2c_2\lambda x + \lambda b_2}{c_2 x^2 + b_2 x + a_2} + \frac{(c_2\lambda\, x^2 + \lambda x b_2 + \lambda a_2)(2c_2 x + b_2)}{(c_2 x^2 + b_2 x + a_2)^2} + \frac{(b_1 x + a_1)(c_2}{(c_2 x^2}\right.
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
\begin{aligned}
&u(x) \\
&= c_3 \operatorname{hypergeom}\left(\left[-\frac{c_2 - b_1 + \sqrt{c_2^2 + (-4a_0\lambda - 2b_1)\, c_2 + b_1^2}}{2c_2}, \frac{-c_2 + b_1 + \sqrt{c_2^2 + (-4a_0\lambda - 2b_1)\, c_2 + b_1^2}}{2c_2}\right],\right. \\
&\quad + c_4 \left(2\sqrt{\frac{-4c_2 a_2 + b_2^2}{c_2^2}}\, x\, c_2^2 + \sqrt{\frac{-4c_2 a_2 + b_2^2}{c_2^2}}\, b_2 c_2 - 4c_2 a_2 + b_2^2\right)^{\frac{c_2\left(c_2 - \frac{b_1}{2}\right)\sqrt{\frac{-4c_2 a_2 + b_2^2}{c_2^2}} + c_2 a_1 - \frac{b_1 b_2}{2}}{\sqrt{\frac{-4c_2 a_2 + b_2^2}{c_2^2}}\, c_2^2}} \operatorname{hypergeom}\Big(
\end{aligned}
$$

The above shows that

Expression too large to display

Using the above in (1) gives the solution

Expression too large to display

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

Expression too large to display

Summary

The solution(s) found are the following

Expression too large to display (1)

Verification of solutions

Expression too large to display

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(b__1*x+a__1)*(diff(y(x), x))
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            <- heuristic approach successful
         <- hypergeometric successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 2160

```
dsolve((c__2*x^2+b__2*x+a__2)*(diff(y(x),x)+lambda*y(x)^2)+(b__1*x+a__1)*y(x)+a__0=0,y(x), s
```

Expression too large to display

✓ Solution by Mathematica

Time used: 14.836 (sec). Leaf size: 1986

```
DSolve[(c2*x^2+b2*x+a2)*(y'[x]+\[Lambda]*y[x]^2)+(b1*x+a1)*y[x]+a0==0,y[x],x,IncludeSingular
```

Too large to display

2.65 problem 65

Internal problem ID [10395]
Internal file name [`OUTPUT/9342_Monday_June_06_2022_02_14_05_PM_28407073/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 65.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{x^3y' - a\,x^3y^2 - \left(b\,x^2 + c\right)y = sx}$$

2.65.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,x^3y^2 + b\,x^2y + yc + sx}{x^3} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,y^2 + \frac{by}{x} + \frac{yc}{x^3} + \frac{s}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{s}{x^2}$, $f_1(x) = \frac{b\,x^2+c}{x^3}$ and $f_2(x) = a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= \frac{(b\, x^2 + c)\, a}{x^3} \\ f_2^2 f_0 &= \frac{a^2 s}{x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$au''(x) - \frac{(b\, x^2 + c)\, au'(x)}{x^3} + \frac{a^2 su(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{4\left(\frac{((b-1)x^2+c)\left(1-\sqrt{-4as+b^2+2b+1}+b\right)c_1 \operatorname{KummerM}\left(\frac{\sqrt{-4as+b^2+2b+1}}{4}+\frac{b}{4}+\frac{1}{4}, 1+\frac{\sqrt{-4as+b^2+2b+1}}{2}, \frac{c}{2x^2}\right)}{2} + x^2\left(\frac{1}{2} + \frac{(b-1)\sqrt{-4as+b}}{2}\right.\right.}{}$$

The above shows that

$$u'(x) = \frac{8s^2\left(c_1\left(1 - \sqrt{-4as + b^2 + 2b + 1} + b\right) \operatorname{KummerM}\left(\frac{\sqrt{-4as+b^2+2b+1}}{4} + \frac{b}{4} + \frac{1}{4}, 1 + \frac{\sqrt{-4as+b^2+2b+1}}{2}, \frac{c}{2x^2}\right) + 4\right.}{\left(1 - \sqrt{-4as + b^2 + 2b + 1} + b\right)^2 \left(\sqrt{-4\right.}$$

Using the above in (1) gives the solution

$$y = -\frac{2s^2\left(c_1\left(1 - \sqrt{\right.\right.}}{\left(1 - \sqrt{-4as + b^2 + 2b + 1} + b\right)\left(\sqrt{-4as + b^2 + 2b + 1} + b + 1\right)\left(\frac{((b-1)x^2+c)\left(1-\sqrt{-4as+b^2+2b+1}+b\right)c_1 \operatorname{Kum}}{}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y =$$

$$-\frac{}{\left(\frac{((b-1)x^2+c)\left(1-\sqrt{-4as+b^2+2b+1}+b\right)c_3 \operatorname{KummerM}\left(\frac{\sqrt{-4as+b^2+2b+1}}{4}+\frac{b}{4}+\frac{1}{4}, 1+\frac{\sqrt{-4as+b^2+2b+1}}{2}, \frac{c}{2x^2}\right)}{2} + x^2\left(\frac{1}{2}+\frac{(b-1)\sqrt{-4as-}}{2}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = \tag{1}$$

$$-\frac{}{\left(\frac{((b-1)x^2+c)\left(1-\sqrt{-4as+b^2+2b+1}+b\right)c_3 \operatorname{KummerM}\left(\frac{\sqrt{-4as+b^2+2b+1}}{4}+\frac{b}{4}+\frac{1}{4}, 1+\frac{\sqrt{-4as+b^2+2b+1}}{2}, \frac{c}{2x^2}\right)}{2} + x^2\left(\frac{1}{2}+\frac{(b-1)\sqrt{-4as-}}{2}\right.\right.}$$

Verification of solutions

$$y =$$

$$-\frac{}{\left(\frac{((b-1)x^2+c)\left(1-\sqrt{-4as+b^2+2b+1}+b\right)c_3 \operatorname{KummerM}\left(\frac{\sqrt{-4as+b^2+2b+1}}{4}+\frac{b}{4}+\frac{1}{4}, 1+\frac{\sqrt{-4as+b^2+2b+1}}{2}, \frac{c}{2x^2}\right)}{2} + x^2\left(\frac{1}{2}+\frac{(b-1)\sqrt{-4as-}}{2}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (b*x^2+c)*(diff(y(x), x))/x^3-
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
         <- Kummer successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 435

```
dsolve(x^3*diff(y(x),x)=a*x^3*y(x)^2+(b*x^2+c)*y(x)+s*x,y(x), singsol=all)
```

$$y(x) = -\frac{\left(\frac{\left(1-\sqrt{-4as+b^2+2b+1}+b\right)\operatorname{KummerM}\left(\frac{\sqrt{-4as+b^2+2b+1}}{4}\right.}{4}\right.}{2a\left(\frac{((b-1)x^2+c)\left(1-\sqrt{-4as+b^2+2b+1}+b\right)\operatorname{KummerM}\left(\frac{\sqrt{-4as+b^2+2b+1}}{4}+\frac{b}{4}+\frac{1}{4},1+\frac{\sqrt{-4as+b^2+2b+1}}{2},\frac{c}{2x^2}\right)}{2}+\left(\frac{1}{2}+\frac{(b-1)\sqrt{-4as+b}}{2}\right.\right.}$$

✓ Solution by Mathematica

Time used: 3.199 (sec). Leaf size: 907

```
DSolve[x^3*y'[x]==a*x^3*y[x]^2+(b*x^2+c)*y[x]+s*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{-\left(\left(\sqrt{-4as+b^2+2b+1}-b-1\right)c^{\frac{1}{2}\sqrt{-4as+b^2+2b+1}}\left(\frac{1}{x}\right)^{\sqrt{-4as+b^2+2b+1}}\operatorname{Hypergeometric1F1}\left(\frac{1}{4}\left(-b+\sqrt{b}\right.\right.\right.}{}$$

$$y(x) \to \frac{\frac{c\left(b\left(\sqrt{-4as+b^2+2b+1}+4\right)+3\sqrt{-4as+b^2+2b+1}-4as+b^2+3\right)\operatorname{Hypergeometric1F1}\left(\frac{1}{4}\left(-b-\sqrt{b^2+2b-4as+1}+3\right),2-\frac{1}{2}\sqrt{b^2+2b-4as+1},-\frac{c}{2x^2}\right.}{\operatorname{Hypergeometric1F1}\left(\frac{1}{4}\left(-b-\sqrt{b^2+2b-4as+1}-1\right),1-\frac{1}{2}\sqrt{b^2+2b-4as+1},-\frac{c}{2x^2}\right)}}{2ax^3\left(4as-b^2-2b+3\right)}$$

2.66 problem 66

Internal problem ID [10396]
Internal file name [`OUTPUT/9343_Monday_June_06_2022_02_14_08_PM_49204472/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 66.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{x^3y' - a\,x^3y^2 - x(bx + c)\,y = \alpha x + \beta}$$

2.66.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{a\,x^3y^2 + b\,x^2y + ycx + \alpha x + \beta}{x^3} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,y^2 + \frac{by}{x} + \frac{yc}{x^2} + \frac{\alpha}{x^2} + \frac{\beta}{x^3}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\alpha x+\beta}{x^3}$, $f_1(x) = \frac{b\,x^2+cx}{x^3}$ and $f_2(x) = a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= \frac{(b\,x^2 + cx)\,a}{x^3} \\ f_2^2f_0 &= \frac{a^2(\alpha x + \beta)}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$au''(x) - \frac{(b\,x^2 + cx)\,au'(x)}{x^3} + \frac{a^2(\alpha x + \beta)\,u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) &= x^{\frac{b}{2}+\frac{1}{2}-\frac{\sqrt{-4\alpha a+b^2+2b+1}}{2}}\mathrm{e}^{-\frac{c}{x}}\left(\mathrm{KummerU}\left(\frac{\sqrt{-4\alpha a + b^2 + 2b + 1}\,c + (3 + b)\,c - 2\beta a}{2c}, 1\right.\right. \\ &\left.+ \sqrt{-4\alpha a + b^2 + 2b + 1}, \frac{c}{x}\right)c_2 \\ &+ \mathrm{KummerM}\left(\frac{\sqrt{-4\alpha a + b^2 + 2b + 1}\,c + (3 + b)\,c - 2\beta a}{2c}, 1\right. \\ &\left.\left.+ \sqrt{-4\alpha a + b^2 + 2b + 1}, \frac{c}{x}\right)c_1\right) \end{aligned}$$

The above shows that

$$u'(x) = -\frac{\left(x((\alpha a + b + 2)\,c^2 - a\beta(3 + b)\,c + a^2\beta^2)\,c_2\,\mathrm{KummerU}\left(\frac{\sqrt{-4\alpha a+b^2+2b+1}\,c+(b+5)c-2\beta a}{2c}, 1 + \sqrt{-4\alpha a + b^2 + \cdots}\right.\right.}{\cdots}$$

Using the above in (1) gives the solution

$$y = \frac{\left(x((\alpha a + b + 2)\,c^2 - a\beta(3 + b)\,c + a^2\beta^2)\,c_2\,\mathrm{KummerU}\left(\frac{\sqrt{-4\alpha a+b^2+2b+1}\,c+(b+5)c-2\beta a}{2c}, 1 + \sqrt{-4\alpha a + b^2 + 2\cdots}\right.\right.}{\cdots}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x\left(\left(\alpha a+b+2\right)c^2-a\beta(3+b)\,c+a^2\beta^2\right)\mathrm{KummerU}\left(\frac{\sqrt{-4\alpha a+b^2+2b+1}\,c+(b+5)c-2\beta a}{2c}, 1+\sqrt{-4\alpha a+b^2+2b+}\right.}{}$$

Summary

The solution(s) found are the following

$$y = \frac{x\left(\left(\alpha a+b+2\right)c^2-a\beta(3+b)\,c+a^2\beta^2\right)\mathrm{KummerU}\left(\frac{\sqrt{-4\alpha a+b^2+2b+1}\,c+(b+5)c-2\beta a}{2c}, 1+\sqrt{-4\alpha a+b^2+2b+}\right.}{} \quad (1)$$

Verification of solutions

$$y = \frac{x\left(\left(\alpha a+b+2\right)c^2-a\beta(3+b)\,c+a^2\beta^2\right)\mathrm{KummerU}\left(\frac{\sqrt{-4\alpha a+b^2+2b+1}\,c+(b+5)c-2\beta a}{2c}, 1+\sqrt{-4\alpha a+b^2+2b+}\right.}{}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (b*x+c)*(diff(y(x), x))/x^2-a*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
         <- Kummer successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 438

```
dsolve(x^3*diff(y(x),x)=a*x^3*y(x)^2+x*(b*x+c)*y(x)+alpha*x+beta,y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(a\alpha+b+2\right)c^2-a\beta(b+3)\,c+a^2\beta^2\right)xc_1\,\mathrm{KummerU}\left(\frac{\sqrt{-4a\alpha+b^2+2b+1}\,c+(b+5)c-2\beta a}{2c},1+\sqrt{-4a\alpha+b^2+2b}\right.}{}$$

✓ Solution by Mathematica

Time used: 2.395 (sec). Leaf size: 908

```
DSolve[x^3*y'[x]==a*x^3*y[x]^2+x*(b*x+c)*y[x]+\[Alpha]*x+\[Beta],y[x],x,IncludeSingularSolut
```

$$y(x) \to -\frac{\frac{c^{\sqrt{-4a\alpha+b^2+2b+1}}\left(\frac{1}{x}\right)^{\sqrt{-4a\alpha+b^2+2b+1}+1}\left(c\left(\sqrt{-4a\alpha+b^2+2b+1}-b-1\right)+2a\beta\right)\mathrm{Hypergeometric1F1}\left(\frac{1}{2}\left(-b+\frac{2a\beta}{c}+\sqrt{b^2+2b-4a\alpha+1}+1\right),\sqrt{b^2}\right.}{\sqrt{-4a\alpha+b^2+2b+1}+1}}{}$$

$$y(x) \to \frac{\frac{\left(c\left(b\left(\sqrt{-4a\alpha+b^2+2b+1}+3\right)+2\left(-2a\alpha+\sqrt{-4a\alpha+b^2+2b+1}+1\right)+b^2\right)-2a\beta\left(\sqrt{-4a\alpha+b^2+2b+1}+1\right)\right)\mathrm{Hypergeometric1F1}\left(\frac{2a\beta-c\left(b+\sqrt{b^2+2b}\right.}{2c}\right.}{\mathrm{Hypergeometric1F1}\left(\frac{2a\beta-c\left(b+\sqrt{b^2+2b-4a\alpha+1}+1\right)}{2c},1-\sqrt{b^2+2b-4a\alpha+1},-\frac{c}{x}\right)}}{2ax^2\left(4a\alpha-b^2-2b\right)}$$

2.67 problem 67

Internal problem ID [10397]
Internal file name [`OUTPUT/9344_Monday_June_06_2022_02_14_16_PM_45730703/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 67.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{x(x^2+a)\left(y'+\lambda y^2\right)+\left(b\,x^2+c\right)y=-sx}$$

2.67.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= -\frac{y^2\lambda\,x^3+y^2a\lambda x+b\,x^2y+yc+sx}{x\,(x^2+a)} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{x^2y^2\lambda}{x^2+a}-\frac{y^2a\lambda}{x^2+a}-\frac{xby}{x^2+a}-\frac{yc}{x\,(x^2+a)}-\frac{s}{x^2+a}$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=-\frac{s}{x^2+a}$, $f_1(x)=-\frac{b\,x^2+c}{x(x^2+a)}$ and $f_2(x)=-\frac{\lambda\,x^3+a\lambda x}{x(x^2+a)}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u}\\ &= \frac{-u'}{-\frac{(\lambda\,x^3+a\lambda x)u}{x(x^2+a)}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{3\lambda x^2 + \lambda a}{x\,(x^2+a)} + \frac{\lambda x^3 + a\lambda x}{x^2\,(x^2+a)} + \frac{2\lambda x^3 + 2a\lambda x}{(x^2+a)^2} \\
f_1 f_2 &= \frac{(b\,x^2 + c)\,(\lambda x^3 + a\lambda x)}{x^2\,(x^2+a)^2} \\
f_2^2 f_0 &= -\frac{(\lambda x^3 + a\lambda x)^2\, s}{x^2\,(x^2+a)^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{(\lambda x^3 + a\lambda x)\, u''(x)}{x\,(x^2+a)} - \left(-\frac{3\lambda x^2 + \lambda a}{x\,(x^2+a)} + \frac{\lambda x^3 + a\lambda x}{x^2\,(x^2+a)} + \frac{2\lambda x^3 + 2a\lambda x}{(x^2+a)^2} + \frac{(b\,x^2+c)\,(\lambda x^3 + a\lambda x)}{x^2\,(x^2+a)^2}\right) u'(x) -$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = (x^2 + a)^{\frac{(2-b)a+c}{2a}} \left(x^{\frac{a-c}{a}} \operatorname{hypergeom}\left(\left[-\frac{b}{4}+\frac{5}{4}+\frac{\sqrt{b^2-4\lambda s-2b+1}}{4}, -\frac{b}{4}+\frac{5}{4}-\frac{\sqrt{b^2-4\lambda s-2b+1}}{4}\right], \left[\frac{3a-}{2a}\right.\right.\right.$$
$$\left.\left. -\frac{x^2}{a}\right) c_1$$
$$+\operatorname{hypergeom}\left(\left[-\frac{\sqrt{b^2-4\lambda s-2b+1}}{4}-\frac{b}{4}+\frac{3}{4}+\frac{c}{2a}, \frac{\sqrt{b^2-4\lambda s-2b+1}}{4}-\frac{b}{4}+\frac{3}{4}+\frac{c}{2a}\right], \left[\frac{1}{2}+\frac{c}{2a}\right],\right.$$
$$\left.\left. -\frac{x^2}{a}\right) c_2\right)$$

The above shows that

$$u'(x) = \frac{3\left((x^2+a)\, x^2\left((-\lambda s + b - 2)\, a^2 + c(-3+b)\, a - c^2\right)\left(a - \frac{c}{3}\right) c_2 \operatorname{hypergeom}\left(\left[\frac{\sqrt{b^2-4\lambda s-2b+1}}{4} - \frac{b}{4} + \frac{7}{4} + \frac{c}{2a}\right.\right.\right.}{}$$

Using the above in (1) gives the solution

$$y = \frac{3\left(\left(x^2+a\right)x^2\left(\left(-\lambda s+b-2\right)a^2+c(-3+b)\,a-c^2\right)\left(a-\frac{c}{3}\right)c_2 \operatorname{hypergeom}\left(\left[\frac{\sqrt{b^2-4\lambda s-2b+1}}{4}-\frac{b}{4}+\frac{7}{4}+\frac{c}{2a}\right.\right.\right.}{}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1}=c_3$ the following solution

$$y = \frac{\left(\left(-\lambda s+b-2\right)a^2+c(-3+b)\,a-c^2\right)\left(a-\frac{c}{3}\right)\left(a\,x^{\frac{a+c}{a}}+x^{\frac{3a+c}{a}}\right)\operatorname{hypergeom}\left(\left[-\frac{\sqrt{b^2-4\lambda s-2b+1}\,a+ab-7a-2c}{4a},\right.\right.}{}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\left(-\lambda s+b-2\right)a^2+c(-3+b)\,a-c^2\right)\left(a-\frac{c}{3}\right)\left(a\,x^{\frac{a+c}{a}}+x^{\frac{3a+c}{a}}\right)\operatorname{hypergeom}\left(\left[-\frac{\sqrt{b^2-4\lambda s-2b+1}\,a+ab-7a-2c}{4a},\right.\right.}{} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\left(-\lambda s+b-2\right)a^2+c(-3+b)\,a-c^2\right)\left(a-\frac{c}{3}\right)\left(a\,x^{\frac{a+c}{a}}+x^{\frac{3a+c}{a}}\right)\operatorname{hypergeom}\left(\left[-\frac{\sqrt{b^2-4\lambda s-2b+1}\,a+ab-7a-2c}{4a},\right.\right.}{}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(b*x^2+c)*(diff(y(x), x))/(x*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            <- heuristic approach successful
         <- hypergeometric successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 613

```
dsolve(x*(x^2+a)*(diff(y(x),x)+lambda*y(x)^2)+(b*x^2+c)*y(x)+s*x=0,y(x), singsol=all)
```

$y(x)$

$$= \frac{\left(a-\frac{c}{3}\right)\left(\left(-\lambda s+b-2\right)a^2+c(b-3)\,a-c^2\right)\left(a\,x^{\frac{a+c}{a}}+x^{\frac{3a+c}{a}}\right)c_1\,\text{hypergeom}\left(\left[-\frac{\sqrt{b^2-4\lambda s-2b+1}\,a+ab-7a-2c}{4a}\right.\right.}{}$$

✓ Solution by Mathematica

Time used: 2.874 (sec). Leaf size: 862

```
DSolve[x*(x^2+a)*(y'[x]+\[Lambda]*y[x]^2)+(b*x^2+c)*y[x]+s*x==0,y[x],x,IncludeSingularSoluti
```

$y(x)$

$$\to \frac{a^{\frac{1}{2}\left(\frac{c}{a}-3\right)}(a-c)x^{-\frac{c}{a}}\,\text{Hypergeometric2F1}\left(\frac{ba-\sqrt{b^2-2b-4s\lambda+1}a+a-2c}{4a},\frac{a\left(b+\sqrt{b^2-2b-4s\lambda+1}+1\right)-2c}{4a},\frac{3}{2}-\frac{c}{2a},-\frac{x^2}{a}\right)+}{\lambda a^{\frac{1}{2}\left(\frac{c}{a}-1\right)}x^{1-\frac{c}{a}}\,\text{Hypergeometric2F1}\left(\frac{ba-\sqrt{b^2-2b-4s\lambda-}}{4a}\right.}$$

$y(x) \to$

$$-\frac{sx\,\text{Hypergeometric2F1}\left(\frac{1}{4}\left(b-\sqrt{b^2-2b-4s\lambda+1}+3\right),\frac{1}{4}\left(b+\sqrt{b^2-2b-4s\lambda+1}+3\right),\frac{1}{2}\left(\frac{c}{a}+3\right),\right.}{(a+c)\,\text{Hypergeometric2F1}\left(\frac{1}{4}\left(b-\sqrt{b^2-2b-4s\lambda+1}-1\right),\frac{1}{4}\left(b+\sqrt{b^2-2b-4s\lambda+1}-1\right),\frac{a+c}{2a},\right.}$$

$y(x) \to$

$$-\frac{sx\,\text{Hypergeometric2F1}\left(\frac{1}{4}\left(b-\sqrt{b^2-2b-4s\lambda+1}+3\right),\frac{1}{4}\left(b+\sqrt{b^2-2b-4s\lambda+1}+3\right),\frac{1}{2}\left(\frac{c}{a}+3\right),\right.}{(a+c)\,\text{Hypergeometric2F1}\left(\frac{1}{4}\left(b-\sqrt{b^2-2b-4s\lambda+1}-1\right),\frac{1}{4}\left(b+\sqrt{b^2-2b-4s\lambda+1}-1\right),\frac{a+c}{2a},\right.}$$

2.68 problem 68

Internal problem ID [10398]
Internal file name [`OUTPUT/9345_Monday_June_06_2022_02_14_18_PM_29258920/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 68.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{x^2(x+a)\left(y'+\lambda y^2\right)+x(bx+c)\,y=-\alpha x-\beta}$$

2.68.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y^2 a\lambda\, x^2+y^2\lambda\, x^3+b\, x^2 y+ycx+\alpha x+\beta}{x^2\,(x+a)} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2 a\lambda}{x+a}-\frac{x\, y^2\lambda}{x+a}-\frac{by}{x+a}-\frac{yc}{x\,(x+a)}-\frac{\alpha}{x\,(x+a)}-\frac{\beta}{x^2\,(x+a)}$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=-\frac{\alpha x+\beta}{x^2(x+a)}$, $f_1(x)=-\frac{b\, x^2+cx}{x^2(x+a)}$ and $f_2(x)=-\frac{\lambda\, x^2 a+\lambda\, x^3}{x^2(x+a)}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{(\lambda\, x^2 a+\lambda\, x^3)u}{x^2(x+a)}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$f_2' = -\frac{2a\lambda x + 3\lambda\, x^2}{x^2\,(x+a)} + \frac{2\lambda\, x^2a + 2\lambda\, x^3}{x^3\,(x+a)} + \frac{\lambda\, x^2a + \lambda\, x^3}{x^2\,(x+a)^2}$$
$$f_1f_2 = \frac{(b\,x^2+cx)\,(\lambda\, x^2a+\lambda\, x^3)}{x^4\,(x+a)^2}$$
$$f_2^2f_0 = -\frac{(\lambda\, x^2a+\lambda\, x^3)^2\,(\alpha x+\beta)}{x^6\,(x+a)^3}$$

Substituting the above terms back in equation (2) gives

$$-\frac{(\lambda\, x^2a+\lambda\, x^3)\, u''(x)}{x^2\,(x+a)} - \left(-\frac{2a\lambda x + 3\lambda\, x^2}{x^2\,(x+a)} + \frac{2\lambda\, x^2a + 2\lambda\, x^3}{x^3\,(x+a)} + \frac{\lambda\, x^2a + \lambda\, x^3}{x^2\,(x+a)^2} + \frac{(b\,x^2+cx)\,(\lambda\, x^2a+\lambda\, x^3)}{x^4\,(x+a)^2}\right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = (x+a)^{\frac{(-b+1)a+c}{a}}\left(c_2x^{-\frac{-a+c+\sqrt{a^2+(-4\beta\lambda-2c)a+c^2}}{2a}}\operatorname{hypergeom}\left(\left[-\frac{\sqrt{-4\alpha\lambda+b^2-2b+1}\,a+ab+\sqrt{a^2+(-4\beta\lambda}}{2a}\right.\right.\right.$$
$$\left.-\frac{x}{a}\right)$$
$$+c_1x^{\frac{a-c+\sqrt{a^2+(-4\beta\lambda-2c)a+c^2}}{2a}}\operatorname{hypergeom}\left(\left[\frac{-ab+\sqrt{-4\alpha\lambda+b^2-2b+1}\,a+2a+c+\sqrt{a^2+(-4\beta\lambda-2c)}}{2a}\right.\right.$$
$$\left.\left.-\frac{x}{a}\right)\right)$$

The above shows that

Expression too large to display

Using the above in (1) gives the solution

Expression too large to display

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

Expression too large to display

Summary
The solution(s) found are the following

Expression too large to display (1)

Verification of solutions

Expression too large to display

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(b*x+c)*(diff(y(x), x))/(x*(a
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            <- heuristic approach successful
         <- hypergeometric successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 1508

```
dsolve(x^2*(x+a)*(diff(y(x),x)+lambda*y(x)^2)+x*(b*x+c)*y(x)+alpha*x+beta=0,y(x), singsol=al
```

Expression too large to display

✓ Solution by Mathematica

Time used: 5.239 (sec). Leaf size: 1770

```
DSolve[x^2*(x+a)*(y'[x]+\[Lambda]*y[x]^2)+x*(b*x+c)*y[x]+\[Alpha]*x+\[Beta]==0,y[x],x,Includ
```

$y(x)$

$$\rightarrow \frac{2a\left(a-c+\sqrt{a^2-2(c+2\beta\lambda)a+c^2}\right)\text{Hypergeometric2F1}\left(\frac{-c+a\left(b-\sqrt{b^2-2b-4\alpha\lambda+1}\right)+\sqrt{a^2-2(c+2\beta\lambda)a+c^2}}{2a},\frac{-c\cdots}{}\right.}{}$$

$y(x)$

$$\rightarrow \frac{\frac{a(c^2-2a(2\beta\lambda+c))\left(\sqrt{a^2-2a(2\beta\lambda+c)+c^2}-a+c\right)}{x}-\frac{\left(2\alpha a^3\lambda+a^2\left(2\alpha\lambda\sqrt{a^2-2a(2\beta\lambda+c)+c^2}+4b\beta\lambda+bc-2\beta\lambda\right)-a\left(bc\sqrt{a^2-2a(2\beta\lambda+c)+c^2}+2\cdots\right.\right.}{\text{Hy}\cdots}}{}$$

2.69 problem 69

Internal problem ID [10399]
Internal file name [`OUTPUT/9346_Monday_June_06_2022_02_14_39_PM_18970698/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Riccati]
```

$$\boxed{(a\,x^2+bx+e)\,(y'x-y)-y^2=-x^2}$$

2.69.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(a\,x^2+bx+e)\,((u'(x)\,x+u(x))\,x-u(x)\,x)-u(x)^2\,x^2=-x^2$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= \frac{u^2-1}{a\,x^2+bx+e}
\end{aligned}$$

Where $f(x)=\frac{1}{a\,x^2+bx+e}$ and $g(u)=u^2-1$. Integrating both sides gives

$$\frac{1}{u^2-1}\,du=\frac{1}{a\,x^2+bx+e}\,dx$$

$$\int \frac{1}{u^2-1}\,du = \int \frac{1}{a\,x^2+bx+e}\,dx$$

$$-\operatorname{arctanh}(u) = \frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}} + c_2$$

The solution is

$$-\operatorname{arctanh}(u(x)) - \frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}} - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\operatorname{arctanh}\left(\frac{y}{x}\right) - \frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}} - c_2 = 0$$

$$-\operatorname{arctanh}\left(\frac{y}{x}\right) - \frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}} - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\operatorname{arctanh}\left(\frac{y}{x}\right) - \frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}} - c_2 = 0 \tag{1}$$

Verification of solutions

$$-\operatorname{arctanh}\left(\frac{y}{x}\right) - \frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}} - c_2 = 0$$

Verified OK.

2.69.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,x^2y+bxy+ye-x^2+y^2}{(a\,x^2+bx+e)\,x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{xay}{a\,x^2+bx+e} + \frac{by}{a\,x^2+bx+e} + \frac{ye}{(a\,x^2+bx+e)\,x} - \frac{x}{a\,x^2+bx+e} + \frac{y^2}{(a\,x^2+bx+e)\,x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{x}{a\,x^2+bx+e}$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{(a\,x^2+bx+e)x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{(a\,x^2+bx+e)x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2xa+b}{(a\,x^2+bx+e)^2\,x} - \frac{1}{(a\,x^2+bx+e)\,x^2} \\ f_1 f_2 &= \frac{1}{(a\,x^2+bx+e)\,x^2} \\ f_2^2 f_0 &= -\frac{1}{(a\,x^2+bx+e)^3\,x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{(a\,x^2+bx+e)\,x} + \frac{(2xa+b)\,u'(x)}{(a\,x^2+bx+e)^2\,x} - \frac{u(x)}{(a\,x^2+bx+e)^3\,x} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sinh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right) + c_2 \cosh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right)$$

The above shows that

$$u'(x) = \frac{c_1 \cosh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right) + c_2 \sinh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right)}{a\,x^2+bx+e}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(c_1 \cosh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right) + c_2 \sinh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right)\right)x}{c_1 \sinh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right) + c_2 \cosh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(c_3 \cosh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right) + \sinh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right)\right)x}{c_3 \sinh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right) + \cosh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(c_3 \cosh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right) + \sinh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right)\right)x}{c_3 \sinh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right) + \cosh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(c_3 \cosh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right) + \sinh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right)\right)x}{c_3 \sinh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right) + \cosh\left(\frac{2\arctan\left(\frac{2xa+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 58

```
dsolve((a*x^2+b*x+e)*(x*diff(y(x),x)-y(x))-y(x)^2+x^2=0,y(x), singsol=all)
```

$$y(x) = -\tanh\left(\frac{c_1\sqrt{4ea-b^2}+2\arctan\left(\frac{2ax+b}{\sqrt{4ea-b^2}}\right)}{\sqrt{4ea-b^2}}\right)x$$

✓ Solution by Mathematica

Time used: 1.973 (sec). Leaf size: 116

```
DSolve[(a*x^2+b*x+e)*(x*y'[x]-y[x])-y[x]^2+x^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x\left(-1+\exp\left(\frac{4\arctan\left(\frac{2ax+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}+2c_1\right)\right)}{1+\exp\left(\frac{4\arctan\left(\frac{2ax+b}{\sqrt{4ae-b^2}}\right)}{\sqrt{4ae-b^2}}+2c_1\right)}$$

$$y(x) \to -x$$

$$y(x) \to x$$

2.70 problem 70

Internal problem ID [10400]
Internal file name [`OUTPUT/9347_Monday_June_06_2022_02_14_40_PM_15338845/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 70.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{x^2(x^2+a)\left(y'+\lambda y^2\right)+x\left(b\,x^2+c\right)y=-s}$$

2.70.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y^2\lambda\,x^4+y^2a\lambda\,x^2+b\,x^3y+ycx+s}{x^2\,(x^2+a)} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{x^2y^2\lambda}{x^2+a}-\frac{y^2a\lambda}{x^2+a}-\frac{xby}{x^2+a}-\frac{yc}{x\,(x^2+a)}-\frac{s}{x^2\,(x^2+a)}$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=-\frac{s}{x^2(x^2+a)}$, $f_1(x)=-\frac{b\,x^3+cx}{x^2(x^2+a)}$ and $f_2(x)=-\frac{\lambda\,x^4+\lambda\,x^2a}{x^2(x^2+a)}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{-\frac{(\lambda\,x^4+\lambda\,x^2a)u}{x^2(x^2+a)}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = -\frac{4\lambda x^3 + 2a\lambda x}{x^2 (x^2+a)} + \frac{2\lambda x^4 + 2\lambda x^2 a}{x^3 (x^2+a)} + \frac{2\lambda x^4 + 2\lambda x^2 a}{x (x^2+a)^2}$$

$$f_1 f_2 = \frac{(b x^3 + cx)(\lambda x^4 + \lambda x^2 a)}{x^4 (x^2+a)^2}$$

$$f_2^2 f_0 = -\frac{(\lambda x^4 + \lambda x^2 a)^2 s}{x^6 (x^2+a)^3}$$

Substituting the above terms back in equation (2) gives

$$-\frac{(\lambda x^4 + \lambda x^2 a) u''(x)}{x^2 (x^2+a)} - \left(-\frac{4\lambda x^3 + 2a\lambda x}{x^2 (x^2+a)} + \frac{2\lambda x^4 + 2\lambda x^2 a}{x^3 (x^2+a)} + \frac{2\lambda x^4 + 2\lambda x^2 a}{x (x^2+a)^2} + \frac{(b x^3 + cx)(\lambda x^4 + \lambda x^2 c}{x^4 (x^2+a)^2}\right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = (x^2 + a)^{\frac{(2-b)a+c}{2a}} \left(c_2 x^{-\frac{-a+c+\sqrt{a^2+(-4\lambda s-2c)a+c^2}}{2a}} \operatorname{hypergeom}\left(\left[-\frac{-3a-c+\sqrt{a^2+(-4\lambda s-2c)a+c^2}}{4a}, \frac{-2ab+}{} \right.\right.\right.$$
$$\left.\left. -\frac{x^2}{a}\right)$$
$$+ c_1 x^{\frac{a-c+\sqrt{a^2+(-4\lambda s-2c)a+c^2}}{2a}} \operatorname{hypergeom}\left(\left[\frac{3a+c+\sqrt{a^2+(-4\lambda s-2c)a+c^2}}{4a}, \frac{-2ab+5a+c+\sqrt{a^2+(}}{4a}\right.\right.$$
$$\left.\left.\left. -\frac{x^2}{a}\right)\right)$$

The above shows that

Expression too large to display

Using the above in (1) gives the solution

Expression too large to display

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

Expression too large to display

Summary

The solution(s) found are the following

Expression too large to display (1)

Verification of solutions

Expression too large to display

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(b*x^2+c)*(diff(y(x), x))/(x*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            <- heuristic approach successful
         <- hypergeometric successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 1329

```
dsolve(x^2*(x^2+a)*(diff(y(x),x)+lambda*y(x)^2)+x*(b*x^2+c)*y(x)+s=0,y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 7.158 (sec). Leaf size: 1470

```
DSolve[x^2*(x^2+a)*(y'[x]+\[Lambda]*y[x]^2)+x*(b*x^2+c)*y[x]+s==0,y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{\left(a-c-\sqrt{a^2-2(c+2s\lambda)a+c^2}\right)c_1\left(\left(-2ba+a+c+\sqrt{a^2-2(c+2s\lambda)a+c^2}\right)\text{Hypergeometric2F1}\left(-\frac{-5a+c+\sqrt{a^2-2(c+2s\lambda)a+c^2}}{4a},-\frac{-a(2b+3)+c+\sqrt{a}}{4}\right.\right.}{}$$

$$y(x) \to \frac{x\left(a^3(-b)+a^2\left(b\sqrt{a^2-2a(c+2\lambda s)+c^2}-4(b-1)\lambda s+c\right)+a\left(bc\left(\sqrt{a^2-2a(c+2\lambda s)+c^2}+c\right)-c\right.\right.}{2a^2\lambda\left(3a^2+2a\right.}$$

$$-\frac{\sqrt{a^2-2a(c+2\lambda s)+c^2}-a+c}{2a\lambda x}$$

2.71 problem 71

Internal problem ID [10401]
Internal file name [`OUTPUT/9348_Monday_June_06_2022_02_14_54_PM_33630119/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 71.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{a(x^2-1)\left(y'+\lambda y^2\right)+bx(x^2-1)\,y=-c\,x^2-dx-s}$$

2.71.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= -\frac{y^2a\lambda\,x^2+b\,x^3y-y^2a\lambda-bxy+c\,x^2+dx+s}{a\,(x^2-1)}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2\lambda\,x^2}{x^2-1}-\frac{b\,x^3y}{a\,(x^2-1)}+\frac{\lambda\,y^2}{x^2-1}+\frac{bxy}{a\,(x^2-1)}-\frac{c\,x^2}{a\,(x^2-1)}-\frac{dx}{a\,(x^2-1)}-\frac{s}{a\,(x^2-1)}$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=-\frac{c\,x^2+dx+s}{a(x^2-1)}$, $f_1(x)=-\frac{b\,x^3-bx}{a(x^2-1)}$ and $f_2(x)=-\frac{\lambda\,x^2a-\lambda a}{a(x^2-1)}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2u}\\ &= \frac{-u'}{-\frac{(\lambda\,x^2a-\lambda a)u}{a(x^2-1)}}\end{aligned}\tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{2\lambda x}{x^2-1} + \frac{2(\lambda\,x^2a-\lambda a)\,x}{a\,(x^2-1)^2} \\
f_1f_2 &= \frac{(b\,x^3-bx)\,(\lambda\,x^2a-\lambda a)}{a^2\,(x^2-1)^2} \\
f_2^2f_0 &= -\frac{(\lambda\,x^2a-\lambda a)^2\,(c\,x^2+dx+s)}{a^3\,(x^2-1)^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{(\lambda\,x^2a-\lambda a)\,u''(x)}{a\,(x^2-1)} - \left(-\frac{2\lambda x}{x^2-1} + \frac{2(\lambda\,x^2a-\lambda a)\,x}{a\,(x^2-1)^2} + \frac{(b\,x^3-bx)\,(\lambda\,x^2a-\lambda a)}{a^2\,(x^2-1)^2}\right)u'(x) - \frac{(\lambda\,x^2a-\lambda a)^2\,(}{a^3\,(}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(-x^3+x)\,b\_Y'(x)}{a\,(x^2-1)} - \frac{(-c\,x^2-dx-s)\,\lambda\_Y(x)}{a\,(x^2-1)}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(-x^3+x)\,b\_Y'(x)}{a\,(x^2-1)} - \frac{(-c\,x^2-dx-s)\,\lambda\_Y(x)}{a\,(x^2-1)}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = \frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(-x^3+x)b\_Y'(x)}{a(x^2-1)} - \frac{(-c\,x^2-dx-s)\lambda\_Y(x)}{a(x^2-1)}\right\}, \{\_Y(x)\}\right)\right)a(x^2-1)}{(\lambda\,x^2a-\lambda a)\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(-x^3+x)b\_Y'(x)}{a(x^2-1)} - \frac{(-c\,x^2-dx-s)\lambda\_Y(x)}{a(x^2-1)}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\frac{\_Y''(x) a(x^2-1)+b(x^3-x) \_Y'(x)+(c x^2+d x+s) \lambda \_Y(x)}{a(x^2-1)}\right\}, \{\_Y(x)\}\right)}{\lambda \operatorname{DESol}\left(\left\{\frac{\_Y''(x) a(x^2-1)+b(x^3-x) \_Y'(x)+(c x^2+d x+s) \lambda \_Y(x)}{a(x^2-1)}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\frac{\_Y''(x) a(x^2-1)+b(x^3-x) \_Y'(x)+(c x^2+d x+s) \lambda \_Y(x)}{a(x^2-1)}\right\}, \{\_Y(x)\}\right)}{\lambda \operatorname{DESol}\left(\left\{\frac{\_Y''(x) a(x^2-1)+b(x^3-x) \_Y'(x)+(c x^2+d x+s) \lambda \_Y(x)}{a(x^2-1)}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\frac{\_Y''(x) a(x^2-1)+b(x^3-x) \_Y'(x)+(c x^2+d x+s) \lambda \_Y(x)}{a(x^2-1)}\right\}, \{\_Y(x)\}\right)}{\lambda \operatorname{DESol}\left(\left\{\frac{\_Y''(x) a(x^2-1)+b(x^3-x) \_Y'(x)+(c x^2+d x+s) \lambda \_Y(x)}{a(x^2-1)}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -b*x*(diff(y(x), x))/a-lambda*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying a solution in terms of MeijerG functions
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         -> Trying a solution in terms of special functions:
```

✗ Solution by Maple

```
dsolve(a*(x^2-1)*(diff(y(x),x)+lambda*y(x)^2)+b*x*(x^2-1)*y(x)+c*x^2+d*x+s=0,y(x), singsol=a
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[a*(x^2-1)*(y'[x]+\[Lambda]*y[x]^2)+b*x*(x^2-1)*y[x]+c*x^2+d*x+s==0,y[x],x,IncludeSing
```

Not solved

2.72 problem 72

Internal problem ID [10402]
Internal file name [`OUTPUT/9349_Monday_June_06_2022_02_14_56_PM_31336711/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 72.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{x^{n+1}y' - x^{2n}y^2a - yx^nb = c\,x^m + d}$$

2.72.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \left(x^{2n}a\,y^2 + x^nby + c\,x^m + d\right)x^{-n-1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x^n a\,y^2}{x} + \frac{by}{x} + \frac{x^{-n}c\,x^m}{x} + \frac{x^{-n}d}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = (c\,x^m + d)\,x^{-n-1}$, $f_1(x) = x^nb\,x^{-n-1}$ and $f_2(x) = x^{2n}a\,x^{-n-1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{x^{2n}a\,x^{-n-1}u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2x^{2n}na\,x^{-n-1}}{x} - \frac{x^{2n}a\,x^{-n-1}(n+1)}{x} \\ f_1f_2 &= x^nb\,x^{-2n-2}x^{2n}a \\ f_2^2f_0 &= x^{4n}a^2x^{-3n-3}(c\,x^m + d) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^{2n}a\,x^{-n-1}u''(x) - \left(\frac{2x^{2n}na\,x^{-n-1}}{x} - \frac{x^{2n}a\,x^{-n-1}(n+1)}{x} + x^nb\,x^{-2n-2}x^{2n}a\right)u'(x) + x^{4n}a^2x^{-3n-3}(c\,x^m +$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{\frac{b}{2}}x^{\frac{n}{2}}\left(\text{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right)c_2 + \text{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right)c_1\right)$$

The above shows that

$$u'(x) = \frac{x^{-1+\frac{b}{2}+\frac{n}{2}}\left(-2\left(\text{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}+1, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right)c_1 + \text{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}+1, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right)c_2\right)\right.}{}$$

Using the above in (1) gives the solution

$$y = -\frac{x^{-1+\frac{b}{2}+\frac{n}{2}}\left(-2\left(\text{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}+1, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right)c_1 + \text{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}+1, \frac{2\sqrt{ca}\,x^{\frac{m}{2}}}{m}\right)c\right.\right.}{2a\left(\text{BesselY}\left(\sqrt{-4\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x^{-n}\left(-2\left(\mathrm{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}+1, \frac{2\sqrt{ca}\, x^{\frac{m}{2}}}{m}\right) c_3 + \mathrm{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}+1, \frac{2\sqrt{ca}\, x^{\frac{m}{2}}}{m}\right)\right)\sqrt{ca}\,}{2a\left(\mathrm{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m},\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^{-n}\left(-2\left(\mathrm{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}+1, \frac{2\sqrt{ca}\, x^{\frac{m}{2}}}{m}\right) c_3 + \mathrm{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}+1, \frac{2\sqrt{ca}\, x^{\frac{m}{2}}}{m}\right)\right)\sqrt{ca}\,}{2a\left(\mathrm{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m},\right.\right.} \quad (1)$$

Verification of solutions

$$y = -\frac{x^{-n}\left(-2\left(\mathrm{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}+1, \frac{2\sqrt{ca}\, x^{\frac{m}{2}}}{m}\right) c_3 + \mathrm{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}+1, \frac{2\sqrt{ca}\, x^{\frac{m}{2}}}{m}\right)\right)\sqrt{ca}\,}{2a\left(\mathrm{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m},\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (b+n-1)*(diff(y(x), x))/x-x^(n
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         <- Bessel successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 284

```
dsolve(x^(n+1)*diff(y(x),x)=a*x^(2*n)*y(x)^2+b*x^n*y(x)+c*x^m+d,y(x), singsol=all)
```

$$y(x) = \frac{\left(\sqrt{ac}\left(\mathrm{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}+1, \frac{2\sqrt{ac}\,x^{\frac{m}{2}}}{m}\right)c_1 + \mathrm{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}+1, \frac{2\sqrt{ac}\,x^{\frac{m}{2}}}{m}\right)\right)x^{\frac{m}{2}} - \frac{(\mathrm{B}}{}\right.}{a\left(\mathrm{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bn+n^2}}{m}, \frac{2\sqrt{ac}\,x^{\frac{m}{2}}}{m}\right)c_1 + \mathrm{Be}\right.}$$

✓ Solution by Mathematica

Time used: 3.153 (sec). Leaf size: 2576

```
DSolve[x^(n+1)*y'[x]==a*x^(2*n)*y[x]^2+b*x^n*y[x]+c*x^m+d,y[x],x,IncludeSingularSolutions ->
```

Too large to display

2.73 problem 73

Internal problem ID [10403]
Internal file name [`OUTPUT/9350_Monday_June_06_2022_02_14_58_PM_25408529/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 73.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{x\left(a\,x^{k}+b\right)y'-\alpha\,x^{n}y^{2}-\left(\beta-an\,x^{k}\right)y=\gamma\,x^{-n}}$$

2.73.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= -\frac{x^{k}any-\alpha\,x^{n}y^{2}-\gamma\,x^{-n}-\beta y}{x\left(a\,x^{k}+b\right)}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=-\frac{x^{k}any}{x\left(a\,x^{k}+b\right)}+\frac{\alpha\,x^{n}y^{2}}{x\left(a\,x^{k}+b\right)}+\frac{\gamma\,x^{-n}}{x\left(a\,x^{k}+b\right)}+\frac{\beta y}{x\left(a\,x^{k}+b\right)}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=\frac{\gamma\,x^{-n}}{x(a\,x^{k}+b)}$, $f_1(x)=-\frac{an\,x^{k}-\beta}{x(a\,x^{k}+b)}$ and $f_2(x)=\frac{\alpha\,x^{n}}{x(a\,x^{k}+b)}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2u}\\ &= \frac{-u'}{\frac{\alpha\,x^{n}u}{x(a\,x^{k}+b)}}\end{aligned}\tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = \frac{\alpha\, x^n n}{x^2 (a\, x^k + b)} - \frac{\alpha\, x^n}{x^2 (a\, x^k + b)} - \frac{\alpha\, x^n a k\, x^k}{x^2 (a\, x^k + b)^2}$$
$$f_1 f_2 = -\frac{(an\, x^k - \beta)\, \alpha\, x^n}{x^2 (a\, x^k + b)^2}$$
$$f_2^2 f_0 = \frac{\alpha^2 x^{2n} \gamma\, x^{-n}}{x^3 (a\, x^k + b)^3}$$

Substituting the above terms back in equation (2) gives

$$\frac{\alpha\, x^n u''(x)}{x (a\, x^k + b)} - \left(\frac{\alpha\, x^n n}{x^2 (a\, x^k + b)} - \frac{\alpha\, x^n}{x^2 (a\, x^k + b)} - \frac{\alpha\, x^n a k\, x^k}{x^2 (a\, x^k + b)^2} - \frac{(an\, x^k - \beta)\, \alpha\, x^n}{x^2 (a\, x^k + b)^2}\right) u'(x) + \frac{\alpha^2 x^{2n} \gamma\, x^{-n} u}{x^3 (a\, x^k + b}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 x^{\frac{k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\, a+bn+\beta}{2b}} (a\, x^k + b)^{-\frac{k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\, a+bn+\beta}{2bk}} + c_2 x^{\frac{-k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\, a+bn+\beta}{2b}} (a\, x^k + b)^{\frac{k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\, a-bn-\beta}{2bk}}$$

The above shows that

$$u'(x) = \frac{-x^{\frac{-k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\, a+bn+\beta}{2b}} c_2 \left(k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\, a - bn - \beta\right) (a\, x^k + b)^{\frac{k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\, a-bn-\beta}{2bk}} + c_1}{2x (a\, x^k +}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-x^{\frac{-k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\, a+bn+\beta}{2b}} c_2 \left(k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\, a - bn - \beta\right) (a\, x^k + b)^{\frac{k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\, a-bn-\beta}{2bk}}\right.}{2\alpha \left(c_1 x^{\frac{k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\, a+bn+\beta}{2b}} (a\, x^k + b)^{-\frac{k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\, a+bn}{2bk}}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(x^{\frac{-k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\,a+bn+\beta}{2b}}\left(k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\,a-bn-\beta\right)\left(a\,x^k+b\right)^{\frac{k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\,a-bn-\beta}{2bk}} - c_3 x}{2\alpha\left(x^{\frac{k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\,a}{2b}}\left(a\,x^k+b\right)^{-\frac{\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}}{2b}}}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(x^{\frac{-k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\,a+bn+\beta}{2b}}\left(k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\,a-bn-\beta\right)\left(a\,x^k+b\right)^{\frac{k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\,a-bn-\beta}{2bk}} - c_3 x}{2\alpha\left(x^{\frac{k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\,a}{2b}}\left(a\,x^k+b\right)^{-\frac{\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}}{2b}}} \quad (1)$$

Verification of solutions

$$y = \frac{\left(x^{\frac{-k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\,a+bn+\beta}{2b}}\left(k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\,a-bn-\beta\right)\left(a\,x^k+b\right)^{\frac{k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\,a-bn-\beta}{2bk}} - c_3 x}{2\alpha\left(x^{\frac{k\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}\,a}{2b}}\left(a\,x^k+b\right)^{-\frac{\sqrt{\frac{b^2n^2+2\beta nb-4\alpha\gamma+\beta^2}{k^2a^2}}}{2b}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 138

```
dsolve(x*(a*x^k+b)*diff(y(x),x)=alpha*x^n*y(x)^2+(beta-a*n*x^k)*y(x)+gamma*x^(-n),y(x), sing
```

$$y(x) = -\frac{x^{-n}\left(\tanh\left(\frac{((-bn-\beta)\ln(a\,x^k+b)+((bn+\beta)\ln(x)+c_1 b)k)\sqrt{(bn+\beta)^2(n^2b^2+2b\beta n-4\gamma\alpha+\beta^2)}}{2kb(bn+\beta)^2}\right)\sqrt{(bn+\beta)^2\left(n^2b^2+2b\beta n-\right.}}{2\alpha\,(bn+\beta)}$$

✓ Solution by Mathematica

Time used: 4.641 (sec). Leaf size: 663

```
DSolve[x*(a*x^k+b)*y'[x]==\[Alpha]*x^n*y[x]^2+(\[Beta]-a*n*x^k)*y[x]+\[Gamma]*x^(-n),y[x],x,
```

$$y(x) \to \frac{x^{-n}\left(b\left(n\left(-\exp\left(-\frac{(\log(ax^k+b)+\log(b)-k\log(x)+\log(k))\left(\sqrt{\alpha}\sqrt{\gamma}\sqrt{\frac{-4\alpha\gamma+b^2n^2+\beta^2+2b\beta n}{\alpha\gamma}}+bn+\beta\right)}{2bk}\right)\right)-c_1 n\exp\left(-\frac{(1}{}\right.\right.\right.}{}$$

$$y(x) \to \frac{x^{-n}\left(\sqrt{\alpha}\sqrt{\gamma}\sqrt{\frac{(bn+\beta)^2}{\alpha\gamma}-4}-bn-\beta\right)}{2\alpha}$$

2.74 problem 74

Internal problem ID [10404]
Internal file name [`OUTPUT/9351_Monday_June_06_2022_02_15_01_PM_58658321/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 74.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{x^2(x^n a-1)\left(y'+\lambda y^2\right)+(p\,x^n+q)\,xy=-r\,x^n-s}$$

2.74.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= -\frac{x^n y^2 a\lambda\,x^2-y^2\lambda\,x^2+x^n ypx+yqx+r\,x^n+s}{x^2\,(x^n a-1)}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=-\frac{x^n y^2 a\lambda}{x^n a-1}+\frac{y^2\lambda}{x^n a-1}-\frac{x^n yp}{x\,(x^n a-1)}-\frac{yq}{x\,(x^n a-1)}-\frac{r\,x^n}{x^2\,(x^n a-1)}-\frac{s}{x^2\,(x^n a-1)}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=-\frac{r\,x^n+s}{x^2(x^n a-1)}$, $f_1(x)=-\frac{x^n px+qx}{x^2(x^n a-1)}$ and $f_2(x)=-\frac{x^n a\lambda\,x^2-\lambda\,x^2}{x^2(x^n a-1)}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{-\frac{(x^n a\lambda\,x^2-\lambda\,x^2)u}{x^2(x^n a-1)}}\end{aligned}\tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$f_2' = -\frac{x^nnxa\lambda + 2x^na\lambda x - 2\lambda x}{x^2(x^na-1)} + \frac{2x^na\lambda\,x^2 - 2\lambda\,x^2}{x^3(x^na-1)} + \frac{(x^na\lambda\,x^2 - \lambda\,x^2)\,x^nna}{x^3(x^na-1)^2}$$

$$f_1f_2 = \frac{(x^npx+qx)\,(x^na\lambda\,x^2 - \lambda\,x^2)}{x^4(x^na-1)^2}$$

$$f_2^2f_0 = -\frac{(x^na\lambda\,x^2 - \lambda\,x^2)^2\,(r\,x^n+s)}{x^6(x^na-1)^3}$$

Substituting the above terms back in equation (2) gives

$$-\frac{(x^na\lambda\,x^2 - \lambda\,x^2)\,u''(x)}{x^2(x^na-1)} - \left(-\frac{x^nnxa\lambda + 2x^na\lambda x - 2\lambda x}{x^2(x^na-1)} + \frac{2x^na\lambda\,x^2 - 2\lambda\,x^2}{x^3(x^na-1)} + \frac{(x^na\lambda\,x^2 - \lambda\,x^2)\,x^nna}{x^3(x^na-1)^2} + \right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{\frac{1}{2}+\frac{q}{2}}\left(c_2x^{-\frac{\sqrt{4\lambda s+q^2+2q+1}}{2}}\,\text{hypergeom}\left(\left[\frac{-a\sqrt{4\lambda s+q^2+2q+1}+aq+\sqrt{a^2+(-4\lambda r-2p)\,a+p^2}+p}{2an}, -\right.\right.\right.$$
$$\left. + c_1x^{\frac{\sqrt{4\lambda s+q^2+2q+1}}{2}}\,\text{hypergeom}\left(\left[\frac{a\sqrt{4\lambda s+q^2+2q+1}+aq+\sqrt{a^2+(-4\lambda r-2p)\,a+p^2}+p}{2an}, \frac{a\sqrt{4\lambda s+}}{}\right.\right.\right.$$

The above shows that

$$u'(x) = -\frac{x^{\frac{q}{2}-\frac{1}{2}}\left(x^{n-\frac{\sqrt{4\lambda s+q^2+2q+1}}{2}}c_2\left(\left(\frac{a\,q^2}{2}+\frac{(-an+a+p)q}{2}+a\lambda s+\lambda r-\frac{pn}{2}+\frac{p}{2}\right)\sqrt{4\lambda s+q^2+2q+1}-\frac{a\,q^3}{2}+\left(\frac{1}{2}an\right.\right.\right.}{}$$

Using the above in (1) gives the solution

Expression too large to display

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$\text{Expression too large to display}$$

Summary

The solution(s) found are the following

$$\text{Expression too large to display} \tag{1}$$

Verification of solutions

$$\text{Expression too large to display}$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(x^(n-1)*p*x+q)*(diff(y(x), x
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            <- heuristic approach successful
         <- hypergeometric successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 1222

```
dsolve(x^2*(a*x^n-1)*(diff(y(x),x)+lambda*y(x)^2)+(p*x^n+q)*x*y(x)+r*x^n+s=0,y(x), singsol=a
```

Expression too large to display

✓ Solution by Mathematica

Time used: 7.968 (sec). Leaf size: 2419

```
DSolve[x^2*(a*x^n-1)*(y'[x]+\[Lambda]*y[x]^2)+(p*x^n+q)*x*y[x]+r*x^n+s==0,y[x],x,IncludeSing
```

Too large to display

2.75 problem 75

Internal problem ID [10405]
Internal file name [`OUTPUT/9352_Monday_June_06_2022_02_15_36_PM_48106836/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 75.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{(x^n a + b\,x^m + c)\,y' - cy^2 + b\,x^{m-1}y = a\,x^{-2+n}}$$

2.75.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{b\,x^{m-1}y - c\,y^2 - a\,x^{-2+n}}{x^n a + b\,x^m + c} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{b\,x^m y}{(x^n a + b\,x^m + c)\,x} + \frac{c\,y^2}{x^n a + b\,x^m + c} + \frac{a\,x^n}{(x^n a + b\,x^m + c)\,x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{a\,x^{-2+n}}{x^n a + b\,x^m + c}$, $f_1(x) = -\frac{x^{m-1}b}{x^n a + b\,x^m + c}$ and $f_2(x) = \frac{c}{x^n a + b\,x^m + c}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{cu}{x^n a + b\,x^m + c}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{c\left(\frac{x^n n a}{x} + \frac{b\, x^m m}{x}\right)}{\left(x^n a + b\, x^m + c\right)^2} \\ f_1 f_2 &= -\frac{x^{m-1} b c}{\left(x^n a + b\, x^m + c\right)^2} \\ f_2^2 f_0 &= \frac{c^2 a\, x^{-2+n}}{\left(x^n a + b\, x^m + c\right)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{c u''(x)}{x^n a + b\, x^m + c} - \left(-\frac{c\left(\frac{x^n n a}{x} + \frac{b\, x^m m}{x}\right)}{\left(x^n a + b\, x^m + c\right)^2} - \frac{x^{m-1} b c}{\left(x^n a + b\, x^m + c\right)^2}\right) u'(x) + \frac{c^2 a\, x^{-2+n} u(x)}{\left(x^n a + b\, x^m + c\right)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\_Y'(x)\left(-x^n n a - b m\, x^m - b\, x^m\right)}{\left(x^n a + b\, x^m + c\right) x} + \frac{c a\, x^{-2+n} \_Y(x)}{\left(x^n a + b\, x^m + c\right)^2}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\_Y'(x)\left(-x^n n a - b m\, x^m - b\, x^m\right)}{\left(x^n a + b\, x^m + c\right) x} + \frac{c a\, x^{-2+n} \_Y(x)}{\left(x^n a + b\, x^m + c\right)^2}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\_Y'(x)(-x^n n a - b m\, x^m - b\, x^m)}{(x^n a + b\, x^m + c) x} + \frac{c a\, x^{-2+n} \_Y(x)}{(x^n a + b\, x^m + c)^2}\right\}, \{\_Y(x)\}\right)\right) \left(x^n a + b\, x^m + c\right)}{c\, \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\_Y'(x)(-x^n n a - b m\, x^m - b\, x^m)}{(x^n a + b\, x^m + c) x} + \frac{c a\, x^{-2+n} \_Y(x)}{(x^n a + b\, x^m + c)^2}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$y =$

$$-\frac{(x^n a + b x^m + c) \left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{x^2(x^n a+b x^m+c)^2 \_\_Y''(x)+(x^n a+b x^m+c)(b(m+1)x^m+x^n n a)x\_\_Y}{x^2(x^n a+b x^m+c)^2}\right.\right.\right.}{c\,\mathrm{DESol}\left(\left\{\frac{(x^{2m}b^2+a^2x^{2n}+2b(x^n a+c)x^m+2x^n ac+c^2)x^2\_\_Y''(x)+\_\_Y'(x)b^2x(m+1)x^{2m}+\_\_Y'(x)x^{2n}a^2nx+((a(1+m+n)x^n+c(r}{x^2(x^n a+b x^m+c)^2}\right.\right.}$$

Summary

The solution(s) found are the following

$y =$ (1)

$$-\frac{(x^n a + b x^m + c) \left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{x^2(x^n a+b x^m+c)^2 \_\_Y''(x)+(x^n a+b x^m+c)(b(m+1)x^m+x^n n a)x\_\_Y}{x^2(x^n a+b x^m+c)^2}\right.\right.\right.}{c\,\mathrm{DESol}\left(\left\{\frac{(x^{2m}b^2+a^2x^{2n}+2b(x^n a+c)x^m+2x^n ac+c^2)x^2\_\_Y''(x)+\_\_Y'(x)b^2x(m+1)x^{2m}+\_\_Y'(x)x^{2n}a^2nx+((a(1+m+n)x^n+c(r}{x^2(x^n a+b x^m+c)^2}\right.\right.}$$

Verification of solutions

$y =$

$$-\frac{(x^n a + b x^m + c) \left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{x^2(x^n a+b x^m+c)^2 \_\_Y''(x)+(x^n a+b x^m+c)(b(m+1)x^m+x^n n a)x\_\_Y}{x^2(x^n a+b x^m+c)^2}\right.\right.\right.}{c\,\mathrm{DESol}\left(\left\{\frac{(x^{2m}b^2+a^2x^{2n}+2b(x^n a+c)x^m+2x^n ac+c^2)x^2\_\_Y''(x)+\_\_Y'(x)b^2x(m+1)x^{2m}+\_\_Y'(x)x^{2n}a^2nx+((a(1+m+n)x^n+c(r}{x^2(x^n a+b x^m+c)^2}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(a*n*x^n+b*m*x^m+x^(m-1)*b*x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 1236

```
dsolve((a*x^n+b*x^m+c)*diff(y(x),x)=c*y(x)^2-b*x^(m-1)*y(x)+a*x^(n-2),y(x), singsol=all)
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(a*x^n+b*x^m+c)*y'[x]==c*y[x]^2-b*x^(m-1)*y[x]+a*x^(n-2),y[x],x,IncludeSingularSoluti
```

Not solved

2.76 problem 76

Internal problem ID [10406]
Internal file name [OUTPUT/9353_Monday_June_06_2022_02_16_17_PM_53182186/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 76.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{(x^n a + b\,x^m + c)\,y' - a\,x^{-2+n}y^2 - b\,x^{m-1}y = c}$$

2.76.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,x^{-2+n}y^2 + b\,x^{m-1}y + c}{x^n a + b\,x^m + c} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,x^n y^2}{(x^n a + b\,x^m + c)\,x^2} + \frac{b\,x^m y}{(x^n a + b\,x^m + c)\,x} + \frac{c}{x^n a + b\,x^m + c}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{c}{x^n a + b\,x^m + c}$, $f_1(x) = \frac{x^{m-1}b}{x^n a + b\,x^m + c}$ and $f_2(x) = \frac{a\,x^{-2+n}}{x^n a + b\,x^m + c}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{a\,x^{-2+n}u}{x^n a + b\,x^m + c}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$f_2' = -\frac{a\,x^{-2+n}\left(\frac{x^n na}{x} + \frac{b\,x^m m}{x}\right)}{(x^na + b\,x^m + c)^2} + \frac{a\,x^{-2+n}(-2+n)}{(x^na + b\,x^m + c)\,x}$$
$$f_1f_2 = \frac{x^{m-1}ba\,x^{-2+n}}{(x^na + b\,x^m + c)^2}$$
$$f_2^2f_0 = \frac{a^2x^{-4+2n}c}{(x^na + b\,x^m + c)^3}$$

Substituting the above terms back in equation (2) gives

$$\frac{a\,x^{-2+n}u''(x)}{x^na + b\,x^m + c} - \left(-\frac{a\,x^{-2+n}\left(\frac{x^n na}{x} + \frac{b\,x^m m}{x}\right)}{(x^na + b\,x^m + c)^2} + \frac{a\,x^{-2+n}(-2+n)}{(x^na + b\,x^m + c)\,x} + \frac{x^{m-1}ba\,x^{-2+n}}{(x^na + b\,x^m + c)^2}\right)u'(x) + \frac{a^2x^{-4}}{(x^na +}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{\_Y''(x)\,(x^na + b\,x^m + c)^2\,x + ac\,x^{-2+n}\_Y(x)\,x + 2\left(\frac{b(m-n+1)x^m}{2} + x^na - \frac{c(-2+n)}{2}\right)(x^na + b\,x}{x\,(x^na + b\,x^m + c)^2}\right.\right.$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\_Y''(x)\,(x^na + b\,x^m + c)^2\,x + ac\,x^{n-1}\_Y(x) + 2\left(\frac{b(m-n+1)x^m}{2} + x^na - \frac{c(-2+n)}{2}\right)(x^na + b\,}{x\,(x^na + b\,x^m + c)^2}\right.\right.$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\_Y''(x)(x^na+b\,x^m+c)^2x+ac\,x^{n-1}\_Y(x)+2\left(\frac{b(m-n+1)x^m}{2}+x^na-\frac{c(-2+n)}{2}\right)(x^na+b\,x^m+c)\_Y'(x)}{x(x^na+b\,x^m+c)^2}\right\},\{\_Y(x\right.\right.}{a\,\mathrm{DESol}\left(\left\{\frac{\_Y''(x)(x^na+b\,x^m+c)^2x+ac\,x^{-2+n}\_Y(x)x+2\left(\frac{b(m-n+1)x^m}{2}+x^na-\frac{c(-2+n)}{2}\right)(x^na+b\,x^m+c)\_Y'(}{x(x^na+b\,x^m+c)^2}\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x^{2-n}(x^n a + b\,x^m + c)\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{x^{1+2m}b^2\_\_Y''(x) + x^{1+2n}a^2\_\_Y''(x) + 2ab\,x^{1+m+n}\_\_Y''(x) + 2c\left(a\,x^{n+1} + x^{m+1}b + \frac{cx}{2}\right. \cdots}{\cdots}\right.\right.\right.}{a\,\mathrm{DESol}\left(\left\{\frac{x^2(x^n a + b\,x^m + c)^2\_\_Y''(x) + 2\left(\frac{b(m-n+1)}{2} \cdots\right.}{\cdots}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^{2-n}(x^n a + b\,x^m + c)\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{x^{1+2m}b^2\_\_Y''(x) + x^{1+2n}a^2\_\_Y''(x) + 2ab\,x^{1+m+n}\_\_Y''(x) + 2c\left(a\,x^{n+1} + x^{m+1}b + \frac{cx}{2}\right. \cdots}{\cdots}\right.\right.\right.}{a\,\mathrm{DESol}\left(\left\{\frac{x^2(x^n a + b\,x^m + c)^2\_\_Y''(x) + 2\left(\frac{b(m-n+1)}{2} \cdots\right.}{\cdots}\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{x^{2-n}(x^n a + b\,x^m + c)\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{x^{1+2m}b^2\_\_Y''(x) + x^{1+2n}a^2\_\_Y''(x) + 2ab\,x^{1+m+n}\_\_Y''(x) + 2c\left(a\,x^{n+1} + x^{m+1}b + \frac{cx}{2}\right. \cdots}{\cdots}\right.\right.\right.}{a\,\mathrm{DESol}\left(\left\{\frac{x^2(x^n a + b\,x^m + c)^2\_\_Y''(x) + 2\left(\frac{b(m-n+1)}{2} \cdots\right.}{\cdots}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(b*m*x^m-x^m*n*b-x^(m-1)*b*x+
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
```

✗ Solution by Maple

```
dsolve((a*x^n+b*x^m+c)*diff(y(x),x)=a*x^(n-2)*y(x)^2+b*x^(m-1)*y(x)+c,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(a*x^n+b*x^m+c)*y'[x]==a*x^(n-2)*y[x]^2+b*x^(m-1)*y[x]+c,y[x],x,IncludeSingularSoluti
```

Not solved

2.77 problem 77

Internal problem ID [10407]
Internal file name [`OUTPUT/9354_Monday_June_06_2022_02_16_53_PM_78079711/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 77.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

Unable to solve or complete the solution.

$$\boxed{(x^n a + b\,x^m + c)\,y' - \alpha\,x^k y^2 - \beta\,x^s y = -\alpha\,\lambda^2 x^k + \beta\lambda\,x^s}$$

2.77.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\alpha\,x^k y^2 + \beta\,x^s y - \alpha\,\lambda^2 x^k + \beta\lambda\,x^s}{x^n a + b\,x^m + c} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{\alpha\,\lambda^2 x^k}{x^n a + b\,x^m + c} + \frac{\alpha\,x^k y^2}{x^n a + b\,x^m + c} + \frac{\beta\lambda\,x^s}{x^n a + b\,x^m + c} + \frac{\beta\,x^s y}{x^n a + b\,x^m + c}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{-\alpha\,\lambda^2 x^k + \beta\lambda\,x^s}{x^n a + b\,x^m + c}$, $f_1(x) = \frac{\beta\,x^s}{x^n a + b\,x^m + c}$ and $f_2(x) = \frac{\alpha\,x^k}{x^n a + b\,x^m + c}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{\alpha\,x^k u}{x^n a + b\,x^m + c}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{\alpha k\,x^k}{x\,(x^n a + b\,x^m + c)} - \frac{\alpha\,x^k \left(\frac{x^n n a}{x} + \frac{b\,x^m m}{x}\right)}{(x^n a + b\,x^m + c)^2} \\ f_1 f_2 &= \frac{\beta\,x^s \alpha\,x^k}{(x^n a + b\,x^m + c)^2} \\ f_2^2 f_0 &= \frac{\alpha^2 x^{2k} \left(-\alpha\,\lambda^2 x^k + \beta\lambda\,x^s\right)}{(x^n a + b\,x^m + c)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\alpha\,x^k u''(x)}{x^n a + b\,x^m + c} - \left(\frac{\alpha k\,x^k}{x\,(x^n a + b\,x^m + c)} - \frac{\alpha\,x^k \left(\frac{x^n n a}{x} + \frac{b\,x^m m}{x}\right)}{(x^n a + b\,x^m + c)^2} + \frac{\beta\,x^s \alpha\,x^k}{(x^n a + b\,x^m + c)^2}\right) u'(x) + \frac{\alpha^2 x^{2k} (-\alpha\,\lambda^2}{(x^n a +}$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (beta*x^s*x+x^n*a*k-a*n*x^n+x^
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 164

```
dsolve((a*x^n+b*x^m+c)*diff(y(x),x)=alpha*x^k*y(x)^2+beta*x^s*y(x)-alpha*lambda^2*x^k+beta*l
```

$$y(x) = \frac{-\alpha\left(\int \frac{x^{k} \mathrm{e}^{-\left(\int \frac{2\alpha\, x^{k}\lambda - x^{s}\beta}{a\, x^{n}+b\, x^{m}+c}dx\right)}}{a\, x^{n}+b\, x^{m}+c}dx\right)\lambda - \lambda c_{1} - \mathrm{e}^{-\left(\int \frac{2\alpha\, x^{k}\lambda - x^{s}\beta}{a\, x^{n}+b\, x^{m}+c}dx\right)}}{c_{1}+\alpha\left(\int \frac{x^{k} \mathrm{e}^{-\left(\int \frac{2\alpha\, x^{k}\lambda - x^{s}\beta}{a\, x^{n}+b\, x^{m}+c}dx\right)}}{a\, x^{n}+b\, x^{m}+c}dx\right)}$$

✓ Solution by Mathematica

Time used: 13.649 (sec). Leaf size: 389

```
DSolve[(a*x^n+b*x^m+c)*y'[x]==\[Alpha]*x^k*y[x]^2+\[Beta]*x^s*y[x]-\[Alpha]*\[Lambda]^2*x^k+
```

$$\text{Solve}\left[\int_1^x \frac{\exp\left(-\int_1^{K[2]} -\frac{\beta K[1]^s-2\alpha\lambda K[1]^k}{bK[1]^m+aK[1]^n+c}dK[1]\right)\left(-\alpha\lambda K[2]^k+\alpha y(x)K[2]^k+\beta K[2]^s\right)}{(k-s)\alpha\beta\left(bK[2]^m+aK[2]^n+c\right)(\lambda+y(x))}dK[2]\right.$$

$$+\int_1^{y(x)}\left(-\int_1^x\left(\frac{\exp\left(-\int_1^{K[2]} -\frac{\beta K[1]^s-2\alpha\lambda K[1]^k}{bK[1]^m+aK[1]^n+c}dK[1]\right)K[2]^k}{(k-s)\beta\left(bK[2]^m+aK[2]^n+c\right)(\lambda+K[3])}-\frac{\exp\left(-\int_1^{K[2]} -\frac{\beta K[1]^s-2\alpha\lambda K[1]^k}{bK[1]^m+aK[1]^n+c}dK[1]\right)(-\alpha}{(k-s)\alpha\beta\left(bK[2]^m+aK[2\right.}\right.\right.$$

$$\left.\left.-\frac{\exp\left(-\int_1^x -\frac{\beta K[1]^s-2\alpha\lambda K[1]^k}{bK[1]^m+aK[1]^n+c}dK[1]\right)}{(k-s)\alpha\beta(\lambda+K[3])^2}\right)dK[3]=c_1,y(x)\right]$$

2.78 problem 78

Internal problem ID [10408]
Internal file name [`OUTPUT/9355_Monday_June_06_2022_02_18_02_PM_79181218/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. 1.2.2. Equations Containing Power Functions
Problem number: 78.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Riccati]
```

$$\boxed{(x^n a + b\,x^m + c)\,(y'x - y) + s\,x^k\left(y^2 - \lambda\,x^2\right) = 0}$$

2.78.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(x^n a + b\,x^m + c)\,((u'(x)\,x + u(x))\,x - u(x)\,x) + s\,x^k\left(u(x)^2\,x^2 - \lambda\,x^2\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{s\,x^k(u^2-\lambda)}{x^n a + b\,x^m + c} \end{aligned}$$

Where $f(x) = -\frac{s\,x^k}{x^n a + b\,x^m + c}$ and $g(u) = u^2 - \lambda$. Integrating both sides gives

$$\frac{1}{u^2-\lambda}\,du = -\frac{s\,x^k}{x^n a + b\,x^m + c}\,dx$$

$$\int \frac{1}{u^2-\lambda}du = \int -\frac{s\,x^k}{x^n a + b\,x^m + c}dx$$
$$-\frac{\operatorname{arctanh}\left(\frac{u}{\sqrt{\lambda}}\right)}{\sqrt{\lambda}} = \int -\frac{s\,x^k}{x^n a + b\,x^m + c}dx + c_2$$

The solution is

$$-\frac{\operatorname{arctanh}\left(\frac{u(x)}{\sqrt{\lambda}}\right)}{\sqrt{\lambda}} - \left(\int -\frac{s\,x^k}{x^n a + b\,x^m + c}dx\right) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{\operatorname{arctanh}\left(\frac{y}{x\sqrt{\lambda}}\right)}{\sqrt{\lambda}} - \left(\int -\frac{s\,x^k}{x^n a + b\,x^m + c}dx\right) - c_2 = 0$$
$$-\frac{\operatorname{arctanh}\left(\frac{y}{x\sqrt{\lambda}}\right)}{\sqrt{\lambda}} + s\left(\int \frac{x^k}{x^n a + b\,x^m + c}dx\right) - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\frac{\operatorname{arctanh}\left(\frac{y}{x\sqrt{\lambda}}\right)}{\sqrt{\lambda}} + s\left(\int \frac{x^k}{x^n a + b\,x^m + c}dx\right) - c_2 = 0 \tag{1}$$

Verification of solutions

$$-\frac{\operatorname{arctanh}\left(\frac{y}{x\sqrt{\lambda}}\right)}{\sqrt{\lambda}} + s\left(\int \frac{x^k}{x^n a + b\,x^m + c}dx\right) - c_2 = 0$$

Verified OK.

2.78.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{-x^k\lambda s\,x^2 + y^2x^k s - a\,x^n y - b\,x^m y - yc}{x\,(x^n a + b\,x^m + c)} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x\,x^k\lambda s}{x^n a + b\,x^m + c} - \frac{y^2 x^k s}{x\,(x^n a + b\,x^m + c)} + \frac{a\,x^n y}{x\,(x^n a + b\,x^m + c)} + \frac{b\,x^m y}{(x^n a + b\,x^m + c)\,x} + \frac{yc}{x\,(x^n a + b\,x^m + c)}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{x\,x^k\lambda s}{x^n a+b\,x^m+c}$, $f_1(x) = -\frac{-x^n a-b\,x^m-c}{x(x^n a+b\,x^m+c)}$ and $f_2(x) = -\frac{s\,x^k}{x(x^n a+b\,x^m+c)}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{s\,x^k u}{x(x^n a+b\,x^m+c)}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{sk\,x^k}{x^2\,(x^n a+b\,x^m+c)} + \frac{s\,x^k}{x^2\,(x^n a+b\,x^m+c)} + \frac{s\,x^k\left(\frac{x^n na}{x}+\frac{b\,x^m m}{x}\right)}{x\,(x^n a+b\,x^m+c)^2} \\ f_1 f_2 &= \frac{(-x^n a-b\,x^m-c)\,s\,x^k}{x^2\,(x^n a+b\,x^m+c)^2} \\ f_2^2 f_0 &= \frac{s^3 x^{3k}\lambda}{x\,(x^n a+b\,x^m+c)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{s\,x^k u''(x)}{x\,(x^n a+b\,x^m+c)} - \left(-\frac{sk\,x^k}{x^2\,(x^n a+b\,x^m+c)} + \frac{s\,x^k}{x^2\,(x^n a+b\,x^m+c)} + \frac{s\,x^k\left(\frac{x^n na}{x}+\frac{b\,x^m m}{x}\right)}{x\,(x^n a+b\,x^m+c)^2} + \frac{(-x^n a -}{x^2\,(x^n a +}\right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = {} & c_1 \mathrm{e}^{is\left(\int x^k\sqrt{-\frac{\lambda}{a^2x^{2n}+2a\,x^{m+n}b+2x^n ac+x^{2m}b^2+2bc\,x^m+c^2}}dx\right)} \\ & + c_2 \mathrm{e}^{-is\left(\int x^k\sqrt{-\frac{\lambda}{a^2x^{2n}+2a\,x^{m+n}b+2x^n ac+x^{2m}b^2+2bc\,x^m+c^2}}dx\right)} \end{aligned}$$

The above shows that

$$\begin{aligned} & u'(x) \\ & = is\,x^k\sqrt{-\frac{\lambda}{a^2x^{2n}+2a\,x^{m+n}b+2x^n ac+x^{2m}b^2+2bc\,x^m+c^2}}\left(c_1 \mathrm{e}^{is\left(\int x^k\sqrt{-\frac{\lambda}{a^2x^{2n}+2a\,x^{m+n}b+2x^n ac+x^{2m}b^2+2bc\,x^m+c^2}}dx\right)}\right. \\ & \left. \quad - c_2 \mathrm{e}^{-is\left(\int x^k\sqrt{-\frac{\lambda}{a^2x^{2n}+2a\,x^{m+n}b+2x^n ac+x^{2m}b^2+2bc\,x^m+c^2}}dx\right)}\right) \end{aligned}$$

Using the above in (1) gives the solution

$$y = \frac{i\sqrt{-\frac{\lambda}{a^{2} x^{2 n}+2 a \,x^{m+n} b+2 x^{n} a c+x^{2 m} b^{2}+2 b c \,x^{m}+c^{2}}}\, \left(c_{1} e^{i s \left(\int x^{k} \sqrt{-\frac{\lambda}{a^{2} x^{2 n}+2 a \,x^{m+n} b+2 x^{n} a c+x^{2 m} b^{2}+2 b c \,x^{m}+c^{2}}}d x \right)}-c_{2} e^{-i s \left(\int x^{k} \sqrt{-\cdots}\right.}\cdots\right.}{c_{1} e^{i s \left(\int x^{k} \sqrt{-\frac{\lambda}{a^{2} x^{2 n}+2 a \,x^{m+n} b+2 x^{n} a c+x^{2 m} b^{2}+2 b c \,x^{m}+c^{2}}}d x \right)}+c_{2} e^{-i s \left(\int x^{k} \sqrt{-\frac{\lambda}{a^{2} x^{2 n}+2 a \,x^{m-\cdots}}}\right.}\cdots}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{i\sqrt{-\frac{\lambda}{a^{2} x^{2 n}+2 a \,x^{m+n} b+2 x^{n} a c+x^{2 m} b^{2}+2 b c \,x^{m}+c^{2}}}\, \left(c_{3} e^{i s \left(\int x^{k} \sqrt{-\frac{\lambda}{a^{2} x^{2 n}+2 a \,x^{m+n} b+2 x^{n} a c+x^{2 m} b^{2}+2 b c \,x^{m}+c^{2}}}d x \right)}-e^{-i s \left(\int x^{k} \sqrt{-\frac{\cdots}{a\cdots}}\right.}\cdots\right.}{c_{3} e^{i s \left(\int x^{k} \sqrt{-\frac{\lambda}{a^{2} x^{2 n}+2 a \,x^{m+n} b+2 x^{n} a c+x^{2 m} b^{2}+2 b c \,x^{m}+c^{2}}}d x \right)}+e^{-i s \left(\int x^{k} \sqrt{-\frac{\cdots}{a^{2} x^{2 n}+2 a \,x^{m+n}\cdots}}\right.}\cdots}$$

Summary

The solution(s) found are the following

$$y = \frac{i\sqrt{-\frac{\lambda}{a^{2} x^{2 n}+2 a \,x^{m+n} b+2 x^{n} a c+x^{2 m} b^{2}+2 b c \,x^{m}+c^{2}}}\, \left(c_{3} e^{i s \left(\int x^{k} \sqrt{-\frac{\lambda}{a^{2} x^{2 n}+2 a \,x^{m+n} b+2 x^{n} a c+x^{2 m} b^{2}+2 b c \,x^{m}+c^{2}}}d x \right)}-e^{-i s \left(\int x^{k} \sqrt{-\frac{\cdots}{a\cdots}}\right.}\cdots\right.}{c_{3} e^{i s \left(\int x^{k} \sqrt{-\frac{\lambda}{a^{2} x^{2 n}+2 a \,x^{m+n} b+2 x^{n} a c+x^{2 m} b^{2}+2 b c \,x^{m}+c^{2}}}d x \right)}+e^{-i s \left(\int x^{k} \sqrt{-\frac{\cdots}{a^{2} x^{2 n}+2 a \,x^{m+n}\cdots}}\right.}\cdots} \tag{1}$$

Verification of solutions

$$y = \frac{i\sqrt{-\frac{\lambda}{a^{2} x^{2 n}+2 a \,x^{m+n} b+2 x^{n} a c+x^{2 m} b^{2}+2 b c \,x^{m}+c^{2}}}\, \left(c_{3} e^{i s \left(\int x^{k} \sqrt{-\frac{\lambda}{a^{2} x^{2 n}+2 a \,x^{m+n} b+2 x^{n} a c+x^{2 m} b^{2}+2 b c \,x^{m}+c^{2}}}d x \right)}-e^{-i s \left(\int x^{k} \sqrt{-\frac{\cdots}{a\cdots}}\right.}\cdots\right.}{c_{3} e^{i s \left(\int x^{k} \sqrt{-\frac{\lambda}{a^{2} x^{2 n}+2 a \,x^{m+n} b+2 x^{n} a c+x^{2 m} b^{2}+2 b c \,x^{m}+c^{2}}}d x \right)}+e^{-i s \left(\int x^{k} \sqrt{-\frac{\cdots}{a^{2} x^{2 n}+2 a \,x^{m+n}\cdots}}\right.}\cdots}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 37

```
dsolve((a*x^n+b*x^m+c)*(x*diff(y(x),x)-y(x))+s*x^k*(y(x)^2-lambda*x^2)=0,y(x), singsol=all)
```

$$y(x) = \tanh\left(s\sqrt{\lambda}\left(\int \frac{x^k}{a\,x^n + b\,x^m + c}dx + c_1\right)\right)x\sqrt{\lambda}$$

✓ Solution by Mathematica

Time used: 22.652 (sec). Leaf size: 53

```
DSolve[(a*x^n+b*x^m+c)*(x*y'[x]-y[x])+s*x^k*(y[x]^2-\[Lambda]*x^2)==0,y[x],x,IncludeSingular
```

$$y(x) \to \sqrt{\lambda}(-x)\tanh\left(\sqrt{\lambda}\left(\int_1^x -\frac{sK[1]^k}{bK[1]^m + aK[1]^n + c}dK[1] + c_1\right)\right)$$

3 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions

3.1 problem 1

Internal problem ID [10409]
Internal file name [OUTPUT/9356_Monday_June_06_2022_02_18_04_PM_49329698/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a y^2 = b\,\mathrm{e}^{\lambda x}}$$

3.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,y^2 + b\,\mathrm{e}^{\lambda x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,y^2 + b\,\mathrm{e}^{\lambda x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\,\mathrm{e}^{\lambda x}$, $f_1(x) = 0$ and $f_2(x) = a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= \mathrm{e}^{\lambda x}a^2b \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$au''(x) + \mathrm{e}^{\lambda x}a^2bu(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \,\mathrm{BesselJ}\left(0, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) + c_2 \,\mathrm{BesselY}\left(0, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)$$

The above shows that

$$u'(x) = \sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}\left(-\mathrm{BesselJ}\left(1, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)c_1 - \mathrm{BesselY}\left(1, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)c_2\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}\left(-\mathrm{BesselJ}\left(1, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)c_1 - \mathrm{BesselY}\left(1, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)c_2\right)}{\sqrt{a}\left(c_1 \,\mathrm{BesselJ}\left(0, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) + c_2 \,\mathrm{BesselY}\left(0, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}\left(\mathrm{BesselJ}\left(1, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)c_3 + \mathrm{BesselY}\left(1, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)\right)}{\sqrt{a}\left(c_3 \,\mathrm{BesselJ}\left(0, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) + \mathrm{BesselY}\left(0, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}} \left(\mathrm{BesselJ}\left(1, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) c_3 + \mathrm{BesselY}\left(1, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)\right)}{\sqrt{a}\, \left(c_3 \,\mathrm{BesselJ}\left(0, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) + \mathrm{BesselY}\left(0, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}} \left(\mathrm{BesselJ}\left(1, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) c_3 + \mathrm{BesselY}\left(1, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)\right)}{\sqrt{a}\, \left(c_3 \,\mathrm{BesselJ}\left(0, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) + \mathrm{BesselY}\left(0, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -a*b*exp(lambda*x)*y(x), y(x)`
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            <- Bessel successful
         <- special function solution successful
         Change of variables used:
            [x = ln(t)/lambda]
         Linear ODE actually solved:
            a*b*u(t)+lambda^2*diff(u(t),t)+lambda^2*t*diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 96

```
dsolve(diff(y(x),x)=a*y(x)^2+b*exp(lambda*x),y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{b}\,\mathrm{e}^{\frac{x\lambda}{2}} \left(\mathrm{BesselY}\left(1, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x\lambda}{2}}}{\lambda}\right) c_1 + \mathrm{BesselJ}\left(1, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x\lambda}{2}}}{\lambda}\right)\right)}{\sqrt{a}\,\left(c_1 \,\mathrm{BesselY}\left(0, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x\lambda}{2}}}{\lambda}\right) + \mathrm{BesselJ}\left(0, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x\lambda}{2}}}{\lambda}\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.551 (sec). Leaf size: 266

```
DSolve[y'[x]==a*y[x]^2+b*Exp[\[Lambda]*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{be^{\lambda x}}\left(2\,\mathrm{BesselY}\left(1, \frac{2\sqrt{a}\sqrt{be^{x\lambda}}}{\lambda}\right) + c_1\,\mathrm{BesselJ}\left(1, \frac{2\sqrt{a}\sqrt{be^{x\lambda}}}{\lambda}\right)\right)}{\sqrt{a}\left(2\,\mathrm{BesselY}\left(0, \frac{2\sqrt{a}\sqrt{be^{x\lambda}}}{\lambda}\right) + c_1\,\mathrm{BesselJ}\left(0, \frac{2\sqrt{a}\sqrt{be^{x\lambda}}}{\lambda}\right)\right)}$$

$$y(x) \to \frac{\sqrt{be^{\lambda x}}\,\mathrm{BesselJ}\left(1, \frac{2\sqrt{a}\sqrt{be^{x\lambda}}}{\lambda}\right)}{\sqrt{a}\,\mathrm{BesselJ}\left(0, \frac{2\sqrt{a}\sqrt{be^{x\lambda}}}{\lambda}\right)}$$

$$y(x) \to \frac{\sqrt{be^{\lambda x}}\,\mathrm{BesselJ}\left(1, \frac{2\sqrt{a}\sqrt{be^{x\lambda}}}{\lambda}\right)}{\sqrt{a}\,\mathrm{BesselJ}\left(0, \frac{2\sqrt{a}\sqrt{be^{x\lambda}}}{\lambda}\right)}$$

3.2 problem 2

Internal problem ID [10410]
Internal file name [OUTPUT/9357_Monday_June_06_2022_02_18_05_PM_73417862/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 = a\lambda\,\mathrm{e}^{\lambda x} - a^2\mathrm{e}^{2\lambda x}}$$

3.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + a\lambda\,\mathrm{e}^{\lambda x} - a^2\mathrm{e}^{2\lambda x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\lambda\,\mathrm{e}^{\lambda x} - a^2\mathrm{e}^{2\lambda x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\lambda\,\mathrm{e}^{\lambda x} - a^2\mathrm{e}^{2\lambda x}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= a\lambda\,\mathrm{e}^{\lambda x} - a^2\mathrm{e}^{2\lambda x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(a\lambda\,\mathrm{e}^{\lambda x} - a^2\mathrm{e}^{2\lambda x}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}} \left(c_1 + \mathrm{expIntegral}_1\left(-\frac{2a\,\mathrm{e}^{\lambda x}}{\lambda}\right) c_2\right)$$

The above shows that

$$u'(x) = -\left(\mathrm{e}^{\lambda x}\,\mathrm{expIntegral}_1\left(-\frac{2a\,\mathrm{e}^{\lambda x}}{\lambda}\right) c_2 a + \mathrm{e}^{\frac{2a\,\mathrm{e}^{\lambda x}}{\lambda}}\lambda c_2 + \mathrm{e}^{\lambda x} c_1 a\right) \mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}$$

Using the above in (1) gives the solution

$$y = \frac{\mathrm{e}^{\lambda x}\,\mathrm{expIntegral}_1\left(-\frac{2a\,\mathrm{e}^{\lambda x}}{\lambda}\right) c_2 a + \mathrm{e}^{\frac{2a\,\mathrm{e}^{\lambda x}}{\lambda}}\lambda c_2 + \mathrm{e}^{\lambda x} c_1 a}{c_1 + \mathrm{expIntegral}_1\left(-\frac{2a\,\mathrm{e}^{\lambda x}}{\lambda}\right) c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\mathrm{expIntegral}_1\left(-\frac{2a\,\mathrm{e}^{\lambda x}}{\lambda}\right) \mathrm{e}^{\lambda x} a + \mathrm{e}^{\frac{2a\,\mathrm{e}^{\lambda x}}{\lambda}}\lambda + \mathrm{e}^{\lambda x} c_3 a}{c_3 + \mathrm{expIntegral}_1\left(-\frac{2a\,\mathrm{e}^{\lambda x}}{\lambda}\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{\operatorname{expIntegral}_1\left(-\frac{2a\,e^{\lambda x}}{\lambda}\right) e^{\lambda x} a + e^{\frac{2a\,e^{\lambda x}}{\lambda}} \lambda + e^{\lambda x} c_3 a}{c_3 + \operatorname{expIntegral}_1\left(-\frac{2a\,e^{\lambda x}}{\lambda}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\operatorname{expIntegral}_1\left(-\frac{2a\,e^{\lambda x}}{\lambda}\right) e^{\lambda x} a + e^{\frac{2a\,e^{\lambda x}}{\lambda}} \lambda + e^{\lambda x} c_3 a}{c_3 + \operatorname{expIntegral}_1\left(-\frac{2a\,e^{\lambda x}}{\lambda}\right)}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-a*lambda*exp(lambda*x)+a^2*e
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
         <- Kovacics algorithm successful
         Change of variables used:
            [x = ln(t)/lambda]
         Linear ODE actually solved:
            (-a^2*t+a*lambda)*u(t)+lambda^2*diff(u(t),t)+lambda^2*t*diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 63

```
dsolve(diff(y(x),x)=y(x)^2+a*lambda*exp(lambda*x)-a^2*exp(2*lambda*x),y(x), singsol=all)
```

$$y(x) = \frac{e^{x\lambda} \operatorname{expIntegral}_1\left(-\frac{2a\,e^{x\lambda}}{\lambda}\right) c_1 a + e^{\frac{2a\,e^{x\lambda}}{\lambda}} c_1 \lambda + e^{x\lambda} a}{\operatorname{expIntegral}_1\left(-\frac{2a\,e^{x\lambda}}{\lambda}\right) c_1 + 1}$$

✓ Solution by Mathematica

Time used: 2.507 (sec). Leaf size: 79

```
DSolve[y'[x]==y[x]^2+a*\[Lambda]*Exp[\[Lambda]*x]-a^2*Exp[2*\[Lambda]*x],y[x],x,IncludeSingu
```

$$y(x) \to \frac{ae^{\lambda x} \operatorname{ExpIntegralEi}\left(\frac{2ae^{x\lambda}}{\lambda}\right) + \lambda\left(-e^{\frac{2ae^{\lambda x}}{\lambda}}\right) + ac_1 e^{\lambda x}}{\operatorname{ExpIntegralEi}\left(\frac{2ae^{x\lambda}}{\lambda}\right) + c_1}$$

$$y(x) \to ae^{\lambda x}$$

3.3 problem 3

Internal problem ID [10411]
Internal file name [`OUTPUT/9358_Monday_June_06_2022_02_18_06_PM_56615204/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - \sigma y^2 = a + b\,\mathrm{e}^{\lambda x} + c\,\mathrm{e}^{2\lambda x}}$$

3.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \sigma\, y^2 + a + b\,\mathrm{e}^{\lambda x} + c\,\mathrm{e}^{2\lambda x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \sigma\, y^2 + a + b\,\mathrm{e}^{\lambda x} + c\,\mathrm{e}^{2\lambda x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a + b\,\mathrm{e}^{\lambda x} + c\,\mathrm{e}^{2\lambda x}$, $f_1(x) = 0$ and $f_2(x) = \sigma$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\sigma u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \sigma^2 (a + b\,\mathrm{e}^{\lambda x} + c\,\mathrm{e}^{2\lambda x}) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\sigma u''(x) + \sigma^2 (a + b\,\mathrm{e}^{\lambda x} + c\,\mathrm{e}^{2\lambda x}) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{-\frac{\lambda x}{2}} \left(\mathrm{WhittakerM}\left(-\frac{i\sqrt{\sigma}\, b}{2\lambda\sqrt{c}}, \frac{i\sqrt{\sigma}\,\sqrt{a}}{\lambda}, \frac{2i\sqrt{\sigma}\,\sqrt{c}\,\mathrm{e}^{\lambda x}}{\lambda} \right) c_1 + \mathrm{WhittakerW}\left(-\frac{i\sqrt{\sigma}\, b}{2\lambda\sqrt{c}}, \frac{i\sqrt{\sigma}\,\sqrt{a}}{\lambda}, \frac{2i\sqrt{\sigma}\,\sqrt{c}\,\mathrm{e}^{\lambda x}}{\lambda} \right) c_2 \right)$$

The above shows that

$$u'(x) = \frac{\mathrm{e}^{-\frac{\lambda x}{2}} \left(c_1 \,\mathrm{WhittakerM}\left(-\frac{i\sqrt{\sigma}\, b - 2\lambda\sqrt{c}}{2\lambda\sqrt{c}}, \frac{i\sqrt{\sigma}\,\sqrt{a}}{\lambda}, \frac{2i\sqrt{\sigma}\,\sqrt{c}\,\mathrm{e}^{\lambda x}}{\lambda} \right) \left(i\left(\sqrt{c}\,\sqrt{a} - \frac{b}{2}\right)\sqrt{\sigma} + \frac{\lambda\sqrt{c}}{2} \right) - \mathrm{WhittakerW}\left(-\frac{i\sqrt{}}{} \right.\right.}{}$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 \,\mathrm{WhittakerM}\left(-\frac{i\sqrt{\sigma}\, b - 2\lambda\sqrt{c}}{2\lambda\sqrt{c}}, \frac{i\sqrt{\sigma}\,\sqrt{a}}{\lambda}, \frac{2i\sqrt{\sigma}\,\sqrt{c}\,\mathrm{e}^{\lambda x}}{\lambda} \right) \left(i\left(\sqrt{c}\,\sqrt{a} - \frac{b}{2}\right)\sqrt{\sigma} + \frac{\lambda\sqrt{c}}{2} \right) - \mathrm{WhittakerW}\left(-\frac{i\sqrt{\sigma}\, b -}{2\lambda\sqrt{}} \right.}{\sqrt{c}\,\sigma \left(\mathrm{WhittakerM}\left(- \right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 \,\mathrm{WhittakerM}\left(-\frac{i\sqrt{\sigma}\, b - 2\lambda\sqrt{c}}{2\lambda\sqrt{c}}, \frac{i\sqrt{\sigma}\,\sqrt{a}}{\lambda}, \frac{2i\sqrt{\sigma}\,\sqrt{c}\,\mathrm{e}^{\lambda x}}{\lambda} \right) \left(i\left(\sqrt{c}\,\sqrt{a} - \frac{b}{2}\right)\sqrt{\sigma} + \frac{\lambda\sqrt{c}}{2} \right) + \lambda \,\mathrm{WhittakerW}\left(-\frac{i\sqrt{\sigma}\, b}{2\lambda} \right.}{\sqrt{c}\,\sigma \left(\mathrm{WhittakerM}\left(-\frac{i}{2\lambda} \right.\right.}$$

Summary
The solution(s) found are the following

$$y = \frac{-c_3 \operatorname{WhittakerM}\left(-\frac{i\sqrt{\sigma}\, b-2\lambda\sqrt{c}}{2\lambda\sqrt{c}}, \frac{i\sqrt{\sigma}\,\sqrt{a}}{\lambda}, \frac{2i\sqrt{\sigma}\,\sqrt{c}\,\mathrm{e}^{\lambda x}}{\lambda}\right)\left(i\left(\sqrt{c}\,\sqrt{a}-\frac{b}{2}\right)\sqrt{\sigma}+\frac{\lambda\sqrt{c}}{2}\right)+\lambda \operatorname{WhittakerW}\left(-\frac{i\sqrt{\sigma}\, b}{2\lambda}\right.}{\sqrt{c}\,\sigma\left(\operatorname{WhittakerM}\left(-\frac{i\sqrt{}}{2\lambda}\right.\right.} \tag{1}$$

Verification of solutions

$$y = \frac{-c_3 \operatorname{WhittakerM}\left(-\frac{i\sqrt{\sigma}\, b-2\lambda\sqrt{c}}{2\lambda\sqrt{c}}, \frac{i\sqrt{\sigma}\,\sqrt{a}}{\lambda}, \frac{2i\sqrt{\sigma}\,\sqrt{c}\,\mathrm{e}^{\lambda x}}{\lambda}\right)\left(i\left(\sqrt{c}\,\sqrt{a}-\frac{b}{2}\right)\sqrt{\sigma}+\frac{\lambda\sqrt{c}}{2}\right)+\lambda \operatorname{WhittakerW}\left(-\frac{i\sqrt{\sigma}\, b}{2\lambda}\right.}{\sqrt{c}\,\sigma\left(\operatorname{WhittakerM}\left(-\frac{i\sqrt{}}{2\lambda}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -sigma*(a+b*exp(lambda*x)+c*ex
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
               <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
            <- Whittaker successful
         <- special function solution successful
         Change of variables used:
            [x = ln(t)/lambda]
         Linear ODE actually solved:
            (c*sigma*t^2+b*sigma*t+a*sigma)*u(t)+lambda^2*t*diff(u(t),t)+lambda^2*t^2*diff(d
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 348

```
dsolve(diff(y(x),x)=sigma*y(x)^2+a+b*exp(lambda*x)+c*exp(2*lambda*x),y(x), singsol=all)
```

$$y(x) = \frac{-\operatorname{WhittakerM}\left(-\frac{i\sqrt{\sigma}\,b-2\lambda\sqrt{c}}{2\lambda\sqrt{c}}, \frac{i\sqrt{a}\,\sqrt{\sigma}}{\lambda}, \frac{2i\sqrt{\sigma}\,\sqrt{c}\,e^{x\lambda}}{\lambda}\right)\left(i\left(\sqrt{c}\,\sqrt{a}-\frac{b}{2}\right)\sqrt{\sigma}+\frac{\lambda\sqrt{c}}{2}\right)+\lambda c_1 \operatorname{WhittakerW}\left(-\frac{i\sqrt{\sigma}\,b \cdots}{2\lambda \cdots}\right.}{\sqrt{c}\,\sigma\left(\operatorname{WhittakerW}\left(-\frac{i\cdots}{2\lambda\cdots}\right.\right.}$$

✓ Solution by Mathematica

Time used: 3.251 (sec). Leaf size: 1081

```
DSolve[y'[x]==sigma*y[x]^2+a+b*Exp[\[Lambda]*x]+c*Exp[2*\[Lambda]*x],y[x],x,IncludeSingularS
```

$$y(x) \to -\frac{i\left(c_1\lambda\left(\sqrt{a}-\sqrt{c}e^{\lambda x}\right)\operatorname{HypergeometricU}\left(\frac{\frac{i\sqrt{\sigma}b}{\sqrt{c}}+\lambda+2i\sqrt{a}\sqrt{\sigma}}{2\lambda}, \frac{2i\sqrt{a}\sqrt{\sigma}}{\lambda}+1, \frac{2i\sqrt{c}e^{x\lambda}\sqrt{\sigma}}{\lambda}\right)-ic_1e^{\lambda x}\left(b\sqrt{\sigma}+\sqrt{c}(\cdots\right.\right.}{\lambda\sqrt{\sigma}\left(c_1 \operatorname{Hyperge}\cdots\right.}$$

$$y(x) \to \frac{-\frac{e^{\lambda x}\left(b\sqrt{\sigma}+\sqrt{c}\left(2\sqrt{a}\sqrt{\sigma}-i\lambda\right)\right)\operatorname{HypergeometricU}\left(\frac{\frac{i\sqrt{\sigma}b}{\sqrt{c}}+3\lambda+2i\sqrt{a}\sqrt{\sigma}}{2\lambda}, \frac{2i\sqrt{a}\sqrt{\sigma}}{\lambda}+2, \frac{2i\sqrt{c}e^{x\lambda}\sqrt{\sigma}}{\lambda}\right)}{\lambda\operatorname{HypergeometricU}\left(\frac{\frac{i\sqrt{\sigma}b}{\sqrt{c}}+\lambda+2i\sqrt{a}\sqrt{\sigma}}{2\lambda}, \frac{2i\sqrt{a}\sqrt{\sigma}}{\lambda}+1, \frac{2i\sqrt{c}e^{x\lambda}\sqrt{\sigma}}{\lambda}\right)}-i\left(\sqrt{a}-\sqrt{c}e^{\lambda x}\right)}{\sqrt{\sigma}}$$

$$y(x) \to \frac{-\frac{e^{\lambda x}\left(b\sqrt{\sigma}+\sqrt{c}\left(2\sqrt{a}\sqrt{\sigma}-i\lambda\right)\right)\operatorname{HypergeometricU}\left(\frac{\frac{i\sqrt{\sigma}b}{\sqrt{c}}+3\lambda+2i\sqrt{a}\sqrt{\sigma}}{2\lambda}, \frac{2i\sqrt{a}\sqrt{\sigma}}{\lambda}+2, \frac{2i\sqrt{c}e^{x\lambda}\sqrt{\sigma}}{\lambda}\right)}{\lambda\operatorname{HypergeometricU}\left(\frac{\frac{i\sqrt{\sigma}b}{\sqrt{c}}+\lambda+2i\sqrt{a}\sqrt{\sigma}}{2\lambda}, \frac{2i\sqrt{a}\sqrt{\sigma}}{\lambda}+1, \frac{2i\sqrt{c}e^{x\lambda}\sqrt{\sigma}}{\lambda}\right)}-i\left(\sqrt{a}-\sqrt{c}e^{\lambda x}\right)}{\sqrt{\sigma}}$$

3.4 problem 4

Internal problem ID [10412]
Internal file name [`OUTPUT/9359_Monday_June_06_2022_02_18_10_PM_32604344/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \sigma y^2 - a y = b\,\mathrm{e}^{x} + c}$$

3.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \sigma\,y^2 + ya + b\,\mathrm{e}^{x} + c \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \sigma\,y^2 + ya + b\,\mathrm{e}^{x} + c$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\,\mathrm{e}^{x} + c$, $f_1(x) = a$ and $f_2(x) = \sigma$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\sigma u} \qquad (1) \end{aligned}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \sigma a \\ f_2^2 f_0 &= \sigma^2 (b\, \mathrm{e}^x + c) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\sigma u''(x) - \sigma a u'(x) + \sigma^2 (b\,\mathrm{e}^x + c)\, u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\frac{xa}{2}} \left(\mathrm{BesselJ}\left(\sqrt{a^2 - 4\sigma c}, 2\sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\right) c_1 + \mathrm{BesselY}\left(\sqrt{a^2 - 4\sigma c}, 2\sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\right) c_2\right)$$

The above shows that

$$\begin{aligned} u'(x) = &-c_1 \sqrt{b}\,\sqrt{\sigma}\,\mathrm{e}^{\frac{(1+a)x}{2}}\,\mathrm{BesselJ}\left(\sqrt{a^2 - 4\sigma c} + 1, 2\sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\right) \\ &- c_2 \sqrt{b}\,\sqrt{\sigma}\,\mathrm{e}^{\frac{(1+a)x}{2}}\,\mathrm{BesselY}\left(\sqrt{a^2 - 4\sigma c} + 1, 2\sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\right) \\ &+ \frac{\mathrm{e}^{\frac{xa}{2}} \left(\mathrm{BesselJ}\left(\sqrt{a^2 - 4\sigma c}, 2\sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\right) c_1 + \mathrm{BesselY}\left(\sqrt{a^2 - 4\sigma c}, 2\sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\right) c_2\right) \left(\sqrt{a^2 - 4\sigma c} + a\right)}{2} \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-c_1 \sqrt{b}\,\sqrt{\sigma}\,\mathrm{e}^{\frac{(1+a)x}{2}}\,\mathrm{BesselJ}\left(\sqrt{a^2 - 4\sigma c} + 1, 2\sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\right) - c_2 \sqrt{b}\,\sqrt{\sigma}\,\mathrm{e}^{\frac{(1+a)x}{2}}\,\mathrm{BesselY}\left(\sqrt{a^2 - 4\sigma c} + 1\right.\right.}{\sigma \left(\mathrm{BesselJ}\left(\sqrt{a^2 - 4\sigma c}, 2\sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\right) c_1 + \mathrm{Be}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2c_3 \sqrt{b}\,\sqrt{\sigma}\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{BesselJ}\left(\sqrt{a^2 - 4\sigma c} + 1, 2\sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\right) + 2\,\mathrm{BesselY}\left(\sqrt{a^2 - 4\sigma c} + 1, 2\sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\right) \sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}}{2\sigma \left(\mathrm{BesselJ}\left(\sqrt{a^2 - 4\sigma c}, 2\sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\right) c_3 + \right.}$$

Summary
The solution(s) found are the following

$$y = \frac{2c_3\sqrt{b}\,\sqrt{\sigma}\,e^{\frac{x}{2}}\,\text{BesselJ}\left(\sqrt{a^2-4\sigma c}+1, 2\sqrt{\sigma}\,\sqrt{b}\,e^{\frac{x}{2}}\right) + 2\,\text{BesselY}\left(\sqrt{a^2-4\sigma c}+1, 2\sqrt{\sigma}\,\sqrt{b}\,e^{\frac{x}{2}}\right)\sqrt{\sigma}\,\sqrt{b}\,e^{\frac{x}{2}}}{2\sigma\left(\text{BesselJ}\left(\sqrt{a^2-4\sigma c}, 2\sqrt{\sigma}\,\sqrt{b}\,e^{\frac{x}{2}}\right)c_3 +\right.} \tag{1}$$

Verification of solutions

$$y = \frac{2c_3\sqrt{b}\,\sqrt{\sigma}\,e^{\frac{x}{2}}\,\text{BesselJ}\left(\sqrt{a^2-4\sigma c}+1, 2\sqrt{\sigma}\,\sqrt{b}\,e^{\frac{x}{2}}\right) + 2\,\text{BesselY}\left(\sqrt{a^2-4\sigma c}+1, 2\sqrt{\sigma}\,\sqrt{b}\,e^{\frac{x}{2}}\right)\sqrt{\sigma}\,\sqrt{b}\,e^{\frac{x}{2}}}{2\sigma\left(\text{BesselJ}\left(\sqrt{a^2-4\sigma c}, 2\sqrt{\sigma}\,\sqrt{b}\,e^{\frac{x}{2}}\right)c_3 +\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (diff(y(x), x))*a-sigma*(b*exp
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            <- Bessel successful
         <- special function solution successful
         Change of variables used:
            [x = ln(t)]
         Linear ODE actually solved:
            (b*sigma*t+c*sigma)*u(t)+(-a*t+t)*diff(u(t),t)+t^2*diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 200

```
dsolve(diff(y(x),x)=sigma*y(x)^2+a*y(x)+b*exp(x)+c,y(x), singsol=all)
```

$$y(x) = -\frac{-2\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{BesselJ}\left(\sqrt{a^2-4\sigma c}+1, 2\sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\right)\sigma - 2\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{BesselY}\left(\sqrt{a^2-4\sigma c}+1, 2\sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\right)c_1\sigma}{\sigma^{\frac{3}{2}}\left(2\,\mathrm{BesselY}\left(\sqrt{a^2-4\sigma c}, 2\sqrt{\sigma}\,\sqrt{b}\,\mathrm{e}^{\frac{x}{2}}\right)c\right.}$$

✓ Solution by Mathematica

Time used: 0.971 (sec). Leaf size: 546

```
DSolve[y'[x]==sigma*y[x]^2+a*y[x]+b*Exp[x]+c,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{a\sqrt{b\sigma e^x}\,\mathrm{Gamma}\left(\sqrt{a^2-4c\sigma}+1\right)\mathrm{BesselJ}\left(\sqrt{a^2-4c\sigma}, 2\sqrt{be^x\sigma}\right) + b\sigma e^x\,\mathrm{Gamma}\left(\sqrt{a^2-4c\sigma}+1\right)\mathrm{Be}}{}$$

$$y(x) \to \frac{\frac{\sqrt{b\sigma e^x}\left(\mathrm{BesselJ}\left(1-\sqrt{a^2-4c\sigma}, 2\sqrt{be^x\sigma}\right) - \mathrm{BesselJ}\left(-\sqrt{a^2-4c\sigma}-1, 2\sqrt{be^x\sigma}\right)\right)}{\mathrm{BesselJ}\left(-\sqrt{a^2-4c\sigma}, 2\sqrt{be^x\sigma}\right)} - a}{2\sigma}$$

3.5 problem 5

Internal problem ID [10413]
Internal file name [`OUTPUT/9360_Monday_June_06_2022_02_18_11_PM_50198122/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - yb = a(\lambda - b)\,\mathrm{e}^{\lambda x} - a^2\mathrm{e}^{2\lambda x}}$$

3.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -a\,\mathrm{e}^{\lambda x}b + a\lambda\,\mathrm{e}^{\lambda x} - a^2\mathrm{e}^{2\lambda x} + by + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a\,\mathrm{e}^{\lambda x}b + a\lambda\,\mathrm{e}^{\lambda x} - a^2\mathrm{e}^{2\lambda x} + by + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a\,\mathrm{e}^{\lambda x}b + a\lambda\,\mathrm{e}^{\lambda x} - a^2\mathrm{e}^{2\lambda x}$, $f_1(x) = b$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= b \\ f_2^2 f_0 &= -a\,\mathrm{e}^{\lambda x} b + a\lambda\,\mathrm{e}^{\lambda x} - a^2 \mathrm{e}^{2\lambda x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - bu'(x) + \left(-a\,\mathrm{e}^{\lambda x} b + a\lambda\,\mathrm{e}^{\lambda x} - a^2 \mathrm{e}^{2\lambda x}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\left(\int \mathrm{e}^{\frac{b\lambda x + 2\,\mathrm{e}^{\lambda x} a}{\lambda}} dx\right) c_1 + c_2\right) \mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}$$

The above shows that

$$u'(x) = -a\left(\left(\int \mathrm{e}^{\frac{b\lambda x + 2\,\mathrm{e}^{\lambda x} a}{\lambda}} dx\right) c_1 + c_2\right) \mathrm{e}^{\frac{\lambda^2 x - \mathrm{e}^{\lambda x} a}{\lambda}} + c_1 \mathrm{e}^{\frac{b\lambda x + \mathrm{e}^{\lambda x} a}{\lambda}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-a\left(\left(\int \mathrm{e}^{\frac{b\lambda x + 2\,\mathrm{e}^{\lambda x} a}{\lambda}} dx\right) c_1 + c_2\right) \mathrm{e}^{\frac{\lambda^2 x - \mathrm{e}^{\lambda x} a}{\lambda}} + c_1 \mathrm{e}^{\frac{b\lambda x + \mathrm{e}^{\lambda x} a}{\lambda}}\right) \mathrm{e}^{\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}}{\left(\int \mathrm{e}^{\frac{b\lambda x + 2\,\mathrm{e}^{\lambda x} a}{\lambda}} dx\right) c_1 + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\mathrm{e}^{\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}} \left(a\left(\left(\int \mathrm{e}^{\frac{b\lambda x + 2\,\mathrm{e}^{\lambda x} a}{\lambda}} dx\right) c_3 + 1\right) \mathrm{e}^{\frac{\lambda^2 x - \mathrm{e}^{\lambda x} a}{\lambda}} - c_3 \mathrm{e}^{\frac{b\lambda x + \mathrm{e}^{\lambda x} a}{\lambda}}\right)}{\left(\int \mathrm{e}^{\frac{b\lambda x + 2\,\mathrm{e}^{\lambda x} a}{\lambda}} dx\right) c_3 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{\frac{a\,e^{\lambda x}}{\lambda}} \left(a\left(\left(\int e^{\frac{b\lambda x+2\,e^{\lambda x} a}{\lambda}}dx\right) c_3 + 1\right) e^{\frac{\lambda^2 x - e^{\lambda x} a}{\lambda}} - c_3 e^{\frac{b\lambda x + e^{\lambda x} a}{\lambda}}\right)}{\left(\int e^{\frac{b\lambda x+2\,e^{\lambda x} a}{\lambda}}dx\right) c_3 + 1} \tag{1}$$

Verification of solutions

$$y = \frac{e^{\frac{a\,e^{\lambda x}}{\lambda}} \left(a\left(\left(\int e^{\frac{b\lambda x+2\,e^{\lambda x} a}{\lambda}}dx\right) c_3 + 1\right) e^{\frac{\lambda^2 x - e^{\lambda x} a}{\lambda}} - c_3 e^{\frac{b\lambda x + e^{\lambda x} a}{\lambda}}\right)}{\left(\int e^{\frac{b\lambda x+2\,e^{\lambda x} a}{\lambda}}dx\right) c_3 + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (diff(y(x), x))*b+(a*b*exp(lam
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      <- linear_1 successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 80

```
dsolve(diff(y(x),x)=y(x)^2+b*y(x)+a*(lambda-b)*exp(lambda*x)-a^2*exp(2*lambda*x),y(x), sings
```

$$y(x) = \frac{e^{x\lambda} a \left(\int e^{\frac{b\lambda x+2\,e^{x\lambda} a}{\lambda}} dx\right) + e^{x\lambda} c_1 a - e^{\frac{b\lambda x+2\,e^{x\lambda} a}{\lambda}}}{\int e^{\frac{b\lambda x+2\,e^{x\lambda} a}{\lambda}} dx + c_1}$$

✓ Solution by Mathematica

Time used: 3.226 (sec). Leaf size: 191

```
DSolve[y'[x]==y[x]^2+b*y[x]+a*(\[Lambda]-b)*Exp[\[Lambda]*x]-a^2*Exp[2*\[Lambda]*x],y[x],x,I
```

$$y(x) \to \frac{-2^{b/\lambda}\left(b - ae^{\lambda x}\right)\left(\frac{ae^{\lambda x}}{\lambda}\right)^{b/\lambda} L_{-\frac{b}{\lambda}}^{\frac{b}{\lambda}}\left(\frac{2ae^{x\lambda}}{\lambda}\right) + ae^{\lambda x}\left(2^{\frac{b+\lambda}{\lambda}}\left(\frac{ae^{\lambda x}}{\lambda}\right)^{b/\lambda} L_{-\frac{b+\lambda}{\lambda}}^{\frac{b+\lambda}{\lambda}}\left(\frac{2ae^{x\lambda}}{\lambda}\right) + c_1\right)}{2^{b/\lambda}\left(\frac{ae^{\lambda x}}{\lambda}\right)^{b/\lambda} L_{-\frac{b}{\lambda}}^{\frac{b}{\lambda}}\left(\frac{2ae^{x\lambda}}{\lambda}\right) + c_1}$$

$$y(x) \to ae^{\lambda x}$$

3.6 problem 6

Internal problem ID [10414]
Internal file name [OUTPUT/9361_Monday_June_06_2022_02_18_12_PM_24173088/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 - a\,\mathrm{e}^{\lambda x} y = -a\,\mathrm{e}^{\lambda x} b - b^2}$$

3.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + a\,\mathrm{e}^{\lambda x} y - a\,\mathrm{e}^{\lambda x} b - b^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\,\mathrm{e}^{\lambda x} y - a\,\mathrm{e}^{\lambda x} b - b^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a\,\mathrm{e}^{\lambda x} b - b^2$, $f_1(x) = \mathrm{e}^{\lambda x} a$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= \mathrm{e}^{\lambda x}a \\ f_2^2f_0 &= -a\,\mathrm{e}^{\lambda x}b - b^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \mathrm{e}^{\lambda x}au'(x) + \left(-a\,\mathrm{e}^{\lambda x}b - b^2\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\frac{-\lambda^2x+\mathrm{e}^{\lambda x}a}{2\lambda}}\left(\mathrm{WhittakerM}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)c_1 + \mathrm{WhittakerW}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)c_2\right)$$

The above shows that

$$u'(x) = 2\left(c_2\left(\mathrm{e}^{\lambda x}a + \frac{3b}{2} - \lambda\right)\mathrm{WhittakerW}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right) - \frac{bc_1\left(-2\,\mathrm{e}^{\frac{a\,\mathrm{e}^{\lambda x}}{2\lambda}}\left(\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)^{\frac{2b+\lambda}{2\lambda}} + \mathrm{WhittakerM}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)\right)}{2}\right)\mathrm{e}^{\frac{-\lambda^2x+\mathrm{e}^{\lambda x}a}{2\lambda}}$$

Using the above in (1) gives the solution

$$y = -\frac{2\left(c_2\left(\mathrm{e}^{\lambda x}a + \frac{3b}{2} - \lambda\right)\mathrm{WhittakerW}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right) - \frac{bc_1\left(-2\,\mathrm{e}^{\frac{a\,\mathrm{e}^{\lambda x}}{2\lambda}}\left(\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)^{\frac{2b+\lambda}{2\lambda}} + \mathrm{WhittakerM}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)\right)}{2}\right)}{\mathrm{WhittakerM}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)c_1 + \mathrm{WhittakerW}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-2\,\mathrm{e}^{\lambda x} a - 3b + 2\lambda\right) \mathrm{WhittakerW}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right) - 2c_3\left(\mathrm{e}^{\frac{a\,\mathrm{e}^{\lambda x}}{2\lambda}} \left(\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)^{\frac{2b+\lambda}{2\lambda}} - \frac{\mathrm{WhittakerM}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)}{2}\right)}{\mathrm{WhittakerM}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right) c_3 + \mathrm{WhittakerW}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(-2\,\mathrm{e}^{\lambda x} a - 3b + 2\lambda\right) \mathrm{WhittakerW}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right) - 2c_3\left(\mathrm{e}^{\frac{a\,\mathrm{e}^{\lambda x}}{2\lambda}} \left(\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)^{\frac{2b+\lambda}{2\lambda}} - \frac{\mathrm{WhittakerM}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)}{2}\right)}{\mathrm{WhittakerM}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right) c_3 + \mathrm{WhittakerW}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(-2\,\mathrm{e}^{\lambda x} a - 3b + 2\lambda\right) \mathrm{WhittakerW}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right) - 2c_3\left(\mathrm{e}^{\frac{a\,\mathrm{e}^{\lambda x}}{2\lambda}} \left(\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)^{\frac{2b+\lambda}{2\lambda}} - \frac{\mathrm{WhittakerM}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)}{2}\right)}{\mathrm{WhittakerM}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right) c_3 + \mathrm{WhittakerW}\left(\frac{-2b+\lambda}{2\lambda}, \frac{b}{\lambda}, \frac{a\,\mathrm{e}^{\lambda x}}{\lambda}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 73

```
dsolve(diff(y(x),x)=y(x)^2+a*exp(lambda*x)*y(x)-a*b*exp(lambda*x)-b^2,y(x), singsol=all)
```

$$y(x) = \frac{-b\left(\int \mathrm{e}^{\frac{2b\lambda x+\mathrm{e}^{x\lambda}a}{\lambda}}dx\right)+c_1b+\mathrm{e}^{\frac{2b\lambda x+\mathrm{e}^{x\lambda}a}{\lambda}}}{-\left(\int \mathrm{e}^{\frac{2b\lambda x+\mathrm{e}^{x\lambda}a}{\lambda}}dx\right)+c_1}$$

✓ Solution by Mathematica

Time used: 0.944 (sec). Leaf size: 115

```
DSolve[y'[x]==y[x]^2+a*Exp[\[Lambda]*x]*y[x]-a*b*Exp[\[Lambda]*x]-b^2,y[x],x,IncludeSingular
```

$$y(x) \to \frac{b\left(-2\lambda e^{\frac{ae^{\lambda x}}{\lambda}}\left(-\frac{ae^{\lambda x}}{\lambda}\right)^{\frac{2b}{\lambda}}+2b\Gamma\left(\frac{2b}{\lambda},0,-\frac{ae^{x\lambda}}{\lambda}\right)+c_1\lambda(-1)^{b/\lambda}\right)}{2b\Gamma\left(\frac{2b}{\lambda},0,-\frac{ae^{x\lambda}}{\lambda}\right)+c_1\lambda(-1)^{b/\lambda}}$$

$$y(x) \to b$$

3.7 problem 7

Internal problem ID [10415]
Internal file name [`OUTPUT/9362_Monday_June_06_2022_02_18_15_PM_89074729/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = a\,\mathrm{e}^{2\lambda x}\left(\mathrm{e}^{\lambda x} + b\right)^n - \frac{\lambda^2}{4}}$$

3.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + a\,\mathrm{e}^{2\lambda x}\left(\mathrm{e}^{\lambda x} + b\right)^n - \frac{\lambda^2}{4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\,\mathrm{e}^{2\lambda x}\left(\mathrm{e}^{\lambda x} + b\right)^n - \frac{\lambda^2}{4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\,\mathrm{e}^{2\lambda x}\left(\mathrm{e}^{\lambda x} + b\right)^n - \frac{\lambda^2}{4}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= a\, e^{2\lambda x} \left(e^{\lambda x} + b\right)^n - \frac{\lambda^2}{4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(a\, e^{2\lambda x} \left(e^{\lambda x} + b\right)^n - \frac{\lambda^2}{4}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{-\csc\left(\frac{\pi(n+3)}{2+n}\right) c_1 \operatorname{BesselI}\left(-\frac{1}{2+n}, 2\sqrt{-\frac{a\left(e^{\lambda x}+b\right)^{2+n}}{\lambda^2(2+n)^2}}\right) \pi \left(-\frac{a\left(e^{\lambda x}+b\right)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{1}{4+2n}} e^{-\frac{\lambda x}{2}} + \Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{a\left(e^{\lambda x}+b\right)^{2+n}}{\lambda^2(2+n)^2}\right.}{(2+n)\,\Gamma\left(\frac{n+3}{2+n}\right)}$$

The above shows that

$$u'(x) = -\frac{\lambda\left(-2\Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{a\left(e^{\lambda x}+b\right)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} c_2 (2+n)^2 \left(e^{\frac{\lambda x}{2}} b + e^{\frac{3\lambda x}{2}}\right) \operatorname{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{a\left(e^{\lambda x}+b\right)^{2+n}}{\lambda^2(2+n)^2}}\right) + \Gamma\left(\frac{n+3}{2+n}\right.\right.}{}$$

Using the above in (1) gives the solution

$$y = \frac{\lambda\left(-2\Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{a\left(e^{\lambda x}+b\right)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} c_2 (2+n)^2 \left(e^{\frac{\lambda x}{2}} b + e^{\frac{3\lambda x}{2}}\right) \operatorname{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{a\left(e^{\lambda x}+b\right)^{2+n}}{\lambda^2(2+n)^2}}\right) + \Gamma\left(\frac{n+3}{2+n}\right)^2\right.}{2\left(e^{\lambda x}+b\right)\left(-\csc\left(\frac{\pi(n}{2}\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-2\Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{a\left(e^{\lambda x}+b\right)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} (2+n)^2 \left(b\,e^{\lambda x}+e^{2\lambda x}\right) \mathrm{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{a\left(e^{\lambda x}+b\right)^{2+n}}{\lambda^2(2+n)^2}}\right) + \Gamma\left(\frac{n+3}{2+n}\right)^2 \left(- \cdots\right.\right.}{2\left(e^{\lambda x}+b\right)\left(-c_3 \csc\left(\frac{\pi(\cdots}{2\cdots}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-2\Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{a\left(e^{\lambda x}+b\right)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} (2+n)^2 \left(b\,e^{\lambda x}+e^{2\lambda x}\right) \mathrm{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{a\left(e^{\lambda x}+b\right)^{2+n}}{\lambda^2(2+n)^2}}\right) + \Gamma\left(\frac{n+3}{2+n}\right)^2 \left(- \cdots\right.\right.}{2\left(e^{\lambda x}+b\right)\left(-c_3 \csc\left(\frac{\pi(\cdots}{2\cdots}\right.\right.} \quad (1)$$

Verification of solutions

$$y = \frac{\left(-2\Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{a\left(e^{\lambda x}+b\right)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} (2+n)^2 \left(b\,e^{\lambda x}+e^{2\lambda x}\right) \mathrm{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{a\left(e^{\lambda x}+b\right)^{2+n}}{\lambda^2(2+n)^2}}\right) + \Gamma\left(\frac{n+3}{2+n}\right)^2 \left(- \cdots\right.\right.}{2\left(e^{\lambda x}+b\right)\left(-c_3 \csc\left(\frac{\pi(\cdots}{2\cdots}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-a*exp(2*lambda*x)*(exp(lambd
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying an equivalence, under non-integer power transformations,
            to LODEs admitting Liouvillian solutions.
            -> Trying a Liouvillian solution using Kovacics algorithm
            <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
               <- hyper3 successful: received ODE is equivalent to the 0F1 ODE
            <- Whittaker successful
         <- special function solution successful
         Change of variables used:
            [x = ln(t)/lambda]
         Linear ODE actually solved:
            (4*a*t^2*(t+b)^n-lambda^2)*u(t)+4*lambda^2*t*diff(u(t),t)+4*lambda^2*t^2*diff(di
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 1342

```
dsolve(diff(y(x),x)=y(x)^2+a*exp(2*lambda*x)*(exp(lambda*x)+b)^n-1/4*lambda^2,y(x), singsol=
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+a*Exp[2*\[Lambda]*x]*(Exp[\[Lambda]*x]+b)^n-1/4*\[Lambda]^2,y[x],x,Incl
```

Not solved

3.8 problem 8

Internal problem ID [10416]
Internal file name [`OUTPUT/9363_Monday_June_06_2022_02_18_57_PM_81881186/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = a\,e^{8\lambda x} + b\,e^{6\lambda x} + c\,e^{4\lambda x} - \lambda^2}$$

3.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + a\,e^{8\lambda x} + b\,e^{6\lambda x} + c\,e^{4\lambda x} - \lambda^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\,e^{8\lambda x} + b\,e^{6\lambda x} + c\,e^{4\lambda x} - \lambda^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\,e^{8\lambda x} + b\,e^{6\lambda x} + c\,e^{4\lambda x} - \lambda^2$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= a\,\mathrm{e}^{8\lambda x} + b\,\mathrm{e}^{6\lambda x} + c\,\mathrm{e}^{4\lambda x} - \lambda^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(a\,\mathrm{e}^{8\lambda x} + b\,\mathrm{e}^{6\lambda x} + c\,\mathrm{e}^{4\lambda x} - \lambda^2\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = {} & c_1\mathrm{e}^{-\frac{i\mathrm{e}^{4\lambda x}a+4\lambda^2x\sqrt{a}+i\mathrm{e}^{2\lambda x}b}{4\lambda\sqrt{a}}}\,\mathrm{hypergeom}\left(\left[\frac{8\lambda\,a^{\frac{3}{2}} + 4ica - ib^2}{32\lambda\,a^{\frac{3}{2}}}\right], \left[\frac{1}{2}\right], \frac{i\left(2\,\mathrm{e}^{2\lambda x}a + b\right)^2}{8\lambda\,a^{\frac{3}{2}}}\right) \\ & + c_2\,\mathrm{hypergeom}\left(\left[\frac{24\lambda\,a^{\frac{3}{2}} + 4ica - ib^2}{32\lambda\,a^{\frac{3}{2}}}\right], \left[\frac{3}{2}\right], \frac{i\left(2\,\mathrm{e}^{2\lambda x}a + b\right)^2}{8\lambda\,a^{\frac{3}{2}}}\right)\left(2a\,\mathrm{e}^{-\frac{-4\lambda^2x\sqrt{a}+i\mathrm{e}^{4\lambda x}a+i\mathrm{e}^{2\lambda x}b}{4\lambda\sqrt{a}}} \right. \\ & \left. + \mathrm{e}^{-\frac{i\mathrm{e}^{4\lambda x}a+4\lambda^2x\sqrt{a}+i\mathrm{e}^{2\lambda x}b}{4\lambda\sqrt{a}}}b\right) \end{aligned}$$

The above shows that

Expression too large to display

Using the above in (1) gives the solution

Expression too large to display

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

Expression too large to display

Summary
The solution(s) found are the following

Expression too large to display (1)

Verification of solutions

Expression too large to display

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-a*exp(8*lambda*x)-b*exp(6*la
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
               <- hyper3 successful: indirect Equivalence to 0F1 under \`\`^ @ Moebius\`\` i
            <- hypergeometric successful
         <- special function solution successful
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 1078

```
dsolve(diff(y(x),x)=y(x)^2+a*exp(8*lambda*x)+b*exp(6*lambda*x)+c*exp(4*lambda*x)-lambda^2,y(
```

Expression too large to display

✓ Solution by Mathematica

Time used: 4.991 (sec). Leaf size: 1282

```
DSolve[y'[x]==y[x]^2+a*Exp[8*\[Lambda]*x]+b*Exp[6*\[Lambda]*x]+c*Exp[4*\[Lambda]*x]-\[Lambda
```

$$y(x) \to \frac{-e^{2x\lambda}\,\text{Hypergeometric1F1}\left(\frac{-ib^2+4iac+40a^{3/2}\lambda}{32a^{3/2}\lambda},\frac{3}{2},\frac{i(2e^{2x\lambda}a+b)^2}{8a^{3/2}\lambda}\right)b^3-2ae^{4x\lambda}\,\text{Hypergeometric1F1}\left(\frac{-ib^2+4iac+\cdots}{32a^{3/\cdots}}\right.}{\cdots}$$

$$y(x) \to \frac{\left(\frac{1}{8}+\frac{i}{8}\right)e^{2\lambda x}\left(8a^{3/2}\lambda+4iac-ib^2\right)\text{HermiteH}\left(\frac{i(b^2-4ac+24ia^{3/2}\lambda)}{16a^{3/2}\lambda},\frac{\left(\frac{1}{4}+\frac{i}{4}\right)(2e^{2x\lambda}a+b)}{a^{3/4}\sqrt{\lambda}}\right)}{a^{5/4}\sqrt{\lambda}\,\text{HermiteH}\left(\frac{i(b^2-4ac+8ia^{3/2}\lambda)}{16a^{3/2}\lambda},\frac{\left(\frac{1}{4}+\frac{i}{4}\right)(2e^{2x\lambda}a+b)}{a^{3/4}\sqrt{\lambda}}\right)} + \frac{ibe^{2\lambda x}}{2\sqrt{a}}+i\sqrt{a}e^{4\lambda x}+\lambda$$

3.9 problem 9

Internal problem ID [10417]
Internal file name [`OUTPUT/9364_Monday_June_06_2022_02_18_59_PM_79365723/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - a\,e^{kx}y^2 = b\,e^{sx}}$$

3.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,e^{kx}y^2 + b\,e^{sx} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,e^{kx}y^2 + b\,e^{sx}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\,e^{sx}$, $f_1(x) = 0$ and $f_2(x) = a\,e^{kx}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{a\,e^{kx}u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= ak\,\mathrm{e}^{kx} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \mathrm{e}^{2kx}\mathrm{e}^{sx} a^2 b \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$a\,\mathrm{e}^{kx} u''(x) - ak\,\mathrm{e}^{kx} u'(x) + \mathrm{e}^{2kx}\mathrm{e}^{sx} a^2 b u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = -\frac{\mathrm{e}^{-\frac{sx}{2}}\left(\mathrm{e}^{\frac{x(k+s)}{2}} \operatorname{BesselJ}\left(\frac{k+2s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right)\sqrt{a}\,\sqrt{b}\,c_1 + \operatorname{BesselY}\left(\frac{k+2s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right)\mathrm{e}^{\frac{x(k+s)}{2}}\sqrt{a}\,\sqrt{b}\,c_2 +}{\sqrt{b}\,\sqrt{a}}$$

The above shows that

$$u'(x) = \left(-c_1 \operatorname{BesselJ}\left(\frac{s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right) - c_2 \operatorname{BesselY}\left(\frac{s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right)\right)\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(2k+s)}{2}}$$

Using the above in (1) gives the solution

$$y = \frac{\left(-c_1 \operatorname{BesselJ}\left(\frac{s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right) - c_2 \operatorname{BesselY}\left(\frac{s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(}{}}}{k+s}\right.\right.}{\mathrm{e}^{\frac{x(k+s)}{2}} \operatorname{BesselJ}\left(\frac{k+2s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right)\sqrt{a}\,\sqrt{b}\,c_1 + \operatorname{BesselY}\left(\frac{k+2s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right)\mathrm{e}^{\frac{x(k+s)}{2}}\sqrt{a}\,\sqrt{b}\,c_2 - s\left(c_1 \mathrm{B}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{b\,\mathrm{e}^{sx}\left(c_3\,\mathrm{BesselJ}\left(\frac{s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right)+\mathrm{BesselY}\left(\frac{s}{k+s}, \right.\right.}{\mathrm{e}^{\frac{x(k+s)}{2}}\,\mathrm{BesselJ}\left(\frac{k+2s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right)\sqrt{a}\,\sqrt{b}\,c_3+\mathrm{BesselY}\left(\frac{k+2s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right)\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}-s\left(c_3\,\mathrm{B}\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{b\,\mathrm{e}^{sx}\left(c_3\,\mathrm{BesselJ}\left(\frac{s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right)+\mathrm{BesselY}\left(\frac{s}{k+s}, \right.\right.}{\mathrm{e}^{\frac{x(k+s)}{2}}\,\mathrm{BesselJ}\left(\frac{k+2s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right)\sqrt{a}\,\sqrt{b}\,c_3+\mathrm{BesselY}\left(\frac{k+2s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right)\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}-s\left(c_3\,\mathrm{B}\right.} \quad (1)$$

Verification of solutions

$$y = -\frac{b\,\mathrm{e}^{sx}\left(c_3\,\mathrm{BesselJ}\left(\frac{s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right)+\mathrm{BesselY}\left(\frac{s}{k+s}, \right.\right.}{\mathrm{e}^{\frac{x(k+s)}{2}}\,\mathrm{BesselJ}\left(\frac{k+2s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right)\sqrt{a}\,\sqrt{b}\,c_3+\mathrm{BesselY}\left(\frac{k+2s}{k+s}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}}{k+s}\right)\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(k+s)}{2}}-s\left(c_3\,\mathrm{B}\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = k*(diff(y(x), x))-a*exp(k*x)*b
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            <- Bessel successful
         <- special function solution successful
         Change of variables used:
            [x = ln(t)/(s+k)]
         Linear ODE actually solved:
            a*b*u(t)+(k*s+s^2)*diff(u(t),t)+(k^2*t+2*k*s*t+s^2*t)*diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 228

```
dsolve(diff(y(x),x)=a*exp(k*x)*y(x)^2+b*exp(s*x),y(x), singsol=all)
```

$y(x) =$

$$-\frac{b\,\mathrm{e}^{sx}\left(\mathrm{BesselY}\left(\frac{s}{s+k}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(s+k)}{2}}}{s+k}\right)c_1 + \mathrm{BesselJ}\left(\frac{s}{s+k},\right.\right.}{\mathrm{BesselJ}\left(\frac{2s+k}{s+k}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(s+k)}{2}}}{s+k}\right)\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(s+k)}{2}} + \sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(s+k)}{2}}\,\mathrm{BesselY}\left(\frac{2s+k}{s+k}, \frac{2\sqrt{a}\,\sqrt{b}\,\mathrm{e}^{\frac{x(s+k)}{2}}}{s+k}\right)c_1 - s\left(\mathrm{Be}\right.}$$

✓ Solution by Mathematica

Time used: 6.491 (sec). Leaf size: 1097

```
DSolve[y'[x]==a*Exp[k*x]*y[x]^2+b*Exp[s*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-kx}\left(-kK_{\frac{k\log\left(e^{k+s}\right)}{(k+s)^2}}\left(2\sqrt{-\frac{ab\left(\left(e^{k+s}\right)^x\right)^{\frac{k+s}{\log\left(e^{k+s}\right)}}\log^2\left(e^{k+s}\right)}{(k+s)^4}}\right)-c_1k(-1)^{\frac{k\log\left(e^{k+s}\right)}{(k+s)^2}}\text{BesselI}\left(\frac{k\log\left(e^{k+s}\right)}{(k+s)^2},2\sqrt{-\frac{ab}{\cdots}}\right.\right.}{\cdots}$$

$$y(x) \to \frac{e^{-kx}\left(-\frac{(k+s)\sqrt{-\frac{ab\log^2\left(e^{k+s}\right)\left(\left(e^{k+s}\right)^x\right)^{\frac{k+s}{\log\left(e^{k+s}\right)}}}{(k+s)^4}}\left(\text{BesselI}\left(\frac{k\log\left(e^{k+s}\right)}{(k+s)^2}-1,2\sqrt{-\frac{ab\left(\left(e^{k+s}\right)^x\right)^{\frac{k+s}{\log\left(e^{k+s}\right)}}\log^2\left(e^{k+s}\right)}{(k+s)^4}}\right)+\text{BesselI}\left(\frac{k\log}{(\cdots}\right.\right.}{\text{BesselI}\left(\frac{k\log\left(e^{k+s}\right)}{(k+s)^2},2\sqrt{-\frac{ab\left(\left(e^{k+s}\right)^x\right)^{\frac{k+s}{\log\left(e^{k+s}\right)}}\log^2\left(e^{k+s}\right)}{(k+s)^4}}\right)}\right.}{2a}$$

$$y(x) \to \frac{e^{-kx}\left(-\frac{(k+s)\sqrt{-\frac{ab\log^2\left(e^{k+s}\right)\left(\left(e^{k+s}\right)^x\right)^{\frac{k+s}{\log\left(e^{k+s}\right)}}}{(k+s)^4}}\left(\text{BesselI}\left(\frac{k\log\left(e^{k+s}\right)}{(k+s)^2}-1,2\sqrt{-\frac{ab\left(\left(e^{k+s}\right)^x\right)^{\frac{k+s}{\log\left(e^{k+s}\right)}}\log^2\left(e^{k+s}\right)}{(k+s)^4}}\right)+\text{BesselI}\left(\frac{k\log}{(\cdots}\right.\right.}{\text{BesselI}\left(\frac{k\log\left(e^{k+s}\right)}{(k+s)^2},2\sqrt{-\frac{ab\left(\left(e^{k+s}\right)^x\right)^{\frac{k+s}{\log\left(e^{k+s}\right)}}\log^2\left(e^{k+s}\right)}{(k+s)^4}}\right)}\right.}{2a}$$

3.10 problem 10

Internal problem ID [10418]
Internal file name [`OUTPUT/9365_Monday_June_06_2022_02_19_01_PM_15885292/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - b\,\mathrm{e}^{x\mu} y^2 = a\lambda\,\mathrm{e}^{\lambda x} - a^2 b\,\mathrm{e}^{(\mu+2\lambda)x}}$$

3.10.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= b\,\mathrm{e}^{x\mu} y^2 + a\lambda\,\mathrm{e}^{\lambda x} - a^2 b\,\mathrm{e}^{(\mu+2\lambda)x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = b\,\mathrm{e}^{x\mu} y^2 + a\lambda\,\mathrm{e}^{\lambda x} - a^2 b\,\mathrm{e}^{2\lambda x}\mathrm{e}^{x\mu}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\lambda\,\mathrm{e}^{\lambda x} - a^2 b\,\mathrm{e}^{(\mu+2\lambda)x}$, $f_1(x) = 0$ and $f_2(x) = b\,\mathrm{e}^{x\mu}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{b\,\mathrm{e}^{x\mu} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= b\mu\, e^{x\mu} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= b^2 e^{2x\mu}\left(a\lambda\, e^{\lambda x} - a^2 b\, e^{(\mu+2\lambda)x}\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$b\, e^{x\mu} u''(x) - b\mu\, e^{x\mu} u'(x) + b^2 e^{2x\mu}\left(a\lambda\, e^{\lambda x} - a^2 b\, e^{(\mu+2\lambda)x}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \text{DESol}\left(\left\{-e^{2x(\lambda+\mu)}\_Y(x)\, a^2 b^2 + e^{x(\lambda+\mu)}\_Y(x)\, ab\lambda - \mu\_Y'(x) + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\text{DESol}\left(\left\{-e^{2x(\lambda+\mu)}\_Y(x)\, a^2 b^2 + e^{x(\lambda+\mu)}\_Y(x)\, ab\lambda - \mu\_Y'(x) + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\text{DESol}\left(\left\{-e^{2x(\lambda+\mu)}\_Y(x)\, a^2 b^2 + e^{x(\lambda+\mu)}\_Y(x)\, ab\lambda - \mu\_Y'(x) + \_Y''(x)\right\}, \{\_Y(x)\}\right)\right) e^{-x\mu}}{b\,\text{DESol}\left(\left\{-e^{2x(\lambda+\mu)}\_Y(x)\, a^2 b^2 + e^{x(\lambda+\mu)}\_Y(x)\, ab\lambda - \mu\_Y'(x) + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\text{DESol}\left(\left\{-e^{2x(\lambda+\mu)}\_Y(x)\, a^2 b^2 + e^{x(\lambda+\mu)}\_Y(x)\, ab\lambda - \mu\_Y'(x) + \_Y''(x)\right\}, \{\_Y(x)\}\right)\right) e^{-x\mu}}{b\,\text{DESol}\left(\left\{-e^{2x(\lambda+\mu)}\_Y(x)\, a^2 b^2 + e^{x(\lambda+\mu)}\_Y(x)\, ab\lambda - \mu\_Y'(x) + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{-\mathrm{e}^{2x(\lambda+\mu)} \_Y(x)\, a^2 b^2 + \mathrm{e}^{x(\lambda+\mu)} \_Y(x)\, ab\lambda - \mu \_Y'(x) + \_Y''(x)\right\}, \{\_Y(x)\}\right)\right) \mathrm{e}^{-x\mu}}{b\, \mathrm{DESol}\left(\left\{-\mathrm{e}^{2x(\lambda+\mu)} \_Y(x)\, a^2 b^2 + \mathrm{e}^{x(\lambda+\mu)} \_Y(x)\, ab\lambda - \mu \_Y'(x) + \_Y''(x)\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{-\mathrm{e}^{2x(\lambda+\mu)} \_Y(x)\, a^2 b^2 + \mathrm{e}^{x(\lambda+\mu)} \_Y(x)\, ab\lambda - \mu \_Y'(x) + \_Y''(x)\right\}, \{\_Y(x)\}\right)\right) \mathrm{e}^{-x\mu}}{b\, \mathrm{DESol}\left(\left\{-\mathrm{e}^{2x(\lambda+\mu)} \_Y(x)\, a^2 b^2 + \mathrm{e}^{x(\lambda+\mu)} \_Y(x)\, ab\lambda - \mu \_Y'(x) + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = mu*(diff(y(x), x))-b*exp(x*mu)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(b*exp(x*mu)*y(x)^2+y(x)+x^2*(a*lambda*ex
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=b*exp(mu*x)*y(x)^2+a*lambda*exp(lambda*x)-a^2*b*exp((mu+2*lambda)*x),y(x
```

No solution found

✓ Solution by Mathematica

Time used: 8.808 (sec). Leaf size: 844

```
DSolve[y'[x]==b*Exp[\[Mu]*x]*y[x]^2+a*\[Lambda]*Exp[\[Lambda]*x]-a^2*b*Exp[(\[Mu]+2*\[Lambda
```

$$y(x) \to \frac{e^{\mu(-x)}\left(-2ab\log\left(e^{\lambda+\mu}\right)\left(\left(e^{\lambda+\mu}\right)^{x}\right)^{\frac{\lambda+\mu}{\log\left(e^{\lambda+\mu}\right)}}\left(2(\lambda+\mu)L_{-\frac{\log\left(e^{\lambda+\mu}\right)}{2(\lambda+\mu)}-\frac{3}{2}}^{\frac{\mu\log\left(e^{\lambda+\mu}\right)}{(\lambda+\mu)^2}+1}\left(-\frac{2ab\left(\left(e^{\lambda+\mu}\right)^{x}\right)^{\frac{\lambda+\mu}{\log\left(e^{\lambda+\mu}\right)}}\log\left(e^{\lambda+\mu}\right)}{(\lambda+\mu)^2}\right)+c\right.\right.}{}$$

$$y(x) \to -\frac{ae^{\mu(-x)}\log\left(e^{\lambda+\mu}\right)\left(\log\left(e^{\lambda+\mu}\right)+\lambda+\mu\right)\left(\left(e^{\lambda+\mu}\right)^{x}\right)^{\frac{\lambda+\mu}{\log\left(e^{\lambda+\mu}\right)}}\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{\log\left(e^{\lambda+\mu}\right)}{\lambda+\mu}+3\right),\frac{2(\lambda+\mu}{}\right.}{(\lambda+\mu)^2\,\text{HypergeometricU}\left(\frac{\lambda+\mu+\log\left(e^{\lambda+\mu}\right)}{2(\lambda+\mu)},\frac{\mu\log\left(e^{\lambda+\mu}\right)}{(\lambda+\mu)^2}+1,-\frac{2ab\left(\left(e^{\lambda+\mu}\right)^{x}\right)^{\overline{\log(}}}{(\lambda+}\right.}$$

$$-\frac{e^{\mu(-x)}\left(\log\left(e^{\lambda+\mu}\right)\left(2ab\left(\left(e^{\lambda+\mu}\right)^{x}\right)^{\frac{\lambda+\mu}{\log\left(e^{\lambda+\mu}\right)}}+\mu\right)+\mu(\lambda+\mu)\right)}{2b(\lambda+\mu)}$$

3.11 problem 11

Internal problem ID [10419]
Internal file name [OUTPUT/9366_Monday_June_06_2022_02_19_03_PM_98824791/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Riccati]
```

$$\boxed{y' - a\,e^{\lambda x} y^2 - yb = c\,e^{-\lambda x}}$$

3.11.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= e^{\lambda x} a\, y^2 + by + c\,e^{-\lambda x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \left(\mathrm{e}^{\lambda x} a\, y^2 + by + c\,\mathrm{e}^{-\lambda x}\right)(b_3 - a_2) - \left(\mathrm{e}^{\lambda x} a\, y^2 + by + c\,\mathrm{e}^{-\lambda x}\right)^2 a_3 \\ & - \left(\lambda\,\mathrm{e}^{\lambda x} a\, y^2 - c\lambda\,\mathrm{e}^{-\lambda x}\right)(xa_2 + ya_3 + a_1) - \left(2a\,\mathrm{e}^{\lambda x} y + b\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} & -\mathrm{e}^{2\lambda x} a^2 y^4 a_3 - 2\,\mathrm{e}^{\lambda x}\mathrm{e}^{-\lambda x} ac\, y^2 a_3 - 2\,\mathrm{e}^{\lambda x} ab\, y^3 a_3 - \mathrm{e}^{\lambda x} a\lambda x\, y^2 a_2 - \mathrm{e}^{\lambda x} a\lambda\, y^3 a_3 \\ & - \mathrm{e}^{\lambda x} a\lambda\, y^2 a_1 - 2\,\mathrm{e}^{\lambda x} axyb_2 - \mathrm{e}^{\lambda x} a\, y^2 a_2 - \mathrm{e}^{\lambda x} a\, y^2 b_3 - \mathrm{e}^{-2\lambda x} c^2 a_3 \\ & - 2\,\mathrm{e}^{-\lambda x} bcya_3 + \mathrm{e}^{-\lambda x} c\lambda x a_2 + \mathrm{e}^{-\lambda x} c\lambda y a_3 - b^2 y^2 a_3 - 2\,\mathrm{e}^{\lambda x} ayb_1 \\ & + \mathrm{e}^{-\lambda x} c\lambda a_1 - \mathrm{e}^{-\lambda x} ca_2 + \mathrm{e}^{-\lambda x} cb_3 - bxb_2 - bya_2 - bb_1 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} & -\mathrm{e}^{2\lambda x} a^2 y^4 a_3 - 2\,\mathrm{e}^{\lambda x}\mathrm{e}^{-\lambda x} ac\, y^2 a_3 - 2\,\mathrm{e}^{\lambda x} ab\, y^3 a_3 - \mathrm{e}^{\lambda x} a\lambda x\, y^2 a_2 - \mathrm{e}^{\lambda x} a\lambda\, y^3 a_3 \\ & - \mathrm{e}^{\lambda x} a\lambda\, y^2 a_1 - 2\,\mathrm{e}^{\lambda x} axyb_2 - \mathrm{e}^{\lambda x} a\, y^2 a_2 - \mathrm{e}^{\lambda x} a\, y^2 b_3 - \mathrm{e}^{-2\lambda x} c^2 a_3 \\ & - 2\,\mathrm{e}^{-\lambda x} bcya_3 + \mathrm{e}^{-\lambda x} c\lambda x a_2 + \mathrm{e}^{-\lambda x} c\lambda y a_3 - b^2 y^2 a_3 - 2\,\mathrm{e}^{\lambda x} ayb_1 \\ & + \mathrm{e}^{-\lambda x} c\lambda a_1 - \mathrm{e}^{-\lambda x} ca_2 + \mathrm{e}^{-\lambda x} cb_3 - bxb_2 - bya_2 - bb_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} & -\mathrm{e}^{2\lambda x} a^2 y^4 a_3 - 2ac\, y^2 a_3 - 2\,\mathrm{e}^{\lambda x} ab\, y^3 a_3 - \mathrm{e}^{\lambda x} a\lambda x\, y^2 a_2 - \mathrm{e}^{\lambda x} a\lambda\, y^3 a_3 \\ & - \mathrm{e}^{\lambda x} a\lambda\, y^2 a_1 - 2\,\mathrm{e}^{\lambda x} axyb_2 - \mathrm{e}^{\lambda x} a\, y^2 a_2 - \mathrm{e}^{\lambda x} a\, y^2 b_3 - \mathrm{e}^{-2\lambda x} c^2 a_3 \\ & - 2\,\mathrm{e}^{-\lambda x} bcya_3 + \mathrm{e}^{-\lambda x} c\lambda x a_2 + \mathrm{e}^{-\lambda x} c\lambda y a_3 - b^2 y^2 a_3 - 2\,\mathrm{e}^{\lambda x} ayb_1 \\ & + \mathrm{e}^{-\lambda x} c\lambda a_1 - \mathrm{e}^{-\lambda x} ca_2 + \mathrm{e}^{-\lambda x} cb_3 - bxb_2 - bya_2 - bb_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \mathrm{e}^{\lambda x}, \mathrm{e}^{-2\lambda x}, \mathrm{e}^{-\lambda x}, \mathrm{e}^{2\lambda x}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \mathrm{e}^{\lambda x} = v_3, \mathrm{e}^{-2\lambda x} = v_4, \mathrm{e}^{-\lambda x} = v_5, \mathrm{e}^{2\lambda x} = v_6\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-v_6 a^2 v_2^4 a_3 - 2v_3 ab v_2^3 a_3 - v_3 a\lambda v_1 v_2^2 a_2 - v_3 a\lambda v_2^3 a_3 - v_3 a\lambda v_2^2 a_1 - 2ac v_2^2 a_3 \\
&- v_3 a v_2^2 a_2 - 2v_3 a v_1 v_2 b_2 - v_3 a v_2^2 b_3 - b^2 v_2^2 a_3 - 2v_5 bc v_2 a_3 + v_5 c\lambda v_1 a_2 + v_5 c\lambda v_2 a_3 \\
&- 2v_3 a v_2 b_1 - v_4 c^2 a_3 + v_5 c\lambda a_1 - b v_2 a_2 - b v_1 b_2 - v_5 c a_2 + v_5 c b_3 - b b_1 + b_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4, v_5, v_6\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-v_3 a\lambda v_1 v_2^2 a_2 - 2v_3 a v_1 v_2 b_2 + v_5 c\lambda v_1 a_2 - b v_1 b_2 - v_6 a^2 v_2^4 a_3 \\
&+ (-2a_3 ab - \lambda a_3 a) v_2^3 v_3 + (-a\lambda a_1 - a_2 a - b_3 a) v_2^2 v_3 \\
&+ \left(-2a_3 ac - a_3 b^2\right) v_2^2 - 2v_3 a v_2 b_1 + (-2a_3 bc + \lambda a_3 c) v_2 v_5 \\
&- b v_2 a_2 - v_4 c^2 a_3 + (c\lambda a_1 - c a_2 + c b_3) v_5 - b b_1 + b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
c\lambda a_2 &= 0 \\
-2ab_1 &= 0 \\
-2ab_2 &= 0 \\
-bb_2 &= 0 \\
-c^2 a_3 &= 0 \\
-a_2 b &= 0 \\
-a_3 a^2 &= 0 \\
-\lambda a_2 a &= 0 \\
-bb_1 + b_2 &= 0 \\
-2a_3 ac - a_3 b^2 &= 0 \\
-2a_3 ab - \lambda a_3 a &= 0 \\
-2a_3 bc + \lambda a_3 c &= 0 \\
-a\lambda a_1 - a_2 a - b_3 a &= 0 \\
c\lambda a_1 - c a_2 + c b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -\lambda a_1
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= -\lambda y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -\lambda y - \left(\mathrm{e}^{\lambda x} a\, y^2 + b y + c\,\mathrm{e}^{-\lambda x}\right)(1) \\
&= -\mathrm{e}^{\lambda x} a\, y^2 - b y - \lambda y - c\,\mathrm{e}^{-\lambda x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\mathrm{e}^{\lambda x} a\, y^2 - b y - \lambda y - c\,\mathrm{e}^{-\lambda x}} dy
\end{aligned}
$$

Which results in

$$S = -\frac{2\,\mathrm{e}^{\lambda x} \arctan\left(\frac{2\,\mathrm{e}^{2\lambda x} a y + b\,\mathrm{e}^{\lambda x} + \lambda\,\mathrm{e}^{\lambda x}}{\sqrt{-\mathrm{e}^{2\lambda x} b^2 - 2\,\mathrm{e}^{2\lambda x} b\lambda - \mathrm{e}^{2\lambda x}\lambda^2 + 4c\,\mathrm{e}^{2\lambda x} a}}\right)}{\sqrt{-\mathrm{e}^{2\lambda x} b^2 - 2\,\mathrm{e}^{2\lambda x} b\lambda - \mathrm{e}^{2\lambda x}\lambda^2 + 4c\,\mathrm{e}^{2\lambda x} a}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \mathrm{e}^{\lambda x} a\, y^2 + by + c\,\mathrm{e}^{-\lambda x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{\mathrm{e}^{\lambda x}\lambda y}{\mathrm{e}^{2\lambda x} a\, y^2 + y\,(b+\lambda)\,\mathrm{e}^{\lambda x} + c} \\
S_y &= -\frac{\mathrm{e}^{\lambda x}}{\mathrm{e}^{2\lambda x} a\, y^2 + y\,(b+\lambda)\,\mathrm{e}^{\lambda x} + c}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2\arctan\left(\frac{2a\,\mathrm{e}^{\lambda x}y+b+\lambda}{\sqrt{4ca-b^2-2b\lambda-\lambda^2}}\right)}{\sqrt{4ca-b^2-2b\lambda-\lambda^2}} = c_1 - x$$

Which simplifies to

$$-\frac{2\arctan\left(\frac{2a\,\mathrm{e}^{\lambda x}y+b+\lambda}{\sqrt{4ca-b^2-2b\lambda-\lambda^2}}\right)}{\sqrt{4ca-b^2-2b\lambda-\lambda^2}} = c_1 - x$$

Which gives

$$y = -\frac{\left(\tan\left(\frac{c_1\sqrt{4ca-b^2-2b\lambda-\lambda^2}}{2} - \frac{x\sqrt{4ca-b^2-2b\lambda-\lambda^2}}{2}\right)\sqrt{4ca-b^2-2b\lambda-\lambda^2} + b + \lambda\right)\mathrm{e}^{-\lambda x}}{2a}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\tan\left(\frac{c_1\sqrt{4ca-b^2-2b\lambda-\lambda^2}}{2} - \frac{x\sqrt{4ca-b^2-2b\lambda-\lambda^2}}{2}\right)\sqrt{4ca-b^2-2b\lambda-\lambda^2} + b + \lambda\right)\mathrm{e}^{-\lambda x}}{2a} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\tan\left(\frac{c_1\sqrt{4ca-b^2-2b\lambda-\lambda^2}}{2} - \frac{x\sqrt{4ca-b^2-2b\lambda-\lambda^2}}{2}\right)\sqrt{4ca-b^2-2b\lambda-\lambda^2} + b + \lambda\right)\mathrm{e}^{-\lambda x}}{2a}$$

Verified OK.

3.11.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \mathrm{e}^{\lambda x} a\, y^2 + by + c\,\mathrm{e}^{-\lambda x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \mathrm{e}^{\lambda x} a\, y^2 + by + c\,\mathrm{e}^{-\lambda x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = c\,\mathrm{e}^{-\lambda x}$, $f_1(x) = b$ and $f_2(x) = \mathrm{e}^{\lambda x} a$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\mathrm{e}^{\lambda x} a u}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= a\lambda\,\mathrm{e}^{\lambda x} \\
f_1 f_2 &= a\,\mathrm{e}^{\lambda x} b \\
f_2^2 f_0 &= \mathrm{e}^{2\lambda x} a^2 c\,\mathrm{e}^{-\lambda x}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\mathrm{e}^{\lambda x} a u''(x) - \left(a\lambda\,\mathrm{e}^{\lambda x} + a\,\mathrm{e}^{\lambda x} b\right) u'(x) + \mathrm{e}^{2\lambda x} a^2 c\,\mathrm{e}^{-\lambda x} u(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 \mathrm{e}^{\frac{\left(b+\lambda+\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}} + c_2 \mathrm{e}^{\frac{\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}}
$$

The above shows that

$$
u'(x) = \frac{c_2\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)\mathrm{e}^{\frac{\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}}}{2} + \frac{c_1 \mathrm{e}^{\frac{\left(b+\lambda+\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}}\left(b+\lambda+\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)}{2}
$$

Using the above in (1) gives the solution

$$
y = -\frac{\left(\frac{c_2\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)\mathrm{e}^{\frac{\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}}}{2} + \frac{c_1 \mathrm{e}^{\frac{\left(b+\lambda+\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}}\left(b+\lambda+\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)}{2}\right)\mathrm{e}^{-\lambda x}}{a\left(c_1 \mathrm{e}^{\frac{\left(b+\lambda+\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}} + c_2 \mathrm{e}^{\frac{\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}}\right)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right) \mathrm{e}^{\frac{\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}} + c_3 \mathrm{e}^{\frac{\left(b+\lambda+\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}} \left(b+\lambda+\sqrt{-4ca\right.\right.}{2a\left(c_3 \mathrm{e}^{\frac{\left(b+\lambda+\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}} + \mathrm{e}^{\frac{\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right) \mathrm{e}^{\frac{\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}} + c_3 \mathrm{e}^{\frac{\left(b+\lambda+\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}} \left(b+\lambda+\sqrt{-4ca\right.\right.}{2a\left(c_3 \mathrm{e}^{\frac{\left(b+\lambda+\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}} + \mathrm{e}^{\frac{\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}}\right)} \quad (1)$$

Verification of solutions

$$y = -\frac{\left(\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right) \mathrm{e}^{\frac{\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}} + c_3 \mathrm{e}^{\frac{\left(b+\lambda+\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}} \left(b+\lambda+\sqrt{-4ca\right.\right.}{2a\left(c_3 \mathrm{e}^{\frac{\left(b+\lambda+\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}} + \mathrm{e}^{\frac{\left(b+\lambda-\sqrt{-4ca+b^2+2b\lambda+\lambda^2}\right)x}{2}}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 96

```
dsolve(diff(y(x),x)=a*exp(lambda*x)*y(x)^2+b*y(x)+c*exp(-lambda*x),y(x), singsol=all)
```

$$y(x) = -\frac{\left(-\sqrt{(b+\lambda)^2 (4ac-b^2-2\lambda b-\lambda^2)}\tan\left(\frac{((b+\lambda)x+c_1)\sqrt{(b+\lambda)^2 (4ac-b^2-2\lambda b-\lambda^2)}}{2(b+\lambda)^2}\right)+(b+\lambda)^2\right)e^{-x\lambda}}{2a(b+\lambda)}$$

✓ Solution by Mathematica

Time used: 0.927 (sec). Leaf size: 188

```
DSolve[y'[x]==a*Exp[\[Lambda]*x]*y[x]^2+b*y[x]+c*Exp[-\[Lambda]*x],y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{e^{\lambda(-x)}\left(-\sqrt{-4ac+b^2+2b\lambda+\lambda^2}+\frac{2}{\frac{1}{\sqrt{-4ac+b^2+2b\lambda+\lambda^2}}+c_1 e^{x\sqrt{-4ac+b^2+2b\lambda+\lambda^2}}}-b-\lambda\right)}{2a}$$

$$y(x) \to -\frac{e^{\lambda(-x)}\left(b\left(\sqrt{-4ac+b^2+2b\lambda+\lambda^2}+2\lambda\right)+\lambda\left(\sqrt{-4ac+b^2+2b\lambda+\lambda^2}+\lambda\right)-4ac+b^2\right)}{2a\sqrt{-4ac+b^2+2b\lambda+\lambda^2}}$$

3.12 problem 12

Internal problem ID [10420]
Internal file name [`OUTPUT/9367_Monday_June_06_2022_02_19_04_PM_92605250/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\,\mathrm{e}^{x\mu}y^2 - \lambda y = -a\,b^2\mathrm{e}^{(\mu+2\lambda)x}}$$

3.12.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,\mathrm{e}^{x\mu}y^2 + \lambda y - a\,b^2\mathrm{e}^{(\mu+2\lambda)x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,\mathrm{e}^{x\mu}y^2 + \lambda y - a\,b^2\mathrm{e}^{2\lambda x}\mathrm{e}^{x\mu}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a\,b^2\mathrm{e}^{(\mu+2\lambda)x}$, $f_1(x) = \lambda$ and $f_2(x) = a\,\mathrm{e}^{x\mu}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{a\,\mathrm{e}^{x\mu}u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= a\mu\, \mathrm{e}^{x\mu} \\ f_1 f_2 &= \lambda a\, \mathrm{e}^{x\mu} \\ f_2^2 f_0 &= -a^3 \mathrm{e}^{2x\mu} b^2 \mathrm{e}^{(\mu+2\lambda)x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$a\, \mathrm{e}^{x\mu} u''(x) - (a\mu\, \mathrm{e}^{x\mu} + \lambda a\, \mathrm{e}^{x\mu}) u'(x) - a^3 \mathrm{e}^{2x\mu} b^2 \mathrm{e}^{(\mu+2\lambda)x} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right) + c_2 \cos\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right)$$

The above shows that

$$u'(x) = \frac{ab\, \mathrm{e}^{2x(\lambda+\mu)} \left(-c_1 \cos\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right) + c_2 \sin\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right)\right)}{\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}$$

Using the above in (1) gives the solution

$$y = -\frac{b\, \mathrm{e}^{2x(\lambda+\mu)} \left(-c_1 \cos\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right) + c_2 \sin\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right)\right) \mathrm{e}^{-x\mu}}{\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}} \left(c_1 \sin\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right) + c_2 \cos\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{b\, \mathrm{e}^{(\mu+2\lambda)x} \left(c_3 \cos\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right) - \sin\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right)\right)}{\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}} \left(c_3 \sin\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right) + \cos\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right)\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{b\,\mathrm{e}^{(\mu+2\lambda)x}\left(c_3\cos\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right)-\sin\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right)\right)}{\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}\left(c_3\sin\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right)+\cos\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{b\,\mathrm{e}^{(\mu+2\lambda)x}\left(c_3\cos\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right)-\sin\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right)\right)}{\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}\left(c_3\sin\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right)+\cos\left(\frac{ab\sqrt{-\mathrm{e}^{2\lambda x}\mathrm{e}^{2x\mu}}}{\lambda+\mu}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (lambda+mu)*(diff(y(x), x))+a^
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      <- linear_1 successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 79

```
dsolve(diff(y(x),x)=a*exp(mu*x)*y(x)^2+lambda*y(x)-a*b^2*exp((mu+2*lambda)*x),y(x), singsol=
```

$$y(x) = -\frac{b\left(c_1 \sinh\left(\frac{ab\,\mathrm{e}^{x(\lambda+\mu)}}{\lambda+\mu}\right)+\cosh\left(\frac{ab\,\mathrm{e}^{x(\lambda+\mu)}}{\lambda+\mu}\right)\right)\mathrm{e}^{x\lambda}}{c_1 \cosh\left(\frac{ab\,\mathrm{e}^{x(\lambda+\mu)}}{\lambda+\mu}\right)+\sinh\left(\frac{ab\,\mathrm{e}^{x(\lambda+\mu)}}{\lambda+\mu}\right)}$$

✓ Solution by Mathematica

Time used: 2.706 (sec). Leaf size: 286

```
DSolve[y'[x]==a*Exp[\[Mu]*x]*y[x]^2+\[Lambda]*y[x]-a*b^2*Exp[(\[Mu]+2*\[Lambda])*x],y[x],x,I
```

$$y(x) \to \boxed{-\frac{\tan\left(\frac{ab^2 e^{x(2\lambda+\mu)}\sqrt{-\frac{e^{-2x\lambda}}{b^2}}}{\lambda+\mu}-c_1\right)}{\sqrt{-\frac{e^{-2x\lambda}}{b^2}}} \text{ if condition}}$$

3.13 problem 13

Internal problem ID [10421]
Internal file name [`OUTPUT/9368_Monday_June_06_2022_02_19_05_PM_49258887/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - e^{\lambda x} y^2 - a\, e^{x\mu} y = a\lambda\, e^{(\mu-\lambda)x}}$$

3.13.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= e^{\lambda x} y^2 + a\, e^{x\mu} y + a\lambda\, e^{(\mu-\lambda)x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = e^{\lambda x} y^2 + a\, e^{x\mu} y + a\lambda\, e^{-\lambda x} e^{x\mu}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\lambda\, e^{(\mu-\lambda)x}$, $f_1(x) = a\, e^{x\mu}$ and $f_2(x) = e^{\lambda x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{e^{\lambda x} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \lambda\,\mathrm{e}^{\lambda x} \\ f_1f_2 &= a\,\mathrm{e}^{x\mu}\mathrm{e}^{\lambda x} \\ f_2^2f_0 &= \mathrm{e}^{2\lambda x}a\lambda\,\mathrm{e}^{(\mu-\lambda)x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\mathrm{e}^{\lambda x}u''(x) - \left(a\,\mathrm{e}^{x\mu}\mathrm{e}^{\lambda x} + \lambda\,\mathrm{e}^{\lambda x}\right)u'(x) + \mathrm{e}^{2\lambda x}a\lambda\,\mathrm{e}^{(\mu-\lambda)x}u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1\mathrm{e}^{\lambda x} + c_2\,\mathrm{hypergeom}\left(\left[-\frac{\lambda}{\mu}\right], \left[\frac{\mu-\lambda}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)$$

The above shows that

$$u'(x) = -\frac{\left(c_2\,\mathrm{hypergeom}\left(\left[\frac{\mu-\lambda}{\mu}\right], \left[\frac{-\lambda+2\mu}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)a\,\mathrm{e}^{x\mu} - \mathrm{e}^{\lambda x}c_1(\mu-\lambda)\right)\lambda}{\mu-\lambda}$$

Using the above in (1) gives the solution

$$y = \frac{\left(c_2\,\mathrm{hypergeom}\left(\left[\frac{\mu-\lambda}{\mu}\right], \left[\frac{-\lambda+2\mu}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)a\,\mathrm{e}^{x\mu} - \mathrm{e}^{\lambda x}c_1(\mu-\lambda)\right)\lambda\,\mathrm{e}^{-\lambda x}}{(\mu-\lambda)\left(c_1\mathrm{e}^{\lambda x} + c_2\,\mathrm{hypergeom}\left(\left[-\frac{\lambda}{\mu}\right], \left[\frac{\mu-\lambda}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\mathrm{hypergeom}\left(\left[\frac{\mu-\lambda}{\mu}\right], \left[\frac{-\lambda+2\mu}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)a\,\mathrm{e}^{(\mu-\lambda)x} - c_3(\mu-\lambda)\right)\lambda}{(\mu-\lambda)\left(c_3\mathrm{e}^{\lambda x} + \mathrm{hypergeom}\left(\left[-\frac{\lambda}{\mu}\right], \left[\frac{\mu-\lambda}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)\right)}$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(\operatorname{hypergeom}\left(\left[\frac{\mu-\lambda}{\mu}\right], \left[\frac{-\lambda+2\mu}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right) a\,\mathrm{e}^{(\mu-\lambda)x} - c_3(\mu-\lambda)\right)\lambda}{(\mu-\lambda)\left(c_3\mathrm{e}^{\lambda x} + \operatorname{hypergeom}\left(\left[-\frac{\lambda}{\mu}\right], \left[\frac{\mu-\lambda}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)\right)} \tag{1}
$$

Verification of solutions

$$
y = \frac{\left(\operatorname{hypergeom}\left(\left[\frac{\mu-\lambda}{\mu}\right], \left[\frac{-\lambda+2\mu}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right) a\,\mathrm{e}^{(\mu-\lambda)x} - c_3(\mu-\lambda)\right)\lambda}{(\mu-\lambda)\left(c_3\mathrm{e}^{\lambda x} + \operatorname{hypergeom}\left(\left[-\frac{\lambda}{\mu}\right], \left[\frac{\mu-\lambda}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)\right)}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a*exp(x*mu)+lambda)*(diff(y(x
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               <- Kummer successful
            <- special function solution successful
               Solution using Kummer functions still has integrals. Trying a hypergeometric
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
               <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
               -> Trying to convert hypergeometric functions to elementary form...
               <- elementary form could result into a too large expression - returning speci
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 97

```
dsolve(diff(y(x),x)=exp(lambda*x)*y(x)^2+a*exp(mu*x)*y(x)+a*lambda*exp((mu-lambda)*x),y(x),
```

$$y(x) = \frac{\lambda\left(a c_1 \mathrm{e}^{(\mu-\lambda)x} \operatorname{hypergeom}\left(\left[\frac{\mu-\lambda}{\mu}\right], \left[\frac{-\lambda+2\mu}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right) + \lambda - \mu\right)}{(\mu-\lambda)\left(c_1 \operatorname{hypergeom}\left(\left[-\frac{\lambda}{\mu}\right], \left[\frac{\mu-\lambda}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right) + \mathrm{e}^{x\lambda}\right)}$$

✓ Solution by Mathematica

Time used: 4.392 (sec). Leaf size: 148

```
DSolve[y'[x]==Exp[\[Lambda]*x]*y[x]^2+a*Exp[\[Mu]*x]*y[x]+a*\[Lambda]*Exp[(\[Mu]-\[Lambda])*
```

$$y(x) \to -\frac{e^{\lambda(-x)}\left(-\lambda\left(-\frac{ae^{\mu x}}{\mu}\right)^{\lambda/\mu}\Gamma\left(-\frac{\lambda}{\mu}, -\frac{ae^{x\mu}}{\mu}\right) + \mu e^{\frac{ae^{\mu x}}{\mu}} + c_1\lambda(e^{\mu x})^{\lambda/\mu}\right)}{-\left(-\frac{ae^{\mu x}}{\mu}\right)^{\lambda/\mu}\Gamma\left(-\frac{\lambda}{\mu}, -\frac{ae^{x\mu}}{\mu}\right) + c_1\left(e^{\mu x}\right)^{\lambda/\mu}}$$

$$y(x) \to \lambda\left(-e^{\lambda(-x)}\right)$$

3.14 problem 14

Internal problem ID [10422]
Internal file name [`OUTPUT/9369_Monday_June_06_2022_02_19_06_PM_308271/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + e^{\lambda x} y^2 \lambda - a\,e^{x\mu} y = -a\,e^{(\mu-\lambda)x}}$$

3.14.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\lambda\,e^{\lambda x} y^2 + a\,e^{x\mu} y - a\,e^{(\mu-\lambda)x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\lambda\,e^{\lambda x} y^2 + a\,e^{x\mu} y - a\,e^{-\lambda x} e^{x\mu}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a\,e^{(\mu-\lambda)x}$, $f_1(x) = a\,e^{x\mu}$ and $f_2(x) = -\lambda\,e^{\lambda x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\lambda\,e^{\lambda x} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\mathrm{e}^{\lambda x}\lambda^2 \\ f_1f_2 &= -a\,\mathrm{e}^{x\mu}\lambda\,\mathrm{e}^{\lambda x} \\ f_2^2 f_0 &= -\lambda^2\mathrm{e}^{2\lambda x}a\,\mathrm{e}^{(\mu-\lambda)x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\lambda\,\mathrm{e}^{\lambda x}u''(x) - \left(-a\,\mathrm{e}^{x\mu}\lambda\,\mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x}\lambda^2\right)u'(x) - \lambda^2\mathrm{e}^{2\lambda x}a\,\mathrm{e}^{(\mu-\lambda)x}u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1\mathrm{e}^{\lambda x} + c_2\,\mathrm{hypergeom}\left(\left[-\frac{\lambda}{\mu}\right], \left[\frac{\mu-\lambda}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)$$

The above shows that

$$u'(x) = -\frac{\left(c_2\,\mathrm{hypergeom}\left(\left[\frac{\mu-\lambda}{\mu}\right], \left[\frac{-\lambda+2\mu}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)a\,\mathrm{e}^{x\mu} - \mathrm{e}^{\lambda x}c_1(\mu-\lambda)\right)\lambda}{\mu-\lambda}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(c_2\,\mathrm{hypergeom}\left(\left[\frac{\mu-\lambda}{\mu}\right], \left[\frac{-\lambda+2\mu}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)a\,\mathrm{e}^{x\mu} - \mathrm{e}^{\lambda x}c_1(\mu-\lambda)\right)\mathrm{e}^{-\lambda x}}{(\mu-\lambda)\left(c_1\mathrm{e}^{\lambda x} + c_2\,\mathrm{hypergeom}\left(\left[-\frac{\lambda}{\mu}\right], \left[\frac{\mu-\lambda}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\,\mathrm{hypergeom}\left(\left[\frac{\mu-\lambda}{\mu}\right], \left[\frac{-\lambda+2\mu}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)a\,\mathrm{e}^{(\mu-\lambda)x} + c_3(\mu-\lambda)}{(\mu-\lambda)\left(c_3\mathrm{e}^{\lambda x} + \mathrm{hypergeom}\left(\left[-\frac{\lambda}{\mu}\right], \left[\frac{\mu-\lambda}{\mu}\right], \frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{-\operatorname{hypergeom}\left(\left[\frac{\mu-\lambda}{\mu}\right], \left[\frac{-\lambda+2\mu}{\mu}\right], \frac{a\,e^{x\mu}}{\mu}\right) a\,e^{(\mu-\lambda)x} + c_3(\mu-\lambda)}{(\mu-\lambda)\left(c_3 e^{\lambda x} + \operatorname{hypergeom}\left(\left[-\frac{\lambda}{\mu}\right], \left[\frac{\mu-\lambda}{\mu}\right], \frac{a\,e^{x\mu}}{\mu}\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-\operatorname{hypergeom}\left(\left[\frac{\mu-\lambda}{\mu}\right], \left[\frac{-\lambda+2\mu}{\mu}\right], \frac{a\,e^{x\mu}}{\mu}\right) a\,e^{(\mu-\lambda)x} + c_3(\mu-\lambda)}{(\mu-\lambda)\left(c_3 e^{\lambda x} + \operatorname{hypergeom}\left(\left[-\frac{\lambda}{\mu}\right], \left[\frac{\mu-\lambda}{\mu}\right], \frac{a\,e^{x\mu}}{\mu}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 96

```
dsolve(diff(y(x),x)=-lambda*exp(lambda*x)*y(x)^2+a*exp(mu*x)*y(x)-a*exp((mu-lambda)*x),y(x),
```

$$y(x)=\frac{ac_1\mathrm{e}^{(\mu-\lambda)x}\,\mathrm{hypergeom}\left(\left[\frac{\mu-\lambda}{\mu}\right],\left[\frac{-\lambda+2\mu}{\mu}\right],\frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)+\lambda-\mu}{(\lambda-\mu)\left(c_1\,\mathrm{hypergeom}\left(\left[-\frac{\lambda}{\mu}\right],\left[\frac{\mu-\lambda}{\mu}\right],\frac{a\,\mathrm{e}^{x\mu}}{\mu}\right)+\mathrm{e}^{x\lambda}\right)}$$

✓ Solution by Mathematica

Time used: 4.358 (sec). Leaf size: 147

```
DSolve[y'[x]==-\[Lambda]*Exp[\[Lambda]*x]*y[x]^2+a*Exp[\[Mu]*x]*y[x]-a*Exp[(\[Mu]-\[Lambda])
```

$$y(x)\to\frac{e^{\lambda(-x)}\left(-\lambda\left(-\frac{ae^{\mu x}}{\mu}\right)^{\lambda/\mu}\Gamma\left(-\frac{\lambda}{\mu},-\frac{ae^{x\mu}}{\mu}\right)+\mu e^{\frac{ae^{\mu x}}{\mu}}+c_1\lambda(e^{\mu x})^{\lambda/\mu}\right)}{\lambda\left(-\left(-\frac{ae^{\mu x}}{\mu}\right)^{\lambda/\mu}\Gamma\left(-\frac{\lambda}{\mu},-\frac{ae^{x\mu}}{\mu}\right)+c_1\left(e^{\mu x}\right)^{\lambda/\mu}\right)}$$

$$y(x)\to e^{\lambda(-x)}$$

3.15 problem 15

Internal problem ID [10423]
Internal file name [`OUTPUT/9370_Monday_June_06_2022_02_19_07_PM_49470304/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\,e^{x\mu}y^2 - ab\,e^{x(\lambda+\mu)}y = -b\lambda\,e^{\lambda x}}$$

3.15.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,e^{x\mu}y^2 + ab\,e^{x(\lambda+\mu)}y - b\lambda\,e^{\lambda x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,e^{x\mu}y^2 + ab\,e^{\lambda x}e^{x\mu}y - b\lambda\,e^{\lambda x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -b\lambda\,e^{\lambda x}$, $f_1(x) = e^{x(\lambda+\mu)}ab$ and $f_2(x) = a\,e^{x\mu}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{a\,e^{x\mu}u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - \left(f_2' + f_1 f_2\right) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= a\mu\, e^{x\mu} \\ f_1 f_2 &= e^{x(\lambda+\mu)} a^2 b\, e^{x\mu} \\ f_2^2 f_0 &= -e^{2x\mu} e^{\lambda x} a^2 b \lambda \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$a\, e^{x\mu} u''(x) - \left(a\mu\, e^{x\mu} + e^{x(\lambda+\mu)} a^2 b\, e^{x\mu}\right) u'(x) - e^{2x\mu} e^{\lambda x} a^2 b \lambda u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = {} & 4\left(\mu + \frac{\lambda}{2}\right)^2 e^{\frac{e^{x(\lambda+\mu)} ab - 2(\lambda+\mu)\left(\frac{3\lambda}{2}+\mu\right)x}{2\lambda+2\mu}} c_2 \,\mathrm{WhittakerM}\left(\frac{\lambda+2\mu}{2\lambda+2\mu}, \frac{2\lambda+3\mu}{2\lambda+2\mu}, \frac{ab\, e^{x(\lambda+\mu)}}{\lambda+\mu}\right) \\ & + (\lambda+\mu)\left((\lambda+2\mu)\, e^{\frac{e^{x(\lambda+\mu)} ab - 2(\lambda+\mu)\left(\frac{3\lambda}{2}+\mu\right)x}{2\lambda+2\mu}} \right. \\ & \left. + ba\, e^{\frac{e^{x(\lambda+\mu)} ab - x\lambda(\lambda+\mu)}{2\lambda+2\mu}}\right) c_2 \,\mathrm{WhittakerM}\left(-\frac{\lambda}{2\lambda+2\mu}, \frac{2\lambda+3\mu}{2\lambda+2\mu}, \frac{ab\, e^{x(\lambda+\mu)}}{\lambda+\mu}\right) \\ & + c_1 e^{\frac{ab\, e^{x(\lambda+\mu)}}{\lambda+\mu}} \end{aligned}$$

The above shows that

$$u'(x) = 6\left(-\frac{2\left(\mu+\frac{\lambda}{2}\right)(\lambda+\mu)\,\mathrm{e}^{\frac{\mathrm{e}^{x(\lambda+\mu)}ab-2(\lambda+\mu)\left(\frac{3\lambda}{2}+\mu\right)x}{2\lambda+2\mu}}}{3} + a\,\mathrm{e}^{\frac{\mathrm{e}^{x(\lambda+\mu)}ab-x\lambda(\lambda+\mu)}{2\lambda+2\mu}}b\left(\frac{2\lambda}{3}+\mu\right)\right)\left(\mu+\frac{\lambda}{2}\right)c_2\,\mathrm{WhittakerM}\left(\frac{\lambda+2\mu}{2\lambda+2\mu}, \frac{2\lambda+3\mu}{2\lambda+2\mu}, \frac{ab\,\mathrm{e}^{x(\lambda+\mu)}}{\lambda+\mu}\right) + \left(\left(-\lambda^2-3\lambda\mu-2\mu^2\right)\mathrm{e}^{\frac{\mathrm{e}^{x(\lambda+\mu)}ab-2(\lambda+\mu)\left(\frac{3\lambda}{2}+\mu\right)x}{2\lambda+2\mu}} + \left(a\,\mathrm{e}^{\frac{\mathrm{e}^{x(\lambda+\mu)}ab+2(\lambda+\mu)x\left(\mu+\frac{\lambda}{2}\right)}{2\lambda+2\mu}}b + 2\left(\mu+\frac{\lambda}{2}\right)\mathrm{e}^{\frac{\mathrm{e}^{x(\lambda+\mu)}ab-x\lambda(\lambda+\mu)}{2\lambda+2\mu}}\right)ab\right)(\lambda+\mu)\,c_2\,\mathrm{WhittakerM}\left(-\frac{\lambda}{2\lambda+2\mu}, \frac{2\lambda+3\mu}{2\lambda+2\mu}, \frac{ab\,\mathrm{e}^{x(\lambda+\mu)}}{\lambda+\mu}\right) + 12\,\mathrm{e}^{-\frac{x(3\lambda+2\mu)}{2}}\left(\mu+\frac{\lambda}{2}\right)^2\left(\frac{2\lambda}{3}+\mu\right)c_2\left(\frac{ab\,\mathrm{e}^{x(\lambda+\mu)}}{\lambda+\mu}\right)^{\frac{3\lambda+4\mu}{2\lambda+2\mu}} + abc_1\mathrm{e}^{\frac{\mathrm{e}^{x(\lambda+\mu)}ab+x(\lambda+\mu)^2}{\lambda+\mu}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(6\left(-\frac{2\left(\mu+\frac{\lambda}{2}\right)(\lambda+\mu)\mathrm{e}^{\frac{\mathrm{e}^{x(\lambda+\mu)}ab-2(\lambda+\mu)\left(\frac{3\lambda}{2}+\mu\right)x}{2\lambda+2\mu}}}{3} + a\,\mathrm{e}^{\frac{\mathrm{e}^{x(\lambda+\mu)}ab-x\lambda(\lambda+\mu)}{2\lambda+2\mu}}b\left(\frac{2\lambda}{3}+\mu\right)\right)\left(\mu+\frac{\lambda}{2}\right)c_2\,\mathrm{WhittakerM}\left(\frac{\lambda+}{2\lambda+}\right.\right.}{a\left(4\left(\mu+\frac{\lambda}{2}\right)^2\mathrm{e}^{\frac{\mathrm{e}^{x(\lambda+}}{}}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\mathrm{e}^{-x\mu}\left(6\left(-\frac{2\left(\mu+\frac{\lambda}{2}\right)(\lambda+\mu)\mathrm{e}^{\frac{\mathrm{e}^{x(\lambda+\mu)}ab-2(\lambda+\mu)\left(\frac{3\lambda}{2}+\mu\right)x}{2\lambda+2\mu}}}{3} + a\,\mathrm{e}^{\frac{\mathrm{e}^{x(\lambda+\mu)}ab-x\lambda(\lambda+\mu)}{2\lambda+2\mu}}b\left(\frac{2\lambda}{3}+\mu\right)\right)\left(\mu+\frac{\lambda}{2}\right)\mathrm{WhittakerM}\left(\right.\right.}{\left(4\left(\mu+\frac{\lambda}{2}\right)^2\mathrm{e}^{\frac{\mathrm{e}^{x(\lambda+\mu)}ab\cdots}{}}\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-x\mu}\left(6\left(-\frac{2\left(\mu+\frac{\lambda}{2}\right)(\lambda+\mu)e^{\frac{e^{x(\lambda+\mu)}ab-2(\lambda+\mu)\left(\frac{3\lambda}{2}+\mu\right)x}{2\lambda+2\mu}}}{3}+a\,e^{\frac{e^{x(\lambda+\mu)}ab-x\lambda(\lambda+\mu)}{2\lambda+2\mu}}b\left(\frac{2\lambda}{3}+\mu\right)\right)\left(\mu+\frac{\lambda}{2}\right)\mathrm{WhittakerM}\left(\cdots\right.\right.}{\left(4\left(\mu+\frac{\lambda}{2}\right)^{2}e^{\frac{e^{x(\lambda+\mu)}ab\cdots}{\cdots}}\cdots\right.} \quad (1)$$

Verification of solutions

$$y = -\frac{e^{-x\mu}\left(6\left(-\frac{2\left(\mu+\frac{\lambda}{2}\right)(\lambda+\mu)e^{\frac{e^{x(\lambda+\mu)}ab-2(\lambda+\mu)\left(\frac{3\lambda}{2}+\mu\right)x}{2\lambda+2\mu}}}{3}+a\,e^{\frac{e^{x(\lambda+\mu)}ab-x\lambda(\lambda+\mu)}{2\lambda+2\mu}}b\left(\frac{2\lambda}{3}+\mu\right)\right)\left(\mu+\frac{\lambda}{2}\right)\mathrm{WhittakerM}\left(\cdots\right.\right.}{\left(4\left(\mu+\frac{\lambda}{2}\right)^{2}e^{\frac{e^{x(\lambda+\mu)}ab\cdots}{\cdots}}\cdots\right.}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (b*a*exp(lambda*x+mu*x)+mu)*(d
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
         <- Kovacics algorithm successful
         Change of variables used:
            [x = ln(t)/(lambda+mu)]
         Linear ODE actually solved:
            -a*b*lambda*u(t)+(-a*b*lambda*t-a*b*mu*t+lambda^2+lambda*mu)*diff(u(t),t)+(lambd
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 629

```
dsolve(diff(y(x),x)=a*exp(mu*x)*y(x)^2+a*b*exp((lambda+mu)*x)*y(x)-b*lambda*exp(lambda*x),y(
```

$$y(x) = \frac{-6\left(-\frac{2(\lambda+\mu)\left(\mu+\frac{\lambda}{2}\right)e^{\frac{ab\,e^{x(\lambda+\mu)}-4(\lambda+\mu)x\left(\mu+\frac{3\lambda}{4}\right)}{2\lambda+2\mu}}}{3}+a\,e^{\frac{ab\,e^{x(\lambda+\mu)}-2(\lambda+\mu)x\left(\mu+\frac{\lambda}{2}\right)}{2\lambda+2\mu}}b\left(\frac{2\lambda}{3}+\mu\right)\right)c_1\left(\mu+\frac{\lambda}{2}\right)\mathrm{WhittakerM}\left(\right.}{\left(4\,e^{\frac{ab\,e^{x(\lambda+\mu)}-2(\lambda+\mu)x\left(\frac{3\lambda}{2}+\mu\right.}{2\lambda+2\mu}}\right.}$$

✓ Solution by Mathematica

Time used: 12.587 (sec). Leaf size: 902

```
DSolve[y'[x]==a*Exp[\[Mu]*x]*y[x]^2+a*b*Exp[(\[Lambda]+\[Mu])*x]*y[x]-b*\[Lambda]*Exp[\[Lamb
```

$$y(x) \to \frac{e^{\mu(-x)}\left(ab\log\left(e^{\lambda+\mu}\right)\left(\left(e^{\lambda+\mu}\right)^x\right)^{\frac{\lambda+\mu}{\log\left(e^{\lambda+\mu}\right)}}\left(2(\lambda+\mu)L_{-\frac{\log\left(e^{\lambda+\mu}\right)}{2(\lambda+\mu)}-\frac{3}{2}}^{\frac{\mu\log\left(e^{\lambda+\mu}\right)}{(\lambda+\mu)^2}+1}\left(\frac{ab\left(\left(e^{\lambda+\mu}\right)^x\right)^{\frac{\lambda+\mu}{\log\left(e^{\lambda+\mu}\right)}}\log\left(e^{\lambda+\mu}\right)}{(\lambda+\mu)^2}\right)+c_1\left(\log\right.\right.\right.}{}$$

$$y(x) \to \frac{be^{\mu(-x)}\log\left(e^{\lambda+\mu}\right)\left(\log\left(e^{\lambda+\mu}\right)+\lambda+\mu\right)\left(\left(e^{\lambda+\mu}\right)^x\right)^{\frac{\lambda+\mu}{\log\left(e^{\lambda+\mu}\right)}}\mathrm{HypergeometricU}\left(\frac{1}{2}\left(\frac{\log\left(e^{\lambda+\mu}\right)}{\lambda+\mu}+3\right),\frac{\mu\log\left(e^{\lambda+\right.}}{(\lambda+\mu)^2}\right.}{2(\lambda+\mu)^2\,\mathrm{HypergeometricU}\left(\frac{\lambda+\mu+\log\left(e^{\lambda+\mu}\right)}{2(\lambda+\mu)},\frac{\mu\log\left(e^{\lambda+\mu}\right)}{(\lambda+\mu)^2}+1,\frac{ab\left(\left(e^{\lambda+\mu}\right)^x\right)^{\frac{\lambda+\mu}{\log\left(e^{\lambda+\mu}\right)}}}{(\lambda+\mu)^2}\right.}$$

$$-\frac{e^{\mu(-x)}\left((\lambda+\mu)\left(ab\left(\left(e^{\lambda+\mu}\right)^x\right)^{\frac{\lambda+\mu}{\log\left(e^{\lambda+\mu}\right)}}+\mu\right)+\log\left(e^{\lambda+\mu}\right)\left(\mu-ab\left(\left(e^{\lambda+\mu}\right)^x\right)^{\frac{\lambda+\mu}{\log\left(e^{\lambda+\mu}\right)}}\right)\right)}{2a(\lambda+\mu)}$$

3.16 problem 16

Internal problem ID [10424]
Internal file name [`OUTPUT/9371_Monday_June_06_2022_02_19_38_PM_36966880/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\,\mathrm{e}^{kx} y^2 - yb = c\,\mathrm{e}^{sx} + d\,\mathrm{e}^{-kx}}$$

3.16.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,\mathrm{e}^{kx} y^2 + by + c\,\mathrm{e}^{sx} + d\,\mathrm{e}^{-kx} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,\mathrm{e}^{kx} y^2 + by + c\,\mathrm{e}^{sx} + d\,\mathrm{e}^{-kx}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = c\,\mathrm{e}^{sx} + d\,\mathrm{e}^{-kx}$, $f_1(x) = b$ and $f_2(x) = a\,\mathrm{e}^{kx}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{a\,\mathrm{e}^{kx} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= ak\,e^{kx} \\ f_1f_2 &= ba\,e^{kx} \\ f_2^2f_0 &= a^2e^{2kx}\left(c\,e^{sx} + d\,e^{-kx}\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$a\,e^{kx}u''(x) - \left(ba\,e^{kx} + ak\,e^{kx}\right)u'(x) + a^2e^{2kx}\left(c\,e^{sx} + d\,e^{-kx}\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = e^{\frac{(b+k)x}{2}}\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bk+k^2}}{k+s}, \frac{2\sqrt{c}\,\sqrt{a}\,e^{\frac{x(k+s)}{2}}}{k+s}\right)c_1 + \operatorname{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bk+k^2}}{k+s}, \frac{2\sqrt{c}\,\sqrt{a}\,e^{\frac{x(k+s)}{2}}}{k+s}\right)c_2\right)$$

The above shows that

$$u'(x) = -\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bk+k^2}+k+s}{k+s}, \frac{2\sqrt{c}\,\sqrt{a}\,e^{\frac{x(k+s)}{2}}}{k+s}\right)c_1 + \operatorname{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bk+k^2}+k+s}{k+s}, \frac{2\sqrt{c}\,\sqrt{a}\,e^{\frac{x(k+s)}{2}}}{k+s}\right)c_2\right)\sqrt{a}\,\sqrt{c}\,e^{\frac{x(b+s+2k)}{2}} + \frac{e^{\frac{(b+k)x}{2}}\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bk+k^2}}{k+s}, \frac{2\sqrt{c}\,\sqrt{a}\,e^{\frac{x(k+s)}{2}}}{k+s}\right)c_1 + \operatorname{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bk+k^2}}{k+s}, \frac{2\sqrt{c}\,\sqrt{a}\,e^{\frac{x(k+s)}{2}}}{k+s}\right)c_2\right)(}{2}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bk+k^2}+k+s}{k+s}, \frac{2\sqrt{c}\,\sqrt{a}\,e^{\frac{x(k+s)}{2}}}{k+s}\right)c_1 + \operatorname{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bk+k^2}+k+s}{k+s}, \frac{2\sqrt{c}\,\sqrt{a}\,e^{\frac{x(k+s)}{2}}}{k+s}\right)\right.\right.}{a\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bk+}}{k+s}\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{e^{-\frac{x(3k+b)}{2}} \left(-2\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bk+k^2}+k+s}{k+s}, \frac{2\sqrt{c}\,\sqrt{a}\,e^{\frac{x(k+s)}{2}}}{k+s}\right) c_3 + \operatorname{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bk+k^2}+k+s}{k+s}, \frac{2\sqrt{c}\,\sqrt{\ldots}}{\ldots}\right.\right.\right.}{2a\left(\operatorname{BesselJ}\left(\ldots\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-\frac{x(3k+b)}{2}} \left(-2\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bk+k^2}+k+s}{k+s}, \frac{2\sqrt{c}\,\sqrt{a}\,e^{\frac{x(k+s)}{2}}}{k+s}\right) c_3 + \operatorname{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bk+k^2}+k+s}{k+s}, \frac{2\sqrt{c}\,\sqrt{\ldots}}{\ldots}\right.\right.\right.}{2a\left(\operatorname{BesselJ}\left(\ldots\right.\right.} \quad (1)$$

Verification of solutions

$$y = -\frac{e^{-\frac{x(3k+b)}{2}} \left(-2\left(\operatorname{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2bk+k^2}+k+s}{k+s}, \frac{2\sqrt{c}\,\sqrt{a}\,e^{\frac{x(k+s)}{2}}}{k+s}\right) c_3 + \operatorname{BesselY}\left(\frac{\sqrt{-4ad+b^2+2bk+k^2}+k+s}{k+s}, \frac{2\sqrt{c}\,\sqrt{\ldots}}{\ldots}\right.\right.\right.}{2a\left(\operatorname{BesselJ}\left(\ldots\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (b+k)*(diff(y(x), x))-a*exp(k*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            <- Bessel successful
         <- special function solution successful
         Change of variables used:
            [x = ln(t)/(s+k)]
         Linear ODE actually solved:
            (a*c*t+a*d)*u(t)+(-b*k*t-b*s*t+k*s*t+s^2*t)*diff(u(t),t)+(k^2*t^2+2*k*s*t^2+s^2*
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 332

```
dsolve(diff(y(x),x)=a*exp(k*x)*y(x)^2+b*y(x)+c*exp(s*x)+d*exp(-k*x),y(x), singsol=all)
```

$$y(x) = \frac{\left(\sqrt{c}\,a\left(\operatorname{BesselY}\left(\frac{\sqrt{-4ad+b^2+2kb+k^2}+s+k}{s+k}, \frac{2\sqrt{c}\,\sqrt{a}\,\mathrm{e}^{\frac{x(s+k)}{2}}}{s+k}\right)c_1 + \operatorname{BesselJ}\left(\frac{\sqrt{-4ad+b^2+2kb+k^2}+s+k}{s+k}, \frac{2\sqrt{c}\,\sqrt{a}\,\mathrm{e}^{\frac{x(s+k)}{2}}}{s+k}\right.\right.\right.}{a^{\frac{3}{2}}\left(\operatorname{BesselY}\left(\frac{\sqrt{-4ad+b^2+2kb+k^2}}{s+k}, \frac{2\sqrt{c}\,\sqrt{a}\,\mathrm{e}^{\frac{x}{}}}{s+k}\right.\right.}$$

✓ Solution by Mathematica

Time used: 18.386 (sec). Leaf size: 1636

```
DSolve[y'[x]==a*Exp[k*x]*y[x]^2+b*y[x]+c*Exp[s*x]+d*Exp[-k*x],y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{e^{-kx}\left(-\left((b+k)K_{\frac{\sqrt{\left(b^2+2kb+k^2-4ad\right)(k+s)^4\log^2\left(e^{k+s}\right)}}{(k+s)^4}}\left(2\sqrt{-\frac{ac\left(\left(e^{k+s}\right)^x\right)^{\frac{k+s}{\log\left(e^{k+s}\right)}}\log^2\left(e^{k+s}\right)}{(k+s)^4}}\right)\right)+(-1)^{\frac{k^4+4sk^3+6s^2k}{}}\right.}{}$$

$$y(x) \to \frac{e^{-kx}\left(-(b+k)(k+s)^3\sqrt{-\frac{ac\log^2\left(e^{k+s}\right)\left(\left(e^{k+s}\right)^x\right)^{\frac{k+s}{\log\left(e^{k+s}\right)}}}{(k+s)^4}}\operatorname{BesselI}\left(\frac{\sqrt{(b^2+2kb+k^2-4ad)(k+s)^4\log^2\left(e^{k+s}\right)}}{(k+s)^4}, 2\sqrt{-\frac{ac(}{}}\right.\right.}{}$$

3.17 problem 17

Internal problem ID [10425]
Internal file name [`OUTPUT/9372_Monday_June_06_2022_02_19_40_PM_7352493/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$\boxed{y' - a\,\mathrm{e}^{(\mu+2\lambda)x} y^2 - \left(\mathrm{e}^{x(\lambda+\mu)} b - \lambda\right) y = c\,\mathrm{e}^{x\mu}}$$

3.17.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,\mathrm{e}^{(\mu+2\lambda)x} y^2 + \mathrm{e}^{x(\lambda+\mu)} b y + c\,\mathrm{e}^{x\mu} - \lambda y \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,\mathrm{e}^{2\lambda x}\mathrm{e}^{x\mu} y^2 + \mathrm{e}^{\lambda x}\mathrm{e}^{x\mu} b y + c\,\mathrm{e}^{x\mu} - \lambda y$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = c\,\mathrm{e}^{x\mu}$, $f_1(x) = \mathrm{e}^{x(\lambda+\mu)} b - \lambda$ and $f_2(x) = \mathrm{e}^{(\mu+2\lambda)x} a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\mathrm{e}^{(\mu+2\lambda)x} a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - \left(f_2' + f_1 f_2\right) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= (\mu + 2\lambda)\,\mathrm{e}^{(\mu+2\lambda)x} a \\ f_1 f_2 &= \left(\mathrm{e}^{x(\lambda+\mu)} b - \lambda\right) \mathrm{e}^{(\mu+2\lambda)x} a \\ f_2^2 f_0 &= \mathrm{e}^{2(\mu+2\lambda)x} a^2 c\,\mathrm{e}^{x\mu} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\mathrm{e}^{(\mu+2\lambda)x} a u''(x) - \left((\mu + 2\lambda)\,\mathrm{e}^{(\mu+2\lambda)x} a + \left(\mathrm{e}^{x(\lambda+\mu)} b - \lambda\right) \mathrm{e}^{(\mu+2\lambda)x} a\right) u'(x) + \mathrm{e}^{2(\mu+2\lambda)x} a^2 c\,\mathrm{e}^{x\mu} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 79

```
dsolve(diff(y(x),x)=a*exp((2*lambda+mu)*x)*y(x)^2+(b*exp((lambda+mu)*x)-lambda)*y(x)+c*exp(m
```

$$y(x) = \frac{\mathrm{e}^{-x\lambda}\left(\sqrt{4a\,b^2c-b^4}\,\tan\left(\frac{(\mathrm{e}^{x(\lambda+\mu)}b+(\lambda+\mu)c_1)\sqrt{4a\,b^2c-b^4}}{2b^2(\lambda+\mu)}\right)-b^2\right)}{2ab}$$

✓ Solution by Mathematica

Time used: 6.375 (sec). Leaf size: 349

```
DSolve[y'[x]==a*Exp[(2*\[Lambda]+\[Mu])*x]*y[x]^2+(b*Exp[(\[Lambda]+\[Mu])*x]-\[Lambda])*y[x
```

$$y(x) \to \frac{e^{\lambda(-x)}\left(b^2e^{x(\lambda+\mu)}\left(\pi+ic_1\left(e^{\sqrt{\frac{(b^2-4ac)e^{2x(\lambda+\mu)}}{(\lambda+\mu)^2}}}-1\right)\right)\right)-b(\lambda+\mu)\sqrt{\frac{(b^2-4ac)e^{2x(\lambda+\mu)}}{(\lambda+\mu)^2}}\left(\pi-ic_1\left(e^{\sqrt{\frac{(b^2-4ac)e^{2}}{(\lambda+\mu)}}}\right.\right.}{2a(\lambda+\mu)\sqrt{\frac{(b^2-4ac)e^{2x(\lambda+\mu)}}{(\lambda+\mu)^2}}\left(\pi-ic_1\left(e^{\sqrt{\frac{(b^2-4ac)e^{2x(}}{(\lambda+\mu)^2}}}\right.\right.}$$

$$y(x) \to \frac{e^{\lambda(-x)}\left(-(\lambda+\mu)e^{-x(\lambda+\mu)}\sqrt{\frac{(b^2-4ac)e^{2x(\lambda+\mu)}}{(\lambda+\mu)^2}}\tanh\left(\frac{1}{2}\sqrt{\frac{(b^2-4ac)e^{2x(\lambda+\mu)}}{(\lambda+\mu)^2}}\right)-b\right)}{2a}$$

3.18 problem 18

Internal problem ID [10426]
Internal file name [`OUTPUT/9373_Monday_June_06_2022_02_19_41_PM_60059753/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\,\mathrm{e}^{kx} y^2 - yb = c\,\mathrm{e}^{knx} + d\,\mathrm{e}^{k(1+2n)x}}$$

3.18.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,\mathrm{e}^{kx} y^2 + by + c\,\mathrm{e}^{knx} + d\,\mathrm{e}^{k(1+2n)x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,\mathrm{e}^{kx} y^2 + by + c\,\mathrm{e}^{knx} + d\,\mathrm{e}^{2knx}\mathrm{e}^{kx}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = c\,\mathrm{e}^{knx} + d\,\mathrm{e}^{k(1+2n)x}$, $f_1(x) = b$ and $f_2(x) = a\,\mathrm{e}^{kx}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{a\,\mathrm{e}^{kx} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= ak\,e^{kx} \\ f_1f_2 &= ba\,e^{kx} \\ f_2^2f_0 &= a^2e^{2kx}\left(c\,e^{knx} + d\,e^{k(1+2n)x}\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$a\,e^{kx}u''(x) - \left(ba\,e^{kx} + ak\,e^{kx}\right)u'(x) + a^2e^{2kx}\left(c\,e^{knx} + d\,e^{k(1+2n)x}\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \text{DESol}\left(\left\{e^{2kx(n+1)}\_Y(x)\,ad + e^{kx(n+1)}\_Y(x)\,ac + \_Y''(x) + (-b-k)\_Y'(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\text{DESol}\left(\left\{e^{2kx(n+1)}\_Y(x)\,ad + e^{kx(n+1)}\_Y(x)\,ac + \_Y''(x) + (-b-k)\_Y'(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\text{DESol}\left(\left\{e^{2kx(n+1)}\_Y(x)\,ad + e^{kx(n+1)}\_Y(x)\,ac + \_Y''(x) + (-b-k)\_Y'(x)\right\}, \{\_Y(x)\}\right)\right)e^{-kx}}{a\,\text{DESol}\left(\left\{e^{2kx(n+1)}\_Y(x)\,ad + e^{kx(n+1)}\_Y(x)\,ac + \_Y''(x) + (-b-k)\_Y'(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\text{DESol}\left(\left\{e^{2kx(n+1)}\_Y(x)\,ad + e^{kx(n+1)}\_Y(x)\,ac + \_Y''(x) + (-b-k)\_Y'(x)\right\}, \{\_Y(x)\}\right)\right)e^{-kx}}{a\,\text{DESol}\left(\left\{e^{2kx(n+1)}\_Y(x)\,ad + e^{kx(n+1)}\_Y(x)\,ac + \_Y''(x) + (-b-k)\_Y'(x)\right\}, \{\_Y(x)\}\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\mathrm{e}^{2kx(n+1)} \_Y(x)\, ad + \mathrm{e}^{kx(n+1)} \_Y(x)\, ac + \_Y''(x) + (-b-k)\_Y'(x)\right\}, \{\_Y(x)\}\right)\right) \mathrm{e}^{-kx}}{a\, \mathrm{DESol}\left(\left\{\mathrm{e}^{2kx(n+1)} \_Y(x)\, ad + \mathrm{e}^{kx(n+1)} \_Y(x)\, ac + \_Y''(x) + (-b-k)\_Y'(x)\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\mathrm{e}^{2kx(n+1)} \_Y(x)\, ad + \mathrm{e}^{kx(n+1)} \_Y(x)\, ac + \_Y''(x) + (-b-k)\_Y'(x)\right\}, \{\_Y(x)\}\right)\right) \mathrm{e}^{-kx}}{a\, \mathrm{DESol}\left(\left\{\mathrm{e}^{2kx(n+1)} \_Y(x)\, ad + \mathrm{e}^{kx(n+1)} \_Y(x)\, ac + \_Y''(x) + (-b-k)\_Y'(x)\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (b+k)*(diff(y(x), x))-a*exp(k*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(a*exp(k*x)*y(x)^2+y(x)+y(x)*b*x+x^2*(c*e
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=a*exp(k*x)*y(x)^2+b*y(x)+c*exp(k*n*x)+d*exp(k*(2*n+1)*x),y(x), singsol=a
```

No solution found

✓ Solution by Mathematica

Time used: 27.598 (sec). Leaf size: 2503

```
DSolve[y'[x]==a*Exp[k*x]*y[x]^2+b*y[x]+c*Exp[k*n*x]+d*Exp[k*(2*n+1)*x],y[x],x,IncludeSingula
```

Too large to display

3.19 problem 19

Internal problem ID [10427]
Internal file name [`OUTPUT/9374_Monday_June_06_2022_02_19_43_PM_62152854/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$\boxed{y' - e^{x\mu}\left(y - b\,e^{\lambda x}\right)^2 = b\lambda\,e^{\lambda x}}$$

3.19.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= e^{x\mu}e^{2\lambda x}b^2 - 2\,e^{\lambda x}e^{x\mu}by + e^{x\mu}y^2 + b\lambda\,e^{\lambda x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = e^{x\mu}e^{2\lambda x}b^2 - 2\,e^{\lambda x}e^{x\mu}by + e^{x\mu}y^2 + b\lambda\,e^{\lambda x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = e^{x\mu}e^{2\lambda x}b^2 + b\lambda\,e^{\lambda x}$, $f_1(x) = -2b\,e^{\lambda x}e^{x\mu}$ and $f_2(x) = e^{x\mu}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{e^{x\mu}u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \mu\,\mathrm{e}^{x\mu} \\ f_1f_2 &= -2b\,\mathrm{e}^{\lambda x}\mathrm{e}^{2x\mu} \\ f_2^2f_0 &= \mathrm{e}^{2x\mu}\left(\mathrm{e}^{x\mu}\mathrm{e}^{2\lambda x}b^2 + b\lambda\,\mathrm{e}^{\lambda x}\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\mathrm{e}^{x\mu}u''(x) - \left(\mu\,\mathrm{e}^{x\mu} - 2b\,\mathrm{e}^{\lambda x}\mathrm{e}^{2x\mu}\right)u'(x) + \mathrm{e}^{2x\mu}\left(\mathrm{e}^{x\mu}\mathrm{e}^{2\lambda x}b^2 + b\lambda\,\mathrm{e}^{\lambda x}\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\frac{-2\,\mathrm{e}^{x(\lambda+\mu)}b + x\mu(\lambda+\mu)}{2\lambda+2\mu}}\left(c_1\sinh\left(\frac{x\mu}{2}\right) + c_2\cosh\left(\frac{x\mu}{2}\right)\right)$$

The above shows that

$$\begin{aligned} u'(x) = -\Bigg(\left(\mathrm{e}^{x(\lambda+\mu)}c_2b - \frac{\mu(c_1+c_2)}{2}\right)\cosh\left(\frac{x\mu}{2}\right) \\ + \left(\mathrm{e}^{x(\lambda+\mu)}bc_1 - \frac{\mu(c_1+c_2)}{2}\right)\sinh\left(\frac{x\mu}{2}\right)\Bigg)\mathrm{e}^{\frac{-2\,\mathrm{e}^{x(\lambda+\mu)}b + x\mu(\lambda+\mu)}{2\lambda+2\mu}} \end{aligned}$$

Using the above in (1) gives the solution

$$y = \frac{\left(\left(\mathrm{e}^{x(\lambda+\mu)}c_2b - \frac{\mu(c_1+c_2)}{2}\right)\cosh\left(\frac{x\mu}{2}\right) + \left(\mathrm{e}^{x(\lambda+\mu)}bc_1 - \frac{\mu(c_1+c_2)}{2}\right)\sinh\left(\frac{x\mu}{2}\right)\right)\mathrm{e}^{-x\mu}}{c_1\sinh\left(\frac{x\mu}{2}\right) + c_2\cosh\left(\frac{x\mu}{2}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\left(\mathrm{e}^{x(\lambda+\mu)}b - \frac{\mu(c_3+1)}{2}\right)\cosh\left(\frac{x\mu}{2}\right) + \left(\mathrm{e}^{x(\lambda+\mu)}bc_3 - \frac{\mu(c_3+1)}{2}\right)\sinh\left(\frac{x\mu}{2}\right)\right)\mathrm{e}^{-x\mu}}{c_3\sinh\left(\frac{x\mu}{2}\right) + \cosh\left(\frac{x\mu}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\left(e^{x(\lambda+\mu)}b - \frac{\mu(c_3+1)}{2}\right)\cosh\left(\frac{x\mu}{2}\right) + \left(e^{x(\lambda+\mu)}bc_3 - \frac{\mu(c_3+1)}{2}\right)\sinh\left(\frac{x\mu}{2}\right)\right)e^{-x\mu}}{c_3\sinh\left(\frac{x\mu}{2}\right) + \cosh\left(\frac{x\mu}{2}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\left(e^{x(\lambda+\mu)}b - \frac{\mu(c_3+1)}{2}\right)\cosh\left(\frac{x\mu}{2}\right) + \left(e^{x(\lambda+\mu)}bc_3 - \frac{\mu(c_3+1)}{2}\right)\sinh\left(\frac{x\mu}{2}\right)\right)e^{-x\mu}}{c_3\sinh\left(\frac{x\mu}{2}\right) + \cosh\left(\frac{x\mu}{2}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular polynomial solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(diff(y(x),x)=exp(mu*x)*(y(x)-b*exp(lambda*x))^2+b*lambda*exp(lambda*x),y(x), singsol=
```

$$y(x) = \frac{\left(e^{x(\lambda+\mu)}c_1 b\mu + b\,e^{x\lambda} - c_1\mu^2\right)e^{-x\mu}}{c_1\mu + e^{-x\mu}}$$

✓ Solution by Mathematica

Time used: 1.524 (sec). Leaf size: 40

```
DSolve[y'[x]==Exp[\[Mu]*x]*(y[x]-b*Exp[\[Lambda]*x])^2+b*\[Lambda]*Exp[\[Lambda]*x],y[x],x,I
```

$$y(x) \to be^{\lambda x} + \frac{\mu}{-e^{\mu x} + c_1 \mu}$$
$$y(x) \to be^{\lambda x}$$

3.20 problem 20

Internal problem ID [10428]
Internal file name [`OUTPUT/9375_Monday_June_06_2022_02_19_44_PM_77437593/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{\left(e^{\lambda x} a+b\,e^{x \mu}+c\right) y'-y^{2}-k\,e^{\nu x} y=-m^{2}+k m\,e^{\nu x}}$$

3.20.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= \frac{y^{2}+k\,e^{\nu x} y-m^{2}+k m\,e^{\nu x}}{e^{\lambda x} a+b\,e^{x \mu}+c}
\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{k m\,e^{\nu x}}{e^{\lambda x} a+b\,e^{x \mu}+c}+\frac{k\,e^{\nu x} y}{e^{\lambda x} a+b\,e^{x \mu}+c}-\frac{m^{2}}{e^{\lambda x} a+b\,e^{x \mu}+c}+\frac{y^{2}}{e^{\lambda x} a+b\,e^{x \mu}+c}$$

With Riccati ODE standard form

$$y' = f_0(x)+ f_1(x)y+f_2(x)y^{2}$$

Shows that $f_0(x)=\frac{-m^{2}+k m\,e^{\nu x}}{e^{\lambda x} a+b\,e^{x \mu}+c}$, $f_1(x)=\frac{k\,e^{\nu x}}{e^{\lambda x} a+b\,e^{x \mu}+c}$ and $f_2(x)=\frac{1}{e^{\lambda x} a+b\,e^{x \mu}+c}$. Let

$$\begin{aligned}
y &= \frac{-u'}{f_2 u}\\
&= \frac{-u'}{\frac{u}{e^{\lambda x} a+b\,e^{x \mu}+c}} \qquad (1)
\end{aligned}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$f_2' = -\frac{a\lambda\,\mathrm{e}^{\lambda x} + b\mu\,\mathrm{e}^{x\mu}}{(\mathrm{e}^{\lambda x}a + b\,\mathrm{e}^{x\mu} + c)^2}$$
$$f_1f_2 = \frac{k\,\mathrm{e}^{\nu x}}{(\mathrm{e}^{\lambda x}a + b\,\mathrm{e}^{x\mu} + c)^2}$$
$$f_2^2f_0 = \frac{-m^2 + km\,\mathrm{e}^{\nu x}}{(\mathrm{e}^{\lambda x}a + b\,\mathrm{e}^{x\mu} + c)^3}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{\mathrm{e}^{\lambda x}a + b\,\mathrm{e}^{x\mu} + c} - \left(-\frac{a\lambda\,\mathrm{e}^{\lambda x} + b\mu\,\mathrm{e}^{x\mu}}{(\mathrm{e}^{\lambda x}a + b\,\mathrm{e}^{x\mu} + c)^2} + \frac{k\,\mathrm{e}^{\nu x}}{(\mathrm{e}^{\lambda x}a + b\,\mathrm{e}^{x\mu} + c)^2}\right)u'(x) + \frac{(-m^2 + km\,\mathrm{e}^{\nu x})\,u(x)}{(\mathrm{e}^{\lambda x}a + b\,\mathrm{e}^{x\mu} + c)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{(2\_Y''(x) + \_Y'(x)\,(\lambda + \mu))\,ab\,\mathrm{e}^{x(\lambda+\mu)} - ka\_Y'(x)\,\mathrm{e}^{x(\lambda+\nu)} - k\_Y'(x)\,b\,\mathrm{e}^{x(\mu+\nu)} + a^2(\_Y'(x)\,\lambda\right.\right.$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{(2\_Y''(x) + \_Y'(x)\,(\lambda + \mu))\,ab\,\mathrm{e}^{x(\lambda+\mu)} - ka\_Y'(x)\,\mathrm{e}^{x(\lambda+\nu)} - k\_Y'(x)\,b\,\mathrm{e}^{x(\mu+\nu)} + a^2(\_Y'(\right.\right.$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(2\_Y''(x)+\_Y'(x)(\lambda+\mu)\right)ab\,\mathrm{e}^{x(\lambda+\mu)}-ka\_Y'(x)\mathrm{e}^{x(\lambda+\nu)}-k\_Y'(x)b\,\mathrm{e}^{x(\mu+\nu)}+a^2\left(\_Y'(x)\lambda+\_Y''(x)\right)\mathrm{e}^{2\lambda x}+}{}\right.\right.\right.}{\mathrm{DESol}\left(\left\{\frac{\left(2\_Y''(x)+\_Y'(x)(\lambda+\mu)\right)ab\,\mathrm{e}^{x(\lambda+\mu)}-ka\_Y'(x)\mathrm{e}^{x(\lambda+\nu)}-k\_Y'(x)b\,\mathrm{e}^{x(\mu+\nu)}+a^2\left(\_Y'(x)\lambda+\_Y''(\right.}{}\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(2\_\_Y''(x)+\_\_Y'(x)(\lambda+\mu)\right)ab\,\mathrm{e}^{x(\lambda+\mu)}-ka\_\_Y'(x)\mathrm{e}^{x(\lambda+\nu)}-k\_\_Y'(x)b\,\mathrm{e}^{x(\mu+\nu)}+a^{2}\left(\_\_Y'(x)\lambda+\_\_Y''(x)\right)\mathrm{e}^{2\lambda x}+ \cdots}{\cdots}\right.\right.\right.}{\mathrm{DESol}\left(\left\{\frac{\left(2\_\_Y''(x)+\_\_Y'(x)(\lambda+\mu)\right)ab\,\mathrm{e}^{x(\lambda+\mu)}-ka\_\_Y'(x)\mathrm{e}^{x(\lambda+\nu)}-k\_\_Y'(x)b\,\mathrm{e}^{x(\mu+\nu)}+a^{2}\left(\_\_Y'(x)\lambda+\_\_Y''(\cdots\right.}{\cdots}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(2\_\_Y''(x)+\_\_Y'(x)(\lambda+\mu)\right)ab\,\mathrm{e}^{x(\lambda+\mu)}-ka\_\_Y'(x)\mathrm{e}^{x(\lambda+\nu)}-k\_\_Y'(x)b\,\mathrm{e}^{x(\mu+\nu)}+a^{2}\left(\_\_Y'(x)\lambda+\_\_Y''(x)\right)\mathrm{e}^{2\lambda x}+ \cdots}{\cdots}\right.\right.\right.}{\mathrm{DESol}\left(\left\{\frac{\left(2\_\_Y''(x)+\_\_Y'(x)(\lambda+\mu)\right)ab\,\mathrm{e}^{x(\lambda+\mu)}-ka\_\_Y'(x)\mathrm{e}^{x(\lambda+\nu)}-k\_\_Y'(x)b\,\mathrm{e}^{x(\mu+\nu)}+a^{2}\left(\_\_Y'(x)\lambda+\_\_Y''(\cdots\right.}{\cdots}\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(2\_\_Y''(x)+\_\_Y'(x)(\lambda+\mu)\right)ab\,\mathrm{e}^{x(\lambda+\mu)}-ka\_\_Y'(x)\mathrm{e}^{x(\lambda+\nu)}-k\_\_Y'(x)b\,\mathrm{e}^{x(\mu+\nu)}+a^{2}\left(\_\_Y'(x)\lambda+\_\_Y''(x)\right)\mathrm{e}^{2\lambda x}+ \cdots}{\cdots}\right.\right.\right.}{\mathrm{DESol}\left(\left\{\frac{\left(2\_\_Y''(x)+\_\_Y'(x)(\lambda+\mu)\right)ab\,\mathrm{e}^{x(\lambda+\mu)}-ka\_\_Y'(x)\mathrm{e}^{x(\lambda+\nu)}-k\_\_Y'(x)b\,\mathrm{e}^{x(\mu+\nu)}+a^{2}\left(\_\_Y'(x)\lambda+\_\_Y''(\cdots\right.}{\cdots}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 202

```
dsolve((a*exp(lambda*x)+b*exp(mu*x)+c)*diff(y(x),x)=y(x)^2+k*exp(nu*x)*y(x)-m^2+k*m*exp(nu*x
```

$$y(x) = \frac{-m\left(\int \frac{e^{k\left(\int \frac{e^{\nu x}}{e^{x\lambda}a+b\,e^{x\mu}+c}dx\right)-2m\left(\int \frac{1}{e^{x\lambda}a+b\,e^{x\mu}+c}dx\right)}}{e^{x\lambda}a+b\,e^{x\mu}+c}dx\right)+c_1 m-e^{k\left(\int \frac{e^{\nu x}}{e^{x\lambda}a+b\,e^{x\mu}+c}dx\right)-2m\left(\int \frac{1}{e^{x\lambda}a+b\,e^{x\mu}+c}dx\right)}}{\int \frac{e^{k\left(\int \frac{e^{\nu x}}{e^{x\lambda}a+b\,e^{x\mu}+c}dx\right)-2m\left(\int \frac{1}{e^{x\lambda}a+b\,e^{x\mu}+c}dx\right)}}{e^{x\lambda}a+b\,e^{x\mu}+c}dx-c_1}$$

✓ Solution by Mathematica

Time used: 16.545 (sec). Leaf size: 358

```
DSolve[(a*Exp[\[Lambda]*x]+b*Exp[\[Mu]*x]+c)*y'[x]==y[x]^2+k*Exp[\[Nu]*x]*y[x]-m^2+k*m*Exp[\
```

$$\text{Solve}\left[\int_1^x -\frac{\exp\left(-\int_1^{K[2]} -\frac{e^{\nu K[1]}k-2m}{e^{\lambda K[1]}a+be^{\mu K[1]}+c}dK[1]\right)\left(e^{\nu K[2]}k-m+y(x)\right)}{\left(e^{\lambda K[2]}a+be^{\mu K[2]}+c\right)k\nu(m+y(x))}dK[2]\right.$$
$$+\int_1^{y(x)}\left(\frac{\exp\left(-\int_1^x -\frac{e^{\nu K[1]}k-2m}{e^{\lambda K[1]}a+be^{\mu K[1]}+c}dK[1]\right)}{k\nu(m+K[3])^2}\right.$$
$$-\int_1^x\left(\frac{\exp\left(-\int_1^{K[2]} -\frac{e^{\nu K[1]}k-2m}{e^{\lambda K[1]}a+be^{\mu K[1]}+c}dK[1]\right)\left(e^{\nu K[2]}k-m+K[3]\right)}{\left(e^{\lambda K[2]}a+be^{\mu K[2]}+c\right)k\nu(m+K[3])^2}-\frac{\exp\left(-\int_1^{K[2]} -\frac{e^{\nu K[1]}k-2m}{e^{\lambda K[1]}a+be^{\mu K[1]}+c}dK[1]\right.}{\left(e^{\lambda K[2]}a+be^{\mu K[2]}+c\right)k\nu(m+K[3]}\right.$$

3.21 problem 21

Internal problem ID [10429]
Internal file name [`OUTPUT/9376_Monday_June_06_2022_02_20_48_PM_50645941/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3. Equations Containing Exponential Functions
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{\left(e^{\lambda x} a + b\,e^{x\mu} + c\right)\left(y' - y^2\right) = -a\,\lambda^2 e^{\lambda x} - b\,\mu^2 e^{x\mu}}$$

3.21.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{e^{\lambda x} a\,y^2 - a\,\lambda^2 e^{\lambda x} + b\,e^{x\mu} y^2 - b\,\mu^2 e^{x\mu} + c\,y^2}{e^{\lambda x} a + b\,e^{x\mu} + c} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{a\,\lambda^2 e^{\lambda x}}{e^{\lambda x} a + b\,e^{x\mu} + c} + \frac{e^{\lambda x} a\,y^2}{e^{\lambda x} a + b\,e^{x\mu} + c} - \frac{b\,\mu^2 e^{x\mu}}{e^{\lambda x} a + b\,e^{x\mu} + c} + \frac{b\,e^{x\mu} y^2}{e^{\lambda x} a + b\,e^{x\mu} + c} + \frac{c\,y^2}{e^{\lambda x} a + b\,e^{x\mu} + c}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{-a\,\lambda^2 e^{\lambda x} - b\,\mu^2 e^{x\mu}}{e^{\lambda x} a + b\,e^{x\mu} + c}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= \frac{-a\,\lambda^2 \mathrm{e}^{\lambda x} - b\,\mu^2 \mathrm{e}^{x\mu}}{\mathrm{e}^{\lambda x}a + b\,\mathrm{e}^{x\mu} + c} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{\left(-a\,\lambda^2 \mathrm{e}^{\lambda x} - b\,\mu^2 \mathrm{e}^{x\mu}\right) u(x)}{\mathrm{e}^{\lambda x}a + b\,\mathrm{e}^{x\mu} + c} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\left(\int \frac{1}{\left(\mathrm{e}^{\lambda x}a + b\,\mathrm{e}^{x\mu} + c\right)^2}dx\right) c_1 + c_2\right) \left(\mathrm{e}^{\lambda x}a + b\,\mathrm{e}^{x\mu} + c\right)$$

The above shows that

$$u'(x) = \frac{\left((\lambda+\mu)\,\mathrm{e}^{x(\lambda+\mu)}ab + a^2\lambda\,\mathrm{e}^{2\lambda x} + \mathrm{e}^{2x\mu}b^2\mu + c\left(a\lambda\,\mathrm{e}^{\lambda x} + b\mu\,\mathrm{e}^{x\mu}\right)\right) c_1\left(\int \frac{1}{\left(\mathrm{e}^{\lambda x}a+b\,\mathrm{e}^{x\mu}+c\right)^2}dx\right) + abc_2(\lambda+\mu)\,\mathrm{e}^{x(\lambda}}{\mathrm{e}^{\lambda x}a + b\,\mathrm{e}^{x\mu} + c}$$

Using the above in (1) gives the solution

$$y = -\frac{\left((\lambda+\mu)\,\mathrm{e}^{x(\lambda+\mu)}ab + a^2\lambda\,\mathrm{e}^{2\lambda x} + \mathrm{e}^{2x\mu}b^2\mu + c\left(a\lambda\,\mathrm{e}^{\lambda x} + b\mu\,\mathrm{e}^{x\mu}\right)\right) c_1\left(\int \frac{1}{\left(\mathrm{e}^{\lambda x}a+b\,\mathrm{e}^{x\mu}+c\right)^2}dx\right) + abc_2(\lambda+\mu)\,\mathrm{e}}{\left(\mathrm{e}^{\lambda x}a + b\,\mathrm{e}^{x\mu} + c\right)^2\left(\left(\int \frac{1}{\left(\mathrm{e}^{\lambda x}a+b\,\mathrm{e}^{x\mu}+c\right)^2}dx\right) c_1 +\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\left((\lambda+\mu)\,\mathrm{e}^{x(\lambda+\mu)}ab + a^2\lambda\,\mathrm{e}^{2\lambda x} + \mathrm{e}^{2x\mu}b^2\mu + c\left(a\lambda\,\mathrm{e}^{\lambda x} + b\mu\,\mathrm{e}^{x\mu}\right)\right) c_3\left(\int \frac{1}{\left(\mathrm{e}^{\lambda x}a+b\,\mathrm{e}^{x\mu}+c\right)^2}dx\right) - (\lambda+\mu)\,\mathrm{e}^{x(\lambda+\mu}}{\left(\mathrm{e}^{\lambda x}a + b\,\mathrm{e}^{x\mu} + c\right)^2\left(\left(\int \frac{1}{\left(\mathrm{e}^{\lambda x}a+b\,\mathrm{e}^{x\mu}+c\right)^2}dx\right) c_3 + 1\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\left(\left(\lambda+\mu\right) \mathrm{e}^{x(\lambda+\mu)} a b + a^{2} \lambda \,\mathrm{e}^{2 \lambda x} + \mathrm{e}^{2 x \mu} b^{2} \mu + c \left(a \lambda \,\mathrm{e}^{\lambda x} + b \mu \,\mathrm{e}^{x \mu}\right)\right) c_{3} \left(\int \frac{1}{\left(\mathrm{e}^{\lambda x} a + b \,\mathrm{e}^{x \mu} + c\right)^{2}}d x\right) - \left(\lambda+\mu\right) \mathrm{e}^{x(\lambda+\mu}}{\left(\mathrm{e}^{\lambda x} a + b \,\mathrm{e}^{x \mu} + c\right)^{2} \left(\left(\int \frac{1}{\left(\mathrm{e}^{\lambda x} a + b \,\mathrm{e}^{x \mu} + c\right)^{2}}d x\right) c_{3} + 1\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-\left(\left(\lambda+\mu\right) \mathrm{e}^{x(\lambda+\mu)} a b + a^{2} \lambda \,\mathrm{e}^{2 \lambda x} + \mathrm{e}^{2 x \mu} b^{2} \mu + c \left(a \lambda \,\mathrm{e}^{\lambda x} + b \mu \,\mathrm{e}^{x \mu}\right)\right) c_{3} \left(\int \frac{1}{\left(\mathrm{e}^{\lambda x} a + b \,\mathrm{e}^{x \mu} + c\right)^{2}}d x\right) - \left(\lambda+\mu\right) \mathrm{e}^{x(\lambda+\mu}}{\left(\mathrm{e}^{\lambda x} a + b \,\mathrm{e}^{x \mu} + c\right)^{2} \left(\left(\int \frac{1}{\left(\mathrm{e}^{\lambda x} a + b \,\mathrm{e}^{x \mu} + c\right)^{2}}d x\right) c_{3} + 1\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a*lambda^2*exp(lambda*x)+mu^2
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      <- linear_1 successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 176

```
dsolve((a*exp(lambda*x)+b*exp(mu*x)+c)*(diff(y(x),x)-y(x)^2)+a*lambda^2*exp(lambda*x)+b*mu^2
```

$$y(x) = \frac{\left(-ab(\lambda+\mu)\,e^{x(\lambda+\mu)} - a^2\lambda\,e^{2x\lambda} - e^{2x\mu}b^2\mu - c\left(a\lambda\,e^{x\lambda} + b\mu\,e^{x\mu}\right)\right)\left(\int \frac{1}{\left(e^{x\lambda}a+b\,e^{x\mu}+c\right)^2}dx\right) - abc_1(\lambda+\mu)\,e^{x(\lambda}}{\left(e^{x\lambda}a+b\,e^{x\mu}+c\right)^2\left(c_1+\int \frac{1}{\left(e^{x\lambda}a+b\,e^{x\mu}+c\right)^2}dx\right)}$$

✓ Solution by Mathematica

Time used: 24.922 (sec). Leaf size: 393

```
DSolve[(a*Exp[\[Lambda]*x]+b*Exp[\[Mu]*x]+c)*(y'[x]-y[x]^2)+a*\[Lambda]^2*Exp[\[Lambda]*x]+b
```

$$\text{Solve}\left[\int_1^x -\frac{-ae^{\lambda K[1]}\lambda^2 - be^{\mu K[1]}\mu^2 + ae^{\lambda K[1]}y(x)^2 + be^{\mu K[1]}y(x)^2 + cy(x)^2}{\left(e^{\lambda K[1]}a + be^{\mu K[1]} + c\right)\left(ae^{\lambda K[1]}\lambda + be^{\mu K[1]}\mu + ae^{\lambda K[1]}y(x) + be^{\mu K[1]}y(x) + cy(x)\right)^2}dK[1] + \int_1^{y(x)}\left(\frac{1}{\left(ae^{x\lambda}\lambda + be^{x\mu}\mu + ae^{x\lambda}K[2] + be^{x\mu}K[2] + cK[2]\right)^2} - \int_1^x\left(\frac{2\left(-ae^{\lambda K[1]}\lambda^2 - be^{\mu K[1]}\mu^2 + ae^{\lambda K[1]}K[2]^2 + be^{\mu K[1]}K[2]^2 + cK[2]^2\right)}{\left(ae^{\lambda K[1]}\lambda + be^{\mu K[1]}\mu + ae^{\lambda K[1]}K[2] + be^{\mu K[1]}K[2] + cK[2]\right)^3} - \frac{2ae^{\lambda K[1}}{\left(e^{\lambda K[1]}a + be^{\mu K[1]} + c\right)\left(ae^{\lambda K[}\right.}\right.\right.\right.$$

4 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions

4.1 problem 22

Internal problem ID [10430]
Internal file name [`OUTPUT/9377_Monday_June_06_2022_02_20_50_PM_93195118/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 - ax\,\mathrm{e}^{\lambda x} y = \mathrm{e}^{\lambda x} a}$$

4.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + \mathrm{e}^{\lambda x} axy + \mathrm{e}^{\lambda x} a \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \mathrm{e}^{\lambda x} axy + \mathrm{e}^{\lambda x} a$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \mathrm{e}^{\lambda x} a$, $f_1(x) = ax\,\mathrm{e}^{\lambda x}$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= ax\, \mathrm{e}^{\lambda x} \\ f_2^2 f_0 &= \mathrm{e}^{\lambda x} a \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - ax\, \mathrm{e}^{\lambda x} u'(x) + \mathrm{e}^{\lambda x} a u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{x \left(c_2 \lambda^2 + \left(\int \frac{\mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}}}{x^2} dx\right) c_1\right)}{\lambda^2}$$

The above shows that

$$u'(x) = \frac{c_2 \lambda^2 x + c_1 \left(\int \frac{\mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}}}{x^2} dx\right) x + \mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}} c_1}{\lambda^2 x}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2 \lambda^2 x + c_1 \left(\int \frac{\mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}}}{x^2} dx\right) x + \mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}} c_1}{x^2 \left(c_2 \lambda^2 + \left(\int \frac{\mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}}}{x^2} dx\right) c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\lambda^2 x - c_3 \left(\int \frac{\mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}}}{x^2} dx\right) x - \mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}} c_3}{x^2 \left(\lambda^2 + \left(\int \frac{\mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}}}{x^2} dx\right) c_3\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\lambda^2 x - c_3 \left(\int \frac{\mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}}}{x^2} dx\right) x - \mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}} c_3}{x^2 \left(\lambda^2 + \left(\int \frac{\mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}}}{x^2} dx\right) c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-\lambda^2 x - c_3 \left(\int \frac{\mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}}}{x^2} dx\right) x - \mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}} c_3}{x^2 \left(\lambda^2 + \left(\int \frac{\mathrm{e}^{\frac{(\lambda x-1)\mathrm{e}^{\lambda x} a}{\lambda^2}}}{x^2} dx\right) c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 85

```
dsolve(diff(y(x),x)=y(x)^2+a*x*exp(lambda*x)*y(x)+a*exp(lambda*x),y(x), singsol=all)
```

$$y(x) = \frac{-c_1\lambda^2 x + \left(\int \frac{\mathrm{e}^{\frac{\mathrm{e}^{x\lambda} a(x\lambda-1)}{\lambda^2}}}{x^2} dx\right) x + \mathrm{e}^{\frac{\mathrm{e}^{x\lambda} a(x\lambda-1)}{\lambda^2}}}{x^2 \left(c_1\lambda^2 - \left(\int \frac{\mathrm{e}^{\frac{\mathrm{e}^{x\lambda} a(x\lambda-1)}{\lambda^2}}}{x^2} dx\right)\right)}$$

✓ Solution by Mathematica

Time used: 2.132 (sec). Leaf size: 110

```
DSolve[y'[x]==y[x]^2+a*x*Exp[\[Lambda]*x]*y[x]+a*Exp[\[Lambda]*x],y[x],x,IncludeSingularSolu
```

$$y(x) \to -\frac{x \int_1^x \frac{e^{\frac{ae^{\lambda K[1]}(\lambda K[1]-1)}{\lambda^2}}}{K[1]^2} dK[1] + e^{\frac{ae^{\lambda x}(\lambda x-1)}{\lambda^2}} + c_1 x}{x^2 \left(\int_1^x \frac{e^{\frac{ae^{\lambda K[1]}(\lambda K[1]-1)}{\lambda^2}}}{K[1]^2} dK[1] + c_1\right)}$$

$$y(x) \to -\frac{1}{x}$$

4.2 problem 23

Internal problem ID [10431]
Internal file name [`OUTPUT/9378_Monday_June_06_2022_02_20_51_PM_54210900/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Riccati]`

$$\boxed{y' - a\,\mathrm{e}^{\lambda x} y^2 = b\,\mathrm{e}^{-\lambda x}}$$

4.2.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \mathrm{e}^{\lambda x} a\,y^2 + b\,\mathrm{e}^{-\lambda x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \left(\mathrm{e}^{\lambda x} a\, y^2 + b\, \mathrm{e}^{-\lambda x}\right)(b_3 - a_2) - \left(\mathrm{e}^{\lambda x} a\, y^2 + b\, \mathrm{e}^{-\lambda x}\right)^2 a_3 - \left(\mathrm{e}^{\lambda x} a\lambda\, y^2 - b\lambda\, \mathrm{e}^{-\lambda x}\right)(x a_2 + y a_3 + a_1) - 2a\, \mathrm{e}^{\lambda x} y(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\mathrm{e}^{2\lambda x} a^2 y^4 a_3 - 2\,\mathrm{e}^{\lambda x} \mathrm{e}^{-\lambda x} ab\, y^2 a_3 - \mathrm{e}^{\lambda x} a\lambda x\, y^2 a_2 - \mathrm{e}^{\lambda x} a\lambda\, y^3 a_3 - \mathrm{e}^{\lambda x} a\lambda\, y^2 a_1 - 2\,\mathrm{e}^{\lambda x} axyb_2 - \mathrm{e}^{\lambda x} a\, y^2 a_2 - \mathrm{e}^{\lambda x} a\, y^2 b_3 - \mathrm{e}^{-2\lambda x} b^2 a_3 + \mathrm{e}^{-\lambda x} b\lambda x a_2 + \mathrm{e}^{-\lambda x} b\lambda y a_3 - 2\,\mathrm{e}^{\lambda x} ayb_1 + \mathrm{e}^{-\lambda x} b\lambda a_1 - \mathrm{e}^{-\lambda x} ba_2 + \mathrm{e}^{-\lambda x} bb_3 + b_2 = 0$$

Setting the numerator to zero gives

$$-\mathrm{e}^{2\lambda x} a^2 y^4 a_3 - 2\,\mathrm{e}^{\lambda x} \mathrm{e}^{-\lambda x} ab\, y^2 a_3 - \mathrm{e}^{\lambda x} a\lambda x\, y^2 a_2 - \mathrm{e}^{\lambda x} a\lambda\, y^3 a_3 - \mathrm{e}^{\lambda x} a\lambda\, y^2 a_1 - 2\,\mathrm{e}^{\lambda x} axyb_2 - \mathrm{e}^{\lambda x} a\, y^2 a_2 - \mathrm{e}^{\lambda x} a\, y^2 b_3 - \mathrm{e}^{-2\lambda x} b^2 a_3 + \mathrm{e}^{-\lambda x} b\lambda x a_2 + \mathrm{e}^{-\lambda x} b\lambda y a_3 - 2\,\mathrm{e}^{\lambda x} ayb_1 + \mathrm{e}^{-\lambda x} b\lambda a_1 - \mathrm{e}^{-\lambda x} ba_2 + \mathrm{e}^{-\lambda x} bb_3 + b_2 = 0 \tag{6E}$$

Simplifying the above gives

$$-\mathrm{e}^{2\lambda x} a^2 y^4 a_3 - 2ab\, y^2 a_3 - \mathrm{e}^{\lambda x} a\lambda x\, y^2 a_2 - \mathrm{e}^{\lambda x} a\lambda\, y^3 a_3 - \mathrm{e}^{\lambda x} a\lambda\, y^2 a_1 - 2\,\mathrm{e}^{\lambda x} axyb_2 - \mathrm{e}^{\lambda x} a\, y^2 a_2 - \mathrm{e}^{\lambda x} a\, y^2 b_3 - \mathrm{e}^{-2\lambda x} b^2 a_3 + \mathrm{e}^{-\lambda x} b\lambda x a_2 + \mathrm{e}^{-\lambda x} b\lambda y a_3 - 2\,\mathrm{e}^{\lambda x} ayb_1 + \mathrm{e}^{-\lambda x} b\lambda a_1 - \mathrm{e}^{-\lambda x} ba_2 + \mathrm{e}^{-\lambda x} bb_3 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \mathrm{e}^{\lambda x}, \mathrm{e}^{-2\lambda x}, \mathrm{e}^{-\lambda x}, \mathrm{e}^{2\lambda x}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \mathrm{e}^{\lambda x} = v_3, \mathrm{e}^{-2\lambda x} = v_4, \mathrm{e}^{-\lambda x} = v_5, \mathrm{e}^{2\lambda x} = v_6\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-v_6a^2v_2^4a_3 - v_3a\lambda v_1v_2^2a_2 - v_3a\lambda v_2^3a_3 - v_3a\lambda v_2^2a_1 - 2abv_2^2a_3 \\
&- v_3av_2^2a_2 - 2v_3av_1v_2b_2 - v_3av_2^2b_3 + v_5b\lambda v_1a_2 + v_5b\lambda v_2a_3 \\
&- 2v_3av_2b_1 - v_4b^2a_3 + v_5b\lambda a_1 - v_5ba_2 + v_5bb_3 + b_2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5, v_6\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-v_3a\lambda v_1v_2^2a_2 - 2v_3av_1v_2b_2 + v_5b\lambda v_1a_2 - v_6a^2v_2^4a_3 - v_3a\lambda v_2^3a_3 \\
&+ (-a\lambda a_1 - aa_2 - ab_3)\, v_2^2v_3 - 2abv_2^2a_3 - 2v_3av_2b_1 \\
&+ v_5b\lambda v_2a_3 - v_4b^2a_3 + (b\lambda a_1 - ba_2 + bb_3)\, v_5 + b_2 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_2 &= 0 \\
b\lambda a_2 &= 0 \\
\lambda a_3 b &= 0 \\
-2ab_1 &= 0 \\
-2ab_2 &= 0 \\
-b^2a_3 &= 0 \\
-a_3a^2 &= 0 \\
-a\lambda a_2 &= 0 \\
-\lambda a_3 a &= 0 \\
-2a_3ab &= 0 \\
-a\lambda a_1 - aa_2 - ab_3 &= 0 \\
b\lambda a_1 - ba_2 + bb_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= a_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -\lambda a_1
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 1$$
$$\eta = -\lambda y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= -\lambda y - \left(\mathrm{e}^{\lambda x} a\, y^2 + b\,\mathrm{e}^{-\lambda x}\right)(1) \\ &= -\mathrm{e}^{\lambda x} a\, y^2 - \lambda y - b\,\mathrm{e}^{-\lambda x} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\mathrm{e}^{\lambda x} a\, y^2 - \lambda y - b\,\mathrm{e}^{-\lambda x}} dy\end{aligned}$$

Which results in

$$S = -\frac{2\,\mathrm{e}^{\lambda x} \arctan\left(\frac{2\,\mathrm{e}^{2\lambda x} a y + \lambda\,\mathrm{e}^{\lambda x}}{\sqrt{4\,\mathrm{e}^{2\lambda x} a b - \mathrm{e}^{2\lambda x} \lambda^2}}\right)}{\sqrt{4\,\mathrm{e}^{2\lambda x} a b - \mathrm{e}^{2\lambda x} \lambda^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = e^{\lambda x} a\, y^2 + b\, e^{-\lambda x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{e^{\lambda x} \lambda y}{e^{2\lambda x} a\, y^2 + e^{\lambda x} \lambda y + b} \\ S_y &= -\frac{e^{\lambda x}}{e^{2\lambda x} a\, y^2 + e^{\lambda x} \lambda y + b} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2\arctan\left(\frac{2a\, e^{\lambda x} y + \lambda}{\sqrt{4ab - \lambda^2}}\right)}{\sqrt{4ab - \lambda^2}} = c_1 - x$$

Which simplifies to

$$-\frac{2\arctan\left(\frac{2a\, e^{\lambda x} y + \lambda}{\sqrt{4ab - \lambda^2}}\right)}{\sqrt{4ab - \lambda^2}} = c_1 - x$$

Which gives

$$y = -\frac{\left(\tan\left(\frac{c_1\sqrt{4ab-\lambda^2}}{2} - \frac{x\sqrt{4ab-\lambda^2}}{2}\right)\sqrt{4ab-\lambda^2}+\lambda\right)\mathrm{e}^{-\lambda x}}{2a}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\tan\left(\frac{c_1\sqrt{4ab-\lambda^2}}{2} - \frac{x\sqrt{4ab-\lambda^2}}{2}\right)\sqrt{4ab-\lambda^2}+\lambda\right)\mathrm{e}^{-\lambda x}}{2a} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\tan\left(\frac{c_1\sqrt{4ab-\lambda^2}}{2} - \frac{x\sqrt{4ab-\lambda^2}}{2}\right)\sqrt{4ab-\lambda^2}+\lambda\right)\mathrm{e}^{-\lambda x}}{2a}$$

Verified OK.

4.2.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \mathrm{e}^{\lambda x} a\, y^2 + b\,\mathrm{e}^{-\lambda x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \mathrm{e}^{\lambda x} a\, y^2 + b\,\mathrm{e}^{-\lambda x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\,\mathrm{e}^{-\lambda x}$, $f_1(x) = 0$ and $f_2(x) = \mathrm{e}^{\lambda x} a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\mathrm{e}^{\lambda x} a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= a\lambda\,\mathrm{e}^{\lambda x} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= \mathrm{e}^{2\lambda x} a^2 b\,\mathrm{e}^{-\lambda x}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\mathrm{e}^{\lambda x} a u''(x) - a\lambda\,\mathrm{e}^{\lambda x} u'(x) + \mathrm{e}^{2\lambda x} a^2 b\,\mathrm{e}^{-\lambda x} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \mathrm{e}^{\frac{\left(\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}} + c_2 \mathrm{e}^{-\frac{\left(-\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}}$$

The above shows that

$$u'(x) = \frac{c_2\left(\lambda - \sqrt{-4ab+\lambda^2}\right)\mathrm{e}^{-\frac{\left(-\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}}}{2} + \frac{c_1 \mathrm{e}^{\frac{\left(\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}}\left(\lambda+\sqrt{-4ab+\lambda^2}\right)}{2}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{c_2\left(\lambda-\sqrt{-4ab+\lambda^2}\right)\mathrm{e}^{-\frac{\left(-\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}}}{2} + \frac{c_1\mathrm{e}^{\frac{\left(\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}}\left(\lambda+\sqrt{-4ab+\lambda^2}\right)}{2}\right)\mathrm{e}^{-\lambda x}}{a\left(c_1\mathrm{e}^{\frac{\left(\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}} + c_2\mathrm{e}^{-\frac{\left(-\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\left(\lambda-\sqrt{-4ab+\lambda^2}\right)\mathrm{e}^{-\frac{\left(-\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}} + c_3\mathrm{e}^{\frac{\left(\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}}\left(\lambda+\sqrt{-4ab+\lambda^2}\right)\right)\mathrm{e}^{-\lambda x}}{2a\left(c_3\mathrm{e}^{\frac{\left(\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}} + \mathrm{e}^{-\frac{\left(-\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}}\right)}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{\left(\left(\lambda-\sqrt{-4ab+\lambda^2}\right)\mathrm{e}^{-\frac{\left(-\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}} + c_3\mathrm{e}^{\frac{\left(\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}}\left(\lambda+\sqrt{-4ab+\lambda^2}\right)\right)\mathrm{e}^{-\lambda x}}{2a\left(c_3\mathrm{e}^{\frac{\left(\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}} + \mathrm{e}^{-\frac{\left(-\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\left(\lambda-\sqrt{-4ab+\lambda^2}\right)\mathrm{e}^{-\frac{\left(-\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}}+c_3\mathrm{e}^{\frac{\left(\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}}\left(\lambda+\sqrt{-4ab+\lambda^2}\right)\right)\mathrm{e}^{-\lambda x}}{2a\left(c_3\mathrm{e}^{\frac{\left(\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}}+\mathrm{e}^{-\frac{\left(-\lambda+\sqrt{-4ab+\lambda^2}\right)x}{2}}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 64

```
dsolve(diff(y(x),x)=a*exp(lambda*x)*y(x)^2+b*exp(-lambda*x),y(x), singsol=all)
```

$$y(x) = -\frac{\left(\lambda^2-\tan\left(\frac{\sqrt{4ab\,\lambda^2-\lambda^4}\,(x\lambda+c_1)}{2\lambda^2}\right)\sqrt{4ab\,\lambda^2-\lambda^4}\right)\mathrm{e}^{-x\lambda}}{2a\lambda}$$

✓ Solution by Mathematica

Time used: 0.624 (sec). Leaf size: 123

```
DSolve[y'[x]==a*Exp[\[Lambda]*x]*y[x]^2+b*Exp[-\[Lambda]*x],y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^{\lambda(-x)}\left(-\sqrt{\lambda^2-4ab}+\frac{2}{\frac{1}{\sqrt{\lambda^2-4ab}}+c_1e^{x\sqrt{\lambda^2-4ab}}}-\lambda\right)}{2a}$$

$$y(x) \to \frac{e^{\lambda(-x)}\left(4ab-\lambda\left(\sqrt{\lambda^2-4ab}+\lambda\right)\right)}{2a\sqrt{\lambda^2-4ab}}$$

4.3 problem 24

Internal problem ID [10432]
Internal file name [`OUTPUT/9379_Monday_June_06_2022_02_20_52_PM_89615702/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\,\mathrm{e}^{\lambda x} y^2 = b\,x^{n-1} n - a\,b^2 \mathrm{e}^{\lambda x} x^{2n}}$$

4.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \mathrm{e}^{\lambda x} a\,y^2 + b\,x^{n-1} n - a\,b^2 \mathrm{e}^{\lambda x} x^{2n} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a\,b^2 \mathrm{e}^{\lambda x} x^{2n} + \mathrm{e}^{\lambda x} a\,y^2 + \frac{b\,x^n n}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\,x^{n-1} n - a\,b^2 \mathrm{e}^{\lambda x} x^{2n}$, $f_1(x) = 0$ and $f_2(x) = \mathrm{e}^{\lambda x} a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\mathrm{e}^{\lambda x} a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= a\lambda\,\mathrm{e}^{\lambda x} \\ f_1f_2 &= 0 \\ f_2^2f_0 &= \mathrm{e}^{2\lambda x}a^2\left(b\,x^{n-1}n - a\,b^2\mathrm{e}^{\lambda x}x^{2n}\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\mathrm{e}^{\lambda x}au''(x) - a\lambda\,\mathrm{e}^{\lambda x}u'(x) + \mathrm{e}^{2\lambda x}a^2\left(b\,x^{n-1}n - a\,b^2\mathrm{e}^{\lambda x}x^{2n}\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{-x^{2n}\mathrm{e}^{2\lambda x}\_Y(x)\,a^2b^2 + \mathrm{e}^{\lambda x}x^{n-1}\_Y(x)\,abn - \_Y'(x)\,\lambda + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{-x^{2n}\mathrm{e}^{2\lambda x}\_Y(x)\,a^2b^2 + \mathrm{e}^{\lambda x}x^{n-1}\_Y(x)\,abn - \_Y'(x)\,\lambda + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{-x^{2n}\mathrm{e}^{2\lambda x}\_Y(x)\,a^2b^2 + \mathrm{e}^{\lambda x}x^{n-1}\_Y(x)\,abn - \_Y'(x)\,\lambda + \_Y''(x)\right\}, \{\_Y(x)\}\right)\right)\mathrm{e}^{-\lambda x}}{a\,\mathrm{DESol}\left(\left\{-x^{2n}\mathrm{e}^{2\lambda x}\_Y(x)\,a^2b^2 + \mathrm{e}^{\lambda x}x^{n-1}\_Y(x)\,abn - \_Y'(x)\,\lambda + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{-x^{2n}\mathrm{e}^{2\lambda x}\_Y(x)\,a^2b^2 + \mathrm{e}^{\lambda x}x^{n-1}\_Y(x)\,abn - \_Y'(x)\,\lambda + \_Y''(x)\right\}, \{\_Y(x)\}\right)\right)\mathrm{e}^{-\lambda x}}{a\,\mathrm{DESol}\left(\left\{-x^{2n}\mathrm{e}^{2\lambda x}\_Y(x)\,a^2b^2 + \mathrm{e}^{\lambda x}x^{n-1}\_Y(x)\,abn - \_Y'(x)\,\lambda + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{-x^{2n}\mathrm{e}^{2\lambda x}\_Y(x)\,a^2b^2+\mathrm{e}^{\lambda x}x^{n-1}\_Y(x)\,abn-\_Y'(x)\,\lambda+\_Y''(x)\right\},\{\_Y(x)\}\right)\right)\mathrm{e}^{-\lambda x}}{a\operatorname{DESol}\left(\left\{-x^{2n}\mathrm{e}^{2\lambda x}\_Y(x)\,a^2b^2+\mathrm{e}^{\lambda x}x^{n-1}\_Y(x)\,abn-\_Y'(x)\,\lambda+\_Y''(x)\right\},\{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{-x^{2n}\mathrm{e}^{2\lambda x}\_Y(x)\,a^2b^2+\mathrm{e}^{\lambda x}x^{n-1}\_Y(x)\,abn-\_Y'(x)\,\lambda+\_Y''(x)\right\},\{\_Y(x)\}\right)\right)\mathrm{e}^{-\lambda x}}{a\operatorname{DESol}\left(\left\{-x^{2n}\mathrm{e}^{2\lambda x}\_Y(x)\,a^2b^2+\mathrm{e}^{\lambda x}x^{n-1}\_Y(x)\,abn-\_Y'(x)\,\lambda+\_Y''(x)\right\},\{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (diff(y(x), x))*lambda-exp(lam
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
            checking if the LODE is missing y
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=a*exp(lambda*x)*y(x)^2+b*n*x^(n-1)-a*b^2*exp(lambda*x)*x^(2*n),y(x), sin
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==a*Exp[\[Lambda]*x]*y[x]^2+b*n*x^(n-1)-a*b^2*Exp[\[Lambda]*x]*x^(2*n),y[x],x,In
```

Not solved

4.4 problem 25

Internal problem ID [10433]
Internal file name [`OUTPUT/9380_Monday_June_06_2022_02_20_55_PM_76225887/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \mathrm{e}^{\lambda x} y^2 - a\, x^n y = a\lambda\, x^n \mathrm{e}^{-\lambda x}}$$

4.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \mathrm{e}^{\lambda x} y^2 + a\, x^n y + a\lambda\, x^n \mathrm{e}^{-\lambda x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \mathrm{e}^{\lambda x} y^2 + a\, x^n y + a\lambda\, x^n \mathrm{e}^{-\lambda x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\lambda\, x^n \mathrm{e}^{-\lambda x}$, $f_1(x) = x^n a$ and $f_2(x) = \mathrm{e}^{\lambda x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\mathrm{e}^{\lambda x} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \lambda\, \mathrm{e}^{\lambda x} \\ f_1 f_2 &= \mathrm{e}^{\lambda x} x^n a \\ f_2^2 f_0 &= \mathrm{e}^{2\lambda x} a\lambda\, x^n \mathrm{e}^{-\lambda x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\mathrm{e}^{\lambda x} u''(x) - \left(\mathrm{e}^{\lambda x} x^n a + \lambda\, \mathrm{e}^{\lambda x}\right) u'(x) + \mathrm{e}^{2\lambda x} a\lambda\, x^n \mathrm{e}^{-\lambda x} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\int \frac{\left(\int \mathrm{e}^{\frac{a\, x^{n+1} - \lambda x(n+1)}{n+1}} dx\right)\lambda - c_1 \lambda + \mathrm{e}^{\frac{a\, x^{n+1} - \lambda x(n+1)}{n+1}}}{\int \mathrm{e}^{\frac{x(x^n a - \lambda(n+1))}{n+1}} dx - c_1} dx} c_2$$

The above shows that

$u'(x)$

$$= \frac{c_2 \left(\left(\int \mathrm{e}^{\frac{x(x^n a - \lambda(n+1))}{n+1}} dx\right)\lambda - c_1\lambda + \mathrm{e}^{\frac{x(x^n a - \lambda(n+1))}{n+1}}\right) \mathrm{e}^{\int \frac{\left(\int \mathrm{e}^{\frac{x(x^n a - \lambda(n+1))}{n+1}} dx\right)\lambda - c_1 \lambda + \mathrm{e}^{\frac{x(x^n a - \lambda(n+1))}{n+1}}}{\int \mathrm{e}^{\frac{x(x^n a - \lambda(n+1))}{n+1}} dx - c_1} dx}}{\int \mathrm{e}^{\frac{x(x^n a - \lambda(n+1))}{n+1}} dx - c_1}$$

Using the above in (1) gives the solution

$y =$

$$-\frac{\left(\left(\int \mathrm{e}^{\frac{x(x^n a - \lambda(n+1))}{n+1}} dx\right)\lambda - c_1\lambda + \mathrm{e}^{\frac{x(x^n a - \lambda(n+1))}{n+1}}\right) \mathrm{e}^{\int \frac{\left(\int \mathrm{e}^{\frac{x(x^n a - \lambda(n+1))}{n+1}} dx\right)\lambda - c_1 \lambda + \mathrm{e}^{\frac{x(x^n a - \lambda(n+1))}{n+1}}}{\int \mathrm{e}^{\frac{x(x^n a - \lambda(n+1))}{n+1}} dx - c_1} dx} \mathrm{e}^{-\lambda x} \mathrm{e}^{\int - \frac{\int \lambda\, \mathrm{e}^{\frac{a\, x}{}}}{}}}{\int \mathrm{e}^{\frac{x(x^n a - \lambda(n+1))}{n+1}} dx - c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{e^{-\lambda x} \left(\left(\int e^{\frac{a\,x^{n+1}-\lambda x(n+1)}{n+1}} dx \right) \lambda - \lambda c_3 + e^{\frac{a\,x^{n+1}-\lambda x(n+1)}{n+1}} \right)}{\int e^{\frac{x(x^n a-\lambda(n+1))}{n+1}} dx - c_3}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-\lambda x} \left(\left(\int e^{\frac{a\,x^{n+1}-\lambda x(n+1)}{n+1}} dx \right) \lambda - \lambda c_3 + e^{\frac{a\,x^{n+1}-\lambda x(n+1)}{n+1}} \right)}{\int e^{\frac{x(x^n a-\lambda(n+1))}{n+1}} dx - c_3} \tag{1}$$

Verification of solutions

$$y = -\frac{e^{-\lambda x} \left(\left(\int e^{\frac{a\,x^{n+1}-\lambda x(n+1)}{n+1}} dx \right) \lambda - \lambda c_3 + e^{\frac{a\,x^{n+1}-\lambda x(n+1)}{n+1}} \right)}{\int e^{\frac{x(x^n a-\lambda(n+1))}{n+1}} dx - c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x^n*a+lambda)*(diff(y(x), x))
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 89

```
dsolve(diff(y(x),x)=exp(lambda*x)*y(x)^2+a*x^(n)*y(x)+a*lambda*x^n*exp(-lambda*x),y(x), sing
```

$$y(x) = -\frac{e^{-x\lambda}\left(\left(\int e^{\frac{x^{n+1}a - x\lambda(n+1)}{n+1}}dx\right)\lambda + \lambda c_1 + e^{\frac{x^{n+1}a - x\lambda(n+1)}{n+1}}\right)}{c_1 + \int e^{\frac{x(a\,x^n - (n+1)\lambda)}{n+1}}dx}$$

✓ Solution by Mathematica

Time used: 1.93 (sec). Leaf size: 254

```
DSolve[y'[x]==Exp[\[Lambda]*x]*y[x]^2+a*x^(n)*y[x]+a*\[Lambda]*x^n*Exp[-\[Lambda]*x],y[x],x,
```

$$\text{Solve}\left[\int_1^{y(x)}\left(\frac{e^{\frac{ax^{n+1}}{n+1}}}{(\lambda + e^{x\lambda}K[2])^2}\right.\right.$$
$$-\int_1^x\left(\frac{2e^{\frac{aK[1]^{n+1}}{n+1}}\left(a\lambda K[1]^n + ae^{\lambda K[1]}K[2]K[1]^n + e^{2\lambda K[1]}K[2]^2\right)}{(\lambda + e^{\lambda K[1]}K[2])^3} - \frac{e^{\frac{aK[1]^{n+1}}{n+1} - \lambda K[1]}\left(ae^{\lambda K[1]}K[1]^n + 2e^{2\lambda K[1]}K\right.}{(\lambda + e^{\lambda K[1]}K[2])^2}\right.$$
$$\left.+\int_1^x -\frac{e^{\frac{aK[1]^{n+1}}{n+1} - \lambda K[1]}\left(a\lambda K[1]^n + ae^{\lambda K[1]}y(x)K[1]^n + e^{2\lambda K[1]}y(x)^2\right)}{(\lambda + e^{\lambda K[1]}y(x))^2}dK[1] = c_1, y(x)\right]$$

4.5 problem 26

Internal problem ID [10434]
Internal file name [`OUTPUT/9381_Monday_June_06_2022_02_20_59_PM_24609640/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + e^{\lambda x} y^2 \lambda - a\, x^n y\, e^{\lambda x} = -x^n a}$$

4.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\lambda\, e^{\lambda x} y^2 + a\, x^n y\, e^{\lambda x} - x^n a \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\lambda\, e^{\lambda x} y^2 + a\, x^n y\, e^{\lambda x} - x^n a$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -x^n a$, $f_1(x) = e^{\lambda x} x^n a$ and $f_2(x) = -\lambda\, e^{\lambda x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\lambda\, e^{\lambda x} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\mathrm{e}^{\lambda x}\lambda^2 \\ f_1f_2 &= -\mathrm{e}^{2\lambda x}x^n a\lambda \\ f_2^2f_0 &= -\mathrm{e}^{2\lambda x}x^n a\,\lambda^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\lambda\,\mathrm{e}^{\lambda x}u''(x) - \left(-\mathrm{e}^{2\lambda x}x^n a\lambda - \mathrm{e}^{\lambda x}\lambda^2\right)u'(x) - \mathrm{e}^{2\lambda x}x^n a\,\lambda^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{\mathrm{e}^{\lambda x}\left(\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}x^n dx\right)}dx\right)c_2 + c_1\lambda\right)}{\lambda}$$

The above shows that

$$u'(x) = \frac{\mathrm{e}^{\lambda x}\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}x^n dx\right)}dx\right)c_2\lambda + \mathrm{e}^{\lambda x}c_1\lambda^2 + c_2\mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x}x^n dx\right)}}{\lambda}$$

Using the above in (1) gives the solution

$$y = \frac{\left(\mathrm{e}^{\lambda x}\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}x^n dx\right)}dx\right)c_2\lambda + \mathrm{e}^{\lambda x}c_1\lambda^2 + c_2\mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x}x^n dx\right)}\right)\mathrm{e}^{-2\lambda x}}{\lambda\left(\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}x^n dx\right)}dx\right)c_2 + c_1\lambda\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\mathrm{e}^{\lambda x}\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}x^n dx\right)}dx\right)\lambda + \mathrm{e}^{\lambda x}c_3\lambda^2 + \mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x}x^n dx\right)}\right)\mathrm{e}^{-2\lambda x}}{\lambda\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}x^n dx\right)}dx + \lambda c_3\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\mathrm{e}^{\lambda x}\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}x^n dx\right)}dx\right)\lambda + \mathrm{e}^{\lambda x}c_3\lambda^2 + \mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x}x^n dx\right)}\right)\mathrm{e}^{-2\lambda x}}{\lambda\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}x^n dx\right)}dx + \lambda c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(e^{\lambda x}\left(\int e^{-\lambda x+a\left(\int e^{\lambda x}x^n dx\right)}dx\right)\lambda + e^{\lambda x}c_3\lambda^2 + e^{a\left(\int e^{\lambda x}x^n dx\right)}\right)e^{-2\lambda x}}{\lambda\left(\int e^{-\lambda x+a\left(\int e^{\lambda x}x^n dx\right)}dx + \lambda c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a*x^n*exp(lambda*x)+lambda)*(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         <- linear_1 successful
         Change of variables used:
            [x = ln(t)/lambda]
         Linear ODE actually solved:
            a*(ln(t)/lambda)^n*u(t)-a*(ln(t)/lambda)^n*t*diff(u(t),t)+t*lambda*diff(diff(u(t
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 92

```
dsolve(diff(y(x),x)=-lambda*exp(lambda*x)*y(x)^2+a*x^(n)*exp(lambda*x)*y(x)-a*x^n,y(x), sing
```

$$y(x) = \frac{e^{-x\lambda}\left(\int e^{-x\lambda+a\left(\int e^{x\lambda}x^n dx\right)}dx\right)c_1\lambda+\lambda^2 e^{-x\lambda}+c_1 e^{-2x\lambda+a\left(\int e^{x\lambda}x^n dx\right)}}{\lambda\left(\left(\int e^{-x\lambda+a\left(\int e^{x\lambda}x^n dx\right)}dx\right)c_1+\lambda\right)}$$

✓ Solution by Mathematica

Time used: 6.627 (sec). Leaf size: 185

```
DSolve[y'[x]==-\[Lambda]*Exp[\[Lambda]*x]*y[x]^2+a*x^(n)*Exp[\[Lambda]*x]*y[x]-a*x^n,y[x],x,
```

$$y(x) \to \frac{e^{-2\lambda x}\left(e^{\lambda x}\int_1^{e^{x\lambda}}\frac{\exp\left(\frac{a\Gamma(n+1,-\log(K[1]))(-\log(K[1]))^{-n}\left(\frac{\log(K[1])}{\lambda}\right)^n}{\lambda}\right)}{K[1]^2}dK[1]+\exp\left(\frac{a\left(-\log\left(e^{\lambda x}\right)\right)^{-n}\left(\frac{\log\left(e^{\lambda x}\right)}{\lambda}\right)^n\Gamma\left(n+1,-\log\right.}{\lambda}\right.}{\int_1^{e^{x\lambda}}\frac{\exp\left(\frac{a\Gamma(n+1,-\log(K[1]))(-\log(K[1]))^{-n}\left(\frac{\log(K[1])}{\lambda}\right)^n}{\lambda}\right)}{K[1]^2}dK[1]+c_1}$$

$$y(x) \to e^{\lambda(-x)}$$

4.6 problem 27

Internal problem ID [10435]
Internal file name [`OUTPUT/9382_Monday_June_06_2022_02_21_00_PM_76993873/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - a\,\mathrm{e}^{\lambda x} y^2 + ab\,x^n \mathrm{e}^{\lambda x} y = b\,x^{n-1} n}$$

4.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \mathrm{e}^{\lambda x} a\,y^2 - ab\,x^n \mathrm{e}^{\lambda x} y + b\,x^{n-1} n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \mathrm{e}^{\lambda x} a\,y^2 - ab\,x^n \mathrm{e}^{\lambda x} y + \frac{b\,x^n n}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\,x^{n-1} n$, $f_1(x) = -ab\,x^n \mathrm{e}^{\lambda x}$ and $f_2(x) = \mathrm{e}^{\lambda x} a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\mathrm{e}^{\lambda x} a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = a\lambda \,\mathrm{e}^{\lambda x}$$
$$f_1 f_2 = -a^2 b\, x^n \mathrm{e}^{2\lambda x}$$
$$f_2^2 f_0 = \mathrm{e}^{2\lambda x} a^2 b\, x^{n-1} n$$

Substituting the above terms back in equation (2) gives

$$\mathrm{e}^{\lambda x} a u''(x) - \left(-a^2 b\, x^n \mathrm{e}^{2\lambda x} + a\lambda\,\mathrm{e}^{\lambda x}\right) u'(x) + \mathrm{e}^{2\lambda x} a^2 b\, x^{n-1} n u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{-ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} \left(c_1 + \left(\int \mathrm{e}^{\lambda x + ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} dx\right) \lambda c_2\right)$$

The above shows that

$$u'(x) = -ab\, x^n \mathrm{e}^{\lambda x - ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} \left(c_1 + \left(\int \mathrm{e}^{\lambda x + ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} dx\right) \lambda c_2\right) + c_2 \lambda\,\mathrm{e}^{\lambda x}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-ab\, x^n \mathrm{e}^{\lambda x - ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} \left(c_1 + \left(\int \mathrm{e}^{\lambda x + ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} dx\right) \lambda c_2\right) + c_2 \lambda\,\mathrm{e}^{\lambda x}\right) \mathrm{e}^{-\lambda x} \mathrm{e}^{\int ab\, x^n \mathrm{e}^{\lambda x} dx}}{a \left(c_1 + \left(\int \mathrm{e}^{\lambda x + ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} dx\right) \lambda c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\mathrm{e}^{-\lambda x + ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} \left(ab\, x^n \mathrm{e}^{\lambda x - ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} \left(c_3 + \lambda\left(\int \mathrm{e}^{\lambda x + ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} dx\right)\right) - \lambda\,\mathrm{e}^{\lambda x}\right)}{a \left(c_3 + \lambda \left(\int \mathrm{e}^{\lambda x + ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} dx\right)\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-\lambda x + ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} \left(ab\, x^n \mathrm{e}^{\lambda x - ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} \left(c_3 + \lambda\left(\int \mathrm{e}^{\lambda x + ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} dx\right)\right) - \lambda\,\mathrm{e}^{\lambda x}\right)}{a \left(c_3 + \lambda \left(\int \mathrm{e}^{\lambda x + ab\left(\int \mathrm{e}^{\lambda x} x^n dx\right)} dx\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{e^{-\lambda x+ab\left(\int e^{\lambda x} x^n dx\right)} \left(ab\, x^n e^{\lambda x-ab\left(\int e^{\lambda x} x^n dx\right)} \left(c_3 + \lambda\left(\int e^{\lambda x+ab\left(\int e^{\lambda x} x^n dx\right)} dx\right)\right) - \lambda\, e^{\lambda x}\right)}{a\left(c_3 + \lambda\left(\int e^{\lambda x+ab\left(\int e^{\lambda x} x^n dx\right)} dx\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-exp(lambda*x)*x^n*a*b+lambda
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            <- 2nd order exact_linear successful
         Change of variables used:
            [x = ln(t)/lambda]
         Linear ODE actually solved:
            b*(ln(t)/lambda)^n/ln(t)*lambda*n*a*u(t)+t*(ln(t)/lambda)^n*a*b*lambda*diff(u(t)
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 88

```
dsolve(diff(y(x),x)=a*exp(lambda*x)*y(x)^2-a*b*x^(n)*exp(lambda*x)*y(x)+b*n*x^(n-1),y(x), si
```

$$y(x) = \frac{x^n\lambda\left(\int e^{x\lambda+ab\left(\int e^{x\lambda}x^n dx\right)}dx\right)c_1ab + x^n ab - c_1\lambda\, e^{ab\left(\int e^{x\lambda}x^n dx\right)}}{a\left(\lambda\left(\int e^{x\lambda+ab\left(\int e^{x\lambda}x^n dx\right)}dx\right)c_1+1\right)}$$

✓ Solution by Mathematica

Time used: 63.132 (sec). Leaf size: 188

```
DSolve[y'[x]==a*Exp[\[Lambda]*x]*y[x]^2-a*b*x^(n)*Exp[\[Lambda]*x]*y[x]+b*n*x^(n-1),y[x],x,I
```

$$y(x) \to \frac{abc_1\left(\frac{\log(e^{\lambda x})}{\lambda}\right)^n\int_1^{e^{x\lambda}}\exp\left(\frac{ab\Gamma(n+1,-\log(K[1]))(-\log(K[1]))^{-n}\left(\frac{\log(K[1])}{\lambda}\right)^n}{\lambda}\right)dK[1] - c_1\lambda\exp\left(\frac{ab(-\log(e^{\lambda x}))^{-n}\left(\frac{\log(\cdots}{\cdots}\right.}{\cdots}\right.}{a + ac_1\int_1^{e^{x\lambda}}\exp\left(\frac{ab\Gamma(n+1,-\log(K[1]))(-\log(K[1]))^{-n}\left(\frac{\log(K[1])}{\lambda}\right)^n}{\lambda}\right)dK[1\cdots}$$

4.7 problem 28

Internal problem ID [10436]
Internal file name [OUTPUT/9383_Monday_June_06_2022_02_21_02_PM_33130912/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - a\,x^n y^2 = b\lambda\,\mathrm{e}^{\lambda x} - a\,b^2 x^n \mathrm{e}^{2\lambda x}}$$

4.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,x^n y^2 + b\lambda\,\mathrm{e}^{\lambda x} - a\,b^2 x^n \mathrm{e}^{2\lambda x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,x^n y^2 + b\lambda\,\mathrm{e}^{\lambda x} - a\,b^2 x^n \mathrm{e}^{2\lambda x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\lambda\,\mathrm{e}^{\lambda x} - a\,b^2 x^n \mathrm{e}^{2\lambda x}$, $f_1(x) = 0$ and $f_2(x) = x^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^n a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{x^n na}{x} \\
f_1f_2 &= 0 \\
f_2^2f_0 &= x^{2n}a^2\left(b\lambda\,\mathrm{e}^{\lambda x} - a\,b^2x^n\mathrm{e}^{2\lambda x}\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^n au''(x) - \frac{x^n nau'(x)}{x} + x^{2n}a^2\left(b\lambda\,\mathrm{e}^{\lambda x} - a\,b^2x^n\mathrm{e}^{2\lambda x}\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{-x^{1+2n}\mathrm{e}^{2\lambda x}\_Y(x)\,a^2b^2 + \mathrm{e}^{\lambda x}x^{n+1}\_Y(x)\,ab\lambda + \_Y''(x)\,x - n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-x^{1+2n}\mathrm{e}^{2\lambda x}\_Y(x)\,a^2b^2 + \mathrm{e}^{\lambda x}x^{n+1}\_Y(x)\,ab\lambda + \_Y''(x)\,x - n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-x^{1+2n}\mathrm{e}^{2\lambda x}\_Y(x)a^2b^2+\mathrm{e}^{\lambda x}x^{n+1}\_Y(x)ab\lambda+\_Y''(x)x-n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)\right)x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{-x^{1+2n}\mathrm{e}^{2\lambda x}\_Y(x)a^2b^2+\mathrm{e}^{\lambda x}x^{n+1}\_Y(x)ab\lambda+\_Y''(x)x-n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-x^{1+2n}\mathrm{e}^{2\lambda x}\_Y(x)a^2b^2+\mathrm{e}^{\lambda x}x^{n+1}\_Y(x)ab\lambda+\_Y''(x)x-n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)\right)x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{-x^{1+2n}\mathrm{e}^{2\lambda x}\_Y(x)a^2b^2+\mathrm{e}^{\lambda x}x^{n+1}\_Y(x)ab\lambda+\_Y''(x)x-n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{-x^{1+2n}\mathrm{e}^{2\lambda x}\_Y(x)a^2b^2+\mathrm{e}^{\lambda x}x^{n+1}\_Y(x)ab\lambda+\_Y''(x)x-n\_Y'(x)}{x}\right\},\{\_Y(x)\}\right)\right)x^{-n}}{a\operatorname{DESol}\left(\left\{\frac{-x^{1+2n}\mathrm{e}^{2\lambda x}\_Y(x)a^2b^2+\mathrm{e}^{\lambda x}x^{n+1}\_Y(x)ab\lambda+\_Y''(x)x-n\_Y'(x)}{x}\right\},\{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{-x^{1+2n}\mathrm{e}^{2\lambda x}\_Y(x)a^2b^2+\mathrm{e}^{\lambda x}x^{n+1}\_Y(x)ab\lambda+\_Y''(x)x-n\_Y'(x)}{x}\right\},\{\_Y(x)\}\right)\right)x^{-n}}{a\operatorname{DESol}\left(\left\{\frac{-x^{1+2n}\mathrm{e}^{2\lambda x}\_Y(x)a^2b^2+\mathrm{e}^{\lambda x}x^{n+1}\_Y(x)ab\lambda+\_Y''(x)x-n\_Y'(x)}{x}\right\},\{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = n*(diff(y(x), x))/x+a*x^n*b*(x
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=a*x^n*y(x)^2+b*lambda*exp(lambda*x)-a*b^2*x^n*exp(2*lambda*x),y(x), sing
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==a*x^n*y[x]^2+b*\[Lambda]*Exp[\[Lambda]*x]-a*b^2*x^n*Exp[2*\[Lambda]*x],y[x],x,
```

Not solved

4.8 problem 29

Internal problem ID [10437]
Internal file name [`OUTPUT/9384_Monday_June_06_2022_02_21_04_PM_3481108/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\,x^n y^2 - \lambda y = -a\,b^2 x^n \mathrm{e}^{2\lambda x}}$$

4.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,x^n y^2 + \lambda y - a\,b^2 x^n \mathrm{e}^{2\lambda x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,x^n y^2 + \lambda y - a\,b^2 x^n \mathrm{e}^{2\lambda x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a\,b^2 x^n \mathrm{e}^{2\lambda x}$, $f_1(x) = \lambda$ and $f_2(x) = x^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^n a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{x^n na}{x} \\ f_1f_2 &= \lambda a\,x^n \\ f_2^2f_0 &= -x^{3n}a^3b^2\mathrm{e}^{2\lambda x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^n au''(x) - \left(\frac{x^n na}{x} + \lambda a\,x^n\right)u'(x) - x^{3n}a^3b^2\mathrm{e}^{2\lambda x}u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = &-c_1\sinh\left(\frac{x^n ab\big((\Gamma(n,-\lambda x)\,n - \Gamma(n+1))\,(-\lambda x)^{-n} + \mathrm{e}^{\lambda x}\big)}{\lambda}\right) \\ &+ c_2\cosh\left(\frac{x^n ab\big((\Gamma(n,-\lambda x)\,n - \Gamma(n+1))\,(-\lambda x)^{-n} + \mathrm{e}^{\lambda x}\big)}{\lambda}\right) \end{aligned}$$

The above shows that

$$\begin{aligned} u'(x) = ab\,x^n\mathrm{e}^{\lambda x}\Bigg(&c_2\sinh\left(\frac{x^n ab\big((\Gamma(n,-\lambda x)\,n - \Gamma(n+1))\,(-\lambda x)^{-n} + \mathrm{e}^{\lambda x}\big)}{\lambda}\right) \\ &- c_1\cosh\left(\frac{x^n ab\big((\Gamma(n,-\lambda x)\,n - \Gamma(n+1))\,(-\lambda x)^{-n} + \mathrm{e}^{\lambda x}\big)}{\lambda}\right)\Bigg) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{b\,\mathrm{e}^{\lambda x}\left(c_2\sinh\left(\frac{x^n ab\left((\Gamma(n,-\lambda x)n-\Gamma(n+1))(-\lambda x)^{-n}+\mathrm{e}^{\lambda x}\right)}{\lambda}\right) - c_1\cosh\left(\frac{x^n ab\left((\Gamma(n,-\lambda x)n-\Gamma(n+1))(-\lambda x)^{-n}+\mathrm{e}^{\lambda x}\right)}{\lambda}\right)\right)}{-c_1\sinh\left(\frac{x^n ab\left((\Gamma(n,-\lambda x)n-\Gamma(n+1))(-\lambda x)^{-n}+\mathrm{e}^{\lambda x}\right)}{\lambda}\right) + c_2\cosh\left(\frac{x^n ab\left((\Gamma(n,-\lambda x)n-\Gamma(n+1))(-\lambda x)^{-n}+\mathrm{e}^{\lambda x}\right)}{\lambda}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{b\,e^{\lambda x}\left(\sinh\left(\frac{(-\lambda x)^{-n} x^n a b\left(e^{\lambda x}(-\lambda x)^n+\Gamma(n,-\lambda x)n-\Gamma(n+1)\right)}{\lambda}\right)-c_3\cosh\left(\frac{(-\lambda x)^{-n} x^n a b\left(e^{\lambda x}(-\lambda x)^n+\Gamma(n,-\lambda x)n-\Gamma(n+1)\right)}{\lambda}\right)\right)}{c_3\sinh\left(\frac{(-\lambda x)^{-n} x^n a b\left(e^{\lambda x}(-\lambda x)^n+\Gamma(n,-\lambda x)n-\Gamma(n+1)\right)}{\lambda}\right)-\cosh\left(\frac{(-\lambda x)^{-n} x^n a b\left(e^{\lambda x}(-\lambda x)^n+\Gamma(n,-\lambda x)n-\Gamma(n+1)\right)}{\lambda}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{b\,e^{\lambda x}\left(\sinh\left(\frac{(-\lambda x)^{-n} x^n a b\left(e^{\lambda x}(-\lambda x)^n+\Gamma(n,-\lambda x)n-\Gamma(n+1)\right)}{\lambda}\right)-c_3\cosh\left(\frac{(-\lambda x)^{-n} x^n a b\left(e^{\lambda x}(-\lambda x)^n+\Gamma(n,-\lambda x)n-\Gamma(n+1)\right)}{\lambda}\right)\right)}{c_3\sinh\left(\frac{(-\lambda x)^{-n} x^n a b\left(e^{\lambda x}(-\lambda x)^n+\Gamma(n,-\lambda x)n-\Gamma(n+1)\right)}{\lambda}\right)-\cosh\left(\frac{(-\lambda x)^{-n} x^n a b\left(e^{\lambda x}(-\lambda x)^n+\Gamma(n,-\lambda x)n-\Gamma(n+1)\right)}{\lambda}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{b\,e^{\lambda x}\left(\sinh\left(\frac{(-\lambda x)^{-n} x^n a b\left(e^{\lambda x}(-\lambda x)^n+\Gamma(n,-\lambda x)n-\Gamma(n+1)\right)}{\lambda}\right)-c_3\cosh\left(\frac{(-\lambda x)^{-n} x^n a b\left(e^{\lambda x}(-\lambda x)^n+\Gamma(n,-\lambda x)n-\Gamma(n+1)\right)}{\lambda}\right)\right)}{c_3\sinh\left(\frac{(-\lambda x)^{-n} x^n a b\left(e^{\lambda x}(-\lambda x)^n+\Gamma(n,-\lambda x)n-\Gamma(n+1)\right)}{\lambda}\right)-\cosh\left(\frac{(-\lambda x)^{-n} x^n a b\left(e^{\lambda x}(-\lambda x)^n+\Gamma(n,-\lambda x)n-\Gamma(n+1)\right)}{\lambda}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 62

```
dsolve(diff(y(x),x)=a*x^n*y(x)^2+lambda*y(x)-a*b^2*x^n*exp(2*lambda*x),y(x), singsol=all)
```

$$y(x) = \tanh\left(\frac{-ab\,x^n\left(n\Gamma(n,-x\lambda)-\Gamma(n+1)\right)(-x\lambda)^{-n}-ba\,e^{x\lambda}x^n+i\lambda c_1}{\lambda}\right)b\,e^{x\lambda}$$

✓ Solution by Mathematica

Time used: 1.69 (sec). Leaf size: 57

```
DSolve[y'[x]==a*x^n*y[x]^2+\[Lambda]*y[x]-a*b^2*x^n*Exp[2*\[Lambda]*x],y[x],x,IncludeSingula
```

$$y(x) \to \sqrt{-b^2}e^{\lambda x}\tan\left(\frac{a\sqrt{-b^2}x^n(\lambda(-x))^{-n}\Gamma(n+1,-x\lambda)}{\lambda}+c_1\right)$$

4.9 problem 30

Internal problem ID [10438]
Internal file name [`OUTPUT/9385_Monday_June_06_2022_02_21_05_PM_40185526/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\,x^n y^2 + ab\,x^n \mathrm{e}^{\lambda x} y = b\lambda\,\mathrm{e}^{\lambda x}}$$

4.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,x^n y^2 - ab\,x^n \mathrm{e}^{\lambda x} y + b\lambda\,\mathrm{e}^{\lambda x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,x^n y^2 - ab\,x^n \mathrm{e}^{\lambda x} y + b\lambda\,\mathrm{e}^{\lambda x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\lambda\,\mathrm{e}^{\lambda x}$, $f_1(x) = -ab\,x^n \mathrm{e}^{\lambda x}$ and $f_2(x) = x^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^n a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{x^n na}{x} \\ f_1f_2 &= -a^2b\,x^{2n}\mathrm{e}^{\lambda x} \\ f_2^2f_0 &= \mathrm{e}^{\lambda x}x^{2n}a^2b\lambda \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^n au''(x) - \left(-a^2b\,x^{2n}\mathrm{e}^{\lambda x} + \frac{x^n na}{x}\right)u'(x) + \mathrm{e}^{\lambda x}x^{2n}a^2b\lambda u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{ab\,\mathrm{e}^{\lambda x}(\lambda\_Y(x) + \_Y'(x))\,x^{n+1} + \_Y''(x)\,x - n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{ab\,\mathrm{e}^{\lambda x}(\lambda\_Y(x) + \_Y'(x))\,x^{n+1} + \_Y''(x)\,x - n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{ab\,\mathrm{e}^{\lambda x}(\lambda\_Y(x)+\_Y'(x))x^{n+1}+\_Y''(x)x-n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)\right)x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{ab\,\mathrm{e}^{\lambda x}(\lambda\_Y(x)+\_Y'(x))x^{n+1}+\_Y''(x)x-n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{ab\,\mathrm{e}^{\lambda x}(\lambda\_Y(x)+\_Y'(x))x^{n+1}+\_Y''(x)x-n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)\right)x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{ab\,\mathrm{e}^{\lambda x}(\lambda\_Y(x)+\_Y'(x))x^{n+1}+\_Y''(x)x-n\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{a b\,\mathrm{e}^{\lambda x}\left(\lambda \_\_Y(x)+\_\_Y'(x)\right)x^{n+1}+\_\_Y''(x)x-n\_\_Y'(x)}{x}\right\},\{\_\_Y(x)\}\right)\right)x^{-n}}{a\operatorname{DESol}\left(\left\{\frac{a b\,\mathrm{e}^{\lambda x}\left(\lambda \_\_Y(x)+\_\_Y'(x)\right)x^{n+1}+\_\_Y''(x)x-n\_\_Y'(x)}{x}\right\},\{\_\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{a b\,\mathrm{e}^{\lambda x}\left(\lambda \_\_Y(x)+\_\_Y'(x)\right)x^{n+1}+\_\_Y''(x)x-n\_\_Y'(x)}{x}\right\},\{\_\_Y(x)\}\right)\right)x^{-n}}{a\operatorname{DESol}\left(\left\{\frac{a b\,\mathrm{e}^{\lambda x}\left(\lambda \_\_Y(x)+\_\_Y'(x)\right)x^{n+1}+\_\_Y''(x)x-n\_\_Y'(x)}{x}\right\},\{\_\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(exp(lambda*x)*x^n*a*b*x-n)*(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
            checking if the LODE is missing y
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=a*x^n*y(x)^2-a*b*x^n*exp(lambda*x)*y(x)+b*lambda*exp(lambda*x),y(x), sin
```

No solution found

✓ Solution by Mathematica

Time used: 53.05 (sec). Leaf size: 190

```
DSolve[y'[x]==a*x^n*y[x]^2-a*b*x^n*Exp[\[Lambda]*x]*y[x]+b*\[Lambda]*Exp[\[Lambda]*x],y[x],x
```

$$y(x) \to \frac{be^{2\lambda x}\left(\int_1^{e^{x\lambda}} \frac{\exp\left(\frac{ab\Gamma(n+1,-\log(K[1]))(-\log(K[1]))^{-n}\left(\frac{\log(K[1])}{\lambda}\right)^n}{\lambda}\right)}{K[1]^2}dK[1] + c_1\right)}{e^{\lambda x}\int_1^{e^{x\lambda}} \frac{\exp\left(\frac{ab\Gamma(n+1,-\log(K[1]))(-\log(K[1]))^{-n}\left(\frac{\log(K[1])}{\lambda}\right)^n}{\lambda}\right)}{K[1]^2}dK[1] + \exp\left(\frac{ab(-\log(e^{\lambda x}))^{-n}\left(\frac{\log(e^{\lambda x})}{\lambda}\right)^n\Gamma(n+1,-\log(e^{x\lambda}))}{\lambda}\right.}$$

$$y(x) \to be^{\lambda x}$$

4.10 problem 31

Internal problem ID [10439]
Internal file name [`OUTPUT/9386_Monday_June_06_2022_02_21_08_PM_74422371/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + (k+1)\,x^k y^2 - a\,x^{k+1} \mathrm{e}^{\lambda x} y = -\mathrm{e}^{\lambda x} a}$$

4.10.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -x^k y^2 k + a\,x^{k+1} \mathrm{e}^{\lambda x} y - x^k y^2 - \mathrm{e}^{\lambda x} a \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -x^k y^2 k + a\,x^k x\,\mathrm{e}^{\lambda x} y - x^k y^2 - \mathrm{e}^{\lambda x} a$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\mathrm{e}^{\lambda x} a$, $f_1(x) = x^{k+1} \mathrm{e}^{\lambda x} a$ and $f_2(x) = -x^k k - x^k$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(-x^k k - x^k)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$f_2' = -\frac{k^2x^k}{x} - \frac{k\,x^k}{x}$$
$$f_1f_2 = x^{k+1}\mathrm{e}^{\lambda x}a\left(-x^kk - x^k\right)$$
$$f_2^2f_0 = -\left(-x^kk - x^k\right)^2\mathrm{e}^{\lambda x}a$$

Substituting the above terms back in equation (2) gives

$$\left(-x^kk - x^k\right)u''(x) - \left(-\frac{k^2x^k}{x} - \frac{k\,x^k}{x} + x^{k+1}\mathrm{e}^{\lambda x}a\left(-x^kk - x^k\right)\right)u'(x) - \left(-x^kk - x^k\right)^2\mathrm{e}^{\lambda x}au(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{k+1}\left(\left(\int x^{-2k-2}\mathrm{e}^{\int\left(x^{k+1}\mathrm{e}^{\lambda x}a+\frac{k}{x}\right)dx}dx\right)c_2 + c_1\right)$$

The above shows that

$$u'(x) = c_2x^{-k-1}\mathrm{e}^{\int\frac{a\,x^{k+2}\mathrm{e}^{\lambda x}+k}{x}dx} + x^k(k+1)\left(\left(\int \mathrm{e}^{\int\frac{a\,x^{k+2}\mathrm{e}^{\lambda x}+k}{x}dx}x^{-2k-2}dx\right)c_2 + c_1\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(c_2x^{-k-1}\mathrm{e}^{\int\frac{a\,x^{k+2}\mathrm{e}^{\lambda x}+k}{x}dx} + x^k(k+1)\left(\left(\int \mathrm{e}^{\int\frac{a\,x^{k+2}\mathrm{e}^{\lambda x}+k}{x}dx}x^{-2k-2}dx\right)c_2 + c_1\right)\right)x^{-k-1}}{\left(-x^kk - x^k\right)\left(\left(\int x^{-2k-2}\mathrm{e}^{\int\left(x^{k+1}\mathrm{e}^{\lambda x}a+\frac{k}{x}\right)dx}dx\right)c_2 + c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(k+1)\left(\int x^{-2k-2}\mathrm{e}^{\int\left(x^{k+1}\mathrm{e}^{\lambda x}a+\frac{k}{x}\right)dx}dx + c_3\right)x^{-k-1} + x^{-3k-2}\mathrm{e}^{\int\left(x^{k+1}\mathrm{e}^{\lambda x}a+\frac{k}{x}\right)dx}}{(k+1)\left(\int \mathrm{e}^{\int\frac{a\,x^{k+2}\mathrm{e}^{\lambda x}+k}{x}dx}x^{-2k-2}dx + c_3\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{(k+1)\left(\int x^{-2k-2} e^{\int \left(x^{k+1} e^{\lambda x} a+\frac{k}{x}\right)dx} dx + c_3\right) x^{-k-1} + x^{-3k-2} e^{\int \left(x^{k+1} e^{\lambda x} a+\frac{k}{x}\right)dx}}{(k+1)\left(\int e^{\int \frac{a\,x^{k+2} e^{\lambda x}+k}{x} dx} x^{-2k-2} dx + c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{(k+1)\left(\int x^{-2k-2} e^{\int \left(x^{k+1} e^{\lambda x} a+\frac{k}{x}\right)dx} dx + c_3\right) x^{-k-1} + x^{-3k-2} e^{\int \left(x^{k+1} e^{\lambda x} a+\frac{k}{x}\right)dx}}{(k+1)\left(\int e^{\int \frac{a\,x^{k+2} e^{\lambda x}+k}{x} dx} x^{-2k-2} dx + c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x^(1+k)*exp(lambda*x)*a*x+k)*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 184

```
dsolve(diff(y(x),x)=-(k+1)*x^k*y(x)^2+a*x^(k+1)*exp(lambda*x)*y(x)-a*exp(lambda*x),y(x), sin
```

$$y(x) = \frac{x^{-1-k}\left(x^{1+k}e^{\int \frac{x^{1+k}e^{x\lambda}ax-2k-2}{x}dx} + \left(\int x^k e^{a\left(\int x^{1+k}e^{x\lambda}dx\right)-2\left(\int \frac{1}{x}dx\right)(1+k)}dx\right)k + \int x^k e^{a\left(\int x^{1+k}e^{x\lambda}dx\right)-2\left(\int \frac{1}{x}dx\right)(1+k)}d\right.}{\left(\int x^k e^{a\left(\int x^{1+k}e^{x\lambda}dx\right)-2\left(\int \frac{1}{x}dx\right)(1+k)}dx\right)k + \int x^k e^{a\left(\int x^{1+k}e^{x\lambda}dx\right)-2\left(\int \frac{1}{x}dx\right)(1+k)}dx - c_1}$$

✓ Solution by Mathematica

Time used: 86.249 (sec). Leaf size: 280

```
DSolve[y'[x]==-(k+1)*x^k*y[x]^2+a*x^(k+1)*Exp[\[Lambda]*x]*y[x]-a*Exp[\[Lambda]*x],y[x],x,In
```

$$y(x) \to \frac{a\lambda \exp\left(\frac{a(-\log(e^{\lambda x}))^{-k}\left(\frac{\log(e^{\lambda x})}{\lambda}\right)^k \Gamma(k+2,-\log(e^{x\lambda}))}{\lambda^2}\right)\left(1 + c_1\int_1^{e^{x\lambda}} \text{ex}\right.}{ac_1\lambda\left(\frac{\log(e^{\lambda x})}{\lambda}\right)^{k+1}\exp\left(\frac{a(-\log(e^{\lambda x}))^{-k}\left(\frac{\log(e^{\lambda x})}{\lambda}\right)^k \Gamma(k+2,-\log(e^{x\lambda}))}{\lambda^2}\right)\int_1^{e^{x\lambda}}\exp\left(-\frac{a\Gamma(k+2,-\log(K[1]))(-\log(K[1]))^{-}}{\lambda^2}\right.}$$

4.11 problem 32

Internal problem ID [10440]
Internal file name [`OUTPUT/9387_Monday_June_06_2022_02_21_12_PM_78437714/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\,x^n y^2 + a\,x^n \left(b\,\mathrm{e}^{\lambda x} + c\right) y = c\,x^n}$$

4.11.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -ab\,x^n \mathrm{e}^{\lambda x} y - x^n a c y + a\,x^n y^2 + c\,x^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -ab\,x^n \mathrm{e}^{\lambda x} y - x^n a c y + a\,x^n y^2 + c\,x^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = c\,x^n$, $f_1(x) = -ab\,x^n \mathrm{e}^{\lambda x} - x^n a c$ and $f_2(x) = x^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^n a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{x^n na}{x} \\ f_1f_2 &= \left(-ab\,x^n\mathrm{e}^{\lambda x} - x^n ac\right)x^n a \\ f_2^2f_0 &= x^{3n}a^2c \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^n au''(x) - \left(\left(-ab\,x^n\mathrm{e}^{\lambda x} - x^n ac\right)x^n a + \frac{x^n na}{x}\right)u'(x) + x^{3n}a^2cu(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{ac\,x^{1+2n}\_Y(x) + \_Y''(x)\,x + \_Y'(x)\left(x^{n+1}\left(b\,\mathrm{e}^{\lambda x}+c\right)a - n\right)}{x}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{ac\,x^{1+2n}\_Y(x) + \_Y''(x)\,x + \_Y'(x)\left(x^{n+1}\left(b\,\mathrm{e}^{\lambda x}+c\right)a - n\right)}{x}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{ac\,x^{1+2n}\_Y(x)+\_Y''(x)x+\_Y'(x)\left(x^{n+1}\left(b\,\mathrm{e}^{\lambda x}+c\right)a-n\right)}{x}\right\}, \{\_Y(x)\}\right)\right)x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{ac\,x^{1+2n}\_Y(x)+\_Y''(x)x+\_Y'(x)\left(x^{n+1}\left(b\,\mathrm{e}^{\lambda x}+c\right)a-n\right)}{x}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{ac\,x^{1+2n}\_Y(x)+\_Y''(x)x+\_Y'(x)\left(x^{n+1}\left(b\,\mathrm{e}^{\lambda x}+c\right)a-n\right)}{x}\right\}, \{\_Y(x)\}\right)\right)x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{ac\,x^{1+2n}\_Y(x)+\_Y''(x)x+\_Y'(x)\left(x^{n+1}\left(b\,\mathrm{e}^{\lambda x}+c\right)a-n\right)}{x}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{a c\, x^{1+2n} \_\_Y(x)+\_\_Y''(x) x+\_\_Y'(x)\left(x^{n+1}\left(b\,\mathrm{e}^{\lambda x}+c\right) a-n\right)}{x}\right\}, \{\_\_Y(x)\}\right)\right) x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{a c\, x^{1+2n} \_\_Y(x)+\_\_Y''(x) x+\_\_Y'(x)\left(x^{n+1}\left(b\,\mathrm{e}^{\lambda x}+c\right) a-n\right)}{x}\right\}, \{\_\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{a c\, x^{1+2n} \_\_Y(x)+\_\_Y''(x) x+\_\_Y'(x)\left(x^{n+1}\left(b\,\mathrm{e}^{\lambda x}+c\right) a-n\right)}{x}\right\}, \{\_\_Y(x)\}\right)\right) x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{a c\, x^{1+2n} \_\_Y(x)+\_\_Y''(x) x+\_\_Y'(x)\left(x^{n+1}\left(b\,\mathrm{e}^{\lambda x}+c\right) a-n\right)}{x}\right\}, \{\_\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(exp(lambda*x)*x^n*a*b*x+a*x^
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=a*x^n*y(x)^2-a*x^n*(b*exp(lambda*x)+c)*y(x)+c*x^n,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==a*x^n*y[x]^2-a*x^n*(b*Exp[\[Lambda]*x]+c)*y[x]+c*x^n,y[x],x,IncludeSingularSol
```

Not solved

4.12 problem 33

Internal problem ID [10441]
Internal file name [`OUTPUT/9388_Monday_June_06_2022_02_21_15_PM_99041241/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - a\,x^n \mathrm{e}^{2\lambda x} y^2 - \left(b\,x^n \mathrm{e}^{\lambda x} - \lambda\right) y = c\,x^n}$$

4.12.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,x^n \mathrm{e}^{2\lambda x} y^2 + x^n \mathrm{e}^{\lambda x} b y + c\,x^n - \lambda y \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,x^n \mathrm{e}^{2\lambda x} y^2 + x^n \mathrm{e}^{\lambda x} b y + c\,x^n - \lambda y$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = c\,x^n$, $f_1(x) = b\,x^n \mathrm{e}^{\lambda x} - \lambda$ and $f_2(x) = x^n \mathrm{e}^{2\lambda x} a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^n \mathrm{e}^{2\lambda x} a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{x^n n\, e^{2\lambda x} a}{x} + 2\, e^{2\lambda x} x^n a \lambda \\ f_1 f_2 &= \left(b\, x^n e^{\lambda x} - \lambda\right) x^n e^{2\lambda x} a \\ f_2^2 f_0 &= x^{3n} e^{4\lambda x} a^2 c \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^n e^{2\lambda x} a u''(x) - \left(\frac{x^n n\, e^{2\lambda x} a}{x} + 2\, e^{2\lambda x} x^n a \lambda + \left(b\, x^n e^{\lambda x} - \lambda\right) x^n e^{2\lambda x} a\right) u'(x) + x^{3n} e^{4\lambda x} a^2 c u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = e^{-\frac{\int \left(\tan\left(\frac{\sqrt{4a\, b^2 c - b^4}\, \left(\Gamma(n, -\lambda x) x^n b (-\lambda x)^{-n} n - x^n (-\lambda x)^{-n} \Gamma(n+1) b + b\, x^n e^{\lambda x} + c_1 \lambda\right)}{2 b^2 \lambda}\right) \sqrt{4a\, b^2 c - b^4} - b^2\right) x^n e^{\lambda x} dx}{2b}} c_2$$

The above shows that

$$u'(x) = -\frac{c_2 x^n e^{\lambda x - \frac{\int \left(\tan\left(\frac{\sqrt{4a\, b^2 c - b^4}\, \left(\Gamma(n, -\lambda x) x^n b (-\lambda x)^{-n} n - x^n (-\lambda x)^{-n} \Gamma(n+1) b + b\, x^n e^{\lambda x} + c_1 \lambda\right)}{2 b^2 \lambda}\right) \sqrt{4a\, b^2 c - b^4} - b^2\right) x^n e^{\lambda x} dx}{2b}} \left(\tan\left(\frac{\sqrt{4a\, b^2 c - b^4}}{}\right.\right.}{2b}$$

Using the above in (1) gives the solution

$$y = \frac{e^{\lambda x - \frac{\int \left(\tan\left(\frac{\sqrt{4a\, b^2 c - b^4}\, \left(\Gamma(n, -\lambda x) x^n b (-\lambda x)^{-n} n - x^n (-\lambda x)^{-n} \Gamma(n+1) b + b\, x^n e^{\lambda x} + c_1 \lambda\right)}{2 b^2 \lambda}\right) \sqrt{4a\, b^2 c - b^4} - b^2\right) x^n e^{\lambda x} dx}{2b}} \left(\tan\left(\frac{\sqrt{4a\, b^2 c - b^4}\, \left(\Gamma(n, -\lambda\right.}{}\right.\right.}{}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{e^{-\lambda x} \left(\tan\left(\frac{\sqrt{4a\,b^2c-b^4}\,\left(\Gamma(n,-\lambda x)x^n b(-\lambda x)^{-n}n-x^n(-\lambda x)^{-n}\Gamma(n+1)b+b\,x^n e^{\lambda x}+\lambda c_3\right)}{2b^2\lambda}\right)\sqrt{4a\,b^2c-b^4}-b^2\right)}{2ab}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-\lambda x} \left(\tan\left(\frac{\sqrt{4a\,b^2c-b^4}\,\left(\Gamma(n,-\lambda x)x^n b(-\lambda x)^{-n}n-x^n(-\lambda x)^{-n}\Gamma(n+1)b+b\,x^n e^{\lambda x}+\lambda c_3\right)}{2b^2\lambda}\right)\sqrt{4a\,b^2c-b^4}-b^2\right)}{2ab} \tag{1}$$

Verification of solutions

$$y = \frac{e^{-\lambda x} \left(\tan\left(\frac{\sqrt{4a\,b^2c-b^4}\,\left(\Gamma(n,-\lambda x)x^n b(-\lambda x)^{-n}n-x^n(-\lambda x)^{-n}\Gamma(n+1)b+b\,x^n e^{\lambda x}+\lambda c_3\right)}{2b^2\lambda}\right)\sqrt{4a\,b^2c-b^4}-b^2\right)}{2ab}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 114

```
dsolve(diff(y(x),x)=a*x^n*exp(2*lambda*x)*y(x)^2+(b*x^n*exp(lambda*x)-lambda)*y(x)+c*x^n,y(x
```

$$y(x) = \frac{\left(\tan\left(\frac{\sqrt{4a\,b^2c-b^4}\left(x^n(-x\lambda)^{-n}\Gamma(n,-x\lambda)bn-x^n(-x\lambda)^{-n}\Gamma(n+1)b+e^{x\lambda}x^n b+\lambda c_1\right)}{2b^2\lambda}\right)\sqrt{4a\,b^2c-b^4}-b^2\right)e^{-x\lambda}}{2ab}$$

✓ Solution by Mathematica

Time used: 3.112 (sec). Leaf size: 102

```
DSolve[y'[x]==a*x^n*Exp[2*\[Lambda]*x]*y[x]^2+(b*x^n*Exp[\[Lambda]*x]-\[Lambda])*y[x]+c*x^n,
```

$$\text{Solve}\left[\int_1^{\sqrt{\frac{ae^{2x\lambda}}{c}}y(x)}\frac{1}{K[1]^2-\sqrt{\frac{b^2}{ac}}K[1]+1}dK[1]=\frac{cx^n e^{\lambda(-x)}(\lambda(-x))^{-n}\sqrt{\frac{ae^{2\lambda x}}{c}}\Gamma(n+1,-x\lambda)}{\lambda}+c_1,y(x)\right]$$

4.13 problem 34

Internal problem ID [10442]
Internal file name [`OUTPUT/9389_Monday_June_06_2022_02_21_19_PM_91254245/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$\boxed{y' - a\,\mathrm{e}^{\lambda x}(y - b\,x^n - c)^2 = b\,x^{n-1}n}$$

4.13.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,b^2\mathrm{e}^{\lambda x}x^{2n} + 2x^n\mathrm{e}^{\lambda x}abc - 2ab\,x^n\mathrm{e}^{\lambda x}y + \mathrm{e}^{\lambda x}a\,c^2 - 2\,\mathrm{e}^{\lambda x}acy + \mathrm{e}^{\lambda x}a\,y^2 + b\,x^{n-1}n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,b^2\mathrm{e}^{\lambda x}x^{2n} + 2x^n\mathrm{e}^{\lambda x}abc - 2ab\,x^n\mathrm{e}^{\lambda x}y + \mathrm{e}^{\lambda x}a\,c^2 - 2\,\mathrm{e}^{\lambda x}acy + \mathrm{e}^{\lambda x}a\,y^2 + \frac{b\,x^n n}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\,b^2\mathrm{e}^{\lambda x}x^{2n} + 2x^n\mathrm{e}^{\lambda x}abc + \mathrm{e}^{\lambda x}a\,c^2 + b\,x^{n-1}n$, $f_1(x) = -2ab\,x^n\mathrm{e}^{\lambda x} - 2\,\mathrm{e}^{\lambda x}ac$ and $f_2(x) = \mathrm{e}^{\lambda x}a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\mathrm{e}^{\lambda x}au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= a\lambda\, e^{\lambda x} \\ f_1 f_2 &= \left(-2ab\, x^n e^{\lambda x} - 2\, e^{\lambda x} ac\right) e^{\lambda x} a \\ f_2^2 f_0 &= e^{2\lambda x} a^2 \left(a\, b^2 e^{\lambda x} x^{2n} + 2x^n e^{\lambda x} abc + e^{\lambda x} a\, c^2 + b\, x^{n-1} n\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$e^{\lambda x} a u''(x) - \left(a\lambda\, e^{\lambda x} + \left(-2ab\, x^n e^{\lambda x} - 2\, e^{\lambda x} ac\right) e^{\lambda x} a\right) u'(x) + e^{2\lambda x} a^2 \left(a\, b^2 e^{\lambda x} x^{2n} + 2x^n e^{\lambda x} abc + e^{\lambda x} a\, c^2 + b\, x$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = e^{-\frac{\left(\int \left(2a\left(b\, x^n + c\right) e^{\lambda x} - \lambda\right) dx\right)}{2}} \left(c_1 \sinh\left(\frac{\lambda x}{2}\right) + c_2 \cosh\left(\frac{\lambda x}{2}\right)\right)$$

The above shows that

$$\begin{aligned} u'(x) = -\Bigg(&\left(a c_2 (b\, x^n + c)\, e^{\lambda x} - \frac{\lambda(c_1 + c_2)}{2}\right) \cosh\left(\frac{\lambda x}{2}\right) \\ &+ \sinh\left(\frac{\lambda x}{2}\right) \left(a c_1 (b\, x^n + c)\, e^{\lambda x} - \frac{\lambda(c_1 + c_2)}{2}\right)\Bigg)\, e^{-\frac{\left(\int \left(2a\left(b\, x^n + c\right) e^{\lambda x} - \lambda\right) dx\right)}{2}} \end{aligned}$$

Using the above in (1) gives the solution

$$y = \frac{\left(\left(a c_2 (b\, x^n + c)\, e^{\lambda x} - \frac{\lambda(c_1 + c_2)}{2}\right) \cosh\left(\frac{\lambda x}{2}\right) + \sinh\left(\frac{\lambda x}{2}\right) \left(a c_1 (b\, x^n + c)\, e^{\lambda x} - \frac{\lambda(c_1 + c_2)}{2}\right)\right) e^{-\lambda x}}{a \left(c_1 \sinh\left(\frac{\lambda x}{2}\right) + c_2 \cosh\left(\frac{\lambda x}{2}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-\lambda(c_3 + 1)\, e^{-\lambda x} + 2(b\, x^n + c)\, a\right) \cosh\left(\frac{\lambda x}{2}\right) + 2\left(-\frac{\lambda(c_3+1) e^{-\lambda x}}{2} + (b\, x^n + c)\, c_3 a\right) \sinh\left(\frac{\lambda x}{2}\right)}{2\left(c_3 \sinh\left(\frac{\lambda x}{2}\right) + \cosh\left(\frac{\lambda x}{2}\right)\right) a}$$

Summary
The solution(s) found are the following

$$
y = \frac{\left(-\lambda (c_3+1)\,\mathrm{e}^{-\lambda x}+2(b\,x^{n}+c)\,a\right)\cosh\left(\frac{\lambda x}{2}\right)+2\left(-\frac{\lambda (c_3+1)\,\mathrm{e}^{-\lambda x}}{2}+(b\,x^{n}+c)\,c_3 a\right)\sinh\left(\frac{\lambda x}{2}\right)}{2\left(c_3\sinh\left(\frac{\lambda x}{2}\right)+\cosh\left(\frac{\lambda x}{2}\right)\right)a} \tag{1}
$$

Verification of solutions

$$
y = \frac{\left(-\lambda (c_3+1)\,\mathrm{e}^{-\lambda x}+2(b\,x^{n}+c)\,a\right)\cosh\left(\frac{\lambda x}{2}\right)+2\left(-\frac{\lambda (c_3+1)\,\mathrm{e}^{-\lambda x}}{2}+(b\,x^{n}+c)\,c_3 a\right)\sinh\left(\frac{\lambda x}{2}\right)}{2\left(c_3\sinh\left(\frac{\lambda x}{2}\right)+\cosh\left(\frac{\lambda x}{2}\right)\right)a}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular polynomial solution successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 50

```
dsolve(diff(y(x),x)=a*exp(lambda*x)*(y(x)-b*x^n-c)^2+b*n*x^(n-1),y(x), singsol=all)
```

$$
y(x) = \frac{a c_1 \lambda (b\,x^{n}+c)\,\mathrm{e}^{x\lambda}+x^{n} a b - c_1 \lambda^{2}+a c}{(\lambda c_1 \mathrm{e}^{x\lambda}+1)\,a}
$$

✓ Solution by Mathematica

Time used: 1.563 (sec). Leaf size: 40

```
DSolve[y'[x]==a*Exp[\[Lambda]*x]*(y[x]-b*x^n-c)^2+b*n*x^(n-1),y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{\lambda}{-ae^{\lambda x} + c_1\lambda} + bx^n + c$$
$$y(x) \to bx^n + c$$

4.14 problem 35

Internal problem ID [10443]
Internal file name [`OUTPUT/9390_Monday_June_06_2022_02_21_21_PM_25838751/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y'x - a\,\mathrm{e}^{\lambda x} y^2 - ky = a\,b^2 x^{2k} \mathrm{e}^{\lambda x}}$$

4.14.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{\mathrm{e}^{\lambda x} a\,y^2 + ky + a\,b^2 x^{2k} \mathrm{e}^{\lambda x}}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,b^2 x^{2k} \mathrm{e}^{\lambda x}}{x} + \frac{\mathrm{e}^{\lambda x} a\,y^2}{x} + \frac{ky}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{a\,b^2 x^{2k} \mathrm{e}^{\lambda x}}{x}$, $f_1(x) = \frac{k}{x}$ and $f_2(x) = \frac{a\,\mathrm{e}^{\lambda x}}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{a\,\mathrm{e}^{\lambda x} u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{a\,\mathrm{e}^{\lambda x}}{x^2} + \frac{a\lambda\,\mathrm{e}^{\lambda x}}{x} \\
f_1 f_2 &= \frac{ka\,\mathrm{e}^{\lambda x}}{x^2} \\
f_2^2 f_0 &= \frac{a^3 \mathrm{e}^{3\lambda x} b^2 x^{2k}}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{a\,\mathrm{e}^{\lambda x} u''(x)}{x} - \left(-\frac{a\,\mathrm{e}^{\lambda x}}{x^2} + \frac{a\lambda\,\mathrm{e}^{\lambda x}}{x} + \frac{ka\,\mathrm{e}^{\lambda x}}{x^2}\right) u'(x) + \frac{a^3 \mathrm{e}^{3\lambda x} b^2 x^{2k} u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
u(x) = c_1 \sin &\left(ab\, x^k (-\lambda x)^{-k} \left(\Gamma(k) - \Gamma(k, -\lambda x)\right)\right) \\
&+ c_2 \cos \left(ab\, x^k (-\lambda x)^{-k} \left(\Gamma(k) - \Gamma(k, -\lambda x)\right)\right)
\end{aligned}$$

The above shows that

$$\begin{aligned}
u'(x) = \Big(c_1 \cos &\left(ab\, x^k (-\lambda x)^{-k} \left(\Gamma(k) - \Gamma(k, -\lambda x)\right)\right) \\
&- c_2 \sin \left(ab\, x^k (-\lambda x)^{-k} \left(\Gamma(k) - \Gamma(k, -\lambda x)\right)\right)\Big)\, ab\, x^{k-1} \mathrm{e}^{\lambda x}
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(c_1 \cos \left(ab\, x^k (-\lambda x)^{-k} \left(\Gamma(k) - \Gamma(k, -\lambda x)\right)\right) - c_2 \sin \left(ab\, x^k (-\lambda x)^{-k} \left(\Gamma(k) - \Gamma(k, -\lambda x)\right)\right)\right) b\, x^{k-1} x}{c_1 \sin \left(ab\, x^k \left(-\lambda x\right)^{-k} \left(\Gamma\left(k\right) - \Gamma\left(k, -\lambda x\right)\right)\right) + c_2 \cos \left(ab\, x^k \left(-\lambda x\right)^{-k} \left(\Gamma\left(k\right) - \Gamma\left(k, -\lambda x\right)\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{b\,x^{k}\left(-c_3 \cos\left(ab\,x^{k}(-\lambda x)^{-k} \left(\Gamma(k)-\Gamma(k,-\lambda x)\right)\right)+\sin\left(ab\,x^{k}(-\lambda x)^{-k} \left(\Gamma(k)-\Gamma(k,-\lambda x)\right)\right)\right)}{c_3 \sin\left(ab\,x^{k}\left(-\lambda x\right)^{-k} \left(\Gamma\left(k\right)-\Gamma\left(k,-\lambda x\right)\right)\right)+\cos\left(ab\,x^{k}\left(-\lambda x\right)^{-k} \left(\Gamma\left(k\right)-\Gamma\left(k,-\lambda x\right)\right)\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{b\,x^{k}\left(-c_3 \cos\left(ab\,x^{k}(-\lambda x)^{-k} \left(\Gamma(k)-\Gamma(k,-\lambda x)\right)\right)+\sin\left(ab\,x^{k}(-\lambda x)^{-k} \left(\Gamma(k)-\Gamma(k,-\lambda x)\right)\right)\right)}{c_3 \sin\left(ab\,x^{k}\left(-\lambda x\right)^{-k} \left(\Gamma\left(k\right)-\Gamma\left(k,-\lambda x\right)\right)\right)+\cos\left(ab\,x^{k}\left(-\lambda x\right)^{-k} \left(\Gamma\left(k\right)-\Gamma\left(k,-\lambda x\right)\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{b\,x^{k}\left(-c_3 \cos\left(ab\,x^{k}(-\lambda x)^{-k} \left(\Gamma(k)-\Gamma(k,-\lambda x)\right)\right)+\sin\left(ab\,x^{k}(-\lambda x)^{-k} \left(\Gamma(k)-\Gamma(k,-\lambda x)\right)\right)\right)}{c_3 \sin\left(ab\,x^{k}\left(-\lambda x\right)^{-k} \left(\Gamma\left(k\right)-\Gamma\left(k,-\lambda x\right)\right)\right)+\cos\left(ab\,x^{k}\left(-\lambda x\right)^{-k} \left(\Gamma\left(k\right)-\Gamma\left(k,-\lambda x\right)\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 38

```
dsolve(x*diff(y(x),x)=a*exp(lambda*x)*y(x)^2+k*y(x)+a*b^2*x^(2*k)*exp(lambda*x),y(x), singsol
```

$$y(x) = -\tan\left(ab\,x^{k}(\Gamma(k,-x\lambda)-\Gamma(k))\left(-x\lambda\right)^{-k}+c_1\right) b\,x^{k}$$

✓ Solution by Mathematica

Time used: 1.593 (sec). Leaf size: 47

```
DSolve[x*y'[x]==a*Exp[\[Lambda]*x]*y[x]^2+k*y[x]+a*b^2*x^(2*k)*Exp[\[Lambda]*x],y[x],x,Inclu
```

$$y(x) \to \sqrt{b^2} x^k \tan \left(-a\sqrt{b^2} x^k (\lambda(-x))^{-k} \Gamma(k, -x\lambda) + c_1\right)$$

4.15 problem 36

Internal problem ID [10444]
Internal file name [`OUTPUT/9391_Monday_June_06_2022_02_21_22_PM_4099908/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - a\,x^{2n}\mathrm{e}^{\lambda x}y^2 - \left(b\,x^n\mathrm{e}^{\lambda x} - n\right)y = \mathrm{e}^{\lambda x}c}$$

4.15.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,x^{2n}\mathrm{e}^{\lambda x}y^2 + x^n\mathrm{e}^{\lambda x}by + \mathrm{e}^{\lambda x}c - ny}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,x^{2n}\mathrm{e}^{\lambda x}y^2}{x} + \frac{\mathrm{e}^{\lambda x}x^n by}{x} + \frac{\mathrm{e}^{\lambda x}c}{x} - \frac{ny}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\mathrm{e}^{\lambda x}c}{x}$, $f_1(x) = \frac{b\,x^n\mathrm{e}^{\lambda x}-n}{x}$ and $f_2(x) = \frac{x^{2n}\mathrm{e}^{\lambda x}a}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{x^{2n}\mathrm{e}^{\lambda x}au}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = \frac{2x^{2n} n\, e^{\lambda x} a}{x^2} + \frac{x^{2n} \lambda\, e^{\lambda x} a}{x} - \frac{e^{\lambda x} a\, x^{2n}}{x^2}$$
$$f_1 f_2 = \frac{(b\, x^n e^{\lambda x} - n)\, x^{2n} e^{\lambda x} a}{x^2}$$
$$f_2^2 f_0 = \frac{x^{4n} e^{3\lambda x} a^2 c}{x^3}$$

Substituting the above terms back in equation (2) gives

$$\frac{x^{2n} e^{\lambda x} a u''(x)}{x} - \left(\frac{2x^{2n} n\, e^{\lambda x} a}{x^2} + \frac{x^{2n} \lambda\, e^{\lambda x} a}{x} - \frac{e^{\lambda x} a\, x^{2n}}{x^2} + \frac{(b\, x^n e^{\lambda x} - n)\, x^{2n} e^{\lambda x} a}{x^2}\right) u'(x) + \frac{x^{4n} e^{3\lambda x} a^2 c u(x)}{x^3} =$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(c_1 \operatorname{BesselJ}\left(\frac{\sqrt{3}\sqrt{-ca}}{8b}, \frac{\sqrt{3}\sqrt{2}\sqrt{ca\, x^{2n}}\, x^{-n}}{8b}\right) + c_2 \operatorname{BesselY}\left(\frac{\sqrt{3}\sqrt{-ca}}{8b}, \frac{\sqrt{3}\sqrt{2}\sqrt{ca\, x^{2n}}\, x^{-n}}{8b}\right)\right) e^{\frac{\left(\int \left(\frac{b\, x^n e^{\lambda x}}{x} + \lambda + \frac{3n}{x}\right) dx\right)}{2}}$$

The above shows that

$$u'(x) = \frac{e^{\frac{\left(\int \frac{b\, x^n e^{\lambda x} + \lambda x + 3n}{x} dx\right)}{2}} \left(c_1 \operatorname{BesselJ}\left(\frac{\sqrt{3}\sqrt{-ca}}{8b}, \frac{\sqrt{3}\sqrt{2}\sqrt{ca\, x^{2n}}\, x^{-n}}{8b}\right) + c_2 \operatorname{BesselY}\left(\frac{\sqrt{3}\sqrt{-ca}}{8b}, \frac{\sqrt{3}\sqrt{2}\sqrt{ca\, x^{2n}}\, x^{-n}}{8b}\right)\right) (b\, x^n e}{2x}$$

Using the above in (1) gives the solution

$$y = -\frac{e^{\frac{\left(\int \frac{b\, x^n e^{\lambda x} + \lambda x + 3n}{x} dx\right)}{2}} (b\, x^n e^{\lambda x} + \lambda x + 3n)\, x^{-2n} e^{-\lambda x} e^{\int \left(-\frac{b\, x^{n-1} e^{\lambda x}}{2} - \frac{\lambda}{2} - \frac{3n}{2x}\right) dx}}{2a}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left((\lambda x + 3n)\, e^{-\lambda x} + b\, x^{n}\right) x^{-2n}}{2a}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left((\lambda x + 3n)\, e^{-\lambda x} + b\, x^{n}\right) x^{-2n}}{2a} \tag{1}$$

Verification of solutions

$$y = -\frac{\left((\lambda x + 3n)\, e^{-\lambda x} + b\, x^{n}\right) x^{-2n}}{2a}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 86

```
dsolve(x*diff(y(x),x)=a*x^(2*n)*exp(lambda*x)*y(x)^2+(b*x^n*exp(lambda*x)-n)*y(x)+c*exp(lamb
```

$$y(x) = -\frac{\left(\tan\left(\frac{\left(x^{n} b(\Gamma(n,-x\lambda)-\Gamma(n))(-x\lambda)^{-n}-c_{1}\right)\sqrt{4a\,b^{2}c-b^{4}}}{2b^{2}}\right)\sqrt{4a\,b^{2}c-b^{4}}+b^{2}\right) x^{-n}}{2ab}$$

✓ Solution by Mathematica

Time used: 3.62 (sec). Leaf size: 87

```
DSolve[x*y'[x]==a*x^(2*n)*Exp[\[Lambda]*x]*y[x]^2+(b*x^n*Exp[\[Lambda]*x]-n)*y[x]+c*Exp[\[La
```

$$\text{Solve}\left[\int_1^{\sqrt{\frac{ax^{2n}}{c}}y(x)}\frac{1}{K[1]^2-\sqrt{\frac{b^2}{ac}}K[1]+1}dK[1]= -c(\lambda(-x))^{-n}\sqrt{\frac{ax^{2n}}{c}}\Gamma(n,-x\lambda)+c_1,y(x)\right]$$

4.16 problem 37

Internal problem ID [10445]
Internal file name [`OUTPUT/9392_Monday_June_06_2022_02_21_24_PM_52456256/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 = 2a\lambda x\, e^{\lambda x^2} - a^2 e^{2\lambda x^2}}$$

4.16.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + 2a\lambda x\, e^{\lambda x^2} - a^2 e^{2\lambda x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + 2a\lambda x\, e^{\lambda x^2} - a^2 e^{2\lambda x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 2a\lambda x\, e^{\lambda x^2} - a^2 e^{2\lambda x^2}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= 2a\lambda x\,\mathrm{e}^{\lambda\,x^2} - a^2\mathrm{e}^{2\lambda\,x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(2a\lambda x\,\mathrm{e}^{\lambda\,x^2} - a^2\mathrm{e}^{2\lambda\,x^2}\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{-\mathrm{e}^{2\lambda\,x^2}\_Y(x)\,a^2 + 2\,\mathrm{e}^{\lambda\,x^2}\_Y(x)\,a\lambda x + \_Y''(x)\right\},\{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{-\mathrm{e}^{2\lambda\,x^2}\_Y(x)\,a^2 + 2\,\mathrm{e}^{\lambda\,x^2}\_Y(x)\,a\lambda x + \_Y''(x)\right\},\{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{-\mathrm{e}^{2\lambda\,x^2}\_Y(x)\,a^2 + 2\,\mathrm{e}^{\lambda\,x^2}\_Y(x)\,a\lambda x + \_Y''(x)\right\},\{\_Y(x)\}\right)}{\mathrm{DESol}\left(\{-\mathrm{e}^{2\lambda\,x^2}\_Y(x)\,a^2 + 2\,\mathrm{e}^{\lambda\,x^2}\_Y(x)\,a\lambda x + \_Y''(x)\},\{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{-\mathrm{e}^{2\lambda\,x^2}\_Y(x)\,a^2 + 2\,\mathrm{e}^{\lambda\,x^2}\_Y(x)\,a\lambda x + \_Y''(x)\right\},\{\_Y(x)\}\right)}{\mathrm{DESol}\left(\{-\mathrm{e}^{2\lambda\,x^2}\_Y(x)\,a^2 + 2\,\mathrm{e}^{\lambda\,x^2}\_Y(x)\,a\lambda x + \_Y''(x)\},\{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{-\mathrm{e}^{2\lambda\,x^2}\_Y(x)\,a^2 + 2\,\mathrm{e}^{\lambda\,x^2}\_Y(x)\,a\lambda x + \_Y''(x)\right\},\{\_Y(x)\}\right)}{\mathrm{DESol}\left(\{-\mathrm{e}^{2\lambda\,x^2}\_Y(x)\,a^2 + 2\,\mathrm{e}^{\lambda\,x^2}\_Y(x)\,a\lambda x + \_Y''(x)\},\{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{-\mathrm{e}^{2\lambda\,x^{2}}\_Y(x)\,a^{2}+2\,\mathrm{e}^{\lambda\,x^{2}}\_Y(x)\,a\lambda x+\_Y''(x)\right\},\{\_Y(x)\}\right)}{\mathrm{DESol}\left(\{-\mathrm{e}^{2\lambda\,x^{2}}\_Y(x)\,a^{2}+2\,\mathrm{e}^{\lambda\,x^{2}}\_Y(x)\,a\lambda x+\_Y''(x)\},\{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-2*a*lambda*x*exp(x^2*lambda)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(y(x)^2+y(x)+x^2*(2*a*lambda*x*exp(x^2*la
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2+2*a*lambda*x*exp(lambda*x^2)-a^2*exp(2*lambda*x^2),y(x), singsol=
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+2*a*\[Lambda]*x*Exp[\[Lambda]*x^2]-a^2*Exp[2*\[Lambda]*x^2],y[x],x,Incl
```

Not solved

4.17 problem 38

Internal problem ID [10446]
Internal file name [`OUTPUT/9393_Monday_June_06_2022_02_21_27_PM_31744357/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\,\mathrm{e}^{-\lambda\,x^2} y^2 - y\lambda x = a\,b^2}$$

4.17.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,\mathrm{e}^{-\lambda\,x^2} y^2 + \lambda x y + a\,b^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,\mathrm{e}^{-\lambda\,x^2} y^2 + \lambda x y + a\,b^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\,b^2$, $f_1(x) = \lambda x$ and $f_2(x) = \mathrm{e}^{-\lambda\,x^2} a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\mathrm{e}^{-\lambda\,x^2} a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -2\lambda x\, \mathrm{e}^{-\lambda x^2} a \\ f_1 f_2 &= \lambda x\, \mathrm{e}^{-\lambda x^2} a \\ f_2^2 f_0 &= \mathrm{e}^{-2\lambda x^2} a^3 b^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\mathrm{e}^{-\lambda x^2} a u''(x) + \lambda x\, \mathrm{e}^{-\lambda x^2} a u'(x) + \mathrm{e}^{-2\lambda x^2} a^3 b^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(\frac{\sqrt{2}\, ab\sqrt{\pi}\ \mathrm{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right) + c_2 \cos\left(\frac{\sqrt{2}\, ab\sqrt{\pi}\ \mathrm{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right)$$

The above shows that

$$u'(x) = ab\, \mathrm{e}^{-\frac{\lambda x^2}{2}} \left(c_1 \cos\left(\frac{\sqrt{2}\, ab\sqrt{\pi}\ \mathrm{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right) - c_2 \sin\left(\frac{\sqrt{2}\, ab\sqrt{\pi}\ \mathrm{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right)\right)$$

Using the above in (1) gives the solution

$$y = -\frac{b\, \mathrm{e}^{-\frac{\lambda x^2}{2}} \left(c_1 \cos\left(\frac{\sqrt{2}\, ab\sqrt{\pi}\ \mathrm{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right) - c_2 \sin\left(\frac{\sqrt{2}\, ab\sqrt{\pi}\ \mathrm{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right)\right) \mathrm{e}^{\lambda x^2}}{c_1 \sin\left(\frac{\sqrt{2}\, ab\sqrt{\pi}\ \mathrm{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right) + c_2 \cos\left(\frac{\sqrt{2}\, ab\sqrt{\pi}\ \mathrm{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{b\, \mathrm{e}^{\frac{\lambda x^2}{2}} \left(-c_3 \cos\left(\frac{\sqrt{2}\, ab\sqrt{\pi}\ \mathrm{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right) + \sin\left(\frac{\sqrt{2}\, ab\sqrt{\pi}\ \mathrm{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right)\right)}{c_3 \sin\left(\frac{\sqrt{2}\, ab\sqrt{\pi}\ \mathrm{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right) + \cos\left(\frac{\sqrt{2}\, ab\sqrt{\pi}\ \mathrm{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right)}$$

Summary
The solution(s) found are the following

$$
y = \frac{b\,\mathrm{e}^{\frac{\lambda x^{2}}{2}} \left(-c_3 \cos\left(\frac{\sqrt{2}\,ab\sqrt{\pi}\,\operatorname{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right)+\sin\left(\frac{\sqrt{2}\,ab\sqrt{\pi}\,\operatorname{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right)\right)}{c_3 \sin\left(\frac{\sqrt{2}\,ab\sqrt{\pi}\,\operatorname{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right)+\cos\left(\frac{\sqrt{2}\,ab\sqrt{\pi}\,\operatorname{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right)} \tag{1}
$$

Verification of solutions

$$
y = \frac{b\,\mathrm{e}^{\frac{\lambda x^{2}}{2}} \left(-c_3 \cos\left(\frac{\sqrt{2}\,ab\sqrt{\pi}\,\operatorname{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right)+\sin\left(\frac{\sqrt{2}\,ab\sqrt{\pi}\,\operatorname{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right)\right)}{c_3 \sin\left(\frac{\sqrt{2}\,ab\sqrt{\pi}\,\operatorname{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right)+\cos\left(\frac{\sqrt{2}\,ab\sqrt{\pi}\,\operatorname{erf}\left(\frac{x\sqrt{2}\,\sqrt{\lambda}}{2}\right)}{2\sqrt{\lambda}}\right)}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 45

```
dsolve(diff(y(x),x)=a*exp(-lambda*x^2)*y(x)^2+lambda*x*y(x)+a*b^2,y(x), singsol=all)
```

$$
y(x) = \tan\left(\frac{ab\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{\lambda}\,x}{2}\right)-2c_1\sqrt{\lambda}}{2\sqrt{\lambda}}\right) b\,\mathrm{e}^{\frac{x^{2}\lambda}{2}}
$$

✓ Solution by Mathematica

Time used: 2.252 (sec). Leaf size: 63

```
DSolve[y'[x]==a*Exp[-\[Lambda]*x^2]*y[x]^2+\[Lambda]*x*y[x]+a*b^2,y[x],x,IncludeSingularSolu
```

$$y(x) \to \sqrt{b^2} e^{\frac{\lambda x^2}{2}} \tan \left(\frac{\sqrt{\frac{\pi}{2}} a \sqrt{b^2} \operatorname{erf}\left(\frac{\sqrt{\lambda} x}{\sqrt{2}} \right)}{\sqrt{\lambda}} + c_1 \right)$$

4.18 problem 39

Internal problem ID [10447]
Internal file name [`OUTPUT/9394_Monday_June_06_2022_02_21_28_PM_93629603/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\,x^n y^2 - y\lambda x = a\,b^2 x^n \mathrm{e}^{\lambda\,x^2}}$$

4.18.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= a\,x^n y^2 + \lambda x y + a\,b^2 x^n \mathrm{e}^{\lambda\,x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,x^n y^2 + \lambda x y + a\,b^2 x^n \mathrm{e}^{\lambda\,x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\,b^2 x^n \mathrm{e}^{\lambda\,x^2}$, $f_1(x) = \lambda x$ and $f_2(x) = x^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{x^n a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{x^n na}{x} \\ f_1f_2 &= \lambda x\, x^n a \\ f_2^2 f_0 &= x^{3n} a^3 b^2 \mathrm{e}^{\lambda\, x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^n a u''(x) - \left(\frac{x^n na}{x} + \lambda x\, x^n a\right) u'(x) + x^{3n} a^3 b^2 \mathrm{e}^{\lambda\, x^2} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = c_1 \sin&\left(ab\, x^{n+1} 2^{-\frac{1}{2}+\frac{n}{2}} (-\lambda\, x^2)^{-\frac{n}{2}-\frac{1}{2}} \left(\Gamma\left(\frac{n}{2}+\frac{1}{2}\right) - \Gamma\left(\frac{n}{2}+\frac{1}{2}, -\frac{\lambda\, x^2}{2}\right)\right)\right) \\ &+ c_2 \cos\left(ab\, x^{n+1} 2^{-\frac{1}{2}+\frac{n}{2}} (-\lambda\, x^2)^{-\frac{n}{2}-\frac{1}{2}} \left(\Gamma\left(\frac{n}{2}+\frac{1}{2}\right) - \Gamma\left(\frac{n}{2}+\frac{1}{2}, -\frac{\lambda\, x^2}{2}\right)\right)\right) \end{aligned}$$

The above shows that

$$\begin{aligned} &u'(x) \\ &= ab\, x^n \mathrm{e}^{\frac{\lambda\, x^2}{2}} \left(c_1 \cos\left(ab\, x^{n+1} 2^{-\frac{1}{2}+\frac{n}{2}} (-\lambda\, x^2)^{-\frac{n}{2}-\frac{1}{2}} \left(\Gamma\left(\frac{n}{2}+\frac{1}{2}\right) - \Gamma\left(\frac{n}{2}+\frac{1}{2}, -\frac{\lambda\, x^2}{2}\right)\right)\right)\right. \\ &\quad \left. - c_2 \sin\left(ab\, x^{n+1} 2^{-\frac{1}{2}+\frac{n}{2}} (-\lambda\, x^2)^{-\frac{n}{2}-\frac{1}{2}} \left(\Gamma\left(\frac{n}{2}+\frac{1}{2}\right) - \Gamma\left(\frac{n}{2}+\frac{1}{2}, -\frac{\lambda\, x^2}{2}\right)\right)\right)\right) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{b\,\mathrm{e}^{\frac{\lambda\, x^2}{2}} \left(c_1 \cos\left(ab\, x^{n+1} 2^{-\frac{1}{2}+\frac{n}{2}} (-\lambda\, x^2)^{-\frac{n}{2}-\frac{1}{2}} \left(\Gamma\left(\frac{n}{2}+\frac{1}{2}\right) - \Gamma\left(\frac{n}{2}+\frac{1}{2}, -\frac{\lambda x^2}{2}\right)\right)\right) - c_2 \sin\left(ab\, x^{n+1} 2^{-\frac{1}{2}+\frac{n}{2}} (-\right.\right.}{c_1 \sin\left(ab\, x^{n+1} 2^{-\frac{1}{2}+\frac{n}{2}} (-\lambda\, x^2)^{-\frac{n}{2}-\frac{1}{2}} \left(\Gamma\left(\frac{n}{2}+\frac{1}{2}\right) - \Gamma\left(\frac{n}{2}+\frac{1}{2}, -\frac{\lambda x^2}{2}\right)\right)\right) + c_2 \cos\left(ab\, x^{n+1} 2^{-\frac{1}{2}+\frac{n}{2}} (-\lambda\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{b\,\mathrm{e}^{\frac{\lambda x^2}{2}}\left(c_3 \cos\left(ab\,x^{n+1}2^{-\frac{1}{2}+\frac{n}{2}}(-\lambda\,x^2)^{-\frac{n}{2}-\frac{1}{2}}\left(\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)-\Gamma\left(\frac{n}{2}+\frac{1}{2},-\frac{\lambda x^2}{2}\right)\right)\right)-\sin\left(ab\,x^{n+1}2^{-\frac{1}{2}+\frac{n}{2}}(-\lambda\right.\right.}{c_3 \sin\left(ab\,x^{n+1}2^{-\frac{1}{2}+\frac{n}{2}}(-\lambda\,x^2)^{-\frac{n}{2}-\frac{1}{2}}\left(\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)-\Gamma\left(\frac{n}{2}+\frac{1}{2},-\frac{\lambda x^2}{2}\right)\right)\right)+\cos\left(ab\,x^{n+1}2^{-\frac{1}{2}+\frac{n}{2}}(-\lambda\,x^2\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{b\,\mathrm{e}^{\frac{\lambda x^2}{2}}\left(c_3 \cos\left(ab\,x^{n+1}2^{-\frac{1}{2}+\frac{n}{2}}(-\lambda\,x^2)^{-\frac{n}{2}-\frac{1}{2}}\left(\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)-\Gamma\left(\frac{n}{2}+\frac{1}{2},-\frac{\lambda x^2}{2}\right)\right)\right)-\sin\left(ab\,x^{n+1}2^{-\frac{1}{2}+\frac{n}{2}}(-\lambda\right.\right.}{c_3 \sin\left(ab\,x^{n+1}2^{-\frac{1}{2}+\frac{n}{2}}(-\lambda\,x^2)^{-\frac{n}{2}-\frac{1}{2}}\left(\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)-\Gamma\left(\frac{n}{2}+\frac{1}{2},-\frac{\lambda x^2}{2}\right)\right)\right)+\cos\left(ab\,x^{n+1}2^{-\frac{1}{2}+\frac{n}{2}}(-\lambda\,x^2\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{b\,\mathrm{e}^{\frac{\lambda x^2}{2}}\left(c_3 \cos\left(ab\,x^{n+1}2^{-\frac{1}{2}+\frac{n}{2}}(-\lambda\,x^2)^{-\frac{n}{2}-\frac{1}{2}}\left(\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)-\Gamma\left(\frac{n}{2}+\frac{1}{2},-\frac{\lambda x^2}{2}\right)\right)\right)-\sin\left(ab\,x^{n+1}2^{-\frac{1}{2}+\frac{n}{2}}(-\lambda\right.\right.}{c_3 \sin\left(ab\,x^{n+1}2^{-\frac{1}{2}+\frac{n}{2}}(-\lambda\,x^2)^{-\frac{n}{2}-\frac{1}{2}}\left(\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)-\Gamma\left(\frac{n}{2}+\frac{1}{2},-\frac{\lambda x^2}{2}\right)\right)\right)+\cos\left(ab\,x^{n+1}2^{-\frac{1}{2}+\frac{n}{2}}(-\lambda\,x^2\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 90

```
dsolve(diff(y(x),x)=a*x^n*y(x)^2+lambda*x*y(x)+a*b^2*x^n*exp(lambda*x^2),y(x), singsol=all)
```

$$y(x) = -\tan\left(-ab2^{\frac{n}{2}-\frac{1}{2}}x^{n+1}\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\left(-x^2\lambda\right)^{-\frac{n}{2}-\frac{1}{2}} + ab2^{\frac{n}{2}-\frac{1}{2}}x^{n+1}\left(-x^2\lambda\right)^{-\frac{n}{2}-\frac{1}{2}}\Gamma\left(\frac{n}{2}+\frac{1}{2},-\frac{x^2\lambda}{2}\right)+c_1\right) b\,\mathrm{e}^{\frac{x^2\lambda}{2}}$$

✓ Solution by Mathematica

Time used: 2.366 (sec). Leaf size: 83

```
DSolve[y'[x]==a*x^n*y[x]^2+\[Lambda]*x*y[x]+a*b^2*x^n*Exp[\[Lambda]*x^2],y[x],x,IncludeSingu
```

$$y(x) \to \sqrt{b^2}e^{\frac{\lambda x^2}{2}}\tan\left(a\sqrt{b^2}\lambda 2^{\frac{n-1}{2}}x^{n+3}\left(\lambda\left(-x^2\right)\right)^{-\frac{n}{2}-\frac{3}{2}}\Gamma\left(\frac{n+1}{2},-\frac{x^2\lambda}{2}\right)+c_1\right)$$

4.19 problem 40

Internal problem ID [10448]
Internal file name [`OUTPUT/9395_Monday_June_06_2022_02_21_29_PM_90333248/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.3-2. Equations with power and exponential functions
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{x^4(y'-y^2) = a + b\,\mathrm{e}^{\frac{k}{x}} + c\,\mathrm{e}^{\frac{2k}{x}}}$$

4.19.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^4y^2 + b\,\mathrm{e}^{\frac{k}{x}} + c\,\mathrm{e}^{\frac{2k}{x}} + a}{x^4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \frac{b\,\mathrm{e}^{\frac{k}{x}}}{x^4} + \frac{c\,\mathrm{e}^{\frac{2k}{x}}}{x^4} + \frac{a}{x^4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{a+b\,\mathrm{e}^{\frac{k}{x}}+c\,\mathrm{e}^{\frac{2k}{x}}}{x^4}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{a + b\,\mathrm{e}^{\frac{k}{x}} + c\,\mathrm{e}^{\frac{2k}{x}}}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{\left(a + b\,\mathrm{e}^{\frac{k}{x}} + c\,\mathrm{e}^{\frac{2k}{x}}\right) u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = x\,\mathrm{e}^{-\frac{k}{2x}} \Bigg(&\mathrm{WhittakerM}\left(-\frac{ib}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \frac{2i\sqrt{c}\,\mathrm{e}^{\frac{k}{x}}}{k}\right) c_1 \\ &+ \mathrm{WhittakerW}\left(-\frac{ib}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \frac{2i\sqrt{c}\,\mathrm{e}^{\frac{k}{x}}}{k}\right) c_2 \Bigg) \end{aligned}$$

The above shows that

$$u'(x) = -\frac{\mathrm{e}^{-\frac{k}{2x}} \left(\left(\left(i\sqrt{a} + \frac{k}{2}\right)\sqrt{c} - \frac{ib}{2}\right) c_1\,\mathrm{WhittakerM}\left(-\frac{ib-2k\sqrt{c}}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \frac{2i\sqrt{c}\,\mathrm{e}^{\frac{k}{x}}}{k}\right) - \mathrm{WhittakerW}\left(-\frac{ib-2k\sqrt{c}}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \right.\right.}{}$$

Using the above in (1) gives the solution

$$y = \frac{\left(\left(i\sqrt{a} + \frac{k}{2}\right)\sqrt{c} - \frac{ib}{2}\right) c_1\,\mathrm{WhittakerM}\left(-\frac{ib-2k\sqrt{c}}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \frac{2i\sqrt{c}\,\mathrm{e}^{\frac{k}{x}}}{k}\right) - \mathrm{WhittakerW}\left(-\frac{ib-2k\sqrt{c}}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \frac{2i\sqrt{c}\,\mathrm{e}^{\frac{k}{x}}}{k}\right) c}{\sqrt{c}\,x^2 \left(\mathrm{WhittakerM}\left(-\frac{ib}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{-\frac{((-2i\sqrt{a}-k)\sqrt{c}+ib)c_3 \operatorname{WhittakerM}\left(-\frac{ib-2k\sqrt{c}}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \frac{2i\sqrt{c}\,e^{\frac{k}{x}}}{k}\right)}{2} - k \operatorname{WhittakerW}\left(-\frac{ib-2k\sqrt{c}}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \frac{2i\sqrt{c}\,e^{\frac{k}{x}}}{k}\right)\sqrt{c} + (ic\,e^{\frac{k}{x}}}{\sqrt{c}\,x^2 (\operatorname{WhittakerM}\left(-\frac{ib}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \frac{2i\sqrt{c}\,e^{\frac{k}{x}}}{k}\right)}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{-\frac{((-2i\sqrt{a}-k)\sqrt{c}+ib)c_3 \operatorname{WhittakerM}\left(-\frac{ib-2k\sqrt{c}}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \frac{2i\sqrt{c}\,e^{\frac{k}{x}}}{k}\right)}{2} - k \operatorname{WhittakerW}\left(-\frac{ib-2k\sqrt{c}}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \frac{2i\sqrt{c}\,e^{\frac{k}{x}}}{k}\right)\sqrt{c} + (ic\,e^{\frac{k}{x}}}{\sqrt{c}\,x^2 (\operatorname{WhittakerM}\left(-\frac{ib}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \frac{2i\sqrt{c}\,e^{\frac{k}{x}}}{k}\right)} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{-\frac{((-2i\sqrt{a}-k)\sqrt{c}+ib)c_3 \operatorname{WhittakerM}\left(-\frac{ib-2k\sqrt{c}}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \frac{2i\sqrt{c}\,e^{\frac{k}{x}}}{k}\right)}{2} - k \operatorname{WhittakerW}\left(-\frac{ib-2k\sqrt{c}}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \frac{2i\sqrt{c}\,e^{\frac{k}{x}}}{k}\right)\sqrt{c} + (ic\,e^{\frac{k}{x}}}{\sqrt{c}\,x^2 (\operatorname{WhittakerM}\left(-\frac{ib}{2k\sqrt{c}}, \frac{i\sqrt{a}}{k}, \frac{2i\sqrt{c}\,e^{\frac{k}{x}}}{k}\right)}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(a+b*exp(k/x)+c*exp(2*k/x))*y
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying an equivalence, under non-integer power transformations,
            to LODEs admitting Liouvillian solutions.
            -> Trying a Liouvillian solution using Kovacics algorithm
            <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
               <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
            <- Whittaker successful
         <- special function solution successful
         Change of variables used:
            [x = 1/ln(t)]
         Linear ODE actually solved:
            (ln(t)*a+ln(t)*b*t^k+ln(t)*c*(t^k)^2)*u(t)+(t*ln(t)+2*t)*diff(u(t),t)+ln(t)*t^2*
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 302

```
dsolve(x^4*(diff(y(x),x)-y(x)^2)=a+b*exp(k/x)+c*exp(2*k/x),y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(i\sqrt{a}+\frac{k}{2}\right)\sqrt{c}-\frac{ib}{2}\right)\text{WhittakerM}\left(-\frac{ib-2k\sqrt{c}}{2k\sqrt{c}},\frac{i\sqrt{a}}{k},\frac{2i\sqrt{c}\,\mathrm{e}^{\frac{k}{x}}}{k}\right)-c_1k\,\text{WhittakerW}\left(-\frac{ib-2k\sqrt{c}}{2k\sqrt{c}},\frac{i\sqrt{a}}{k},\frac{2i\sqrt{c}\,\mathrm{e}^{\frac{k}{x}}}{k}\right)}{\sqrt{c}\,x^2\left(\text{WhittakerW}\left(-\frac{ib}{2k\sqrt{c}},\frac{i\sqrt{a}}{k},\frac{2i\sqrt{}}{}\right.\right.}$$

✓ Solution by Mathematica

Time used: 4.039 (sec). Leaf size: 940

```
DSolve[x^4*(y'[x]-y[x]^2)==a+b*Exp[k/x]+c*Exp[2*k/x],y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{e^{k/x}\log\left(e^{k/x}\right)\left(c_1\left(b+\sqrt{c}(2\sqrt{a}-ik)\right)\text{HypergeometricU}\left(\frac{\frac{ib}{\sqrt{c}}+3k+2i\sqrt{a}}{2k},\frac{2i\sqrt{a}}{k}+2,\frac{2i\sqrt{c}e^{k/x}}{k}\right)-2i\sqrt{c}kL\right.}{kx^2\log\left(e^{k/x}\right)}$$

$$y(x) \to \frac{\frac{e^{k/x}\left(b+\sqrt{c}(2\sqrt{a}-ik)\right)\text{HypergeometricU}\left(\frac{\frac{ib}{\sqrt{c}}+3k+2i\sqrt{a}}{2k},\frac{2i\sqrt{a}}{k}+2,\frac{2i\sqrt{c}e^{k/x}}{k}\right)}{k\,\text{HypergeometricU}\left(\frac{\frac{ib}{\sqrt{c}}+k+2i\sqrt{a}}{2k},\frac{2i\sqrt{a}}{k}+1,\frac{2i\sqrt{c}e^{k/x}}{k}\right)}+i\left(\sqrt{a}-\sqrt{c}e^{k/x}\right)-\frac{k}{\log\left(e^{k/x}\right)}}{x^2}$$

$$y(x) \to \frac{\frac{e^{k/x}\left(b+\sqrt{c}(2\sqrt{a}-ik)\right)\text{HypergeometricU}\left(\frac{\frac{ib}{\sqrt{c}}+3k+2i\sqrt{a}}{2k},\frac{2i\sqrt{a}}{k}+2,\frac{2i\sqrt{c}e^{k/x}}{k}\right)}{k\,\text{HypergeometricU}\left(\frac{\frac{ib}{\sqrt{c}}+k+2i\sqrt{a}}{2k},\frac{2i\sqrt{a}}{k}+1,\frac{2i\sqrt{c}e^{k/x}}{k}\right)}+i\left(\sqrt{a}-\sqrt{c}e^{k/x}\right)-\frac{k}{\log\left(e^{k/x}\right)}}{x^2}$$

5 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine

5.1 problem 1

Internal problem ID [10449]
Internal file name [OUTPUT/9396_Monday_June_06_2022_02_21_32_PM_22140854/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 = -a^2 + a\lambda \sinh(\lambda x) - a^2 \sinh(\lambda x)^2}$$

5.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 - a^2 + a\lambda \sinh(\lambda x) - a^2 \sinh(\lambda x)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - a^2 + a\lambda \sinh(\lambda x) - a^2 \sinh(\lambda x)^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2 + a\lambda \sinh(\lambda x) - a^2 \sinh(\lambda x)^2$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= -a^2 + a\lambda \sinh(\lambda x) - a^2 \sinh(\lambda x)^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-a^2 + a\lambda \sinh(\lambda x) - a^2 \sinh(\lambda x)^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\frac{a\sinh(\lambda x)}{\lambda}}\left(c_1 \,\mathrm{HeunC}\left(\frac{4ia}{\lambda}, -\frac{1}{2}, -\frac{1}{2}, \frac{2ia}{\lambda}, -\frac{8ia-3\lambda}{8\lambda}, -\frac{i\sinh(\lambda x)}{2} + \frac{1}{2}\right) + c_2 \sinh\left(\frac{i\pi}{4} + \frac{\lambda x}{2}\right) \mathrm{HeunC}\left(\frac{4ia}{\lambda}, \frac{1}{2}, -\frac{1}{2}, \frac{2ia}{\lambda}, -\frac{8ia-3\lambda}{8\lambda}, -\frac{i\sinh(\lambda x)}{2} + \frac{1}{2}\right)\right)$$

The above shows that

$$u'(x) = -\frac{\mathrm{e}^{\frac{a\sinh(\lambda x)}{\lambda}}\left(c_2\left(-2a\sinh\left(\frac{i\pi}{4} + \frac{\lambda x}{2}\right)\cosh(\lambda x) - \lambda\cosh\left(\frac{i\pi}{4} + \frac{\lambda x}{2}\right)\right)\mathrm{HeunC}\left(\frac{4ia}{\lambda}, \frac{1}{2}, -\frac{1}{2}, \frac{2ia}{\lambda}, -\frac{8ia-3\lambda}{8\lambda}, -\frac{i\sinh(}{2}\right.\right.}{}$$

Using the above in (1) gives the solution

$$y = \frac{c_2\left(-2a\sinh\left(\frac{i\pi}{4} + \frac{\lambda x}{2}\right)\cosh(\lambda x) - \lambda\cosh\left(\frac{i\pi}{4} + \frac{\lambda x}{2}\right)\right)\mathrm{HeunC}\left(\frac{4ia}{\lambda}, \frac{1}{2}, -\frac{1}{2}, \frac{2ia}{\lambda}, -\frac{8ia-3\lambda}{8\lambda}, -\frac{i\sinh(\lambda x)}{2} + \frac{1}{2}\right) + }{2c_1\,\mathrm{Heun}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-2a\sinh\left(\frac{i\pi}{4} + \frac{\lambda x}{2}\right)\cosh(\lambda x) - \lambda\cosh\left(\frac{i\pi}{4} + \frac{\lambda x}{2}\right)\right)\mathrm{HeunC}\left(\frac{4ia}{\lambda}, \frac{1}{2}, -\frac{1}{2}, \frac{2ia}{\lambda}, -\frac{8ia-3\lambda}{8\lambda}, -\frac{i\sinh(\lambda x)}{2} + \frac{1}{2}\right) + \left(-\right.}{2\sinh\left(\frac{i\pi}{4}\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-2a\sinh\left(\frac{i\pi}{4}+\frac{\lambda x}{2}\right)\cosh\left(\lambda x\right)-\lambda\cosh\left(\frac{i\pi}{4}+\frac{\lambda x}{2}\right)\right)\operatorname{HeunC}\left(\frac{4ia}{\lambda},\frac{1}{2},-\frac{1}{2},\frac{2ia}{\lambda},-\frac{8ia-3\lambda}{8\lambda},-\frac{i\sinh(\lambda x)}{2}+\frac{1}{2}\right)+\left(-\right.}{2\sinh\left(\frac{i\pi}{4}\right.} \tag{1}$$

Verification of solutions

$$y = \frac{\left(-2a\sinh\left(\frac{i\pi}{4}+\frac{\lambda x}{2}\right)\cosh\left(\lambda x\right)-\lambda\cosh\left(\frac{i\pi}{4}+\frac{\lambda x}{2}\right)\right)\operatorname{HeunC}\left(\frac{4ia}{\lambda},\frac{1}{2},-\frac{1}{2},\frac{2ia}{\lambda},-\frac{8ia-3\lambda}{8\lambda},-\frac{i\sinh(\lambda x)}{2}+\frac{1}{2}\right)+\left(-\right.}{2\sinh\left(\frac{i\pi}{4}\right.}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a^2-a*lambda*sinh(lambda*x)+a
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 318

```
dsolve(diff(y(x),x)=y(x)^2-a^2+a*lambda*sinh(lambda*x)-a^2*sinh(lambda*x)^2,y(x), singsol=al
```

$$y(x) = \frac{\left(-2\sinh\left(\frac{i\pi}{4}+\frac{x\lambda}{2}\right)\cosh(x\lambda)c_1a-\cosh\left(\frac{i\pi}{4}+\frac{x\lambda}{2}\right)c_1\lambda\right)\mathrm{HeunC}\left(\frac{4ia}{\lambda},\frac{1}{2},-\frac{1}{2},\frac{2ia}{\lambda},-\frac{8ia-3\lambda}{8\lambda},-\frac{i\sinh(x\lambda)}{2}+\frac{1}{2}\right)}{2\sinh\left(\frac{i\pi}{4}\right.}$$

✓ Solution by Mathematica

Time used: 11.807 (sec). Leaf size: 162

```
DSolve[y'[x]==y[x]^2-a^2+a*\[Lambda]*Sinh[\[Lambda]*x]-a^2*Sinh[\[Lambda]*x]^2,y[x],x,Includ
```

$$y(x) \to \frac{e^{\lambda(-x)}\left(a\left(e^{2\lambda x}+1\right)\int_1^{e^{x\lambda}}\frac{e^{\frac{a\left(K[1]^2-1\right)}{\lambda K[1]}}}{K[1]}dK[1]-2\lambda e^{\frac{ae^{\lambda(-x)}\left(e^{2\lambda x}-1\right)}{\lambda}+\lambda x}+ac_1e^{2\lambda x}+ac_1\right)}{2\left(\int_1^{e^{x\lambda}}\frac{e^{\frac{a\left(K[1]^2-1\right)}{\lambda K[1]}}}{K[1]}dK[1]+c_1\right)}$$

$$y(x) \to \frac{1}{2}ae^{\lambda(-x)}\left(e^{2\lambda x}+1\right)$$

5.2 problem 2

Internal problem ID [10450]
Internal file name [`OUTPUT/9397_Monday_June_06_2022_02_21_36_PM_50210798/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - a\sinh(\beta x)\, y = ab\sinh(\beta x) - b^2}$$

5.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + a\sinh(\beta x)\, y + ab\sinh(\beta x) - b^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\sinh(\beta x)\, y + ab\sinh(\beta x) - b^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = ab\sinh(\beta x) - b^2$, $f_1(x) = \sinh(\beta x)\, a$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \sinh(\beta x)\, a \\ f_2^2 f_0 &= ab \sinh(\beta x) - b^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \sinh(\beta x)\, a u'(x) + \left(ab \sinh(\beta x) - b^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = e^{bx} \left(c_1 + c_2 \beta \left(\int e^{\frac{-2b\beta x + a\cosh(\beta x)}{\beta}} dx\right)\right)$$

The above shows that

$$u'(x) = e^{bx} \left(\left(\int e^{\frac{-2b\beta x + a\cosh(\beta x)}{\beta}} dx\right) c_2 b \beta + c_2 \beta\, e^{\frac{-2b\beta x + a\cosh(\beta x)}{\beta}} + c_1 b\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\int e^{\frac{-2b\beta x + a\cosh(\beta x)}{\beta}} dx\right) c_2 b \beta + c_2 \beta\, e^{\frac{-2b\beta x + a\cosh(\beta x)}{\beta}} + c_1 b}{c_1 + c_2 \beta \left(\int e^{\frac{-2b\beta x + a\cosh(\beta x)}{\beta}} dx\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\left(\int e^{\frac{-2b\beta x + a\cosh(\beta x)}{\beta}} dx\right) b \beta - e^{\frac{-2b\beta x + a\cosh(\beta x)}{\beta}} \beta - b c_3}{c_3 + \beta \left(\int e^{\frac{-2b\beta x + a\cosh(\beta x)}{\beta}} dx\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\left(\int e^{\frac{-2b\beta x+a\cosh(\beta x)}{\beta}}dx\right)b\beta - e^{\frac{-2b\beta x+a\cosh(\beta x)}{\beta}}\beta - bc_3}{c_3+\beta\left(\int e^{\frac{-2b\beta x+a\cosh(\beta x)}{\beta}}dx\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-\left(\int e^{\frac{-2b\beta x+a\cosh(\beta x)}{\beta}}dx\right)b\beta - e^{\frac{-2b\beta x+a\cosh(\beta x)}{\beta}}\beta - bc_3}{c_3+\beta\left(\int e^{\frac{-2b\beta x+a\cosh(\beta x)}{\beta}}dx\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 73

```
dsolve(diff(y(x),x)=y(x)^2+a*sinh(beta*x)*y(x)+a*b*sinh(beta*x)-b^2,y(x), singsol=all)
```

$$y(x) = \frac{b\left(\int e^{\frac{-2b\beta x+a\cosh(x\beta)}{\beta}}dx\right) - c_1 b + e^{\frac{-2b\beta x+a\cosh(x\beta)}{\beta}}}{-\left(\int e^{\frac{-2b\beta x+a\cosh(x\beta)}{\beta}}dx\right)+c_1}$$

✓ Solution by Mathematica

Time used: 9.168 (sec). Leaf size: 183

```
DSolve[y'[x]==y[x]^2+a*Sinh[\[Beta]*x]*y[x]+a*b*Sinh[\[Beta]*x]-b^2,y[x],x,IncludeSingularSo
```

$$\text{Solve}\left[\int_1^x -\frac{e^{\frac{a\cosh(\beta K[1])}{\beta}-2bK[1]}(-b+a\sinh(\beta K[1])+y(x))}{a\beta(b+y(x))}dK[1] + \int_1^{y(x)}\left(\frac{e^{\frac{a\cosh(x\beta)}{\beta}-2bx}}{a\beta(b+K[2])^2} - \int_1^x\left(\frac{e^{\frac{a\cosh(\beta K[1])}{\beta}-2bK[1]}(-b+K[2]+a\sinh(\beta K[1]))}{a\beta(b+K[2])^2} - \frac{e^{\frac{a\cosh(\beta K[1])}{\beta}-2bK[1]}}{a\beta(b+K[2])}\right)dK[1]\right)dK[2] = c_1, y(x)\right]$$

5.3 problem 3

Internal problem ID [10451]
Internal file name [`OUTPUT/9398_Monday_June_06_2022_02_21_38_PM_36739669/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - ax \sinh(bx)^m y = a \sinh(bx)^m}$$

5.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + ax \sinh(bx)^m y + a \sinh(bx)^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + ax \sinh(bx)^m y + a \sinh(bx)^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a \sinh(bx)^m$, $f_1(x) = xa \sinh(bx)^m$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= xa\sinh\left(bx\right)^m \\ f_2^2f_0 &= a\sinh\left(bx\right)^m \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - xa\sinh\left(bx\right)^m u'(x) + a\sinh\left(bx\right)^m u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{x\left(c_2\left(\int \frac{e^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh(bx)}{x^2}dx\right) + c_1b\right)}{b}$$

The above shows that

$$u'(x) = \frac{c_2e^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh\left(bx\right) + c_2\left(\int \frac{e^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh(bx)}{x^2}dx\right)x + c_1bx}{bx}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2e^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh\left(bx\right) + c_2\left(\int \frac{e^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh(bx)}{x^2}dx\right)x + c_1bx}{x^2\left(c_2\left(\int \frac{e^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh(bx)}{x^2}dx\right) + c_1b\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-e^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh\left(bx\right) - \left(\int \frac{e^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh(bx)}{x^2}dx\right)x - c_3bx}{x^2\left(\int \frac{e^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh(bx)}{x^2}dx + bc_3\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{-\mathrm{e}^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh(bx) - \left(\int \frac{\mathrm{e}^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh(bx)}{x^2}dx\right)x - c_3 bx}{x^2\left(\int \frac{\mathrm{e}^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh(bx)}{x^2}dx + bc_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-\mathrm{e}^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh(bx) - \left(\int \frac{\mathrm{e}^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh(bx)}{x^2}dx\right)x - c_3 bx}{x^2\left(\int \frac{\mathrm{e}^{\int (xa\sinh(bx)^m - \tanh(bx)b)dx}\cosh(bx)}{x^2}dx + bc_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 85

```
dsolve(diff(y(x),x)=y(x)^2+a*x*sinh(b*x)^m*y(x)+a*sinh(b*x)^m,y(x), singsol=all)
```

$$y(x) = \frac{-\mathrm{e}^{\int \frac{\sinh(bx)^m x^2 a - 2}{x}dx} x - \left(\int \mathrm{e}^{\int \frac{\sinh(bx)^m x^2 a - 2}{x}dx}dx\right) + c_1}{\left(-c_1 + \int \mathrm{e}^{\int \frac{\sinh(bx)^m x^2 a - 2}{x}dx}dx\right)x}$$

✓ Solution by Mathematica

Time used: 7.437 (sec). Leaf size: 379

```
DSolve[y'[x]==y[x]^2+a*x*Sinh[b*x]^m*y[x]+a*Sinh[b*x]^m,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\frac{\int_1^x \frac{\exp\left(-\frac{a\left(-e^{-bK[1]}+e^{bK[1]}\right)^m\left(2-2e^{2bK[1]}\right)^{-m}\left({}_3F_2\left(-m,-\frac{m}{2},-\frac{m}{2};1-\frac{m}{2},1-\frac{m}{2};e^{2bK[1]}\right)+bm\,\text{Hypergeometric2F1}\left(-m,-\frac{m}{2},1-\frac{m}{2},e^{2bK[1]}\right)K[1\right.}{b^2m^2}\right.}{K[1]^2}}{x\left(\int_1^x \frac{\exp\left(-\frac{a\left(-e^{-bK[1]}+e^{bK[1]}\right)^m\left(2-2e^{2bK[1]}\right)^{-m}\left({}_3F_2\left(-m,-\frac{m}{2},-\frac{m}{2};1-\frac{m}{2}\right.\right.}{b^2}\right.}{K[1]}\right.}$$

$$y(x) \to -\frac{1}{x}$$

5.4 problem 4

Internal problem ID [10452]
Internal file name [`OUTPUT/9399_Monday_June_06_2022_02_21_43_PM_13728466/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \sinh(\lambda x) y^2 = -\lambda \sinh(\lambda x)^3}$$

5.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \sinh(\lambda x) \lambda y^2 - \lambda \sinh(\lambda x)^3 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \sinh(\lambda x) \lambda y^2 - \lambda \sinh(\lambda x)^3$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\lambda \sinh(\lambda x)^3$, $f_1(x) = 0$ and $f_2(x) = \lambda \sinh(\lambda x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\lambda \sinh(\lambda x) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \lambda^2 \cosh(\lambda x) \\ f_1f_2 &= 0 \\ f_2^2f_0 &= -\lambda^3 \sinh(\lambda x)^5 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\lambda \sinh(\lambda x)\,u''(x) - \lambda^2 \cosh(\lambda x)\,u'(x) - \lambda^3 \sinh(\lambda x)^5\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{-\frac{\cosh(\lambda x)^2}{2}}\left(c_1 + c_2 \,\mathrm{erfi}\left(\cosh(\lambda x)\right)\right)$$

The above shows that

$$u'(x) = -\frac{\left(\sqrt{\pi}\,\cosh(\lambda x)\left(c_1 + c_2\,\mathrm{erfi}\left(\cosh(\lambda x)\right)\right)\mathrm{e}^{-\frac{\cosh(\lambda x)^2}{2}} - 2c_2\mathrm{e}^{\frac{\cosh(\lambda x)^2}{2}}\right)\lambda \sinh(\lambda x)}{\sqrt{\pi}}$$

Using the above in (1) gives the solution

$$y = \frac{\left(\sqrt{\pi}\,\cosh(\lambda x)\left(c_1 + c_2\,\mathrm{erfi}\left(\cosh(\lambda x)\right)\right)\mathrm{e}^{-\frac{\cosh(\lambda x)^2}{2}} - 2c_2\mathrm{e}^{\frac{\cosh(\lambda x)^2}{2}}\right)\mathrm{e}^{\frac{\cosh(2\lambda x)}{4}+\frac{1}{4}}}{\sqrt{\pi}\,\left(c_1 + c_2\,\mathrm{erfi}\left(\cosh(\lambda x)\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-2\,\mathrm{e}^{\cosh(\lambda x)^2} + \cosh(\lambda x)\sqrt{\pi}\left(c_3 + \mathrm{erfi}\left(\cosh(\lambda x)\right)\right)}{\sqrt{\pi}\,\left(c_3 + \mathrm{erfi}\left(\cosh(\lambda x)\right)\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{-2\,\mathrm{e}^{\cosh(\lambda x)^2} + \cosh(\lambda x)\sqrt{\pi}\left(c_3 + \mathrm{erfi}\left(\cosh(\lambda x)\right)\right)}{\sqrt{\pi}\,\left(c_3 + \mathrm{erfi}\left(\cosh(\lambda x)\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-2\,\mathrm{e}^{\cosh(\lambda x)^2} + \cosh(\lambda x)\sqrt{\pi}\,(c_3 + \operatorname{erfi}(\cosh(\lambda x)))}{\sqrt{\pi}\,(c_3 + \operatorname{erfi}(\cosh(\lambda x)))}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = lambda*cosh(lambda*x)*(diff(y(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
         <- Kovacics algorithm successful
         Change of variables used:
            [x = arccosh(t)/lambda]
         Linear ODE actually solved:
            -16*(t-1)^(1/2)*(t+1)^(1/2)*(t^4-2*t^2+1)*u(t)+16*(t-1)^(1/2)*(t+1)^(1/2)*(t^2-1
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 51

```
dsolve(diff(y(x),x)=lambda*sinh(lambda*x)*y(x)^2-lambda*sinh(lambda*x)^3,y(x), singsol=all)
```

$$y(x) = -\frac{2\left(e^{\frac{\cosh(2x\lambda)}{2}+\frac{1}{2}}c_1 - \frac{\cosh(x\lambda)\sqrt{\pi}\,(\operatorname{erfi}(\cosh(x\lambda))c_1+1)}{2}\right)}{\sqrt{\pi}\,\left(\operatorname{erfi}\left(\cosh\left(x\lambda\right)\right)c_1+1\right)}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==\[Lambda]*Sinh[\[Lambda]*x]*y[x]^2-\[Lambda]*Sinh[\[Lambda]*x]^3,y[x],x,Includ
```

Not solved

5.5 problem 5

Internal problem ID [10453]
Internal file name [`OUTPUT/9400_Monday_June_06_2022_02_21_45_PM_17419007/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$y' - \left(a \sinh \left(\lambda x\right)^{2}-\lambda \right) y^{2} = -a \sinh \left(\lambda x\right)^{2}+\lambda -a$$

5.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \sinh \left(\lambda x\right)^{2} a \,y^{2}-a \sinh \left(\lambda x\right)^{2}-\lambda \,y^{2}-a +\lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \sinh \left(\lambda x\right)^{2} a \,y^{2}-a \sinh \left(\lambda x\right)^{2}-\lambda \,y^{2}-a +\lambda$$

With Riccati ODE standard form

$$y' = f_0(x)+ f_1(x)y+f_2(x)y^{2}$$

Shows that $f_0(x)=-a \sinh \left(\lambda x\right)^{2}+\lambda -a$, $f_1(x)=0$ and $f_2(x)=a \sinh \left(\lambda x\right)^{2}-\lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\left(a \sinh \left(\lambda x\right)^{2}-\lambda \right) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2a\sinh(\lambda x)\,\lambda\cosh(\lambda x) \\ f_1f_2 &= 0 \\ f_2^2f_0 &= \left(a\sinh(\lambda x)^2 - \lambda\right)^2\left(-a\sinh(\lambda x)^2 + \lambda - a\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\left(a\sinh(\lambda x)^2 - \lambda\right)u''(x) - 2a\sinh(\lambda x)\,\lambda\cosh(\lambda x)\,u'(x) + \left(a\sinh(\lambda x)^2 - \lambda\right)^2\left(-a\sinh(\lambda x)^2 + \lambda - a\right)u($$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = -2\,\mathrm{e}^{-\frac{\cosh(2\lambda x)a}{4\lambda}}\sinh(\lambda x)\left(c_2\lambda\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a - \operatorname{csch}(\lambda x)^2\lambda\right)dx\right) - \frac{c_1}{2}\right)$$

The above shows that

$$\begin{aligned} u'(x) = &\left(\sinh(2\lambda x)\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a - \operatorname{csch}(\lambda x)^2\lambda\right)dx\right)c_2\lambda\,\mathrm{e}^{-\frac{\cosh(2\lambda x)a}{4\lambda}}\right. \\ &\left.- \frac{\sinh(2\lambda x)\,c_1\mathrm{e}^{-\frac{\cosh(2\lambda x)a}{4\lambda}}}{2} + 2c_2\lambda\,\mathrm{e}^{\frac{\cosh(2\lambda x)a}{4\lambda}}\right)\operatorname{csch}(\lambda x)\left(a\sinh(\lambda x)^2 - \lambda\right) \end{aligned}$$

Using the above in (1) gives the solution

$$y = \frac{\left(\sinh(2\lambda x)\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a - \operatorname{csch}(\lambda x)^2\lambda\right)dx\right)c_2\lambda\,\mathrm{e}^{-\frac{\cosh(2\lambda x)a}{4\lambda}} - \frac{\sinh(2\lambda x)c_1\mathrm{e}^{-\frac{\cosh(2\lambda x)a}{4\lambda}}}{2} + 2c_2\lambda\,\mathrm{e}^{\frac{\cosh(2\lambda x)a}{4\lambda}}\right)}{2\sinh(\lambda x)\left(c_2\lambda\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a - \operatorname{csch}(\lambda x)^2\lambda\right)dx\right) - \frac{c_1}{2}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-2\,\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\operatorname{csch}(\lambda x)^2\lambda - 2\lambda\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a - \operatorname{csch}(\lambda x)^2\lambda\right)dx\right)\coth(\lambda x) + c_3\coth(\lambda x)}{-2\lambda\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a - \operatorname{csch}(\lambda x)^2\lambda\right)dx\right) + c_3}$$

Summary
The solution(s) found are the following

$$y = \frac{-2\,\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\,\mathrm{csch}\left(\lambda x\right)^{2}\lambda-2\lambda\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a-\mathrm{csch}\left(\lambda x\right)^{2}\lambda\right)dx\right)\coth\left(\lambda x\right)+c_{3}\coth\left(\lambda x\right)}{-2\lambda\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a-\mathrm{csch}\left(\lambda x\right)^{2}\lambda\right)dx\right)+c_{3}} \tag{1}$$

Verification of solutions

$$y = \frac{-2\,\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\,\mathrm{csch}\left(\lambda x\right)^{2}\lambda-2\lambda\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a-\mathrm{csch}\left(\lambda x\right)^{2}\lambda\right)dx\right)\coth\left(\lambda x\right)+c_{3}\coth\left(\lambda x\right)}{-2\lambda\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a-\mathrm{csch}\left(\lambda x\right)^{2}\lambda\right)dx\right)+c_{3}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = 2*sinh(lambda*x)*a*lambda*cosh
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               -> Kummer
                  -> hyper3: Equivalence to 1F1 under a power @ Moebius
               -> hypergeometric
                  -> heuristic approach
                  -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
               -> Mathieu
                  -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 104

```
dsolve(diff(y(x),x)=(a*sinh(lambda*x)^2-lambda)*y(x)^2-a*sinh(lambda*x)^2+lambda-a,y(x), sin
```

$$y(x) = \frac{2\coth(x\lambda)\lambda\left(\int -e^{\frac{a\cosh(2x\lambda)}{2\lambda}}\left(a-\operatorname{csch}(x\lambda)^2\lambda\right)dx\right)c_1+2\operatorname{csch}(x\lambda)^2 e^{\frac{a\cosh(2x\lambda)}{2\lambda}}c_1\lambda-\coth(x\lambda)}{2\lambda\left(\int -e^{\frac{a\cosh(2x\lambda)}{2\lambda}}\left(a-\operatorname{csch}(x\lambda)^2\lambda\right)dx\right)c_1-1}$$

✓ Solution by Mathematica

Time used: 50.151 (sec). Leaf size: 211

```
DSolve[y'[x]==(a*Sinh[\[Lambda]*x]^2-\[Lambda])*y[x]^2-a*Sinh[\[Lambda]*x]^2+\[Lambda]-a,y[x
```

$$y(x) \to \frac{\operatorname{csch}^2(\lambda x)\left(c_1\sinh(2\lambda x)\int_1^x e^{\frac{a\sinh^2(\lambda K[1])}{\lambda}}\operatorname{csch}^2(\lambda K[1])\left(\lambda-a\sinh^2(\lambda K[1])\right)dK[1]+2c_1e^{\frac{a\sinh^2(\lambda x)}{\lambda}}+\sinh(\right.}{2+2c_1\int_1^x e^{\frac{a\sinh^2(\lambda K[1])}{\lambda}}\operatorname{csch}^2(\lambda K[1])\left(\lambda-a\sinh^2(\lambda K[1])\right)dK[1]}$$

$$y(x) \to \frac{1}{2}\operatorname{csch}^2(\lambda x)\left(\frac{2e^{\frac{a\sinh^2(\lambda x)}{\lambda}}}{\int_1^x e^{\frac{a\sinh^2(\lambda K[1])}{\lambda}}\operatorname{csch}^2(\lambda K[1])\left(\lambda-a\sinh^2(\lambda K[1])\right)dK[1]}+\sinh(2\lambda x)\right)$$

5.6 problem 6

Internal problem ID [10454]
Internal file name [`OUTPUT/9401_Monday_June_06_2022_02_22_19_PM_6503264/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$\boxed{(a\sinh(\lambda x)+b)\,y'-y^2-c\sinh(x\mu)\,y=-d^2+cd\sinh(x\mu)}$$

5.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= \frac{y^2+c\sinh(x\mu)\,y-d^2+cd\sinh(x\mu)}{a\sinh(\lambda x)+b} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=\frac{cd\sinh(x\mu)}{a\sinh(\lambda x)+b}+\frac{c\sinh(x\mu)\,y}{a\sinh(\lambda x)+b}-\frac{d^2}{a\sinh(\lambda x)+b}+\frac{y^2}{a\sinh(\lambda x)+b}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x) = \frac{-d^2+cd\sinh(x\mu)}{a\sinh(\lambda x)+b}$, $f_1(x) = \frac{c\sinh(x\mu)}{a\sinh(\lambda x)+b}$ and $f_2(x) = \frac{1}{a\sinh(\lambda x)+b}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{a\sinh(\lambda x)+b}}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{a\lambda \cosh(\lambda x)}{(a\sinh(\lambda x)+b)^2} \\
f_1 f_2 &= \frac{c\sinh(x\mu)}{(a\sinh(\lambda x)+b)^2} \\
f_2^2 f_0 &= \frac{-d^2+cd\sinh(x\mu)}{(a\sinh(\lambda x)+b)^3}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{a\sinh(\lambda x)+b} - \left(-\frac{a\lambda\cosh(\lambda x)}{(a\sinh(\lambda x)+b)^2} + \frac{c\sinh(x\mu)}{(a\sinh(\lambda x)+b)^2}\right) u'(x) + \frac{(-d^2+cd\sinh(x\mu))\, u(x)}{(a\sinh(\lambda x)+b)^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 253

```
dsolve((a*sinh(lambda*x)+b)*diff(y(x),x)=y(x)^2+c*sinh(mu*x)*y(x)-d^2+c*d*sinh(mu*x),y(x), s
```

$$y(x) = \frac{-d\left(\int \frac{e^{\frac{c\left(\int \frac{\sinh(x\mu)}{\sinh(x\lambda)a+b}dx\right)\lambda\sqrt{a^2+b^2}+4d\,\operatorname{arctanh}\left(\frac{-\tanh\left(\frac{x\lambda}{2}\right)b+a}{\sqrt{a^2+b^2}}\right)}{\lambda\sqrt{a^2+b^2}}}}{\sinh(x\lambda)a+b}dx\right)+dc_1-e^{\frac{c\left(\int \frac{\sinh(x\mu)}{\sinh(x\lambda)a+b}dx\right)\lambda\sqrt{a^2+b^2}+4d\,\operatorname{arctanh}\left(\frac{-\tanh\left(\frac{x\lambda}{2}\right)}{\sqrt{a^2+}}\right.}{\lambda\sqrt{a^2+b^2}}}}{\int \frac{e^{\frac{c\left(\int \frac{\sinh(x\mu)}{\sinh(x\lambda)a+b}dx\right)\lambda\sqrt{a^2+b^2}+4d\,\operatorname{arctanh}\left(\frac{-\tanh\left(\frac{x\lambda}{2}\right)b+a}{\sqrt{a^2+b^2}}\right)}{\lambda\sqrt{a^2+b^2}}}}{\sinh(x\lambda)a+b}dx-c_1}$$

✓ Solution by Mathematica

Time used: 28.506 (sec). Leaf size: 289

```
DSolve[(a*Sinh[\[Lambda]*x]+b)*y'[x]==y[x]^2+c*Sinh[\[Mu]*x]*y[x]-d^2+c*d*Sinh[\[Mu]*x],y[x]
```

$$\text{Solve}\left[\int_1^x -\frac{\exp\left(-\int_1^{K[2]}\frac{2d-c\sinh(\mu K[1])}{b+a\sinh(\lambda K[1])}dK[1]\right)(-d+c\sinh(\mu K[2])+y(x))}{c\mu(b+a\sinh(\lambda K[2]))(d+y(x))}dK[2]\right.$$
$$+\int_1^{y(x)}\left(\frac{\exp\left(-\int_1^x\frac{2d-c\sinh(\mu K[1])}{b+a\sinh(\lambda K[1])}dK[1]\right)}{c\mu(d+K[3])^2}\right.$$
$$-\int_1^x\left(\frac{\exp\left(-\int_1^{K[2]}\frac{2d-c\sinh(\mu K[1])}{b+a\sinh(\lambda K[1])}dK[1]\right)(-d+K[3]+c\sinh(\mu K[2]))}{c\mu(d+K[3])^2(b+a\sinh(\lambda K[2]))}-\frac{\exp\left(-\int_1^{K[2]}\frac{2d-c\sinh(\mu K[1])}{b+a\sinh(\lambda K[1])}dK[1\right.}{c\mu(d+K[3])(b+a\sinh(\lambda K[2])}\right.$$

5.7 problem 7

Internal problem ID [10455]
Internal file name [OUTPUT/9402_Monday_June_06_2022_02_23_34_PM_5846101/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{(a\sinh(\lambda x)+b)\left(y'-y^2\right)=-a\,\lambda^2\sinh(\lambda x)}$$

5.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\sinh(\lambda x)\,y^2a-a\,\lambda^2\sinh(\lambda x)+y^2b}{a\sinh(\lambda x)+b} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=-\frac{a\,\lambda^2\sinh(\lambda x)}{a\sinh(\lambda x)+b}+\frac{\sinh(\lambda x)\,y^2a}{a\sinh(\lambda x)+b}+\frac{y^2b}{a\sinh(\lambda x)+b}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=-\frac{a\,\lambda^2\sinh(\lambda x)}{a\sinh(\lambda x)+b}$, $f_1(x)=0$ and $f_2(x)=1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = 0$$
$$f_1 f_2 = 0$$
$$f_2^2 f_0 = -\frac{a\,\lambda^2 \sinh(\lambda x)}{a \sinh(\lambda x) + b}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \frac{a\,\lambda^2 \sinh(\lambda x)\, u(x)}{a \sinh(\lambda x) + b} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = &-a c_1 \cosh\left(\frac{\lambda x}{2}\right) (a^2+b^2)^{\frac{3}{2}} \left(a \sinh\left(\frac{\lambda x}{2}\right) + b \cosh\left(\frac{\lambda x}{2}\right)\right) \\ &- 2\left(\sinh\left(\frac{\lambda x}{2}\right) a \cosh\left(\frac{\lambda x}{2}\right) + \frac{b}{2}\right) \left(\operatorname{arctanh}\left(\frac{-b \tanh\left(\frac{\lambda x}{2}\right) + a}{\sqrt{a^2+b^2}}\right) a^2 b^2 c_1 \right. \\ &\left. + \operatorname{arctanh}\left(\frac{-b \tanh\left(\frac{\lambda x}{2}\right) + a}{\sqrt{a^2+b^2}}\right) b^4 c_1 - c_2\right) \end{aligned}$$

The above shows that

$$u'(x) = -\frac{4\left(\left(\cosh\left(\frac{\lambda x}{2}\right)^2 + \sinh\left(\frac{\lambda x}{2}\right)^2\right) \left(\operatorname{arctanh}\left(\frac{-b \tanh\left(\frac{\lambda x}{2}\right)+a}{\sqrt{a^2+b^2}}\right) a^2 b^2 c_1 + \operatorname{arctanh}\left(\frac{-b \tanh\left(\frac{\lambda x}{2}\right)+a}{\sqrt{a^2+b^2}}\right) b^4 c_1 - c_2\right)\right.}{}$$

Using the above in (1) gives the solution

$$y = \frac{4\left(\left(\cosh\left(\frac{\lambda x}{2}\right)^2 + \sinh\left(\frac{\lambda x}{2}\right)^2\right) \left(\operatorname{arctanh}\left(\frac{-b \tanh\left(\frac{\lambda x}{2}\right)+a}{\sqrt{a^2+b^2}}\right) a^2 b^2 c_1 + \operatorname{arctanh}\left(\frac{-b \tanh\left(\frac{\lambda x}{2}\right)+a}{\sqrt{a^2+b^2}}\right) b^4 c_1 - c_2\right) a\right.}{\sqrt{a^2+b^2}\, \left(-2 \tanh\left(\frac{\lambda x}{2}\right)^2 b + 4 \tanh\left(\frac{\lambda x}{2}\right) a + 2b\right) \left(-a c_1 \cosh\left(\frac{\lambda x}{2}\right) (a\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{4\left(\left(\operatorname{arctanh}\left(\frac{-b\tanh\left(\frac{\lambda x}{2}\right)+a}{\sqrt{a^2+b^2}}\right)a^2b^2c_3 + \operatorname{arctanh}\left(\frac{-b\tanh\left(\frac{\lambda x}{2}\right)+a}{\sqrt{a^2+b^2}}\right)b^4c_3 - 1\right)a\left(\cosh\left(\frac{\lambda x}{2}\right)^2 - \frac{1}{2}\right)\sqrt{\quad}\right.}{\sqrt{a^2+b^2}\left(2ac_3\cosh\left(\frac{\lambda x}{2}\right)\left(a^2+b^2\right)^{\frac{3}{2}}\left(a\sinh\left(\frac{\lambda x}{2}\right)+b\cosh\left(\frac{\lambda x}{2}\right)\right)+4\left(\sinh\left(\frac{\lambda x}{2}\right)a\cosh\left(\frac{\lambda x}{2}\right)+\frac{b}{2}\right)\left(\mathrm{arc}\right.\right.}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{4\left(\left(\operatorname{arctanh}\left(\frac{-b\tanh\left(\frac{\lambda x}{2}\right)+a}{\sqrt{a^2+b^2}}\right)a^2b^2c_3 + \operatorname{arctanh}\left(\frac{-b\tanh\left(\frac{\lambda x}{2}\right)+a}{\sqrt{a^2+b^2}}\right)b^4c_3 - 1\right)a\left(\cosh\left(\frac{\lambda x}{2}\right)^2 - \frac{1}{2}\right)\sqrt{\quad}\right.}{\sqrt{a^2+b^2}\left(2ac_3\cosh\left(\frac{\lambda x}{2}\right)\left(a^2+b^2\right)^{\frac{3}{2}}\left(a\sinh\left(\frac{\lambda x}{2}\right)+b\cosh\left(\frac{\lambda x}{2}\right)\right)+4\left(\sinh\left(\frac{\lambda x}{2}\right)a\cosh\left(\frac{\lambda x}{2}\right)+\frac{b}{2}\right)\left(\mathrm{arc}\right.\right.} \tag{1}$$

<u>Verification of solutions</u>

$$y = -\frac{4\left(\left(\operatorname{arctanh}\left(\frac{-b\tanh\left(\frac{\lambda x}{2}\right)+a}{\sqrt{a^2+b^2}}\right)a^2b^2c_3 + \operatorname{arctanh}\left(\frac{-b\tanh\left(\frac{\lambda x}{2}\right)+a}{\sqrt{a^2+b^2}}\right)b^4c_3 - 1\right)a\left(\cosh\left(\frac{\lambda x}{2}\right)^2 - \frac{1}{2}\right)\sqrt{\quad}\right.}{\sqrt{a^2+b^2}\left(2ac_3\cosh\left(\frac{\lambda x}{2}\right)\left(a^2+b^2\right)^{\frac{3}{2}}\left(a\sinh\left(\frac{\lambda x}{2}\right)+b\cosh\left(\frac{\lambda x}{2}\right)\right)+4\left(\sinh\left(\frac{\lambda x}{2}\right)a\cosh\left(\frac{\lambda x}{2}\right)+\frac{b}{2}\right)\left(\mathrm{arc}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = lambda^2*sinh(lambda*x)*a*y(x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      <- linear_1 successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 250

```
dsolve((a*sinh(lambda*x)+b)*(diff(y(x),x)-y(x)^2)+a*lambda^2*sinh(lambda*x)=0,y(x), singsol=
```

$$y(x) = -\frac{4\left(\left(\operatorname{arctanh}\left(\frac{-\tanh\left(\frac{x\lambda}{2}\right)b+a}{\sqrt{a^2+b^2}}\right)a^2b^2+\operatorname{arctanh}\left(\frac{-\tanh\left(\frac{x\lambda}{2}\right)b+a}{\sqrt{a^2+b^2}}\right)b^4-c_1\right)a\left(\cosh\left(\frac{x\lambda}{2}\right)^2-\frac{1}{2}\right)\sqrt{a^2}}{\sqrt{a^2+b^2}\left(2a\cosh\left(\frac{x\lambda}{2}\right)\left(a^2+b^2\right)^{\frac{3}{2}}\left(a\sinh\left(\frac{x\lambda}{2}\right)+b\cosh\left(\frac{x\lambda}{2}\right)\right)+4\left(\operatorname{arctanh}\left(\frac{-\tanh\left(\frac{x\lambda}{2}\right)b+a}{\sqrt{a^2+b^2}}\right)a^2b^2+a}$$

✓ Solution by Mathematica

Time used: 24.532 (sec). Leaf size: 202

```
DSolve[(a*Sinh[\[Lambda]*x]+b)*(y'[x]-y[x]^2)+a*\[Lambda]^2*Sinh[\[Lambda]*x]==0,y[x],x,Incl
```

$$y(x) \to -\frac{\lambda\left(\sqrt{-a^2-b^2}(b-a\sinh(\lambda x))+a\cosh(\lambda x)\left(2b\arctan\left(\frac{a-b\tanh\left(\frac{\lambda x}{2}\right)}{\sqrt{-a^2-b^2}}\right)-c_1\lambda(-a^2-b^2)^{3/2}\right)\right)}{-a\sqrt{-a^2-b^2}\cosh(\lambda x)+(a\sinh(\lambda x)+b)\left(2b\arctan\left(\frac{a-b\tanh\left(\frac{\lambda x}{2}\right)}{\sqrt{-a^2-b^2}}\right)-c_1\lambda\left(-a^2-b^2\right)^{3/2}\right)}$$

$$y(x) \to -\frac{a\lambda\cosh(\lambda x)}{a\sinh(\lambda x)+b}$$

5.8 problem 8

Internal problem ID [10456]
Internal file name [`OUTPUT/9403_Monday_June_06_2022_02_24_26_PM_28185864/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - \alpha y^2 = \beta + \gamma \cosh(x)}$$

5.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \alpha y^2 + \beta + \gamma \cosh(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \alpha y^2 + \beta + \gamma \cosh(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \beta + \gamma \cosh(x)$, $f_1(x) = 0$ and $f_2(x) = \alpha$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\alpha u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \alpha^2(\beta + \gamma \cosh(x)) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\alpha u''(x) + \alpha^2(\beta + \gamma \cosh(x)) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{MathieuC}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right) + c_2 \operatorname{MathieuS}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right)$$

The above shows that

$$u'(x) = \frac{i\left(c_2 \operatorname{MathieuSPrime}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right) + c_1 \operatorname{MathieuCPrime}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right)\right)}{2}$$

Using the above in (1) gives the solution

$$y = -\frac{i\left(c_2 \operatorname{MathieuSPrime}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right) + c_1 \operatorname{MathieuCPrime}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right)\right)}{2\alpha\left(c_1 \operatorname{MathieuC}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right) + c_2 \operatorname{MathieuS}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{i\left(\operatorname{MathieuSPrime}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right) + c_3 \operatorname{MathieuCPrime}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right)\right)}{2\alpha\left(c_3 \operatorname{MathieuC}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right) + \operatorname{MathieuS}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right)\right)}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{i\left(\operatorname{MathieuSPrime}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right) + c_3 \operatorname{MathieuCPrime}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right)\right)}{2\alpha\left(c_3 \operatorname{MathieuC}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right) + \operatorname{MathieuS}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right)\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{i\left(\mathrm{MathieuSPrime}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right) + c_3\,\mathrm{MathieuCPrime}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right)\right)}{2\alpha\left(c_3\,\mathrm{MathieuC}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right) + \mathrm{MathieuS}\left(-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -alpha*(beta+gamma*cosh(x))*y(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Kummer
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            -> Mathieu
               -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
               Equivalence transformation and function parameters: {z = 1/2*t+1/2}, {kappa =
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 70

```
dsolve(diff(y(x),x)=alpha*y(x)^2+beta+gamma*cosh(x),y(x), singsol=all)
```

$$y(x) = -\frac{i\left(c_1 \operatorname{MathieuSPrime}\left(-4\alpha\beta, 2\gamma\alpha, \frac{ix}{2}\right) + \operatorname{MathieuCPrime}\left(-4\alpha\beta, 2\gamma\alpha, \frac{ix}{2}\right)\right)}{2\alpha\left(c_1 \operatorname{MathieuS}\left(-4\alpha\beta, 2\gamma\alpha, \frac{ix}{2}\right) + \operatorname{MathieuC}\left(-4\alpha\beta, 2\gamma\alpha, \frac{ix}{2}\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.543 (sec). Leaf size: 140

```
DSolve[y'[x]==\[Alpha]*y[x]^2+\[Beta]+\[Gamma]*Cosh[x],y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to -\frac{ic_1 \text{MathieuCPrime}\left[-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right] - i\text{MathieuSPrime}\left[-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right]}{2\alpha c_1 \text{MathieuC}\left[-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right] - 2\alpha \text{MathieuS}\left[-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right]}$$

$$y(x) \to -\frac{i\text{MathieuCPrime}\left[-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right]}{2\alpha \text{MathieuC}\left[-4\alpha\beta, 2\alpha\gamma, \frac{ix}{2}\right]}$$

5.9 problem 9

Internal problem ID [10457]
Internal file name [`OUTPUT/9404_Monday_June_06_2022_02_24_28_PM_71061260/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - a\cosh(\beta x)\, y = ab\cosh(\beta x) - b^2}$$

5.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + a\cosh(\beta x)\, y + ab\cosh(\beta x) - b^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\cosh(\beta x)\, y + ab\cosh(\beta x) - b^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = ab\cosh(\beta x) - b^2$, $f_1(x) = a\cosh(\beta x)$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= a \cosh(\beta x) \\ f_2^2 f_0 &= ab \cosh(\beta x) - b^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - a \cosh(\beta x) u'(x) + \left(ab \cosh(\beta x) - b^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = {} & c_1 \operatorname{HeunD}\left(-\frac{2a}{\beta}, \frac{4b(a-b)}{\beta^2}, \frac{4a}{\beta}, \frac{4b(a+b)}{\beta^2}, \coth\left(\frac{\beta x}{2}\right)\right) \\ & + c_2 \operatorname{HeunD}\left(\frac{2a}{\beta}, \frac{4b(a-b)}{\beta^2}, \frac{4a}{\beta}, \frac{4b(a+b)}{\beta^2}, \coth\left(\frac{\beta x}{2}\right)\right) \mathrm{e}^{\frac{a \sinh(\beta x)}{\beta}} \end{aligned}$$

The above shows that

$$\begin{aligned} u'(x) = {} & -\frac{\operatorname{csch}\left(\frac{\beta x}{2}\right)^2 c_1 \beta \operatorname{HeunDPrime}\left(-\frac{2a}{\beta}, \frac{4b(a-b)}{\beta^2}, \frac{4a}{\beta}, \frac{4b(a+b)}{\beta^2}, \coth\left(\frac{\beta x}{2}\right)\right)}{2} \\ & + \left(-\frac{\operatorname{csch}\left(\frac{\beta x}{2}\right)^2 \operatorname{HeunDPrime}\left(\frac{2a}{\beta}, \frac{4b(a-b)}{\beta^2}, \frac{4a}{\beta}, \frac{4b(a+b)}{\beta^2}, \coth\left(\frac{\beta x}{2}\right)\right) \beta}{2} \right. \\ & \left. + a \operatorname{HeunD}\left(\frac{2a}{\beta}, \frac{4b(a-b)}{\beta^2}, \frac{4a}{\beta}, \frac{4b(a+b)}{\beta^2}, \coth\left(\frac{\beta x}{2}\right)\right) \cosh(\beta x)\right) c_2 \mathrm{e}^{\frac{a \sinh(\beta x)}{\beta}} \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{-\frac{\operatorname{csch}\left(\frac{\beta x}{2}\right)^2 c_1 \beta \operatorname{HeunDPrime}\left(-\frac{2a}{\beta}, \frac{4b(a-b)}{\beta^2}, \frac{4a}{\beta}, \frac{4b(a+b)}{\beta^2}, \coth\left(\frac{\beta x}{2}\right)\right)}{2} + \left(-\frac{\operatorname{csch}\left(\frac{\beta x}{2}\right)^2 \operatorname{HeunDPrime}\left(\frac{2a}{\beta}, \frac{4b(a-b)}{\beta^2}, \frac{4a}{\beta}, \frac{4b(a+b)}{\beta^2}, \coth\left(\frac{\beta x}{2}\right)\right)}{2}\right)}{c_1 \operatorname{HeunD}\left(-\frac{2a}{\beta}, \frac{4b(a-b)}{\beta^2}, \frac{4a}{\beta}, \frac{4b(a+b)}{\beta^2}, \coth\left(\frac{\beta x}{2}\right)\right) + c_2 \operatorname{HeunD}\left(\frac{2a}{\beta}, \ldots\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-2\,\mathrm{e}^{\frac{a\sinh(\beta x)}{\beta}} a\,\mathrm{HeunD}\left(\frac{2a}{\beta}, \frac{4b(a-b)}{\beta^2}, \frac{4a}{\beta}, \frac{4b(a+b)}{\beta^2}, \coth\left(\frac{\beta x}{2}\right)\right)\cosh(\beta x) + \beta\,\mathrm{csch}\left(\frac{\beta x}{2}\right)^2\left(\mathrm{e}^{\frac{a\sinh(\beta x)}{\beta}}\,\mathrm{HeunDPrime}\right.}{2\,\mathrm{HeunD}\left(\frac{2a}{\beta}, \frac{4b(a-b)}{\beta^2}, \frac{4a}{\beta}, \frac{4b(a+b)}{\beta^2}, \coth\left(\frac{\beta x}{2}\right)\right)\mathrm{e}^{\frac{a\sinh(\beta x)}{\beta}} + 2c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{-2\,\mathrm{e}^{\frac{a\sinh(\beta x)}{\beta}} a\,\mathrm{HeunD}\left(\frac{2a}{\beta}, \frac{4b(a-b)}{\beta^2}, \frac{4a}{\beta}, \frac{4b(a+b)}{\beta^2}, \coth\left(\frac{\beta x}{2}\right)\right)\cosh(\beta x) + \beta\,\mathrm{csch}\left(\frac{\beta x}{2}\right)^2\left(\mathrm{e}^{\frac{a\sinh(\beta x)}{\beta}}\,\mathrm{HeunDPrime}\right.}{2\,\mathrm{HeunD}\left(\frac{2a}{\beta}, \frac{4b(a-b)}{\beta^2}, \frac{4a}{\beta}, \frac{4b(a+b)}{\beta^2}, \coth\left(\frac{\beta x}{2}\right)\right)\mathrm{e}^{\frac{a\sinh(\beta x)}{\beta}} + 2c_3} \tag{1}$$

Verification of solutions

$$y = \frac{-2\,\mathrm{e}^{\frac{a\sinh(\beta x)}{\beta}} a\,\mathrm{HeunD}\left(\frac{2a}{\beta}, \frac{4b(a-b)}{\beta^2}, \frac{4a}{\beta}, \frac{4b(a+b)}{\beta^2}, \coth\left(\frac{\beta x}{2}\right)\right)\cosh(\beta x) + \beta\,\mathrm{csch}\left(\frac{\beta x}{2}\right)^2\left(\mathrm{e}^{\frac{a\sinh(\beta x)}{\beta}}\,\mathrm{HeunDPrime}\right.}{2\,\mathrm{HeunD}\left(\frac{2a}{\beta}, \frac{4b(a-b)}{\beta^2}, \frac{4a}{\beta}, \frac{4b(a+b)}{\beta^2}, \coth\left(\frac{\beta x}{2}\right)\right)\mathrm{e}^{\frac{a\sinh(\beta x)}{\beta}} + 2c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 73

```
dsolve(diff(y(x),x)=y(x)^2+a*cosh(beta*x)*y(x)+a*b*cosh(beta*x)-b^2,y(x), singsol=all)
```

$$y(x) = \frac{b\left(\int \mathrm{e}^{\frac{-2b\beta x+\sinh(x\beta)a}{\beta}}dx\right) - c_1 b + \mathrm{e}^{\frac{-2b\beta x+\sinh(x\beta)a}{\beta}}}{-\left(\int \mathrm{e}^{\frac{-2b\beta x+\sinh(x\beta)a}{\beta}}dx\right) + c_1}$$

✓ Solution by Mathematica

Time used: 9.815 (sec). Leaf size: 242

```
DSolve[y'[x]==y[x]^2+a*Cosh[\[Beta]*x]*y[x]+a*b*Cosh[\[Beta]*x]-b^2,y[x],x,IncludeSingularSo
```

$$y(x) \to -\frac{b\int_1^{e^{x\beta}} e^{\frac{a\left(K[1]^2-1\right)}{2\beta K[1]}} K[1]^{-\frac{2b}{\beta}-1} dK[1] + \beta e^{\frac{ae^{\beta(-x)}\left(e^{2\beta x}-1\right)}{2\beta}}\left(e^{\beta x}\right)^{-\frac{2b}{\beta}} + bc_1}{\int_1^{e^{x\beta}} e^{\frac{a\left(K[1]^2-1\right)}{2\beta K[1]}} K[1]^{-\frac{2b}{\beta}-1} dK[1] + c_1}$$

$$y(x) \to -b$$

$$y(x) \to -\frac{\beta e^{\frac{ae^{\beta(-x)}\left(e^{2\beta x}-1\right)}{2\beta}}\left(e^{\beta x}\right)^{-\frac{2b}{\beta}}}{\int_1^{e^{x\beta}} e^{\frac{a\left(K[1]^2-1\right)}{2\beta K[1]}} K[1]^{-\frac{2b}{\beta}-1} dK[1]} - b$$

5.10 problem 10

Internal problem ID [10458]
Internal file name [OUTPUT/9405_Monday_June_06_2022_02_24_32_PM_77442611/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 - ax\cosh(bx)^m y = a\cosh(bx)^m}$$

5.10.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + ax\cosh(bx)^m y + a\cosh(bx)^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + ax\cosh(bx)^m y + a\cosh(bx)^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\cosh(bx)^m$, $f_1(x) = \cosh(bx)^m ax$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= \cosh(bx)^m\,ax \\ f_2^2f_0 &= a\cosh(bx)^m \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \cosh(bx)^m\,axu'(x) + a\cosh(bx)^m\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x\left(c_1\left(\int e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}dx\right) + c_2\right)$$

The above shows that

$$u'(x) = c_1\left(\int e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}dx\right) + c_2 + xc_1e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}$$

Using the above in (1) gives the solution

$$y = -\frac{c_1\left(\int e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}dx\right) + c_2 + xc_1e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}}{x\left(c_1\left(\int e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}dx\right) + c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3\left(\int e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}dx\right) - 1 - xc_3e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}}{x\left(c_3\left(\int e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}dx\right) + 1\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_3\left(\int e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}dx\right)-1-xc_3 e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}}{x\left(c_3\left(\int e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}dx\right)+1\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-c_3\left(\int e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}dx\right)-1-xc_3 e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}}{x\left(c_3\left(\int e^{\int \frac{\cosh(bx)^m a\,x^2-2}{x}dx}dx\right)+1\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 85

```
dsolve(diff(y(x),x)=y(x)^2+a*x*cosh(b*x)^m*y(x)+a*cosh(b*x)^m,y(x), singsol=all)
```

$$y(x) = \frac{-e^{\int \frac{\cosh(bx)^m x^2 a-2}{x}dx}x-\left(\int e^{\int \frac{\cosh(bx)^m x^2 a-2}{x}dx}dx\right)+c_1}{\left(-c_1+\int e^{\int \frac{\cosh(bx)^m x^2 a-2}{x}dx}dx\right)x}$$

✓ Solution by Mathematica

Time used: 7.557 (sec). Leaf size: 394

```
DSolve[y'[x]==y[x]^2+a*x*Cosh[b*x]^m*y[x]+a*Cosh[b*x]^m,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to$$

$$-\frac{\int_1^x \frac{\exp\left(-\frac{2^{-m}a\left(e^{-bK[1]}+e^{bK[1]}\right)^m\left(1+e^{2bK[1]}\right)^{-m}\left({}_3F_2\left(-m,-\frac{m}{2},-\frac{m}{2};1-\frac{m}{2},1-\frac{m}{2};-e^{2bK[1]}\right)+bm\,\text{Hypergeometric2F1}\left(-m,-\frac{m}{2},1-\frac{m}{2},-e^{2bK[1]}\right.\right.}{b^2m^2}\right.}{K[1]^2}}{x\left(\int_1^x \frac{\exp\left(-\frac{2^{-m}a\left(e^{-bK[1]}+e^{bK[1]}\right)^m\left(1+e^{2bK[1]}\right)^{-m}\left({}_3F_2\left(-m,-\frac{m}{2},-\frac{m}{2};\right.\right.}{}\right.}{}\right.}$$

$$y(x) \to -\frac{1}{x}$$

5.11 problem 11

Internal problem ID [10459]
Internal file name [`OUTPUT/9406_Monday_June_06_2022_02_24_34_PM_92845045/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \left(a \cosh\left(\lambda x\right)^{2}-\lambda\right) y^{2} = -a \cosh\left(\lambda x\right)^{2}+a+\lambda}$$

5.11.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= \cosh\left(\lambda x\right)^{2} a\, y^{2}-a \cosh\left(\lambda x\right)^{2}-\lambda\, y^{2}+a+\lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \cosh\left(\lambda x\right)^{2} a\, y^{2}-a \cosh\left(\lambda x\right)^{2}-\lambda\, y^{2}+a+\lambda$$

With Riccati ODE standard form

$$y' = f_0(x)+ f_1(x)y+f_2(x)y^{2}$$

Shows that $f_0(x)=-a \cosh\left(\lambda x\right)^{2}+a+\lambda$, $f_1(x)=0$ and $f_2(x)=a \cosh\left(\lambda x\right)^{2}-\lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{\left(a \cosh\left(\lambda x\right)^{2}-\lambda\right) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= 2a \sinh(\lambda x)\, \lambda \cosh(\lambda x) \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= \left(a \cosh(\lambda x)^2 - \lambda\right)^2 \left(-a \cosh(\lambda x)^2 + a + \lambda\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\left(a \cosh(\lambda x)^2 - \lambda\right) u''(x) - 2a \sinh(\lambda x)\, \lambda \cosh(\lambda x)\, u'(x) + \left(a \cosh(\lambda x)^2 - \lambda\right)^2 \left(-a \cosh(\lambda x)^2 + a + \lambda\right) u$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = -2\, \mathrm{e}^{-\frac{\cosh(2\lambda x) a}{4\lambda}} \left(c_2 \lambda \left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x) a}{2\lambda}} \left(a - \operatorname{sech}(\lambda x)^2 \lambda\right) dx\right) - \frac{c_1}{2}\right) \cosh(\lambda x)$$

The above shows that

$$\begin{aligned}
u'(x) = {} & \left(a \cosh(\lambda x)^2\right. \\
& \left. - \lambda\right) \operatorname{sech}(\lambda x) \left(\sinh(2\lambda x) \left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x) a}{2\lambda}} \left(a - \operatorname{sech}(\lambda x)^2 \lambda\right) dx\right) c_2 \lambda\, \mathrm{e}^{-\frac{\cosh(2\lambda x) a}{4\lambda}}\right. \\
& \left. - \frac{\sinh(2\lambda x)\, c_1 \mathrm{e}^{-\frac{\cosh(2\lambda x) a}{4\lambda}}}{2} + 2 c_2 \lambda\, \mathrm{e}^{\frac{\cosh(2\lambda x) a}{4\lambda}}\right)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = \frac{\operatorname{sech}(\lambda x) \left(\sinh(2\lambda x) \left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x) a}{2\lambda}} \left(a - \operatorname{sech}(\lambda x)^2 \lambda\right) dx\right) c_2 \lambda\, \mathrm{e}^{-\frac{\cosh(2\lambda x) a}{4\lambda}} - \frac{\sinh(2\lambda x) c_1 \mathrm{e}^{-\frac{\cosh(2\lambda x) a}{4\lambda}}}{2} + 2 c_2 \lambda\, \mathrm{e}\right.}{2 \left(c_2 \lambda \left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x) a}{2\lambda}} \left(a - \operatorname{sech}(\lambda x)^2 \lambda\right) dx\right) - \frac{c_1}{2}\right) \cosh(\lambda x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-2\,\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\,\mathrm{sech}\left(\lambda x\right)^{2}\lambda - 2\lambda\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a-\mathrm{sech}\left(\lambda x\right)^{2}\lambda\right)dx\right)\tanh\left(\lambda x\right) + c_3\tanh\left(\lambda x\right)}{-2\lambda\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a-\mathrm{sech}\left(\lambda x\right)^{2}\lambda\right)dx\right) + c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{-2\,\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\,\mathrm{sech}\left(\lambda x\right)^{2}\lambda - 2\lambda\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a-\mathrm{sech}\left(\lambda x\right)^{2}\lambda\right)dx\right)\tanh\left(\lambda x\right) + c_3\tanh\left(\lambda x\right)}{-2\lambda\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a-\mathrm{sech}\left(\lambda x\right)^{2}\lambda\right)dx\right) + c_3} \tag{1}$$

Verification of solutions

$$y = \frac{-2\,\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\,\mathrm{sech}\left(\lambda x\right)^{2}\lambda - 2\lambda\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a-\mathrm{sech}\left(\lambda x\right)^{2}\lambda\right)dx\right)\tanh\left(\lambda x\right) + c_3\tanh\left(\lambda x\right)}{-2\lambda\left(\int -\mathrm{e}^{\frac{\cosh(2\lambda x)a}{2\lambda}}\left(a-\mathrm{sech}\left(\lambda x\right)^{2}\lambda\right)dx\right) + c_3}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = 2*a*cosh(lambda*x)*lambda*sinh
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               -> Kummer
                  -> hyper3: Equivalence to 1F1 under a power @ Moebius
               -> hypergeometric
                  -> heuristic approach
                  -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
               -> Mathieu
                  -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 104

```
dsolve(diff(y(x),x)=(a*cosh(lambda*x)^2-lambda)*y(x)^2+a+lambda-a*cosh(lambda*x)^2,y(x), sin
```

$$y(x) = \frac{2\tanh(x\lambda)\lambda\left(\int -e^{\frac{a\cosh(2x\lambda)}{2\lambda}}\left(a-\operatorname{sech}(x\lambda)^2\lambda\right)dx\right)c_1+2\operatorname{sech}(x\lambda)^2 e^{\frac{a\cosh(2x\lambda)}{2\lambda}}c_1\lambda-\tanh(x\lambda)}{2\lambda\left(\int -e^{\frac{a\cosh(2x\lambda)}{2\lambda}}\left(a-\operatorname{sech}(x\lambda)^2\lambda\right)dx\right)c_1-1}$$

✓ Solution by Mathematica

Time used: 49.81 (sec). Leaf size: 211

```
DSolve[y'[x]==(a*Cosh[\[Lambda]*x]^2-\[Lambda])*y[x]^2+a+\[Lambda]-a*Cosh[\[Lambda]*x]^2,y[x
```

$$y(x) \to \frac{\operatorname{sech}^2(\lambda x)\left(c_1\sinh(2\lambda x)\int_1^x e^{\frac{a\cosh^2(\lambda K[1])}{\lambda}}\left(\lambda-a\cosh^2(\lambda K[1])\right)\operatorname{sech}^2(\lambda K[1])dK[1]+2c_1 e^{\frac{a\cosh^2(\lambda x)}{\lambda}}+\sinh(\right.}{2+2c_1\int_1^x e^{\frac{a\cosh^2(\lambda K[1])}{\lambda}}\left(\lambda-a\cosh^2(\lambda K[1])\right)\operatorname{sech}^2(\lambda K[1])dK[1]}$$

$$y(x) \to \frac{1}{2}\operatorname{sech}^2(\lambda x)\left(\frac{2e^{\frac{a\cosh^2(\lambda x)}{\lambda}}}{\int_1^x e^{\frac{a\cosh^2(\lambda K[1])}{\lambda}}\left(\lambda-a\cosh^2(\lambda K[1])\right)\operatorname{sech}^2(\lambda K[1])dK[1]}+\sinh(2\lambda x)\right)$$

5.12 problem 12

Internal problem ID [10460]
Internal file name [OUTPUT/9407_Monday_June_06_2022_02_24_39_PM_73916692/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{2y' - (a-\lambda+a\cosh(\lambda x))\,y^2 = a+\lambda-a\cosh(\lambda x)}$$

5.12.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\cosh(\lambda x)\,a\,y^2}{2} + \frac{a\,y^2}{2} - \frac{\lambda\,y^2}{2} + \frac{a}{2} + \frac{\lambda}{2} - \frac{a\cosh(\lambda x)}{2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{\cosh(\lambda x)\,a\,y^2}{2} + \frac{a\,y^2}{2} - \frac{\lambda\,y^2}{2} + \frac{a}{2} + \frac{\lambda}{2} - \frac{a\cosh(\lambda x)}{2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{a}{2} + \frac{\lambda}{2} - \frac{a\cosh(\lambda x)}{2}$, $f_1(x) = 0$ and $f_2(x) = \frac{a}{2} - \frac{\lambda}{2} + \frac{a\cosh(\lambda x)}{2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\left(\frac{a}{2} - \frac{\lambda}{2} + \frac{a\cosh(\lambda x)}{2}\right) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{a\lambda \sinh(\lambda x)}{2} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= \left(\frac{a}{2} - \frac{\lambda}{2} + \frac{a\cosh(\lambda x)}{2}\right)^2 \left(\frac{a}{2} + \frac{\lambda}{2} - \frac{a\cosh(\lambda x)}{2}\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\left(\frac{a}{2} - \frac{\lambda}{2} + \frac{a\cosh(\lambda x)}{2}\right) u''(x) - \frac{a\lambda \sinh(\lambda x) u'(x)}{2} + \left(\frac{a}{2} - \frac{\lambda}{2} + \frac{a\cosh(\lambda x)}{2}\right)^2 \left(\frac{a}{2} + \frac{\lambda}{2} - \frac{a\cosh(\lambda x)}{2}\right) u$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = -\frac{\cosh\left(\frac{\lambda x}{2}\right) e^{-\frac{\cosh(\lambda x)a}{2\lambda}} \left(c_2 \lambda \left(\int e^{\frac{\cosh(\lambda x)a}{\lambda}} \left(-2a + \operatorname{sech}\left(\frac{\lambda x}{2}\right)^2 \lambda\right) dx\right) - 2c_1\right)}{2}$$

The above shows that

$$u'(x) = \frac{\operatorname{sech}\left(\frac{\lambda x}{2}\right) (a - \lambda + a\cosh(\lambda x)) \left(4 e^{\frac{\cosh(\lambda x)a}{2\lambda}} c_2 \lambda + \sinh(\lambda x) \left(\int e^{\frac{\cosh(\lambda x)a}{\lambda}} \left(-2a + \operatorname{sech}\left(\frac{\lambda x}{2}\right)^2 \lambda\right) dx\right) c_2 \lambda e^{-}\right.}{8}$$

Using the above in (1) gives the solution

$$y = \frac{\operatorname{sech}\left(\frac{\lambda x}{2}\right) (a - \lambda + a\cosh(\lambda x)) \left(4 e^{\frac{\cosh(\lambda x)a}{2\lambda}} c_2 \lambda + \sinh(\lambda x) \left(\int e^{\frac{\cosh(\lambda x)a}{\lambda}} \left(-2a + \operatorname{sech}\left(\frac{\lambda x}{2}\right)^2 \lambda\right) dx\right) c_2 \lambda e^{-}\right.}{4 \left(\frac{a}{2} - \frac{\lambda}{2} + \frac{a\cosh(\lambda x)}{2}\right) \cosh\left(\frac{\lambda x}{2}\right) \left(c_2 \lambda \left(\int e^{\frac{\cosh(\lambda x)a}{\lambda}} \left(-2a + \operatorname{sech}\left(\frac{\lambda x}{2}\right)^2 \lambda\right) d\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\operatorname{sech}\left(\frac{\lambda x}{2}\right)^2 \left(4 e^{\frac{\cosh(\lambda x)a}{\lambda}} \lambda + \lambda \left(\int e^{\frac{\cosh(\lambda x)a}{\lambda}} \left(-2a + \operatorname{sech}\left(\frac{\lambda x}{2}\right)^2 \lambda\right) dx\right) \sinh(\lambda x) - 2c_3 \sinh(\lambda x)\right)}{2\lambda \left(\int e^{\frac{\cosh(\lambda x)a}{\lambda}} \left(-2a + \operatorname{sech}\left(\frac{\lambda x}{2}\right)^2 \lambda\right) dx\right) - 4c_3}$$

Summary
The solution(s) found are the following

$$y = \frac{\operatorname{sech}\left(\frac{\lambda x}{2}\right)^{2} \left(4\,{\mathrm e}^{\frac{\cosh(\lambda x) a}{\lambda}} \lambda +\lambda \left(\int {\mathrm e}^{\frac{\cosh(\lambda x) a}{\lambda}} \left(-2 a +\operatorname{sech}\left(\frac{\lambda x}{2}\right)^{2} \lambda \right)d x \right) \sinh\left(\lambda x \right)-2 c_{3} \sinh\left(\lambda x \right)\right)}{2 \lambda \left(\int {\mathrm e}^{\frac{\cosh(\lambda x) a}{\lambda}} \left(-2 a +\operatorname{sech}\left(\frac{\lambda x}{2}\right)^{2} \lambda \right)d x \right)-4 c_{3}} \tag{1}$$

Verification of solutions

$$y = \frac{\operatorname{sech}\left(\frac{\lambda x}{2}\right)^{2} \left(4\,{\mathrm e}^{\frac{\cosh(\lambda x) a}{\lambda}} \lambda +\lambda \left(\int {\mathrm e}^{\frac{\cosh(\lambda x) a}{\lambda}} \left(-2 a +\operatorname{sech}\left(\frac{\lambda x}{2}\right)^{2} \lambda \right)d x \right) \sinh\left(\lambda x \right)-2 c_{3} \sinh\left(\lambda x \right)\right)}{2 \lambda \left(\int {\mathrm e}^{\frac{\cosh(\lambda x) a}{\lambda}} \left(-2 a +\operatorname{sech}\left(\frac{\lambda x}{2}\right)^{2} \lambda \right)d x \right)-4 c_{3}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = sinh(lambda*x)*a*lambda*(diff(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               -> Kummer
                  -> hyper3: Equivalence to 1F1 under a power @ Moebius
               -> hypergeometric
                  -> heuristic approach
                  -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
               -> Mathieu
                  -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 101

```
dsolve(2*diff(y(x),x)=(a-lambda+a*cosh(lambda*x))*y(x)^2+a+lambda-a*cosh(lambda*x),y(x), sin
```

$$y(x) = \frac{\tanh\left(\frac{x\lambda}{2}\right)\lambda\left(\int \mathrm{e}^{\frac{a\cosh(x\lambda)}{\lambda}}\left(-2a+\operatorname{sech}\left(\frac{x\lambda}{2}\right)^2\lambda\right)dx\right)c_1+2\operatorname{sech}\left(\frac{x\lambda}{2}\right)^2\mathrm{e}^{\frac{a\cosh(x\lambda)}{\lambda}}c_1\lambda-2\tanh\left(\frac{x\lambda}{2}\right)}{\lambda\left(\int \mathrm{e}^{\frac{a\cosh(x\lambda)}{\lambda}}\left(-2a+\operatorname{sech}\left(\frac{x\lambda}{2}\right)^2\lambda\right)dx\right)c_1-2}$$

✓ Solution by Mathematica

Time used: 59.899 (sec). Leaf size: 338

```
DSolve[2*y'[x]==(a-\[Lambda]+a*Cosh[\[Lambda]*x])*y[x]^2+a+\[Lambda]-a*Cosh[\[Lambda]*x],y[x
```

$$y(x) \to \frac{\operatorname{sech}^2\left(\frac{\lambda x}{2}\right)\left(c_1\sinh(\lambda x)\int_1^x -e^{\frac{2a\cosh^2\left(\frac{1}{2}\lambda K[1]\right)}{\lambda}}(\cosh(\lambda K[1])a+a-\lambda)\operatorname{sech}^2\left(\frac{1}{2}\lambda K[1]\right)dK[1]+4c_1e^{\frac{2a\cosh^2\left(\frac{\lambda x}{2}\right)}{\lambda}}\right.}{2+2c_1\int_1^x -e^{\frac{2a\cosh^2\left(\frac{1}{2}\lambda K[1]\right)}{\lambda}}(\cosh(\lambda K[1])a+a-\lambda)\operatorname{sech}^2\left(\frac{1}{2}\lambda K[1]\right)dK[1]}$$

$$y(x) \to \frac{1}{2}\operatorname{sech}^2\left(\frac{\lambda x}{2}\right)\left(\frac{4e^{\frac{2a\cosh^2\left(\frac{\lambda x}{2}\right)}{\lambda}}}{\int_1^x -e^{\frac{2a\cosh^2\left(\frac{1}{2}\lambda K[1]\right)}{\lambda}}(\cosh(\lambda K[1])a+a-\lambda)\operatorname{sech}^2\left(\frac{1}{2}\lambda K[1]\right)dK[1]}+\sinh(\lambda x)\right)$$

$$y(x) \to \frac{1}{2}\operatorname{sech}^2\left(\frac{\lambda x}{2}\right)\left(\frac{4e^{\frac{2a\cosh^2\left(\frac{\lambda x}{2}\right)}{\lambda}}}{\int_1^x -e^{\frac{2a\cosh^2\left(\frac{1}{2}\lambda K[1]\right)}{\lambda}}(\cosh(\lambda K[1])a+a-\lambda)\operatorname{sech}^2\left(\frac{1}{2}\lambda K[1]\right)dK[1]}+\sinh(\lambda x)\right)$$

5.13 problem 13

Internal problem ID [10461]
Internal file name [`OUTPUT/9408_Monday_June_06_2022_02_24_42_PM_90381490/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = -\lambda^2 + a\cosh(\lambda x)^n \sinh(\lambda x)^{-n-4}}$$

5.13.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 - \lambda^2 + a\cosh(\lambda x)^n \sinh(\lambda x)^{-n-4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - \lambda^2 + \frac{a\cosh(\lambda x)^n \sinh(\lambda x)^{-n}}{\sinh(\lambda x)^4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\lambda^2 + a\cosh(\lambda x)^n \sinh(\lambda x)^{-n-4}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -\lambda^2 + a \cosh(\lambda x)^n \sinh(\lambda x)^{-n-4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-\lambda^2 + a \cosh(\lambda x)^n \sinh(\lambda x)^{-n-4}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\cosh(\lambda x)^n \sinh(\lambda x)^{-n-4} \_Y(x)\, a - \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\cosh(\lambda x)^n \sinh(\lambda x)^{-n-4} \_Y(x)\, a - \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\cosh(\lambda x)^n \sinh(\lambda x)^{-n-4} \_Y(x)\, a - \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\cosh(\lambda x)^n \sinh(\lambda x)^{-n-4} \_Y(x)\, a - \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\cosh(\lambda x)^n \sinh(\lambda x)^{-n-4} \_Y(x)\, a - \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\cosh(\lambda x)^n \sinh(\lambda x)^{-n-4} \_Y(x)\, a - \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\cosh(\lambda x)^n \sinh(\lambda x)^{-n-4} \_Y(x)\, a - \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\cosh(\lambda x)^n \sinh(\lambda x)^{-n-4} \_Y(x)\, a - \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\cosh(\lambda x)^n \sinh(\lambda x)^{-n-4} \_Y(x)\, a - \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\operatorname{DESol}\left(\left\{\cosh(\lambda x)^n \sinh(\lambda x)^{-n-4} \_Y(x)\, a - \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (lambda^2-a*cosh(lambda*x)^n*s
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            -> Mathieu
               -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2-lambda^2+a*cosh(lambda*x)^n*sinh(lambda*x)^(-n-4),y(x), singsol=a
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2-\[Lambda]^2+a*Cosh[\[Lambda]*x]^n*Sinh[\[Lambda]*x]^(-n-4),y[x],x,Inclu
```

Not solved

5.14 problem 14

Internal problem ID [10462]
Internal file name [`OUTPUT/9409_Monday_June_06_2022_02_27_10_PM_68507380/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \sinh(\lambda x) y^2 a = b \sinh(\lambda x) \cosh(\lambda x)^n}$$

5.14.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \sinh(\lambda x) y^2 a + b \sinh(\lambda x) \cosh(\lambda x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \sinh(\lambda x) y^2 a + b \sinh(\lambda x) \cosh(\lambda x)^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b \sinh(\lambda x) \cosh(\lambda x)^n$, $f_1(x) = 0$ and $f_2(x) = a \sinh(\lambda x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{a \sinh(\lambda x) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= a\lambda \cosh(\lambda x) \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= a^2 \sinh(\lambda x)^3\, b\cosh(\lambda x)^n \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$a\sinh(\lambda x)\,u''(x) - a\lambda\cosh(\lambda x)\,u'(x) + a^2\sinh(\lambda x)^3\, b\cosh(\lambda x)^n\, u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sqrt{\cosh(\lambda x)}\left(c_1 \operatorname{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right) + c_2 \operatorname{BesselY}\left(\frac{1}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right)\right)$$

The above shows that

$$u'(x) = -\frac{\left(\sqrt{a}\,\sqrt{b}\,\operatorname{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right)\cosh(\lambda x)^{1+\frac{n}{2}}\,c_1 + \operatorname{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right)\sqrt{a}\,\sqrt{b}\,\mathrm{c}\right.}{\sqrt{\mathrm{c}}}$$

Using the above in (1) gives the solution

$$y = \frac{\sqrt{a}\,\sqrt{b}\,\operatorname{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right)\cosh(\lambda x)^{1+\frac{n}{2}}\,c_1 + \operatorname{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right)\sqrt{a}\,\sqrt{b}\,\cosh}{\cosh(\lambda x)\,a\left(c_1 \operatorname{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right) + \right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(-\sqrt{b}\left(\text{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right) c_3 + \text{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right)\right) \cosh(\lambda x)^{1+\frac{n}{2}} \sqrt{a} + \lambda}{\left(c_3 \text{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right) + \text{Bessel}}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(-\sqrt{b}\left(\text{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right) c_3 + \text{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right)\right) \cosh(\lambda x)^{1+\frac{n}{2}} \sqrt{a} + \lambda}{\left(c_3 \text{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right) + \text{Bessel}} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(-\sqrt{b}\left(\text{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right) c_3 + \text{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right)\right) \cosh(\lambda x)^{1+\frac{n}{2}} \sqrt{a} + \lambda}{\left(c_3 \text{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(\lambda x)^{1+\frac{n}{2}}}{\lambda(2+n)}\right) + \text{Bessel}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = lambda*cosh(lambda*x)*(diff(y(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying an equivalence, under non-integer power transformations,
            to LODEs admitting Liouvillian solutions.
            -> Trying a Liouvillian solution using Kovacics algorithm
            <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            <- Bessel successful
         <- special function solution successful
         Change of variables used:
            [x = arccosh(t)/lambda]
         Linear ODE actually solved:
            4*(t-1)^(1/2)*(t+1)^(1/2)*t^n*a*b*(t^2-1)*u(t)+4*(t-1)^(1/2)*(t+1)^(1/2)*lambda^
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 245

```
dsolve(diff(y(x),x)=a*sinh(lambda*x)*y(x)^2+b*sinh(lambda*x)*cosh(lambda*x)^n,y(x), singsol=
```

$$y(x) = \frac{\operatorname{sech}(x\lambda)\left(-\lambda\sqrt{a}\left(\operatorname{BesselY}\left(\frac{1}{n+2}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(x\lambda)^{\frac{n}{2}+1}}{\lambda(n+2)}\right)c_1 + \operatorname{BesselJ}\left(\frac{1}{n+2}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(x\lambda)^{\frac{n}{2}+1}}{\lambda(n+2)}\right)\right) + \left(\operatorname{BesselY}\right.\right.}{a^{\frac{3}{2}}\left(\operatorname{BesselY}\left(\frac{1}{n+2}, \frac{2\sqrt{a}\,\sqrt{b}\,\cosh(x\lambda)^{\frac{n}{2}+1}}{\lambda(n+2)}\right)c_1 + \operatorname{Besse}\right.}$$

✓ Solution by Mathematica

Time used: 1.376 (sec). Leaf size: 667

```
DSolve[y'[x]==a*Sinh[\[Lambda]*x]*y[x]^2+b*Sinh[\[Lambda]*x]*Cosh[\[Lambda]*x]^n,y[x],x,Incl
```

$$y(x) \to \frac{\sqrt{a}\sqrt{b}c_1\operatorname{Gamma}\left(\frac{n+1}{n+2}\right)\cosh^{\frac{n}{2}}(\lambda x)\operatorname{BesselJ}\left(\frac{n+1}{n+2}, \frac{2\sqrt{a}\sqrt{b}\cosh^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right) - \operatorname{sech}(\lambda x)\left(\operatorname{Gamma}\left(1+\frac{1}{n+2}\right)\left(\sqrt{a}\right.\right.}{}$$

$$y(x) \to \frac{\frac{\sqrt{a}\sqrt{b}\cosh^{\frac{n}{2}}(\lambda x)\left(\operatorname{BesselJ}\left(\frac{n+1}{n+2}, \frac{2\sqrt{a}\sqrt{b}\cosh^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right) - \operatorname{BesselJ}\left(-\frac{n+3}{n+2}, \frac{2\sqrt{a}\sqrt{b}\cosh^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right)\right)}{\operatorname{BesselJ}\left(-\frac{1}{n+2}, \frac{2\sqrt{a}\sqrt{b}\cosh^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right)} - \lambda\operatorname{sech}(\lambda x)}{2a}$$

5.15 problem 15

Internal problem ID [10463]
Internal file name [`OUTPUT/9410_Monday_June_06_2022_02_27_12_PM_40133814/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 \cosh(\lambda x)\, a = b \cosh(\lambda x) \sinh(\lambda x)^n}$$

5.15.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \cosh(\lambda x)\, a\, y^2 + b \cosh(\lambda x) \sinh(\lambda x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \cosh(\lambda x)\, a\, y^2 + b \cosh(\lambda x) \sinh(\lambda x)^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = b \cosh(\lambda x) \sinh(\lambda x)^n$, $f_1(x) = 0$ and $f_2(x) = a \cosh(\lambda x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{a \cosh(\lambda x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= a\lambda \sinh(\lambda x) \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= a^2 \cosh(\lambda x)^3\, b \sinh(\lambda x)^n \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$a \cosh(\lambda x)\, u''(x) - a\lambda \sinh(\lambda x)\, u'(x) + a^2 \cosh(\lambda x)^3\, b \sinh(\lambda x)^n\, u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{-\csc\left(\frac{\pi(n+3)}{2+n}\right) c_1 \operatorname{BesselI}\left(-\frac{1}{2+n}, 2\sqrt{-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) \pi \left(-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{1}{4+2n}} + c_2 \sinh(\lambda x) \operatorname{BesselI}\left(\frac{1}{2+n}, 2\right.}{(2+n)\,\Gamma\left(\frac{n+3}{2+n}\right)}$$

The above shows that

$$u'(x) = \frac{\left(\Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} c_2 \cosh(\lambda x)(2+n) \operatorname{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) + \cosh(\lambda x)\, c_2 \operatorname{Bess}\right.}{}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} c_2 \cosh(\lambda x)(2+n) \operatorname{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) + \cosh(\lambda x)\, c_2 \operatorname{Be}\right.}{a\cosh(\lambda x)\left(-\csc\left(\frac{\pi(n+3)}{2+n}\right) c_1 \operatorname{BesselI}\left(-\frac{1}{2+n}, 2\sqrt{-\frac{ab\sinh(\lambda x)^{2}}{\lambda^2(2+n)^2}}\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\lambda(2+n)\left(\Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} (2+n)\,\mathrm{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) - \operatorname{csch}(\lambda x)\,\pi c_3 \csc\left(\right.\right.}{\left(-\csc\left(\frac{\pi(n+3)}{2+n}\right) c_3\,\mathrm{BesselI}\left(-\frac{1}{2+n}, 2\sqrt{-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right)\pi\left(-\frac{ab\sinh(\lambda x)}{\lambda^2(2+n)}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\lambda(2+n)\left(\Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} (2+n)\,\mathrm{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) - \operatorname{csch}(\lambda x)\,\pi c_3 \csc\left(\right.\right.}{\left(-\csc\left(\frac{\pi(n+3)}{2+n}\right) c_3\,\mathrm{BesselI}\left(-\frac{1}{2+n}, 2\sqrt{-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right)\pi\left(-\frac{ab\sinh(\lambda x)}{\lambda^2(2+n)}\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\lambda(2+n)\left(\Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} (2+n)\,\mathrm{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) - \operatorname{csch}(\lambda x)\,\pi c_3 \csc\left(\right.\right.}{\left(-\csc\left(\frac{\pi(n+3)}{2+n}\right) c_3\,\mathrm{BesselI}\left(-\frac{1}{2+n}, 2\sqrt{-\frac{ab\sinh(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right)\pi\left(-\frac{ab\sinh(\lambda x)}{\lambda^2(2+n)}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = lambda*sinh(lambda*x)*(diff(y(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying an equivalence, under non-integer power transformations,
            to LODEs admitting Liouvillian solutions.
            -> Trying a Liouvillian solution using Kovacics algorithm
            <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Kummer
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
               <- hyper3 successful: received ODE is equivalent to the 0F1 ODE
            <- Kummer successful
         <- special function solution successful
         Change of variables used:
            [x = arccosh(t)/lambda]
         Linear ODE actually solved:
            4*a*b*(t^2-1)^(1/2*n)*t^3*u(t)+4*lambda^2*diff(u(t),t)+(4*lambda^2*t^3-4*lambda^
      <- change of variables successful
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 951

```
dsolve(diff(y(x),x)=a*cosh(lambda*x)*y(x)^2+b*cosh(lambda*x)*sinh(lambda*x)^n,y(x), singsol=
```

Expression too large to display

✓ Solution by Mathematica

Time used: 1.277 (sec). Leaf size: 633

```
DSolve[y'[x]==a*Cosh[\[Lambda]*x]*y[x]^2+b*Cosh[\[Lambda]*x]*Sinh[\[Lambda]*x]^n,y[x],x,Incl
```

$$y(x) \to \frac{\operatorname{csch}(\lambda x)\left(-\lambda \operatorname{Gamma}\left(1+\frac{1}{n+2}\right) \operatorname{BesselJ}\left(\frac{1}{n+2}, \frac{2\sqrt{a}\sqrt{b}\sinh^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right)+\sqrt{a}\sqrt{b}\sinh^{\frac{n}{2}+1}(\lambda x)\left(\operatorname{Gamma}\left(1+\right.\right.\right.}{}$$

$$y(x) \to \frac{\frac{\sqrt{a}\sqrt{b}\sinh^{\frac{n}{2}}(\lambda x)\left(\operatorname{BesselJ}\left(\frac{n+1}{n+2}, \frac{2\sqrt{a}\sqrt{b}\sinh^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right)-\operatorname{BesselJ}\left(-\frac{n+3}{n+2}, \frac{2\sqrt{a}\sqrt{b}\sinh^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right)\right)}{\operatorname{BesselJ}\left(-\frac{1}{n+2}, \frac{2\sqrt{a}\sqrt{b}\sinh^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right)}-\lambda\operatorname{csch}(\lambda x)}{2a}$$

5.16 problem 16

Internal problem ID [10464]
Internal file name [`OUTPUT/9411_Monday_June_06_2022_02_27_14_PM_53299429/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$
\boxed{(a\cosh(\lambda x)+b)\,y'-y^2-c\cosh(x\mu)\,y=-d^2+cd\cosh(x\mu)}
$$

5.16.1 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y)\\
&= \frac{y^2+c\cosh(x\mu)\,y-d^2+cd\cosh(x\mu)}{a\cosh(\lambda x)+b}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y'=\frac{cd\cosh(x\mu)}{a\cosh(\lambda x)+b}+\frac{c\cosh(x\mu)\,y}{a\cosh(\lambda x)+b}-\frac{d^2}{a\cosh(\lambda x)+b}+\frac{y^2}{a\cosh(\lambda x)+b}
$$

With Riccati ODE standard form

$$
y'=f_0(x)+f_1(x)y+f_2(x)y^2
$$

Shows that $f_0(x) = \frac{-d^2+cd\cosh(x\mu)}{a\cosh(\lambda x)+b}$, $f_1(x) = \frac{c\cosh(x\mu)}{a\cosh(\lambda x)+b}$ and $f_2(x) = \frac{1}{a\cosh(\lambda x)+b}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{a\cosh(\lambda x)+b}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{a\lambda \sinh(\lambda x)}{(a\cosh(\lambda x)+b)^2} \\
f_1 f_2 &= \frac{c\cosh(x\mu)}{(a\cosh(\lambda x)+b)^2} \\
f_2^2 f_0 &= \frac{-d^2 + cd\cosh(x\mu)}{(a\cosh(\lambda x)+b)^3}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{a\cosh(\lambda x)+b} - \left(-\frac{a\lambda\sinh(\lambda x)}{(a\cosh(\lambda x)+b)^2} + \frac{c\cosh(x\mu)}{(a\cosh(\lambda x)+b)^2}\right) u'(x) + \frac{(-d^2+cd\cosh(x\mu))\, u(x)}{(a\cosh(\lambda x)+b)^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 268

```
dsolve((a*cosh(lambda*x)+b)*diff(y(x),x)=y(x)^2+c*cosh(mu*x)*y(x)-d^2+c*d*cosh(mu*x),y(x), s
```

$$y(x) = \frac{-d\left(\int \frac{e^{\frac{c\left(\int \frac{\cosh(x\mu)}{a\cosh(x\lambda)+b}dx\right)\sqrt{a^2-b^2}\,\lambda-4d\arctan\left(\frac{(a-b)\tanh\left(\frac{x\lambda}{2}\right)}{\sqrt{a^2-b^2}}\right)}{\sqrt{a^2-b^2}\,\lambda}}}{a\cosh(x\lambda)+b}dx\right)+dc_1-e^{\frac{c\left(\int \frac{\cosh(x\mu)}{a\cosh(x\lambda)+b}dx\right)\sqrt{a^2-b^2}\,\lambda-4d\arctan\left(\frac{(a-b)\tanh\left(\frac{x\lambda}{2}\right)}{\sqrt{a^2-b}}\right.}{\sqrt{a^2-b^2}\,\lambda}}}{\int \frac{e^{\frac{c\left(\int \frac{\cosh(x\mu)}{a\cosh(x\lambda)+b}dx\right)\sqrt{a^2-b^2}\,\lambda-4d\arctan\left(\frac{(a-b)\tanh\left(\frac{x\lambda}{2}\right)}{\sqrt{a^2-b^2}}\right)}{\sqrt{a^2-b^2}\,\lambda}}}{a\cosh(x\lambda)+b}dx-c_1}$$

✓ Solution by Mathematica

Time used: 24.309 (sec). Leaf size: 289

```
DSolve[(a*Cosh[\[Lambda]*x]+b)*y'[x]==y[x]^2+c*Cosh[\[Mu]*x]*y[x]-d^2+c*d*Cosh[\[Mu]*x],y[x]
```

$$\text{Solve}\left[\int_1^x -\frac{\exp\left(-\int_1^{K[2]}\frac{2d-c\cosh(\mu K[1])}{b+a\cosh(\lambda K[1])}dK[1]\right)(-d+c\cosh(\mu K[2])+y(x))}{c\mu(b+a\cosh(\lambda K[2]))(d+y(x))}dK[2]\right.$$
$$+\int_1^{y(x)}\left(\frac{\exp\left(-\int_1^x\frac{2d-c\cosh(\mu K[1])}{b+a\cosh(\lambda K[1])}dK[1]\right)}{c\mu(d+K[3])^2}\right.$$
$$-\int_1^x\left(\frac{\exp\left(-\int_1^{K[2]}\frac{2d-c\cosh(\mu K[1])}{b+a\cosh(\lambda K[1])}dK[1]\right)(-d+c\cosh(\mu K[2])+K[3])}{c\mu(b+a\cosh(\lambda K[2]))(d+K[3])^2}-\frac{\exp\left(-\int_1^{K[2]}\frac{2d-c\cosh(\mu K[1])}{b+a\cosh(\lambda K[1])}dK[1\right.}{c\mu(b+a\cosh(\lambda K[2]))(d+K[3}\right.$$

5.17 problem 17

Internal problem ID [10465]
Internal file name [OUTPUT/9412_Monday_June_06_2022_02_28_26_PM_5814984/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-1. Equations with hyperbolic sine and cosine
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

[_Riccati]

$$\boxed{(a\cosh(\lambda x)+b)\left(y'-y^2\right)=-a\,\lambda^2\cosh(\lambda x)}$$

5.17.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\cosh(\lambda x)\,a\,y^2-a\,\lambda^2\cosh(\lambda x)+y^2b}{a\cosh(\lambda x)+b} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=-\frac{a\,\lambda^2\cosh(\lambda x)}{a\cosh(\lambda x)+b}+\frac{\cosh(\lambda x)\,a\,y^2}{a\cosh(\lambda x)+b}+\frac{y^2b}{a\cosh(\lambda x)+b}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=-\frac{a\,\lambda^2\cosh(\lambda x)}{a\cosh(\lambda x)+b}$, $f_1(x)=0$ and $f_2(x)=1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -\frac{a\,\lambda^2 \cosh(\lambda x)}{a\cosh(\lambda x) + b} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \frac{a\,\lambda^2 \cosh(\lambda x)\,u(x)}{a\cosh(\lambda x) + b} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = -2c_1\left(a\cosh\left(\frac{\lambda x}{2}\right)^2 - \frac{a}{2} + \frac{b}{2}\right) b \arctan\left(\frac{\tanh\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right) \\ + \sinh\left(\frac{\lambda x}{2}\right)\cosh\left(\frac{\lambda x}{2}\right)\sqrt{a^2-b^2}\,c_1 a + 2c_2\left(a\cosh\left(\frac{\lambda x}{2}\right)^2 - \frac{a}{2} + \frac{b}{2}\right) \end{aligned}$$

The above shows that

$$u'(x) = \frac{\left(-2c_1 a\cosh\left(\frac{\lambda x}{2}\right)\sinh\left(\frac{\lambda x}{2}\right) b\arctan\left(\frac{\tanh\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right)\sqrt{a^2-b^2} + 2\sinh\left(\frac{\lambda x}{2}\right) c_2 a\cosh\left(\frac{\lambda x}{2}\right)\sqrt{a^2-b^2} + c_1}{\sqrt{a^2-b^2}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-2c_1 a\cosh\left(\frac{\lambda x}{2}\right)\sinh\left(\frac{\lambda x}{2}\right) b\arctan\left(\frac{\tanh\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right)\sqrt{a^2-b^2} + 2\sinh\left(\frac{\lambda x}{2}\right) c_2 a\cosh\left(\frac{\lambda x}{2}\right)\sqrt{a^2-b^2} +}{\sqrt{a^2-b^2}\left(-2c_1\left(a\cosh\left(\frac{\lambda x}{2}\right)^2 - \frac{a}{2} + \frac{b}{2}\right) b\arctan\left(\frac{\tanh\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right) + \sinh\left(\frac{\lambda x}{2}\right)\cosh\left(\frac{\lambda x}{2}\right)\sqrt{a^2 -}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\lambda\left(-2c_3 a\cosh\left(\frac{\lambda x}{2}\right)\sinh\left(\frac{\lambda x}{2}\right) b\arctan\left(\frac{\tanh\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right)\sqrt{a^2-b^2} + 2\sinh\left(\frac{\lambda x}{2}\right)\cosh\left(\frac{\lambda x}{2}\right)\sqrt{a^2-b^2}\, a + c_3(}{\sqrt{a^2-b^2}\,\left(2c_3\left(a\cosh\left(\frac{\lambda x}{2}\right)^2 - \frac{a}{2} + \frac{b}{2}\right) b\arctan\left(\frac{\tanh\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right) - \sinh\left(\frac{\lambda x}{2}\right)\cosh\left(\frac{\lambda x}{2}\right)\sqrt{a^2-b}}$$

Summary

The solution(s) found are the following

$$y = \frac{\lambda\left(-2c_3 a\cosh\left(\frac{\lambda x}{2}\right)\sinh\left(\frac{\lambda x}{2}\right) b\arctan\left(\frac{\tanh\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right)\sqrt{a^2-b^2} + 2\sinh\left(\frac{\lambda x}{2}\right)\cosh\left(\frac{\lambda x}{2}\right)\sqrt{a^2-b^2}\, a + c_3(}{\sqrt{a^2-b^2}\,\left(2c_3\left(a\cosh\left(\frac{\lambda x}{2}\right)^2 - \frac{a}{2} + \frac{b}{2}\right) b\arctan\left(\frac{\tanh\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right) - \sinh\left(\frac{\lambda x}{2}\right)\cosh\left(\frac{\lambda x}{2}\right)\sqrt{a^2-b}} \tag{1}$$

Verification of solutions

$$y = \frac{\lambda\left(-2c_3 a\cosh\left(\frac{\lambda x}{2}\right)\sinh\left(\frac{\lambda x}{2}\right) b\arctan\left(\frac{\tanh\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right)\sqrt{a^2-b^2} + 2\sinh\left(\frac{\lambda x}{2}\right)\cosh\left(\frac{\lambda x}{2}\right)\sqrt{a^2-b^2}\, a + c_3(}{\sqrt{a^2-b^2}\,\left(2c_3\left(a\cosh\left(\frac{\lambda x}{2}\right)^2 - \frac{a}{2} + \frac{b}{2}\right) b\arctan\left(\frac{\tanh\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right) - \sinh\left(\frac{\lambda x}{2}\right)\cosh\left(\frac{\lambda x}{2}\right)\sqrt{a^2-b}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = a*lambda^2*cosh(lambda*x)*y(x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      <- linear_1 successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 204

```
dsolve((a*cosh(lambda*x)+b)*(diff(y(x),x)-y(x)^2)+a*lambda^2*cosh(lambda*x)=0,y(x), singsol=
```

$$y(x) = \frac{\lambda\left(-2\arctan\left(\frac{(a-b)\tanh\left(\frac{x\lambda}{2}\right)}{\sqrt{a^2-b^2}}\right)\sqrt{a^2-b^2}\,ab\cosh\left(\frac{x\lambda}{2}\right)\sinh\left(\frac{x\lambda}{2}\right)+2\sqrt{a^2-b^2}\,c_1a\cosh\left(\frac{x\lambda}{2}\right)\sinh\left(\frac{x\lambda}{2}\right)+(a}{\sqrt{a^2-b^2}\left(2\left(\cosh\left(\frac{x\lambda}{2}\right)^2a-\frac{a}{2}+\frac{b}{2}\right)b\arctan\left(\frac{(a-b)\tanh\left(\frac{x\lambda}{2}\right)}{\sqrt{a^2-b^2}}\right)-\sqrt{a^2-b^2}\,a\cosh\left(\frac{x\lambda}{2}\right)\sinh\left(\frac{x\lambda}{2}\right)-}$$

✓ Solution by Mathematica

Time used: 7.749 (sec). Leaf size: 246

```
DSolve[(a*Cosh[\[Lambda]*x]+b)*(y'[x]-y[x]^2)+a*\[Lambda]^2*Cosh[\[Lambda]*x]==0,y[x],x,Incl
```

$$y(x) \to$$

$$-\frac{\lambda\left(a\sinh(\lambda x)\left(2b\arctan\left(\frac{(b-a)\tanh\left(\frac{\lambda x}{2}\right)}{\sqrt{a^2-b^2}}\right)+c_1\lambda(a^2-b^2)^{3/2}\right)+a\sqrt{a^2-b^2}\cosh(\lambda x)+}{b\left(2b\arctan\left(\frac{(b-a)\tanh\left(\frac{\lambda x}{2}\right)}{\sqrt{a^2-b^2}}\right)+c_1\lambda\left(a^2-b^2\right)^{3/2}\right)+a\cosh(\lambda x)\left(2b\arctan\left(\frac{(b-a)\tanh\left(\frac{\lambda x}{2}\right)}{\sqrt{a^2-b^2}}\right)+c_1\lambda\left(a^2\right.\right.}$$

$$y(x) \to -\frac{a\lambda\sinh(\lambda x)}{a\cosh(\lambda x)+b}$$

6 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-2. Equations with hyperbolic tangent and cotangent.

6.1 problem 18

Internal problem ID [10466]
Internal file name [`OUTPUT/9413_Monday_June_06_2022_02_28_29_PM_65894832/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-2. Equations with hyperbolic tangent and cotangent.
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = \lambda a - a(a+\lambda)\tanh(\lambda x)^2}$$

6.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -a^2 \tanh(\lambda x)^2 - a\tanh(\lambda x)^2 \lambda + \lambda a + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a^2 \tanh(\lambda x)^2 - a\tanh(\lambda x)^2 \lambda + \lambda a + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2 \tanh(\lambda x)^2 - a\tanh(\lambda x)^2 \lambda + \lambda a$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= 0\\
f_1 f_2 &= 0\\
f_2^2 f_0 &= -a^2 \tanh(\lambda x)^2 - a \tanh(\lambda x)^2 \lambda + \lambda a
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-a^2 \tanh(\lambda x)^2 - a \tanh(\lambda x)^2 \lambda + \lambda a\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right) + c_2 \operatorname{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right)$$

The above shows that

$$\begin{aligned}
u'(x) = &- \operatorname{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right) c_1 \lambda\\
&- \operatorname{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right) c_2 \lambda\\
&+ \tanh(\lambda x) \left(c_1 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right)\right.\\
&\left.\qquad + c_2 \operatorname{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right)\right) (a+\lambda)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{-\operatorname{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right) c_1 \lambda - \operatorname{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right) c_2 \lambda + \tanh(\lambda x) \left(c_1 \operatorname{LegendreP}\right.}{c_1 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right) + c_2 \operatorname{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\operatorname{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right) c_3 \lambda + \operatorname{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right) \lambda - \tanh(\lambda x) \left(c_3 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda},\right.\right.}{c_3 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right) + \operatorname{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \tanh(\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\operatorname{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right) c_3\lambda + \operatorname{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right)\lambda - \tanh(\lambda x)\left(c_3 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda},\right.\right.}{c_3 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right) + \operatorname{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \tanh(\right.} \quad (1)$$

Verification of solutions

$$y = \frac{\operatorname{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right) c_3\lambda + \operatorname{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right)\lambda - \tanh(\lambda x)\left(c_3 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda},\right.\right.}{c_3 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \tanh(\lambda x)\right) + \operatorname{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \tanh(\right.}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a^2*tanh(lambda*x)^2+a*tanh(l
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               <- Legendre successful
            <- special function solution successful
               -> Trying to convert hypergeometric functions to elementary form...
               <- elementary form could result into a too large expression - returning speci
            <- Kovacics algorithm successful
         Change of variables used:
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 122

```
dsolve(diff(y(x),x)=y(x)^2+a*lambda-a*(a+lambda)*tanh(lambda*x)^2,y(x), singsol=all)
```

$$y(x) = \frac{\text{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \tanh(x\lambda)\right)\lambda + \text{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \tanh(x\lambda)\right)c_1\lambda - \tanh(x\lambda)\left(c_1 \text{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \right.\right.}{c_1 \text{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \tanh(x\lambda)\right) + \text{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \tanh(\right.}$$

✓ Solution by Mathematica

Time used: 8.574 (sec). Leaf size: 177

```
DSolve[y'[x]==y[x]^2+a*\[Lambda]-a*(a+\[Lambda])*Tanh[\[Lambda]*x]^2,y[x],x,IncludeSingularS
```

$$y(x) \to \frac{a\left(-\lambda\left(e^{2\lambda x}-1\right)\text{Hypergeometric2F1}\left(-\frac{2a}{\lambda}, -\frac{a}{\lambda}, 1-\frac{a}{\lambda}, -e^{2x\lambda}\right) - 2\lambda\left(e^{2\lambda x}+1\right)^{\frac{2a}{\lambda}+1} + ac_1\left(e^{2\lambda x}-1\right)\left(e^{2\lambda\right.\right.}{\left(e^{2\lambda x}+1\right)\left(-\lambda\,\text{Hypergeometric2F1}\left(-\frac{2a}{\lambda}, -\frac{a}{\lambda}, 1-\frac{a}{\lambda}, -e^{2x\lambda}\right) + ac_1\left(e^{2\lambda x}\right)^{a/\lambda}\right)}$$

$$y(x) \to \frac{a\left(e^{2\lambda x}-1\right)}{e^{2\lambda x}+1}$$

6.2 problem 19

Internal problem ID [10467]
Internal file name [`OUTPUT/9414_Monday_June_06_2022_02_28_31_PM_34807093/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-2. Equations with hyperbolic tangent and cotangent.
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = 3\lambda a - \lambda^2 - a(a+\lambda)\tanh(\lambda x)^2}$$

6.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -a^2\tanh(\lambda x)^2 - a\tanh(\lambda x)^2\lambda + 3\lambda a - \lambda^2 + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a^2\tanh(\lambda x)^2 - a\tanh(\lambda x)^2\lambda + 3\lambda a - \lambda^2 + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2\tanh(\lambda x)^2 - a\tanh(\lambda x)^2\lambda + 3\lambda a - \lambda^2$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= -a^2 \tanh(\lambda x)^2 - a \tanh(\lambda x)^2 \lambda + 3\lambda a - \lambda^2
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-a^2 \tanh(\lambda x)^2 - a \tanh(\lambda x)^2 \lambda + 3\lambda a - \lambda^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \,\mathrm{LegendreP}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right) + c_2 \,\mathrm{LegendreQ}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right)$$

The above shows that

$$\begin{aligned}
u'(x) = &-2 \,\mathrm{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right) c_1 \lambda \\
&- 2 \,\mathrm{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right) c_2 \lambda \\
&+ \tanh(\lambda x) \left(c_1 \,\mathrm{LegendreP}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right) \right. \\
&\left. + c_2 \,\mathrm{LegendreQ}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right)\right) (a+\lambda)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{-2 \,\mathrm{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right) c_1 \lambda - 2 \,\mathrm{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right) c_2 \lambda + \tanh(\lambda x) \left(c_1 \,\mathrm{Leg}\right.}{c_1 \,\mathrm{LegendreP}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right) + c_2 \,\mathrm{LegendreQ}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2\,\mathrm{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right) c_3\lambda + 2\,\mathrm{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right)\lambda - \tanh(\lambda x)\left(c_3\,\mathrm{Legendre}\right.}{c_3\,\mathrm{LegendreP}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right) + \mathrm{LegendreQ}\left(\frac{a}{\lambda}, \frac{a-}{\lambda}\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{2\,\mathrm{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right) c_3\lambda + 2\,\mathrm{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right)\lambda - \tanh(\lambda x)\left(c_3\,\mathrm{Legendre}\right.}{c_3\,\mathrm{LegendreP}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right) + \mathrm{LegendreQ}\left(\frac{a}{\lambda}, \frac{a-}{\lambda}\right.} \tag{1}$$

Verification of solutions

$$y = \frac{2\,\mathrm{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right) c_3\lambda + 2\,\mathrm{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right)\lambda - \tanh(\lambda x)\left(c_3\,\mathrm{Legendre}\right.}{c_3\,\mathrm{LegendreP}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(\lambda x)\right) + \mathrm{LegendreQ}\left(\frac{a}{\lambda}, \frac{a-}{\lambda}\right.}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a^2*tanh(lambda*x)^2+a*tanh(l
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               <- Legendre successful
            <- special function solution successful
               -> Trying to convert hypergeometric functions to elementary form...
               <- elementary form could result into a too large expression - returning speci
            <- Kovacics algorithm successful
         Change of variables used:
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 148

```
dsolve(diff(y(x),x)=y(x)^2+3*a*lambda-lambda^2-a*(a+lambda)*tanh(lambda*x)^2,y(x), singsol=a
```

$$y(x) = \frac{2\,\mathrm{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(x\lambda)\right)\lambda + 2\,\mathrm{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(x\lambda)\right)c_1\lambda - \tanh(x\lambda)\left(c_1\,\mathrm{Legendre}\right.}{c_1\,\mathrm{LegendreQ}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \tanh(x\lambda)\right) + \mathrm{LegendreP}\left(\frac{a}{\lambda}, \frac{a-}{\lambda}\right.}$$

✓ Solution by Mathematica

Time used: 12.804 (sec). Leaf size: 631

```
DSolve[y'[x]==y[x]^2+3*a*\[Lambda]-\[Lambda]^2-a*(a+\[Lambda])*Tanh[\[Lambda]*x]^2,y[x],x,In
```

$$y(x) \to \frac{-\lambda(a-2\lambda)\left(e^{2\lambda x}-1\right)\left(e^{2\lambda x}+1\right)^{\frac{2a}{\lambda}}\left(\frac{1}{e^{2\lambda x}-1}+1\right)^{a/\lambda}\left(a\left(4e^{2\lambda x}+e^{4\lambda x}-1\right)+\lambda-\lambda e^{4\lambda x}\right)\mathrm{AppellF1}\left(1-\right.}{}$$

$$y(x) \to \frac{a\left(e^{2\lambda x}-1\right)^2-\lambda\left(e^{2\lambda x}+1\right)^2}{e^{4\lambda x}-1}$$

6.3 problem 20

Internal problem ID [10468]
Internal file name [OUTPUT/9415_Monday_June_06_2022_02_28_33_PM_45959373/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-2. Equations with hyperbolic tangent and cotangent.
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 - ax\tanh(bx)^m y = a\tanh(bx)^m}$$

6.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + ax\tanh(bx)^m y + a\tanh(bx)^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + ax\tanh(bx)^m y + a\tanh(bx)^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\tanh(bx)^m$, $f_1(x) = \tanh(bx)^m ax$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= \tanh(bx)^m ax \\ f_2^2 f_0 &= a\tanh(bx)^m \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \tanh(bx)^m axu'(x) + a\tanh(bx)^m u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x\left(c_1\left(\int e^{\int \frac{\tanh(bx)^m a x^2 - 2}{x}dx}dx\right) + c_2\right)$$

The above shows that

$$u'(x) = c_1\left(\int e^{\int \frac{\tanh(bx)^m a x^2 - 2}{x}dx}dx\right) + c_2 + xc_1 e^{\int \frac{\tanh(bx)^m a x^2 - 2}{x}dx}$$

Using the above in (1) gives the solution

$$y = -\frac{c_1\left(\int e^{\int \frac{\tanh(bx)^m a x^2 - 2}{x}dx}dx\right) + c_2 + xc_1 e^{\int \frac{\tanh(bx)^m a x^2 - 2}{x}dx}}{x\left(c_1\left(\int e^{\int \frac{\tanh(bx)^m a x^2 - 2}{x}dx}dx\right) + c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3\left(\int e^{\int \frac{\tanh(bx)^m a x^2 - 2}{x}dx}dx\right) - 1 - xc_3 e^{\int \frac{\tanh(bx)^m a x^2 - 2}{x}dx}}{x\left(c_3\left(\int e^{\int \frac{\tanh(bx)^m a x^2 - 2}{x}dx}dx\right) + 1\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{-c_3\left(\int e^{\int \frac{\tanh(bx)^m a\,x^2-2}{x}dx}dx\right)-1-xc_3 e^{\int \frac{\tanh(bx)^m a\,x^2-2}{x}dx}}{x\left(c_3\left(\int e^{\int \frac{\tanh(bx)^m a\,x^2-2}{x}dx}dx\right)+1\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-c_3\left(\int e^{\int \frac{\tanh(bx)^m a\,x^2-2}{x}dx}dx\right)-1-xc_3 e^{\int \frac{\tanh(bx)^m a\,x^2-2}{x}dx}}{x\left(c_3\left(\int e^{\int \frac{\tanh(bx)^m a\,x^2-2}{x}dx}dx\right)+1\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 85

```
dsolve(diff(y(x),x)=y(x)^2+a*x*tanh(b*x)^m*y(x)+a*tanh(b*x)^m,y(x), singsol=all)
```

$$y(x) = \frac{-e^{\int \frac{a\tanh(bx)^m x^2-2}{x}dx}x-\left(\int e^{\int \frac{a\tanh(bx)^m x^2-2}{x}dx}dx\right)+c_1}{\left(-c_1+\int e^{\int \frac{a\tanh(bx)^m x^2-2}{x}dx}dx\right)x}$$

✓ Solution by Mathematica

Time used: 12.331 (sec). Leaf size: 126

```
DSolve[y'[x]==y[x]^2+a*x*Tanh[b*x]^m*y[x]+a*Tanh[b*x]^m,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\frac{\exp\left(-\int_1^x -aK[1]\tanh^m(bK[1])dK[1]\right) + x\int_1^x \frac{\exp\left(-\int_1^{K[2]} -aK[1]\tanh^m(bK[1])dK[1]\right)}{K[2]^2}dK[2] + c_1 x}{x^2\left(\int_1^x \frac{\exp\left(-\int_1^{K[2]} -aK[1]\tanh^m(bK[1])dK[1]\right)}{K[2]^2}dK[2] + c_1\right)}$$

$$y(x) \to -\frac{1}{x}$$

6.4 problem 21

Internal problem ID [10469]
Internal file name [`OUTPUT/9416_Monday_June_06_2022_02_28_36_PM_97894842/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-2. Equations with hyperbolic tangent and cotangent.
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$\boxed{(a\tanh(\lambda x)+b)\,y'-y^2-c\tanh(x\mu)\,y=-d^2+cd\tanh(x\mu)}$$

6.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= \frac{y^2+c\tanh(x\mu)\,y-d^2+cd\tanh(x\mu)}{a\tanh(\lambda x)+b}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=\frac{cd\tanh(x\mu)}{a\tanh(\lambda x)+b}+\frac{c\tanh(x\mu)\,y}{a\tanh(\lambda x)+b}-\frac{d^2}{a\tanh(\lambda x)+b}+\frac{y^2}{a\tanh(\lambda x)+b}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x) = \frac{-d^2+cd\tanh(x\mu)}{a\tanh(\lambda x)+b}$, $f_1(x) = \frac{c\tanh(x\mu)}{a\tanh(\lambda x)+b}$ and $f_2(x) = \frac{1}{a\tanh(\lambda x)+b}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{a\tanh(\lambda x)+b}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{a\lambda\left(1-\tanh(\lambda x)^2\right)}{(a\tanh(\lambda x)+b)^2} \\ f_1 f_2 &= \frac{c\tanh(x\mu)}{(a\tanh(\lambda x)+b)^2} \\ f_2^2 f_0 &= \frac{-d^2+cd\tanh(x\mu)}{(a\tanh(\lambda x)+b)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{a\tanh(\lambda x)+b} - \left(-\frac{a\lambda\left(1-\tanh(\lambda x)^2\right)}{(a\tanh(\lambda x)+b)^2} + \frac{c\tanh(x\mu)}{(a\tanh(\lambda x)+b)^2}\right) u'(x) + \frac{(-d^2+cd\tanh(x\mu))\, u(x)}{(a\tanh(\lambda x)+b)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 302

```
dsolve((a*tanh(lambda*x)+b)*diff(y(x),x)=y(x)^2+c*tanh(mu*x)*y(x)-d^2+c*d*tanh(mu*x),y(x), s
```

$$y(x) = \frac{-e^{c\left(\int \frac{\tanh(x\mu)}{a\tanh(x\lambda)+b}dx\right)}(\tanh(x\lambda)+1)^{\frac{d}{\lambda(a-b)}}(\tanh(x\lambda)-1)^{\frac{d}{\lambda(a+b)}}(a\tanh(x\lambda)+b)^{-\frac{2ad}{\lambda(a^2-b^2)}} - d\left(\int (a\tanh(\right.}{\int (a\tanh(x\lambda)+b)^{\frac{(-a^2+b^2)\lambda-2ad}{\lambda(a^2-b^2)}}(\tanh(x\lambda)-1)^{\frac{d}{\lambda(a+b)}}(\mathrm{ta}}$$

✓ Solution by Mathematica

Time used: 163.692 (sec). Leaf size: 800

```
DSolve[(a*Tanh[\[Lambda]*x]+b)*y'[x]==y[x]^2+c*Tanh[\[Mu]*x]*y[x]-d^2+c*d*Tanh[\[Mu]*x],y[x]
```

$$\mathrm{Solve}\left[\int_1^x \frac{e^{-\int_1^{K[2]}\frac{\mathrm{sech}(\mu K[1])(2d\cosh(\lambda K[1]-\mu K[1])+2d\cosh(\lambda K[1]+\mu K[1])+c\sinh(\lambda K[1]-\mu K[1])-c\sinh(\lambda K[1]+\mu K[1]))}{2(b\cosh(\lambda K[1])+a\sinh(\lambda K[1]))}dK[1]}(d\cosh(\lambda K[2]}{c\mu(b\cosh(\lambda K[2]-\mu K[2])+b\,\mathrm{co}}\right.$$

$$+\int_1^{y(x)}\left(\frac{e^{-\int_1^{x}\frac{\mathrm{sech}(\mu K[1])(2d\cosh(\lambda K[1]-\mu K[1])+2d\cosh(\lambda K[1]+\mu K[1])+c\sinh(\lambda K[1]-\mu K[1])-c\sinh(\lambda K[1]+\mu K[1]))}{2(b\cosh(\lambda K[1])+a\sinh(\lambda K[1]))}dK[1]}}{c\mu(d+K[3])^2}\right.$$

$$-\int_1^x\left(\frac{e^{-\int_1^{K[2]}\frac{\mathrm{sech}(\mu K[1])(2d\cosh(\lambda K[1]-\mu K[1])+2d\cosh(\lambda K[1]+\mu K[1])+c\sinh(\lambda K[1]-\mu K[1])-c\sinh(\lambda K[1]+\mu K[1]))}{2(b\cosh(\lambda K[1])+a\sinh(\lambda K[1]))}dK[1]}(-\cosh(\lambda K[2]-}{c\mu(d+K[3])(b\cosh(\lambda K[2]-\mu K[2])+b\cosh(\lambda K[2]+\mu K[2])+a\sinh(\lambda K[2]-\mu K[2])+}\right.$$

6.5 problem 22

Internal problem ID [10470]
Internal file name [`OUTPUT/9417_Monday_June_06_2022_02_29_55_PM_36680957/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-2. Equations with hyperbolic tangent and cotangent.
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 = \lambda a - a(a + \lambda) \coth(\lambda x)^2}$$

6.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -a^2 \coth(\lambda x)^2 - a \coth(\lambda x)^2 \lambda + \lambda a + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a^2 \coth(\lambda x)^2 - a \coth(\lambda x)^2 \lambda + \lambda a + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2 \coth(\lambda x)^2 - a \coth(\lambda x)^2 \lambda + \lambda a$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= -a^2\coth(\lambda x)^2 - a\coth(\lambda x)^2\lambda + \lambda a \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-a^2\coth(\lambda x)^2 - a\coth(\lambda x)^2\lambda + \lambda a\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right) + c_2 \operatorname{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right)$$

The above shows that

$$\begin{aligned} u'(x) = &-\operatorname{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right)c_1\lambda - \operatorname{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right)c_2\lambda \\ &+ \coth(\lambda x)\left(c_1 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right)\right. \\ &\left.+ c_2 \operatorname{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right)\right)(a+\lambda) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{-\operatorname{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right)c_1\lambda - \operatorname{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right)c_2\lambda + \coth(\lambda x)\left(c_1 \operatorname{LegendreP}\right.}{c_1 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right) + c_2 \operatorname{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda},\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\operatorname{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right)c_3\lambda + \operatorname{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right)\lambda - \coth(\lambda x)\left(c_3 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda},\right.\right.}{c_3 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right) + \operatorname{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \coth(\right.}$$

Summary
The solution(s) found are the following

$$y = \frac{\operatorname{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right) c_3\lambda + \operatorname{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right)\lambda - \coth(\lambda x)\left(c_3 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, c\right.\right.}{c_3 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right) + \operatorname{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \coth(\lambda\right.} \quad (1)$$

Verification of solutions

$$y = \frac{\operatorname{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right) c_3\lambda + \operatorname{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right)\lambda - \coth(\lambda x)\left(c_3 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, c\right.\right.}{c_3 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \coth(\lambda x)\right) + \operatorname{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \coth(\lambda\right.}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a^2*coth(lambda*x)^2+a*coth(l
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               <- Legendre successful
            <- special function solution successful
               -> Trying to convert hypergeometric functions to elementary form...
               <- elementary form could result into a too large expression - returning speci
            <- Kovacics algorithm successful
         Change of variables used:
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 122

```
dsolve(diff(y(x),x)=y(x)^2+a*lambda-a*(a+lambda)*coth(lambda*x)^2,y(x), singsol=all)
```

$$y(x) = \frac{\text{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \coth(x\lambda)\right)\lambda + \text{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a}{\lambda}, \coth(x\lambda)\right)c_1\lambda - \coth(x\lambda)\left(c_1 \text{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \right.\right.}{c_1 \text{LegendreQ}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \coth(x\lambda)\right) + \text{LegendreP}\left(\frac{a}{\lambda}, \frac{a}{\lambda}, \coth(\right.}$$

✓ Solution by Mathematica

Time used: 8.402 (sec). Leaf size: 175

```
DSolve[y'[x]==y[x]^2+a*\[Lambda]-a*(a+\[Lambda])*Coth[\[Lambda]*x]^2,y[x],x,IncludeSingularS
```

$$y(x) \to \frac{a\left(-\lambda(e^{2\lambda x}+1)\,\text{Hypergeometric2F1}\left(-\frac{2a}{\lambda}, -\frac{a}{\lambda}, 1-\frac{a}{\lambda}, e^{2x\lambda}\right) + 2\lambda\left(1-e^{2\lambda x}\right)^{\frac{2a}{\lambda}+1} + ac_1\left(e^{2\lambda x}+1\right)\left(e^{2\lambda x}\right)\right.}{\left(e^{2\lambda x}-1\right)\left(-\lambda\,\text{Hypergeometric2F1}\left(-\frac{2a}{\lambda}, -\frac{a}{\lambda}, 1-\frac{a}{\lambda}, e^{2x\lambda}\right) + ac_1\left(e^{2\lambda x}\right)^{a/\lambda}\right)}$$

$$y(x) \to \frac{a\left(e^{2\lambda x}+1\right)}{e^{2\lambda x}-1}$$

6.6 problem 23

Internal problem ID [10471]
Internal file name [OUTPUT/9418_Monday_June_06_2022_02_29_57_PM_21366693/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-2. Equations with hyperbolic tangent and cotangent.
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = 3\lambda a - \lambda^2 - a(a+\lambda)\coth(\lambda x)^2}$$

6.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -a^2\coth(\lambda x)^2 - a\coth(\lambda x)^2\lambda + 3\lambda a - \lambda^2 + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a^2\coth(\lambda x)^2 - a\coth(\lambda x)^2\lambda + 3\lambda a - \lambda^2 + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2\coth(\lambda x)^2 - a\coth(\lambda x)^2\lambda + 3\lambda a - \lambda^2$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -a^2 \coth(\lambda x)^2 - a \coth(\lambda x)^2 \lambda + 3\lambda a - \lambda^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-a^2 \coth(\lambda x)^2 - a \coth(\lambda x)^2 \lambda + 3\lambda a - \lambda^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) + c_2 \operatorname{LegendreQ}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right)$$

The above shows that

$$\begin{aligned} u'(x) = &-2 \operatorname{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) c_1 \lambda \\ &- 2 \operatorname{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) c_2 \lambda \\ &+ \coth(\lambda x) \left(c_1 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) \right. \\ &\left. + c_2 \operatorname{LegendreQ}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right)\right) (a+\lambda) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{-2 \operatorname{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) c_1 \lambda - 2 \operatorname{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) c_2 \lambda + \coth(\lambda x) \left(c_1 \operatorname{Lege}\right.}{c_1 \operatorname{LegendreP}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) + c_2 \operatorname{LegendreQ}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2\,\text{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) c_3 \lambda + 2\,\text{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) \lambda - \coth(\lambda x) \left(c_3\,\text{LegendreP}\right.}{c_3\,\text{LegendreP}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) + \text{LegendreQ}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{2\,\text{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) c_3 \lambda + 2\,\text{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) \lambda - \coth(\lambda x) \left(c_3\,\text{LegendreP}\right.}{c_3\,\text{LegendreP}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) + \text{LegendreQ}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}\right.} \tag{1}$$

Verification of solutions

$$y = \frac{2\,\text{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) c_3 \lambda + 2\,\text{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) \lambda - \coth(\lambda x) \left(c_3\,\text{LegendreP}\right.}{c_3\,\text{LegendreP}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(\lambda x)\right) + \text{LegendreQ}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}\right.}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a^2*coth(lambda*x)^2+a*coth(l
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               <- Legendre successful
            <- special function solution successful
               -> Trying to convert hypergeometric functions to elementary form...
               <- elementary form could result into a too large expression - returning speci
            <- Kovacics algorithm successful
         Change of variables used:
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 148

```
dsolve(diff(y(x),x)=y(x)^2-lambda^2+3*a*lambda-a*(a+lambda)*coth(lambda*x)^2,y(x), singsol=a
```

$$y(x) = \frac{2\,\mathrm{LegendreP}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(x\lambda)\right)\lambda + 2\,\mathrm{LegendreQ}\left(\frac{a+\lambda}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(x\lambda)\right)c_1\lambda - \coth(x\lambda)\left(c_1\,\mathrm{LegendreQ}\right.}{c_1\,\mathrm{LegendreQ}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}, \coth(x\lambda)\right) + \mathrm{LegendreP}\left(\frac{a}{\lambda}, \frac{a-\lambda}{\lambda}\right.}$$

✓ Solution by Mathematica

Time used: 14.312 (sec). Leaf size: 659

```
DSolve[y'[x]==y[x]^2-\[Lambda]^2+3*a*\[Lambda]-a*(a+\[Lambda])*Coth[\[Lambda]*x]^2,y[x],x,In
```

$$y(x) \to \frac{-\lambda(a-2\lambda)\left(e^{2\lambda x}+1\right)\left(1-e^{2\lambda x}\right)^{\frac{2a}{\lambda}}\left(\frac{e^{2\lambda x}}{e^{2\lambda x}+1}\right)^{a/\lambda}\left(a\left(-4e^{2\lambda x}+e^{4\lambda x}-1\right)+\lambda-\lambda e^{4\lambda x}\right)\mathrm{AppellF1}\left(1-\frac{a}{\lambda},\right.}{}$$

$$y(x) \to \frac{a\left(e^{2\lambda x}+1\right)^2-\lambda\left(e^{2\lambda x}-1\right)^2}{e^{4\lambda x}-1}$$

6.7 problem 24

Internal problem ID [10472]
Internal file name [OUTPUT/9419_Monday_June_06_2022_02_30_00_PM_88007401/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-2. Equations with hyperbolic tangent and cotangent.
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - ax \coth (bx)^m y = a \coth (bx)^m}$$

6.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + ax \coth (bx)^m y + a \coth (bx)^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + ax \coth (bx)^m y + a \coth (bx)^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a \coth (bx)^m$, $f_1(x) = a \coth (bx)^m x$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= a \coth(bx)^m x \\ f_2^2 f_0 &= a \coth(bx)^m \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - a \coth(bx)^m x u'(x) + a \coth(bx)^m u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x \left(\left(\int e^{\int \frac{\coth(bx)^m a\,x^2 - 2}{x} dx} dx \right) c_1 + c_2 \right)$$

The above shows that

$$u'(x) = \left(\int e^{\int \frac{\coth(bx)^m a\,x^2 - 2}{x} dx} dx \right) c_1 + c_2 + x\,e^{\int \frac{\coth(bx)^m a\,x^2 - 2}{x} dx} c_1$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\int e^{\int \frac{\coth(bx)^m a\,x^2 - 2}{x} dx} dx \right) c_1 + c_2 + x\,e^{\int \frac{\coth(bx)^m a\,x^2 - 2}{x} dx} c_1}{x \left(\left(\int e^{\int \frac{\coth(bx)^m a\,x^2 - 2}{x} dx} dx \right) c_1 + c_2 \right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\left(\int e^{\int \frac{\coth(bx)^m a\,x^2 - 2}{x} dx} dx \right) c_3 - 1 - x\,e^{\int \frac{\coth(bx)^m a\,x^2 - 2}{x} dx} c_3}{x \left(\left(\int e^{\int \frac{\coth(bx)^m a\,x^2 - 2}{x} dx} dx \right) c_3 + 1 \right)}$$

Summary
The solution(s) found are the following

$$y = \frac{-\left(\int e^{\int \frac{\coth(bx)^m a\, x^2-2}{x}dx}dx\right) c_3 - 1 - x\, e^{\int \frac{\coth(bx)^m a\, x^2-2}{x}dx} c_3}{x\left(\left(\int e^{\int \frac{\coth(bx)^m a\, x^2-2}{x}dx}dx\right) c_3 + 1\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-\left(\int e^{\int \frac{\coth(bx)^m a\, x^2-2}{x}dx}dx\right) c_3 - 1 - x\, e^{\int \frac{\coth(bx)^m a\, x^2-2}{x}dx} c_3}{x\left(\left(\int e^{\int \frac{\coth(bx)^m a\, x^2-2}{x}dx}dx\right) c_3 + 1\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 85

```
dsolve(diff(y(x),x)=y(x)^2+a*x*coth(b*x)^m*y(x)+a*coth(b*x)^m,y(x), singsol=all)
```

$$y(x) = \frac{-e^{\int \frac{a \coth(bx)^m x^2-2}{x}dx} x - \left(\int e^{\int \frac{a \coth(bx)^m x^2-2}{x}dx}dx\right) + c_1}{\left(-c_1 + \int e^{\int \frac{a \coth(bx)^m x^2-2}{x}dx}dx\right) x}$$

✓ Solution by Mathematica

Time used: 11.817 (sec). Leaf size: 126

```
DSolve[y'[x]==y[x]^2+a*x*Coth[b*x]^m*y[x]+a*Coth[b*x]^m,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\frac{\exp\left(-\int_1^x -a\coth^m(bK[1])K[1]dK[1]\right) + x\int_1^x \frac{\exp\left(-\int_1^{K[2]} -a\coth^m(bK[1])K[1]dK[1]\right)}{K[2]^2}dK[2] + c_1x}{x^2\left(\int_1^x \frac{\exp\left(-\int_1^{K[2]} -a\coth^m(bK[1])K[1]dK[1]\right)}{K[2]^2}dK[2] + c_1\right)}$$

$$y(x) \to -\frac{1}{x}$$

6.8 problem 25

Internal problem ID [10473]
Internal file name [`OUTPUT/9420_Monday_June_06_2022_02_30_03_PM_79434428/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-2. Equations with hyperbolic tangent and cotangent.
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$\boxed{(a\coth(\lambda x)+b)\,y'-y^2-c\coth(x\mu)\,y=-d^2+cd\coth(x\mu)}$$

6.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= \frac{y^2+c\coth(x\mu)\,y-d^2+cd\coth(x\mu)}{a\coth(\lambda x)+b}
\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=\frac{cd\coth(x\mu)}{a\coth(\lambda x)+b}+\frac{c\coth(x\mu)\,y}{a\coth(\lambda x)+b}-\frac{d^2}{a\coth(\lambda x)+b}+\frac{y^2}{a\coth(\lambda x)+b}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x) = \frac{-d^2+cd\coth(x\mu)}{a\coth(\lambda x)+b}$, $f_1(x) = \frac{c\coth(x\mu)}{a\coth(\lambda x)+b}$ and $f_2(x) = \frac{1}{a\coth(\lambda x)+b}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{a\coth(\lambda x)+b}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{a\lambda\left(1-\coth(\lambda x)^2\right)}{(a\coth(\lambda x)+b)^2} \\
f_1 f_2 &= \frac{c\coth(x\mu)}{(a\coth(\lambda x)+b)^2} \\
f_2^2 f_0 &= \frac{-d^2+cd\coth(x\mu)}{(a\coth(\lambda x)+b)^3}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{a\coth(\lambda x)+b} - \left(-\frac{a\lambda\left(1-\coth(\lambda x)^2\right)}{(a\coth(\lambda x)+b)^2} + \frac{c\coth(x\mu)}{(a\coth(\lambda x)+b)^2}\right) u'(x) + \frac{(-d^2+cd\coth(x\mu))\, u(x)}{(a\coth(\lambda x)+b)^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 302

```
dsolve((a*coth(lambda*x)+b)*diff(y(x),x)=y(x)^2+c*coth(mu*x)*y(x)-d^2+c*d*coth(mu*x),y(x), s
```

$$y(x) = \frac{-e^{c\left(\int \frac{\coth(x\mu)}{a\coth(x\lambda)+b}dx\right)}(\coth(x\lambda)+1)^{\frac{d}{\lambda(a-b)}}(\coth(x\lambda)-1)^{\frac{d}{\lambda(a+b)}}(a\coth(x\lambda)+b)^{-\frac{2ad}{\lambda(a^2-b^2)}} - d\left(\int (a\coth(x\right.}{\int (a\coth(x\lambda)+b)^{\frac{(-a^2+b^2)\lambda-2ad}{\lambda(a^2-b^2)}}(\coth(x\lambda)-1)^{\frac{d}{\lambda(a+b)}}(\cot}$$

✓ Solution by Mathematica

Time used: 153.106 (sec). Leaf size: 808

```
DSolve[(a*Coth[\[Lambda]*x]+b)*y'[x]==y[x]^2+c*Coth[\[Mu]*x]*y[x]-d^2+c*d*Coth[\[Mu]*x],y[x]
```

$$\text{Solve}\Bigg[\int_1^x - \frac{e^{-\int_1^{K[2]} \frac{\text{csch}(\mu K[1])(-2d\cosh(\lambda K[1]-\mu K[1])+2d\cosh(\lambda K[1]+\mu K[1])-c\sinh(\lambda K[1]-\mu K[1])-c\sinh(\lambda K[1]+\mu K[1]))}{2(a\cosh(\lambda K[1])+b\sinh(\lambda K[1]))}dK[1]}(d\cosh(\lambda K[2]-\mu K[2}{c\mu(b\cosh(\lambda K[2]-\mu K[2])-b\cosh(\lambda K[}$$

$$+\int_1^{y(x)}\left(-\int_1^x\left(\frac{e^{-\int_1^{K[2]} \frac{\text{csch}(\mu K[1])(-2d\cosh(\lambda K[1]-\mu K[1])+2d\cosh(\lambda K[1]+\mu K[1])-c\sinh(\lambda K[1]-\mu K[1])-c\sinh(\lambda K[1]+\mu K[1]))}{2(a\cosh(\lambda K[1])+b\sinh(\lambda K[1]))}dK[1]}(d\cos}{c\mu(d+K[3])^2(b\cosh(\lambda K}\right.\right.$$

$$-\frac{e^{-\int_1^x \frac{\text{csch}(\mu K[1])(-2d\cosh(\lambda K[1]-\mu K[1])+2d\cosh(\lambda K[1]+\mu K[1])-c\sinh(\lambda K[1]-\mu K[1])-c\sinh(\lambda K[1]+\mu K[1]))}{2(a\cosh(\lambda K[1])+b\sinh(\lambda K[1]))}dK[1]}}{c\mu(d+K[3])^2}\Bigg)dK[3]=c_1, y(x)\Bigg]$$

6.9 problem 26

Internal problem ID [10474]
Internal file name [OUTPUT/9421_Monday_June_06_2022_02_31_39_PM_41931427/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-2. Equations with hyperbolic tangent and cotangent.
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = -2\tanh(\lambda x)^2 \lambda^2 - 2\lambda^2 \coth(\lambda x)^2}$$

6.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 - 2\tanh(\lambda x)^2 \lambda^2 - 2\lambda^2 \coth(\lambda x)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - 2\tanh(\lambda x)^2 \lambda^2 - 2\lambda^2 \coth(\lambda x)^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -2\tanh(\lambda x)^2 \lambda^2 - 2\lambda^2 \coth(\lambda x)^2$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -2\tanh(\lambda x)^2 \lambda^2 - 2\lambda^2 \coth(\lambda x)^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-2\tanh(\lambda x)^2 \lambda^2 - 2\lambda^2 \coth(\lambda x)^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = \operatorname{sech}(\lambda x) \operatorname{csch}(\lambda x) \big(&c_2 \ln(\coth(\lambda x) - 1) - c_2 \ln(\coth(\lambda x) + 1) + c_1 \\ &+ 2\sinh(\lambda x)\cosh(\lambda x)\left(2\cosh(\lambda x)^2 - 1\right) c_2\big) \end{aligned}$$

The above shows that

$$\begin{aligned} u'(x) = -2\lambda \operatorname{sech}(\lambda x)^2 \operatorname{csch}(\lambda x)^2 \Bigg(&c_2\left(\cosh(\lambda x)^2 - \frac{1}{2}\right)\ln(\coth(\lambda x) - 1) \\ &+ c_2\left(-\cosh(\lambda x)^2 + \frac{1}{2}\right)\ln(\coth(\lambda x) + 1) - 4\cosh(\lambda x)^5 \sinh(\lambda x) c_2 \\ &+ 4c_2 \cosh(\lambda x)^3 \sinh(\lambda x) + \cosh(\lambda x)^2 c_1 + c_2 \cosh(\lambda x)\sinh(\lambda x) - \frac{c_1}{2}\Bigg) \end{aligned}$$

Using the above in (1) gives the solution

$$y = \frac{2\lambda \operatorname{sech}(\lambda x) \operatorname{csch}(\lambda x) \left(c_2\left(\cosh(\lambda x)^2 - \frac{1}{2}\right)\ln(\coth(\lambda x) - 1) + c_2\left(-\cosh(\lambda x)^2 + \frac{1}{2}\right)\ln(\coth(\lambda x) + 1)\right.}{c_2 \ln(\coth(\lambda x) - 1) - c_2 \ln(\coth(\lambda x) + 1) + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\operatorname{sech}(\lambda x) \operatorname{csch}(\lambda x) \left(8\cosh(\lambda x)^5 \sinh(\lambda x) - 8\sinh(\lambda x)\cosh(\lambda x)^3 - 2\cosh(\lambda x)^2 \ln(\coth(\lambda x) - 1) + 2\right.}{-4\sinh(\lambda x)\cosh(\lambda x)^3 + 2\cosh(\lambda x) \mathrm{s}}$$

Summary

The solution(s) found are the following

$$y = \frac{\operatorname{sech}(\lambda x)\operatorname{csch}(\lambda x)\left(8\cosh(\lambda x)^5\sinh(\lambda x) - 8\sinh(\lambda x)\cosh(\lambda x)^3 - 2\cosh(\lambda x)^2\ln(\coth(\lambda x)-1) + 2}{-4\sinh(\lambda x)\cosh(\lambda x)^3 + 2\cosh(\lambda x)\,\mathrm{s}} \tag{1}$$

Verification of solutions

$$y = \frac{\operatorname{sech}(\lambda x)\operatorname{csch}(\lambda x)\left(8\cosh(\lambda x)^5\sinh(\lambda x) - 8\sinh(\lambda x)\cosh(\lambda x)^3 - 2\cosh(\lambda x)^2\ln(\coth(\lambda x)-1) + 2}{-4\sinh(\lambda x)\cosh(\lambda x)^3 + 2\cosh(\lambda x)\,\mathrm{s}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (2*lambda^2*tanh(lambda*x)^2+2
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
         <- Kovacics algorithm successful
         Change of variables used:
            [x = arccoth(t)/lambda]
         Linear ODE actually solved:
            (-2*t^4-2)*u(t)+(2*t^5-2*t^3)*diff(u(t),t)+(t^6-2*t^4+t^2)*diff(diff(u(t),t),t)
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 143

```
dsolve(diff(y(x),x)=y(x)^2-2*lambda^2*tanh(lambda*x)^2-2*lambda^2*coth(lambda*x)^2,y(x), sin
```

$$y(x) = \frac{2\left(-\frac{1}{2} + c_1\left(-\cosh(x\lambda)^2 + \frac{1}{2}\right)\ln(\coth(x\lambda) - 1) + c_1\left(\cosh(x\lambda)^2 - \frac{1}{2}\right)\ln(\coth(x\lambda) + 1) + 4\cosh(x\lambda)^5 c}{-4\cosh(x\lambda)^3 c_1 \sinh(x\lambda) + 2\sinh(x\lambda)\cosh(x\lambda) c_1 +}$$

✓ Solution by Mathematica

Time used: 7.989 (sec). Leaf size: 132

```
DSolve[y'[x]==y[x]^2-2*\[Lambda]^2*Tanh[\[Lambda]*x]^2-2*\[Lambda]^2*Coth[\[Lambda]*x]^2,y[x
```

$$y(x) \to -\frac{2\lambda\left(e^{12\lambda x} + 2e^{4\lambda x}\left(e^{4\lambda x} + 1\right)\log\left(e^{4\lambda x}\right) + (-7 + c_1)\left(-e^{4\lambda x}\right) - (7 + c_1)e^{8\lambda x} - 1\right)}{\left(e^{4\lambda x} - 1\right)\left(e^{8\lambda x} - 2e^{4\lambda x}\log\left(e^{4\lambda x}\right) + c_1 e^{4\lambda x} - 1\right)}$$

$$y(x) \to \frac{2\lambda\left(e^{4\lambda x} + 1\right)}{e^{4\lambda x} - 1}$$

6.10 problem 27

Internal problem ID [10475]
Internal file name [`OUTPUT/9422_Monday_June_06_2022_02_31_42_PM_84409171/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.4-2. Equations with hyperbolic tangent and cotangent.
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = -2ab + \lambda a + b\lambda - a(a+\lambda)\tanh(\lambda x)^2 - b(b+\lambda)\coth(\lambda x)^2}$$

6.10.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -b^2\coth(\lambda x)^2 - b\coth(\lambda x)^2\lambda - a^2\tanh(\lambda x)^2 - a\tanh(\lambda x)^2\lambda - 2ab + \lambda a + b\lambda + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -b^2\coth(\lambda x)^2 - b\coth(\lambda x)^2\lambda - a^2\tanh(\lambda x)^2 - a\tanh(\lambda x)^2\lambda - 2ab + \lambda a + b\lambda + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -b^2\coth(\lambda x)^2 - b\coth(\lambda x)^2\lambda - a^2\tanh(\lambda x)^2 - a\tanh(\lambda x)^2\lambda - 2ab + \lambda a + b\lambda$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0\\ f_1 f_2 &= 0\\ f_2^2 f_0 &= -b^2 \coth(\lambda x)^2 - b\coth(\lambda x)^2 \lambda - a^2 \tanh(\lambda x)^2 - a\tanh(\lambda x)^2 \lambda - 2ab + \lambda a + b\lambda \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-b^2 \coth(\lambda x)^2 - b\coth(\lambda x)^2 \lambda - a^2 \tanh(\lambda x)^2 - a\tanh(\lambda x)^2 \lambda - 2ab + \lambda a + b\lambda\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = \operatorname{csch}(\lambda x)^{\frac{-a-b}{\lambda}} \Bigg(&c_1 \coth(\lambda x)^{-\frac{a}{\lambda}} \left(-\operatorname{csch}(\lambda x)^2\right)^{\frac{a+b}{\lambda}}\\ &+ c_2 \coth(\lambda x)^{\frac{a+\lambda}{\lambda}} \operatorname{hypergeom}\left(\left[1, \frac{1}{2} - \frac{b}{\lambda}\right], \left[\frac{3}{2} + \frac{a}{\lambda}\right], \coth(\lambda x)^2\right)\Bigg) \end{aligned}$$

The above shows that

$$u'(x) = \frac{4\left(\frac{\coth(\lambda x)^{\frac{a+\lambda}{\lambda}} c_2((b-\lambda)\coth(\lambda x)+\tanh(\lambda x)(a+\lambda))\left(\frac{3\lambda}{2}+a\right) \operatorname{hypergeom}\left(\left[1,\frac{1}{2}-\frac{b}{\lambda}\right],\left[\frac{3}{2}+\frac{a}{\lambda}\right],\coth(\lambda x)^2\right)}{2} + \coth(\lambda x)^{\frac{a+\lambda}{\lambda}} c_2\left(-\frac{\lambda}{2} + b\right.\right.}{}$$

Using the above in (1) gives the solution

$$y = -\frac{4\left(\frac{\coth(\lambda x)^{\frac{a+\lambda}{\lambda}} c_2((b-\lambda)\coth(\lambda x)+\tanh(\lambda x)(a+\lambda))\left(\frac{3\lambda}{2}+a\right) \operatorname{hypergeom}\left(\left[1,\frac{1}{2}-\frac{b}{\lambda}\right],\left[\frac{3}{2}+\frac{a}{\lambda}\right],\coth(\lambda x)^2\right)}{2} + \coth(\lambda x)^{\frac{a+\lambda}{\lambda}} c_2\left(-\frac{\lambda}{2}\right.\right.}{(2a +}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-4 \coth(\lambda x)^{\frac{2a+2\lambda}{\lambda}} \lambda \operatorname{csch}(\lambda x)^2 \left(-\frac{\lambda}{2} + b\right) \operatorname{hypergeom}\left(\left[2, -\frac{2b-3\lambda}{2\lambda}\right], \left[\frac{2a+5\lambda}{2\lambda}\right], \coth(\lambda x)^2\right) + \Big(((-3a-3b)}{\Big(c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{-4 \coth(\lambda x)^{\frac{2a+2\lambda}{\lambda}} \lambda \operatorname{csch}(\lambda x)^2 \left(-\frac{\lambda}{2} + b\right) \operatorname{hypergeom}\left(\left[2, -\frac{2b-3\lambda}{2\lambda}\right], \left[\frac{2a+5\lambda}{2\lambda}\right], \coth(\lambda x)^2\right) + \Big(((-3a-3b)}{\Big(c_3} \quad (1)$$

Verification of solutions

$$y = \frac{-4 \coth(\lambda x)^{\frac{2a+2\lambda}{\lambda}} \lambda \operatorname{csch}(\lambda x)^2 \left(-\frac{\lambda}{2} + b\right) \operatorname{hypergeom}\left(\left[2, -\frac{2b-3\lambda}{2\lambda}\right], \left[\frac{2a+5\lambda}{2\lambda}\right], \coth(\lambda x)^2\right) + \Big(((-3a-3b)}{\Big(c_3}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (b^2*coth(lambda*x)^2+b*coth(l
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               -> Whittaker
                  -> hyper3: Equivalence to 1F1 under a power @ Moebius
               -> hypergeometric
                  -> heuristic approach
                  <- heuristic approach successful
               <- hypergeometric successful
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 289

```
dsolve(diff(y(x),x)=y(x)^2+a*lambda+b*lambda-2*a*b-a*(a+lambda)*tanh(lambda*x)^2-b*(b+lambda
```

$y(x)$

$$= \frac{-4c_1\lambda\left(b-\frac{\lambda}{2}\right)\coth(x\lambda)^{\frac{2a+2\lambda}{\lambda}}\operatorname{csch}(x\lambda)^2\operatorname{hypergeom}\left(\left[2,-\frac{2b-3\lambda}{2\lambda}\right],\left[\frac{2a+5\lambda}{2\lambda}\right],\coth(x\lambda)^2\right)-2c_1\left(\left(\left(\frac{3a}{2}+\frac{3b}{2}\right.\right.\right.}{\left(\mathrm{hyp}\right.}$$

✓ Solution by Mathematica

Time used: 40.238 (sec). Leaf size: 493

```
DSolve[y'[x]==y[x]^2+a*\[Lambda]+b*\[Lambda]-2*a*b-a*(a+\[Lambda])*Tanh[\[Lambda]*x]^2-b*(b+
```

$y(x) \rightarrow$

$$-\frac{(a+b)\left(e^{2\lambda x}\right)^{\frac{a+b}{\lambda}}\left(\frac{2\lambda\left(a\left(e^{2\lambda x}-1\right)^2+b\left(e^{2\lambda x}+1\right)^2\right)\left(e^{2\lambda x}\right)^{-\frac{a+b}{\lambda}}\operatorname{AppellF1}\left(-\frac{a+b}{\lambda},-\frac{2b}{\lambda},-\frac{2a}{\lambda},-\frac{a+b-\lambda}{\lambda},e^{2x\lambda},-e^{2x\lambda}\right)}{(a+b)\left(e^{2\lambda x}-1\right)\left(e^{2\lambda x}+1\right)}+4\lambda\left(e^{2\lambda x}\right)^{-}\right.}{}$$

$$y(x) \rightarrow \frac{a\left(e^{2\lambda x}-1\right)^2+b\left(e^{2\lambda x}+1\right)^2}{e^{4\lambda x}-1}$$

7 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-1. Equations Containing Logarithmic Functions

7.1 problem 1

Internal problem ID [10476]
Internal file name [`OUTPUT/9423_Monday_June_06_2022_02_31_52_PM_82473112/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-1. Equations Containing Logarithmic Functions
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\ln(x)^n y^2 = bm\, x^{m-1} - a\, b^2 x^{2m} \ln(x)^n}$$

7.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\ln(x)^n y^2 + bm\, x^{m-1} - a\, b^2 x^{2m} \ln(x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a\, b^2 x^{2m} \ln(x)^n + a\ln(x)^n y^2 + \frac{b\, x^m m}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = bm\, x^{m-1} - a\, b^2 x^{2m} \ln(x)^n$, $f_1(x) = 0$ and $f_2(x) = \ln(x)^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\ln(x)^n a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = \frac{\ln(x)^n na}{x \ln(x)}$$
$$f_1 f_2 = 0$$
$$f_2^2 f_0 = \ln(x)^{2n} a^2 \left(bm\, x^{m-1} - a\, b^2 x^{2m} \ln(x)^n\right)$$

Substituting the above terms back in equation (2) gives

$$\ln(x)^n au''(x) - \frac{\ln(x)^n nau'(x)}{x \ln(x)} + \ln(x)^{2n} a^2 \left(bm\, x^{m-1} - a\, b^2 x^{2m} \ln(x)^n\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{x \ln(x)} + a\_Y(x) \left(\ln(x)^n bm\, x^{m-1} - a\, b^2 x^{2m} \ln(x)^{2n}\right)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{x \ln(x)} + a\_Y(x) \left(\ln(x)^n bm\, x^{m-1} - a\, b^2 x^{2m} \ln(x)^{2n}\right)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{x \ln(x)} + a\_Y(x) \left(\ln(x)^n bm\, x^{m-1} - a\, b^2 x^{2m} \ln(x)^{2n}\right)\right\}, \{\_Y(x)\}\right)\right) \ln(x)}{a\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{x \ln(x)} + a\_Y(x) \left(\ln(x)^n bm\, x^{m-1} - a\, b^2 x^{2m} \ln(x)^{2n}\right)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{x \ln(x)} + a\_Y(x) \left(\ln(x)^n bm\, x^{m-1} - a\, b^2 x^{2m} \ln(x)^{2n}\right)\right\}, \{\_Y(x)\}\right)\right) \ln(x)}{a\, \mathrm{DESol}\left(\left\{\frac{-a^2b^2\_Y(x)\ln(x)^{1+2n}x^{1+2m} + abm\, x^m\_Y(x)\ln(x)^{n+1} + \_Y''(x)\ln(x)x - n\_Y'(x)}{\ln(x)x}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{x \ln(x)} + a\_Y(x) \left(\ln(x)^n bm\, x^{m-1} - a\, b^2 x^{2m} \ln(x)^{2n}\right)\right\}, \{\_Y(x)\}\right)\right) \ln(x)}{a\, \mathrm{DESol}\left(\left\{\frac{-a^2b^2\_Y(x)\ln(x)^{1+2n}x^{1+2m} + abm\, x^m\_Y(x)\ln(x)^{n+1} + \_Y''(x)\ln(x)x - n\_Y'(x)}{\ln(x)x}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{x \ln(x)} + a\_Y(x) \left(\ln(x)^n bm\, x^{m-1} - a\, b^2 x^{2m} \ln(x)^{2n}\right)\right\}, \{\_Y(x)\}\right)\right) \ln(x)}{a\, \mathrm{DESol}\left(\left\{\frac{-a^2b^2\_Y(x)\ln(x)^{1+2n}x^{1+2m} + abm\, x^m\_Y(x)\ln(x)^{n+1} + \_Y''(x)\ln(x)x - n\_Y'(x)}{\ln(x)x}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = n*(diff(y(x), x))/(x*ln(x))+a*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
            checking if the LODE is missing y
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=a*(ln(x))^n*y(x)^2+b*m*x^(m-1)-a*b^2*x^(2*m)*(ln(x))^n,y(x), singsol=all
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==a*(Log[x])^n*y[x]^2+b*m*x^(m-1)-a*b^2*x^(2*m)*(Log[x])^n,y[x],x,IncludeSingula
```

Not solved

7.2 problem 2

Internal problem ID [10477]
Internal file name [`OUTPUT/9424_Monday_June_06_2022_02_31_55_PM_71859029/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-1. Equations Containing Logarithmic Functions
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - a y^2 = b\ln(x) + c}$$

7.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,y^2 + b\ln(x) + c}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,y^2}{x} + \frac{b\ln(x)}{x} + \frac{c}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{b\ln(x)+c}{x}$, $f_1(x) = 0$ and $f_2(x) = \frac{a}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{au}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{a}{x^2} \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= \frac{a^2(b\ln(x) + c)}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{au''(x)}{x} + \frac{au'(x)}{x^2} + \frac{a^2(b\ln(x) + c)\, u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{AiryAi}\left(-\frac{(ab)^{\frac{1}{3}}\, (b\ln(x) + c)}{b}\right) + c_2 \operatorname{AiryBi}\left(-\frac{(ab)^{\frac{1}{3}}\, (b\ln(x) + c)}{b}\right)$$

The above shows that

$$u'(x) = \frac{\left(-\operatorname{AiryAi}\left(1, -\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right) c_1 - \operatorname{AiryBi}\left(1, -\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right) c_2\right)(ab)^{\frac{1}{3}}}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-\operatorname{AiryAi}\left(1, -\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right) c_1 - \operatorname{AiryBi}\left(1, -\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right) c_2\right)(ab)^{\frac{1}{3}}}{a\left(c_1 \operatorname{AiryAi}\left(-\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right) + c_2 \operatorname{AiryBi}\left(-\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\operatorname{AiryAi}\left(1, -\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right) c_3 + \operatorname{AiryBi}\left(1, -\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right)\right)(ab)^{\frac{1}{3}}}{a\left(c_3 \operatorname{AiryAi}\left(-\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right) + \operatorname{AiryBi}\left(-\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right)\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\operatorname{AiryAi}\left(1, -\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right)c_3 + \operatorname{AiryBi}\left(1, -\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right)\right)(ab)^{\frac{1}{3}}}{a\left(c_3 \operatorname{AiryAi}\left(-\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right) + \operatorname{AiryBi}\left(-\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\operatorname{AiryAi}\left(1, -\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right)c_3 + \operatorname{AiryBi}\left(1, -\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right)\right)(ab)^{\frac{1}{3}}}{a\left(c_3 \operatorname{AiryAi}\left(-\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right) + \operatorname{AiryBi}\left(-\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(diff(y(x), x))/x-a*(ln(x)*b+
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            <- Bessel successful
         <- special function solution successful
         Change of variables used:
            [x = exp(t)]
         Linear ODE actually solved:
            (a*b*t+a*c)*u(t)+diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 91

```
dsolve(x*diff(y(x),x)=a*y(x)^2+b*ln(x)+c,y(x), singsol=all)
```

$$y(x) = \frac{(ab)^{\frac{1}{3}} \left(\mathrm{AiryBi}\left(1, -\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right) c_1 + \mathrm{AiryAi}\left(1, -\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right)\right)}{a\left(c_1 \,\mathrm{AiryBi}\left(-\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right) + \mathrm{AiryAi}\left(-\frac{(ab)^{\frac{1}{3}}(b\ln(x)+c)}{b}\right)\right)}$$

✓ Solution by Mathematica

Time used: 1.682 (sec). Leaf size: 149

```
DSolve[x*y'[x]==a*y[x]^2+b*Log[x]+c,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{b\left(\mathrm{AiryBiPrime}\left(-\frac{a(c+b\log(x))}{(-ab)^{2/3}}\right) + c_1 \,\mathrm{AiryAiPrime}\left(-\frac{a(c+b\log(x))}{(-ab)^{2/3}}\right)\right)}{(-ab)^{2/3}\left(\mathrm{AiryBi}\left(-\frac{a(c+b\log(x))}{(-ab)^{2/3}}\right) + c_1 \,\mathrm{AiryAi}\left(-\frac{a(c+b\log(x))}{(-ab)^{2/3}}\right)\right)}$$

$$y(x) \to \frac{b\,\mathrm{AiryAiPrime}\left(-\frac{a(c+b\log(x))}{(-ab)^{2/3}}\right)}{(-ab)^{2/3}\,\mathrm{AiryAi}\left(-\frac{a(c+b\log(x))}{(-ab)^{2/3}}\right)}$$

7.3 problem 3

Internal problem ID [10478]
Internal file name [`OUTPUT/9425_Monday_June_06_2022_02_31_57_PM_8846351/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-1. Equations Containing Logarithmic Functions
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - a y^2 = b \ln(x)^k + c \ln(x)^{2k+2}}$$

7.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,y^2 + b\ln(x)^k + c\ln(x)^{2k+2}}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,y^2}{x} + \frac{b\ln(x)^k}{x} + \frac{c\ln(x)^{2k}\ln(x)^2}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{b\ln(x)^k + c\ln(x)^{2k+2}}{x}$, $f_1(x) = 0$ and $f_2(x) = \frac{a}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{au}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{a}{x^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{a^2 \left(b \ln(x)^k + c \ln(x)^{2k+2}\right)}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{a u''(x)}{x} + \frac{a u'(x)}{x^2} + \frac{a^2 \left(b \ln(x)^k + c \ln(x)^{2k+2}\right) u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) \\ = \mathrm{e}^{-\frac{i\sqrt{c}\,\sqrt{a}\,\ln(x)^{k+2}}{k+2}} &\left(\mathrm{hypergeom}\left(\left[\frac{(k+1)\sqrt{c} + i\sqrt{a}\, b}{\sqrt{c}\,(4+2k)}\right], \left[\frac{k+1}{k+2}\right], \frac{2i\sqrt{c}\,\sqrt{a}\,\ln(x)^{k+2}}{k+2}\right) c_1\right. \\ &\left. + \mathrm{hypergeom}\left(\left[\frac{(3+k)\sqrt{c} + i\sqrt{a}\, b}{\sqrt{c}\,(4+2k)}\right], \left[\frac{3+k}{k+2}\right], \frac{2i\sqrt{c}\,\sqrt{a}\,\ln(x)^{k+2}}{k+2}\right) c_2 \ln(x)\right) \end{aligned}$$

The above shows that

$$u'(x) = -\frac{\left(-\ln(x)^{k+1} (3+k)\left(i(k+1)\sqrt{c}\,\sqrt{a} - ab\right) c_1\, \mathrm{hypergeom}\left(\left[\frac{(5+3k)\sqrt{c}+i\sqrt{a}\,b}{\sqrt{c}\,(4+2k)}\right], \left[\frac{3+2k}{k+2}\right], \frac{2i\sqrt{c}\,\sqrt{a}\,\ln(x)^{k+2}}{k+2}\right) + \right.}{}$$

Using the above in (1) gives the solution

$$y = \frac{-\ln(x)^{k+1} (3+k)\left(i(k+1)\sqrt{c}\,\sqrt{a} - ab\right) c_1\, \mathrm{hypergeom}\left(\left[\frac{(5+3k)\sqrt{c}+i\sqrt{a}\,b}{\sqrt{c}\,(4+2k)}\right], \left[\frac{3+2k}{k+2}\right], \frac{2i\sqrt{c}\,\sqrt{a}\,\ln(x)^{k+2}}{k+2}\right) + (k}{}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\ln(x)^{k+1}(3+k)\left(i(k+1)\sqrt{c}\sqrt{a}-ab\right)c_3 \operatorname{hypergeom}\left(\left[\frac{(5+3k)\sqrt{c}+i\sqrt{a}\,b}{\sqrt{c}\,(4+2k)}\right],\left[\frac{3+2k}{k+2}\right],\frac{2i\sqrt{c}\sqrt{a}\,\ln(x)^{k+2}}{k+2}\right)+(k\ \cdots}{\cdots}$$

Summary

The solution(s) found are the following

$$y = \frac{-\ln(x)^{k+1}(3+k)\left(i(k+1)\sqrt{c}\sqrt{a}-ab\right)c_3 \operatorname{hypergeom}\left(\left[\frac{(5+3k)\sqrt{c}+i\sqrt{a}\,b}{\sqrt{c}\,(4+2k)}\right],\left[\frac{3+2k}{k+2}\right],\frac{2i\sqrt{c}\sqrt{a}\,\ln(x)^{k+2}}{k+2}\right)+(k\ \cdots}{\cdots} \tag{1}$$

Verification of solutions

$$y = \frac{-\ln(x)^{k+1}(3+k)\left(i(k+1)\sqrt{c}\sqrt{a}-ab\right)c_3 \operatorname{hypergeom}\left(\left[\frac{(5+3k)\sqrt{c}+i\sqrt{a}\,b}{\sqrt{c}\,(4+2k)}\right],\left[\frac{3+2k}{k+2}\right],\frac{2i\sqrt{c}\sqrt{a}\,\ln(x)^{k+2}}{k+2}\right)+(k\ \cdots}{\cdots}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(diff(y(x), x))/x-a*(b*ln(x)^
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying an equivalence, under non-integer power transformations,
            to LODEs admitting Liouvillian solutions.
            -> Trying a Liouvillian solution using Kovacics algorithm
            <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
               <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
            <- Whittaker successful
         <- special function solution successful
         Change of variables used:
            [x = exp(t)]
         Linear ODE actually solved:
            a*(b*t^k+c*t^(2+2*k))*u(t)+diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 480

```
dsolve(x*diff(y(x),x)=a*y(x)^2+b*(ln(x))^k+c*(ln(x))^(2*k+2),y(x), singsol=all)
```

$$y(x) = \frac{-\ln(x)^{1+k}(k+3)\left(i\sqrt{c}(1+k)\sqrt{a}-ab\right)\text{hypergeom}\left(\left[\frac{(3k+5)\sqrt{c}+i\sqrt{a}\,b}{\sqrt{c}\,(2k+4)}\right],\left[\frac{2k+3}{k+2}\right],\frac{2i\sqrt{a}\,\sqrt{c}\,\ln(x)^{k+2}}{k+2}\right)+\left(-(i\right.}{}$$

✓ Solution by Mathematica

Time used: 3.775 (sec). Leaf size: 807

```
DSolve[x*y'[x]==a*y[x]^2+b*(Log[x])^k+c*(Log[x])^(2*k+2),y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\frac{\log^{k+1}(x)\left(\sqrt{c}c_1(k+2)\sqrt{-(k+2)^2}\,\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{\sqrt{a}b}{\sqrt{c}\sqrt{-(k+2)^2}}+\frac{k+1}{k+2}\right),\frac{k+1}{k+2},\frac{2\sqrt{a}\sqrt{c}\log^{k+2}(x)}{\sqrt{-(k+2)^2}}\right)+\right.}{\sqrt{a}(}$$

$$y(x) \to \frac{\log^{k+1}(x)\left(-\frac{\left(\sqrt{a}b(k+2)+\sqrt{c}\sqrt{-(k+2)^2}(k+1)\right)\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{\sqrt{a}b}{\sqrt{c}\sqrt{-(k+2)^2}}+\frac{3k+5}{k+2}\right),\frac{2k+3}{k+2},\frac{2\sqrt{a}\sqrt{c}\log^{k+2}(x)}{\sqrt{-(k+2)^2}}\right)}{\text{HypergeometricU}\left(\frac{1}{2}\left(\frac{\sqrt{a}b}{\sqrt{c}\sqrt{-(k+2)^2}}+\frac{k+1}{k+2}\right),\frac{k+1}{k+2},\frac{2\sqrt{a}\sqrt{c}\log^{k+2}(x)}{\sqrt{-(k+2)^2}}\right)}-\sqrt{c}\sqrt{-(\right.}{\sqrt{a}(k+2)^2}$$

7.4 problem 4

Internal problem ID [10479]
Internal file name [OUTPUT/9426_Monday_June_06_2022_02_32_00_PM_24545857/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-1. Equations Containing Logarithmic Functions
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - y^2x = -a^2x\ln\left(\beta x\right)^2 + a}$$

7.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{a^2x\ln\left(\beta x\right)^2 - x\,y^2 - a}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a^2\ln\left(\beta x\right)^2 + y^2 + \frac{a}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{a^2x\ln(\beta x)^2 - a}{x}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= -\frac{a^2x\ln(\beta x)^2 - a}{x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \frac{(a^2x\ln(\beta x)^2 - a)\,u(x)}{x} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(a^2x\ln(\beta x)^2 - a)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(a^2x\ln(\beta x)^2 - a)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(a^2x\ln(\beta x)^2 - a)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(a^2x\ln(\beta x)^2 - a)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\_Y''(x)x - a(\ln(\beta x)^2ax - 1)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{\_Y''(x)x - a(\ln(\beta x)^2ax - 1)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\_Y''(x)x - a\left(\ln(\beta x)^2 a x - 1\right)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}{\operatorname{DESol}\left(\left\{\frac{\_Y''(x)x - a\left(\ln(\beta x)^2 a x - 1\right)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\_Y''(x)x - a\left(\ln(\beta x)^2 a x - 1\right)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}{\operatorname{DESol}\left(\left\{\frac{\_Y''(x)x - a\left(\ln(\beta x)^2 a x - 1\right)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = a*(ln(beta*x)^2*a*x-1)*y(x)/x,
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(y(x)^2+y(x)+x^2*(-ln(beta*x)^2*a^2+a/x))
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x)=x*y(x)^2-a^2*x*(ln(beta*x))^2+a,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y'[x]==x*y[x]^2-a^2*x*(Log[\[Beta]*x])^2+a,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

7.5 problem 5

Internal problem ID [10480]
Internal file name [`OUTPUT/9427_Monday_June_06_2022_02_32_05_PM_43314102/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-1. Equations Containing Logarithmic Functions
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - y^2x = -a^2x \ln(\beta x)^{2k} + ak \ln(\beta x)^{k-1}}$$

7.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{-x\,y^2 + a^2x \ln(\beta x)^{2k} - ak \ln(\beta x)^{k-1}}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - a^2 \ln(\beta x)^{2k} + \frac{ak \ln(\beta x)^k}{x \ln(\beta x)}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{a^2x \ln(\beta x)^{2k} - ak \ln(\beta x)^{k-1}}{x}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -\frac{a^2 x \ln(\beta x)^{2k} - ak \ln(\beta x)^{k-1}}{x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \frac{\left(a^2 x \ln(\beta x)^{2k} - ak \ln(\beta x)^{k-1}\right) u(x)}{x} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \text{DESol}\left(\left\{\_Y''(x) - \frac{\left(a^2 x \ln(\beta x)^{2k} - ak \ln(\beta x)^{k-1}\right) \_Y(x)}{x}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \text{DESol}\left(\left\{\_Y''(x) - \frac{\left(a^2 x \ln(\beta x)^{2k} - ak \ln(\beta x)^{k-1}\right) \_Y(x)}{x}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \text{DESol}\left(\left\{\_Y''(x) - \frac{\left(a^2 x \ln(\beta x)^{2k} - ak \ln(\beta x)^{k-1}\right) \_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}{\text{DESol}\left(\left\{\_Y''(x) - \frac{\left(a^2 x \ln(\beta x)^{2k} - ak \ln(\beta x)^{k-1}\right) \_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \text{DESol}\left(\left\{\_Y''(x) - \frac{\left(a^2 x \ln(\beta x)^{2k} - ak \ln(\beta x)^{k-1}\right) \_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}{\text{DESol}\left(\left\{\frac{-\ln(\beta x)^{1+2k} \_Y(x) a^2 x + \ln(\beta x)^k \_Y(x) ak + \_Y''(x) \ln(\beta x) x}{\ln(\beta x) x}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(a^2 x \ln(\beta x)^{2k} - a k \ln(\beta x)^{k-1}\right)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{-\ln(\beta x)^{1+2k}\_Y(x) a^2 x + \ln(\beta x)^k \_Y(x) a k + \_Y''(x)\ln(\beta x) x}{\ln(\beta x) x}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(a^2 x \ln(\beta x)^{2k} - a k \ln(\beta x)^{k-1}\right)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{-\ln(\beta x)^{1+2k}\_Y(x) a^2 x + \ln(\beta x)^k \_Y(x) a k + \_Y''(x)\ln(\beta x) x}{\ln(\beta x) x}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = a*(ln(beta*x)^(2*k)*a*x-ln(bet
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(y(x)^2+y(x)+x^2*(-ln(beta*x)^(2*k)*a^2+a
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x)=x*y(x)^2-a^2*x*(ln(beta*x))^(2*k)+a*k*(ln(beta*x))^(k-1),y(x), singsol
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y'[x]==x*y[x]^2-a^2*x*(Log[\[Beta]*x])^(2*k)+a*k*(Log[\[Beta]*x])^(k-1),y[x],x,Incl
```

Not solved

7.6 problem 6

Internal problem ID [10481]
Internal file name [`OUTPUT/9428_Monday_June_06_2022_02_32_08_PM_61700309/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-1. Equations Containing Logarithmic Functions
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - a\,x^n y^2 = b - a\,b^2 x^n \ln(x)^2}$$

7.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{a\,b^2 x^n \ln(x)^2 - a\,x^n y^2 - b}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{a\,b^2 x^n \ln(x)^2}{x} + \frac{a\,x^n y^2}{x} + \frac{b}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{-b+a\,b^2 x^n \ln(x)^2}{x}$, $f_1(x) = 0$ and $f_2(x) = \frac{a\,x^n}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{a\,x^n u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{a\,x^n n}{x^2} - \frac{a\,x^n}{x^2} \\
f_1f_2 &= 0 \\
f_2^2f_0 &= -\frac{a^2x^{2n}\left(-b + a\,b^2x^n \ln\left(x\right)^2\right)}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{a\,x^n u''(x)}{x} - \left(\frac{a\,x^n n}{x^2} - \frac{a\,x^n}{x^2}\right)u'(x) - \frac{a^2x^{2n}\left(-b + a\,b^2x^n \ln\left(x\right)^2\right)u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(n-1)\_Y'(x)}{x} - \frac{a\,x^n\left(-b + a\,b^2x^n\ln\left(x\right)^2\right)\_Y(x)}{x^2}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(n-1)\_Y'(x)}{x} - \frac{a\,x^n\left(-b + a\,b^2x^n\ln\left(x\right)^2\right)\_Y(x)}{x^2}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(n-1)\_Y'(x)}{x} - \frac{a\,x^n\left(-b+a\,b^2x^n\ln(x)^2\right)\_Y(x)}{x^2}\right\}, \{\_Y(x)\}\right)\right)x^{-n}x}{a\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(n-1)\_Y'(x)}{x} - \frac{a\,x^n\left(-b+a\,b^2x^n\ln(x)^2\right)\_Y(x)}{x^2}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x^{1-n}\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\_Y''(x)\,x^2-\ln(x)^2 x^{2n}\_Y(x)\,a^2b^2-(n-1)\_Y'(x)\,x+x^n\_Y(x)\,ab}{x^2}\right\},\{\_Y(x)\}\right)\right)}{a\,\mathrm{DESol}\left(\left\{\frac{\_Y''(x)\,x^2-\ln(x)^2 x^{2n}\_Y(x)\,a^2b^2-(n-1)\_Y'(x)\,x+x^n\_Y(x)\,ab}{x^2}\right\},\{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^{1-n}\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\_Y''(x)\,x^2-\ln(x)^2 x^{2n}\_Y(x)\,a^2b^2-(n-1)\_Y'(x)\,x+x^n\_Y(x)\,ab}{x^2}\right\},\{\_Y(x)\}\right)\right)}{a\,\mathrm{DESol}\left(\left\{\frac{\_Y''(x)\,x^2-\ln(x)^2 x^{2n}\_Y(x)\,a^2b^2-(n-1)\_Y'(x)\,x+x^n\_Y(x)\,ab}{x^2}\right\},\{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{x^{1-n}\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\_Y''(x)\,x^2-\ln(x)^2 x^{2n}\_Y(x)\,a^2b^2-(n-1)\_Y'(x)\,x+x^n\_Y(x)\,ab}{x^2}\right\},\{\_Y(x)\}\right)\right)}{a\,\mathrm{DESol}\left(\left\{\frac{\_Y''(x)\,x^2-\ln(x)^2 x^{2n}\_Y(x)\,a^2b^2-(n-1)\_Y'(x)\,x+x^n\_Y(x)\,ab}{x^2}\right\},\{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (n-1)*(diff(y(x), x))/x+x^(n-1
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x)=a*x^n*y(x)^2+b-a*b^2*x^n*(ln(x))^2,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y'[x]==a*x^n*y[x]^2+b-a*b^2*x^n*(Log[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

7.7 problem 7

Internal problem ID [10482]
Internal file name [`OUTPUT/9429_Monday_June_06_2022_02_32_10_PM_31193013/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-1. Equations Containing Logarithmic Functions
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{x^2y' - x^2y^2 = a\ln(x)^2 + b\ln(x) + c}$$

7.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^2y^2 + a\ln(x)^2 + b\ln(x) + c}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \frac{a\ln(x)^2}{x^2} + \frac{b\ln(x)}{x^2} + \frac{c}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{a\ln(x)^2 + b\ln(x) + c}{x^2}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{a \ln(x)^2 + b \ln(x) + c}{x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{\left(a \ln(x)^2 + b \ln(x) + c\right) u(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) &= x^{-\frac{ib-\sqrt{a}}{2\sqrt{a}}} \mathrm{e}^{-\frac{i\ln(x)^2\sqrt{a}}{2}} \left(2\ln(x) \operatorname{hypergeom}\left(\left[\frac{12a^{\frac{3}{2}} + i(4c-1)\,a - ib^2}{16a^{\frac{3}{2}}}\right], \left[\frac{3}{2}\right], \frac{i(2a\ln(x)+b)^2}{4a^{\frac{3}{2}}}\right) c_2 a \right. \\ &\quad + \operatorname{hypergeom}\left(\left[\frac{12a^{\frac{3}{2}} + i(4c-1)\,a - ib^2}{16a^{\frac{3}{2}}}\right], \left[\frac{3}{2}\right], \frac{i(2a\ln(x)+b)^2}{4a^{\frac{3}{2}}}\right) c_2 b \\ &\quad \left. + c_1 \operatorname{hypergeom}\left(\left[\frac{4a^{\frac{3}{2}} + i(4c-1)\,a - ib^2}{16a^{\frac{3}{2}}}\right], \left[\frac{1}{2}\right], \frac{i(2a\ln(x)+b)^2}{4a^{\frac{3}{2}}}\right)\right) \end{aligned}$$

The above shows that

$$u'(x) = -\frac{2x^{-\frac{ib+\sqrt{a}}{2\sqrt{a}}} \left(-\left(\frac{b\ln(x)(b\ln(x)-4c+1)a^{\frac{5}{2}}}{12} + \frac{(b\ln(x)-c+\frac{1}{4})b^2 a^{\frac{3}{2}}}{12} - \frac{\ln(x)^2(c-\frac{1}{4})a^{\frac{7}{2}}}{3} + \frac{b^4\sqrt{a}}{48} + i\left(a\ln(x)+\frac{b}{2}\right)^2 a^2\right) c_2 \text{hy}\right.}{}$$

Using the above in (1) gives the solution

$$y = \frac{2x^{-\frac{ib+\sqrt{a}}{2\sqrt{a}}} \left(-\left(\frac{b\ln(x)(b\ln(x)-4c+1)a^{\frac{5}{2}}}{12} + \frac{(b\ln(x)-c+\frac{1}{4})b^2 a^{\frac{3}{2}}}{12} - \frac{\ln(x)^2(c-\frac{1}{4})a^{\frac{7}{2}}}{3} + \frac{b^4\sqrt{a}}{48} + i\left(a\ln(x)+\frac{b}{2}\right)^2 a^2\right) c_2 \text{hype}\right.}{}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-\frac{b\ln(x)(b\ln(x)-4c+1)a^{\frac{5}{2}}}{12} - \frac{(b\ln(x)-c+\frac{1}{4})b^2a^{\frac{3}{2}}}{12} + \frac{\ln(x)^2(c-\frac{1}{4})a^{\frac{7}{2}}}{3} - \frac{b^4\sqrt{a}}{48} - i\left(a\ln(x)+\frac{b}{2}\right)^2a^2\right)\text{hypergeom}\left(\left[\frac{28a}{}\right.\right.}{}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(-\frac{b\ln(x)(b\ln(x)-4c+1)a^{\frac{5}{2}}}{12} - \frac{(b\ln(x)-c+\frac{1}{4})b^2a^{\frac{3}{2}}}{12} + \frac{\ln(x)^2(c-\frac{1}{4})a^{\frac{7}{2}}}{3} - \frac{b^4\sqrt{a}}{48} - i\left(a\ln(x)+\frac{b}{2}\right)^2a^2\right)\text{hypergeom}\left(\left[\frac{28a}{}\right.\right.}{} \tag{1}$$

Verification of solutions

$$y = \frac{\left(-\frac{b\ln(x)(b\ln(x)-4c+1)a^{\frac{5}{2}}}{12} - \frac{(b\ln(x)-c+\frac{1}{4})b^2a^{\frac{3}{2}}}{12} + \frac{\ln(x)^2(c-\frac{1}{4})a^{\frac{7}{2}}}{3} - \frac{b^4\sqrt{a}}{48} - i\left(a\ln(x)+\frac{b}{2}\right)^2a^2\right)\text{hypergeom}\left(\left[\frac{28a}{}\right.\right.}{}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(a*ln(x)^2+ln(x)*b+c)*y(x)/x^
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Kummer
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
               <- hyper3 successful: indirect Equivalence to 0F1 under \`\`^ @ Moebius\`\` i
            <- hypergeometric successful
         <- special function solution successful
         Change of variables used:
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 477

```
dsolve(x^2*diff(y(x),x)=x^2*y(x)^2+a*(ln(x))^2+b*ln(x)+c,y(x), singsol=all)
```

$$y(x) = \frac{\left(-\frac{b\ln(x)(b\ln(x)-4c+1)a^{\frac{5}{2}}}{12} - \frac{\left(b\ln(x)-c+\frac{1}{4}\right)b^2a^{\frac{3}{2}}}{12} + \frac{\left(c-\frac{1}{4}\right)\ln(x)^2a^{\frac{7}{2}}}{3} - \frac{\sqrt{a}\,b^4}{48} - i\left(a\ln(x)+\frac{b}{2}\right)^2a^2\right)c_1\,\text{hypergeom}\left(\left[\right.\right.}{}$$

✓ Solution by Mathematica

Time used: 1.151 (sec). Leaf size: 868

```
DSolve[x^2*y'[x]==x^2*y[x]^2+a*(Log[x])^2+b*Log[x]+c,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{ib\,\text{ParabolicCylinderD}\left(\frac{-ib^2-4a^{3/2}+ia(4c-1)}{8a^{3/2}}, -\frac{\left(\frac{1}{2}-\frac{i}{2}\right)(b+2a\log(x))}{a^{3/4}}\right) + 2ia\log(x)\,\text{ParabolicCylinderD}\left(\frac{-ib^2-4}{}\right.}{}$$

$$y(x) \to -\frac{-\frac{2\sqrt[4]{-1}\sqrt{2}\sqrt[4]{a}\,\text{ParabolicCylinderD}\left(\frac{ib^2+4a^{3/2}-ia(4c-1)}{8a^{3/2}}, \frac{\left(\frac{1}{2}+\frac{i}{2}\right)(b+2a\log(x))}{a^{3/4}}\right)}{\text{ParabolicCylinderD}\left(\frac{ib^2-4a^{3/2}-ia(4c-1)}{8a^{3/2}}, \frac{\left(\frac{1}{2}+\frac{i}{2}\right)(b+2a\log(x))}{a^{3/4}}\right)} + \frac{ib}{\sqrt{a}} + 2i\sqrt{a}\log(x) + 1}{2x}$$

$$y(x) \to -\frac{-\frac{2\sqrt[4]{-1}\sqrt{2}\sqrt[4]{a}\,\text{ParabolicCylinderD}\left(\frac{ib^2+4a^{3/2}-ia(4c-1)}{8a^{3/2}}, \frac{\left(\frac{1}{2}+\frac{i}{2}\right)(b+2a\log(x))}{a^{3/4}}\right)}{\text{ParabolicCylinderD}\left(\frac{ib^2-4a^{3/2}-ia(4c-1)}{8a^{3/2}}, \frac{\left(\frac{1}{2}+\frac{i}{2}\right)(b+2a\log(x))}{a^{3/4}}\right)} + \frac{ib}{\sqrt{a}} + 2i\sqrt{a}\log(x) + 1}{2x}$$

7.8 problem 8

Internal problem ID [10483]
Internal file name [`OUTPUT/9430_Monday_June_06_2022_02_32_12_PM_7672516/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-1. Equations Containing Logarithmic Functions
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{x^2y' - x^2y^2 = a(b\ln(x) + c)^n + \frac{1}{4}}$$

7.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{4x^2y^2 + 4a(b\ln(x) + c)^n + 1}{4x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \frac{a(b\ln(x) + c)^n}{x^2} + \frac{1}{4x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{4a(b\ln(x)+c)^n+1}{4x^2}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= \frac{4a(b\ln(x)+c)^n + 1}{4x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{(4a(b\ln(x)+c)^n + 1)\,u(x)}{4x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \text{DESol}\left(\left\{\_Y''(x) + \frac{(4a(b\ln(x)+c)^n + 1)\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\text{DESol}\left(\left\{\_Y''(x) + \frac{(4a(b\ln(x)+c)^n + 1)\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\text{DESol}\left(\left\{\_Y''(x) + \frac{(4a(b\ln(x)+c)^n+1)\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}{\text{DESol}\left(\left\{\_Y''(x) + \frac{(4a(b\ln(x)+c)^n+1)\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\text{DESol}\left(\left\{\frac{4(b\ln(x)+c)^n\_Y(x)a + 4\_Y''(x)x^2 + \_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}{\text{DESol}\left(\left\{\frac{4(b\ln(x)+c)^n\_Y(x)a + 4\_Y''(x)x^2 + \_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{4(b\ln(x)+c)^n \_Y(x)a+4\_Y''(x)x^2+\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}{\operatorname{DESol}\left(\left\{\frac{4(b\ln(x)+c)^n \_Y(x)a+4\_Y''(x)x^2+\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{4(b\ln(x)+c)^n \_Y(x)a+4\_Y''(x)x^2+\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}{\operatorname{DESol}\left(\left\{\frac{4(b\ln(x)+c)^n \_Y(x)a+4\_Y''(x)x^2+\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(1/4)*(4*a*(ln(x)*b+c)^n+1)*y
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(y(x)^2+y(x)+x^2*(a*(ln(x)*b+c)^n/x^2+(1/
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
      trying separable
      trying inverse linear
      trying homogeneous types:
      trying Chini
      differential order: 1; looking for linear symmetries
      trying exact
      Looking for potential symmetries
```

✗ Solution by Maple

```
dsolve(x^2*diff(y(x),x)=x^2*y(x)^2+a*(b*ln(x)+c)^n+1/4,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*y'[x]==x^2*y[x]^2+a*(b*Log[x]+c)^n+1/4,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

7.9 problem 9

Internal problem ID [10484]
Internal file name [`OUTPUT/9431_Monday_June_06_2022_02_32_14_PM_30986494/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-1. Equations Containing Logarithmic Functions
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{x^2 \ln(xa)\left(y' - y^2\right) = 1}$$

7.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2 \ln(xa)\, x^2 + 1}{x^2 \ln(xa)} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \frac{1}{x^2 \ln(xa)}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{1}{x^2 \ln(xa)}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= \frac{1}{x^2\ln(xa)} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{u(x)}{x^2\ln(xa)} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = -\ln(xa)\operatorname{expIntegral}_1(-\ln(xa))\,c_2 - c_2ax + c_1\ln(xa)$$

The above shows that

$$u'(x) = \frac{-\operatorname{expIntegral}_1(-\ln(xa))\,c_2 + c_1}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{-\operatorname{expIntegral}_1(-\ln(xa))\,c_2 + c_1}{x(-\ln(xa)\operatorname{expIntegral}_1(-\ln(xa))\,c_2 - c_2ax + c_1\ln(xa))}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\operatorname{expIntegral}_1(-\ln(xa)) + c_3}{x(\operatorname{expIntegral}_1(-\ln(xa))\ln(xa) - c_3\ln(xa) + xa)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\operatorname{expIntegral}_1(-\ln(xa)) + c_3}{x(\operatorname{expIntegral}_1(-\ln(xa))\ln(xa) - c_3\ln(xa) + xa)} \tag{1}$$

Verification of solutions

$$y = \frac{-\operatorname{expIntegral}_1(-\ln(xa)) + c_3}{x(\operatorname{expIntegral}_1(-\ln(xa))\ln(xa) - c_3\ln(xa) + xa)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -y(x)/(x^2*ln(a*x)), y(x)`
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         <- linear_1 successful
         Change of variables used:
            [x = exp(t)/a]
         Linear ODE actually solved:
            u(t)-t*diff(u(t),t)+t*diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
dsolve(x^2*ln(a*x)*(diff(y(x),x)-y(x)^2)=1,y(x), singsol=all)
```

$$y(x) = \frac{-c_1 \operatorname{expIntegral}_1(-\ln(ax)) + 1}{x\left((c_1 \operatorname{expIntegral}_1(-\ln(ax)) - 1)\ln(ax) + c_1 a x\right)}$$

✓ Solution by Mathematica

Time used: 0.616 (sec). Leaf size: 74

```
DSolve[x^2*Log[a*x]*(y'[x]-y[x]^2)==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{a + c_1 \operatorname{LogIntegral}(ax)}{-c_1 x \operatorname{LogIntegral}(ax)\log(ax) + a c_1 x^2 - a x \log(ax)}$$

$$y(x) \to \frac{\operatorname{LogIntegral}(ax)}{a x^2 - x \operatorname{LogIntegral}(ax)\log(ax)}$$

8 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2

8.1 problem 10

Internal problem ID [10485]
Internal file name [`OUTPUT/9432_Monday_June_06_2022_02_32_15_PM_13889787/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - a\ln(\beta x)\, y = -ab\ln(\beta x) - b^2}$$

8.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + a\ln(\beta x)\, y - ab\ln(\beta x) - b^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\ln(\beta x)\, y - ab\ln(\beta x) - b^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -ab\ln(\beta x) - b^2$, $f_1(x) = \ln(\beta x)\, a$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \ln(\beta x) a \\ f_2^2 f_0 &= -ab \ln(\beta x) - b^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \ln(\beta x) a u'(x) + \left(-ab \ln(\beta x) - b^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = -\left(\int (\beta x)^{xa} \mathrm{e}^{-x(a-2b)} dx + c_1\right) c_2 \mathrm{e}^{-bx}$$

The above shows that

$$u'(x) = c_2 \left(b \,\mathrm{e}^{-bx} \left(\int (\beta x)^{xa} \mathrm{e}^{-x(a-2b)} dx\right) + b \,\mathrm{e}^{-bx} c_1 - \mathrm{e}^{-x(a-b)} (\beta x)^{xa}\right)$$

Using the above in (1) gives the solution

$$y = \frac{\left(b \,\mathrm{e}^{-bx} \left(\int (\beta x)^{xa} \mathrm{e}^{-x(a-2b)} dx\right) + b \,\mathrm{e}^{-bx} c_1 - \mathrm{e}^{-x(a-b)} (\beta x)^{xa}\right) \mathrm{e}^{bx}}{\int (\beta x)^{xa} \mathrm{e}^{-x(a-2b)} dx + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\int (\beta x)^{xa} \mathrm{e}^{-x(a-2b)} dx + c_3\right) b - (\beta x)^{xa} \mathrm{e}^{-x(a-2b)}}{\int (\beta x)^{xa} \mathrm{e}^{-x(a-2b)} dx + c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\int (\beta x)^{xa} \mathrm{e}^{-x(a-2b)} dx + c_3\right) b - (\beta x)^{xa} \mathrm{e}^{-x(a-2b)}}{\int (\beta x)^{xa} \mathrm{e}^{-x(a-2b)} dx + c_3} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\int (\beta x)^{xa} \mathrm{e}^{-x(a-2b)} dx + c_3\right) b - (\beta x)^{xa} \mathrm{e}^{-x(a-2b)}}{\int (\beta x)^{xa} \mathrm{e}^{-x(a-2b)} dx + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 74

```
dsolve(diff(y(x),x)=y(x)^2+a*ln(beta*x)*y(x)-a*b*ln(beta*x)-b^2,y(x), singsol=all)
```

$$y(x) = \frac{\left(\int (x\beta)^{ax} \mathrm{e}^{-(a-2b)x} dx - c_1\right) b - (x\beta)^{ax} \mathrm{e}^{-(a-2b)x}}{\int (x\beta)^{ax} \mathrm{e}^{-(a-2b)x} dx - c_1}$$

✓ Solution by Mathematica

Time used: 1.584 (sec). Leaf size: 187

```
DSolve[y'[x]==y[x]^2+a*Log[\[Beta]*x]*y[x]-a*b*Log[\[Beta]*x]-b^2,y[x],x,IncludeSingularSolu
```

$$\text{Solve}\left[\int_1^x \frac{e^{2bK[1]-aK[1]}(\beta K[1])^{aK[1]}(b+a\log(\beta K[1])+y(x))}{a(b-y(x))}dK[1] + \int_1^{y(x)}\left(\frac{e^{2bx-ax}(x\beta)^{ax}}{a(K[2]-b)^2} - \int_1^x\left(\frac{e^{2bK[1]-aK[1]}(b+K[2]+a\log(\beta K[1]))(\beta K[1])^{aK[1]}}{a(b-K[2])^2} + \frac{e^{2bK[1]-aK[1]}(\beta K[1])^{aK[1]}}{a(b-K[2])}\right)dK[1]\right)dK[2]=c_1,\right.$$

8.2 problem 11

Internal problem ID [10486]
Internal file name [`OUTPUT/9433_Monday_June_06_2022_02_32_17_PM_25607459/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - ax \ln (bx)^m y = a \ln (bx)^m}$$

8.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + ax \ln (bx)^m y + a \ln (bx)^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + ax \ln (bx)^m y + a \ln (bx)^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a \ln (bx)^m$, $f_1(x) = a \ln (bx)^m x$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= a \ln(bx)^m x \\ f_2^2 f_0 &= a \ln(bx)^m \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - a \ln(bx)^m x u'(x) + a \ln(bx)^m u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x\left(\left(\int e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} dx\right) c_1 + c_2\right)$$

The above shows that

$$u'(x) = \left(\int e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} dx\right) c_1 + c_2 + x\, e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} c_1$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\int e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} dx\right) c_1 + c_2 + x\, e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} c_1}{x\left(\left(\int e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} dx\right) c_1 + c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\left(\int e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} dx\right) c_3 - 1 - x\, e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} c_3}{x\left(\left(\int e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} dx\right) c_3 + 1\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\left(\int e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} dx\right) c_3 - 1 - x\, e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} c_3}{x \left(\left(\int e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} dx\right) c_3 + 1\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-\left(\int e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} dx\right) c_3 - 1 - x\, e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} c_3}{x \left(\left(\int e^{\int \frac{\ln(bx)^m a\, x^2 - 2}{x} dx} dx\right) c_3 + 1\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 85

```
dsolve(diff(y(x),x)=y(x)^2+a*x*(ln(b*x))^m*y(x)+a*(ln(b*x))^m,y(x), singsol=all)
```

$$y(x) = \frac{-e^{\int \frac{a \ln(bx)^m x^2 - 2}{x} dx} x - \left(\int e^{\int \frac{a \ln(bx)^m x^2 - 2}{x} dx} dx\right) + c_1}{\left(-c_1 + \int e^{\int \frac{a \ln(bx)^m x^2 - 2}{x} dx} dx\right) x}$$

✓ Solution by Mathematica

Time used: 3.589 (sec). Leaf size: 181

```
DSolve[y'[x]==y[x]^2+a*x*(Log[b*x])^m*y[x]+a*(Log[b*x])^m,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to$$

$$-\frac{x\int_1^x \frac{\exp\left(\frac{2^{-m-1}a\Gamma(m+1,-2\log(bK[1]))(-\log(bK[1]))^{-m}\log^m(bK[1])}{b^2}\right)}{K[1]^2}dK[1]+\exp\left(\frac{a2^{-m-1}(-\log(bx))^{-m}\log^m(bx)\Gamma(m+1,-2\log(\right.}{b^2}}{x^2\left(\int_1^x \frac{\exp\left(\frac{2^{-m-1}a\Gamma(m+1,-2\log(bK[1]))(-\log(bK[1]))^{-m}\log^m(bK[1])}{b^2}\right)}{K[1]^2}dK[1]+c_1\right)}$$

$$y(x) \to -\frac{1}{x}$$

8.3 problem 12

Internal problem ID [10487]
Internal file name [`OUTPUT/9434_Monday_June_06_2022_02_32_18_PM_89796833/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\,x^n y^2 + ab\,x^{n+1} \ln(x)\, y = b \ln(x) + b}$$

8.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,x^n y^2 - ab\,x^{n+1} \ln(x)\, y + b \ln(x) + b \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,x^n y^2 - ab\,x^n x \ln(x)\, y + b \ln(x) + b$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\ln(x) + b$, $f_1(x) = -a\ln(x)\,x^{n+1}b$ and $f_2(x) = x^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^n a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{x^n n a}{x} \\
f_1 f_2 &= -a^2 \ln(x) x^{n+1} b x^n \\
f_2^2 f_0 &= x^{2n} a^2 (b \ln(x) + b)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^n a u''(x) - \left(\frac{x^n n a}{x} - a^2 \ln(x) x^{n+1} b x^n\right) u'(x) + x^{2n} a^2 (b \ln(x) + b) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{a\_Y(x) b(1+\ln(x)) x^{n+1} + \_Y''(x) x + \_Y'(x) (a \ln(x) x^{2+n} b - n)}{x}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{a\_Y(x) b(1+\ln(x)) x^{n+1} + \_Y''(x) x + \_Y'(x) (a \ln(x) x^{2+n} b - n)}{x}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{a\_Y(x) b(1+\ln(x)) x^{n+1} + \_Y''(x) x + \_Y'(x)(a \ln(x) x^{2+n} b - n)}{x}\right\}, \{\_Y(x)\}\right)\right) x^{-n}}{a\, \mathrm{DESol}\left(\left\{\frac{a\_Y(x) b(1+\ln(x)) x^{n+1} + \_Y''(x) x + \_Y'(x)(a \ln(x) x^{2+n} b - n)}{x}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{a\_Y(x) b(1+\ln(x)) x^{n+1} + \_Y''(x) x + \_Y'(x)(a \ln(x) x^{2+n} b - n)}{x}\right\}, \{\_Y(x)\}\right)\right) x^{-n}}{a\, \mathrm{DESol}\left(\left\{\frac{a\_Y(x) b(1+\ln(x)) x^{n+1} + \_Y''(x) x + \_Y'(x)(a \ln(x) x^{2+n} b - n)}{x}\right\}, \{\_Y(x)\}\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{a\_Y(x)b(1+\ln(x))x^{n+1}+\_Y''(x)x+\_Y'(x)\left(a\ln(x)x^{2+n}b-n\right)}{x}\right\},\{\_Y(x)\}\right)\right)x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{a\_Y(x)b(1+\ln(x))x^{n+1}+\_Y''(x)x+\_Y'(x)\left(a\ln(x)x^{2+n}b-n\right)}{x}\right\},\{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{a\_Y(x)b(1+\ln(x))x^{n+1}+\_Y''(x)x+\_Y'(x)\left(a\ln(x)x^{2+n}b-n\right)}{x}\right\},\{\_Y(x)\}\right)\right)x^{-n}}{a\,\mathrm{DESol}\left(\left\{\frac{a\_Y(x)b(1+\ln(x))x^{n+1}+\_Y''(x)x+\_Y'(x)\left(a\ln(x)x^{2+n}b-n\right)}{x}\right\},\{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(x^(n+1)*ln(x)*a*b*x-n)*(diff
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
            checking if the LODE is missing y
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=a*x^n*y(x)^2-a*b*x^(n+1)*ln(x)*y(x)+b*ln(x)+b,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==a*x^n*y[x]^2-a*b*x^(n+1)*Log[x]*y[x]+b*Log[x]+b,y[x],x,IncludeSingularSolution
```

Not solved

8.4 problem 13

Internal problem ID [10488]
Internal file name [`OUTPUT/9435_Monday_June_06_2022_02_32_20_PM_19258479/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + (n+1)\,x^n y^2 - a\,x^{n+1}\ln(x)^m\,y = -a\ln(x)^m}$$

8.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,x^{n+1}\ln(x)^m\,y - x^n y^2 n - x^n y^2 - a\ln(x)^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,x^n x\ln(x)^m\,y - x^n y^2 n - x^n y^2 - a\ln(x)^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a\ln(x)^m$, $f_1(x) = a\ln(x)^m\,x^{n+1}$ and $f_2(x) = -n\,x^n - x^n$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(-n\,x^n - x^n)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{n^2 x^n}{x} - \frac{x^n n}{x} \\
f_1 f_2 &= a \ln(x)^m x^{n+1}(-n\,x^n - x^n) \\
f_2^2 f_0 &= -(-n\,x^n - x^n)^2 a \ln(x)^m
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(-n\,x^n - x^n)\, u''(x) - \left(-\frac{n^2 x^n}{x} - \frac{x^n n}{x} + a \ln(x)^m x^{n+1}(-n\,x^n - x^n)\right) u'(x) - (-n\,x^n - x^n)^2 a \ln(x)^m u(x$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{n+1}\left(\left(\int x^{-2n-2} e^{\int \left(a \ln(x)^m x^{n+1} + \frac{n}{x}\right)dx} dx\right) c_2 + c_1\right)$$

The above shows that

$$u'(x) = x^n (n+1)\left(\left(\int e^{\int \frac{a \ln(x)^m x^{2+n} + n}{x} dx} x^{-2n-2} dx\right) c_2 + c_1\right) + c_2 x^{-n-1} e^{\int \frac{a \ln(x)^m x^{2+n} + n}{x} dx}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(x^n (n+1)\left(\left(\int e^{\int \frac{a \ln(x)^m x^{2+n} + n}{x} dx} x^{-2n-2} dx\right) c_2 + c_1\right) + c_2 x^{-n-1} e^{\int \frac{a \ln(x)^m x^{2+n} + n}{x} dx}\right) x^{-n-1}}{(-n\,x^n - x^n)\left(\left(\int x^{-2n-2} e^{\int \left(a \ln(x)^m x^{n+1} + \frac{n}{x}\right)dx} dx\right) c_2 + c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\int x^{-2n-2} e^{\int \left(a \ln(x)^m x^{n+1} + \frac{n}{x}\right)dx} dx + c_3\right)(n+1)\, x^{-n-1} + x^{-2-3n} e^{\int \left(a \ln(x)^m x^{n+1} + \frac{n}{x}\right)dx}}{(n+1)\left(\int e^{\int \frac{a \ln(x)^m x^{2+n} + n}{x} dx} x^{-2n-2} dx + c_3\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\int x^{-2n-2} e^{\int \left(a \ln(x)^m x^{n+1} + \frac{n}{x}\right) dx} dx + c_3\right) (n+1)\, x^{-n-1} + x^{-2-3n} e^{\int \left(a \ln(x)^m x^{n+1} + \frac{n}{x}\right) dx}}{(n+1) \left(\int e^{\int \frac{a \ln(x)^m x^{2+n} + n}{x} dx} x^{-2n-2} dx + c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\int x^{-2n-2} e^{\int \left(a \ln(x)^m x^{n+1} + \frac{n}{x}\right) dx} dx + c_3\right) (n+1)\, x^{-n-1} + x^{-2-3n} e^{\int \left(a \ln(x)^m x^{n+1} + \frac{n}{x}\right) dx}}{(n+1) \left(\int e^{\int \frac{a \ln(x)^m x^{2+n} + n}{x} dx} x^{-2n-2} dx + c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a*x^(n+1)*ln(x)^m*x+n)*(diff(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
            checking if the LODE is missing y
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 184

```
dsolve(diff(y(x),x)=-(n+1)*x^n*y(x)^2+a*x^(n+1)*(ln(x))^m*y(x)-a*(ln(x))^m,y(x), singsol=all
```

$$y(x) = \frac{x^{-n-1}\left(x^{n+1}e^{\int \frac{a\,x^{n+1}\ln(x)^{m}x-2n-2}{x}dx} + \left(\int x^{n}e^{a\left(\int x^{n+1}\ln(x)^{m}dx\right)-2\left(\int \frac{1}{x}dx\right)(n+1)}dx\right)n + \int x^{n}e^{a\left(\int x^{n+1}\ln(x)^{m}dx\right)-2\left(\int\right.}\right.}{\left(\int x^{n}e^{a\left(\int x^{n+1}\ln(x)^{m}dx\right)-2\left(\int \frac{1}{x}dx\right)(n+1)}dx\right)n + \int x^{n}e^{a\left(\int x^{n+1}\ln(x)^{m}dx\right)-2\left(\int \frac{1}{x}dx\right)(n+1)}dx - c_1}$$

✓ Solution by Mathematica

Time used: 5.364 (sec). Leaf size: 311

```
DSolve[y'[x]==-(n+1)*x^n*y[x]^2+a*x^(n+1)*(Log[x])^m*y[x]-a*(Log[x])^m,y[x],x,IncludeSingula
```

$$y(x) \to \frac{x^{-2(n+1)}\left(c_1(n+1)x^{n+1}\int_1^x \exp\left(\frac{a\Gamma(m+1,-((n+2)\log(K[1])))\log^m(K[1])(-((n+2)\log(K[1])))^{-m}}{n+2} - (n+2)\log(K[1])\right)\right.}{(n+1)\left(1+c_1\int_1^x \exp\left(\frac{a\Gamma(m+1,-((n+2)\log(K[1])))\log^m(K[1])(-((n+2)\log\right.}{n+2}\right.}$$

$$y(x) \to \frac{x^{-2(n+1)}\left(\frac{\exp\left(\frac{a\log^m(x)(-((n+2)\log(x)))^{-m}\Gamma(m+1,-((n+2)\log(x)))}{n+2}\right)}{\int_1^x \exp\left(\frac{a\Gamma(m+1,-((n+2)\log(K[1])))\log^m(K[1])(-((n+2)\log(K[1])))^{-m}}{n+2}-(n+2)\log(K[1])\right)dK[1]} + (n+1)x^{n+1}\right)}{n+1}$$

8.5 problem 14

Internal problem ID [10489]
Internal file name [`OUTPUT/9436_Monday_June_06_2022_02_32_23_PM_88598672/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - a\ln(x)^n y + abx\ln(x)^{n+1} y = b\ln(x) + b}$$

8.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\ln(x)^n a(1 - \ln(x) bx)\\ q(x) &= b(1 + \ln(x))\end{aligned}$$

Hence the ode is

$$y' - \ln(x)^n a(1 - \ln(x) bx) y = b(1 + \ln(x))$$

The integrating factor μ is

$$\mu = e^{\int -\ln(x)^n a(1 - \ln(x) bx)dx}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\,(b(1+\ln(x)))$$

$$\frac{d}{dx}\left(e^{\int -\ln(x)^n a(1-\ln(x)bx)dx}y\right) = \left(e^{\int -\ln(x)^n a(1-\ln(x)bx)dx}\right)(b(1+\ln(x)))$$

$$d\left(e^{\int -\ln(x)^n a(1-\ln(x)bx)dx}y\right) = \left(b(1+\ln(x))\,e^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}\right)\,dx$$

Integrating gives

$$e^{\int -\ln(x)^n a(1-\ln(x)bx)dx}y = \int b(1+\ln(x))\,e^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}\,dx$$

$$e^{\int -\ln(x)^n a(1-\ln(x)bx)dx}y = \int b(1+\ln(x))\,e^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}\,dx + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\int -\ln(x)^n a(1-\ln(x)bx)dx}$ results in

$$y = e^{-a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}\left(\int b(1+\ln(x))\,e^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}dx\right) + c_1 e^{-a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}$$

which simplifies to

$$y = e^{-a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}\left(b\left(\int (1+\ln(x))\,e^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}dx\right) + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{-a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}\left(b\left(\int (1+\ln(x))\,e^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}dx\right) + c_1\right) \tag{1}$$

Verification of solutions

$$y = e^{-a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}\left(b\left(\int (1+\ln(x))\,e^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}dx\right) + c_1\right)$$

Verified OK.

8.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = a\ln(x)^n y - abx\ln(x)^{n+1} y + b\ln(x) + b$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 11: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{\int \ln(x)^n a(1 - \ln(x)bx)dx}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta}dy\\ &= \int \frac{1}{e^{\int \ln(x)^n a(1-\ln(x)bx)dx}}dy\end{aligned}$$

8.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(a\ln(x)^n y - abx\ln(x)^{n+1} y + b\ln(x) + b\right)\mathrm{d}x$$

$$\left(-a\ln(x)^n y + abx\ln(x)^{n+1} y - b\ln(x) - b\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -a\ln(x)^n y + abx\ln(x)^{n+1} y - b\ln(x) - b \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-a\ln(x)^n y + abx\ln(x)^{n+1} y - b\ln(x) - b\right) \\ &= \ln(x)^n a(-1 + \ln(x)\,bx) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\ln(x)^n a + \ln(x)^{n+1} axb\right) - (0)\right) \\ &= \ln(x)^n a(-1 + \ln(x)\,bx) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int \ln(x)^n a(-1+\ln(x)bx)\,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\int \ln(x)^n a(-1+\ln(x)bx)dx} \\
&= \mathrm{e}^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}\left(-a\ln(x)^n y + abx\ln(x)^{n+1} y - b\ln(x) - b\right) \\
&= (ay(-1+\ln(x)\,bx)\ln(x)^n - b(1+\ln(x)))\,\mathrm{e}^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}(1) \\
&= \mathrm{e}^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left((ay(-1+\ln(x)\,bx)\ln(x)^n - b(1+\ln(x)))\,\mathrm{e}^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}\right) + \left(\mathrm{e}^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int (ay(-1+\ln(x)\,bx)\ln(x)^n - b(1+\ln(x)))\,\mathrm{e}^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
\phi = \int^x & (ay(-1+\ln(\_a)\,b\_a)\ln(\_a)^n \\
& - b(1+\ln(\_a)))\,\mathrm{e}^{a\left(\int \ln(\_a)^n(-1+\ln(\_a)b\_a)d\_a\right)}d\_a + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{a\left(\int^{x} \ln(\_a)^{n}(-1+\ln(\_a)b\_a)d\_a\right)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{a\left(\int \ln(x)^{n}(-1+\ln(x)bx)dx\right)}$. Therefore equation (4) becomes

$$e^{a\left(\int \ln(x)^{n}(-1+\ln(x)bx)dx\right)} = e^{a\left(\int^{x} \ln(\_a)^{n}(-1+\ln(\_a)b\_a)d\_a\right)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -e^{a\left(\int^{x} \ln(\_a)^{n}(-1+\ln(\_a)b\_a)d\_a\right)} + e^{a\left(\int \ln(x)^{n}(-1+\ln(x)bx)dx\right)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,dy = \int \left(-e^{a\left(\int^{x} \ln(\_a)^{n}(-1+\ln(\_a)b\_a)d\_a\right)} + e^{a\left(\int \ln(x)^{n}(-1+\ln(x)bx)dx\right)}\right) dy$$

$$f(y) = \int_{0}^{y} \left(-e^{a\left(\int^{x} \ln(\_a)^{n}(-1+\ln(\_a)b\_a)d\_a\right)} + e^{a\left(\int \ln(x)^{n}(-1+\ln(x)bx)dx\right)}\right) d\_a + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^{x} (ay(-1+\ln(\_a)\,b\_a)\ln(\_a)^{n} - b(1+\ln(\_a)))\,e^{a\left(\int \ln(\_a)^{n}(-1+\ln(\_a)b\_a)d\_a\right)} d\_a + \int_{0}^{y} \left(-e^{a\left(\int^{x} \ln(\_a)^{n}(-1+\ln(\_a)b\_a)d\_a\right)} + e^{a\left(\int \ln(x)^{n}(-1+\ln(x)bx)dx\right)}\right) d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^{x} (ay(-1+\ln(\_a)\,b\_a)\ln(\_a)^{n} - b(1+\ln(\_a)))\,e^{a\left(\int \ln(\_a)^{n}(-1+\ln(\_a)b\_a)d\_a\right)} d\_a + \int_{0}^{y} \left(-e^{a\left(\int^{x} \ln(\_a)^{n}(-1+\ln(\_a)b\_a)d\_a\right)} + e^{a\left(\int \ln(x)^{n}(-1+\ln(x)bx)dx\right)}\right) d\_a$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
&\int^{x} (ay(-1+\ln(\_a)\, b\_a)\ln(\_a)^{n} \\
&- b(1+\ln(\_a)))\, \mathrm{e}^{a\left(\int \ln(\_a)^{n}(-1+\ln(\_a)b\_a)d\_a\right)} d\_a \\
&+ \int_{0}^{y} \left(-\mathrm{e}^{a\left(\int^{x} \ln(\_a)^{n}(-1+\ln(\_a)b\_a)d\_a\right)} + \mathrm{e}^{a\left(\int \ln(x)^{n}(-1+\ln(x)bx)dx\right)}\right) d\_a = c_{1}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
&\int^{x} (ay(-1+\ln(\_a)\, b\_a)\ln(\_a)^{n} - b(1+\ln(\_a)))\, \mathrm{e}^{a\left(\int \ln(\_a)^{n}(-1+\ln(\_a)b\_a)d\_a\right)} d\_a \\
&+ \int_{0}^{y} \left(-\mathrm{e}^{a\left(\int^{x} \ln(\_a)^{n}(-1+\ln(\_a)b\_a)d\_a\right)} + \mathrm{e}^{a\left(\int \ln(x)^{n}(-1+\ln(x)bx)dx\right)}\right) d\_a = c_{1}
\end{aligned}
$$

Verified OK.

8.5.4 Maple step by step solution

Let's solve

$y' - a\ln(x)^{n} y + abx\ln(x)^{n+1} y = b\ln(x) + b$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \left(\ln(x)^{n} a - \ln(x)^{n+1} axb\right) y + b\ln(x) + b$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \left(-\ln(x)^{n} a + \ln(x)^{n+1} axb\right) y = b\ln(x) + b$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \left(-\ln(x)^{n} a + \ln(x)^{n+1} axb\right) y\right) = \mu(x)\left(b\ln(x) + b\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \left(-\ln(x)^{n} a + \ln(x)^{n+1} axb\right) y\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\left(-\ln(x)^{n} a + \ln(x)^{n+1} axb\right)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\int \ln(x)^{n} a(-1+\ln(x)bx)dx}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\,(b \ln(x) + b)\, dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\,(b \ln(x) + b)\, dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)(b \ln(x)+b)dx+c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{\int \ln(x)^n a(-1+\ln(x)bx)dx}$

  $y = \frac{\int \mathrm{e}^{\int \ln(x)^n a(-1+\ln(x)bx)dx}(b \ln(x)+b)dx+c_1}{\mathrm{e}^{\int \ln(x)^n a(-1+\ln(x)bx)dx}}$
- Simplify

  $y = \mathrm{e}^{-a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)} \left(b\left(\int (1+\ln(x))\, \mathrm{e}^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)} dx\right) + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 51

```
dsolve(diff(y(x),x)=a*(ln(x))^n*y(x)-a*b*x*(ln(x))^(n+1)*y(x)+b*ln(x)+b,y(x), singsol=all)
```

$$y(x) = \left(b\left(\int \mathrm{e}^{a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}(\ln(x)+1)\, dx\right) + c_1\right) \mathrm{e}^{-a\left(\int \ln(x)^n(-1+\ln(x)bx)dx\right)}$$

✓ Solution by Mathematica

Time used: 0.86 (sec). Leaf size: 124

```
DSolve[y'[x]==a*(Log[x])^n*y[x]-a*b*x*(Log[x])^(n+1)*y[x]+b*Log[x]+b,y[x],x,IncludeSingularS
```

$$y(x) \to \exp \left(a2^{-n-2}(-\log(x))^{-n} \log^n(x) \left(b\Gamma(n+2,-2\log(x)) +2^{n+2}\Gamma(n+1,-\log(x))\right)\right) \left(\int_1^x b \exp \left(-2^{-n-2}a\left(2^{n+2}\Gamma(n+1,-\log(K[1]))+b\Gamma(n+2,-2\log(K[1]))\right) (- \right. \right.$$
$$\left. + 1)dK[1] + c_1\right)$$

8.6 problem 15

Internal problem ID [10490]
Internal file name [`OUTPUT/9437_Monday_June_06_2022_02_32_25_PM_74432929/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$\boxed{y' - a\ln(x)^{k}\left(y - b\,x^{n} - c\right)^{2} = b\,x^{n-1}n}$$

8.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x^{2n}\ln(x)^{k}\,a\,b^{2} + 2x^{n}\ln(x)^{k}\,abc - 2x^{n}\ln(x)^{k}\,aby + \ln(x)^{k}\,a\,c^{2} - 2\ln(x)^{k}\,acy + \ln(x)^{k}\,a\,y^{2} + b\,x^{n-1}n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^{2n}\ln(x)^{k}\,a\,b^{2} + 2x^{n}\ln(x)^{k}\,abc - 2x^{n}\ln(x)^{k}\,aby + \ln(x)^{k}\,a\,c^{2} - 2\ln(x)^{k}\,acy + \ln(x)^{k}\,a\,y^{2} + \frac{b\,x^{n}n}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^{2n}\ln(x)^{k}\,a\,b^{2} + 2x^{n}\ln(x)^{k}\,abc + \ln(x)^{k}\,a\,c^{2} + b\,x^{n-1}n$, $f_1(x) = -2\ln(x)^{k}\,a\,x^{n}b - 2\ln(x)^{k}\,ac$ and $f_2(x) = \ln(x)^{k}\,a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\ln(x)^{k}\,au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{\ln(x)^k ka}{x\ln(x)}\\
f_1f_2 &= \left(-2\ln(x)^k a\,x^n b - 2\ln(x)^k ac\right)\ln(x)^k a\\
f_2^2f_0 &= \ln(x)^{2k} a^2\left(x^{2n}\ln(x)^k a\,b^2 + 2x^n\ln(x)^k abc + \ln(x)^k a\,c^2 + b\,x^{n-1}n\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\ln(x)^k au''(x) - \left(\frac{\ln(x)^k ka}{x\ln(x)} + \left(-2\ln(x)^k a\,x^n b - 2\ln(x)^k ac\right)\ln(x)^k a\right)u'(x) + \ln(x)^{2k} a^2\left(x^{2n}\ln(x)^k a$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{(-\ln(x))^{-k}\left(-c_2\left(x(-\ln(x))^k + \Gamma(k,-\ln(x))\,k - \Gamma(k+1)\right)\ln(x)^{\frac{k}{2}} + \ln(x)^{-\frac{k}{2}} c_1(-\ln(x))^k\right)\mathrm{e}^{-\frac{\left(\int\left(2\ln\right.\right.}{}}}{\sqrt{x}}$$

The above shows that

$$u'(x) = \frac{(-\ln(x))^{-k}\left(\left((x^{n+1}b + cx)(-\ln(x))^k - (b\,x^n + c)(-\Gamma(k,-\ln(x))\,k + \Gamma(k+1))\right)ac_2\ln(x)^{\frac{3k}{2}} - (-\ln\right.}{\sqrt{x}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\left((x^{n+1}b + cx)(-\ln(x))^k - (b\,x^n + c)(-\Gamma(k,-\ln(x))\,k + \Gamma(k+1))\right)ac_2\ln(x)^{\frac{3k}{2}} - (-\ln(x))^k\ln(\right.}{a\left(-c_2\left(x(-\ln(x))^k + \Gamma(k,-\ln(x))\,k - \Gamma(k+1)\right)\ln(x)^{\frac{k}{2}} + \ln(x)^{-\frac{k}{2}} c_1(-\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\ln(x)^{-k}\left(-(-\ln(x))^{k}\ln(x)^{\frac{k}{2}}(x^{n}c_{3}ab + c_{3}ca + 1) + \ln(x)^{\frac{3k}{2}}a\Big((x^{n+1}b + cx)(-\ln(x))^{k} + (\Gamma(k, -\ln(x)\right.}{\left(\left(x(-\ln(x))^{k} + \Gamma(k, -\ln(x))k - \Gamma(k+1)\right)\ln(x)^{\frac{k}{2}} - \ln(x)^{-\frac{k}{2}}c_{3}(-\ln(x))^{k}\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^{-k}\left(-(-\ln(x))^{k}\ln(x)^{\frac{k}{2}}(x^{n}c_{3}ab + c_{3}ca + 1) + \ln(x)^{\frac{3k}{2}}a\Big((x^{n+1}b + cx)(-\ln(x))^{k} + (\Gamma(k, -\ln(x)\right.}{\left(\left(x(-\ln(x))^{k} + \Gamma(k, -\ln(x))k - \Gamma(k+1)\right)\ln(x)^{\frac{k}{2}} - \ln(x)^{-\frac{k}{2}}c_{3}(-\ln(x))^{k}\right.} \tag{1}$$

Verification of solutions

$$y = \frac{\ln(x)^{-k}\left(-(-\ln(x))^{k}\ln(x)^{\frac{k}{2}}(x^{n}c_{3}ab + c_{3}ca + 1) + \ln(x)^{\frac{3k}{2}}a\Big((x^{n+1}b + cx)(-\ln(x))^{k} + (\Gamma(k, -\ln(x)\right.}{\left(\left(x(-\ln(x))^{k} + \Gamma(k, -\ln(x))k - \Gamma(k+1)\right)\ln(x)^{\frac{k}{2}} - \ln(x)^{-\frac{k}{2}}c_{3}(-\ln(x))^{k}\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (d) successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)=a*(ln(x))^k*(y(x)-b*x^n-c)^2+b*n*x^(n-1),y(x), singsol=all)
```

$$y(x) = b\,x^{n} + c + \frac{1}{c_{1} - a\left(\int \ln\left(x\right)^{k} dx\right)}$$

✓ Solution by Mathematica

Time used: 1.572 (sec). Leaf size: 51

```
DSolve[y'[x]==a*(Log[x])^k*(y[x]-b*x^n-c)^2+b*n*x^(n-1),y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{-a(-\log(x))^{-k}\log^{k}(x)\Gamma(k+1,-\log(x))+c_{1}} + bx^{n} + c$$
$$y(x) \to bx^{n} + c$$

8.7 problem 16

Internal problem ID [10491]
Internal file name [`OUTPUT/9438_Monday_June_06_2022_02_32_27_PM_544547/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\ln (x)^n y^2 - b\ln (x)^m y = bc\ln (x)^m - a\,c^2 \ln (x)^n}$$

8.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\ln (x)^n y^2 + b\ln (x)^m y + bc\ln (x)^m - a\,c^2 \ln (x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\ln (x)^n y^2 + b\ln (x)^m y + bc\ln (x)^m - a\,c^2 \ln (x)^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = bc\ln (x)^m - a\,c^2 \ln (x)^n$, $f_1(x) = \ln (x)^m b$ and $f_2(x) = \ln (x)^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\ln (x)^n au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{\ln(x)^n n a}{x \ln(x)} \\
f_1 f_2 &= \ln(x)^m b \ln(x)^n a \\
f_2^2 f_0 &= \ln(x)^{2n} a^2 \left(bc \ln(x)^m - a\, c^2 \ln(x)^n\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\ln(x)^n a u''(x) - \left(\frac{\ln(x)^n n a}{x \ln(x)} + \ln(x)^m b \ln(x)^n a\right) u'(x) + \ln(x)^{2n} a^2 \left(bc \ln(x)^m - a\, c^2 \ln(x)^n\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
u(x) = \mathrm{DESol}\Bigg(\Bigg\{_Y''(x) - _Y'(x)\left(\frac{n}{x \ln(x)} + \ln(x)^m b\right) \\
+ a_Y(x)\left(bc \ln(x)^{m+n} - a\, c^2 \ln(x)^{2n}\right)\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

The above shows that

$$\begin{aligned}
u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\Bigg(\Bigg\{_Y''(x) - _Y'(x)\left(\frac{n}{x \ln(x)} + \ln(x)^m b\right) \\
+ a_Y(x)\left(bc \ln(x)^{m+n} - a\, c^2 \ln(x)^{2n}\right)\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \_Y'(x)\left(\frac{n}{x \ln(x)} + \ln(x)^m b\right) + a\_Y(x)\left(bc \ln(x)^{m+n} - a\, c^2 \ln(x)^{2n}\right)\right\}, \{\_Y(\right.\right.}{a\, \mathrm{DESol}\left(\left\{\_Y''(x) - \_Y'(x)\left(\frac{n}{x \ln(x)} + \ln(x)^m b\right) + a\_Y(x)\left(bc \ln(x)^{m+n} - a\, c^2 \ln(x)^{2n}\right)\right\}, \{\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-\ln(x)\ln(x)^{2n} \_\_Y(x) a^2 c^2 x + \ln(x)\ln(x)^{m+n} \_\_Y(x) abcx - \ln(x)^{m+1} \_\_Y'(x) bx + \_\_Y''(x)\ln(x) x - n \_\_Y'(x)}{\ln(x) x}\right\}, \{\_\right.\right.}{a\,\mathrm{DESol}\left(\left\{\frac{-x\,a^2 c^2 \_\_Y(x)\ln(x)^{1+2n} + abcx \_\_Y(x)\ln(x)^{1+m+n} - \ln(x)^{m+1} \_\_Y'(x) bx + \_\_Y''(x)\ln(x) x - n \_\_Y'(x)}{x\ln(x)}\right\}, \{\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-\ln(x)\ln(x)^{2n} \_\_Y(x) a^2 c^2 x + \ln(x)\ln(x)^{m+n} \_\_Y(x) abcx - \ln(x)^{m+1} \_\_Y'(x) bx + \_\_Y''(x)\ln(x) x - n \_\_Y'(x)}{\ln(x) x}\right\}, \{\_\right.\right.}{a\,\mathrm{DESol}\left(\left\{\frac{-x\,a^2 c^2 \_\_Y(x)\ln(x)^{1+2n} + abcx \_\_Y(x)\ln(x)^{1+m+n} - \ln(x)^{m+1} \_\_Y'(x) bx + \_\_Y''(x)\ln(x) x - n \_\_Y'(x)}{x\ln(x)}\right\}, \{\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-\ln(x)\ln(x)^{2n} \_\_Y(x) a^2 c^2 x + \ln(x)\ln(x)^{m+n} \_\_Y(x) abcx - \ln(x)^{m+1} \_\_Y'(x) bx + \_\_Y''(x)\ln(x) x - n \_\_Y'(x)}{\ln(x) x}\right\}, \{\_\right.\right.}{a\,\mathrm{DESol}\left(\left\{\frac{-x\,a^2 c^2 \_\_Y(x)\ln(x)^{1+2n} + abcx \_\_Y(x)\ln(x)^{1+m+n} - \ln(x)^{m+1} \_\_Y'(x) bx + \_\_Y''(x)\ln(x) x - n \_\_Y'(x)}{x\ln(x)}\right\}, \{\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (ln(x)^m*ln(x)*b*x+n)*(diff(y(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 99

```
dsolve(diff(y(x),x)=a*(ln(x))^n*y(x)^2+b*(ln(x))^m*y(x)+b*c*(ln(x))^m-a*c^2*(ln(x))^n,y(x),
```

$$y(x) = \frac{-ca\left(\int \ln(x)^n \, e^{-\left(\int (2\ln(x)^n ac - \ln(x)^m b)dx\right)} dx\right) - c_1 c - e^{-\left(\int (2\ln(x)^n ac - \ln(x)^m b)dx\right)}}{c_1 + a\left(\int \ln(x)^n \, e^{-\left(\int (2\ln(x)^n ac - \ln(x)^m b)dx\right)} dx\right)}$$

✓ Solution by Mathematica

Time used: 3.927 (sec). Leaf size: 385

```
DSolve[y'[x]==a*(Log[x])^n*y[x]^2+b*(Log[x])^m*y[x]+b*c*(Log[x])^m-a*c^2*(Log[x])^n,y[x],x,I
```

$$\text{Solve}\left[\int_1^x \frac{\exp\left(b\Gamma(m+1,-\log(K[1]))(-\log(K[1]))^{-m}\log^m(K[1]) - 2ac\Gamma(n+1,-\log(K[1]))(-\log(K[\right.}{ab(m-n)(c+y(x))}\right.$$

$$+\int_1^{y(x)}\left(\frac{\exp\left(b\Gamma(m+1,-\log(x))(-\log(x))^{-m}\log^m(x) - 2ac\Gamma(n+1,-\log(x))(-\log(x))^{-n}\log^n(x)\right)}{ab(m-n)(c+K[2])^2}\right.$$

$$-\int_1^x\left(-\frac{\exp\left(b\Gamma(m+1,-\log(K[1]))(-\log(K[1]))^{-m}\log^m(K[1]) - 2ac\Gamma(n+1,-\log(K[1]))(-\log(K[1]\right.}{b(m-n)(c+K[2])}\right.$$

8.8 problem 17

Internal problem ID [10492]
Internal file name [`OUTPUT/9439_Monday_June_06_2022_02_32_31_PM_90703748/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Riccati]
```

$$\boxed{y'x-(ay+b\ln(x))^2=0}$$

8.8.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{(ya+b\ln(x))^2}{x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(ya + b\ln(x))^2 (b_3 - a_2)}{x} - \frac{(ya + b\ln(x))^4 a_3}{x^2} - \left(\frac{2(ya + b\ln(x))\, b}{x^2} - \frac{(ya + b\ln(x))^2}{x^2}\right)(xa_2 + ya_3 + a_1) - \frac{2(ya + b\ln(x))\, a(xb_2 + yb_3 + b_1)}{x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\ln(x)^4 b^4 a_3 + 4\ln(x)^3 a\, b^3 y a_3 + 6\ln(x)^2 a^2 b^2 y^2 a_3 + 4\ln(x)\, a^3 b\, y^3 a_3 + a^4 y^4 a_3 - \ln(x)^2 b^2 x b_3 - \ln(x)^2 b^2}{}$$
$$= 0$$

Setting the numerator to zero gives

$$-\ln(x)^4 b^4 a_3 - 4\ln(x)^3 a\, b^3 y a_3 - 6\ln(x)^2 a^2 b^2 y^2 a_3 - 4\ln(x)\, a^3 b\, y^3 a_3 - a^4 y^4 a_3 + \ln(x)^2 b^2 x b_3 + \ln(x)^2 b^2 y a_3 - 2\ln(x)\, ab\, x^2 b_2 + 2\ln(x)\, ab\, y^2 a_3 - 2a^2 x^2 y b_2 - a^2 x\, y^2 b_3 + a^2 y^3 a_3 + \ln(x)^2 b^2 a_1 - 2\ln(x)\, abx b_1 + 2\ln(x)\, aby a_1 - 2\ln(x)\, b^2 x a_2 - 2\ln(x)\, b^2 y a_3 - 2a^2 xy b_1 + a^2 y^2 a_1 - 2abxy a_2 - 2ab\, y^2 a_3 - 2\ln(x)\, b^2 a_1 - 2aby a_1 + b_2 x^2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \ln(x)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \ln(x) = v_3\}$$

The above PDE (6E) now becomes

$$-a^4 v_2^4 a_3 - 4v_3 a^3 b v_2^3 a_3 - 6v_3^2 a^2 b^2 v_2^2 a_3 - 4v_3^3 a\, b^3 v_2 a_3 - v_3^4 b^4 a_3 + a^2 v_2^3 a_3 - 2a^2 v_1^2 v_2 b_2 - a^2 v_1 v_2^2 b_3 + 2v_3 ab v_2^2 a_3 - 2v_3 ab v_1^2 b_2 + v_3^2 b^2 v_2 a_3 + v_3^2 b^2 v_1 b_3 + a^2 v_2^2 a_1 - 2a^2 v_1 v_2 b_1 + 2v_3 ab v_2 a_1 - 2ab v_1 v_2 a_2 - 2ab v_2^2 a_3 - 2v_3 ab v_1 b_1 + v_3^2 b^2 a_1 - 2v_3 b^2 v_1 a_2 - 2v_3 b^2 v_2 a_3 - 2ab v_2 a_1 - 2v_3 b^2 a_1 + b_2 v_1^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-2a^2v_1^2v_2b_2 - 2v_3abv_1^2b_2 + b_2v_1^2 - a^2v_1v_2^2b_3 + \left(-2a^2b_1 - 2aba_2\right)v_1v_2 \\
&+ v_3^2b^2v_1b_3 + \left(-2abb_1 - 2b^2a_2\right)v_1v_3 - a^4v_2^4a_3 - 4v_3a^3bv_2^3a_3 + a^2v_2^3a_3 \\
&- 6v_3^2a^2b^2v_2^2a_3 + 2v_3abv_2^2a_3 + \left(a^2a_1 - 2aba_3\right)v_2^2 - 4v_3^3a\,b^3v_2a_3 + v_3^2b^2v_2a_3 \\
&+ \left(2aba_1 - 2b^2a_3\right)v_2v_3 - 2abv_2a_1 - v_3^4b^4a_3 + v_3^2b^2a_1 - 2v_3b^2a_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_2 &= 0 \\
a^2a_3 &= 0 \\
b^2a_1 &= 0 \\
b^2a_3 &= 0 \\
b^2b_3 &= 0 \\
-2a^2b_2 &= 0 \\
-a^2b_3 &= 0 \\
-a^4a_3 &= 0 \\
-2b^2a_1 &= 0 \\
-b^4a_3 &= 0 \\
-2aba_1 &= 0 \\
2aba_3 &= 0 \\
-2abb_2 &= 0 \\
-4a\,b^3a_3 &= 0 \\
-6a^2b^2a_3 &= 0 \\
-4a^3ba_3 &= 0 \\
2aba_1 - 2b^2a_3 &= 0 \\
a^2a_1 - 2aba_3 &= 0 \\
-2abb_1 - 2b^2a_2 &= 0 \\
-2a^2b_1 - 2aba_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -\frac{ab_1}{b} \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{ax}{b} \\
\eta &= 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 1 - \left(\frac{(ya + b\ln(x))^2}{x}\right)\left(-\frac{ax}{b}\right) \\
&= \frac{\ln(x)^2\, a\, b^2 + 2\ln(x)\, a^2 by + a^3 y^2 + b}{b} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{\ln(x)^2 a\, b^2 + 2\ln(x) a^2 b y + a^3 y^2 + b}{b}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{b \arctan\left(\frac{2y\, a^3 + 2\ln(x) a^2 b}{2a\sqrt{ab}}\right)}{a\sqrt{ab}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{(ya + b\ln(x))^2}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{b^2}{ax\left(\ln(x)^2 a\, b^2 + 2\ln(x) a^2 b y + a^3 y^2 + b\right)} \\
S_y &= \frac{b}{\ln(x)^2 a\, b^2 + 2\ln(x) a^2 b y + a^3 y^2 + b}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{b}{ax} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{b}{aR}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{b \ln(R)}{a} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{b}\, \arctan\left(\frac{\sqrt{a}\,(ay+b\ln(x))}{\sqrt{b}}\right)}{a^{\frac{3}{2}}} = \frac{b \ln(x)}{a} + c_1$$

Which simplifies to

$$\frac{\sqrt{b}\, \arctan\left(\frac{\sqrt{a}\,(ay+b\ln(x))}{\sqrt{b}}\right)}{a^{\frac{3}{2}}} = \frac{b \ln(x)}{a} + c_1$$

Which gives

$$y = -\frac{b \ln(x)\sqrt{a} - \tan\left(\frac{\sqrt{a}\,(c_1 a+b\ln(x))}{\sqrt{b}}\right)\sqrt{b}}{a^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = -\frac{b \ln(x)\sqrt{a} - \tan\left(\frac{\sqrt{a}\,(c_1 a+b\ln(x))}{\sqrt{b}}\right)\sqrt{b}}{a^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = -\frac{b \ln(x)\sqrt{a} - \tan\left(\frac{\sqrt{a}\,(c_1 a+b\ln(x))}{\sqrt{b}}\right)\sqrt{b}}{a^{\frac{3}{2}}}$$

Verified OK.

8.8.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{(ya + b\ln(x))^2}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{\ln(x)^2 b^2}{x} + \frac{2\ln(x)\, aby}{x} + \frac{a^2 y^2}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\ln(x)^2 b^2}{x}$, $f_1(x) = \frac{2ab\ln(x)}{x}$ and $f_2(x) = \frac{a^2}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{a^2 u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{a^2}{x^2} \\ f_1 f_2 &= \frac{2a^3 b \ln(x)}{x^2} \\ f_2^2 f_0 &= \frac{a^4 \ln(x)^2 b^2}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{a^2 u''(x)}{x} - \left(-\frac{a^2}{x^2} + \frac{2a^3 b \ln(x)}{x^2}\right) u'(x) + \frac{a^4 \ln(x)^2 b^2 u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = e^{\frac{\ln(x)^2 ab}{2}} \left(x^{-\sqrt{-ab}} c_2 + x^{\sqrt{-ab}} c_1\right)$$

The above shows that

$$u'(x) = \frac{e^{\frac{\ln(x)^2 ab}{2}} \left(c_2 \left(a \ln(x) b - \sqrt{-ab}\right) x^{-\sqrt{-ab}} + c_1 x^{\sqrt{-ab}} \left(a \ln(x) b + \sqrt{-ab}\right)\right)}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2 \left(a \ln(x) b - \sqrt{-ab}\right) x^{-\sqrt{-ab}} + c_1 x^{\sqrt{-ab}} \left(a \ln(x) b + \sqrt{-ab}\right)}{a^2 \left(x^{-\sqrt{-ab}} c_2 + x^{\sqrt{-ab}} c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3\left(a\ln\left(x\right)b + \sqrt{-ab}\right)x^{2\sqrt{-ab}} - a\ln\left(x\right)b + \sqrt{-ab}}{a^2\left(1 + x^{2\sqrt{-ab}}c_3\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_3\left(a\ln\left(x\right)b + \sqrt{-ab}\right)x^{2\sqrt{-ab}} - a\ln\left(x\right)b + \sqrt{-ab}}{a^2\left(1 + x^{2\sqrt{-ab}}c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-c_3\left(a\ln\left(x\right)b + \sqrt{-ab}\right)x^{2\sqrt{-ab}} - a\ln\left(x\right)b + \sqrt{-ab}}{a^2\left(1 + x^{2\sqrt{-ab}}c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -b/(a*x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      <- quadrature successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve(x*diff(y(x),x)=(a*y(x)+b*ln(x))^2,y(x), singsol=all)
```

$$y(x) = \frac{-\ln(x)\,ab + \tan\left(\left(\ln(x) + c_1\right)\sqrt{ab}\right)\sqrt{ab}}{a^2}$$

✓ Solution by Mathematica

Time used: 6.524 (sec). Leaf size: 43

```
DSolve[x*y'[x]==(a*y[x]+b*Log[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{b\log(x)}{a} + \sqrt{\frac{b}{a^3}}\tan\left(a^2\sqrt{\frac{b}{a^3}}\log(x) + c_1\right)$$

8.9 problem 18

Internal problem ID [10493]
Internal file name [OUTPUT/9440_Monday_June_06_2022_02_32_32_PM_95367239/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y'x - a\ln(\lambda x)^m y^2 - ky = a\,b^2x^{2k}\ln(\lambda x)^m}$$

8.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\ln(\lambda x)^m y^2 + ky + a\,b^2x^{2k}\ln(\lambda x)^m}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,b^2x^{2k}\ln(\lambda x)^m}{x} + \frac{a\ln(\lambda x)^m y^2}{x} + \frac{ky}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{a\,b^2x^{2k}\ln(\lambda x)^m}{x}$, $f_1(x) = \frac{k}{x}$ and $f_2(x) = \frac{a\ln(\lambda x)^m}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{a\ln(\lambda x)^m u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{a \ln (\lambda x)^m m}{x^2 \ln (\lambda x)} - \frac{a \ln (\lambda x)^m}{x^2} \\ f_1 f_2 &= \frac{k a \ln (\lambda x)^m}{x^2} \\ f_2^2 f_0 &= \frac{a^3 \ln (\lambda x)^{3m} b^2 x^{2k}}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{a \ln (\lambda x)^m u''(x)}{x} - \left(\frac{a \ln (\lambda x)^m m}{x^2 \ln (\lambda x)} - \frac{a \ln (\lambda x)^m}{x^2} + \frac{k a \ln (\lambda x)^m}{x^2}\right) u'(x) + \frac{a^3 \ln (\lambda x)^{3m} b^2 x^{2k} u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \mathrm{e}^{iab\left(\int x^{k-1} \ln(\lambda x)^m dx\right)} + c_2 \mathrm{e}^{-iab\left(\int x^{k-1} \ln(\lambda x)^m dx\right)}$$

The above shows that

$$u'(x) = iab\, x^{k-1} \ln (\lambda x)^m \mathrm{e}^{-iab\left(\int x^{k-1} \ln(\lambda x)^m dx\right)} \left(c_1 \mathrm{e}^{2iab\left(\int x^{k-1} \ln(\lambda x)^m dx\right)} - c_2\right)$$

Using the above in (1) gives the solution

$$y = -\frac{ib\, x^{k-1} \mathrm{e}^{-iab\left(\int x^{k-1} \ln(\lambda x)^m dx\right)} \left(c_1 \mathrm{e}^{2iab\left(\int x^{k-1} \ln(\lambda x)^m dx\right)} - c_2\right) x}{c_1 \mathrm{e}^{iab\left(\int x^{k-1} \ln(\lambda x)^m dx\right)} + c_2 \mathrm{e}^{-iab\left(\int x^{k-1} \ln(\lambda x)^m dx\right)}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{ib\, x^k \left(c_3 \mathrm{e}^{2iab\left(\int x^{k-1} \ln(\lambda x)^m dx\right)} - 1\right)}{c_3 \mathrm{e}^{2iab\left(\int x^{k-1} \ln(\lambda x)^m dx\right)} + 1}$$

Summary
The solution(s) found are the following

$$y = -\frac{ib\,x^k\left(c_3 e^{2iab\left(\int x^{k-1}\ln(\lambda x)^m dx\right)}-1\right)}{c_3 e^{2iab\left(\int x^{k-1}\ln(\lambda x)^m dx\right)}+1} \tag{1}$$

Verification of solutions

$$y = -\frac{ib\,x^k\left(c_3 e^{2iab\left(\int x^{k-1}\ln(\lambda x)^m dx\right)}-1\right)}{c_3 e^{2iab\left(\int x^{k-1}\ln(\lambda x)^m dx\right)}+1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple
Time used: 0.047 (sec). Leaf size: 31

```
dsolve(x*diff(y(x),x)=a*(ln(lambda*x))^m*y(x)^2+k*y(x)+a*b^2*x^(2*k)*(ln(lambda*x))^m,y(x),
```

$$y(x) = -\tan\left(-ab\left(\int x^{-1+k}\ln(x\lambda)^m\,dx\right)+c_1\right)b\,x^k$$

✓ Solution by Mathematica

Time used: 2.161 (sec). Leaf size: 70

```
DSolve[x*y'[x]==a*(Log[\[Lambda]*x])^m*y[x]^2+k*y[x]+a*b^2*x^(2*k)*(Log[\[Lambda]*x])^m,y[x]
```

$$y(x) \to \sqrt{b^2} x^k \tan \left(\frac{a\sqrt{b^2} x^k (\lambda x)^{-k} \log^m(\lambda x)(-k \log(\lambda x))^{-m} \Gamma(m+1, -k \log(x\lambda))}{k} + c_1 \right)$$

8.10 problem 19

Internal problem ID [10494]
Internal file name [`OUTPUT/9441_Monday_June_06_2022_02_32_34_PM_36674882/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$\boxed{y'x - a\,x^n(y + b\ln(x))^2 = -b}$$

8.10.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,b^2x^n\ln(x)^2 + 2\ln(x)\,x^n aby + a\,x^n y^2 - b}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{\ln(x)^2\,x^n a\,b^2}{x} + \frac{2\ln(x)\,x^n aby}{x} + \frac{x^n a\,y^2}{x} - \frac{b}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{-b+a\,b^2x^n\ln(x)^2}{x}$, $f_1(x) = \frac{2ab\,x^n\ln(x)}{x}$ and $f_2(x) = \frac{a\,x^n}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{a\,x^n u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{a\,x^n n}{x^2} - \frac{a\,x^n}{x^2} \\ f_1 f_2 &= \frac{2a^2 b\,x^{2n} \ln(x)}{x^2} \\ f_2^2 f_0 &= \frac{a^2 x^{2n} \left(-b + a\,b^2 x^n \ln(x)^2\right)}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{a\,x^n u''(x)}{x} - \left(\frac{a\,x^n n}{x^2} - \frac{a\,x^n}{x^2} + \frac{2a^2 b\,x^{2n} \ln(x)}{x^2}\right) u'(x) + \frac{a^2 x^{2n} \left(-b + a\,b^2 x^n \ln(x)^2\right) u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{-\frac{ab\,x^n}{n^2}} \left(x^{\frac{ab\,x^n}{n}} c_1 + x^{\frac{n^2 + a\,x^n b}{n}} c_2\right)$$

The above shows that

$$u'(x) = \frac{\left(\ln(x)\,x^{\frac{2n^2 + a\,x^n b}{n}} c_2 ab + \ln(x)\,x^{\frac{n^2 + a\,x^n b}{n}} c_1 ab + x^{\frac{n^2 + a\,x^n b}{n}} c_2 n\right) \mathrm{e}^{-\frac{ab\,x^n}{n^2}}}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\ln(x)\,x^{\frac{2n^2 + a\,x^n b}{n}} c_2 ab + \ln(x)\,x^{\frac{n^2 + a\,x^n b}{n}} c_1 ab + x^{\frac{n^2 + a\,x^n b}{n}} c_2 n\right) x^{-n}}{a\left(x^{\frac{ab\,x^n}{n}} c_1 + x^{\frac{n^2 + a\,x^n b}{n}} c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x^{-\frac{ab\,x^n}{n}} \left(ab\,x^{\frac{n^2 + a\,x^n b}{n}} \ln(x) + x^{\frac{ab\,x^n}{n}} (\ln(x)\,c_3 ab + n)\right)}{a\,(x^n + c_3)}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^{-\frac{ab\,x^{n}}{n}} \left(ab\,x^{\frac{n^{2}+a\,x^{n} b}{n}} \ln\left(x\right)+x^{\frac{ab\,x^{n}}{n}} \left(\ln\left(x\right) c_{3} a b +n \right)\right)}{a \left(x^{n}+c_{3}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{x^{-\frac{ab\,x^{n}}{n}} \left(ab\,x^{\frac{n^{2}+a\,x^{n} b}{n}} \ln\left(x\right)+x^{\frac{ab\,x^{n}}{n}} \left(\ln\left(x\right) c_{3} a b +n \right)\right)}{a \left(x^{n}+c_{3}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (d) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(x*diff(y(x),x)=a*x^n*(y(x)+b*ln(x))^2-b,y(x), singsol=all)
```

$$y(x) = -b\ln\left(x\right)+\frac{n}{c_{1} n -a\,x^{n}}$$

✓ Solution by Mathematica

Time used: 0.649 (sec). Leaf size: 35

```
DSolve[x*y'[x]==a*x^n*(y[x]+b*Log[x])^2-b,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -b\log(x) + \frac{n}{-ax^n + c_1 n}$$
$$y(x) \to -b\log(x)$$

8.11 problem 20

Internal problem ID [10495]
Internal file name [`OUTPUT/9442_Monday_June_06_2022_02_32_35_PM_83494421/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - a\,x^{2n}\ln(x)\,y^2 - (b\,x^n\ln(x) - n)\,y = c\ln(x)}$$

8.11.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,x^{2n}\ln(x)\,y^2 + x^n\ln(x)\,by + c\ln(x) - ny}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,x^{2n}\ln(x)\,y^2}{x} + \frac{x^n\ln(x)\,by}{x} + \frac{c\ln(x)}{x} - \frac{ny}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{c\ln(x)}{x}$, $f_1(x) = \frac{b\,x^n\ln(x)-n}{x}$ and $f_2(x) = \frac{x^{2n}\ln(x)a}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{\frac{x^{2n}\ln(x)au}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2x^{2n}n\ln(x)\,a}{x^2} + \frac{a\,x^{2n}}{x^2} - \frac{\ln(x)\,a\,x^{2n}}{x^2} \\ f_1f_2 &= \frac{(b\,x^n\ln(x) - n)\,x^{2n}\ln(x)\,a}{x^2} \\ f_2^2f_0 &= \frac{x^{4n}\ln(x)^3\,a^2c}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{x^{2n}\ln(x)\,au''(x)}{x} - \left(\frac{2x^{2n}n\ln(x)\,a}{x^2} + \frac{a\,x^{2n}}{x^2} - \frac{\ln(x)\,a\,x^{2n}}{x^2} + \frac{(b\,x^n\ln(x) - n)\,x^{2n}\ln(x)\,a}{x^2}\right)u'(x) + \frac{x^{4n}\ln(}{}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(c_1\,\text{BesselJ}\left(\frac{\sqrt{3}\,\sqrt{-ca}}{8b}, \frac{\sqrt{3}\,\sqrt{2}\,\sqrt{ca\,x^{2n}}\,x^{-n}}{8b}\right) + c_2\,\text{BesselY}\left(\frac{\sqrt{3}\,\sqrt{-ca}}{8b}, \frac{\sqrt{3}\,\sqrt{2}\,\sqrt{ca\,x^{2n}}\,x^{-n}}{8b}\right)\right)\sqrt{\ln(x)}\,x^{\frac{b\,x^n+3n^2}{2n}}\,\mathrm{e}^{-\frac{b\,x^n}{2n^2}}$$

The above shows that

$$u'(x) = \frac{x^{\frac{-2n+b\,x^n+3n^2}{2n}}\,\mathrm{e}^{-\frac{b\,x^n}{2n^2}}\left(1 + \ln(x)^2\,x^nb + 3n\ln(x)\right)\left(c_1\,\text{BesselJ}\left(\frac{\sqrt{3}\,\sqrt{-ca}}{8b}, \frac{\sqrt{3}\,\sqrt{2}\,\sqrt{ca\,x^{2n}}\,x^{-n}}{8b}\right) + c_2\,\text{BesselY}\left(\frac{\sqrt{3}\,\sqrt{}}{8}\right.\right.}{2\sqrt{\ln(x)}}$$

Using the above in (1) gives the solution

$$y = -\frac{x^{\frac{-2n+b\,x^n+3n^2}{2n}}\left(1 + \ln(x)^2\,x^nb + 3n\ln(x)\right)x^{-2n}x\,x^{-\frac{b\,x^n+3n^2}{2n}}}{2\ln(x)^2\,a}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x^{-2n}\left(1+\ln\left(x\right)^{2} x^{n} b+3 n \ln\left(x\right)\right)}{2 a \ln\left(x\right)^{2}}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^{-2n}\left(1+\ln\left(x\right)^{2} x^{n} b+3 n \ln\left(x\right)\right)}{2 a \ln\left(x\right)^{2}} \tag{1}$$

Verification of solutions

$$y = -\frac{x^{-2n}\left(1+\ln\left(x\right)^{2} x^{n} b+3 n \ln\left(x\right)\right)}{2 a \ln\left(x\right)^{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 80

```
dsolve(x*diff(y(x),x)=a*x^(2*n)*ln(x)*y(x)^2+(b*x^n*ln(x)-n)*y(x)+c*ln(x),y(x), singsol=all)
```

$$y(x) = \frac{\left(\tan\left(\frac{(b(n\ln(x)-1)x^{n}+c_{1}n^{2})\sqrt{4a\,b^{2}c-b^{4}}}{2b^{2}n^{2}}\right)\sqrt{4a\,b^{2}c-b^{4}}-b^{2}\right)x^{-n}}{2ab}$$

✓ Solution by Mathematica

Time used: 1.872 (sec). Leaf size: 130

```
DSolve[x*y'[x]==a*x^(2*n)*Log[x]*y[x]^2+(b*x^n*Log[x]-n)*y[x]+c*Log[x],y[x],x,IncludeSingula
```

$$y(x) \to \frac{x^{-n}\left(-b+\frac{\sqrt{b^2-4ac}\left(-e^{\frac{x^n\sqrt{b^2-4ac}(n\log(x)-1)}{n^2}}+c_1\right)}{e^{\frac{x^n\sqrt{b^2-4ac}(n\log(x)-1)}{n^2}}+c_1}\right)}{2a}$$

$$y(x) \to \frac{x^{-n}\left(\sqrt{b^2-4ac}-b\right)}{2a}$$

8.12 problem 21

Internal problem ID [10496]
Internal file name [`OUTPUT/9443_Monday_June_06_2022_02_32_37_PM_81719794/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{x^2y' - y^2a^2x^2 + xy = b^2 \ln(x)^n}$$

8.12.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2a^2x^2 - yx + b^2 \ln(x)^n}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a^2y^2 + \frac{b^2 \ln(x)^n}{x^2} - \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{b^2 \ln(x)^n}{x^2}$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = a^2$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{a^2u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= -\frac{a^2}{x} \\ f_2^2 f_0 &= \frac{a^4 b^2 \ln(x)^n}{x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$a^2 u''(x) + \frac{a^2 u'(x)}{x} + \frac{a^4 b^2 \ln(x)^n u(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\text{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right) c_1 + \text{BesselY}\left(\frac{1}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right) c_2\right) \sqrt{\ln(x)}$$

The above shows that

$$u'(x) = \frac{-\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}\,\text{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right) c_1 - \sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}\,\text{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right) c_2 +}{\sqrt{\ln(x)}\,x}$$

Using the above in (1) gives the solution

$$y = -\frac{-\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}\,\text{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right) c_1 - \sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}\,\text{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right) c}{\ln(x)\,x\,a^2\left(\text{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right) c_1 + \text{BesselY}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\ln(x)^{1+\frac{n}{2}} \left(\operatorname{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right) c_3 + \operatorname{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right)\right)\sqrt{a^2b^2} - \operatorname{BesselJ}\left(\frac{1}{2+n}, 2\sqrt{a \cdots\right.}{\ln(x)\, x\, a^2 \left(\operatorname{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right) c_3 + \operatorname{BesselY}\left(\frac{1}{2+n}, \frac{2\sqrt{a^2b^2}}{2 \cdots}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^{1+\frac{n}{2}} \left(\operatorname{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right) c_3 + \operatorname{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right)\right)\sqrt{a^2b^2} - \operatorname{BesselJ}\left(\frac{1}{2+n}, 2\sqrt{a \cdots\right.}{\ln(x)\, x\, a^2 \left(\operatorname{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right) c_3 + \operatorname{BesselY}\left(\frac{1}{2+n}, \frac{2\sqrt{a^2b^2}}{2 \cdots}\right.\right.} \tag{1}$$

Verification of solutions

$$y = \frac{\ln(x)^{1+\frac{n}{2}} \left(\operatorname{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right) c_3 + \operatorname{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right)\right)\sqrt{a^2b^2} - \operatorname{BesselJ}\left(\frac{1}{2+n}, 2\sqrt{a \cdots\right.}{\ln(x)\, x\, a^2 \left(\operatorname{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{1+\frac{n}{2}}}{2+n}\right) c_3 + \operatorname{BesselY}\left(\frac{1}{2+n}, \frac{2\sqrt{a^2b^2}}{2 \cdots}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(diff(y(x), x))/x-a^2*b^2*ln(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying an equivalence, under non-integer power transformations,
            to LODEs admitting Liouvillian solutions.
            -> Trying a Liouvillian solution using Kovacics algorithm
            <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            <- Bessel successful
         <- special function solution successful
         Change of variables used:
            [x = exp(t)]
         Linear ODE actually solved:
            a^2*b^2*t^n*u(t)+diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 251

```
dsolve(x^2*diff(y(x),x)=a^2*x^2*y(x)^2-x*y(x)+b^2*(ln(x))^n,y(x), singsol=all)
```

$$y(x) = \frac{\ln(x)^{\frac{n}{2}+1} \operatorname{BesselY}\left(\frac{3+n}{n+2}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{\frac{n}{2}+1}}{n+2}\right)\sqrt{a^2b^2}\,c_1 + \operatorname{BesselJ}\left(\frac{3+n}{n+2}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{\frac{n}{2}+1}}{n+2}\right)\sqrt{a^2b^2}\,\ln(x)^{\frac{n}{2}+1} - \text{Bes}}{\left(\operatorname{BesselY}\left(\frac{1}{n+2}, \frac{2\sqrt{a^2b^2}\,\ln(x)^{\frac{n}{2}+1}}{n+2}\right)c_1 + \operatorname{BesselJ}\left(\frac{1}{n+2}, \frac{2\sqrt{a^2b^2}\,\ln(x\right.}{n+2}\right.}$$

✓ Solution by Mathematica

Time used: 45.846 (sec). Leaf size: 1769

```
DSolve[x^2*y'[x]==a^2*x^2*y[x]^2-x*y[x]+b^2*(Log[x])^n,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{}{x\left(2a(n+2)^{\frac{2(n+1)}{n+2}} \operatorname{BesselJ}\left(-\frac{1}{n+2}, \frac{2a\sqrt{\left(b^2\log^{n+1}(x)\right)^{1+\frac{1}{n+1}}}}{\sqrt{b^{\frac{2}{n+1}}(n+2)^2}}\right)\operatorname{Gamma}\left(\frac{2n+3}{n+2}\right)\left(b^2\log^{n+1}(x)\right)^{1+\frac{1}{n+1}} b^{\frac{2}{n+2}} + an\right.}$$

$$y(x) \to \frac{2b^2\sqrt{(n+2)^2b^{\frac{2}{n+1}}}\log^{n+1}(x)\sqrt{(b}}{x\left(-a(n+2)\left(b^2\log^{n+1}(x)\right)^{\frac{1}{n+1}+1}\operatorname{BesselJ}\left(\frac{1}{n+2}, \frac{2a\sqrt{\left(b^2\log^{n+1}(x)\right)^{1+\frac{1}{n+1}}}}{\sqrt{b^{\frac{2}{n+1}}(n+2)^2}}\right) + a(n+2)\left(b^2\log^{n+1}(x)\right)^{\frac{1}{n+1}+1}\right.}$$

8.13 problem 22

Internal problem ID [10497]
Internal file name [OUTPUT/9444_Monday_June_06_2022_02_32_39_PM_53703409/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{(a\ln(x)+b)\,y'-y^2-c\ln(x)^n\,y=-\lambda^2+\lambda c\ln(x)^n}$$

8.13.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2+c\ln(x)^n\,y-\lambda^2+\lambda c\ln(x)^n}{a\ln(x)+b} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=\frac{\lambda c\ln(x)^n}{a\ln(x)+b}+\frac{c\ln(x)^n\,y}{a\ln(x)+b}-\frac{\lambda^2}{a\ln(x)+b}+\frac{y^2}{a\ln(x)+b}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=\frac{-\lambda^2+\lambda c\ln(x)^n}{a\ln(x)+b}$, $f_1(x)=\frac{c\ln(x)^n}{a\ln(x)+b}$ and $f_2(x)=\frac{1}{a\ln(x)+b}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{\frac{u}{a\ln(x)+b}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{a}{(a\ln(x)+b)^2\,x} \\ f_1f_2 &= \frac{c\ln(x)^n}{(a\ln(x)+b)^2} \\ f_2^2f_0 &= \frac{-\lambda^2+\lambda c\ln(x)^n}{(a\ln(x)+b)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{a\ln(x)+b} - \left(-\frac{a}{(a\ln(x)+b)^2\,x} + \frac{c\ln(x)^n}{(a\ln(x)+b)^2}\right)u'(x) + \frac{(-\lambda^2+\lambda c\ln(x)^n)\,u(x)}{(a\ln(x)+b)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(-\frac{a}{x}+\ln(x)^n c\right)\_Y'(x)}{a\ln(x)+b} + \frac{(-\lambda^2+\lambda c\ln(x)^n)\_Y(x)}{(a\ln(x)+b)^2}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(-\frac{a}{x}+\ln(x)^n c\right)\_Y'(x)}{a\ln(x)+b} + \frac{(-\lambda^2+\lambda c\ln(x)^n)\_Y(x)}{(a\ln(x)+b)^2}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(-\frac{a}{x}+\ln(x)^n c\right)\_Y'(x)}{a\ln(x)+b} + \frac{(-\lambda^2+\lambda c\ln(x)^n)\_Y(x)}{(a\ln(x)+b)^2}\right\}, \{\_Y(x)\}\right)\right)(a\ln(x)+b)}{\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(-\frac{a}{x}+\ln(x)^n c\right)\_Y'(x)}{a\ln(x)+b} + \frac{(-\lambda^2+\lambda c\ln(x)^n)\_Y(x)}{(a\ln(x)+b)^2}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{-acx\ln(x)^{n+1}\_\_Y'(x)+\_\_Y''(x)(a\ln(x)+b)^2x+(-b\ln(x)^n cx+(a\ln(x)+b)a)\_\_Y'(x)+\lambda(\ln(x)^n c-\lambda)\_\_Y(x)x}{(a\ln(x)+b)^2x}\right\},\{\right.\right)}{\operatorname{DESol}\left(\left\{\frac{-acx\ln(x)^{n+1}\_\_Y'(x)+\_\_Y''(x)(a\ln(x)+b)^2x+(-b\ln(x)^n cx+(a\ln(x)+b)a)\_\_Y'(x)+\lambda(\ln(x)^n c-\lambda)\_\_Y(x)x}{(a\ln(x)+b)^2x}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{-acx\ln(x)^{n+1}\_\_Y'(x)+\_\_Y''(x)(a\ln(x)+b)^2x+(-b\ln(x)^n cx+(a\ln(x)+b)a)\_\_Y'(x)+\lambda(\ln(x)^n c-\lambda)\_\_Y(x)x}{(a\ln(x)+b)^2x}\right\},\{\right.\right)}{\operatorname{DESol}\left(\left\{\frac{-acx\ln(x)^{n+1}\_\_Y'(x)+\_\_Y''(x)(a\ln(x)+b)^2x+(-b\ln(x)^n cx+(a\ln(x)+b)a)\_\_Y'(x)+\lambda(\ln(x)^n c-\lambda)\_\_Y(x)x}{(a\ln(x)+b)^2x}\right.\right.} \quad (1)$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{-acx\ln(x)^{n+1}\_\_Y'(x)+\_\_Y''(x)(a\ln(x)+b)^2x+(-b\ln(x)^n cx+(a\ln(x)+b)a)\_\_Y'(x)+\lambda(\ln(x)^n c-\lambda)\_\_Y(x)x}{(a\ln(x)+b)^2x}\right\},\{\right.\right)}{\operatorname{DESol}\left(\left\{\frac{-acx\ln(x)^{n+1}\_\_Y'(x)+\_\_Y''(x)(a\ln(x)+b)^2x+(-b\ln(x)^n cx+(a\ln(x)+b)a)\_\_Y'(x)+\lambda(\ln(x)^n c-\lambda)\_\_Y(x)x}{(a\ln(x)+b)^2x}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (ln(x)^n*c*x-a)*(diff(y(x), x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 107

```
dsolve((a*ln(x)+b)*diff(y(x),x)=y(x)^2+c*(ln(x))^n*y(x)-lambda^2+lambda*c*(ln(x))^n,y(x), si
```

$$y(x) = \frac{-\left(\int \frac{e^{\int \frac{\ln(x)^n c-2\lambda}{a\ln(x)+b}dx}}{a\ln(x)+b}dx\right)\lambda - \lambda c_1 - e^{\int \frac{\ln(x)^n c-2\lambda}{a\ln(x)+b}dx}}{c_1 + \int \frac{e^{\int \frac{\ln(x)^n c-2\lambda}{a\ln(x)+b}dx}}{a\ln(x)+b}dx}$$

✓ Solution by Mathematica

Time used: 5.348 (sec). Leaf size: 275

```
DSolve[(a*Log[x]+b)*y'[x]==y[x]^2+c*(Log[x])^n*y[x]-\[Lambda]^2+\[Lambda]*c*(Log[x])^n,y[x],
```

$$\text{Solve}\left[\int_1^x -\frac{\exp\left(-\int_1^{K[2]} \frac{2\lambda - c\log^n(K[1])}{b+a\log(K[1])}dK[1]\right)(c\log^n(K[2]) - \lambda + y(x))}{cn(b+a\log(K[2]))(\lambda+y(x))}dK[2] + \int_1^{y(x)}\left(\frac{\exp\left(-\int_1^x \frac{2\lambda - c\log^n(K[1])}{b+a\log(K[1])}dK[1]\right)}{cn(\lambda+K[3])^2} - \int_1^x\left(\frac{\exp\left(-\int_1^{K[2]} \frac{2\lambda - c\log^n(K[1])}{b+a\log(K[1])}dK[1]\right)(c\log^n(K[2]) - \lambda + K[3])}{cn(\lambda+K[3])^2(b+a\log(K[2]))} - \frac{\exp\left(-\int_1^{K[2]} \frac{2\lambda - c\log^n(K[1])}{b+a\log(K[1])}dK[1]\right)}{cn(\lambda+K[3])(b+a\log(K[2]))}\right)\right.\right.$$

8.14 problem 23

Internal problem ID [10498]
Internal file name [`OUTPUT/9445_Monday_June_06_2022_02_32_43_PM_25188570/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.5-2
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{(a\ln(x)+b)\,y'-\ln(x)^n\,y^2-yc=-\lambda^2\ln(x)^n+c\lambda}$$

8.14.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= \frac{\ln(x)^n\,y^2+yc-\lambda^2\ln(x)^n+c\lambda}{a\ln(x)+b}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=-\frac{\lambda^2\ln(x)^n}{a\ln(x)+b}+\frac{\ln(x)^n\,y^2}{a\ln(x)+b}+\frac{c\lambda}{a\ln(x)+b}+\frac{yc}{a\ln(x)+b}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=\frac{-\lambda^2\ln(x)^n+c\lambda}{a\ln(x)+b}$, $f_1(x)=\frac{c}{a\ln(x)+b}$ and $f_2(x)=\frac{\ln(x)^n}{a\ln(x)+b}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2u}\\ &= \frac{-u'}{\frac{\ln(x)^nu}{a\ln(x)+b}}\end{aligned}\tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{\ln(x)^n n}{x \ln(x) (a \ln(x) + b)} - \frac{\ln(x)^n a}{(a \ln(x) + b)^2 x} \\
f_1 f_2 &= \frac{c \ln(x)^n}{(a \ln(x) + b)^2} \\
f_2^2 f_0 &= \frac{\ln(x)^{2n} (-\lambda^2 \ln(x)^n + c\lambda)}{(a \ln(x) + b)^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\ln(x)^n u''(x)}{a \ln(x) + b} - \left(\frac{\ln(x)^n n}{x \ln(x) (a \ln(x) + b)} - \frac{\ln(x)^n a}{(a \ln(x) + b)^2 x} + \frac{c \ln(x)^n}{(a \ln(x) + b)^2}\right) u'(x) + \frac{\ln(x)^{2n} (-\lambda^2 \ln(x)^n}{(a \ln(x) + b}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
&u(x) \\
&= \mathrm{DESol}\Bigg(\Bigg\{_Y''(x) \\
&\quad - (a \ln(x) + b) _Y'(x) \left(\frac{n}{x \ln(x) (a \ln(x) + b)} - \frac{a}{(a \ln(x) + b)^2 x} + \frac{c}{(a \ln(x) + b)^2}\right) \\
&\quad + \frac{_Y(x) \left(-\ln(x)^{2n} \lambda^2 + \lambda c \ln(x)^n\right)}{(a \ln(x) + b)^2}\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

The above shows that

$$\begin{aligned}
&u'(x) \\
&= \frac{\partial}{\partial x} \mathrm{DESol}\Bigg(\Bigg\{_Y''(x) \\
&\quad - (a \ln(x) + b) _Y'(x) \left(\frac{n}{x \ln(x) (a \ln(x) + b)} - \frac{a}{(a \ln(x) + b)^2 x} + \frac{c}{(a \ln(x) + b)^2}\right) \\
&\quad + \frac{_Y(x) \left(-\ln(x)^{2n} \lambda^2 + \lambda c \ln(x)^n\right)}{(a \ln(x) + b)^2}\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - (a\ln(x)+b)\_Y'(x)\left(\frac{n}{x\ln(x)(a\ln(x)+b)} - \frac{a}{(a\ln(x)+b)^2 x} + \frac{c}{(a\ln(x)+b)^2}\right) + \frac{\_Y(x)\left(-\right.}{(}\right.\right.\right.}{\mathrm{DESol}\left(\left\{\_Y''(x) - (a\ln(x)+b)\_Y'(x)\left(\frac{n}{x\ln(x)(a\ln(x)+b)} - \frac{a}{(a\ln(x)+b)^2 x} + \frac{c}{(a\ln(x)+b)^2}\right) +\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-x\_\_Y(x)\ln(x)^{1+2n}\lambda^2 + cx\lambda\_\_Y(x)\ln(x)^{n+1} + (a\ln(x)+b)\left(x\ln(x)(a\ln(x)+b)\_\_Y''(x) - ((a(n-1)+cx)\ln(x)+bn)\_\right.}{x\ln(x)(a\ln(x)+b)^2}\right.\right.\right.}{\mathrm{DESol}\left(\left\{\frac{-x\_\_Y(x)\ln(x)^{1+2n}\lambda^2 + cx\lambda\_\_Y(x)\ln(x)^{n+1} + (a\ln(x)+b)\left(x\ln(x)(a\ln(x)+b)\_\_Y''(x) - ((a(n-1)+cx)\right.}{x\ln(x)(a\ln(x)+b)^2}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-x\_\_Y(x)\ln(x)^{1+2n}\lambda^2 + cx\lambda\_\_Y(x)\ln(x)^{n+1} + (a\ln(x)+b)\left(x\ln(x)(a\ln(x)+b)\_\_Y''(x) - ((a(n-1)+cx)\ln(x)+bn)\_\right.}{x\ln(x)(a\ln(x)+b)^2}\right.\right.\right.}{\mathrm{DESol}\left(\left\{\frac{-x\_\_Y(x)\ln(x)^{1+2n}\lambda^2 + cx\lambda\_\_Y(x)\ln(x)^{n+1} + (a\ln(x)+b)\left(x\ln(x)(a\ln(x)+b)\_\_Y''(x) - ((a(n-1)+cx)\right.}{x\ln(x)(a\ln(x)+b)^2}\right.\right.} \quad (1)$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-x\_\_Y(x)\ln(x)^{1+2n}\lambda^2 + cx\lambda\_\_Y(x)\ln(x)^{n+1} + (a\ln(x)+b)\left(x\ln(x)(a\ln(x)+b)\_\_Y''(x) - ((a(n-1)+cx)\ln(x)+bn)\_\right.}{x\ln(x)(a\ln(x)+b)^2}\right.\right.\right.}{\mathrm{DESol}\left(\left\{\frac{-x\_\_Y(x)\ln(x)^{1+2n}\lambda^2 + cx\lambda\_\_Y(x)\ln(x)^{n+1} + (a\ln(x)+b)\left(x\ln(x)(a\ln(x)+b)\_\_Y''(x) - ((a(n-1)+cx)\right.}{x\ln(x)(a\ln(x)+b)^2}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (ln(x)*a*n+c*x*ln(x)-ln(x)*a+b
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 124

```
dsolve((a*ln(x)+b)*diff(y(x),x)=(ln(x))^n*y(x)^2+c*y(x)-lambda^2*(ln(x))^n+c*lambda,y(x), si
```

$$y(x) = \frac{-\lambda c_1 - \left(\int \frac{\ln(x)^n e^{-\left(\int \frac{2\ln(x)^n\lambda - c}{a\ln(x)+b}dx\right)}}{a\ln(x)+b}dx\right)\lambda - e^{-\left(\int \frac{2\ln(x)^n\lambda - c}{a\ln(x)+b}dx\right)}}{c_1 + \int \frac{\ln(x)^n e^{-\left(\int \frac{2\ln(x)^n\lambda - c}{a\ln(x)+b}dx\right)}}{a\ln(x)+b}dx}$$

✓ Solution by Mathematica

Time used: 5.137 (sec). Leaf size: 286

```
DSolve[(a*Log[x]+b)*y'[x]==(Log[x])^n*y[x]^2+c*y[x]-\[Lambda]^2*(Log[x])^n+c*\[Lambda],y[x],
```

$$\text{Solve}\left[\int_1^x \frac{\exp\left(-\int_1^{K[2]} -\frac{c-2\lambda\log^n(K[1])}{b+a\log(K[1])}dK[1]\right)(-\lambda\log^n(K[2])+y(x)\log^n(K[2])+c)}{cn(b+a\log(K[2]))(\lambda+y(x))}dK[2]\right.$$
$$+\int_1^{y(x)}\left(-\int_1^x\left(\frac{\exp\left(-\int_1^{K[2]} -\frac{c-2\lambda\log^n(K[1])}{b+a\log(K[1])}dK[1]\right)\log^n(K[2])}{cn(\lambda+K[3])(b+a\log(K[2]))} - \frac{\exp\left(-\int_1^{K[2]} -\frac{c-2\lambda\log^n(K[1])}{b+a\log(K[1])}dK[1]\right)(-\lambda}{cn(\lambda+K[3])^2(b+}\right.\right.$$
$$\left.\left. - \frac{\exp\left(-\int_1^x -\frac{c-2\lambda\log^n(K[1])}{b+a\log(K[1])}dK[1]\right)}{cn(\lambda+K[3])^2}\right)dK[3]=c_1,y(x)\right]$$

9 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-1. Equations with sine

9.1 problem 1

Internal problem ID [10499]
Internal file name [`OUTPUT/9446_Monday_June_06_2022_02_32_55_PM_8580992/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-1. Equations with sine
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \alpha y^2 = \beta + \gamma \sin(\lambda x)}$$

9.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \alpha\, y^2 + \beta + \gamma \sin(\lambda x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \alpha\, y^2 + \beta + \gamma \sin(\lambda x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \beta + \gamma \sin(\lambda x)$, $f_1(x) = 0$ and $f_2(x) = \alpha$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\alpha u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= \alpha^2(\beta + \gamma\sin(\lambda x)) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\alpha u''(x) + \alpha^2(\beta + \gamma\sin(\lambda x))\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{MathieuC}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4} + \frac{\lambda x}{2}\right) + c_2 \operatorname{MathieuS}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4} + \frac{\lambda x}{2}\right)$$

The above shows that

$$u'(x) = \frac{\lambda\left(c_1 \operatorname{MathieuCPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4} + \frac{\lambda x}{2}\right) + c_2 \operatorname{MathieuSPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4} + \frac{\lambda x}{2}\right)\right)}{2}$$

Using the above in (1) gives the solution

$$y = -\frac{\lambda\left(c_1 \operatorname{MathieuCPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4} + \frac{\lambda x}{2}\right) + c_2 \operatorname{MathieuSPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4} + \frac{\lambda x}{2}\right)\right)}{2\alpha\left(c_1 \operatorname{MathieuC}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4} + \frac{\lambda x}{2}\right) + c_2 \operatorname{MathieuS}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4} + \frac{\lambda x}{2}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\lambda\left(c_3 \operatorname{MathieuCPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4} + \frac{\lambda x}{2}\right) + \operatorname{MathieuSPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4} + \frac{\lambda x}{2}\right)\right)}{2\alpha\left(c_3 \operatorname{MathieuC}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4} + \frac{\lambda x}{2}\right) + \operatorname{MathieuS}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4} + \frac{\lambda x}{2}\right)\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\lambda\left(c_3 \,\text{MathieuCPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4}+\frac{\lambda x}{2}\right) + \text{MathieuSPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4}+\frac{\lambda x}{2}\right)\right)}{2\alpha\left(c_3 \,\text{MathieuC}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4}+\frac{\lambda x}{2}\right) + \text{MathieuS}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4}+\frac{\lambda x}{2}\right)\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\lambda\left(c_3 \,\text{MathieuCPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4}+\frac{\lambda x}{2}\right) + \text{MathieuSPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4}+\frac{\lambda x}{2}\right)\right)}{2\alpha\left(c_3 \,\text{MathieuC}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4}+\frac{\lambda x}{2}\right) + \text{MathieuS}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, -\frac{\pi}{4}+\frac{\lambda x}{2}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -alpha*(beta+gamma*sin(lambda*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying an equivalence, under non-integer power transformations,
            to LODEs admitting Liouvillian solutions.
            -> Trying a Liouvillian solution using Kovacics algorithm
            <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Kummer
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            -> Mathieu
               -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
               Equivalence transformation and function parameters: {t = 1/2*t+1/2}, {kappa =
               <- Equivalence to the rational form of Mathieu ODE successful
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 110

```
dsolve(diff(y(x),x)=alpha*y(x)^2+beta+gamma*sin(lambda*x),y(x), singsol=all)
```

$$y(x) = -\frac{\lambda\left(c_1 \operatorname{MathieuSPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\gamma\alpha}{\lambda^2}, -\frac{\pi}{4}+\frac{x\lambda}{2}\right)+\operatorname{MathieuCPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\gamma\alpha}{\lambda^2}, -\frac{\pi}{4}+\frac{x\lambda}{2}\right)\right)}{2\alpha\left(c_1 \operatorname{MathieuS}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\gamma\alpha}{\lambda^2}, -\frac{\pi}{4}+\frac{x\lambda}{2}\right)+\operatorname{MathieuC}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\gamma\alpha}{\lambda^2}, -\frac{\pi}{4}+\frac{x\lambda}{2}\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.612 (sec). Leaf size: 191

```
DSolve[y'[x]==\[Alpha]*y[x]^2+\[Beta]+\[Gamma]*Sin[\[Lambda]*x],y[x],x,IncludeSingularSoluti
```

$$y(x) \to -\frac{\lambda\left(\text{MathieuSPrime}\left[\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{1}{4}(\pi-2\lambda x)\right]+c_1\text{MathieuCPrime}\left[\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{1}{4}(2\lambda x-\pi)\right]\right)}{2\alpha\left(\text{MathieuS}\left[\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{1}{4}(2\lambda x-\pi)\right]+c_1\text{MathieuC}\left[\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{1}{4}(\pi-2\lambda x)\right]\right)}$$

$$y(x) \to \frac{\lambda\text{MathieuCPrime}\left[\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{1}{4}(\pi-2\lambda x)\right]}{2\alpha\text{MathieuC}\left[\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{1}{4}(\pi-2\lambda x)\right]}$$

9.2 problem 2

Internal problem ID [10500]
Internal file name [OUTPUT/9447_Monday_June_06_2022_02_32_58_PM_31295989/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-1. Equations with sine
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = -a^2 + a\lambda \sin(\lambda x) + a^2 \sin(\lambda x)^2}$$

9.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 - a^2 + a\lambda \sin(\lambda x) + a^2 \sin(\lambda x)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - a^2 + a\lambda \sin(\lambda x) + a^2 \sin(\lambda x)^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2 + a\lambda \sin(\lambda x) + a^2 \sin(\lambda x)^2$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= -a^2 + a\lambda\sin(\lambda x) + a^2\sin(\lambda x)^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-a^2 + a\lambda\sin(\lambda x) + a^2\sin(\lambda x)^2\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = {}& \mathrm{e}^{\frac{\sin(\lambda x)a}{\lambda}}\left(c_1\,\mathrm{HeunC}\left(\frac{4a}{\lambda}, -\frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\sin(\lambda x)}{2} + \frac{1}{2}\right)\right. \\ &\left. + c_2\,\mathrm{HeunC}\left(\frac{4a}{\lambda}, \frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\sin(\lambda x)}{2} + \frac{1}{2}\right)\sin\left(\frac{\pi}{4} + \frac{\lambda x}{2}\right)\right) \end{aligned}$$

The above shows that

$$\begin{aligned} u'(x) = {}& \left(c_2\left(\cos(\lambda x)\sin\left(\frac{\pi}{4} + \frac{\lambda x}{2}\right)a + \frac{\lambda\cos\left(\frac{\pi}{4} + \frac{\lambda x}{2}\right)}{2}\right)\mathrm{HeunC}\left(\frac{4a}{\lambda}, \frac{1}{2}, -\frac{1}{2},\right.\right. \\ &\left. -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\sin(\lambda x)}{2} + \frac{1}{2}\right) \\ &+ \cos(\lambda x)\left(\frac{c_2\lambda\,\mathrm{HeunCPrime}\left(\frac{4a}{\lambda}, \frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\sin(\lambda x)}{2} + \frac{1}{2}\right)\sin\left(\frac{\pi}{4} + \frac{\lambda x}{2}\right)}{2}\right. \\ &+ \mathrm{HeunC}\left(\frac{4a}{\lambda}, -\frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\sin(\lambda x)}{2} + \frac{1}{2}\right)c_1a \\ &\left.\left. + \frac{c_1\lambda\,\mathrm{HeunCPrime}\left(\frac{4a}{\lambda}, -\frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\sin(\lambda x)}{2} + \frac{1}{2}\right)}{2}\right)\right)\mathrm{e}^{\frac{\sin(\lambda x)a}{\lambda}} \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2\left(\cos(\lambda x)\sin\left(\frac{\pi}{4} + \frac{\lambda x}{2}\right)a + \frac{\lambda\cos\left(\frac{\pi}{4} + \frac{\lambda x}{2}\right)}{2}\right)\mathrm{HeunC}\left(\frac{4a}{\lambda}, \frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\sin(\lambda x)}{2} + \frac{1}{2}\right) + \cos(\lambda x)\left(\frac{c_2\lambda}{}\right.}{c_1\,\mathrm{HeunC}\left(\frac{4a}{\lambda}, -\frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\left(\cos(\lambda x)\sin\left(\frac{\pi}{4}+\frac{\lambda x}{2}\right)a+\frac{\lambda\cos\left(\frac{\pi}{4}+\frac{\lambda x}{2}\right)}{2}\right)\operatorname{HeunC}\left(\frac{4a}{\lambda},\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\sin(\lambda x)}{2}+\frac{1}{2}\right)-\cos(\lambda x)\left(\frac{\lambda \operatorname{Heun}}{}\right.}{c_3\operatorname{HeunC}\left(\frac{4a}{\lambda},-\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\right.}$$

Summary
The solution(s) found are the following

$$y = \frac{-\left(\cos(\lambda x)\sin\left(\frac{\pi}{4}+\frac{\lambda x}{2}\right)a+\frac{\lambda\cos\left(\frac{\pi}{4}+\frac{\lambda x}{2}\right)}{2}\right)\operatorname{HeunC}\left(\frac{4a}{\lambda},\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\sin(\lambda x)}{2}+\frac{1}{2}\right)-\cos(\lambda x)\left(\frac{\lambda \operatorname{Heun}}{}\right.}{c_3\operatorname{HeunC}\left(\frac{4a}{\lambda},-\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\right.} \tag{1}$$

Verification of solutions

$$y = \frac{-\left(\cos(\lambda x)\sin\left(\frac{\pi}{4}+\frac{\lambda x}{2}\right)a+\frac{\lambda\cos\left(\frac{\pi}{4}+\frac{\lambda x}{2}\right)}{2}\right)\operatorname{HeunC}\left(\frac{4a}{\lambda},\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\sin(\lambda x)}{2}+\frac{1}{2}\right)-\cos(\lambda x)\left(\frac{\lambda \operatorname{Heun}}{}\right.}{c_3\operatorname{HeunC}\left(\frac{4a}{\lambda},-\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\right.}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a^2-a*lambda*sin(lambda*x)-a^
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying an equivalence, under non-integer power transformations,
            to LODEs admitting Liouvillian solutions.
            -> Trying a Liouvillian solution using Kovacics algorithm
               A Liouvillian solution exists
               Reducible group (found an exponential solution)
               Group is reducible, not completely reducible
               Solution has integrals. Trying a special function solution free of integrals.
               -> Trying a solution in terms of special functions:
                  -> Bessel
                  -> elliptic
                  -> Legendre
                  -> Whittaker
                     -> hyper3: Equivalence to 1F1 under a power @ Moebius
                  -> hypergeometric
                     -> heuristic approach
                     -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
                  -> Mathieu
                     -> Equivalence to the rational form of Mathieu ODE under a power @ Moeb
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 289

```
dsolve(diff(y(x),x)=y(x)^2-a^2+a*lambda*sin(lambda*x)+a^2*sin(lambda*x)^2,y(x), singsol=all)
```

$$y(x) = \frac{\left(-2ac_1\cos\left(x\lambda\right)\sin\left(\frac{\pi}{4}+\frac{x\lambda}{2}\right)-c_1\lambda\cos\left(\frac{\pi}{4}+\frac{x\lambda}{2}\right)\right)\mathrm{HeunC}\left(\frac{4a}{\lambda},\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\sin(x\lambda)}{2}+\frac{1}{2}\right)-2\Big(a\,\mathrm{Heu}}{2\,\mathrm{HeunC}\left(\frac{4a}{\lambda},\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\mathrm{sin}\right.}$$

✓ Solution by Mathematica

Time used: 4.337 (sec). Leaf size: 132

```
DSolve[y'[x]==y[x]^2-a^2+a*\[Lambda]*Sin[\[Lambda]*x]+a^2*Sin[\[Lambda]*x]^2,y[x],x,IncludeS
```

$$y(x)\to -\frac{ac_1\cos(\lambda x)\int_1^x e^{-\frac{2a\sin(\lambda K[1])}{\lambda}}dK[1]+a\cos(\lambda x)+c_1e^{-\frac{2a\sin(\lambda x)}{\lambda}}}{1+c_1\int_1^x e^{-\frac{2a\sin(\lambda K[1])}{\lambda}}dK[1]}$$

$$y(x)\to -\frac{e^{-\frac{2a\sin(\lambda x)}{\lambda}}}{\int_1^x e^{-\frac{2a\sin(\lambda K[1])}{\lambda}}dK[1]}-a\cos(\lambda x)$$

9.3 problem 3

Internal problem ID [10501]
Internal file name [`OUTPUT/9448_Monday_June_06_2022_02_33_01_PM_4199649/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-1. Equations with sine
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$\boxed{y' - y^2 = \lambda^2 + c\sin(\lambda x + a)^n \sin(\lambda x + b)^{-n-4}}$$

9.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + \lambda^2 + c\sin(\lambda x + a)^n \sin(\lambda x + b)^{-n-4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \lambda^2 + \frac{c(\sin(\lambda x)\cos(a) + \cos(\lambda x)\sin(a))^n \sin(\lambda x + b)^{-n}}{(\sin(\lambda x)\cos(b) + \cos(\lambda x)\sin(b))^4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \lambda^2 + c\sin(\lambda x + a)^n \sin(\lambda x + b)^{-n-4}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \lambda^2 + c\sin(\lambda x + a)^n \sin(\lambda x + b)^{-n-4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(\lambda^2 + c\sin(\lambda x + a)^n \sin(\lambda x + b)^{-n-4}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2+lambda^2+c*sin(lambda*x+a)^n*sin(lambda*x+b)^(-n-4),y(x), singsol
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+\[Lambda]^2+c*Sin[\[Lambda]*x+a]^n*Sin[\[Lambda]*x+b]^(-n-4),y[x],x,Inc
```

Not solved

9.4 problem 4

Internal problem ID [10502]
Internal file name [`OUTPUT/9449_Monday_June_06_2022_02_36_09_PM_13360096/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-1. Equations with sine
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - a\sin(\beta x)\, y = ab\sin(\beta x) - b^2}$$

9.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + a\sin(\beta x)\, y + ab\sin(\beta x) - b^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\sin(\beta x)\, y + ab\sin(\beta x) - b^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = ab\sin(\beta x) - b^2$, $f_1(x) = \sin(\beta x)\, a$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \qquad (1) \end{aligned}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \sin(\beta x) a \\ f_2^2 f_0 &= ab\sin(\beta x) - b^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \sin(\beta x) a u'(x) + (ab\sin(\beta x) - b^2) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(-ic_2\beta\left(\int e^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}}dx\right) + c_1\right) e^{bx}$$

The above shows that

$$u'(x) = \left(-i\left(\int e^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}}dx\right) c_2 b\beta - ic_2\beta\, e^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}} + c_1 b\right) e^{bx}$$

Using the above in (1) gives the solution

$$y = -\frac{-i\left(\int e^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}}dx\right) c_2 b\beta - ic_2\beta\, e^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}} + c_1 b}{-ic_2\beta\left(\int e^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}}dx\right) + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\left(\int e^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}}dx\right) b\beta - ibc_3 - e^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}}\beta}{\beta\left(\int e^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}}dx\right) + ic_3}$$

Summary

The solution(s) found are the following

$$y = \frac{-\left(\int \mathrm{e}^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}}dx\right)b\beta - ibc_3 - \mathrm{e}^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}}\beta}{\beta\left(\int \mathrm{e}^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}}dx\right) + ic_3} \tag{1}$$

Verification of solutions

$$y = \frac{-\left(\int \mathrm{e}^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}}dx\right)b\beta - ibc_3 - \mathrm{e}^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}}\beta}{\beta\left(\int \mathrm{e}^{\frac{-2b\beta x - a\cos(\beta x)}{\beta}}dx\right) + ic_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 76

```
dsolve(diff(y(x),x)=y(x)^2+a*sin(beta*x)*y(x)+a*b*sin(beta*x)-b^2,y(x), singsol=all)
```

$$y(x) = \frac{b\left(\int \mathrm{e}^{\frac{-2b\beta x - a\cos(x\beta)}{\beta}}dx\right) - c_1 b + \mathrm{e}^{\frac{-2b\beta x - a\cos(x\beta)}{\beta}}}{-\left(\int \mathrm{e}^{\frac{-2b\beta x - a\cos(x\beta)}{\beta}}dx\right) + c_1}$$

✓ Solution by Mathematica

Time used: 9.066 (sec). Leaf size: 187

```
DSolve[y'[x]==y[x]^2+a*Sin[\[Beta]*x]*y[x]+a*b*Sin[\[Beta]*x]-b^2,y[x],x,IncludeSingularSolu
```

$$\text{Solve}\left[\int_1^x -\frac{e^{-\frac{a\cos(\beta K[1])}{\beta}-2bK[1]}(-b+a\sin(\beta K[1])+y(x))}{a\beta(b+y(x))}dK[1]+\int_1^{y(x)}\left(\frac{e^{-2bx-\frac{a\cos(x\beta)}{\beta}}}{a\beta(b+K[2])^2}\right.\right.$$
$$\left.\left.-\int_1^x\left(\frac{e^{-\frac{a\cos(\beta K[1])}{\beta}-2bK[1]}(-b+K[2]+a\sin(\beta K[1]))}{a\beta(b+K[2])^2}-\frac{e^{-\frac{a\cos(\beta K[1])}{\beta}-2bK[1]}}{a\beta(b+K[2])}\right)dK[1]\right)dK[2]=c_1,y(x)\right]$$

9.5 problem 5

Internal problem ID [10503]
Internal file name [`OUTPUT/9450_Monday_June_06_2022_02_36_25_PM_10230958/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-1. Equations with sine
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - a\sin(bx)^m y = a\sin(bx)^m}$$

9.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + a\sin(bx)^m y + a\sin(bx)^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\sin(bx)^m y + a\sin(bx)^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\sin(bx)^m$, $f_1(x) = a\sin(bx)^m$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= a\sin(bx)^m \\ f_2^2 f_0 &= a\sin(bx)^m \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - a\sin(bx)^m\, u'(x) + a\sin(bx)^m\, u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) + a\sin(bx)^m\,(-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + a\sin(bx)^m\,(-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + a\sin(bx)^m\,(-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)}{\mathrm{DESol}\left(\left\{\_Y''(x) + a\sin(bx)^m\,(-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + a\sin(bx)^m\,(-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)}{\mathrm{DESol}\left(\left\{\_Y''(x) + a\sin(bx)^m\,(-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + a\sin(bx)^m\,(-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)}{\mathrm{DESol}\left(\left\{\_Y''(x) + a\sin(bx)^m\,(-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\_Y''(x) + a\sin\left(bx\right)^{m}\left(-\_Y'(x) + \_Y(x)\right)\right\}, \left\{\_Y(x)\right\}\right)}{\mathrm{DESol}\left(\left\{\_Y''\left(x\right) + a\sin\left(bx\right)^{m}\left(-\_Y'\left(x\right) + \_Y\left(x\right)\right)\right\}, \left\{\_Y\left(x\right)\right\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = a*sin(x*b)^m*(diff(y(x), x))-a
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Kummer
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            -> Mathieu
               -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2+a*sin(b*x)^m*y(x)+a*sin(b*x)^m,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+a*Sin[b*x]^m*y[x]+a*Sin[b*x]^m,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

9.6 problem 6

Internal problem ID [10504]
Internal file name [OUTPUT/9451_Monday_June_06_2022_02_36_31_PM_98522936/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-1. Equations with sine
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \sin(\lambda x) y^2 = \lambda \sin(\lambda x)^3}$$

9.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \sin(\lambda x) \lambda y^2 + \lambda \sin(\lambda x)^3 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \sin(\lambda x) \lambda y^2 + \lambda \sin(\lambda x)^3$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \lambda \sin(\lambda x)^3$, $f_1(x) = 0$ and $f_2(x) = \lambda \sin(\lambda x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\lambda \sin(\lambda x) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \lambda^2 \cos(\lambda x) \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \lambda^3 \sin(\lambda x)^5 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\lambda \sin(\lambda x)\, u''(x) - \lambda^2 \cos(\lambda x)\, u'(x) + \lambda^3 \sin(\lambda x)^5 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{-\frac{\cos(\lambda x)^2}{2}} (c_2 \operatorname{erfi}(\cos(\lambda x)) + c_1)$$

The above shows that

$$u'(x) = \frac{\lambda \sin(\lambda x) \left(\sqrt{\pi}\, \cos(\lambda x)\, (c_2 \operatorname{erfi}(\cos(\lambda x)) + c_1)\, \mathrm{e}^{-\frac{\cos(\lambda x)^2}{2}} - 2c_2 \mathrm{e}^{\frac{\cos(\lambda x)^2}{2}}\right)}{\sqrt{\pi}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\sqrt{\pi}\, \cos(\lambda x)\, (c_2 \operatorname{erfi}(\cos(\lambda x)) + c_1)\, \mathrm{e}^{-\frac{\cos(\lambda x)^2}{2}} - 2c_2 \mathrm{e}^{\frac{\cos(\lambda x)^2}{2}}\right) \mathrm{e}^{\frac{\cos(2\lambda x)}{4} + \frac{1}{4}}}{\sqrt{\pi}\, (c_2 \operatorname{erfi}(\cos(\lambda x)) + c_1)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{-2\, \mathrm{e}^{\cos(\lambda x)^2} + \cos(\lambda x) \sqrt{\pi}\, (\operatorname{erfi}(\cos(\lambda x)) + c_3)}{\sqrt{\pi}\, (\operatorname{erfi}(\cos(\lambda x)) + c_3)}$$

Summary

The solution(s) found are the following

$$y = -\frac{-2\, \mathrm{e}^{\cos(\lambda x)^2} + \cos(\lambda x) \sqrt{\pi}\, (\operatorname{erfi}(\cos(\lambda x)) + c_3)}{\sqrt{\pi}\, (\operatorname{erfi}(\cos(\lambda x)) + c_3)} \tag{1}$$

Verification of solutions

$$y = -\frac{-2\,e^{\cos(\lambda x)^2} + \cos(\lambda x)\sqrt{\pi}\,(\mathrm{erfi}(\cos(\lambda x)) + c_3)}{\sqrt{\pi}\,(\mathrm{erfi}(\cos(\lambda x)) + c_3)}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = lambda*cos(lambda*x)*(diff(y(x
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
         <- Kovacics algorithm successful
         Change of variables used:
            [x = arccos(t)/lambda]
         Linear ODE actually solved:
            16*(-t^2+1)^(1/2)*(t^4-2*t^2+1)*u(t)+16*(-t^2+1)^(3/2)*diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
dsolve(diff(y(x),x)=lambda*sin(lambda*x)*y(x)^2+lambda*sin(lambda*x)^3,y(x), singsol=all)
```

$$y(x) = \frac{2\,\mathrm{e}^{\frac{\cos(2x\lambda)}{2}+\frac{1}{2}} c_1 - \cos(x\lambda)\sqrt{\pi}\left(\operatorname{erfi}(\cos(x\lambda))\,c_1+1\right)}{\sqrt{\pi}\,\left(\operatorname{erfi}(\cos(x\lambda))\,c_1+1\right)}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==\[Lambda]*Sin[\[Lambda]*x]*y[x]^2+\[Lambda]*Sin[\[Lambda]*x]^3,y[x],x,IncludeS
```

Not solved

9.7 problem 7

Internal problem ID [10505]
Internal file name [`OUTPUT/9452_Monday_June_06_2022_02_36_32_PM_15666899/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-1. Equations with sine
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{2y' - (\lambda + a - a\sin(\lambda x))\, y^2 = -a + \lambda - a\sin(\lambda x)}$$

9.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{\sin(\lambda x)\, a\, y^2}{2} + \frac{a\, y^2}{2} + \frac{\lambda\, y^2}{2} + \frac{\lambda}{2} - \frac{a}{2} - \frac{a\sin(\lambda x)}{2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{\sin(\lambda x)\, a\, y^2}{2} + \frac{a\, y^2}{2} + \frac{\lambda\, y^2}{2} + \frac{\lambda}{2} - \frac{a}{2} - \frac{a\sin(\lambda x)}{2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\lambda}{2} - \frac{a}{2} - \frac{a\sin(\lambda x)}{2}$, $f_1(x) = 0$ and $f_2(x) = \frac{a}{2} + \frac{\lambda}{2} - \frac{a\sin(\lambda x)}{2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\left(\frac{a}{2} + \frac{\lambda}{2} - \frac{a\sin(\lambda x)}{2}\right) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{a\lambda\cos(\lambda x)}{2} \\
f_1f_2 &= 0 \\
f_2^2f_0 &= \left(\frac{a}{2} + \frac{\lambda}{2} - \frac{a\sin(\lambda x)}{2}\right)^2\left(\frac{\lambda}{2} - \frac{a}{2} - \frac{a\sin(\lambda x)}{2}\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\left(\frac{a}{2} + \frac{\lambda}{2} - \frac{a\sin(\lambda x)}{2}\right)u''(x) + \frac{a\lambda\cos(\lambda x)\,u'(x)}{2} + \left(\frac{a}{2} + \frac{\lambda}{2} - \frac{a\sin(\lambda x)}{2}\right)^2\left(\frac{\lambda}{2} - \frac{a}{2} - \frac{a\sin(\lambda x)}{2}\right)u(x) =$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{\sqrt{\sin(\lambda x)}\,\cos\left(\frac{\pi}{4} + \frac{\lambda x}{2}\right)e^{\frac{-a\sin(\lambda x)+\lambda^2(a+\lambda)\left(\int \frac{\cot(\lambda x)}{a\sin(\lambda x)-a-\lambda}dx\right)}{2\lambda}}\left(c_1 + c_2\left(\int^{\sin(\lambda x)} \frac{((-1+\_a)a-\lambda)e^{-\frac{\_a a}{\lambda}}}{(-1+\_a)^{\frac{3}{2}}\sqrt{\_a+1}}d\_a\right)\right)}{\sqrt{a\sin(\lambda x) - a - \lambda}}$$

The above shows that

$$u'(x) = -\frac{\csc\left(\frac{\pi}{4} + \frac{\lambda x}{2}\right)\sec\left(\frac{\pi}{4} + \frac{\lambda x}{2}\right)^2 e^{\frac{-a\sin(\lambda x)+\lambda^2(a+\lambda)\left(\int \frac{\cot(\lambda x)}{a\sin(\lambda x)-a-\lambda}dx\right)}{2\lambda}}\left(\sin\left(\frac{\pi}{4} + \frac{\lambda x}{2}\right)\left(c_1 + c_2\left(\int^{\sin(\lambda x)} \frac{((-1+\_a)a-}{(-1+\_a)^{\frac{3}{2}}\sqrt{}}\right.\right.\right.}{}$$

Using the above in (1) gives the solution

$$y = \frac{\csc\left(\frac{\pi}{4} + \frac{\lambda x}{2}\right)\sec\left(\frac{\pi}{4} + \frac{\lambda x}{2}\right)^2\left(\sin\left(\frac{\pi}{4} + \frac{\lambda x}{2}\right)\left(c_1 + c_2\left(\int^{\sin(\lambda x)} \frac{((-1+\_a)a-\lambda)e^{-\frac{\_a a}{\lambda}}}{(-1+\_a)^{\frac{3}{2}}\sqrt{\_a+1}}d\_a\right)\right)\left((2a + \lambda)\sin(\lambda x) -\right.\right.}{4\sqrt{}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\sec\left(\frac{\pi}{4}+\frac{\lambda x}{2}\right)^2 \sqrt{2}\left(\sqrt{2}\,\sqrt{\sin(\lambda x)-1}\,\left(a\cos(\lambda x)^2+2\left(a+\frac{\lambda}{2}\right)(\sin(\lambda x)-1)\right)\operatorname{csgn}\left(\sin\left(\frac{\pi}{4}+\frac{\lambda x}{2}\right)\right)(\cos\dots}{}$$

Simplifying the solution $y = -\frac{\sec\left(\frac{\pi}{4}+\frac{\lambda x}{2}\right)^2 \sqrt{2}\left(\sqrt{2}\,\sqrt{\sin(\lambda x)-1}\,\left(a\cos(\lambda x)^2+2\left(a+\frac{\lambda}{2}\right)(\sin(\lambda x)-1)\right)\operatorname{csgn}\left(\sin\left(\frac{\pi}{4}+\frac{\lambda x}{2}\right)\right)\left(\cos(\lambda x)a\dots\right.}{}$

to $y = -\frac{\sec\left(\frac{\pi}{4}+\frac{\lambda x}{2}\right)^2 \sqrt{2}\left(\sqrt{2}\,\sqrt{\sin(\lambda x)-1}\,\left(a\cos(\lambda x)^2+2\left(a+\frac{\lambda}{2}\right)(\sin(\lambda x)-1)\right)\left(\cos(\lambda x)a+\tan\left(\frac{\pi}{4}+\frac{\lambda x}{2}\right)\lambda\right)\left(\int^{\sin(\lambda x)}\frac{((-1+\_a)a-\lambda)e^{-\frac{\_aa}{\lambda}}}{(-1+\_a)^{\frac{3}{2}}\sqrt{\_a+1}}\dots\right.}{4\sqrt{\sin(\lambda x)-1}\,(a\dots}$

Summary

The solution(s) found are the following

$$y = -\frac{\sec\left(\frac{\pi}{4}+\frac{\lambda x}{2}\right)^2 \sqrt{2}\left(\sqrt{2}\,\sqrt{\sin(\lambda x)-1}\,\left(a\cos(\lambda x)^2+2\left(a+\frac{\lambda}{2}\right)(\sin(\lambda x)-1)\right)\left(\cos(\lambda x)a+\tan\left(\frac{\pi}{4}+\dots\right.\right.}{} \tag{1}$$

Verification of solutions

$$y = -\frac{\sec\left(\frac{\pi}{4}+\frac{\lambda x}{2}\right)^2 \sqrt{2}\left(\sqrt{2}\,\sqrt{\sin(\lambda x)-1}\,\left(a\cos(\lambda x)^2+2\left(a+\frac{\lambda}{2}\right)(\sin(\lambda x)-1)\right)\left(\cos(\lambda x)a+\tan\left(\frac{\pi}{4}+\dots\right.\right.}{}$$

Warning, solution could not be verified

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = a*lambda*cos(lambda*x)*(diff(y
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying an equivalence, under non-integer power transformations,
            to LODEs admitting Liouvillian solutions.
            -> Trying a Liouvillian solution using Kovacics algorithm
               A Liouvillian solution exists
               Reducible group (found an exponential solution)
               Group is reducible, not completely reducible
               Solution has integrals. Trying a special function solution free of integrals.
               -> Trying a solution in terms of special functions:
                  -> Bessel
                  -> elliptic
                  -> Legendre
                  -> Whittaker
                     -> hyper3: Equivalence to 1F1 under a power @ Moebius
                  -> hypergeometric
                     -> heuristic approach
                     -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
                  -> Mathieu
                     -> Equivalence to the rational form of Mathieu ODE under a power @ Moeb
                  -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a po
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 220

```
dsolve(2*diff(y(x),x)=(lambda+a-a*sin(lambda*x))*y(x)^2+lambda-a-a*sin(lambda*x),y(x), sings
```

$$y(x) = -\frac{\left(\left(\left(\int^{\sin(x\lambda)} \frac{(a(\_a-1)-\lambda)\mathrm{e}^{\frac{a\_a}{\lambda}}}{(\_a-1)^{\frac{3}{2}}\sqrt{\_a+1}}d\_a\right)c_1+1\right)\sqrt{-\cos\left(\frac{\pi}{4}+\frac{x\lambda}{2}\right)^2}\left(a\cos(x\lambda)+\tan\left(\frac{\pi}{4}+\frac{x\lambda}{2}\right)\lambda\right)\operatorname{csgn}(\sin}{\sqrt{-\cos\left(\frac{\pi}{4}+\frac{x\lambda}{2}\right)^2}\left(\left(\int^{\sin(x\lambda)} \frac{(a(\_a-1)-\lambda)\mathrm{e}^{\frac{a\_a}{\lambda}}}{(\_a-1)^{\frac{3}{2}}\sqrt{\_a+1}}d\_a\right.\right.}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[2*y'[x]==(\[Lambda]+a-a*Sin[\[Lambda]*x])*y[x]^2+\[Lambda]-a-a*Sin[\[Lambda]*x],y[x],
```

Not solved

9.8 problem 8

Internal problem ID [10506]
Internal file name [`OUTPUT/9453_Monday_June_06_2022_02_36_43_PM_56838212/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-1. Equations with sine
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \left(\lambda + \sin\left(\lambda x\right)^{2} a\right) y^{2} = -a + \lambda + \sin\left(\lambda x\right)^{2} a}$$

9.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \sin\left(\lambda x\right)^{2} a\, y^{2} + \sin\left(\lambda x\right)^{2} a + \lambda\, y^{2} - a + \lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \sin\left(\lambda x\right)^{2} a\, y^{2} + \sin\left(\lambda x\right)^{2} a + \lambda\, y^{2} - a + \lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a + \lambda + \sin\left(\lambda x\right)^{2} a$, $f_1(x) = 0$ and $f_2(x) = \lambda + \sin\left(\lambda x\right)^{2} a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\left(\lambda + \sin\left(\lambda x\right)^{2} a\right) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2\sin(\lambda x) a\lambda \cos(\lambda x) \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \left(\lambda + \sin(\lambda x)^2 a\right)^2 \left(-a + \lambda + \sin(\lambda x)^2 a\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\left(\lambda + \sin(\lambda x)^2 a\right) u''(x) - 2\sin(\lambda x) a\lambda\cos(\lambda x) u'(x) + \left(\lambda + \sin(\lambda x)^2 a\right)^2 \left(-a + \lambda + \sin(\lambda x)^2 a\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sin(\lambda x)\, e^{-\frac{\cos(2\lambda x) a}{4\lambda}} \left(c_1 + 2ic_2\lambda\left(\int e^{\frac{\cos(2\lambda x) a}{2\lambda}} \left(\csc(\lambda x)^2 \lambda + a\right) dx\right)\right)$$

The above shows that

$$\begin{aligned} u'(x) = \csc(\lambda x) \Bigg(\lambda &+ \sin(\lambda x)^2 a\Bigg) \left(i\sin(2\lambda x)\left(\int e^{\frac{\cos(2\lambda x) a}{2\lambda}} \left(\csc(\lambda x)^2 \lambda + a\right) dx\right) c_2\lambda\, e^{-\frac{\cos(2\lambda x) a}{4\lambda}}\right. \\ &\left. + 2ic_2\lambda\, e^{\frac{\cos(2\lambda x) a}{4\lambda}} + \frac{\sin(2\lambda x) c_1 e^{-\frac{\cos(2\lambda x) a}{4\lambda}}}{2}\right) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\csc(\lambda x)\left(i\sin(2\lambda x)\left(\int e^{\frac{\cos(2\lambda x) a}{2\lambda}} \left(\csc(\lambda x)^2 \lambda + a\right) dx\right) c_2\lambda\, e^{-\frac{\cos(2\lambda x) a}{4\lambda}} + 2ic_2\lambda\, e^{\frac{\cos(2\lambda x) a}{4\lambda}} + \frac{\sin(2\lambda x) c_1 e^{-\frac{\cos(2\lambda x) a}{4\lambda}}}{2}\right)}{\sin(\lambda x)\left(c_1 + 2ic_2\lambda\left(\int e^{\frac{\cos(2\lambda x) a}{2\lambda}} \left(\csc(\lambda x)^2 \lambda + a\right) dx\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2\,\mathrm{e}^{\frac{\cos(2\lambda x)a}{2\lambda}} \csc\left(\lambda x\right)^{2} \lambda - i c_3 \cot\left(\lambda x\right) + 2\lambda\left(\int \mathrm{e}^{\frac{\cos(2\lambda x)a}{2\lambda}} \left(\csc\left(\lambda x\right)^{2} \lambda + a\right) dx\right) \cot\left(\lambda x\right)}{i c_3 - 2\lambda\left(\int \mathrm{e}^{\frac{\cos(2\lambda x)a}{2\lambda}} \left(\csc\left(\lambda x\right)^{2} \lambda + a\right) dx\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{2\,\mathrm{e}^{\frac{\cos(2\lambda x)a}{2\lambda}} \csc\left(\lambda x\right)^{2} \lambda - i c_3 \cot\left(\lambda x\right) + 2\lambda\left(\int \mathrm{e}^{\frac{\cos(2\lambda x)a}{2\lambda}} \left(\csc\left(\lambda x\right)^{2} \lambda + a\right) dx\right) \cot\left(\lambda x\right)}{i c_3 - 2\lambda\left(\int \mathrm{e}^{\frac{\cos(2\lambda x)a}{2\lambda}} \left(\csc\left(\lambda x\right)^{2} \lambda + a\right) dx\right)} \tag{1}$$

Verification of solutions

$$y = \frac{2\,\mathrm{e}^{\frac{\cos(2\lambda x)a}{2\lambda}} \csc\left(\lambda x\right)^{2} \lambda - i c_3 \cot\left(\lambda x\right) + 2\lambda\left(\int \mathrm{e}^{\frac{\cos(2\lambda x)a}{2\lambda}} \left(\csc\left(\lambda x\right)^{2} \lambda + a\right) dx\right) \cot\left(\lambda x\right)}{i c_3 - 2\lambda\left(\int \mathrm{e}^{\frac{\cos(2\lambda x)a}{2\lambda}} \left(\csc\left(\lambda x\right)^{2} \lambda + a\right) dx\right)}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = 2*sin(lambda*x)*a*lambda*cos(l
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               -> Kummer
                  -> hyper3: Equivalence to 1F1 under a power @ Moebius
               -> hypergeometric
                  -> heuristic approach
                  -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
               -> Mathieu
                  -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 102

```
dsolve(diff(y(x),x)=(lambda+a*sin(lambda*x)^2)*y(x)^2+lambda-a+a*sin(lambda*x)^2,y(x), sings
```

$$y(x) = \frac{2\cot(x\lambda)\lambda\left(\int e^{\frac{a\cos(2x\lambda)}{2\lambda}}\left(\csc(x\lambda)^2\lambda + a\right)dx\right)c_1 + 2\csc(x\lambda)^2 e^{\frac{a\cos(2x\lambda)}{2\lambda}}c_1\lambda - i\cot(x\lambda)}{-2\lambda\left(\int e^{\frac{a\cos(2x\lambda)}{2\lambda}}\left(\csc(x\lambda)^2\lambda + a\right)dx\right)c_1 + i}$$

✓ Solution by Mathematica

Time used: 41.676 (sec). Leaf size: 187

```
DSolve[y'[x]==(\[Lambda]+a*Sin[\[Lambda]*x]^2)*y[x]^2+\[Lambda]-a+a*Sin[\[Lambda]*x]^2,y[x],
```

$$y(x) \to -\frac{2\left(c_1\cot(\lambda x)\int_1^x e^{-\frac{a\sin^2(\lambda K[1])}{\lambda}}(\lambda\csc^2(\lambda K[1]) + a)\,dK[1] + c_1\csc^2(\lambda x)e^{-\frac{a\sin^2(\lambda x)}{\lambda}} + \cot(\lambda x)\right)}{2 + 2c_1\int_1^x e^{-\frac{a\sin^2(\lambda K[1])}{\lambda}}(\lambda\csc^2(\lambda K[1]) + a)\,dK[1]}$$

$$y(x) \to -\frac{\csc^2(\lambda x)e^{-\frac{a\sin^2(\lambda x)}{\lambda}}}{\int_1^x e^{-\frac{a\sin^2(\lambda K[1])}{\lambda}}(\lambda\csc^2(\lambda K[1]) + a)\,dK[1]} - \cot(\lambda x)$$

9.9 problem 9

Internal problem ID [10507]
Internal file name [`OUTPUT/9454_Monday_June_06_2022_02_37_24_PM_58114225/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-1. Equations with sine
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + (k+1)\, x^{k} y^{2} - a\, x^{k+1} \sin\left(x\right)^{m} y = -a \sin\left(x\right)^{m}}$$

9.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\, x^{k+1} \sin\left(x\right)^{m} y - x^{k} y^{2} k - x^{k} y^{2} - a \sin\left(x\right)^{m} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\, x^{k} x \sin\left(x\right)^{m} y - x^{k} y^{2} k - x^{k} y^{2} - a \sin\left(x\right)^{m}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = -a \sin\left(x\right)^{m}$, $f_1(x) = x^{k+1} \sin\left(x\right)^{m} a$ and $f_2(x) = -x^{k} k - x^{k}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\left(-x^{k} k - x^{k}\right) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{k^2 x^k}{x} - \frac{k\, x^k}{x} \\
f_1 f_2 &= x^{k+1} \sin(x)^m\, a\left(-x^k k - x^k\right) \\
f_2^2 f_0 &= -\left(-x^k k - x^k\right)^2 a \sin(x)^m
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\left(-x^k k - x^k\right) u''(x) - \left(-\frac{k^2 x^k}{x} - \frac{k\, x^k}{x} + x^{k+1} \sin(x)^m\, a\left(-x^k k - x^k\right)\right) u'(x) - \left(-x^k k - x^k\right)^2 a \sin(x)^m\, u($$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{k+1}\left(c_1 + c_2\left(\int x^{-2k-2} \mathrm{e}^{\int \left(x^{k+1} \sin(x)^m a + \frac{k}{x}\right) dx} dx\right)\right)$$

The above shows that

$$\begin{aligned}
u'(x) = \Bigg(& c_2 x^{-2k-1} \mathrm{e}^{\int \left(x^{k+1} \sin(x)^m a + \frac{k}{x}\right) dx} \\
& + \left(c_1 + c_2\left(\int x^{-2k-2} \mathrm{e}^{\int \left(x^{k+1} \sin(x)^m a + \frac{k}{x}\right) dx} dx\right)\right)(k+1)\Bigg) x^k
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(c_2 x^{-2k-1} \mathrm{e}^{\int \left(x^{k+1} \sin(x)^m a + \frac{k}{x}\right) dx} + \left(c_1 + c_2\left(\int x^{-2k-2} \mathrm{e}^{\int \left(x^{k+1} \sin(x)^m a + \frac{k}{x}\right) dx} dx\right)\right)(k+1)\right) x^k x^{-k-1}}{\left(-x^k k - x^k\right)\left(c_1 + c_2\left(\int x^{-2k-2} \mathrm{e}^{\int \left(x^{k+1} \sin(x)^m a + \frac{k}{x}\right) dx} dx\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x^{-k-1}\left(x^{-2k-1} \mathrm{e}^{\int \left(x^{k+1} \sin(x)^m a + \frac{k}{x}\right) dx} + \left(c_3 + \int x^{-2k-2} \mathrm{e}^{\int \left(x^{k+1} \sin(x)^m a + \frac{k}{x}\right) dx} dx\right)(k+1)\right)}{(k+1)\left(c_3 + \int \mathrm{e}^{\int \frac{a\, x^{k+2} \sin(x)^m + k}{x} dx} x^{-2k-2} dx\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{x^{-k-1}\left(x^{-2k-1}\mathrm{e}^{\int \left(x^{k+1}\sin(x)^{m} a+\frac{k}{x}\right)dx}+\left(c_3+\int x^{-2k-2}\mathrm{e}^{\int \left(x^{k+1}\sin(x)^{m} a+\frac{k}{x}\right)dx}dx\right)(k+1)\right)}{(k+1)\left(c_3+\int \mathrm{e}^{\int \frac{a\,x^{k+2}\sin(x)^{m}+k}{x}dx}x^{-2k-2}dx\right)} \tag{1}$$

Verification of solutions

$$y = \frac{x^{-k-1}\left(x^{-2k-1}\mathrm{e}^{\int \left(x^{k+1}\sin(x)^{m} a+\frac{k}{x}\right)dx}+\left(c_3+\int x^{-2k-2}\mathrm{e}^{\int \left(x^{k+1}\sin(x)^{m} a+\frac{k}{x}\right)dx}dx\right)(k+1)\right)}{(k+1)\left(c_3+\int \mathrm{e}^{\int \frac{a\,x^{k+2}\sin(x)^{m}+k}{x}dx}x^{-2k-2}dx\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x^(1+k)*sin(x)^m*a*x+k)*(diff
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 174

```
dsolve(diff(y(x),x)=-(k+1)*x^k*y(x)^2+a*x^(k+1)*sin(x)^m*y(x)-a*sin(x)^m,y(x), singsol=all)
```

$$y(x) = \frac{x^{-1-k}\left(x^{1+k}\mathrm{e}^{\int \frac{x^{1+k}\sin(x)^m a x-2k-2}{x}dx} + \left(\int x^k \mathrm{e}^{\int \frac{x^{1+k}\sin(x)^m a x-2k-2}{x}dx}dx\right)k + \int x^k \mathrm{e}^{\int \frac{x^{1+k}\sin(x)^m a x-2k-2}{x}dx}dx - c_1\right)}{\left(\int x^k \mathrm{e}^{\int \frac{a\,x^{k+2}\sin(x)^m-2k-2}{x}dx}dx\right)k + \int x^k \mathrm{e}^{\int \frac{a\,x^{k+2}\sin(x)^m-2k-2}{x}dx}dx - c_1}$$

✓ Solution by Mathematica

Time used: 16.483 (sec). Leaf size: 248

```
DSolve[y'[x]==-(k+1)*x^k*y[x]^2+a*x^(k+1)*Sin[x]^m*y[x]-a*Sin[x]^m,y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{x^{-k-1}\left(c_1 x \exp\left(\int_1^x -\frac{-a\sin^m(K[1])K[1]^{k+2}+k+2}{K[1]}dK[1]\right) + c_1(k+1)\int_1^x \exp\left(\int_1^{K[2]} -\frac{-a\sin^m(K[1])K[1]^{k+2}+k+2}{K[1]}dK\right)\right)}{(k+1)\left(1+c_1\int_1^x \exp\left(\int_1^{K[2]} -\frac{-a\sin^m(K[1])K[1]^{k+2}+k+2}{K[1]}dK[1]\right)dK[2]\right)}$$

$$y(x) \to \frac{x^{-k}\left(\frac{\exp\left(\int_1^x -\frac{-a\sin^m(K[1])K[1]^{k+2}+k+2}{K[1]}dK[1]\right)}{\int_1^x \exp\left(\int_1^{K[2]} -\frac{-a\sin^m(K[1])K[1]^{k+2}+k+2}{K[1]}dK[1]\right)dK[2]} + \frac{k+1}{x}\right)}{k+1}$$

9.10 problem 10

Internal problem ID [10508]
Internal file name [`OUTPUT/9455_Monday_June_06_2022_02_37_33_PM_50920785/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-1. Equations with sine
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$y' - a\sin(\lambda x+\mu)^k (y - b\,x^n - c)^2 = b\,x^{n-1}n$$

9.10.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= x^{2n}\sin(\lambda x+\mu)^k a\,b^2 + 2x^n\sin(\lambda x+\mu)^k abc - 2x^n\sin(\lambda x+\mu)^k aby + \sin(\lambda x+\mu)^k a\,c^2 - 2\sin(\lambda x \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^{2n}(\sin(\lambda x)\cos(\mu)+\cos(\lambda x)\sin(\mu))^k a\,b^2 + 2x^n(\sin(\lambda x)\cos(\mu)+\cos(\lambda x)\sin(\mu))^k abc - 2x^n(\sin(\lambda x$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^{2n}\sin(\lambda x+\mu)^k a\,b^2 + 2x^n\sin(\lambda x+\mu)^k abc + \sin(\lambda x+\mu)^k a\,c^2 + b\,x^{n-1}n$, $f_1(x) = -2\sin(\lambda x+\mu)^k a\,x^n b - 2\sin(\lambda x+\mu)^k ac$ and $f_2(x) = \sin(\lambda x+\mu)^k a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{\sin(\lambda x+\mu)^k au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$
\begin{aligned}
f_2' &= \frac{\sin(\lambda x+\mu)^k k\lambda \cos(\lambda x+\mu)\, a}{\sin(\lambda x+\mu)}\\
f_1f_2 &= \left(-2\sin(\lambda x+\mu)^k a\, x^n b - 2\sin(\lambda x+\mu)^k ac\right)\sin(\lambda x+\mu)^k a\\
f_2^2 f_0 &= \sin(\lambda x+\mu)^{2k} a^2\left(x^{2n}\sin(\lambda x+\mu)^k a\, b^2 + 2x^n \sin(\lambda x+\mu)^k abc + \sin(\lambda x+\mu)^k a\, c^2 + b\, x^{n-1}n\right)
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$\sin(\lambda x+\mu)^k a u''(x) - \left(\frac{\sin(\lambda x+\mu)^k k\lambda\cos(\lambda x+\mu)\, a}{\sin(\lambda x+\mu)} + \left(-2\sin(\lambda x+\mu)^k a\, x^n b - 2\sin(\lambda x+\mu)^k ac\right) \mathrm{s}\right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
\begin{aligned}
u(x) = \mathrm{e}^{-\frac{\left(\int\left(2a(b\,x^n+c)\sin(\lambda x+\mu)^k-\cot(\lambda x+\mu)\lambda(k-1)\right)dx\right)}{2}} \sin(\lambda x\\
+\mu)\left(c_1 \operatorname{LegendreP}\left(\frac{k}{2}-\frac{1}{2}, \frac{k}{2}+\frac{1}{2}, \cos(\lambda x+\mu)\right)\right.\\
\left.+ c_2 \operatorname{LegendreQ}\left(\frac{k}{2}-\frac{1}{2}, \frac{k}{2}+\frac{1}{2}, \cos(\lambda x+\mu)\right)\right)
\end{aligned}
$$

The above shows that

$$
\begin{aligned}
u'(x) = -a\sin(\lambda x+\mu)^k \mathrm{e}^{-\frac{\left(\int\left(2a(b\,x^n+c)\sin(\lambda x+\mu)^k-\cot(\lambda x+\mu)\lambda(k-1)\right)dx\right)}{2}} \sin(\lambda x\\
+\mu)\left(c_1 \operatorname{LegendreP}\left(\frac{k}{2}-\frac{1}{2}, \frac{k}{2}+\frac{1}{2}, \cos(\lambda x+\mu)\right)\right.\\
\left.+ c_2 \operatorname{LegendreQ}\left(\frac{k}{2}-\frac{1}{2}, \frac{k}{2}+\frac{1}{2}, \cos(\lambda x+\mu)\right)\right)(b\,x^n+c)
\end{aligned}
$$

Using the above in (1) gives the solution

$$y = b\,x^n + c$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = b\,x^n + c$$

Summary

The solution(s) found are the following

$$y = b\,x^n + c \tag{1}$$

Verification of solutions

$$y = b\,x^n + c$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (d) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x)=a*sin(lambda*x+mu)^k*(y(x)-b*x^n-c)^2+b*n*x^(n-1),y(x), singsol=all)
```

$$y(x) = b\,x^n + c + \frac{1}{c_1 - a\left(\int (\sin(x\lambda)\cos(\mu) + \cos(x\lambda)\sin(\mu))^k\,dx\right)}$$

✓ Solution by Mathematica

Time used: 5.928 (sec). Leaf size: 93

```
DSolve[y'[x]==a*Sin[\[Lambda]*x+\[Mu]]^k*(y[x]-b*x^n-c)^2+b*n*x^(n-1),y[x],x,IncludeSingular
```

$$y(x) \to \frac{1}{-\frac{a\sqrt{\cos^2(\mu+\lambda x)}\sec(\mu+\lambda x)\sin^{k+1}(\mu+\lambda x)\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{k+1}{2},\frac{k+3}{2},\sin^2(x\lambda+\mu)\right)}{(k+1)\lambda}+c_1} + bx^n + c$$

$$y(x) \to bx^n + c$$

9.11 problem 11

Internal problem ID [10509]
Internal file name [OUTPUT/9456_Monday_June_06_2022_02_38_20_PM_86812107/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-1. Equations with sine
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y'x - a\sin(\lambda x)^m y^2 - ky = a\,b^2x^{2k}\sin(\lambda x)^m}$$

9.11.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= \frac{a\sin(\lambda x)^m y^2 + ky + a\,b^2x^{2k}\sin(\lambda x)^m}{x}
\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,b^2x^{2k}\sin(\lambda x)^m}{x} + \frac{a\sin(\lambda x)^m y^2}{x} + \frac{ky}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{a\,b^2x^{2k}\sin(\lambda x)^m}{x}$, $f_1(x) = \frac{k}{x}$ and $f_2(x) = \frac{a\sin(\lambda x)^m}{x}$. Let

$$\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{a\sin(\lambda x)^m u}{x}}
\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{a \sin(\lambda x)^m m\lambda \cos(\lambda x)}{\sin(\lambda x)\, x} - \frac{a \sin(\lambda x)^m}{x^2} \\
f_1 f_2 &= \frac{ka \sin(\lambda x)^m}{x^2} \\
f_2^2 f_0 &= \frac{a^3 \sin(\lambda x)^{3m} b^2 x^{2k}}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{a \sin(\lambda x)^m u''(x)}{x} - \left(\frac{a \sin(\lambda x)^m m\lambda \cos(\lambda x)}{\sin(\lambda x)\, x} - \frac{a \sin(\lambda x)^m}{x^2} + \frac{ka \sin(\lambda x)^m}{x^2}\right) u'(x) + \frac{a^3 \sin(\lambda x)^{3m} b^2 x^{2k}}{x^3}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \mathrm{e}^{iab\left(\int x^{k-1} \sin(\lambda x)^m dx\right)} + c_2 \mathrm{e}^{-iab\left(\int x^{k-1} \sin(\lambda x)^m dx\right)}$$

The above shows that

$$u'(x) = iab\, x^{k-1} \sin(\lambda x)^m \left(c_1 \mathrm{e}^{iab\left(\int x^{k-1} \sin(\lambda x)^m dx\right)} - c_2 \mathrm{e}^{-iab\left(\int x^{k-1} \sin(\lambda x)^m dx\right)}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{ib\, x^{k-1} \left(c_1 \mathrm{e}^{iab\left(\int x^{k-1} \sin(\lambda x)^m dx\right)} - c_2 \mathrm{e}^{-iab\left(\int x^{k-1} \sin(\lambda x)^m dx\right)}\right) x}{c_1 \mathrm{e}^{iab\left(\int x^{k-1} \sin(\lambda x)^m dx\right)} + c_2 \mathrm{e}^{-iab\left(\int x^{k-1} \sin(\lambda x)^m dx\right)}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{ib\, x^{k} \left(c_3 \mathrm{e}^{iab\left(\int x^{k-1} \sin(\lambda x)^m dx\right)} - \mathrm{e}^{-iab\left(\int x^{k-1} \sin(\lambda x)^m dx\right)}\right)}{c_3 \mathrm{e}^{iab\left(\int x^{k-1} \sin(\lambda x)^m dx\right)} + \mathrm{e}^{-iab\left(\int x^{k-1} \sin(\lambda x)^m dx\right)}}$$

Summary
The solution(s) found are the following

$$y = -\frac{ib\,x^k\left(c_3 e^{iab\left(\int x^{k-1}\sin(\lambda x)^m dx\right)} - e^{-iab\left(\int x^{k-1}\sin(\lambda x)^m dx\right)}\right)}{c_3 e^{iab\left(\int x^{k-1}\sin(\lambda x)^m dx\right)} + e^{-iab\left(\int x^{k-1}\sin(\lambda x)^m dx\right)}} \tag{1}$$

Verification of solutions

$$y = -\frac{ib\,x^k\left(c_3 e^{iab\left(\int x^{k-1}\sin(\lambda x)^m dx\right)} - e^{-iab\left(\int x^{k-1}\sin(\lambda x)^m dx\right)}\right)}{c_3 e^{iab\left(\int x^{k-1}\sin(\lambda x)^m dx\right)} + e^{-iab\left(\int x^{k-1}\sin(\lambda x)^m dx\right)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 31

```
dsolve(x*diff(y(x),x)=a*sin(lambda*x)^m*y(x)^2+k*y(x)+a*b^2*x^(2*k)*sin(lambda*x)^m,y(x), si
```

$$y(x) = -\tan\left(-ab\left(\int x^{-1+k}\sin(x\lambda)^m\,dx\right)+c_1\right)b\,x^k$$

✓ Solution by Mathematica
Time used: 1.774 (sec). Leaf size: 50

```
DSolve[x*y'[x]==a*Sin[\[Lambda]*x]^m*y[x]^2+k*y[x]+a*b^2*x^(2*k)*Sin[\[Lambda]*x]^m,y[x],x,I
```

$$y(x) \to \sqrt{b^2}x^k\tan\left(\sqrt{b^2}\int_1^x aK[1]^{k-1}\sin^m(\lambda K[1])dK[1]+c_1\right)$$

9.12 problem 12

Internal problem ID [10510]
Internal file name [`OUTPUT/9457_Monday_June_06_2022_02_38_23_PM_57207150/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-1. Equations with sine
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$\boxed{(a\sin(\lambda x)+b)\,y'-y^2-c\sin(x\mu)\,y=-d^2+cd\sin(x\mu)}$$

9.12.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= \frac{y^2+c\sin(x\mu)\,y-d^2+cd\sin(x\mu)}{a\sin(\lambda x)+b} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=\frac{cd\sin(x\mu)}{a\sin(\lambda x)+b}+\frac{c\sin(x\mu)\,y}{a\sin(\lambda x)+b}-\frac{d^2}{a\sin(\lambda x)+b}+\frac{y^2}{a\sin(\lambda x)+b}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=\frac{-d^2+cd\sin(x\mu)}{a\sin(\lambda x)+b}$, $f_1(x)=\frac{c\sin(x\mu)}{a\sin(\lambda x)+b}$ and $f_2(x)=\frac{1}{a\sin(\lambda x)+b}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u}\\ &= \frac{-u'}{\frac{u}{a\sin(\lambda x)+b}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{a\lambda\cos(\lambda x)}{(a\sin(\lambda x) + b)^2} \\ f_1f_2 &= \frac{c\sin(x\mu)}{(a\sin(\lambda x) + b)^2} \\ f_2^2f_0 &= \frac{-d^2 + cd\sin(x\mu)}{(a\sin(\lambda x) + b)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{a\sin(\lambda x) + b} - \left(-\frac{a\lambda\cos(\lambda x)}{(a\sin(\lambda x) + b)^2} + \frac{c\sin(x\mu)}{(a\sin(\lambda x) + b)^2}\right)u'(x) + \frac{(-d^2 + cd\sin(x\mu))\,u(x)}{(a\sin(\lambda x) + b)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 265

```
dsolve((a*sin(lambda*x)+b)*diff(y(x),x)=y(x)^2+c*sin(mu*x)*y(x)-d^2+c*d*sin(mu*x),y(x), sing
```

$$y(x) = \frac{-d\left(\int \frac{e^{\frac{c\left(\int \frac{\sin(x\mu)}{a\sin(x\lambda)+b}dx\right)\sqrt{-a^2+b^2}\,\lambda-4d\arctan\left(\frac{\tan\left(\frac{x\lambda}{2}\right)b+a}{\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}\,\lambda}}}{a\sin(x\lambda)+b}dx\right)+dc_1-e^{\frac{c\left(\int \frac{\sin(x\mu)}{a\sin(x\lambda)+b}dx\right)\sqrt{-a^2+b^2}\,\lambda-4d\arctan\left(\frac{\tan\left(\frac{x\lambda}{2}\right)b+a}{\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}\,\lambda}}}{\int \frac{e^{\frac{c\left(\int \frac{\sin(x\mu)}{a\sin(x\lambda)+b}dx\right)\sqrt{-a^2+b^2}\,\lambda-4d\arctan\left(\frac{\tan\left(\frac{x\lambda}{2}\right)b+a}{\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}\,\lambda}}}{a\sin(x\lambda)+b}dx-c_1}$$

✓ Solution by Mathematica

Time used: 15.846 (sec). Leaf size: 289

```
DSolve[(a*Sin[\[Lambda]*x]+b)*y'[x]==y[x]^2+c*Sin[\[Mu]*x]*y[x]-d^2+c*d*Sin[\[Mu]*x],y[x],x,
```

$$\text{Solve}\left[\int_1^x -\frac{\exp\left(-\int_1^{K[2]}\frac{2d-c\sin(\mu K[1])}{b+a\sin(\lambda K[1])}dK[1]\right)(-d+c\sin(\mu K[2])+y(x))}{c\mu(b+a\sin(\lambda K[2]))(d+y(x))}dK[2]\right.$$
$$+\int_1^{y(x)}\left(\frac{\exp\left(-\int_1^x\frac{2d-c\sin(\mu K[1])}{b+a\sin(\lambda K[1])}dK[1]\right)}{c\mu(d+K[3])^2}\right.$$
$$-\int_1^x\left(\frac{\exp\left(-\int_1^{K[2]}\frac{2d-c\sin(\mu K[1])}{b+a\sin(\lambda K[1])}dK[1]\right)(-d+K[3]+c\sin(\mu K[2]))}{c\mu(d+K[3])^2(b+a\sin(\lambda K[2]))}-\frac{\exp\left(-\int_1^{K[2]}\frac{2d-c\sin(\mu K[1])}{b+a\sin(\lambda K[1])}dK[1]\right)}{c\mu(d+K[3])(b+a\sin(\lambda K[2]))}\right)$$

9.13 problem 13

Internal problem ID [10511]
Internal file name [`OUTPUT/9458_Monday_June_06_2022_02_39_36_PM_69357783/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-1. Equations with sine
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{(a\sin(\lambda x)+b)\left(y'-y^2\right)=a\,\lambda^2\sin(\lambda x)}$$

9.13.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2\sin(\lambda x)\,a+a\,\lambda^2\sin(\lambda x)+y^2b}{a\sin(\lambda x)+b} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=\frac{a\,\lambda^2\sin(\lambda x)}{a\sin(\lambda x)+b}+\frac{y^2\sin(\lambda x)\,a}{a\sin(\lambda x)+b}+\frac{y^2b}{a\sin(\lambda x)+b}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=\frac{a\,\lambda^2\sin(\lambda x)}{a\sin(\lambda x)+b}$, $f_1(x)=0$ and $f_2(x)=1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{a\,\lambda^2 \sin(\lambda x)}{a \sin(\lambda x) + b} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{a\,\lambda^2 \sin(\lambda x)\, u(x)}{a \sin(\lambda x) + b} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = {} & 2c_1(a-b)(a+b)\, b^2 \left(\sin\left(\frac{\lambda x}{2}\right) a \cos\left(\frac{\lambda x}{2}\right) + \frac{b}{2}\right) \arctan\left(\frac{b \tan\left(\frac{\lambda x}{2}\right) + a}{\sqrt{-a^2+b^2}}\right) \\ & + a c_1 \cos\left(\frac{\lambda x}{2}\right)(a-b)(a+b)\left(a \sin\left(\frac{\lambda x}{2}\right) + b \cos\left(\frac{\lambda x}{2}\right)\right)\sqrt{-a^2+b^2} \\ & + 2c_2\left(\sin\left(\frac{\lambda x}{2}\right) a \cos\left(\frac{\lambda x}{2}\right) + \frac{b}{2}\right) \end{aligned}$$

The above shows that

$$u'(x) = -\frac{\left(-2c_1\left(\cos\left(\frac{\lambda x}{2}\right)^2 - \frac{1}{2}\right)(a-b)(a+b)\, a\sqrt{-a^2+b^2}\, b^2 \arctan\left(\frac{b\tan\left(\frac{\lambda x}{2}\right)+a}{\sqrt{-a^2+b^2}}\right) - 2\left(\cos\left(\frac{\lambda x}{2}\right)^2 - \frac{1}{2}\right) c_2 a\sqrt{-}\right.}{\sqrt{-a^2+b^2}}$$

Using the above in (1) gives the solution

$$y = \frac{\left(-2c_1\left(\cos\left(\frac{\lambda x}{2}\right)^2 - \frac{1}{2}\right)(a-b)(a+b)\, a\sqrt{-a^2+b^2}\, b^2 \arctan\left(\frac{b\tan\left(\frac{\lambda x}{2}\right)+a}{\sqrt{-a^2+b^2}}\right) - 2\left(\cos\left(\frac{\lambda x}{2}\right)^2 - \frac{1}{2}\right) c_2 a\sqrt{-a}\right.}{\sqrt{-a^2+b^2}\left(2c_1(a-b)(a+b)\, b^2\left(\sin\left(\frac{\lambda x}{2}\right) a \cos\left(\frac{\lambda x}{2}\right) + \frac{b}{2}\right) \arctan\left(\frac{b\tan\left(\frac{\lambda x}{2}\right)+a}{\sqrt{-a^2+b^2}}\right) + a c_1 \cos\left(\frac{\lambda x}{2}\right)(a-b)\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\lambda\left(-2c_3\left(\cos\left(\frac{\lambda x}{2}\right)^2 - \frac{1}{2}\right)(a-b)(a+b)\,a\sqrt{-a^2+b^2}\,b^2 \arctan\left(\frac{b\tan\left(\frac{\lambda x}{2}\right)+a}{\sqrt{-a^2+b^2}}\right) + a\left(-2\cos\left(\frac{\lambda x}{2}\right)^2+1\right)\sqrt{-}\right.}{\sqrt{-a^2+b^2}\left(2c_3(a-b)(a+b)\,b^2\left(\sin\left(\frac{\lambda x}{2}\right)a\cos\left(\frac{\lambda x}{2}\right)+\frac{b}{2}\right)\arctan\left(\frac{b\tan\left(\frac{\lambda x}{2}\right)+a}{\sqrt{-a^2+b^2}}\right) + ac_3\cos\left(\frac{\lambda x}{2}\right)(a-\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\lambda\left(-2c_3\left(\cos\left(\frac{\lambda x}{2}\right)^2 - \frac{1}{2}\right)(a-b)(a+b)\,a\sqrt{-a^2+b^2}\,b^2 \arctan\left(\frac{b\tan\left(\frac{\lambda x}{2}\right)+a}{\sqrt{-a^2+b^2}}\right) + a\left(-2\cos\left(\frac{\lambda x}{2}\right)^2+1\right)\sqrt{-}\right.}{\sqrt{-a^2+b^2}\left(2c_3(a-b)(a+b)\,b^2\left(\sin\left(\frac{\lambda x}{2}\right)a\cos\left(\frac{\lambda x}{2}\right)+\frac{b}{2}\right)\arctan\left(\frac{b\tan\left(\frac{\lambda x}{2}\right)+a}{\sqrt{-a^2+b^2}}\right) + ac_3\cos\left(\frac{\lambda x}{2}\right)(a-\right.} \tag{1}$$

Verification of solutions

$$y = \frac{\lambda\left(-2c_3\left(\cos\left(\frac{\lambda x}{2}\right)^2 - \frac{1}{2}\right)(a-b)(a+b)\,a\sqrt{-a^2+b^2}\,b^2 \arctan\left(\frac{b\tan\left(\frac{\lambda x}{2}\right)+a}{\sqrt{-a^2+b^2}}\right) + a\left(-2\cos\left(\frac{\lambda x}{2}\right)^2+1\right)\sqrt{-}\right.}{\sqrt{-a^2+b^2}\left(2c_3(a-b)(a+b)\,b^2\left(\sin\left(\frac{\lambda x}{2}\right)a\cos\left(\frac{\lambda x}{2}\right)+\frac{b}{2}\right)\arctan\left(\frac{b\tan\left(\frac{\lambda x}{2}\right)+a}{\sqrt{-a^2+b^2}}\right) + ac_3\cos\left(\frac{\lambda x}{2}\right)(a-\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -a*lambda^2*sin(lambda*x)*y(x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      <- linear_1 successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 261

```
dsolve((a*sin(lambda*x)+b)*(diff(y(x),x)-y(x)^2)-a*lambda^2*sin(lambda*x)=0,y(x), singsol=al
```

$$y(x) = \frac{\left(-2(a-b)\,b^2(a+b)\left(\cos\left(\frac{x\lambda}{2}\right)^2-\frac{1}{2}\right)a\sqrt{-a^2+b^2}\,\arctan\left(\frac{\tan\left(\frac{x\lambda}{2}\right)b+a}{\sqrt{-a^2+b^2}}\right)+2c_1\left(\cos\left(\frac{x\lambda}{2}\right)^2-\frac{1}{2}\right)a\sqrt{-a^2+\right.}{\sqrt{-a^2+b^2}\,\left(2\,(a-b)\,b^2\,(a+b)\left(\sin\left(\frac{x\lambda}{2}\right)a\cos\left(\frac{x\lambda}{2}\right)+\frac{b}{2}\right)\arctan\left(\frac{\tan\left(\frac{x\lambda}{2}\right)b+a}{\sqrt{-a^2+b^2}}\right)+a\cos\left(\frac{x\lambda}{2}\right)(a-b)\,(a+\right.}$$

✓ Solution by Mathematica

Time used: 24.795 (sec). Leaf size: 189

```
DSolve[(a*Sin[\[Lambda]*x]+b)*(y'[x]-y[x]^2)-a*\[Lambda]^2*Sin[\[Lambda]*x]==0,y[x],x,Includ
```

$$y(x) \to \frac{\lambda\left(2ab\cos(\lambda x)\arctan\left(\frac{a+b\tan\left(\frac{\lambda x}{2}\right)}{\sqrt{b^2-a^2}}\right)+\sqrt{b^2-a^2}(-ac_1\lambda(a^2-b^2)\cos(\lambda x)-a\sin(\lambda x)+b)\right)}{-2b(a\sin(\lambda x)+b)\arctan\left(\frac{a+b\tan\left(\frac{\lambda x}{2}\right)}{\sqrt{b^2-a^2}}\right)+\sqrt{b^2-a^2}\left(-a\cos(\lambda x)+c_1\lambda\left(a^2-b^2\right)\left(a\sin(\lambda x)+b\right)\right)}$$

$$y(x) \to -\frac{a\lambda\cos(\lambda x)}{a\sin(\lambda x)+b}$$

10 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-2. Equations with cosine.

10.1 problem 14

Internal problem ID [10512]
Internal file name [`OUTPUT/9459_Monday_June_06_2022_02_40_18_PM_5103947/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-2. Equations with cosine.
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \alpha y^2 = \beta + \gamma \cos(\lambda x)}$$

10.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \alpha y^2 + \beta + \gamma \cos(\lambda x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \alpha y^2 + \beta + \gamma \cos(\lambda x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \beta + \gamma \cos(\lambda x)$, $f_1(x) = 0$ and $f_2(x) = \alpha$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\alpha u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \alpha^2(\beta + \gamma \cos(\lambda x)) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\alpha u''(x) + \alpha^2(\beta + \gamma \cos(\lambda x)) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{MathieuC}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right) + c_2 \operatorname{MathieuS}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right)$$

The above shows that

$$u'(x) = \frac{\lambda\left(c_1 \operatorname{MathieuCPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right) + c_2 \operatorname{MathieuSPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right)\right)}{2}$$

Using the above in (1) gives the solution

$$y = -\frac{\lambda\left(c_1 \operatorname{MathieuCPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right) + c_2 \operatorname{MathieuSPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right)\right)}{2\alpha\left(c_1 \operatorname{MathieuC}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right) + c_2 \operatorname{MathieuS}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\lambda\left(c_3 \operatorname{MathieuCPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right) + \operatorname{MathieuSPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right)\right)}{2\alpha\left(c_3 \operatorname{MathieuC}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right) + \operatorname{MathieuS}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right)\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\lambda\left(c_3 \operatorname{MathieuCPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right) + \operatorname{MathieuSPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right)\right)}{2\alpha\left(c_3 \operatorname{MathieuC}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right) + \operatorname{MathieuS}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right)\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\lambda\left(c_3 \operatorname{MathieuCPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right) + \operatorname{MathieuSPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right)\right)}{2\alpha\left(c_3 \operatorname{MathieuC}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right) + \operatorname{MathieuS}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -alpha*(beta+gamma*cos(lambda*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Kummer
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            -> Mathieu
               -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
               Equivalence transformation and function parameters: {z = 1/2*t+1/2}, {kappa =
               <- Equivalence to the rational form of Mathieu ODE successful
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 94

```
dsolve(diff(y(x),x)=alpha*y(x)^2+beta+gamma*cos(lambda*x),y(x), singsol=all)
```

$$y(x) = -\frac{\lambda\left(c_1 \operatorname{MathieuSPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\gamma\alpha}{\lambda^2}, \frac{x\lambda}{2}\right) + \operatorname{MathieuCPrime}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\gamma\alpha}{\lambda^2}, \frac{x\lambda}{2}\right)\right)}{2\alpha\left(c_1 \operatorname{MathieuS}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\gamma\alpha}{\lambda^2}, \frac{x\lambda}{2}\right) + \operatorname{MathieuC}\left(\frac{4\alpha\beta}{\lambda^2}, -\frac{2\gamma\alpha}{\lambda^2}, \frac{x\lambda}{2}\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.577 (sec). Leaf size: 163

```
DSolve[y'[x]==\[Alpha]*y[x]^2+\[Beta]+\[Gamma]*Cos[\[Lambda]*x],y[x],x,IncludeSingularSoluti
```

$$y(x) \to -\frac{\lambda\left(\operatorname{MathieuSPrime}\left[\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right] + c_1 \operatorname{MathieuCPrime}\left[\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right]\right)}{2\alpha\left(\operatorname{MathieuS}\left[\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right] + c_1 \operatorname{MathieuC}\left[\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right]\right)}$$

$$y(x) \to -\frac{\lambda \operatorname{MathieuCPrime}\left[\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right]}{2\alpha \operatorname{MathieuC}\left[\frac{4\alpha\beta}{\lambda^2}, -\frac{2\alpha\gamma}{\lambda^2}, \frac{\lambda x}{2}\right]}$$

10.2 problem 15

Internal problem ID [10513]
Internal file name [`OUTPUT/9460_Monday_June_06_2022_02_40_20_PM_50369115/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-2. Equations with cosine.
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = -a^2 + a\lambda \cos(\lambda x) + a^2 \cos(\lambda x)^2}$$

10.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 - a^2 + a\lambda \cos(\lambda x) + a^2 \cos(\lambda x)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - a^2 + a\lambda \cos(\lambda x) + a^2 \cos(\lambda x)^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2 + a\lambda \cos(\lambda x) + a^2 \cos(\lambda x)^2$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= -a^2 + a\lambda\cos(\lambda x) + a^2\cos(\lambda x)^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-a^2 + a\lambda\cos(\lambda x) + a^2\cos(\lambda x)^2\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\frac{a\cos(\lambda x)}{\lambda}}\left(c_1\,\mathrm{HeunC}\left(\frac{4a}{\lambda}, -\frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\cos(\lambda x)}{2} + \frac{1}{2}\right) + c_2\cos\left(\frac{\lambda x}{2}\right)\mathrm{HeunC}\left(\frac{4a}{\lambda}, \frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\cos(\lambda x)}{2} + \frac{1}{2}\right)\right)$$

The above shows that

$$u'(x) = -2\,\mathrm{e}^{\frac{a\cos(\lambda x)}{\lambda}}\left(\frac{c_2\left(\cos(\lambda x)\,a + a + \frac{\lambda}{2}\right)\mathrm{HeunC}\left(\frac{4a}{\lambda}, \frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\cos(\lambda x)}{2} + \frac{1}{2}\right)}{2} + \frac{(1+\cos(\lambda x))\,c_2\lambda\,\mathrm{HeunCPrime}\left(\frac{4a}{\lambda}, \frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\cos(\lambda x)}{2} + \frac{1}{2}\right)}{4} + c_1\cos\left(\frac{\lambda x}{2}\right)\left(a\,\mathrm{HeunC}\left(\frac{4a}{\lambda}, -\frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\cos(\lambda x)}{2} + \frac{1}{2}\right) + \frac{\lambda\,\mathrm{HeunCPrime}\left(\frac{4a}{\lambda}, -\frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\cos(\lambda x)}{2} + \frac{1}{2}\right)}{2}\right)\right)\sin\left(\frac{\lambda x}{2}\right)$$

Using the above in (1) gives the solution

$$y = \frac{2\left(\frac{c_2\left(\cos(\lambda x)a+a+\frac{\lambda}{2}\right)\mathrm{HeunC}\left(\frac{4a}{\lambda},\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\cos(\lambda x)}{2}+\frac{1}{2}\right)}{2} + \frac{(1+\cos(\lambda x))c_2\lambda\,\mathrm{HeunCPrime}\left(\frac{4a}{\lambda},\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\cos(\lambda x)}{2}+\frac{1}{2}\right)}{4} + \right.}{c_1\,\mathrm{HeunC}\left(\frac{4a}{\lambda}, -\frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\cos(\lambda x)}{2} + \frac{1}{2}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2\left(\frac{\left(\cos(\lambda x)a+a+\frac{\lambda}{2}\right)\operatorname{HeunC}\left(\frac{4a}{\lambda},\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\cos(\lambda x)}{2}+\frac{1}{2}\right)}{2}+\frac{(1+\cos(\lambda x))\operatorname{HeunCPrime}\left(\frac{4a}{\lambda},\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\cos(\lambda x)}{2}+\frac{1}{2}\right)\lambda}{4}+c_3 \cdots}{c_3 \operatorname{HeunC}\left(\frac{4a}{\lambda},-\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\cos(\lambda x)}{2}+\frac{1}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{2\left(\frac{\left(\cos(\lambda x)a+a+\frac{\lambda}{2}\right)\operatorname{HeunC}\left(\frac{4a}{\lambda},\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\cos(\lambda x)}{2}+\frac{1}{2}\right)}{2}+\frac{(1+\cos(\lambda x))\operatorname{HeunCPrime}\left(\frac{4a}{\lambda},\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\cos(\lambda x)}{2}+\frac{1}{2}\right)\lambda}{4}+c_3 \cdots}{c_3 \operatorname{HeunC}\left(\frac{4a}{\lambda},-\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\cos(\lambda x)}{2}+\frac{1}{2}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{2\left(\frac{\left(\cos(\lambda x)a+a+\frac{\lambda}{2}\right)\operatorname{HeunC}\left(\frac{4a}{\lambda},\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\cos(\lambda x)}{2}+\frac{1}{2}\right)}{2}+\frac{(1+\cos(\lambda x))\operatorname{HeunCPrime}\left(\frac{4a}{\lambda},\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\cos(\lambda x)}{2}+\frac{1}{2}\right)\lambda}{4}+c_3 \cdots}{c_3 \operatorname{HeunC}\left(\frac{4a}{\lambda},-\frac{1}{2},-\frac{1}{2},-\frac{2a}{\lambda},\frac{8a+3\lambda}{8\lambda},\frac{\cos(\lambda x)}{2}+\frac{1}{2}\right)}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a^2-a*lambda*cos(lambda*x)-co
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               -> Kummer
                  -> hyper3: Equivalence to 1F1 under a power @ Moebius
               -> hypergeometric
                  -> heuristic approach
                  -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
               -> Mathieu
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 272

```
dsolve(diff(y(x),x)=y(x)^2-a^2+a*lambda*cos(lambda*x)+a^2*cos(lambda*x)^2,y(x), singsol=all)
```

$$y(x) = \frac{\left(2ac_1 \sin(x\lambda)\cos\left(\frac{x\lambda}{2}\right) + c_1\lambda \sin\left(\frac{x\lambda}{2}\right)\right)\mathrm{HeunC}\left(\frac{4a}{\lambda}, \frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \frac{\cos(x\lambda)}{2} + \frac{1}{2}\right) + 2\sin(x\lambda)\left(a\,\mathrm{HeunC}\right.}{2\cos\left(\frac{x\lambda}{2}\right)\mathrm{HeunC}\left(\frac{4a}{\lambda}, \frac{1}{2}, -\frac{1}{2}, -\frac{2a}{\lambda}, \frac{8a+3\lambda}{8\lambda}, \right.}$$

✓ Solution by Mathematica

Time used: 3.942 (sec). Leaf size: 131

```
DSolve[y'[x]==y[x]^2-a^2+a*\[Lambda]*Cos[\[Lambda]*x]+a^2*Cos[\[Lambda]*x]^2,y[x],x,IncludeS
```

$$y(x) \to \frac{ac_1 \sin(\lambda x)\int_1^x e^{-\frac{2a\cos(\lambda K[1])}{\lambda}}dK[1] + a\sin(\lambda x) + c_1\left(-e^{-\frac{2a\cos(\lambda x)}{\lambda}}\right)}{1 + c_1\int_1^x e^{-\frac{2a\cos(\lambda K[1])}{\lambda}}dK[1]}$$

$$y(x) \to a\sin(\lambda x) - \frac{e^{-\frac{2a\cos(\lambda x)}{\lambda}}}{\int_1^x e^{-\frac{2a\cos(\lambda K[1])}{\lambda}}dK[1]}$$

10.3 problem 16

Internal problem ID [10514]
Internal file name [`OUTPUT/9461_Monday_June_06_2022_02_40_22_PM_53437655/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-2. Equations with cosine.
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$\boxed{y' - y^2 = \lambda^2 + c\cos(\lambda x + a)^n \cos(\lambda x + b)^{-n-4}}$$

10.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + \lambda^2 + c\cos(\lambda x + a)^n \cos(\lambda x + b)^{-n-4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \lambda^2 + \frac{c(\cos(\lambda x)\cos(a) - \sin(\lambda x)\sin(a))^n \cos(\lambda x + b)^{-n}}{(\cos(\lambda x)\cos(b) - \sin(\lambda x)\sin(b))^4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \lambda^2 + c\cos(\lambda x + a)^n \cos(\lambda x + b)^{-n-4}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= \lambda^2 + c\cos(\lambda x + a)^n \cos(\lambda x + b)^{-n-4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(\lambda^2 + c\cos(\lambda x + a)^n \cos(\lambda x + b)^{-n-4}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2+lambda^2+c*cos(lambda*x+a)^n*cos(lambda*x+b)^(-n-4),y(x), singsol
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+\[Lambda]^2+c*Cos[\[Lambda]*x+a]^n*Cos[\[Lambda]*x+b]^(-n-4),y[x],x,Inc
```

Not solved

10.4 problem 17

Internal problem ID [10515]
Internal file name [`OUTPUT/9462_Monday_June_06_2022_02_43_31_PM_724295/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-2. Equations with cosine.
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - a\cos(\beta x)\, y = ab\cos(\beta x) - b^2}$$

10.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= y^2 + a\cos(\beta x)\, y + ab\cos(\beta x) - b^2
\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\cos(\beta x)\, y + ab\cos(\beta x) - b^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = ab\cos(\beta x) - b^2$, $f_1(x) = a\cos(\beta x)$ and $f_2(x) = 1$. Let

$$\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{u} \qquad (1)
\end{aligned}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= a\cos(\beta x) \\ f_2^2f_0 &= ab\cos(\beta x) - b^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - a\cos(\beta x)\,u'(x) + \left(ab\cos(\beta x) - b^2\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1\mathrm{e}^{-\frac{b(-2\beta x+\pi)}{2\beta}} + ic_2\mathrm{e}^{bx}\beta\left(\int \mathrm{e}^{\frac{-2b\beta x+\sin(\beta x)a+\pi b}{\beta}}dx\right)$$

The above shows that

$$u'(x) = c_1b\,\mathrm{e}^{-\frac{b(-2\beta x+\pi)}{2\beta}} + ic_2b\,\mathrm{e}^{bx}\beta\left(\int \mathrm{e}^{\frac{-2b\beta x+\sin(\beta x)a+\pi b}{\beta}}dx\right) + i\beta c_2\mathrm{e}^{\frac{\sin(\beta x)a-b(\beta x-\pi)}{\beta}}$$

Using the above in (1) gives the solution

$$y = -\frac{c_1b\,\mathrm{e}^{-\frac{b(-2\beta x+\pi)}{2\beta}} + ic_2b\,\mathrm{e}^{bx}\beta\left(\int \mathrm{e}^{\frac{-2b\beta x+\sin(\beta x)a+\pi b}{\beta}}dx\right) + i\beta c_2\mathrm{e}^{\frac{\sin(\beta x)a-b(\beta x-\pi)}{\beta}}}{c_1\mathrm{e}^{-\frac{b(-2\beta x+\pi)}{2\beta}} + ic_2\mathrm{e}^{bx}\beta\left(\int \mathrm{e}^{\frac{-2b\beta x+\sin(\beta x)a+\pi b}{\beta}}dx\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\beta\,\mathrm{e}^{\frac{\sin(\beta x)a-b(\beta x-\pi)}{\beta}} - b\left(ic_3\mathrm{e}^{-\frac{b(-2\beta x+\pi)}{2\beta}} - \mathrm{e}^{bx}\beta\left(\int \mathrm{e}^{\frac{-2b\beta x+\sin(\beta x)a+\pi b}{\beta}}dx\right)\right)}{ic_3\mathrm{e}^{-\frac{b(-2\beta x+\pi)}{2\beta}} - \mathrm{e}^{bx}\beta\left(\int \mathrm{e}^{\frac{-2b\beta x+\sin(\beta x)a+\pi b}{\beta}}dx\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\beta\, \mathrm{e}^{\frac{\sin(\beta x)a - b(\beta x - \pi)}{\beta}} - b\left(i c_3 \mathrm{e}^{-\frac{b(-2\beta x+\pi)}{2\beta}} - \mathrm{e}^{bx}\beta\left(\int \mathrm{e}^{\frac{-2b\beta x+\sin(\beta x)a+\pi b}{\beta}}dx\right)\right)}{i c_3 \mathrm{e}^{-\frac{b(-2\beta x+\pi)}{2\beta}} - \mathrm{e}^{bx}\beta\left(\int \mathrm{e}^{\frac{-2b\beta x+\sin(\beta x)a+\pi b}{\beta}}dx\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\beta\, \mathrm{e}^{\frac{\sin(\beta x)a - b(\beta x - \pi)}{\beta}} - b\left(i c_3 \mathrm{e}^{-\frac{b(-2\beta x+\pi)}{2\beta}} - \mathrm{e}^{bx}\beta\left(\int \mathrm{e}^{\frac{-2b\beta x+\sin(\beta x)a+\pi b}{\beta}}dx\right)\right)}{i c_3 \mathrm{e}^{-\frac{b(-2\beta x+\pi)}{2\beta}} - \mathrm{e}^{bx}\beta\left(\int \mathrm{e}^{\frac{-2b\beta x+\sin(\beta x)a+\pi b}{\beta}}dx\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 73

```
dsolve(diff(y(x),x)=y(x)^2+a*cos(beta*x)*y(x)+a*b*cos(beta*x)-b^2,y(x), singsol=all)
```

$$y(x) = \frac{b\left(\int \mathrm{e}^{\frac{-2b\beta x+\sin(x\beta)a}{\beta}}dx\right) - c_1 b + \mathrm{e}^{\frac{-2b\beta x+\sin(x\beta)a}{\beta}}}{-\left(\int \mathrm{e}^{\frac{-2b\beta x+\sin(x\beta)a}{\beta}}dx\right) + c_1}$$

✓ Solution by Mathematica

Time used: 9.071 (sec). Leaf size: 183

```
DSolve[y'[x]==y[x]^2+a*Cos[\[Beta]*x]*y[x]+a*b*Cos[\[Beta]*x]-b^2,y[x],x,IncludeSingularSolu
```

$$\text{Solve}\left[\int_1^x \frac{e^{\frac{a\sin(\beta K[1])}{\beta}-2bK[1]}(-b+a\cos(\beta K[1])+y(x))}{a\beta(b+y(x))}dK[1]\right.$$
$$+\int_1^{y(x)}\left(-\int_1^x\left(\frac{e^{\frac{a\sin(\beta K[1])}{\beta}-2bK[1]}}{a\beta(b+K[2])}-\frac{e^{\frac{a\sin(\beta K[1])}{\beta}-2bK[1]}(-b+a\cos(\beta K[1])+K[2])}{a\beta(b+K[2])^2}\right)dK[1]\right.$$
$$\left.\left.-\frac{e^{\frac{a\sin(x\beta)}{\beta}-2bx}}{a\beta(b+K[2])^2}\right)dK[2]=c_1,y(x)\right]$$

10.5 problem 18

Internal problem ID [10516]
Internal file name [`OUTPUT/9463_Monday_June_06_2022_02_44_05_PM_18158228/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-2. Equations with cosine.
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - a\cos(bx)^m y = a\cos(bx)^m}$$

10.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + a\cos(bx)^m y + a\cos(bx)^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\cos(bx)^m y + a\cos(bx)^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\cos(bx)^m$, $f_1(x) = a\cos(bx)^m$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= a \cos(bx)^m \\ f_2^2 f_0 &= a \cos(bx)^m \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - a \cos(bx)^m\, u'(x) + a \cos(bx)^m\, u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) + a \cos(bx)^m\, (-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) + a \cos(bx)^m\, (-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) + a \cos(bx)^m\, (-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)}{\mathrm{DESol}\left(\left\{\_Y''(x) + a \cos(bx)^m\, (-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) + a \cos(bx)^m\, (-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)}{\mathrm{DESol}\left(\left\{\_Y''(x) + a \cos(bx)^m\, (-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) + a \cos(bx)^m\, (-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)}{\mathrm{DESol}\left(\left\{\_Y''(x) + a \cos(bx)^m\, (-\_Y'(x) + \_Y(x))\right\}, \left\{\_Y(x)\right\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\_Y''(x) + a\cos\left(bx\right)^{m}\left(-\_Y'(x) + \_Y(x)\right)\right\}, \left\{\_Y(x)\right\}\right)}{\mathrm{DESol}\left(\left\{\_Y''\left(x\right) + a\cos\left(bx\right)^{m}\left(-\_Y'\left(x\right) + \_Y\left(x\right)\right)\right\}, \left\{\_Y\left(x\right)\right\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = a*cos(x*b)^m*(diff(y(x), x))-a
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Kummer
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            -> Mathieu
               -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2+a*cos(b*x)^m*y(x)+a*cos(b*x)^m,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+a*Cos[b*x]^m*y[x]+a*Cos[b*x]^m,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

10.6 problem 19

Internal problem ID [10517]
Internal file name [`OUTPUT/9464_Monday_June_06_2022_02_44_10_PM_6467390/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-2. Equations with cosine.
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \cos(\lambda x)\, y^2 \lambda = \lambda \cos(\lambda x)^3}$$

10.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \cos(\lambda x)\, \lambda\, y^2 + \lambda \cos(\lambda x)^3 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \cos(\lambda x)\, \lambda\, y^2 + \lambda \cos(\lambda x)^3$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \lambda \cos(\lambda x)^3$, $f_1(x) = 0$ and $f_2(x) = \lambda \cos(\lambda x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\lambda \cos(\lambda x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\sin(\lambda x)\lambda^2 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \lambda^3 \cos(\lambda x)^5 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\lambda \cos(\lambda x) u''(x) + \sin(\lambda x)\lambda^2 u'(x) + \lambda^3 \cos(\lambda x)^5 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{\mathrm{e}^{-\sin(\lambda x)^2 - \frac{\cos(\lambda x)^2}{2}} \left((c_1 - 2c_2)\operatorname{erf}\left(\sqrt{-\sin(\lambda x)^2}\right) + 2c_2\right)\sin(\lambda x)\sqrt{\pi}}{2\sqrt{-\sin(\lambda x)^2}}$$

The above shows that

$$u'(x) = \frac{\lambda\left(-\frac{\sqrt{\pi}\sin(\lambda x)^2 (c_1-2c_2)\operatorname{erf}\left(\sqrt{-\sin(\lambda x)^2}\right)}{2} + \mathrm{e}^{\sin(\lambda x)^2}(c_1 - 2c_2)\sqrt{-\sin(\lambda x)^2} - \sin(\lambda x)^2\sqrt{\pi}\, c_2\right)\mathrm{e}^{\frac{\cos(\lambda x)^2}{2} - 1}\cos}{\sqrt{-\sin(\lambda x)^2}}$$

Using the above in (1) gives the solution

$$y = -\frac{2\left(-\frac{\sqrt{\pi}\sin(\lambda x)^2 (c_1-2c_2)\operatorname{erf}\left(\sqrt{-\sin(\lambda x)^2}\right)}{2} + \mathrm{e}^{\sin(\lambda x)^2}(c_1 - 2c_2)\sqrt{-\sin(\lambda x)^2} - \sin(\lambda x)^2\sqrt{\pi}\, c_2\right)\mathrm{e}^{\frac{\cos(\lambda x)^2}{2} - 1}\mathrm{e}}{\left((c_1 - 2c_2)\operatorname{erf}\left(\sqrt{-\sin(\lambda x)^2}\right) + 2c_2\right)\sin(\lambda x)\sqrt{\pi}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{2\left(-\frac{\sin(\lambda x)\sqrt{\pi}\,(c_3-2)\,\mathrm{erf}\left(\sqrt{-\sin(\lambda x)^2}\right)}{2}+\csc(\lambda x)\,\mathrm{e}^{\sin(\lambda x)^2}(c_3-2)\sqrt{-\sin(\lambda x)^2}-\sin(\lambda x)\sqrt{\pi}\right)}{\left((c_3-2)\,\mathrm{erf}\left(\sqrt{-\sin(\lambda x)^2}\right)+2\right)\sqrt{\pi}}$$

Summary

The solution(s) found are the following

$$y = -\frac{2\left(-\frac{\sin(\lambda x)\sqrt{\pi}\,(c_3-2)\,\mathrm{erf}\left(\sqrt{-\sin(\lambda x)^2}\right)}{2}+\csc(\lambda x)\,\mathrm{e}^{\sin(\lambda x)^2}(c_3-2)\sqrt{-\sin(\lambda x)^2}-\sin(\lambda x)\sqrt{\pi}\right)}{\left((c_3-2)\,\mathrm{erf}\left(\sqrt{-\sin(\lambda x)^2}\right)+2\right)\sqrt{\pi}} \tag{1}$$

Verification of solutions

$$y = -\frac{2\left(-\frac{\sin(\lambda x)\sqrt{\pi}\,(c_3-2)\,\mathrm{erf}\left(\sqrt{-\sin(\lambda x)^2}\right)}{2}+\csc(\lambda x)\,\mathrm{e}^{\sin(\lambda x)^2}(c_3-2)\sqrt{-\sin(\lambda x)^2}-\sin(\lambda x)\sqrt{\pi}\right)}{\left((c_3-2)\,\mathrm{erf}\left(\sqrt{-\sin(\lambda x)^2}\right)+2\right)\sqrt{\pi}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -lambda*sin(lambda*x)*(diff(y(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               -> Kummer
                  -> hyper3: Equivalence to 1F1 under a power @ Moebius
                  <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
               <- Kummer successful
            <- special function solution successful
               -> Trying to convert hypergeometric functions to elementary form...
               <- elementary form is not straightforward to achieve - returning special func
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 95

```
dsolve(diff(y(x),x)=lambda*cos(lambda*x)*y(x)^2+lambda*cos(lambda*x)^3,y(x), singsol=all)
```

$$y(x) = -\frac{4\csc(x\lambda)\left(-\frac{\sqrt{\pi}\sin(x\lambda)^2(c_1-\frac{1}{2})\operatorname{erf}\left(\sqrt{-\sin(x\lambda)^2}\right)}{2}+\left(c_1-\frac{1}{2}\right)e^{\sin(x\lambda)^2}\sqrt{-\sin(x\lambda)^2}+\frac{\sin(x\lambda)^2\sqrt{\pi}\,c_1}{2}\right)}{\sqrt{\pi}\left(\operatorname{erf}\left(\sqrt{-\sin(x\lambda)^2}\right)(2c_1-1)-2c_1\right)}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==\[Lambda]*Cos[\[Lambda]*x]*y[x]^2+\[Lambda]*Cos[\[Lambda]*x]^3,y[x],x,IncludeS
```

Not solved

10.7 problem 20

Internal problem ID [10518]
Internal file name [`OUTPUT/9465_Monday_June_06_2022_02_44_12_PM_8959918/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-2. Equations with cosine.
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{2y' - (\lambda + a - \cos(\lambda x)\, a)\, y^2 = -a + \lambda - \cos(\lambda x)\, a}$$

10.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{\cos(\lambda x)\, a\, y^2}{2} + \frac{a\, y^2}{2} + \frac{\lambda\, y^2}{2} + \frac{\lambda}{2} - \frac{a}{2} - \frac{\cos(\lambda x)\, a}{2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{\cos(\lambda x)\, a\, y^2}{2} + \frac{a\, y^2}{2} + \frac{\lambda\, y^2}{2} + \frac{\lambda}{2} - \frac{a}{2} - \frac{\cos(\lambda x)\, a}{2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\lambda}{2} - \frac{a}{2} - \frac{\cos(\lambda x)a}{2}$, $f_1(x) = 0$ and $f_2(x) = \frac{a}{2} + \frac{\lambda}{2} - \frac{\cos(\lambda x)a}{2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\left(\frac{a}{2} + \frac{\lambda}{2} - \frac{\cos(\lambda x)a}{2}\right) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{a\lambda \sin(\lambda x)}{2} \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= \left(\frac{a}{2} + \frac{\lambda}{2} - \frac{\cos(\lambda x)\, a}{2}\right)^2 \left(\frac{\lambda}{2} - \frac{a}{2} - \frac{\cos(\lambda x)\, a}{2}\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\left(\frac{a}{2} + \frac{\lambda}{2} - \frac{\cos(\lambda x)\, a}{2}\right) u''(x) - \frac{a\lambda \sin(\lambda x)\, u'(x)}{2} + \left(\frac{a}{2} + \frac{\lambda}{2} - \frac{\cos(\lambda x)\, a}{2}\right)^2 \left(\frac{\lambda}{2} - \frac{a}{2} - \frac{\cos(\lambda x)\, a}{2}\right) u(x) =$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{\sin\left(\frac{\lambda x}{2}\right) e^{-\frac{a\cos(\lambda x)}{2\lambda}} \left(i c_2 \lambda \left(\int e^{\frac{a\cos(\lambda x)}{\lambda}} \left(2a + \csc\left(\frac{\lambda x}{2}\right)^2 \lambda\right) dx\right) + 2c_1\right)}{2}$$

The above shows that

$$u'(x) = -\frac{\csc\left(\frac{\lambda x}{2}\right) (\cos(\lambda x)\, a - a - \lambda) \left(i \sin(\lambda x) \left(\int e^{\frac{a\cos(\lambda x)}{\lambda}} \left(2a + \csc\left(\frac{\lambda x}{2}\right)^2 \lambda\right) dx\right) c_2 \lambda e^{-\frac{a\cos(\lambda x)}{2\lambda}} + 4i e^{\frac{a\cos(\lambda x)}{2\lambda}} c}{8}$$

Using the above in (1) gives the solution

$$y = \frac{\csc\left(\frac{\lambda x}{2}\right) (\cos(\lambda x)\, a - a - \lambda) \left(i \sin(\lambda x) \left(\int e^{\frac{a\cos(\lambda x)}{\lambda}} \left(2a + \csc\left(\frac{\lambda x}{2}\right)^2 \lambda\right) dx\right) c_2 \lambda e^{-\frac{a\cos(\lambda x)}{2\lambda}} + 4i e^{\frac{a\cos(\lambda x)}{2\lambda}} c_2 \lambda}{4\left(\frac{a}{2} + \frac{\lambda}{2} - \frac{\cos(\lambda x)a}{2}\right) \sin\left(\frac{\lambda x}{2}\right) \left(i c_2 \lambda \left(\int e^{\frac{a\cos(\lambda x)}{\lambda}} \left(2a + \csc\left(\frac{\lambda x}{2}\right)^2 \lambda\right) dx\right) + 2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{2\left(i \sin(\lambda x)\, c_3 - \frac{\lambda\left(\int e^{\frac{a\cos(\lambda x)}{\lambda}} \left(2a+\csc\left(\frac{\lambda x}{2}\right)^2 \lambda\right) dx\right) \sin(\lambda x)}{2} - 2\lambda e^{\frac{a\cos(\lambda x)}{\lambda}}\right) \csc\left(\frac{\lambda x}{2}\right)^2}{4ic_3 - 2\lambda \left(\int e^{\frac{a\cos(\lambda x)}{\lambda}} \left(2a + \csc\left(\frac{\lambda x}{2}\right)^2 \lambda\right) dx\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{2\left(i\sin\left(\lambda x\right)c_3 - \frac{\lambda\left(\int \mathrm{e}^{\frac{a\cos(\lambda x)}{\lambda}}\left(2a+\csc\left(\frac{\lambda x}{2}\right)^2\lambda\right)dx\right)\sin(\lambda x)}{2} - 2\lambda\,\mathrm{e}^{\frac{a\cos(\lambda x)}{\lambda}}\right)\csc\left(\frac{\lambda x}{2}\right)^2}{4ic_3 - 2\lambda\left(\int \mathrm{e}^{\frac{a\cos(\lambda x)}{\lambda}}\left(2a+\csc\left(\frac{\lambda x}{2}\right)^2\lambda\right)dx\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{2\left(i\sin\left(\lambda x\right)c_3 - \frac{\lambda\left(\int \mathrm{e}^{\frac{a\cos(\lambda x)}{\lambda}}\left(2a+\csc\left(\frac{\lambda x}{2}\right)^2\lambda\right)dx\right)\sin(\lambda x)}{2} - 2\lambda\,\mathrm{e}^{\frac{a\cos(\lambda x)}{\lambda}}\right)\csc\left(\frac{\lambda x}{2}\right)^2}{4ic_3 - 2\lambda\left(\int \mathrm{e}^{\frac{a\cos(\lambda x)}{\lambda}}\left(2a+\csc\left(\frac{\lambda x}{2}\right)^2\lambda\right)dx\right)}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -a*lambda*sin(lambda*x)*(diff(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               -> Kummer
                  -> hyper3: Equivalence to 1F1 under a power @ Moebius
               -> hypergeometric
                  -> heuristic approach
                  -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
               -> Mathieu
                  -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 122

```
dsolve(2*diff(y(x),x)=(lambda+a-a*cos(lambda*x))*y(x)^2+lambda-a-a*cos(lambda*x),y(x), sings
```

$$y(x) = \frac{-\cot\left(\frac{x\lambda}{2}\right)\lambda\left(\int \mathrm{e}^{\frac{a\cos(x\lambda)}{\lambda}}\operatorname{csgn}\left(\sin\left(\frac{x\lambda}{2}\right)\right)\left(\csc\left(\frac{x\lambda}{2}\right)^2\lambda+2a\right)dx\right)c_1-2\csc\left(\frac{x\lambda}{2}\right)^2\operatorname{csgn}\left(\sin\left(\frac{x\lambda}{2}\right)\right)\mathrm{e}^{\frac{a\cos(x\lambda)}{\lambda}}c_1}{\lambda\left(\int \mathrm{e}^{\frac{a\cos(x\lambda)}{\lambda}}\operatorname{csgn}\left(\sin\left(\frac{x\lambda}{2}\right)\right)\left(\csc\left(\frac{x\lambda}{2}\right)^2\lambda+2a\right)dx\right)c_1-2i}$$

✓ Solution by Mathematica

Time used: 34.139 (sec). Leaf size: 234

```
DSolve[2*y'[x]==(\[Lambda]+a-a*Cos[\[Lambda]*x])*y[x]^2+\[Lambda]-a-a*Cos[\[Lambda]*x],y[x],
```

$$y(x) \to -\frac{2\left(c_1\cot\left(\frac{\lambda x}{2}\right)\int_1^x e^{-\frac{2a\sin^2\left(\frac{1}{2}\lambda K[1]\right)}{\lambda}}\left(\lambda\csc^2\left(\frac{1}{2}\lambda K[1]\right)+2a\right)dK[1]+2c_1\csc^2\left(\frac{\lambda x}{2}\right)e^{-\frac{2a\sin^2\left(\frac{\lambda x}{2}\right)}{\lambda}}+\cot\left(\frac{\lambda x}{2}\right)\right)}{2+2c_1\int_1^x e^{-\frac{2a\sin^2\left(\frac{1}{2}\lambda K[1]\right)}{\lambda}}\left(\lambda\csc^2\left(\frac{1}{2}\lambda K[1]\right)+2a\right)dK[1]}$$

$$y(x) \to \frac{1}{2}\csc^2\left(\frac{\lambda x}{2}\right)\left(-\frac{4e^{-\frac{2a\sin^2\left(\frac{\lambda x}{2}\right)}{\lambda}}}{\int_1^x e^{-\frac{2a\sin^2\left(\frac{1}{2}\lambda K[1]\right)}{\lambda}}\left(\lambda\csc^2\left(\frac{1}{2}\lambda K[1]\right)+2a\right)dK[1]}-\sin(\lambda x)\right)$$

10.8 problem 21

Internal problem ID [10519]
Internal file name [`OUTPUT/9466_Monday_June_06_2022_02_44_15_PM_26441355/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-2. Equations with cosine.
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \left(\lambda + a\cos\left(\lambda x\right)^{2}\right) y^{2} = -a + \lambda + a\cos\left(\lambda x\right)^{2}}$$

10.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \cos\left(\lambda x\right)^{2} a\,y^{2} + a\cos\left(\lambda x\right)^{2} + \lambda\,y^{2} - a + \lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \cos\left(\lambda x\right)^{2} a\,y^{2} + a\cos\left(\lambda x\right)^{2} + \lambda\,y^{2} - a + \lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a + \lambda + a\cos\left(\lambda x\right)^{2}$, $f_1(x) = 0$ and $f_2(x) = \lambda + a\cos\left(\lambda x\right)^{2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\left(\lambda + a\cos\left(\lambda x\right)^{2}\right) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -2\sin(\lambda x)\, a\lambda \cos(\lambda x) \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \left(\lambda + a\cos(\lambda x)^2\right)^2 \left(-a + \lambda + a\cos(\lambda x)^2\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\left(\lambda + a\cos(\lambda x)^2\right) u''(x) + 2\sin(\lambda x)\, a\lambda\cos(\lambda x)\, u'(x) + \left(\lambda + a\cos(\lambda x)^2\right)^2 \left(-a + \lambda + a\cos(\lambda x)^2\right) u(x) = ($$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \cos(\lambda x)\, \mathrm{e}^{\frac{\cos(2\lambda x) a}{4\lambda}} \left(c_1 + 2ic_2\lambda \left(\int \mathrm{e}^{-\frac{\cos(2\lambda x) a}{2\lambda}} \left(\sec(\lambda x)^2 \lambda + a\right) dx\right)\right)$$

The above shows that

$$u'(x) = -\frac{\sec(\lambda x)\left(\lambda + a\cos(\lambda x)^2\right)\left(2i\sin(2\lambda x)\left(\int \mathrm{e}^{-\frac{\cos(2\lambda x) a}{2\lambda}} \left(\sec(\lambda x)^2 \lambda + a\right) dx\right) c_2\lambda\, \mathrm{e}^{\frac{\cos(2\lambda x) a}{4\lambda}} + \sin(2\lambda x)\, c_1 \mathrm{e}}{2}$$

Using the above in (1) gives the solution

$$y = \frac{\sec(\lambda x)\left(2i\sin(2\lambda x)\left(\int \mathrm{e}^{-\frac{\cos(2\lambda x) a}{2\lambda}} \left(\sec(\lambda x)^2 \lambda + a\right) dx\right) c_2\lambda\, \mathrm{e}^{\frac{\cos(2\lambda x) a}{4\lambda}} + \sin(2\lambda x)\, c_1 \mathrm{e}^{\frac{\cos(2\lambda x) a}{4\lambda}} - 4i\lambda c_2 \mathrm{e}^{-\frac{\cos}{}}}{2\cos(\lambda x)\left(c_1 + 2ic_2\lambda\left(\int \mathrm{e}^{-\frac{\cos(2\lambda x) a}{2\lambda}} \left(\sec(\lambda x)^2 \lambda + a\right) dx\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{ic_3 \tan(\lambda x) - 2\lambda\left(\int \mathrm{e}^{-\frac{\cos(2\lambda x) a}{2\lambda}} \left(\sec(\lambda x)^2 \lambda + a\right) dx\right) \tan(\lambda x) + 2\sec(\lambda x)^2 \mathrm{e}^{-\frac{\cos(2\lambda x) a}{2\lambda}} \lambda}{ic_3 - 2\lambda\left(\int \mathrm{e}^{-\frac{\cos(2\lambda x) a}{2\lambda}} \left(\sec(\lambda x)^2 \lambda + a\right) dx\right)}$$

Summary
The solution(s) found are the following

$$
y = \frac{i c_3 \tan(\lambda x) - 2\lambda \left(\int e^{-\frac{\cos(2\lambda x) a}{2\lambda}} \left(\sec(\lambda x)^2 \lambda + a\right) dx\right) \tan(\lambda x) + 2\sec(\lambda x)^2 e^{-\frac{\cos(2\lambda x) a}{2\lambda}} \lambda}{i c_3 - 2\lambda \left(\int e^{-\frac{\cos(2\lambda x) a}{2\lambda}} \left(\sec(\lambda x)^2 \lambda + a\right) dx\right)} \tag{1}
$$

Verification of solutions

$$
y = \frac{i c_3 \tan(\lambda x) - 2\lambda \left(\int e^{-\frac{\cos(2\lambda x) a}{2\lambda}} \left(\sec(\lambda x)^2 \lambda + a\right) dx\right) \tan(\lambda x) + 2\sec(\lambda x)^2 e^{-\frac{\cos(2\lambda x) a}{2\lambda}} \lambda}{i c_3 - 2\lambda \left(\int e^{-\frac{\cos(2\lambda x) a}{2\lambda}} \left(\sec(\lambda x)^2 \lambda + a\right) dx\right)}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -2*a*cos(lambda*x)*lambda*sin(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               -> Kummer
                  -> hyper3: Equivalence to 1F1 under a power @ Moebius
               -> hypergeometric
                  -> heuristic approach
                  -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
               -> Mathieu
                  -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 102

```
dsolve(diff(y(x),x)=(lambda+a*cos(lambda*x)^2)*y(x)^2+lambda-a+a*cos(lambda*x)^2,y(x), sings
```

$$y(x) = \frac{2\sec(x\lambda)^2 e^{-\frac{a\cos(2x\lambda)}{2\lambda}} c_1\lambda - 2\tan(x\lambda)\lambda\left(\int e^{-\frac{a\cos(2x\lambda)}{2\lambda}}\left(\sec(x\lambda)^2\lambda+a\right)dx\right)c_1 + i\tan(x\lambda)}{-2\lambda\left(\int e^{-\frac{a\cos(2x\lambda)}{2\lambda}}\left(\sec(x\lambda)^2\lambda+a\right)dx\right)c_1+i}$$

✓ Solution by Mathematica

Time used: 36.333 (sec). Leaf size: 263

```
DSolve[y'[x]==(\[Lambda]+a*Cos[\[Lambda]*x]^2)*y[x]^2+\[Lambda]-a+a*Cos[\[Lambda]*x]^2,y[x],
```

$$y(x) \to \frac{2\left(c_1\tan(\lambda x)\int_1^x e^{-\frac{a\cos^2(\lambda K[1])}{\lambda}}\left(\lambda\sec^2(\lambda K[1])+a\right)dK[1] + c_1\sec^2(\lambda x)\left(-e^{-\frac{a\cos^2(\lambda x)}{\lambda}}\right)+\tan(\lambda x)\right)}{2+2c_1\int_1^x e^{-\frac{a\cos^2(\lambda K[1])}{\lambda}}\left(\lambda\sec^2(\lambda K[1])+a\right)dK[1]}$$

$$y(x) \to \frac{1}{2}\sec^2(\lambda x)\left(\sin(2\lambda x) - \frac{2e^{-\frac{a\cos^2(\lambda x)}{\lambda}}}{\int_1^x e^{-\frac{a\cos^2(\lambda K[1])}{\lambda}}\left(\lambda\sec^2(\lambda K[1])+a\right)dK[1]}\right)$$

$$y(x) \to \frac{1}{2}\sec^2(\lambda x)\left(\sin(2\lambda x) - \frac{2e^{-\frac{a\cos^2(\lambda x)}{\lambda}}}{\int_1^x e^{-\frac{a\cos^2(\lambda K[1])}{\lambda}}\left(\lambda\sec^2(\lambda K[1])+a\right)dK[1]}\right)$$

10.9 problem 22

Internal problem ID [10520]
Internal file name [`OUTPUT/9467_Monday_June_06_2022_02_45_03_PM_33788457/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-2. Equations with cosine.
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + (k+1)\, x^k y^2 - a\, x^{k+1} \cos(x)^m\, y = -a \cos(x)^m}$$

10.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\, x^{k+1} \cos(x)^m\, y - x^k y^2 k - x^k y^2 - a \cos(x)^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\, x^k x \cos(x)^m\, y - x^k y^2 k - x^k y^2 - a \cos(x)^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a\cos(x)^m$, $f_1(x) = a\cos(x)^m\, x^{k+1}$ and $f_2(x) = -x^k k - x^k$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(-x^k k - x^k)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{k^2x^k}{x} - \frac{k\,x^k}{x} \\
f_1f_2 &= a\cos(x)^m\,x^{k+1}\left(-x^kk - x^k\right) \\
f_2^2f_0 &= -\left(-x^kk - x^k\right)^2 a\cos(x)^m
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\left(-x^kk - x^k\right)u''(x) - \left(-\frac{k^2x^k}{x} - \frac{k\,x^k}{x} + a\cos(x)^m\,x^{k+1}\left(-x^kk - x^k\right)\right)u'(x) - \left(-x^kk - x^k\right)^2 a\cos(x)^m\,u($$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{k+1}\left(c_1 + c_2\left(\int x^{-2k-2}\mathrm{e}^{\int\left(a\cos(x)^m x^{k+1}+\frac{k}{x}\right)dx}dx\right)\right)$$

The above shows that

$$u'(x) = x^k\left(c_2x^{-2k-1}\mathrm{e}^{\int\left(a\cos(x)^m x^{k+1}+\frac{k}{x}\right)dx} + \left(c_1 + c_2\left(\int x^{-2k-2}\mathrm{e}^{\int\left(a\cos(x)^m x^{k+1}+\frac{k}{x}\right)dx}dx\right)\right)(k+1)\right)$$

Using the above in (1) gives the solution

$$y = -\frac{x^k\left(c_2x^{-2k-1}\mathrm{e}^{\int\left(a\cos(x)^m x^{k+1}+\frac{k}{x}\right)dx} + \left(c_1 + c_2\left(\int x^{-2k-2}\mathrm{e}^{\int\left(a\cos(x)^m x^{k+1}+\frac{k}{x}\right)dx}dx\right)\right)(k+1)\right)x^{-k-1}}{\left(-x^kk - x^k\right)\left(c_1 + c_2\left(\int x^{-2k-2}\mathrm{e}^{\int\left(a\cos(x)^m x^{k+1}+\frac{k}{x}\right)dx}dx\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x^{-k-1}\left(x^{-2k-1}\mathrm{e}^{\int\left(a\cos(x)^m x^{k+1}+\frac{k}{x}\right)dx} + \left(c_3 + \int x^{-2k-2}\mathrm{e}^{\int\left(a\cos(x)^m x^{k+1}+\frac{k}{x}\right)dx}dx\right)(k+1)\right)}{(k+1)\left(c_3 + \int \mathrm{e}^{\int\frac{a\,x^{k+2}\cos(x)^m+k}{x}dx}x^{-2k-2}dx\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{x^{-k-1}\left(x^{-2k-1}\mathrm{e}^{\int \left(a\cos(x)^{m} x^{k+1}+\frac{k}{x}\right)dx}+\left(c_3+\int x^{-2k-2}\mathrm{e}^{\int \left(a\cos(x)^{m} x^{k+1}+\frac{k}{x}\right)dx}dx\right)(k+1)\right)}{(k+1)\left(c_3+\int \mathrm{e}^{\int \frac{a\,x^{k+2}\cos(x)^{m}+k}{x}dx}x^{-2k-2}dx\right)} \tag{1}$$

Verification of solutions

$$y = \frac{x^{-k-1}\left(x^{-2k-1}\mathrm{e}^{\int \left(a\cos(x)^{m} x^{k+1}+\frac{k}{x}\right)dx}+\left(c_3+\int x^{-2k-2}\mathrm{e}^{\int \left(a\cos(x)^{m} x^{k+1}+\frac{k}{x}\right)dx}dx\right)(k+1)\right)}{(k+1)\left(c_3+\int \mathrm{e}^{\int \frac{a\,x^{k+2}\cos(x)^{m}+k}{x}dx}x^{-2k-2}dx\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (cos(x)^m*x^(1+k)*a*x+k)*(diff
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 174

```
dsolve(diff(y(x),x)=-(k+1)*x^k*y(x)^2+a*x^(k+1)*cos(x)^m*y(x)-a*cos(x)^m,y(x), singsol=all)
```

$$y(x) = \frac{x^{-1-k}\left(x^{1+k}\mathrm{e}^{\int \frac{\cos(x)^m x^{1+k} a x-2k-2}{x}dx} + \left(\int x^k \mathrm{e}^{\int \frac{\cos(x)^m x^{1+k} a x-2k-2}{x}dx}dx\right)k + \int x^k \mathrm{e}^{\int \frac{\cos(x)^m x^{1+k} a x-2k-2}{x}dx}dx - c_1\right)}{\left(\int x^k \mathrm{e}^{\int \frac{a\,x^{k+2}\cos(x)^m-2k-2}{x}dx}dx\right)k + \int x^k \mathrm{e}^{\int \frac{a\,x^{k+2}\cos(x)^m-2k-2}{x}dx}dx - c_1}$$

✓ Solution by Mathematica

Time used: 20.002 (sec). Leaf size: 248

```
DSolve[y'[x]==-(k+1)*x^k*y[x]^2+a*x^(k+1)*Cos[x]^m*y[x]-a*Cos[x]^m,y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{x^{-k-1}\left(c_1 x \exp\left(\int_1^x -\frac{-a\cos^m(K[1])K[1]^{k+2}+k+2}{K[1]}dK[1]\right) + c_1(k+1)\int_1^x \exp\left(\int_1^{K[2]} -\frac{-a\cos^m(K[1])K[1]^{k+2}+k+2}{K[1]}dK[1]\right)dK[2]\ldots\right)}{(k+1)\left(1+c_1\int_1^x \exp\left(\int_1^{K[2]} -\frac{-a\cos^m(K[1])K[1]^{k+2}+k+2}{K[1]}dK[1]\right)dK[2]\right)}$$

$$y(x) \to \frac{x^{-k}\left(\frac{\exp\left(\int_1^x -\frac{-a\cos^m(K[1])K[1]^{k+2}+k+2}{K[1]}dK[1]\right)}{\int_1^x \exp\left(\int_1^{K[2]} -\frac{-a\cos^m(K[1])K[1]^{k+2}+k+2}{K[1]}dK[1]\right)dK[2]} + \frac{k+1}{x}\right)}{k+1}$$

10.10 problem 23

Internal problem ID [10521]
Internal file name [`OUTPUT/9468_Monday_June_06_2022_02_45_15_PM_21042046/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-2. Equations with cosine.
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$\boxed{y' - a\cos(\lambda x+\mu)^k (y - b\,x^n - c)^2 = b\,x^{n-1}n}$$

10.10.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x^{2n}\cos(\lambda x+\mu)^k\,a\,b^2 + 2x^n\cos(\lambda x+\mu)^k\,abc - 2x^n\cos(\lambda x+\mu)^k\,aby + \cos(\lambda x+\mu)^k\,a\,c^2 - 2\cos(\lambda x \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^{2n}(\cos(\mu)\cos(\lambda x) - \sin(\lambda x)\sin(\mu))^k\,a\,b^2 + 2x^n(\cos(\mu)\cos(\lambda x) - \sin(\lambda x)\sin(\mu))^k\,abc - 2x^n(\cos(\mu$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^{2n}\cos(\lambda x+\mu)^k\,a\,b^2 + 2x^n\cos(\lambda x+\mu)^k\,abc + \cos(\lambda x+\mu)^k\,a\,c^2 + b\,x^{n-1}n$, $f_1(x) = -2a\,x^n b\cos(\lambda x+\mu)^k - 2ca\cos(\lambda x+\mu)^k$ and $f_2(x) = a\cos(\lambda x+\mu)^k$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{a\cos(\lambda x+\mu)^k\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$f_2' = -\frac{a\cos(\lambda x+\mu)^k\,k\lambda\sin(\lambda x+\mu)}{\cos(\lambda x+\mu)}$$

$$f_1f_2 = \left(-2a\,x^n b\cos(\lambda x+\mu)^k - 2ca\cos(\lambda x+\mu)^k\right)a\cos(\lambda x+\mu)^k$$

$$f_2^2f_0 = a^2\cos(\lambda x+\mu)^{2k}\left(x^{2n}\cos(\lambda x+\mu)^k\,a\,b^2 + 2x^n\cos(\lambda x+\mu)^k\,abc + \cos(\lambda x+\mu)^k\,a\,c^2 + b\,x^{n-1}n\right)$$

Substituting the above terms back in equation (2) gives

$$a\cos(\lambda x+\mu)^k\,u''(x) - \left(-\frac{a\cos(\lambda x+\mu)^k\,k\lambda\sin(\lambda x+\mu)}{\cos(\lambda x+\mu)} + \left(-2a\,x^n b\cos(\lambda x+\mu)^k - 2ca\cos(\lambda x+\mu)^k\right.\right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(c_1 - ic_2\lambda\left(\int\cos(\lambda x+\mu)^k\,dx\right)\right)\sqrt{\sin(\lambda x+\mu)}\,(\cos(\mu)\cos(\lambda x) - \sin(\lambda x)\sin(\mu))^{-\frac{k}{2}}\,\mathrm{e}^{-\frac{\left(\int\left(2\sec(\lambda x+\mu)a(b\,x^n+c)\cos(\lambda x+\mu)^{k+1}+\lambda(\tan(\lambda x+\mu)k+\cot(\lambda x+\mu))\right)dx\right)}{2}}$$

The above shows that

$$u'(x) = \Bigg(\left(\cos(\mu)\cos(\lambda x) - \sin(\lambda x)\sin(\mu)\right)\left(ic_2a(b\,x^n+c)\,\lambda\left(\int\cos(\lambda x+\mu)^k\,dx\right) - x^nc_1ab - ic_2\lambda - c_1ca\right)\cos(\lambda x+\mu)^{k+1}$$
$$+\frac{k\left(ic_2\lambda\left(\int\cos(\lambda x+\mu)^k\,dx\right) - c_1\right)\left((-\sin(\lambda x)\cos(\mu) - \cos(\lambda x)\sin(\mu))\cos(\lambda x+\mu) + \sin(\lambda x+\mu)\right.}{2}$$
$$+\mu)\,(\cos(\mu)\cos(\lambda x) - \sin(\lambda x)\sin(\mu))^{-\frac{k}{2}-1}\,\mathrm{e}^{-\frac{\left(\int\left(2\sec(\lambda x+\mu)a(b\,x^n+c)\cos(\lambda x+\mu)^{k+1}+\lambda(\tan(\lambda x+\mu)k+\cot(\lambda x+\mu))\right)dx\right)}{2}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\left(\cos(\mu)\cos(\lambda x) - \sin(\lambda x)\sin(\mu)\right)\left(ic_2a(b\,x^n+c)\,\lambda\left(\int\cos(\lambda x+\mu)^k\,dx\right) - x^nc_1ab - ic_2\lambda - c_1ca\right)\right.}{}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{a(b\,x^n + c)\,\lambda\left(\int \cos\left(\lambda x + \mu\right)^k dx\right) + i x^n c_3 a b - \lambda + i c_3 c a}{a\left(\lambda\left(\int \cos\left(\lambda x + \mu\right)^k dx\right) + i c_3\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{a(b\,x^n + c)\,\lambda\left(\int \cos\left(\lambda x + \mu\right)^k dx\right) + i x^n c_3 a b - \lambda + i c_3 c a}{a\left(\lambda\left(\int \cos\left(\lambda x + \mu\right)^k dx\right) + i c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{a(b\,x^n + c)\,\lambda\left(\int \cos\left(\lambda x + \mu\right)^k dx\right) + i x^n c_3 a b - \lambda + i c_3 c a}{a\left(\lambda\left(\int \cos\left(\lambda x + \mu\right)^k dx\right) + i c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (d) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x)=a*cos(lambda*x+mu)^k*(y(x)-b*x^n-c)^2+b*n*x^(n-1),y(x), singsol=all)
```

$$y(x) = b\,x^n + c + \frac{1}{c_1 - a\left(\int \left(\cos\left(x\lambda\right)\cos\left(\mu\right) - \sin\left(x\lambda\right)\sin\left(\mu\right)\right)^k dx\right)}$$

✓ Solution by Mathematica

Time used: 6.016 (sec). Leaf size: 92

```
DSolve[y'[x]==a*Cos[\[Lambda]*x+\[Mu]]^k*(y[x]-b*x^n-c)^2+b*n*x^(n-1),y[x],x,IncludeSingular
```

$$y(x) \to \frac{1}{\frac{a\sqrt{\sin^2(\mu+\lambda x)}\csc(\mu+\lambda x)\cos^{k+1}(\mu+\lambda x)\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{k+1}{2},\frac{k+3}{2},\cos^2(x\lambda+\mu)\right)}{(k+1)\lambda} + c_1} + bx^n + c$$

$$y(x) \to bx^n + c$$

10.11 problem 24

Internal problem ID [10522]
Internal file name [`OUTPUT/9469_Monday_June_06_2022_02_45_54_PM_50478589/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-2. Equations with cosine.
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - a\cos(\lambda x)^m y^2 - ky = a\,b^2x^{2k}\cos(\lambda x)^m}$$

10.11.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= \frac{a\cos(\lambda x)^m y^2 + ky + a\,b^2x^{2k}\cos(\lambda x)^m}{x}
\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,b^2x^{2k}\cos(\lambda x)^m}{x} + \frac{a\cos(\lambda x)^m y^2}{x} + \frac{ky}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{a\,b^2x^{2k}\cos(\lambda x)^m}{x}$, $f_1(x) = \frac{k}{x}$ and $f_2(x) = \frac{a\cos(\lambda x)^m}{x}$. Let

$$\begin{aligned}
y &= \frac{-u'}{f_2 u}\\
&= \frac{-u'}{\frac{a\cos(\lambda x)^m u}{x}} \qquad (1)
\end{aligned}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{a \cos(\lambda x)^m m\lambda \sin(\lambda x)}{\cos(\lambda x) x} - \frac{a \cos(\lambda x)^m}{x^2} \\ f_1 f_2 &= \frac{ka \cos(\lambda x)^m}{x^2} \\ f_2^2 f_0 &= \frac{a^3 \cos(\lambda x)^{3m} b^2 x^{2k}}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{a \cos(\lambda x)^m u''(x)}{x} - \left(-\frac{a \cos(\lambda x)^m m\lambda \sin(\lambda x)}{\cos(\lambda x) x} - \frac{a \cos(\lambda x)^m}{x^2} + \frac{ka \cos(\lambda x)^m}{x^2}\right) u'(x) + \frac{a^3 \cos(\lambda x)^{3m} b^2 x}{x^3}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 e^{iab\left(\int x^{k-1} \cos(\lambda x)^m dx\right)} + c_2 e^{-iab\left(\int x^{k-1} \cos(\lambda x)^m dx\right)}$$

The above shows that

$$u'(x) = iab\, x^{k-1} \cos(\lambda x)^m \left(c_1 e^{iab\left(\int x^{k-1} \cos(\lambda x)^m dx\right)} - c_2 e^{-iab\left(\int x^{k-1} \cos(\lambda x)^m dx\right)}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{ib\, x^{k-1} \left(c_1 e^{iab\left(\int x^{k-1} \cos(\lambda x)^m dx\right)} - c_2 e^{-iab\left(\int x^{k-1} \cos(\lambda x)^m dx\right)}\right) x}{c_1 e^{iab\left(\int x^{k-1} \cos(\lambda x)^m dx\right)} + c_2 e^{-iab\left(\int x^{k-1} \cos(\lambda x)^m dx\right)}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{ib\, x^{k} \left(c_3 e^{iab\left(\int x^{k-1} \cos(\lambda x)^m dx\right)} - e^{-iab\left(\int x^{k-1} \cos(\lambda x)^m dx\right)}\right)}{c_3 e^{iab\left(\int x^{k-1} \cos(\lambda x)^m dx\right)} + e^{-iab\left(\int x^{k-1} \cos(\lambda x)^m dx\right)}}$$

Summary

The solution(s) found are the following

$$y = -\frac{ib\,x^{k}\left(c_{3}\mathrm{e}^{iab\left(\int x^{k-1}\cos(\lambda x)^{m}dx\right)}-\mathrm{e}^{-iab\left(\int x^{k-1}\cos(\lambda x)^{m}dx\right)}\right)}{c_{3}\mathrm{e}^{iab\left(\int x^{k-1}\cos(\lambda x)^{m}dx\right)}+\mathrm{e}^{-iab\left(\int x^{k-1}\cos(\lambda x)^{m}dx\right)}} \tag{1}$$

Verification of solutions

$$y = -\frac{ib\,x^{k}\left(c_{3}\mathrm{e}^{iab\left(\int x^{k-1}\cos(\lambda x)^{m}dx\right)}-\mathrm{e}^{-iab\left(\int x^{k-1}\cos(\lambda x)^{m}dx\right)}\right)}{c_{3}\mathrm{e}^{iab\left(\int x^{k-1}\cos(\lambda x)^{m}dx\right)}+\mathrm{e}^{-iab\left(\int x^{k-1}\cos(\lambda x)^{m}dx\right)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 31

```
dsolve(x*diff(y(x),x)=a*cos(lambda*x)^m*y(x)^2+k*y(x)+a*b^2*x^(2*k)*cos(lambda*x)^m,y(x), si
```

$$y(x) = -\tan\left(-ab\left(\int x^{-1+k}\cos(x\lambda)^{m}\,dx\right)+c_{1}\right)b\,x^{k}$$

✓ Solution by Mathematica

Time used: 1.628 (sec). Leaf size: 50

```
DSolve[x*y'[x]==a*Cos[\[Lambda]*x]^m*y[x]^2+k*y[x]+a*b^2*x^(2*k)*Cos[\[Lambda]*x]^m,y[x],x,I
```

$$y(x) \to \sqrt{b^{2}}x^{k}\tan\left(\sqrt{b^{2}}\int_{1}^{x}a\cos^{m}(\lambda K[1])K[1]^{k-1}dK[1]+c_{1}\right)$$

10.12 problem 25

Internal problem ID [10523]
Internal file name [OUTPUT/9470_Monday_June_06_2022_02_45_57_PM_6452655/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-2. Equations with cosine.
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$\boxed{(\cos(\lambda x) a + b) y' - y^2 - c\cos(x\mu) y = -d^2 + cd\cos(x\mu)}$$

10.12.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2 + c\cos(x\mu) y - d^2 + cd\cos(x\mu)}{\cos(\lambda x) a + b} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{cd\cos(x\mu)}{\cos(\lambda x) a + b} + \frac{c\cos(x\mu) y}{\cos(\lambda x) a + b} - \frac{d^2}{\cos(\lambda x) a + b} + \frac{y^2}{\cos(\lambda x) a + b}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{-d^2+cd\cos(x\mu)}{\cos(\lambda x)a+b}$, $f_1(x) = \frac{c\cos(x\mu)}{\cos(\lambda x)a+b}$ and $f_2(x) = \frac{1}{\cos(\lambda x)a+b}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{\cos(\lambda x)a+b}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{a\lambda\sin(\lambda x)}{(\cos(\lambda x)\,a+b)^2} \\ f_1f_2 &= \frac{c\cos(x\mu)}{(\cos(\lambda x)\,a+b)^2} \\ f_2^2f_0 &= \frac{-d^2+cd\cos(x\mu)}{(\cos(\lambda x)\,a+b)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{\cos(\lambda x)\,a+b} - \left(\frac{a\lambda\sin(\lambda x)}{(\cos(\lambda x)\,a+b)^2} + \frac{c\cos(x\mu)}{(\cos(\lambda x)\,a+b)^2}\right)u'(x) + \frac{(-d^2+cd\cos(x\mu))\,u(x)}{(\cos(\lambda x)\,a+b)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 268

```
dsolve((a*cos(lambda*x)+b)*diff(y(x),x)=y(x)^2+c*cos(mu*x)*y(x)-d^2+c*d*cos(mu*x),y(x), sing
```

$$y(x) = \frac{-d\left(\int \frac{e^{\frac{c\left(\int \frac{\cos(x\mu)}{a\cos(x\lambda)+b}dx\right)\sqrt{a^2-b^2}\,\lambda-4d\,\operatorname{arctanh}\left(\frac{(a-b)\tan\left(\frac{x\lambda}{2}\right)}{\sqrt{a^2-b^2}}\right)}{\sqrt{a^2-b^2}\,\lambda}}}{a\cos(x\lambda)+b}dx\right)+dc_1-e^{\frac{c\left(\int \frac{\cos(x\mu)}{a\cos(x\lambda)+b}dx\right)\sqrt{a^2-b^2}\,\lambda-4d\,\operatorname{arctanh}\left(\frac{(a-b)\tan\left(\frac{x}{}\right.}{\sqrt{a^2-b^2}}\right.}{\sqrt{a^2-b^2}\,\lambda}}}{\int \frac{e^{\frac{c\left(\int \frac{\cos(x\mu)}{a\cos(x\lambda)+b}dx\right)\sqrt{a^2-b^2}\,\lambda-4d\,\operatorname{arctanh}\left(\frac{(a-b)\tan\left(\frac{x\lambda}{2}\right)}{\sqrt{a^2-b^2}}\right)}{\sqrt{a^2-b^2}\,\lambda}}}{a\cos(x\lambda)+b}dx-c_1}$$

✓ Solution by Mathematica

Time used: 12.31 (sec). Leaf size: 289

```
DSolve[(a*Cos[\[Lambda]*x]+b)*y'[x]==y[x]^2+c*Cos[\[Mu]*x]*y[x]-d^2+c*d*Cos[\[Mu]*x],y[x],x,
```

$$\text{Solve}\left[\int_1^x \frac{\exp\left(-\int_1^{K[2]}\frac{2d-c\cos(\mu K[1])}{b+a\cos(\lambda K[1])}dK[1]\right)(-d+c\cos(\mu K[2])+y(x))}{c\mu(b+a\cos(\lambda K[2]))(d+y(x))}dK[2]\right.$$
$$+\int_1^{y(x)}\left(-\int_1^x\left(\frac{\exp\left(-\int_1^{K[2]}\frac{2d-c\cos(\mu K[1])}{b+a\cos(\lambda K[1])}dK[1]\right)}{c\mu(b+a\cos(\lambda K[2]))(d+K[3])}-\frac{\exp\left(-\int_1^{K[2]}\frac{2d-c\cos(\mu K[1])}{b+a\cos(\lambda K[1])}dK[1]\right)(-d+c\cos(\mu K[2]}{c\mu(b+a\cos(\lambda K[2]))(d+K[3])^2}\right.\right.$$
$$\left.\left.-\frac{\exp\left(-\int_1^x\frac{2d-c\cos(\mu K[1])}{b+a\cos(\lambda K[1])}dK[1]\right)}{c\mu(d+K[3])^2}\right)dK[3]=c_1,y(x)\right]$$

10.13 problem 26

Internal problem ID [10524]
Internal file name [`OUTPUT/9471_Monday_June_06_2022_02_47_08_PM_56852219/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-2. Equations with cosine.
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{(\cos(\lambda x)\, a+b)\left(y'-y^2\right)=a\,\lambda^2\cos(\lambda x)}$$

10.13.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\,y^2\cos(\lambda x)+a\,\lambda^2\cos(\lambda x)+y^2b}{\cos(\lambda x)\,a+b} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=\frac{a\,\lambda^2\cos(\lambda x)}{\cos(\lambda x)\,a+b}+\frac{a\,y^2\cos(\lambda x)}{\cos(\lambda x)\,a+b}+\frac{y^2b}{\cos(\lambda x)\,a+b}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=\frac{a\,\lambda^2\cos(\lambda x)}{\cos(\lambda x)a+b}$, $f_1(x)=0$ and $f_2(x)=1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= \frac{a\,\lambda^2 \cos(\lambda x)}{\cos(\lambda x)\, a + b} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{a\,\lambda^2 \cos(\lambda x)\, u(x)}{\cos(\lambda x)\, a + b} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = -2c_1 b\left(\cos\left(\frac{\lambda x}{2}\right)^2 a - \frac{a}{2} + \frac{b}{2}\right) \operatorname{arctanh}\left(\frac{\tan\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right) \\ + \sin\left(\frac{\lambda x}{2}\right) \cos\left(\frac{\lambda x}{2}\right) \sqrt{a^2-b^2}\, c_1 a + 2c_2\left(\cos\left(\frac{\lambda x}{2}\right)^2 a - \frac{a}{2} + \frac{b}{2}\right) \end{aligned}$$

The above shows that

$$u'(x) = \frac{\left(2\sqrt{a^2-b^2}\, \operatorname{arctanh}\left(\frac{\tan\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right) c_1 ab \cos\left(\frac{\lambda x}{2}\right) \sin\left(\frac{\lambda x}{2}\right) - 2\sqrt{a^2-b^2}\, c_2 a \cos\left(\frac{\lambda x}{2}\right) \sin\left(\frac{\lambda x}{2}\right) + c_1(a-b)\right.}{\sqrt{a^2-b^2}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(2\sqrt{a^2-b^2}\, \operatorname{arctanh}\left(\frac{\tan\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right) c_1 ab \cos\left(\frac{\lambda x}{2}\right) \sin\left(\frac{\lambda x}{2}\right) - 2\sqrt{a^2-b^2}\, c_2 a \cos\left(\frac{\lambda x}{2}\right) \sin\left(\frac{\lambda x}{2}\right) + c_1(a-\right.}{\sqrt{a^2-b^2}\, \left(-2c_1 b\left(\cos\left(\frac{\lambda x}{2}\right)^2 a - \frac{a}{2} + \frac{b}{2}\right) \operatorname{arctanh}\left(\frac{\tan\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right) + \sin\left(\frac{\lambda x}{2}\right) \cos\left(\frac{\lambda x}{2}\right) \sqrt{a^2-b^2}\, c_1 a\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\lambda\Bigg(2\sqrt{a^2-b^2}\,\operatorname{arctanh}\left(\frac{\tan\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right)c_3 ab\cos\left(\frac{\lambda x}{2}\right)\sin\left(\frac{\lambda x}{2}\right)-2\sqrt{a^2-b^2}\,\sin\left(\frac{\lambda x}{2}\right)\cos\left(\frac{\lambda x}{2}\right)a+c_3(a-b)}{\sqrt{a^2-b^2}\,\Bigg(2c_3 b\left(\cos\left(\frac{\lambda x}{2}\right)^2 a-\frac{a}{2}+\frac{b}{2}\right)\operatorname{arctanh}\left(\frac{\tan\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right)-\sin\left(\frac{\lambda x}{2}\right)\cos\left(\frac{\lambda x}{2}\right)\sqrt{a^2-b^2}\,c_3 a -}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{\lambda\Bigg(2\sqrt{a^2-b^2}\,\operatorname{arctanh}\left(\frac{\tan\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right)c_3 ab\cos\left(\frac{\lambda x}{2}\right)\sin\left(\frac{\lambda x}{2}\right)-2\sqrt{a^2-b^2}\,\sin\left(\frac{\lambda x}{2}\right)\cos\left(\frac{\lambda x}{2}\right)a+c_3(a-b)}{\sqrt{a^2-b^2}\,\Bigg(2c_3 b\left(\cos\left(\frac{\lambda x}{2}\right)^2 a-\frac{a}{2}+\frac{b}{2}\right)\operatorname{arctanh}\left(\frac{\tan\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right)-\sin\left(\frac{\lambda x}{2}\right)\cos\left(\frac{\lambda x}{2}\right)\sqrt{a^2-b^2}\,c_3 a -} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{\lambda\Bigg(2\sqrt{a^2-b^2}\,\operatorname{arctanh}\left(\frac{\tan\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right)c_3 ab\cos\left(\frac{\lambda x}{2}\right)\sin\left(\frac{\lambda x}{2}\right)-2\sqrt{a^2-b^2}\,\sin\left(\frac{\lambda x}{2}\right)\cos\left(\frac{\lambda x}{2}\right)a+c_3(a-b)}{\sqrt{a^2-b^2}\,\Bigg(2c_3 b\left(\cos\left(\frac{\lambda x}{2}\right)^2 a-\frac{a}{2}+\frac{b}{2}\right)\operatorname{arctanh}\left(\frac{\tan\left(\frac{\lambda x}{2}\right)(a-b)}{\sqrt{a^2-b^2}}\right)-\sin\left(\frac{\lambda x}{2}\right)\cos\left(\frac{\lambda x}{2}\right)\sqrt{a^2-b^2}\,c_3 a -}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -a*lambda^2*cos(lambda*x)*y(x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      <- linear_1 successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 204

```
dsolve((a*cos(lambda*x)+b)*(diff(y(x),x)-y(x)^2)-a*lambda^2*cos(lambda*x)=0,y(x), singsol=al
```

$$y(x) = \frac{\left(2\,\operatorname{arctanh}\left(\frac{(a-b)\tan\left(\frac{x\lambda}{2}\right)}{\sqrt{a^2-b^2}}\right)\sqrt{a^2-b^2}\,ab\cos\left(\frac{x\lambda}{2}\right)\sin\left(\frac{x\lambda}{2}\right)-2\sqrt{a^2-b^2}\,c_1a\cos\left(\frac{x\lambda}{2}\right)\sin\left(\frac{x\lambda}{2}\right)+\left(a\cos\left(\frac{x\lambda}{2}\right.\right.\right.}{\sqrt{a^2-b^2}\left(2\left(a\cos\left(\frac{x\lambda}{2}\right)^2-\frac{a}{2}+\frac{b}{2}\right)b\,\operatorname{arctanh}\left(\frac{(a-b)\tan\left(\frac{x\lambda}{2}\right)}{\sqrt{a^2-b^2}}\right)-\sqrt{a^2-b^2}\,a\cos\left(\frac{x\lambda}{2}\right)\sin\left(\frac{x\lambda}{2}\right)-2c_1\right.}$$

✓ Solution by Mathematica

Time used: 7.903 (sec). Leaf size: 202

```
DSolve[(a*Cos[\[Lambda]*x]+b)*(y'[x]-y[x]^2)-a*\[Lambda]^2*Cos[\[Lambda]*x]==0,y[x],x,Includ
```

$$y(x) \to -\frac{\lambda\left(-2ab\sin(\lambda x)\text{arctanh}\left(\frac{(b-a)\tan\left(\frac{\lambda x}{2}\right)}{\sqrt{a^2-b^2}}\right)+\sqrt{a^2-b^2}(-ac_1\lambda(a^2-b^2)\sin(\lambda x)+a\cos(\lambda x)-b)\right)}{2b(a\cos(\lambda x)+b)\text{arctanh}\left(\frac{(b-a)\tan\left(\frac{\lambda x}{2}\right)}{\sqrt{a^2-b^2}}\right)+\sqrt{a^2-b^2}\left(bc_1\lambda\left(a^2-b^2\right)+ac_1\lambda\left(a^2-b^2\right)\cos(\lambda x)+a\sin(\right.}$$

$$y(x) \to \frac{a\lambda\sin(\lambda x)}{a\cos(\lambda x)+b}$$

11 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-3. Equations with tangent.

11.1 problem 27

Internal problem ID [10525]
Internal file name [`OUTPUT/9472_Monday_June_06_2022_02_47_10_PM_81937076/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-3. Equations with tangent.
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = \lambda a + a(\lambda - a)\tan(\lambda x)^2}$$

11.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -a^2\tan(\lambda x)^2 + a\tan(\lambda x)^2\lambda + \lambda a + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a^2\tan(\lambda x)^2 + a\tan(\lambda x)^2\lambda + \lambda a + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2\tan(\lambda x)^2 + a\tan(\lambda x)^2\lambda + \lambda a$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -a^2 \tan(\lambda x)^2 + a \tan(\lambda x)^2 \lambda + \lambda a \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-a^2 \tan(\lambda x)^2 + a \tan(\lambda x)^2 \lambda + \lambda a\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sqrt{\cos(\lambda x)}\left(c_1 \operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) + c_2 \operatorname{LegendreQ}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right)\right)$$

The above shows that

$$u'(x) = \frac{\sin(\lambda x) \operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) c_1 a + \sin(\lambda x) \operatorname{LegendreQ}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) c_2 a - \lambda(c_1 \operatorname{Leg}}{\sqrt{\cos(\lambda x)}}$$

Using the above in (1) gives the solution

$$y = -\frac{\sin(\lambda x) \operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) c_1 a + \sin(\lambda x) \operatorname{LegendreQ}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) c_2 a - \lambda(c_1 \operatorname{L}}{\cos(\lambda x)\left(c_1 \operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) + c_2 \operatorname{Legend}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\sec(\lambda x)\left(\sin(\lambda x) \operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) c_3 a + \sin(\lambda x) \operatorname{LegendreQ}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) a \right.}{c_3 \operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) + \operatorname{Legendre}}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sec(\lambda x)\left(\sin(\lambda x)\operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right)c_3 a + \sin(\lambda x)\operatorname{LegendreQ}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right)a -}{c_3\operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) + \operatorname{Legendre}} \tag{1}$$

Verification of solutions

$$y = -\frac{\sec(\lambda x)\left(\sin(\lambda x)\operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right)c_3 a + \sin(\lambda x)\operatorname{LegendreQ}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right)a -}{c_3\operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) + \operatorname{Legendre}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a^2*tan(lambda*x)^2-a*tan(lam
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               -> Whittaker
                  -> hyper3: Equivalence to 1F1 under a power @ Moebius
               -> hypergeometric
                  -> heuristic approach
                  <- heuristic approach successful
               <- hypergeometric successful
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 205

```
dsolve(diff(y(x),x)=y(x)^2+a*lambda+a*(lambda-a)*tan(lambda*x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{\left(\sin(x\lambda)\,\mathrm{LegendreP}\left(\frac{2a-\lambda}{2\lambda},\frac{2a-\lambda}{2\lambda},\sin(x\lambda)\right)a+\mathrm{LegendreQ}\left(\frac{2a-\lambda}{2\lambda},\frac{2a-\lambda}{2\lambda},\sin(x\lambda)\right)c_1 a\sin(x\lambda)-\lambda\left(\mathrm{Lege}\right.\right.}{\mathrm{LegendreQ}\left(\frac{2a-\lambda}{2\lambda},\frac{2a-\lambda}{2\lambda},\sin(x\lambda)\right)c_1+\mathrm{Legendre}}$$

✓ Solution by Mathematica

Time used: 2.982 (sec). Leaf size: 259

```
DSolve[y'[x]==y[x]^2+a*\[Lambda]+a*(\[Lambda]-a)*Tan[\[Lambda]*x]^2,y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{2\left(ac_1\sin^2(\lambda x)\,\mathrm{Hypergeometric2F1}\left(\frac{1}{2},\frac{1}{2}-\frac{a}{\lambda},\frac{3}{2}-\frac{a}{\lambda},\cos^2(x\lambda)\right)+(2a-\lambda)\sqrt{\sin^2(\lambda x)}\left(a\sin(\lambda x)\cos^{\frac{2a}{\lambda}-1}\right.\right.}{2(2a-\lambda)\sqrt{\sin^2(\lambda x)}\cos^{\frac{2a}{\lambda}}(\lambda x)+c_1\sin(2\lambda x)\,\mathrm{Hypergeometric2F1}\left(\frac{1}{2},\frac{1}{2}-\frac{a}{\lambda},\frac{3}{2}-\frac{a}{\lambda},\cos^2(x\lambda)\right)}$$

$$y(x) \to \frac{\tan(\lambda x)\left(a\sqrt{\sin^2(\lambda x)}\,\mathrm{Hypergeometric2F1}\left(\frac{1}{2},\frac{1}{2}-\frac{a}{\lambda},\frac{3}{2}-\frac{a}{\lambda},\cos^2(x\lambda)\right)-2a+\lambda\right)}{\sqrt{\sin^2(\lambda x)}\,\mathrm{Hypergeometric2F1}\left(\frac{1}{2},\frac{1}{2}-\frac{a}{\lambda},\frac{3}{2}-\frac{a}{\lambda},\cos^2(x\lambda)\right)}$$

11.2 problem 28

Internal problem ID [10526]
Internal file name [`OUTPUT/9473_Monday_June_06_2022_02_47_13_PM_89469444/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-3. Equations with tangent.
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = 3\lambda a + \lambda^2 + a(\lambda - a)\tan(\lambda x)^2}$$

11.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -a^2 \tan(\lambda x)^2 + a\tan(\lambda x)^2 \lambda + 3\lambda a + \lambda^2 + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a^2 \tan(\lambda x)^2 + a\tan(\lambda x)^2 \lambda + 3\lambda a + \lambda^2 + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2 \tan(\lambda x)^2 + a\tan(\lambda x)^2 \lambda + 3\lambda a + \lambda^2$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -a^2 \tan(\lambda x)^2 + a \tan(\lambda x)^2 \lambda + 3\lambda a + \lambda^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-a^2 \tan(\lambda x)^2 + a \tan(\lambda x)^2 \lambda + 3\lambda a + \lambda^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sqrt{\cos(\lambda x)} \left(c_1 \operatorname{LegendreP}\left(\frac{2a+\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) + c_2 \operatorname{LegendreQ}\left(\frac{2a+\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right)\right)$$

The above shows that

$$u'(x) = \frac{-2 \operatorname{LegendreP}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) c_1 \lambda - 2 \operatorname{LegendreQ}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) c_2 \lambda + \sin(\lambda x) (c_1 \operatorname{Legen}}{\sqrt{\cos(\lambda x)}}$$

Using the above in (1) gives the solution

$$y = -\frac{-2 \operatorname{LegendreP}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) c_1 \lambda - 2 \operatorname{LegendreQ}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) c_2 \lambda + \sin(\lambda x) (c_1 \operatorname{Leg}}{\cos(\lambda x) \left(c_1 \operatorname{LegendreP}\left(\frac{2a+\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) + c_2 \operatorname{Lege}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\sec(\lambda x) \left(-2 \operatorname{LegendreP}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) c_3 \lambda - 2 \operatorname{LegendreQ}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) \lambda + \sin(\lambda x)\right.}{c_3 \operatorname{LegendreP}\left(\frac{2a+\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) + \operatorname{Legend}}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sec(\lambda x)\left(-2\,\mathrm{LegendreP}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right)c_3\lambda - 2\,\mathrm{LegendreQ}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right)\lambda + \sin(\lambda x)\right.}{c_3\,\mathrm{LegendreP}\left(\frac{2a+\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) + \mathrm{Legend}} \tag{1}$$

Verification of solutions

$$y = -\frac{\sec(\lambda x)\left(-2\,\mathrm{LegendreP}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right)c_3\lambda - 2\,\mathrm{LegendreQ}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right)\lambda + \sin(\lambda x)\right.}{c_3\,\mathrm{LegendreP}\left(\frac{2a+\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \sin(\lambda x)\right) + \mathrm{Legend}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a^2*tan(lambda*x)^2-a*tan(lam
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               -> Whittaker
                  -> hyper3: Equivalence to 1F1 under a power @ Moebius
               -> hypergeometric
                  -> heuristic approach
                  <- heuristic approach successful
               <- hypergeometric successful
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 200

```
dsolve(diff(y(x),x)=y(x)^2+lambda^2+3*a*lambda+a*(lambda-a)*tan(lambda*x)^2,y(x), singsol=al
```

$y(x) =$

$$-\frac{\left(-2\,\mathrm{LegendreP}\left(\frac{2a+3\lambda}{2\lambda},\frac{2a-\lambda}{2\lambda},\sin(x\lambda)\right)\lambda-2\,\mathrm{LegendreQ}\left(\frac{2a+3\lambda}{2\lambda},\frac{2a-\lambda}{2\lambda},\sin(x\lambda)\right)c_1\lambda+\sin(x\lambda)\left(\mathrm{Legend}\right.\right.}{\mathrm{LegendreQ}\left(\frac{2a+\lambda}{2\lambda},\frac{2a-\lambda}{2\lambda},\sin(x\lambda)\right)c_1+\mathrm{Legend}}$$

✓ Solution by Mathematica

Time used: 76.241 (sec). Leaf size: 319

```
DSolve[y'[x]==y[x]^2+\[Lambda]^2+3*a*\[Lambda]+a*(\[Lambda]-a)*Tan[\[Lambda]*x]^2,y[x],x,Inc
```

$y(x) \to$

$$-\frac{\sin^{-\frac{a+\lambda}{\lambda}}(2\lambda x)e^{-\frac{a\operatorname{arctanh}(\cos(2\lambda x))}{\lambda}}\left(c_1\sin^{\frac{a}{\lambda}}(2\lambda x)((a+\lambda)\cos(2\lambda x)-a+\lambda)e^{\frac{a\operatorname{arctanh}(\cos(2\lambda x))}{\lambda}}\int_1^x e^{-\frac{(a-\lambda)\operatorname{arctanh}}{\lambda}}\right.}{1+c_1\int_1^x e^{-\frac{(a-\lambda)\operatorname{arctanh}(c}{\lambda}}}$$

$$y(x) \to \csc(2\lambda x)\left(-\frac{\sin^{-\frac{a}{\lambda}}(2\lambda x)e^{-\frac{(a-\lambda)\operatorname{arctanh}(\cos(2\lambda x))}{\lambda}}}{\int_1^x e^{-\frac{(a-\lambda)\operatorname{arctanh}(\cos(2\lambda K[1]))}{\lambda}}\sin^{-\frac{a+\lambda}{\lambda}}(2\lambda K[1])dK[1]}-(a+\lambda)\cos(2\lambda x)\right.$$
$$\left.+a-\lambda\right)$$

11.3 problem 29

Internal problem ID [10527]
Internal file name [`OUTPUT/9474_Monday_June_06_2022_02_47_15_PM_7792648/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-3. Equations with tangent.
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a y^2 - b \tan(x) y = c}$$

11.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= a y^2 + b \tan(x) y + c \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a y^2 + b \tan(x) y + c$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = c$, $f_1(x) = b \tan(x)$ and $f_2(x) = a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$f_2' = 0$$
$$f_1f_2 = b\tan(x)\,a$$
$$f_2^2f_0 = a^2c$$

Substituting the above terms back in equation (2) gives

$$au''(x) - b\tan(x)\,au'(x) + a^2cu(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \cos(x)^{-\frac{b}{2}+\frac{1}{2}}\left(c_1 \operatorname{LegendreP}\left(\frac{\sqrt{4ca+b^2}}{2} - \frac{1}{2}, \frac{b}{2} - \frac{1}{2}, \sin(x)\right) + c_2 \operatorname{LegendreQ}\left(\frac{\sqrt{4ca+b^2}}{2} - \frac{1}{2}, \frac{b}{2} - \frac{1}{2}, \sin(x)\right)\right)$$

The above shows that

$$u'(x) = -\frac{\left(c_1\left(\cos(x)\sin(x)\sqrt{4ca+b^2} + \cos(x)\sin(x) - \tan(x)(\sin(x)-1)(\sin(x)+1)(b-1)\right) \operatorname{LegendreP}}{}$$

Using the above in (1) gives the solution

$$y = \frac{c_1\left(\cos(x)\sin(x)\sqrt{4ca+b^2} + \cos(x)\sin(x) - \tan(x)(\sin(x)-1)(\sin(x)+1)(b-1)\right) \operatorname{LegendreP}\left(\frac{\sqrt{4}}{}\right.}{}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-\left(c_3 \operatorname{LegendreP}\left(\frac{\sqrt{4ca+b^2}}{2} - \frac{1}{2}, \frac{b}{2} - \frac{1}{2}, \sin(x)\right) + \operatorname{LegendreQ}\left(\frac{\sqrt{4ca+b^2}}{2} - \frac{1}{2}, \frac{b}{2} - \frac{1}{2}, \sin(x)\right)\right) \left(\sqrt{4ca+b^2}\right.}{2\left(c_3 \operatorname{LegendreP}\left(\frac{\sqrt{4ca+b^2}}{2} - \frac{1}{2},\right.\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-\left(c_3 \operatorname{LegendreP}\left(\frac{\sqrt{4ca+b^2}}{2} - \frac{1}{2}, \frac{b}{2} - \frac{1}{2}, \sin(x)\right) + \operatorname{LegendreQ}\left(\frac{\sqrt{4ca+b^2}}{2} - \frac{1}{2}, \frac{b}{2} - \frac{1}{2}, \sin(x)\right)\right) \left(\sqrt{4ca+b^2}\right.}{2\left(c_3 \operatorname{LegendreP}\left(\frac{\sqrt{4ca+b^2}}{2} - \frac{1}{2},\right.\right.} \tag{1}$$

Verification of solutions

$$y = \frac{\left(-\left(c_3 \operatorname{LegendreP}\left(\frac{\sqrt{4ca+b^2}}{2} - \frac{1}{2}, \frac{b}{2} - \frac{1}{2}, \sin(x)\right) + \operatorname{LegendreQ}\left(\frac{\sqrt{4ca+b^2}}{2} - \frac{1}{2}, \frac{b}{2} - \frac{1}{2}, \sin(x)\right)\right) \left(\sqrt{4ca+b^2}\right.}{2\left(c_3 \operatorname{LegendreP}\left(\frac{\sqrt{4ca+b^2}}{2} - \frac{1}{2},\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = b*tan(x)*(diff(y(x), x))-a*c*y
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            <- Legendre successful
         <- special function solution successful
         Change of variables used:
            [x = arcsin(t)]
         Linear ODE actually solved:
            a*c*u(t)+(-b*t-t)*diff(u(t),t)+(-t^2+1)*diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 187

```
dsolve(diff(y(x),x)=a*y(x)^2+b*tan(x)*y(x)+c,y(x), singsol=all)
```

$$y(x) = \frac{\sec(x)\left(-\left(\mathrm{LegendreQ}\left(\frac{\sqrt{4ac+b^2}}{2}-\frac{1}{2},-\frac{1}{2}+\frac{b}{2},\sin(x)\right)c_1+\mathrm{LegendreP}\left(\frac{\sqrt{4ac+b^2}}{2}-\frac{1}{2},-\frac{1}{2}+\frac{b}{2},\sin(x)\right)\right)(\ldots}{2\left(\mathrm{LegendreQ}\left(\frac{\sqrt{4ac+b^2}}{2}-\frac{1}{2},\ldots\right.\right.}$$

✓ Solution by Mathematica

Time used: 2.211 (sec). Leaf size: 608

```
DSolve[y'[x]==a*y[x]^2+b*Tan[x]*y[x]+c,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sin(x)\left((-b^3+3b^2+b-3)\,\mathrm{Hypergeometric2F1}\left(\frac{1}{4}\left(-b-\sqrt{b^2+4ac}+2\right),\frac{1}{4}\left(-b+\sqrt{b^2+4ac}+2\right),\frac{3-b}{2}\ldots\right.\right.}{a(b-3)(b+1)\left(\cos(x)\,\mathrm{Hyperg}\ldots\right.}$$

$$y(x) \to -\frac{c\sin(x)\cos(x)\,\mathrm{Hypergeometric2F1}\left(\frac{1}{4}\left(b-\sqrt{b^2+4ac}+4\right),\frac{1}{4}\left(b+\sqrt{b^2+4ac}+4\right),\frac{b+3}{2},\cos^2(x)\right)}{(b+1)\,\mathrm{Hypergeometric2F1}\left(\frac{1}{4}\left(b-\sqrt{b^2+4ac}\right),\frac{1}{4}\left(b+\sqrt{b^2+4ac}\right),\frac{b+1}{2},\cos^2(x)\right)}$$

$$y(x) \to -\frac{c\sin(x)\cos(x)\,\mathrm{Hypergeometric2F1}\left(\frac{1}{4}\left(b-\sqrt{b^2+4ac}+4\right),\frac{1}{4}\left(b+\sqrt{b^2+4ac}+4\right),\frac{b+3}{2},\cos^2(x)\right)}{(b+1)\,\mathrm{Hypergeometric2F1}\left(\frac{1}{4}\left(b-\sqrt{b^2+4ac}\right),\frac{1}{4}\left(b+\sqrt{b^2+4ac}\right),\frac{b+1}{2},\cos^2(x)\right)}$$

11.4 problem 30

Internal problem ID [10528]
Internal file name [`OUTPUT/9475_Monday_June_06_2022_02_47_17_PM_43109865/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-3. Equations with tangent.
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$\boxed{y' - ay^2 - 2ab\tan(x)\,y = b(ab-1)\tan(x)^2}$$

11.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \tan(x)^2 a\,b^2 + 2ab\tan(x)\,y - b\tan(x)^2 + a\,y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \tan(x)^2 a\,b^2 + 2ab\tan(x)\,y - b\tan(x)^2 + a\,y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \tan(x)^2 a\,b^2 - b\tan(x)^2$, $f_1(x) = 2b\tan(x)\,a$ and $f_2(x) = a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 2b \tan(x) a^2 \\ f_2^2 f_0 &= a^2 \left(\tan(x)^2 a b^2 - b \tan(x)^2\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$a u''(x) - 2b \tan(x) a^2 u'(x) + a^2 \left(\tan(x)^2 a b^2 - b \tan(x)^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \cos(x)^{-ab} \left(c_1 \sinh\left(\sqrt{-ab}\, x\right) + c_2 \cosh\left(\sqrt{-ab}\, x\right)\right)$$

The above shows that

$$\begin{aligned} u'(x) = \Big(&\left(a c_2 b \tan(x) + c_1 \sqrt{-ab}\right) \cosh\left(\sqrt{-ab}\, x\right) \\ &+ \sinh\left(\sqrt{-ab}\, x\right) \left(a b c_1 \tan(x) + c_2 \sqrt{-ab}\right)\Big) \cos(x)^{-ab} \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(a c_2 b \tan(x) + c_1 \sqrt{-ab}\right) \cosh\left(\sqrt{-ab}\, x\right) + \sinh\left(\sqrt{-ab}\, x\right) \left(a b c_1 \tan(x) + c_2 \sqrt{-ab}\right)}{a \left(c_1 \sinh\left(\sqrt{-ab}\, x\right) + c_2 \cosh\left(\sqrt{-ab}\, x\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-a b c_3 \tan(x) - \sqrt{-ab}\right) \sinh\left(\sqrt{-ab}\, x\right) - \left(b \tan(x) a + c_3 \sqrt{-ab}\right) \cosh\left(\sqrt{-ab}\, x\right)}{\left(c_3 \sinh\left(\sqrt{-ab}\, x\right) + \cosh\left(\sqrt{-ab}\, x\right)\right) a}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-a b c_3 \tan(x) - \sqrt{-ab}\right) \sinh\left(\sqrt{-ab}\, x\right) - \left(b \tan(x) a + c_3 \sqrt{-ab}\right) \cosh\left(\sqrt{-ab}\, x\right)}{\left(c_3 \sinh\left(\sqrt{-ab}\, x\right) + \cosh\left(\sqrt{-ab}\, x\right)\right) a} \tag{1}$$

Verification of solutions

$$y = \frac{\left(-abc_3 \tan \left(x\right)-\sqrt{-ab}\right) \sinh \left(\sqrt{-ab}\, x\right)-\left(b \tan \left(x\right) a+c_3 \sqrt{-ab}\right) \cosh \left(\sqrt{-ab}\, x\right)}{\left(c_3 \sinh \left(\sqrt{-ab}\, x\right)+\cosh \left(\sqrt{-ab}\, x\right)\right) a}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular polynomial solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 87

```
dsolve(diff(y(x),x)=a*y(x)^2+2*a*b*tan(x)*y(x)+b*(a*b-1)*tan(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{2c_1 ab-2i \tan \left(x\right) a^{\frac{3}{2}} b^{\frac{3}{2}} c_1+i \sqrt{a}\, \sqrt{b}\, \mathrm{e}^{-2i \sqrt{a}\, \sqrt{b}\, x}-\tan \left(x\right) \mathrm{e}^{-2i \sqrt{a}\, \sqrt{b}\, x} ab}{a \left(2i c_1 \sqrt{a}\, \sqrt{b}+\mathrm{e}^{-2i \sqrt{a}\, \sqrt{b}\, x}\right)}$$

✓ Solution by Mathematica

Time used: 12.833 (sec). Leaf size: 37

```
DSolve[y'[x]==a*y[x]^2+2*a*b*Tan[x]*y[x]+b*(a*b-1)*Tan[x]^2,y[x],x,IncludeSingularSolutions
```

$$y(x) \to -b\tan(x) + \sqrt{\frac{b}{a}}\tan\left(ax\sqrt{\frac{b}{a}} + c_1\right)$$

11.5 problem 31

Internal problem ID [10529]
Internal file name [OUTPUT/9476_Monday_June_06_2022_02_47_18_PM_75210480/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-3. Equations with tangent.
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - a\tan(\beta x)\, y = ab\tan(\beta x) - b^2}$$

11.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + a\tan(\beta x)\, y + ab\tan(\beta x) - b^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\tan(\beta x)\, y + ab\tan(\beta x) - b^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = ab\tan(\beta x) - b^2$, $f_1(x) = \tan(\beta x)\, a$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= \tan(\beta x)\,a \\ f_2^2 f_0 &= ab\tan(\beta x) - b^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \tan(\beta x)\,au'(x) + \left(ab\tan(\beta x) - b^2\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = {} & c_1(\tan(\beta x)+i)^{\frac{ib}{2\beta}}(\tan(\beta x)-i)^{-\frac{ib}{2\beta}} \\ & + c_2(\tan(\beta x)+i)^{\frac{-ib+a}{2\beta}}\operatorname{hypergeom}\left(\left[1,\frac{a}{\beta}\right],\left[\frac{-2ib+a+2\beta}{2\beta}\right],\frac{1}{2} \right. \\ & \left. - \frac{i\tan(\beta x)}{2}\right)(\tan(\beta x)-i)^{\frac{ib+a}{2\beta}} \end{aligned}$$

The above shows that

$$\begin{aligned} & u'(x) \\ & = \left(\frac{c_1b(\tan(\beta x)-i)^{-\frac{ib}{2\beta}}(\tan(\beta x)+i)^{\frac{ib}{2\beta}}}{2-2i\tan(\beta x)} - \frac{i(\tan(\beta x)-i)^{-\frac{ib+2\beta}{2\beta}}c_1b(\tan(\beta x)+i)^{\frac{ib}{2\beta}}}{2}\right. \\ & + \frac{(-ib+a)\,c_2(\tan(\beta x)-i)^{\frac{ib+a}{2\beta}}(\tan(\beta x)+i)^{\frac{-ib+a-2\beta}{2\beta}}\operatorname{hypergeom}\left(\left[1,\frac{a}{\beta}\right],\left[\frac{-2ib+a+2\beta}{2\beta}\right],\frac{1}{2}-\frac{i\tan(\beta x)}{2}\right)}{2} \\ & + \frac{c_2(\tan(\beta x)+i)^{\frac{-ib+a}{2\beta}}a\operatorname{hypergeom}\left(\left[2,\frac{a+\beta}{\beta}\right],\left[\frac{-2ib+a+4\beta}{2\beta}\right],\frac{1}{2}-\frac{i\tan(\beta x)}{2}\right)\beta(\tan(\beta x)-i)^{\frac{ib+a}{2\beta}}}{ia+2i\beta+2b} \\ & \left. + \frac{(\tan(\beta x)-i)^{\frac{ib+a-2\beta}{2\beta}}c_2(\tan(\beta x)+i)^{\frac{-ib+a}{2\beta}}\operatorname{hypergeom}\left(\left[1,\frac{a}{\beta}\right],\left[\frac{-2ib+a+2\beta}{2\beta}\right],\frac{1}{2}-\frac{i\tan(\beta x)}{2}\right)(ib+a)}{2}\right)(\\ & + \tan(\beta x)^2) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{c_1 b(\tan(\beta x)-i)^{-\frac{ib}{2\beta}}(\tan(\beta x)+i)^{\frac{ib}{2\beta}}}{2-2i\tan(\beta x)} - \frac{i(\tan(\beta x)-i)^{-\frac{ib+2\beta}{2\beta}} c_1 b(\tan(\beta x)+i)^{\frac{ib}{2\beta}}}{2} + \frac{(-ib+a)c_2(\tan(\beta x)-i)^{\frac{ib+a}{2\beta}}(\tan(\beta x)+i)^{\frac{-ib+a-2\beta}{2\beta}}}{2}}{c_1 (\tan(\beta x)+i}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(1+\tan(\beta x)^2)\left(2\beta(\tan(\beta x)-i)^{\frac{ib+a}{2\beta}} a(\tan(\beta x)+i)^{\frac{-ib+a}{2\beta}} \operatorname{hypergeom}\left(\left[2, \frac{a+\beta}{\beta}\right], \left[\frac{-2ib+a+4\beta}{2\beta}\right], \frac{1}{2} - \frac{i\tan(}{2}\right.\right.}{}$$

Summary

The solution(s) found are the following

$$y = \frac{(1+\tan(\beta x)^2)\left(2\beta(\tan(\beta x)-i)^{\frac{ib+a}{2\beta}} a(\tan(\beta x)+i)^{\frac{-ib+a}{2\beta}} \operatorname{hypergeom}\left(\left[2, \frac{a+\beta}{\beta}\right], \left[\frac{-2ib+a+4\beta}{2\beta}\right], \frac{1}{2} - \frac{i\tan(}{2}\right.\right.}{} \tag{1}$$

Verification of solutions

$$y = \frac{(1+\tan(\beta x)^2)\left(2\beta(\tan(\beta x)-i)^{\frac{ib+a}{2\beta}} a(\tan(\beta x)+i)^{\frac{-ib+a}{2\beta}} \operatorname{hypergeom}\left(\left[2, \frac{a+\beta}{\beta}\right], \left[\frac{-2ib+a+4\beta}{2\beta}\right], \frac{1}{2} - \frac{i\tan(}{2}\right.\right.}{}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 81

```
dsolve(diff(y(x),x)=y(x)^2+a*tan(beta*x)*y(x)+a*b*tan(beta*x)-b^2,y(x), singsol=all)
```

$$y(x) = \frac{-\left(\sec\left(x\beta\right)^2\right)^{\frac{a}{2\beta}} \mathrm{e}^{-2bx} - b\left(\int \left(\sec\left(x\beta\right)^2\right)^{\frac{a}{2\beta}} \mathrm{e}^{-2bx} dx - c_1\right)}{\int \left(\sec\left(x\beta\right)^2\right)^{\frac{a}{2\beta}} \mathrm{e}^{-2bx} dx - c_1}$$

✓ Solution by Mathematica

Time used: 25.611 (sec). Leaf size: 408

```
DSolve[y'[x]==y[x]^2+a*Tan[\[Beta]*x]*y[x]+a*b*Tan[\[Beta]*x]-b^2,y[x],x,IncludeSingularSolu
```

$$y(x) \rightarrow \frac{2^{-\frac{a}{\beta}} \cos^{-\frac{a}{\beta}}(\beta x) \left(ib(a+2ib+2\beta) \text{Hypergeometric2F1}\left(1, -\frac{a-2ib}{2\beta}, \frac{a+2ib+2\beta}{2\beta}, -e^{2ix\beta}\right) (\sin(2\beta x)\csc(\beta x))^{a/}\right.}{(a+2ib)\left(a\beta c_1 e^{2bx}(a+2ib+2\beta)\cos^{\frac{a}{\beta}}(\beta x) - ie^{2i\beta x}\,\text{Hyper}\right.}$$

$$y(x) \rightarrow -b$$

11.6 problem 32

Internal problem ID [10530]
Internal file name [`OUTPUT/9477_Monday_June_06_2022_02_47_20_PM_47973751/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-3. Equations with tangent.
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - ax\tan(bx)^m y = a\tan(bx)^m}$$

11.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + ax\tan(bx)^m y + a\tan(bx)^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + ax\tan(bx)^m y + a\tan(bx)^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\tan(bx)^m$, $f_1(x) = \tan(bx)^m ax$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \qquad (1) \end{aligned}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \tan(bx)^m\, ax \\ f_2^2 f_0 &= a\tan(bx)^m \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \tan(bx)^m\, ax u'(x) + a\tan(bx)^m\, u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = -\frac{x\left(c_2\left(\int \frac{\mathrm{e}^{\int (\tan(bx)^m ax - \cot(bx)b)dx}\sin(bx)}{x^2}dx\right) - c_1 b\right)}{b}$$

The above shows that

$$u'(x) = \frac{-c_2 \mathrm{e}^{\int (\tan(bx)^m ax - \cot(bx)b)dx}\sin(bx) - \left(\int \frac{\mathrm{e}^{\int (\tan(bx)^m ax - \cot(bx)b)dx}\sin(bx)}{x^2}dx\right) c_2 x + c_1 bx}{bx}$$

Using the above in (1) gives the solution

$$y = \frac{-c_2 \mathrm{e}^{\int (\tan(bx)^m ax - \cot(bx)b)dx}\sin(bx) - \left(\int \frac{\mathrm{e}^{\int (\tan(bx)^m ax - \cot(bx)b)dx}\sin(bx)}{x^2}dx\right) c_2 x + c_1 bx}{x^2\left(c_2\left(\int \frac{\mathrm{e}^{\int (\tan(bx)^m ax - \cot(bx)b)dx}\sin(bx)}{x^2}dx\right) - c_1 b\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\mathrm{e}^{\int (\tan(bx)^m ax - \cot(bx)b)dx}\sin(bx) - \left(\int \frac{\mathrm{e}^{\int (\tan(bx)^m ax - \cot(bx)b)dx}\sin(bx)}{x^2}dx\right) x + c_3 bx}{x^2\left(\int \frac{\mathrm{e}^{\int (\tan(bx)^m ax - \cot(bx)b)dx}\sin(bx)}{x^2}dx - bc_3\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\mathrm{e}^{\int (\tan(bx)^m ax-\cot(bx)b)dx} \sin(bx) - \left(\int \frac{\mathrm{e}^{\int (\tan(bx)^m ax-\cot(bx)b)dx} \sin(bx)}{x^2}dx\right) x + c_3 bx}{x^2 \left(\int \frac{\mathrm{e}^{\int (\tan(bx)^m ax-\cot(bx)b)dx} \sin(bx)}{x^2}dx - bc_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-\mathrm{e}^{\int (\tan(bx)^m ax-\cot(bx)b)dx} \sin(bx) - \left(\int \frac{\mathrm{e}^{\int (\tan(bx)^m ax-\cot(bx)b)dx} \sin(bx)}{x^2}dx\right) x + c_3 bx}{x^2 \left(\int \frac{\mathrm{e}^{\int (\tan(bx)^m ax-\cot(bx)b)dx} \sin(bx)}{x^2}dx - bc_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 85

```
dsolve(diff(y(x),x)=y(x)^2+a*x*tan(b*x)^m*y(x)+a*tan(b*x)^m,y(x), singsol=all)
```

$$y(x) = \frac{-\mathrm{e}^{\int \frac{a\tan(bx)^m x^2-2}{x}dx} x - \left(\int \mathrm{e}^{\int \frac{a\tan(bx)^m x^2-2}{x}dx} dx\right) + c_1}{\left(-c_1 + \int \mathrm{e}^{\int \frac{a\tan(bx)^m x^2-2}{x}dx} dx\right) x}$$

✓ Solution by Mathematica

Time used: 8.199 (sec). Leaf size: 126

```
DSolve[y'[x]==y[x]^2+a*x*Tan[b*x]^m*y[x]+a*Tan[b*x]^m,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to$$
$$-\frac{\exp\left(-\int_1^x -aK[1]\tan^m(bK[1])dK[1]\right)+x\int_1^x \frac{\exp\left(-\int_1^{K[2]} -aK[1]\tan^m(bK[1])dK[1]\right)}{K[2]^2}dK[2]+c_1x}{x^2\left(\int_1^x \frac{\exp\left(-\int_1^{K[2]} -aK[1]\tan^m(bK[1])dK[1]\right)}{K[2]^2}dK[2]+c_1\right)}$$

$$y(x) \to -\frac{1}{x}$$

11.7 problem 33

Internal problem ID [10531]
Internal file name [`OUTPUT/9478_Monday_June_06_2022_02_47_58_PM_52864866/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-3. Equations with tangent.
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + (k+1)\, x^k y^2 - a\, x^{k+1} \tan(x)^m\, y = -a \tan(x)^m}$$

11.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\, x^{k+1} \tan(x)^m\, y - x^k y^2 k - x^k y^2 - a \tan(x)^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\, x^k x \tan(x)^m\, y - x^k y^2 k - x^k y^2 - a \tan(x)^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a \tan(x)^m$, $f_1(x) = x^{k+1} \tan(x)^m\, a$ and $f_2(x) = -x^k k - x^k$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(-x^k k - x^k)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{k^2x^k}{x} - \frac{k\,x^k}{x} \\ f_1f_2 &= x^{k+1}\tan(x)^m\,a\left(-x^kk - x^k\right) \\ f_2^2f_0 &= -\left(-x^kk - x^k\right)^2 a\tan(x)^m \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\left(-x^kk - x^k\right)u''(x) - \left(-\frac{k^2x^k}{x} - \frac{k\,x^k}{x} + x^{k+1}\tan(x)^m\,a\left(-x^kk - x^k\right)\right)u'(x) - \left(-x^kk - x^k\right)^2 a\tan(x)^m u$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{k+1}\left(c_1 + c_2\left(\int x^{-2k-2}\mathrm{e}^{\int\left(x^{k+1}\tan(x)^m a+\frac{k}{x}\right)dx}dx\right)\right)$$

The above shows that

$$\begin{aligned} u'(x) = x^k\Bigg(& c_2x^{-2k-1}\mathrm{e}^{\int\left(x^{k+1}\tan(x)^m a+\frac{k}{x}\right)dx} \\ &+ \left(c_1 + c_2\left(\int x^{-2k-2}\mathrm{e}^{\int\left(x^{k+1}\tan(x)^m a+\frac{k}{x}\right)dx}dx\right)\right)(k+1)\Bigg) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{x^k\left(c_2x^{-2k-1}\mathrm{e}^{\int\left(x^{k+1}\tan(x)^m a+\frac{k}{x}\right)dx} + \left(c_1 + c_2\left(\int x^{-2k-2}\mathrm{e}^{\int\left(x^{k+1}\tan(x)^m a+\frac{k}{x}\right)dx}dx\right)\right)(k+1)\right)x^{-k-1}}{\left(-x^kk - x^k\right)\left(c_1 + c_2\left(\int x^{-2k-2}\mathrm{e}^{\int\left(x^{k+1}\tan(x)^m a+\frac{k}{x}\right)dx}dx\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x^{-k-1}\left(x^{-2k-1}\mathrm{e}^{\int\left(x^{k+1}\tan(x)^m a+\frac{k}{x}\right)dx} + \left(c_3 + \int x^{-2k-2}\mathrm{e}^{\int\left(x^{k+1}\tan(x)^m a+\frac{k}{x}\right)dx}dx\right)(k+1)\right)}{(k+1)\left(c_3 + \int \mathrm{e}^{\int\frac{a\,x^{k+2}\tan(x)^m+k}{x}dx}x^{-2k-2}dx\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{x^{-k-1}\left(x^{-2k-1}\mathrm{e}^{\int \left(x^{k+1}\tan(x)^{m} a+\frac{k}{x}\right)dx}+\left(c_3+\int x^{-2k-2}\mathrm{e}^{\int \left(x^{k+1}\tan(x)^{m} a+\frac{k}{x}\right)dx}dx\right)(k+1)\right)}{(k+1)\left(c_3+\int \mathrm{e}^{\int \frac{a\,x^{k+2}\tan(x)^{m}+k}{x}dx}x^{-2k-2}dx\right)} \tag{1}$$

Verification of solutions

$$y = \frac{x^{-k-1}\left(x^{-2k-1}\mathrm{e}^{\int \left(x^{k+1}\tan(x)^{m} a+\frac{k}{x}\right)dx}+\left(c_3+\int x^{-2k-2}\mathrm{e}^{\int \left(x^{k+1}\tan(x)^{m} a+\frac{k}{x}\right)dx}dx\right)(k+1)\right)}{(k+1)\left(c_3+\int \mathrm{e}^{\int \frac{a\,x^{k+2}\tan(x)^{m}+k}{x}dx}x^{-2k-2}dx\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x^(1+k)*tan(x)^m*a*x+k)*(diff
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 174

```
dsolve(diff(y(x),x)=-(k+1)*x^k*y(x)^2+a*x^(k+1)*tan(x)^m*y(x)-a*tan(x)^m,y(x), singsol=all)
```

$$y(x) = \frac{x^{-1-k}\left(x^{1+k}\mathrm{e}^{\int \frac{x^{1+k}\tan(x)^m a x-2k-2}{x}dx} + \left(\int x^k \mathrm{e}^{\int \frac{x^{1+k}\tan(x)^m a x-2k-2}{x}dx}dx\right)k + \int x^k \mathrm{e}^{\int \frac{x^{1+k}\tan(x)^m a x-2k-2}{x}dx}dx - c_1\right.}{\left(\int x^k \mathrm{e}^{\int \frac{a\,x^{k+2}\tan(x)^m-2k-2}{x}dx}dx\right)k + \int x^k \mathrm{e}^{\int \frac{a\,x^{k+2}\tan(x)^m-2k-2}{x}dx}dx - c_1}$$

✓ Solution by Mathematica

Time used: 19.083 (sec). Leaf size: 248

```
DSolve[y'[x]==-(k+1)*x^k*y[x]^2+a*x^(k+1)*Tan[x]^m*y[x]-a*Tan[x]^m,y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{x^{-k-1}\left(c_1 x \exp\left(\int_1^x -\frac{-a\tan^m(K[1])K[1]^{k+2}+k+2}{K[1]}dK[1]\right) + c_1(k+1)\int_1^x \exp\left(\int_1^{K[2]} -\frac{-a\tan^m(K[1])K[1]^{k+2}+k+2}{K[1]}d\right.\right.}{(k+1)\left(1+c_1\int_1^x \exp\left(\int_1^{K[2]} -\frac{-a\tan^m(K[1])K[1]^{k+2}+k+2}{K[1]}dK[1]\right)dK[2]\right)}$$

$$y(x) \to \frac{x^{-k}\left(\frac{\exp\left(\int_1^x -\frac{-a\tan^m(K[1])K[1]^{k+2}+k+2}{K[1]}dK[1]\right)}{\int_1^x \exp\left(\int_1^{K[2]} -\frac{-a\tan^m(K[1])K[1]^{k+2}+k+2}{K[1]}dK[1]\right)dK[2]} + \frac{k+1}{x}\right)}{k+1}$$

11.8 problem 34

Internal problem ID [10532]
Internal file name [`OUTPUT/9479_Monday_June_06_2022_02_48_09_PM_7980074/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-3. Equations with tangent.
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\tan(\lambda x)^n y^2 = -a\,b^2\tan(\lambda x)^{2+n} + b\lambda\tan(\lambda x)^2 + b\lambda}$$

11.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\tan(\lambda x)^n y^2 - a\,b^2\tan(\lambda x)^{2+n} + b\lambda\tan(\lambda x)^2 + b\lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\tan(\lambda x)^n y^2 - a\,b^2\tan(\lambda x)^2\tan(\lambda x)^n + b\lambda\tan(\lambda x)^2 + b\lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a\,b^2\tan(\lambda x)^{2+n} + b\lambda\tan(\lambda x)^2 + b\lambda$, $f_1(x) = 0$ and $f_2(x) = \tan(\lambda x)^n a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\tan(\lambda x)^n au} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{\tan(\lambda x)^n\, n\lambda(1+\tan(\lambda x)^2)\, a}{\tan(\lambda x)} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= \tan(\lambda x)^{2n}\, a^2(-a\, b^2 \tan(\lambda x)^{2+n} + b\lambda \tan(\lambda x)^2 + b\lambda)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\tan(\lambda x)^n\, a u''(x) - \frac{\tan(\lambda x)^n\, n\lambda(1+\tan(\lambda x)^2)\, a u'(x)}{\tan(\lambda x)} + \tan(\lambda x)^{2n}\, a^2(-a\, b^2 \tan(\lambda x)^{2+n} + b\lambda \tan(\lambda x)^2 \cdots$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{a\left(\int \frac{\left(-a\left(\int \tan(\lambda x)^{n+1} \mathrm{e}^{\int \cot(\lambda x)\left(2\tan(\lambda x)^{2+n} ab - \lambda \tan(\lambda x)^2 - \lambda\right) dx} dx\right) b + c_1 b + \mathrm{e}^{\int \cot(\lambda x)\left(2\tan(\lambda x)^{2+n} ab - \lambda\tan(\lambda x)^2 - \lambda\right) dx}\right) \tan(\lambda x)^{n+1}}{a\left(\int \tan(\lambda x)^{n+1} \mathrm{e}^{\int \cot(\lambda x)\left(2\tan(\lambda x)^{2+n} ab - \lambda\tan(\lambda x)^2 - \lambda\right) dx} dx\right) - c_1} dx\right)} c_2$$

The above shows that

$$u'(x) = \frac{\tan(\lambda x)^{n+1}\left(-a\left(\int \tan(\lambda x)^{n+1}\, \mathrm{e}^{\int \left(2ab\tan(\lambda x)^{n+1} - \sec(\lambda x)\csc(\lambda x)\lambda\right) dx} dx\right) b + c_1 b + \mathrm{e}^{\int \left(2ab\tan(\lambda x)^{n+1} - \sec(\lambda x)\csc(\cdots\right.} \cdots}{a\left(\int \tan(\lambda x)^{n+1}\, \mathrm{e}^{\int \cot(\lambda x)\left(2\tan \cdots\right.} \cdots\right.}$$

Using the above in (1) gives the solution

$$y = -\frac{\tan(\lambda x)^{n+1}\left(-a\left(\int \tan(\lambda x)^{n+1}\, \mathrm{e}^{\int \left(2ab\tan(\lambda x)^{n+1} - \sec(\lambda x)\csc(\lambda x)\lambda\right) dx} dx\right) b + c_1 b + \mathrm{e}^{\int \left(2ab\tan(\lambda x)^{n+1} - \sec(\lambda x)\, \mathrm{c}\cdots\right.} \cdots}{\cdots}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\tan(\lambda x)\left(a\left(\int \tan(\lambda x)^{n+1} \mathrm{e}^{\int \left(2ab\tan(\lambda x)^{n+1}-\sec(\lambda x)\csc(\lambda x)\lambda\right)dx}dx\right)b - bc_3 - \mathrm{e}^{\int \left(2ab\tan(\lambda x)^{n+1}-\sec(\lambda x)\csc(\lambda x)\lambda\right)d}}{a\left(\int \tan(\lambda x)^{n+1} \mathrm{e}^{\int \cot(\lambda x)\left(2\tan(\lambda x)^{2+n}ab-\lambda\tan(\lambda x)^2-\lambda\right)dx}dx\right) - c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{\tan(\lambda x)\left(a\left(\int \tan(\lambda x)^{n+1} \mathrm{e}^{\int \left(2ab\tan(\lambda x)^{n+1}-\sec(\lambda x)\csc(\lambda x)\lambda\right)dx}dx\right)b - bc_3 - \mathrm{e}^{\int \left(2ab\tan(\lambda x)^{n+1}-\sec(\lambda x)\csc(\lambda x)\lambda\right)d}}{a\left(\int \tan(\lambda x)^{n+1} \mathrm{e}^{\int \cot(\lambda x)\left(2\tan(\lambda x)^{2+n}ab-\lambda\tan(\lambda x)^2-\lambda\right)dx}dx\right) - c_3} \tag{1}$$

Verification of solutions

$$y = \frac{\tan(\lambda x)\left(a\left(\int \tan(\lambda x)^{n+1} \mathrm{e}^{\int \left(2ab\tan(\lambda x)^{n+1}-\sec(\lambda x)\csc(\lambda x)\lambda\right)dx}dx\right)b - bc_3 - \mathrm{e}^{\int \left(2ab\tan(\lambda x)^{n+1}-\sec(\lambda x)\csc(\lambda x)\lambda\right)d}}{a\left(\int \tan(\lambda x)^{n+1} \mathrm{e}^{\int \cot(\lambda x)\left(2\tan(\lambda x)^{2+n}ab-\lambda\tan(\lambda x)^2-\lambda\right)dx}dx\right) - c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = n*lambda*(1+tan(lambda*x)^2)*(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            -> Mathieu
               -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=a*tan(lambda*x)^n*y(x)^2-a*b^2*tan(lambda*x)^(n+2)+b*lambda*tan(lambda*x
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==a*Tan[\[Lambda]*x]^n*y[x]^2-a*b^2*Tan[\[Lambda]*x]^(n+2)+b*\[Lambda]*Tan[\[Lam
```

Not solved

11.9 problem 35

Internal problem ID [10533]
Internal file name [`OUTPUT/9480_Monday_June_06_2022_02_48_23_PM_47763726/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-3. Equations with tangent.
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

Unable to solve or complete the solution.

$$\boxed{y' - a\tan(\lambda x+\mu)^k (y - b\,x^n - c)^2 = b\,x^{n-1}n}$$

11.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x^{2n}\tan(\lambda x+\mu)^k a\,b^2 + 2x^n\tan(\lambda x+\mu)^k abc - 2x^n\tan(\lambda x+\mu)^k aby + \tan(\lambda x+\mu)^k a\,c^2 - 2\tan(\lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^{2n}\left(\frac{\tan(\mu)}{1-\tan(\mu)\tan(\lambda x)} + \frac{\tan(\lambda x)}{1-\tan(\mu)\tan(\lambda x)}\right)^k a\,b^2 + 2x^n\left(\frac{\tan(\mu)}{1-\tan(\mu)\tan(\lambda x)} + \frac{\tan(\lambda x)}{1-\tan(\mu)\tan}\right.$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^{2n}\tan(\lambda x+\mu)^k a\,b^2 + 2x^n\tan(\lambda x+\mu)^k abc + \tan(\lambda x+\mu)^k a\,c^2 + b\,x^{n-1}n$, $f_1(x) = -2\tan(\lambda x+\mu)^k a\,x^n b - 2\tan(\lambda x+\mu)^k ac$ and $f_2(x) = \tan(\lambda x+\mu)^k a$.

Let

$$y = \frac{-u'}{f_2 u} = \frac{-u'}{\tan(\lambda x+\mu)^k a u} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = \frac{\tan(\lambda x+\mu)^k k\lambda\left(1+\tan(\lambda x+\mu)^2\right) a}{\tan(\lambda x+\mu)}$$

$$f_1 f_2 = \left(-2\tan(\lambda x+\mu)^k a\, x^n b - 2\tan(\lambda x+\mu)^k ac\right)\tan(\lambda x+\mu)^k a$$

$$f_2^2 f_0 = \tan(\lambda x+\mu)^{2k} a^2 \left(x^{2n}\tan(\lambda x+\mu)^k a\, b^2 + 2x^n \tan(\lambda x+\mu)^k abc + \tan(\lambda x+\mu)^k a\, c^2 + b\, x^{n-1} n\right)$$

Substituting the above terms back in equation (2) gives

$$\tan(\lambda x+\mu)^k a u''(x) - \left(\frac{\tan(\lambda x+\mu)^k k\lambda\left(1+\tan(\lambda x+\mu)^2\right) a}{\tan(\lambda x+\mu)} + \left(-2\tan(\lambda x+\mu)^k a\, x^n b - 2\tan(\lambda x+\right.\right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (d) successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
dsolve(diff(y(x),x)=a*tan(lambda*x+mu)^k*(y(x)-b*x^n-c)^2+b*n*x^(n-1),y(x), singsol=all)
```

$$y(x) = b\,x^{n} + c + \frac{1}{c_1 - a\left(\int \left(-\frac{\tan(\mu)+\tan(x\lambda)}{\tan(\mu)\tan(x\lambda)-1}\right)^{k} dx\right)}$$

✓ Solution by Mathematica

Time used: 6.024 (sec). Leaf size: 75

```
DSolve[y'[x]==a*Tan[\[Lambda]*x+mu]^k*(y[x]-b*x^n-c)^2+b*n*x^(n-1),y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{1}{-\frac{a\tan^{k+1}(\mu+\lambda x)\,\text{Hypergeometric2F1}\left(1,\frac{k+1}{2},\frac{k+3}{2},-\tan^2(\mu+x\lambda)\right)}{(k+1)\lambda} + c_1} + bx^n + c$$

$$y(x) \to bx^n + c$$

11.10 problem 36

Internal problem ID [10534]
Internal file name [`OUTPUT/9481_Monday_June_06_2022_02_49_19_PM_25084523/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-3. Equations with tangent.
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - a\tan(\lambda x)^m y^2 - ky = a\,b^2x^{2k}\tan(\lambda x)^m}$$

11.10.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\tan(\lambda x)^m y^2 + ky + a\,b^2x^{2k}\tan(\lambda x)^m}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,b^2x^{2k}\tan(\lambda x)^m}{x} + \frac{a\tan(\lambda x)^m y^2}{x} + \frac{ky}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{a\,b^2x^{2k}\tan(\lambda x)^m}{x}$, $f_1(x) = \frac{k}{x}$ and $f_2(x) = \frac{a\tan(\lambda x)^m}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{a\tan(\lambda x)^m u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{a \tan(\lambda x)^m m\lambda(1+\tan(\lambda x)^2)}{\tan(\lambda x) x} - \frac{a \tan(\lambda x)^m}{x^2} \\ f_1 f_2 &= \frac{ka \tan(\lambda x)^m}{x^2} \\ f_2^2 f_0 &= \frac{a^3 \tan(\lambda x)^{3m} b^2 x^{2k}}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{a \tan(\lambda x)^m u''(x)}{x} - \left(\frac{a \tan(\lambda x)^m m\lambda(1+\tan(\lambda x)^2)}{\tan(\lambda x) x} - \frac{a \tan(\lambda x)^m}{x^2} + \frac{ka \tan(\lambda x)^m}{x^2}\right) u'(x) + \frac{a^3 \tan(\lambda x$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 e^{iab\left(\int x^{k-1} \tan(\lambda x)^m dx\right)} + c_2 e^{-iab\left(\int x^{k-1} \tan(\lambda x)^m dx\right)}$$

The above shows that

$$u'(x) = iab\, x^{k-1} \tan(\lambda x)^m \left(c_1 e^{iab\left(\int x^{k-1} \tan(\lambda x)^m dx\right)} - c_2 e^{-iab\left(\int x^{k-1} \tan(\lambda x)^m dx\right)}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{ib\, x^{k-1} \left(c_1 e^{iab\left(\int x^{k-1} \tan(\lambda x)^m dx\right)} - c_2 e^{-iab\left(\int x^{k-1} \tan(\lambda x)^m dx\right)}\right) x}{c_1 e^{iab\left(\int x^{k-1} \tan(\lambda x)^m dx\right)} + c_2 e^{-iab\left(\int x^{k-1} \tan(\lambda x)^m dx\right)}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{ib\, x^{k} \left(c_3 e^{iab\left(\int x^{k-1} \tan(\lambda x)^m dx\right)} - e^{-iab\left(\int x^{k-1} \tan(\lambda x)^m dx\right)}\right)}{c_3 e^{iab\left(\int x^{k-1} \tan(\lambda x)^m dx\right)} + e^{-iab\left(\int x^{k-1} \tan(\lambda x)^m dx\right)}}$$

Summary
The solution(s) found are the following

$$y = -\frac{ib\,x^k\left(c_3 \mathrm{e}^{iab\left(\int x^{k-1}\tan(\lambda x)^m dx\right)} - \mathrm{e}^{-iab\left(\int x^{k-1}\tan(\lambda x)^m dx\right)}\right)}{c_3 \mathrm{e}^{iab\left(\int x^{k-1}\tan(\lambda x)^m dx\right)} + \mathrm{e}^{-iab\left(\int x^{k-1}\tan(\lambda x)^m dx\right)}} \tag{1}$$

Verification of solutions

$$y = -\frac{ib\,x^k\left(c_3 \mathrm{e}^{iab\left(\int x^{k-1}\tan(\lambda x)^m dx\right)} - \mathrm{e}^{-iab\left(\int x^{k-1}\tan(\lambda x)^m dx\right)}\right)}{c_3 \mathrm{e}^{iab\left(\int x^{k-1}\tan(\lambda x)^m dx\right)} + \mathrm{e}^{-iab\left(\int x^{k-1}\tan(\lambda x)^m dx\right)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 31

```
dsolve(x*diff(y(x),x)=a*tan(lambda*x)^m*y(x)^2+k*y(x)+a*b^2*x^(2*k)*tan(lambda*x)^m,y(x), si
```

$$y(x) = -\tan\left(-ab\left(\int x^{-1+k}\tan(x\lambda)^m\,dx\right) + c_1\right) b\,x^k$$

✓ Solution by Mathematica
Time used: 1.817 (sec). Leaf size: 50

```
DSolve[x*y'[x]==a*Tan[\[Lambda]*x]^m*y[x]^2+k*y[x]+a*b^2*x^(2*k)*Tan[\[Lambda]*x]^m,y[x],x,I
```

$$y(x) \to \sqrt{b^2}x^k \tan\left(\sqrt{b^2}\int_1^x aK[1]^{k-1}\tan^m(\lambda K[1])dK[1] + c_1\right)$$

11.11 problem 37

Internal problem ID [10535]
Internal file name [`OUTPUT/9482_Monday_June_06_2022_02_49_22_PM_50388009/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-3. Equations with tangent.
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$\boxed{(a\tan(\lambda x)+b)\,y'-y^2-k\tan(x\mu)\,y=-d^2+kd\tan(x\mu)}$$

11.11.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= \frac{y^2+k\tan(x\mu)\,y-d^2+kd\tan(x\mu)}{a\tan(\lambda x)+b}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=\frac{kd\tan(x\mu)}{a\tan(\lambda x)+b}+\frac{k\tan(x\mu)\,y}{a\tan(\lambda x)+b}-\frac{d^2}{a\tan(\lambda x)+b}+\frac{y^2}{a\tan(\lambda x)+b}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x) = \frac{-d^2+kd\tan(x\mu)}{a\tan(\lambda x)+b}$, $f_1(x) = \frac{k\tan(x\mu)}{a\tan(\lambda x)+b}$ and $f_2(x) = \frac{1}{a\tan(\lambda x)+b}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{a\tan(\lambda x)+b}}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - \left(f_2' + f_1 f_2\right) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{a\lambda\left(1+\tan\left(\lambda x\right)^2\right)}{\left(a\tan\left(\lambda x\right)+b\right)^2} \\
f_1 f_2 &= \frac{k\tan\left(x\mu\right)}{\left(a\tan\left(\lambda x\right)+b\right)^2} \\
f_2^2 f_0 &= \frac{-d^2+kd\tan\left(x\mu\right)}{\left(a\tan\left(\lambda x\right)+b\right)^3}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{a\tan\left(\lambda x\right)+b} - \left(-\frac{a\lambda\left(1+\tan\left(\lambda x\right)^2\right)}{\left(a\tan\left(\lambda x\right)+b\right)^2} + \frac{k\tan\left(x\mu\right)}{\left(a\tan\left(\lambda x\right)+b\right)^2}\right) u'(x) + \frac{\left(-d^2+kd\tan\left(x\mu\right)\right) u(x)}{\left(a\tan\left(\lambda x\right)+b\right)^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 351

```
dsolve((a*tan(lambda*x)+b)*diff(y(x),x)=y(x)^2+k*tan(mu*x)*y(x)-d^2+k*d*tan(mu*x),y(x), sing
```

$$y(x) = \frac{-\left(\sec(x\lambda)^2\right)^{\frac{ad}{\lambda(a^2+b^2)}} (a\tan(x\lambda)+b)^{-\frac{2ad}{\lambda(a^2+b^2)}} \mathrm{e}^{\frac{\lambda k(a^2+b^2)\left(\int \frac{\tan(x\mu)}{a\tan(x\lambda)+b}dx\right)-2\arctan(\tan(x\lambda))bd}{\lambda(a^2+b^2)}} - d\left(\int (a\tan(x\lambda)+\right.}{\int (a\tan(x\lambda)+b)^{\frac{(-a^2-b^2)\lambda-2ad}{\lambda(a^2+b^2)}} \left(\sec(x\lambda)^2\right)^{\frac{ad}{\lambda(a^2+b^2)}} \mathrm{e}^{\frac{\lambda k(a^2+b^2)}{}}}$$

✓ Solution by Mathematica

Time used: 130.719 (sec). Leaf size: 800

```
DSolve[(a*Tan[\[Lambda]*x]+b)*y'[x]==y[x]^2+k*Tan[\[Mu]*x]*y[x]-d^2+k*d*Tan[\[Mu]*x],y[x],x,
```

$$\mathrm{Solve}\left[\int_1^x \frac{e^{-\int_1^{K[2]} \frac{\sec(\mu K[1])(2d\cos(\lambda K[1]-\mu K[1])+2d\cos(\lambda K[1]+\mu K[1])+k\sin(\lambda K[1]-\mu K[1])-k\sin(\lambda K[1]+\mu K[1]))}{2(b\cos(\lambda K[1])+a\sin(\lambda K[1]))}dK[1]}(d\cos(\lambda K[2]-\mu K}{k\mu(b\cos(\lambda K[2]-\mu K[2])+b\cos(\lambda}\right.$$

$$+\int_1^{y(x)}\left(\frac{e^{-\int_1^x \frac{\sec(\mu K[1])(2d\cos(\lambda K[1]-\mu K[1])+2d\cos(\lambda K[1]+\mu K[1])+k\sin(\lambda K[1]-\mu K[1])-k\sin(\lambda K[1]+\mu K[1]))}{2(b\cos(\lambda K[1])+a\sin(\lambda K[1]))}dK[1]}}{k\mu(d+K[3])^2}\right.$$

$$-\int_1^x\left(\frac{e^{-\int_1^{K[2]} \frac{\sec(\mu K[1])(2d\cos(\lambda K[1]-\mu K[1])+2d\cos(\lambda K[1]+\mu K[1])+k\sin(\lambda K[1]-\mu K[1])-k\sin(\lambda K[1]+\mu K[1]))}{2(b\cos(\lambda K[1])+a\sin(\lambda K[1]))}dK[1]}(-\cos(\lambda K[2]-\mu K[2}{k\mu(d+K[3])(b\cos(\lambda K[2]-\mu K[2])+b\cos(\lambda K[2]+\mu K[2])+a\sin(\lambda K[2]-\mu K[2])+a\sin}\right.$$

12 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-4. Equations with cotangent.

12.1 problem 38

Internal problem ID [10536]
Internal file name [`OUTPUT/9483_Monday_June_06_2022_02_50_30_PM_22071506/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-4. Equations with cotangent.
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = \lambda a + a(\lambda - a)\cot(\lambda x)^2}$$

12.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -a^2\cot(\lambda x)^2 + a\cot(\lambda x)^2\lambda + \lambda a + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a^2\cot(\lambda x)^2 + a\cot(\lambda x)^2\lambda + \lambda a + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2\cot(\lambda x)^2 + a\cot(\lambda x)^2\lambda + \lambda a$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= -a^2\cot(\lambda x)^2 + a\cot(\lambda x)^2\lambda + \lambda a \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-a^2\cot(\lambda x)^2 + a\cot(\lambda x)^2\lambda + \lambda a\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sqrt{\sin(\lambda x)}\left(c_1 \operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right) + c_2 \operatorname{LegendreQ}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right)\right)$$

The above shows that

$$u'(x) = -\frac{\cos(\lambda x)\operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right)c_1 a + \cos(\lambda x)\operatorname{LegendreQ}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right)c_2 a - \lambda(c_1}{\sqrt{\sin(\lambda x)}}$$

Using the above in (1) gives the solution

$$y = \frac{\cos(\lambda x)\operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right)c_1 a + \cos(\lambda x)\operatorname{LegendreQ}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right)c_2 a - \lambda(c_1 \mathrm{Le}}{\sin(\lambda x)\left(c_1\operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right) + c_2\,\mathrm{Legendre}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\cos(\lambda x)\operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right)c_3 a + \cos(\lambda x)\operatorname{LegendreQ}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right)a - \lambda(c_3 \mathrm{Leg}\right.}{c_3\operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right) + \mathrm{LegendreQ}}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\cos\left(\lambda x\right) \mathrm{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos\left(\lambda x\right)\right) c_3 a + \cos\left(\lambda x\right) \mathrm{LegendreQ}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos\left(\lambda x\right)\right) a - \lambda\left(c_3 \mathrm{Leg}\right.\right.}{c_3 \mathrm{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos\left(\lambda x\right)\right) + \mathrm{LegendreQ}} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\cos\left(\lambda x\right) \mathrm{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos\left(\lambda x\right)\right) c_3 a + \cos\left(\lambda x\right) \mathrm{LegendreQ}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos\left(\lambda x\right)\right) a - \lambda\left(c_3 \mathrm{Leg}\right.\right.}{c_3 \mathrm{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos\left(\lambda x\right)\right) + \mathrm{LegendreQ}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a^2*cot(lambda*x)^2-a*cot(lam
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               <- Legendre successful
            <- special function solution successful
               -> Trying to convert hypergeometric functions to elementary form...
               <- elementary form is not straightforward to achieve - returning special func
            <- Kovacics algorithm successful
         Change of variables used:
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 204

```
dsolve(diff(y(x),x)=y(x)^2+a*lambda+a*(lambda-a)*cot(lambda*x)^2,y(x), singsol=all)
```

$$y(x) = \frac{\left(\cos(x\lambda)\operatorname{LegendreP}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(x\lambda)\right) a + \operatorname{LegendreQ}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(x\lambda)\right)\cos(x\lambda) c_1 a - \lambda\big(\mathrm{Legen}\right.}{\operatorname{LegendreQ}\left(\frac{2a-\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(x\lambda)\right) c_1 + \operatorname{LegendreP}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+a*\[Lambda]+a*(\[Lambda]-a)*Cot[\[Lambda]*x]^2,y[x],x,IncludeSingularSo
```

Not solved

12.2 problem 39

Internal problem ID [10537]
Internal file name [`OUTPUT/9484_Monday_June_06_2022_02_50_33_PM_71867773/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-4. Equations with cotangent.
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 = 3\lambda a + \lambda^2 + a(\lambda - a)\cot(\lambda x)^2}$$

12.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -a^2\cot(\lambda x)^2 + a\cot(\lambda x)^2\lambda + 3\lambda a + \lambda^2 + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a^2\cot(\lambda x)^2 + a\cot(\lambda x)^2\lambda + 3\lambda a + \lambda^2 + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2\cot(\lambda x)^2 + a\cot(\lambda x)^2\lambda + 3\lambda a + \lambda^2$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -a^2 \cot(\lambda x)^2 + a \cot(\lambda x)^2 \lambda + 3\lambda a + \lambda^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-a^2 \cot(\lambda x)^2 + a \cot(\lambda x)^2 \lambda + 3\lambda a + \lambda^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sqrt{\sin(\lambda x)}\left(c_1 \operatorname{LegendreP}\left(\frac{2a+\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right) + c_2 \operatorname{LegendreQ}\left(\frac{2a+\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right)\right)$$

The above shows that

$$u'(x) = -\frac{-2 \operatorname{LegendreP}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right) c_1 \lambda - 2 \operatorname{LegendreQ}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right) c_2 \lambda + \cos(\lambda x) (c_1 \operatorname{Leg}}{\sqrt{\sin(\lambda x)}}$$

Using the above in (1) gives the solution

$$y = \frac{-2 \operatorname{LegendreP}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right) c_1 \lambda - 2 \operatorname{LegendreQ}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right) c_2 \lambda + \cos(\lambda x) (c_1 \operatorname{Legen}}{\sin(\lambda x) \left(c_1 \operatorname{LegendreP}\left(\frac{2a+\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right) + c_2 \operatorname{Legend}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\csc(\lambda x) \left(-2 \operatorname{LegendreP}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right) c_3 \lambda - 2 \operatorname{LegendreQ}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right) \lambda + \cos(\lambda x) (\right.}{c_3 \operatorname{LegendreP}\left(\frac{2a+\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right) + \operatorname{Legendre}}$$

Summary
The solution(s) found are the following

$$y = \frac{\csc(\lambda x)\left(-2\,\mathrm{LegendreP}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right)c_3\lambda - 2\,\mathrm{LegendreQ}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right)\lambda + \cos(\lambda x)\left(\right.\right.}{c_3\,\mathrm{LegendreP}\left(\frac{2a+\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right) + \mathrm{Legendre}} \tag{1}$$

Verification of solutions

$$y = \frac{\csc(\lambda x)\left(-2\,\mathrm{LegendreP}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right)c_3\lambda - 2\,\mathrm{LegendreQ}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right)\lambda + \cos(\lambda x)\left(\right.\right.}{c_3\,\mathrm{LegendreP}\left(\frac{2a+\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(\lambda x)\right) + \mathrm{Legendre}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a^2*cot(lambda*x)^2-a*cot(lam
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               <- Legendre successful
            <- special function solution successful
               -> Trying to convert hypergeometric functions to elementary form...
               <- elementary form is not straightforward to achieve - returning special func
            <- Kovacics algorithm successful
         Change of variables used:
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 199

```
dsolve(diff(y(x),x)=y(x)^2+lambda^2+3*a*lambda+a*(lambda-a)*cot(lambda*x)^2,y(x), singsol=al
```

$$y(x) = \frac{\csc(x\lambda)\left(-2\,\mathrm{LegendreP}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(x\lambda)\right)\lambda - 2\,\mathrm{LegendreQ}\left(\frac{2a+3\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(x\lambda)\right)c_1\lambda + \cos(x\lambda)\left(\right.\right.}{\mathrm{LegendreQ}\left(\frac{2a+\lambda}{2\lambda}, \frac{2a-\lambda}{2\lambda}, \cos(x\lambda)\right)c_1 + \mathrm{Legendr}}$$

✓ Solution by Mathematica

Time used: 67.099 (sec). Leaf size: 306

```
DSolve[y'[x]==y[x]^2+\[Lambda]^2+3*a*\[Lambda]+a*(\[Lambda]-a)*Cot[\[Lambda]*x]^2,y[x],x,Inc
```

$$y(x) \to -\frac{\sin^{-\frac{a+\lambda}{\lambda}}(2\lambda x)e^{-\operatorname{arctanh}(\cos(2\lambda x))}\left(c_1 \sin^{\frac{a}{\lambda}}(2\lambda x)((a+\lambda)\cos(2\lambda x)+a-\lambda)e^{\operatorname{arctanh}(\cos(2\lambda x))}\int_1^x e^{\frac{(a-\lambda)\operatorname{arctanh}(\mathrm{co}}{\lambda}}\right.}{1+c_1\int_1^x e^{\frac{(a-\lambda)\operatorname{arctanh}(\cos(}{\lambda}}}$$

$$y(x) \to \csc(2\lambda x)\left(-\frac{\sin^{-\frac{a}{\lambda}}(2\lambda x)e^{\frac{(a-\lambda)\operatorname{arctanh}(\cos(2\lambda x))}{\lambda}}}{\int_1^x e^{\frac{(a-\lambda)\operatorname{arctanh}(\cos(2\lambda K[1]))}{\lambda}}\sin^{-\frac{a+\lambda}{\lambda}}(2\lambda K[1])dK[1]} - (a+\lambda)\cos(2\lambda x) - a + \lambda\right)$$

12.3 problem 40

Internal problem ID [10538]
Internal file name [`OUTPUT/9485_Monday_June_06_2022_02_50_37_PM_8560361/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-4. Equations with cotangent.
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 + 2ab\cot(xa)\,y = -a^2 + b^2}$$

12.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 - 2ab\cot(xa)\,y + b^2 - a^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - 2ab\cot(xa)\,y + b^2 - a^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2 + b^2$, $f_1(x) = -2\cot(xa)\,ab$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= -2\cot(xa)\, ab \\ f_2^2 f_0 &= -a^2 + b^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + 2\cot(xa)\, ab u'(x) + \left(-a^2 + b^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = \sin(xa)^{-b+\frac{1}{2}} \Bigg(& c_1 \operatorname{LegendreP}\left(\frac{-a + 2\sqrt{(b^2-1)\, a^2 + b^2}}{2a}, b - \frac{1}{2}, \cos(xa)\right) \\ & + c_2 \operatorname{LegendreQ}\left(\frac{-a + 2\sqrt{(b^2-1)\, a^2 + b^2}}{2a}, b - \frac{1}{2}, \cos(xa)\right)\Bigg) \end{aligned}$$

The above shows that

$$u'(x) = -\frac{2\Big(c_1\Big(-\cos(xa)\sin(xa)\sqrt{(b^2-1)\, a^2 + b^2} + a\Big(-\frac{\cos(xa)\sin(xa)}{2} + (\cos(xa) - 1)\left(b - \frac{1}{2}\right)\cot(xa)\,(\cos(x \cdots}{\cdots}$$

Using the above in (1) gives the solution

$$y = \frac{2c_1\Big(-\cos(xa)\sin(xa)\sqrt{(b^2-1)\, a^2 + b^2} + a\Big(-\frac{\cos(xa)\sin(xa)}{2} + (\cos(xa) - 1)\left(b - \frac{1}{2}\right)\cot(xa)\,(\cos(xa) \cdots}{\cdots}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\cos\left(xa\right) c_3 \left(ab+\sqrt{\left(b^2-1\right) a^2+b^2}\right) \text{LegendreP}\left(\frac{-a+2\sqrt{\left(b^2-1\right)a^2+b^2}}{2a}, b-\frac{1}{2}, \cos\left(xa\right)\right) + \cos\left(xa\right) \left(ab+\sqrt{}\right.\right.}{}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\cos\left(xa\right) c_3 \left(ab+\sqrt{\left(b^2-1\right) a^2+b^2}\right) \text{LegendreP}\left(\frac{-a+2\sqrt{\left(b^2-1\right)a^2+b^2}}{2a}, b-\frac{1}{2}, \cos\left(xa\right)\right) + \cos\left(xa\right) \left(ab+\sqrt{}\right.\right.}{} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\cos\left(xa\right) c_3 \left(ab+\sqrt{\left(b^2-1\right) a^2+b^2}\right) \text{LegendreP}\left(\frac{-a+2\sqrt{\left(b^2-1\right)a^2+b^2}}{2a}, b-\frac{1}{2}, \cos\left(xa\right)\right) + \cos\left(xa\right) \left(ab+\sqrt{}\right.\right.}{}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -2*a*b*cot(a*x)*(diff(y(x), x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Kummer
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               <- heuristic approach successful
            <- hypergeometric successful
         <- special function solution successful
         Change of variables used:
            [x = 1/a*arcsin(t)]
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 291

```
dsolve(diff(y(x),x)=y(x)^2-2*a*b*cot(a*x)*y(x)+b^2-a^2,y(x), singsol=all)
```

$$y(x) = \frac{\left(\cos\left(ax\right)\left(ab+\sqrt{\left(b^2-1\right)a^2+b^2}\right)\operatorname{LegendreP}\left(\frac{-a+2\sqrt{\left(b^2-1\right)a^2+b^2}}{2a}, b-\frac{1}{2}, \cos\left(ax\right)\right)+c_1\cos\left(ax\right)\left(ab+\sqrt{\ldots}\right.\right.}{}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2-2*a*b*Cot[a*x]*y[x]+b^2-a^2,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

12.4 problem 41

Internal problem ID [10539]
Internal file name [`OUTPUT/9486_Monday_June_06_2022_02_50_49_PM_24821107/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-4. Equations with cotangent.
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - a\cot(\beta x)\, y = ab\cot(\beta x) - b^2}$$

12.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + a\cot(\beta x)\, y + ab\cot(\beta x) - b^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a\cot(\beta x)\, y + ab\cot(\beta x) - b^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = ab\cot(\beta x) - b^2$, $f_1(x) = \cot(\beta x)\, a$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \cot(\beta x) a \\ f_2^2 f_0 &= ab\cot(\beta x) - b^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \cot(\beta x) a u'(x) + (ab\cot(\beta x) - b^2) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = -\frac{e^{bx}\left(ic_2\beta\left(\int \sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\sin(\beta x)^{\frac{a}{\beta}}\cos(\beta x)\,e^{-2bx}dx\right) - 2c_1\right)}{2}$$

The above shows that

$$u'(x) = -\frac{i\left(\int \sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\sin(\beta x)^{\frac{a}{\beta}}\cos(\beta x)\,e^{-2bx}dx\right)c_2 b\beta\,e^{bx}}{2} - \frac{i\cos(\beta x)\sin(\beta x)^{\frac{a}{\beta}}\csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\beta c_2 e^{-bx}}{2} + c_1 b\,e^{bx}$$

Using the above in (1) gives the solution

$$y = \frac{2\left(-\frac{i\left(\int \sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\sin(\beta x)^{\frac{a}{\beta}}\cos(\beta x)e^{-2bx}dx\right)c_2 b\beta\,e^{bx}}{2} - \frac{i\cos(\beta x)\sin(\beta x)^{\frac{a}{\beta}}\csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\beta c_2 e^{-bx}}{2} + c_1 b\,e^{\cdots}\right.}{ic_2\beta\left(\int \sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\sin(\beta x)^{\frac{a}{\beta}}\cos(\beta x)\,e^{-2bx}dx\right) - 2c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\beta\sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\sin(\beta x)^{\frac{a}{\beta}}\cos(\beta x)\,e^{-2bx} - 2ibc_3 - \beta\left(\int \sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\sin(\beta x)^{\cdots}\right.}{\beta\left(\int \sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right)\sin(\beta x)^{\frac{a}{\beta}}\cos(\beta x)\,e^{-2bx}dx\right) + 2ic_3}$$

Summary

The solution(s) found are the following

$$y = \frac{-\beta \sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right) \csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right) \sin(\beta x)^{\frac{a}{\beta}} \cos(\beta x)\, \mathrm{e}^{-2bx} - 2ibc_3 - \beta\left(\int \sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right) \csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right) \sin(\beta x)^{\frac{a}{\beta}}\right.}{\beta\left(\int \sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right) \csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right) \sin(\beta x)^{\frac{a}{\beta}} \cos(\beta x)\, \mathrm{e}^{-2bx} dx\right) + 2ic_3} \tag{1}$$

Verification of solutions

$$y = \frac{-\beta \sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right) \csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right) \sin(\beta x)^{\frac{a}{\beta}} \cos(\beta x)\, \mathrm{e}^{-2bx} - 2ibc_3 - \beta\left(\int \sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right) \csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right) \sin(\beta x)^{\frac{a}{\beta}}\right.}{\beta\left(\int \sec\left(\frac{\pi}{4}+\frac{\beta x}{2}\right) \csc\left(\frac{\pi}{4}+\frac{\beta x}{2}\right) \sin(\beta x)^{\frac{a}{\beta}} \cos(\beta x)\, \mathrm{e}^{-2bx} dx\right) + 2ic_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 81

```
dsolve(diff(y(x),x)=y(x)^2+a*cot(beta*x)*y(x)+a*b*cot(beta*x)-b^2,y(x), singsol=all)
```

$$y(x) = \frac{-\left(\csc\left(x\beta\right)^2\right)^{-\frac{a}{2\beta}} \mathrm{e}^{-2bx} - b\left(\int \left(\csc\left(x\beta\right)^2\right)^{-\frac{a}{2\beta}} \mathrm{e}^{-2bx} dx - c_1\right)}{\int \left(\csc\left(x\beta\right)^2\right)^{-\frac{a}{2\beta}} \mathrm{e}^{-2bx} dx - c_1}$$

✓ Solution by Mathematica

Time used: 26.26 (sec). Leaf size: 462

```
DSolve[y'[x]==y[x]^2+a*Cot[\[Beta]*x]*y[x]+a*b*Cot[\[Beta]*x]-b^2,y[x],x,IncludeSingularSolu
```

$y(x)$

$$\rightarrow \frac{b(ia + 2b - 2i\beta)\left(-ie^{-i\beta x}\left(-1 + e^{2i\beta x}\right)\right)^{a/\beta} \text{Hypergeometric2F1}\left(1, \frac{a+2ib}{2\beta}, -\frac{a-2ib-2\beta}{2\beta}, e^{2ix\beta}\right) + (a - 2ib)}{i(-a + 2ib + 2\beta)\left(-ie^{-i\beta x}\left(-1 + e^{2i\beta x}\right)\right)^{a/\beta} \text{Hypergeometric2F1}\left(1, \frac{a+2ib}{2\beta}, -\frac{a-2ib-2\beta}{2\beta}, e^{2}\right.}$$

$y(x) \rightarrow -b$

12.5 problem 42

Internal problem ID [10540]
Internal file name [`OUTPUT/9487_Monday_June_06_2022_02_51_45_PM_23649721/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-4. Equations with cotangent.
Problem number: 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - ax\cot(bx)^m y = a\cot(bx)^m}$$

12.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + ax\cot(bx)^m y + a\cot(bx)^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + ax\cot(bx)^m y + a\cot(bx)^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\cot(bx)^m$, $f_1(x) = xa\cot(bx)^m$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= xa\cot(bx)^m \\ f_2^2f_0 &= a\cot(bx)^m \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - xa\cot(bx)^m\,u'(x) + a\cot(bx)^m\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x\left(c_1\left(\int e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}dx\right) + c_2\right)$$

The above shows that

$$u'(x) = c_1\left(\int e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}dx\right) + c_2 + xc_1e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}$$

Using the above in (1) gives the solution

$$y = -\frac{c_1\left(\int e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}dx\right) + c_2 + xc_1e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}}{x\left(c_1\left(\int e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}dx\right) + c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3\left(\int e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}dx\right) - 1 - xc_3e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}}{x\left(c_3\left(\int e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}dx\right) + 1\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_3 \left(\int e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}dx\right) - 1 - x c_3 e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}}{x \left(c_3 \left(\int e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}dx\right) + 1\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-c_3 \left(\int e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}dx\right) - 1 - x c_3 e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}}{x \left(c_3 \left(\int e^{\int \frac{\cot(bx)^m a\,x^2-2}{x}dx}dx\right) + 1\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 85

```
dsolve(diff(y(x),x)=y(x)^2+a*x*cot(b*x)^m*y(x)+a*cot(b*x)^m,y(x), singsol=all)
```

$$y(x) = \frac{-e^{\int \frac{a\cot(bx)^m x^2-2}{x}dx} x - \left(\int e^{\int \frac{a\cot(bx)^m x^2-2}{x}dx}dx\right) + c_1}{\left(-c_1 + \int e^{\int \frac{a\cot(bx)^m x^2-2}{x}dx}dx\right) x}$$

✓ Solution by Mathematica

Time used: 8.36 (sec). Leaf size: 126

```
DSolve[y'[x]==y[x]^2+a*x*Cot[b*x]^m*y[x]+a*Cot[b*x]^m,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to -\frac{\exp \left(-\int_1^x -a \cot^m(bK[1])K[1]dK[1]\right) + x\int_1^x \frac{\exp\left(-\int_1^{K[2]} -a\cot^m(bK[1])K[1]dK[1]\right)}{K[2]^2}dK[2] + c_1x}{x^2\left(\int_1^x \frac{\exp\left(-\int_1^{K[2]} -a\cot^m(bK[1])K[1]dK[1]\right)}{K[2]^2}dK[2] + c_1\right)}$$

$$y(x) \to -\frac{1}{x}$$

12.6 problem 43

Internal problem ID [10541]
Internal file name [OUTPUT/9488_Monday_June_06_2022_02_51_48_PM_37851356/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-4. Equations with cotangent.
Problem number: 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + (k+1)\,x^k y^2 - a\,x^{k+1}\cot(x)^m\,y = -a\cot(x)^m}$$

12.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a\,x^{k+1}\cot(x)^m\,y - x^k y^2 k - x^k y^2 - a\cot(x)^m \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a\,x^k x\cot(x)^m\,y - x^k y^2 k - x^k y^2 - a\cot(x)^m$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a\cot(x)^m$, $f_1(x) = a\cot(x)^m\,x^{k+1}$ and $f_2(x) = -x^k k - x^k$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(-x^k k - x^k)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{k^2 x^k}{x} - \frac{k\,x^k}{x} \\ f_1 f_2 &= a\cot(x)^m x^{k+1}(-x^k k - x^k) \\ f_2^2 f_0 &= -(-x^k k - x^k)^2 a\cot(x)^m \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(-x^k k - x^k) u''(x) - \left(-\frac{k^2 x^k}{x} - \frac{k\,x^k}{x} + a\cot(x)^m x^{k+1}(-x^k k - x^k)\right) u'(x) - (-x^k k - x^k)^2 a\cot(x)^m u($$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{k+1}\left(c_1 + c_2\left(\int x^{-2k-2} e^{\int \left(a\cot(x)^m x^{k+1} + \frac{k}{x}\right)dx} dx\right)\right)$$

The above shows that

$$\begin{aligned} u'(x) = &\left(c_2 x^{-2k-1} e^{\int \left(a\cot(x)^m x^{k+1} + \frac{k}{x}\right)dx}\right. \\ &\left. + \left(c_1 + c_2\left(\int x^{-2k-2} e^{\int \left(a\cot(x)^m x^{k+1} + \frac{k}{x}\right)dx} dx\right)\right)(k+1)\right) x^k \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(c_2 x^{-2k-1} e^{\int \left(a\cot(x)^m x^{k+1} + \frac{k}{x}\right)dx} + \left(c_1 + c_2\left(\int x^{-2k-2} e^{\int \left(a\cot(x)^m x^{k+1} + \frac{k}{x}\right)dx} dx\right)\right)(k+1)\right) x^k x^{-k-1}}{(-x^k k - x^k)\left(c_1 + c_2\left(\int x^{-2k-2} e^{\int \left(a\cot(x)^m x^{k+1} + \frac{k}{x}\right)dx} dx\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x^{-k-1}\left(x^{-2k-1} e^{\int \left(a\cot(x)^m x^{k+1} + \frac{k}{x}\right)dx} + \left(c_3 + \int x^{-2k-2} e^{\int \left(a\cot(x)^m x^{k+1} + \frac{k}{x}\right)dx} dx\right)(k+1)\right)}{(k+1)\left(c_3 + \int e^{\int \frac{a\,x^{k+2}\cot(x)^m + k}{x}dx} x^{-2k-2} dx\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{x^{-k-1}\left(x^{-2k-1}\mathrm{e}^{\int \left(a\cot(x)^{m} x^{k+1}+\frac{k}{x}\right)dx}+\left(c_3+\int x^{-2k-2}\mathrm{e}^{\int \left(a\cot(x)^{m} x^{k+1}+\frac{k}{x}\right)dx}dx\right)(k+1)\right)}{(k+1)\left(c_3+\int \mathrm{e}^{\int \frac{a\,x^{k+2}\cot(x)^{m}+k}{x}dx} x^{-2k-2}dx\right)} \tag{1}$$

Verification of solutions

$$y = \frac{x^{-k-1}\left(x^{-2k-1}\mathrm{e}^{\int \left(a\cot(x)^{m} x^{k+1}+\frac{k}{x}\right)dx}+\left(c_3+\int x^{-2k-2}\mathrm{e}^{\int \left(a\cot(x)^{m} x^{k+1}+\frac{k}{x}\right)dx}dx\right)(k+1)\right)}{(k+1)\left(c_3+\int \mathrm{e}^{\int \frac{a\,x^{k+2}\cot(x)^{m}+k}{x}dx} x^{-2k-2}dx\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x^(1+k)*cot(x)^m*a*x+k)*(diff
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 170

```
dsolve(diff(y(x),x)=-(k+1)*x^k*y(x)^2+a*x^(k+1)*cot(x)^m*y(x)-a*cot(x)^m,y(x), singsol=all)
```

$$y(x) = \frac{x^{-1-k}\left(x^{1+k}\mathrm{e}^{\int \frac{x^{1+k}\cot(x)^m a x-2k-2}{x}dx} + \left(\int x^k \mathrm{e}^{\int \frac{x^{1+k}\cot(x)^m a x-2k-2}{x}dx}dx\right)k + \int x^k \mathrm{e}^{\int \frac{x^{1+k}\cot(x)^m a x-2k-2}{x}dx}dx + c_1}{\left(\int x^k \mathrm{e}^{\int \frac{a\,x^{k+2}\cot(x)^m-2k-2}{x}dx}dx\right)k + c_1 + \int x^k \mathrm{e}^{\int \frac{a\,x^{k+2}\cot(x)^m-2k-2}{x}dx}dx}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==-(k+1)*x^k*y[x]^2+a*x^(k+1)*Cot[x]^m*y[x]-a*Cot[x]^m,y[x],x,IncludeSingularSol
```

Not solved

12.7 problem 44

Internal problem ID [10542]
Internal file name [`OUTPUT/9489_Monday_June_06_2022_02_51_58_PM_60994990/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-4. Equations with cotangent.
Problem number: 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

Unable to solve or complete the solution.

$$y' - a\cot(\lambda x+\mu)^k (y - b\,x^n - c)^2 = b\,x^{n-1}n$$

12.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= x^{2n}\cot(\lambda x+\mu)^k a\,b^2 + 2x^n\cot(\lambda x+\mu)^k abc - 2x^n\cot(\lambda x+\mu)^k aby + \cot(\lambda x+\mu)^k a\,c^2 - 2\cot(\lambda x \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^{2n}\left(-\frac{1}{\cot(\mu)+\cot(\lambda x)} + \frac{\cot(\mu)\cot(\lambda x)}{\cot(\mu)+\cot(\lambda x)}\right)^k a\,b^2 + 2x^n\left(-\frac{1}{\cot(\mu)+\cot(\lambda x)} + \frac{\cot(\mu)\cot(\lambda x)}{\cot(\mu)+\cot(\lambda x}\right.$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^{2n}\cot(\lambda x+\mu)^k a\,b^2 + 2x^n\cot(\lambda x+\mu)^k abc + \cot(\lambda x+\mu)^k a\,c^2 + b\,x^{n-1}n$, $f_1(x) = -2a\,x^n b\cot(\lambda x+\mu)^k - 2a\cot(\lambda x+\mu)^k c$ and $f_2(x) = a\cot(\lambda x+\mu)^k$.

Let

$$y = \frac{-u'}{f_2 u}$$
$$= \frac{-u'}{a \cot(\lambda x+\mu)^k u} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = \frac{a \cot(\lambda x+\mu)^k k\lambda(-1-\cot(\lambda x+\mu)^2)}{\cot(\lambda x+\mu)}$$

$$f_1 f_2 = \left(-2a\, x^n b \cot(\lambda x+\mu)^k - 2a \cot(\lambda x+\mu)^k c\right) a \cot(\lambda x+\mu)^k$$

$$f_2^2 f_0 = a^2 \cot(\lambda x+\mu)^{2k} \left(x^{2n} \cot(\lambda x+\mu)^k a\, b^2 + 2x^n \cot(\lambda x+\mu)^k abc + \cot(\lambda x+\mu)^k a\, c^2 + b\, x^{n-1} n\right)$$

Substituting the above terms back in equation (2) gives

$$a \cot(\lambda x+\mu)^k u''(x) - \left(\frac{a \cot(\lambda x+\mu)^k k\lambda(-1-\cot(\lambda x+\mu)^2)}{\cot(\lambda x+\mu)} + \left(-2a\, x^n b \cot(\lambda x+\mu)^k - 2a \cot(\lambda x\right.\right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (d) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(y(x),x)=a*cot(lambda*x+mu)^k*(y(x)-b*x^n-c)^2+b*n*x^(n-1),y(x), singsol=all)
```

$$y(x) = b\,x^{n} + c + \frac{1}{c_{1} - a\left(\int \left(\frac{-1+\cot(\mu)\cot(x\lambda)}{\cot(\mu)+\cot(x\lambda)}\right)^{k} dx\right)}$$

✓ Solution by Mathematica

Time used: 5.758 (sec). Leaf size: 74

```
DSolve[y'[x]==a*Cot[\[Lambda]*x+mu]^k*(y[x]-b*x^n-c)^2+b*n*x^(n-1),y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{1}{\frac{a\cot^{k+1}(\mu+\lambda x)\,\text{Hypergeometric2F1}\left(1,\frac{k+1}{2},\frac{k+3}{2},-\cot^{2}(\mu+x\lambda)\right)}{(k+1)\lambda} + c_{1}} + bx^{n} + c$$

$$y(x) \to bx^{n} + c$$

12.8 problem 45

Internal problem ID [10543]
Internal file name [`OUTPUT/9490_Monday_June_06_2022_02_53_05_PM_83446516/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-4. Equations with cotangent.
Problem number: 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - a\cot(\lambda x)^m y^2 - ky = a\,b^2x^{2k}\cot(\lambda x)^m}$$

12.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= \frac{a\cot(\lambda x)^m y^2 + ky + a\,b^2x^{2k}\cot(\lambda x)^m}{x}
\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,b^2x^{2k}\cot(\lambda x)^m}{x} + \frac{a\cot(\lambda x)^m y^2}{x} + \frac{ky}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{a\,b^2x^{2k}\cot(\lambda x)^m}{x}$, $f_1(x) = \frac{k}{x}$ and $f_2(x) = \frac{a\cot(\lambda x)^m}{x}$. Let

$$\begin{aligned}
y &= \frac{-u'}{f_2 u}\\
&= \frac{-u'}{\frac{a\cot(\lambda x)^m u}{x}} \qquad (1)
\end{aligned}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{a \cot(\lambda x)^m m\lambda(-1 - \cot(\lambda x)^2)}{\cot(\lambda x) x} - \frac{a \cot(\lambda x)^m}{x^2} \\ f_1 f_2 &= \frac{ka \cot(\lambda x)^m}{x^2} \\ f_2^2 f_0 &= \frac{a^3 \cot(\lambda x)^{3m} b^2 x^{2k}}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{a \cot(\lambda x)^m u''(x)}{x} - \left(\frac{a \cot(\lambda x)^m m\lambda(-1 - \cot(\lambda x)^2)}{\cot(\lambda x) x} - \frac{a \cot(\lambda x)^m}{x^2} + \frac{ka \cot(\lambda x)^m}{x^2}\right) u'(x) + \frac{a^3 \cot(\lambda x$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 e^{iab\left(\int x^{k-1} \cot(\lambda x)^m dx\right)} + c_2 e^{-iab\left(\int x^{k-1} \cot(\lambda x)^m dx\right)}$$

The above shows that

$$u'(x) = iab\, x^{k-1} \cot(\lambda x)^m \left(c_1 e^{iab\left(\int x^{k-1} \cot(\lambda x)^m dx\right)} - c_2 e^{-iab\left(\int x^{k-1} \cot(\lambda x)^m dx\right)}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{ib\, x^{k-1} \left(c_1 e^{iab\left(\int x^{k-1} \cot(\lambda x)^m dx\right)} - c_2 e^{-iab\left(\int x^{k-1} \cot(\lambda x)^m dx\right)}\right) x}{c_1 e^{iab\left(\int x^{k-1} \cot(\lambda x)^m dx\right)} + c_2 e^{-iab\left(\int x^{k-1} \cot(\lambda x)^m dx\right)}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{ib\, x^{k} \left(c_3 e^{iab\left(\int x^{k-1} \cot(\lambda x)^m dx\right)} - e^{-iab\left(\int x^{k-1} \cot(\lambda x)^m dx\right)}\right)}{c_3 e^{iab\left(\int x^{k-1} \cot(\lambda x)^m dx\right)} + e^{-iab\left(\int x^{k-1} \cot(\lambda x)^m dx\right)}}$$

Summary

The solution(s) found are the following

$$y = -\frac{ib\,x^{k}\left(c_3 \mathrm{e}^{iab\left(\int x^{k-1}\cot(\lambda x)^{m}dx\right)} - \mathrm{e}^{-iab\left(\int x^{k-1}\cot(\lambda x)^{m}dx\right)}\right)}{c_3 \mathrm{e}^{iab\left(\int x^{k-1}\cot(\lambda x)^{m}dx\right)} + \mathrm{e}^{-iab\left(\int x^{k-1}\cot(\lambda x)^{m}dx\right)}} \tag{1}$$

Verification of solutions

$$y = -\frac{ib\,x^{k}\left(c_3 \mathrm{e}^{iab\left(\int x^{k-1}\cot(\lambda x)^{m}dx\right)} - \mathrm{e}^{-iab\left(\int x^{k-1}\cot(\lambda x)^{m}dx\right)}\right)}{c_3 \mathrm{e}^{iab\left(\int x^{k-1}\cot(\lambda x)^{m}dx\right)} + \mathrm{e}^{-iab\left(\int x^{k-1}\cot(\lambda x)^{m}dx\right)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 31

```
dsolve(x*diff(y(x),x)=a*cot(lambda*x)^m*y(x)^2+k*y(x)+a*b^2*x^(2*k)*cot(lambda*x)^m,y(x), si
```

$$y(x) = -\tan\left(-ab\left(\int x^{-1+k}\cot(x\lambda)^{m}\,dx\right) + c_1\right) b\,x^{k}$$

✓ Solution by Mathematica

Time used: 1.805 (sec). Leaf size: 50

```
DSolve[x*y'[x]==a*Cot[\[Lambda]*x]^m*y[x]^2+k*y[x]+a*b^2*x^(2*k)*Cot[\[Lambda]*x]^m,y[x],x,I
```

$$y(x) \to \sqrt{b^2}x^{k}\tan\left(\sqrt{b^2}\int_1^x a\cot^{m}(\lambda K[1])K[1]^{k-1}dK[1] + c_1\right)$$

12.9 problem 46

Internal problem ID [10544]
Internal file name [OUTPUT/9491_Monday_June_06_2022_02_53_25_PM_4136660/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-4. Equations with cotangent.
Problem number: 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$\boxed{(a\cot(\lambda x)+b)\,y'-y^2-c\cot(x\mu)\,y=-d^2+cd\cot(x\mu)}$$

12.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2+c\cot(x\mu)\,y-d^2+cd\cot(x\mu)}{a\cot(\lambda x)+b} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=\frac{cd\cot(x\mu)}{a\cot(\lambda x)+b}+\frac{c\cot(x\mu)\,y}{a\cot(\lambda x)+b}-\frac{d^2}{a\cot(\lambda x)+b}+\frac{y^2}{a\cot(\lambda x)+b}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x) = \frac{-d^2+cd\cot(x\mu)}{a\cot(\lambda x)+b}$, $f_1(x) = \frac{c\cot(x\mu)}{a\cot(\lambda x)+b}$ and $f_2(x) = \frac{1}{a\cot(\lambda x)+b}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{a\cot(\lambda x)+b}}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{\lambda\left(-1-\cot(\lambda x)^2\right) a}{(a\cot(\lambda x)+b)^2} \\
f_1 f_2 &= \frac{c\cot(x\mu)}{(a\cot(\lambda x)+b)^2} \\
f_2^2 f_0 &= \frac{-d^2+cd\cot(x\mu)}{(a\cot(\lambda x)+b)^3}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{a\cot(\lambda x)+b} - \left(-\frac{\lambda\left(-1-\cot(\lambda x)^2\right) a}{(a\cot(\lambda x)+b)^2} + \frac{c\cot(x\mu)}{(a\cot(\lambda x)+b)^2}\right) u'(x) + \frac{(-d^2+cd\cot(x\mu))\, u(x)}{(a\cot(\lambda x)+b)^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 366

```
dsolve((a*cot(lambda*x)+b)*diff(y(x),x)=y(x)^2+c*cot(mu*x)*y(x)-d^2+c*d*cot(mu*x),y(x), sing
```

$$y(x) = \frac{-\mathrm{e}^{\frac{\lambda c\left(a^2+b^2\right)\left(\int \frac{\cot(x\mu)}{a\cot(x\lambda)+b}dx\right)-2d\left(\operatorname{arccot}(\cot(x\lambda))-\frac{\pi}{2}\right)b}{\lambda\left(a^2+b^2\right)}}\left(\csc(x\lambda)^2\right)^{-\frac{ad}{\lambda\left(a^2+b^2\right)}}\left(a\cot(x\lambda)+b\right)^{\frac{2ad}{\lambda\left(a^2+b^2\right)}} - d\left(\int (a\cot(x\lambda) \cdots\right.}{\int (a\cot(x\lambda)+b)^{\frac{\left(-a^2-b^2\right)\lambda+2ad}{\lambda\left(a^2+b^2\right)}}\mathrm{e}^{\frac{\lambda c\left(a^2+b^2\right)\left(\int \frac{\cot(x\mu)}{a\cot(x\lambda)+b}dx\right)-2d\left(\operatorname{arc}\cdots\right.}{\lambda\left(a^2+b^2\right)}}\cdots}$$

✓ Solution by Mathematica

Time used: 87.594 (sec). Leaf size: 799

```
DSolve[(a*Cot[\[Lambda]*x]+b)*y'[x]==y[x]^2+c*Cot[\[Mu]*x]*y[x]-d^2+c*d*Cot[\[Mu]*x],y[x],x,
```

$$\text{Solve}\left[\int_1^x \frac{e^{-\int_1^{K[2]} -\frac{\csc(\mu K[1])(-2d\cos(\lambda K[1]-\mu K[1])+2d\cos(\lambda K[1]+\mu K[1])+c\sin(\lambda K[1]-\mu K[1])+c\sin(\lambda K[1]+\mu K[1]))}{2(a\cos(\lambda K[1])+b\sin(\lambda K[1]))}dK[1]}(-d\cos(\lambda K[2]- \cdots}{c\mu(b\cos(\lambda K[2]-\mu K[2])-b\cos(\cdots} \right.$$

$$+\int_1^{y(x)}\left(-\int_1^x\left(\frac{e^{-\int_1^{K[2]} -\frac{\csc(\mu K[1])(-2d\cos(\lambda K[1]-\mu K[1])+2d\cos(\lambda K[1]+\mu K[1])+c\sin(\lambda K[1]-\mu K[1])+c\sin(\lambda K[1]+\mu K[1]))}{2(a\cos(\lambda K[1])+b\sin(\lambda K[1]))}dK[1]}(\cos(\lambda K[2 \cdots}{c\mu(d+K[3])(b\cos(\lambda K[2]-\mu K[2])-b\cos(\lambda K[2]+\mu K[2])-a\sin(\lambda K[2]-\mu K \cdots}\right.\right.$$

$$\left.\left. -\frac{e^{-\int_1^x -\frac{\csc(\mu K[1])(-2d\cos(\lambda K[1]-\mu K[1])+2d\cos(\lambda K[1]+\mu K[1])+c\sin(\lambda K[1]-\mu K[1])+c\sin(\lambda K[1]+\mu K[1]))}{2(a\cos(\lambda K[1])+b\sin(\lambda K[1]))}dK[1]}}{c\mu(d+K[3])^2}\right)dK[3]=c_1, y(x)\right]$$

13 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-5. Equations containing combinations of trigonometric functions.

13.1 problem 47

Internal problem ID [10545]
Internal file name [`OUTPUT/9492_Monday_June_06_2022_02_54_38_PM_68260555/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-5. Equations containing combinations of trigonometric functions.
Problem number: 47.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = \lambda^2 + c \sin(\lambda x)^n \cos(\lambda x)^{-n-4}}$$

13.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + \lambda^2 + c \sin(\lambda x)^n \cos(\lambda x)^{-n-4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \lambda^2 + \frac{c \sin(\lambda x)^n \cos(\lambda x)^{-n}}{\cos(\lambda x)^4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \lambda^2 + c \sin(\lambda x)^n \cos(\lambda x)^{-n-4}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \lambda^2 + c \sin(\lambda x)^n \cos(\lambda x)^{-n-4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(\lambda^2 + c \sin(\lambda x)^n \cos(\lambda x)^{-n-4}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\sin(\lambda x)^n \cos(\lambda x)^{-n-4} \_Y(x) c + \_Y(x) \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\sin(\lambda x)^n \cos(\lambda x)^{-n-4} \_Y(x) c + \_Y(x) \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\sin(\lambda x)^n \cos(\lambda x)^{-n-4} \_Y(x) c + \_Y(x) \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\sin(\lambda x)^n \cos(\lambda x)^{-n-4} \_Y(x) c + \_Y(x) \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\sin(\lambda x)^n \cos(\lambda x)^{-n-4} \_Y(x) c + \_Y(x) \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\sin(\lambda x)^n \cos(\lambda x)^{-n-4} \_Y(x) c + \_Y(x) \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\sin(\lambda x)^n \cos(\lambda x)^{-n-4} \_Y(x) c + \_Y(x) \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\sin(\lambda x)^n \cos(\lambda x)^{-n-4} \_Y(x) c + \_Y(x) \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\sin(\lambda x)^n \cos(\lambda x)^{-n-4} \_Y(x)\, c + \_Y(x)\,\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\operatorname{DESol}\left(\left\{\sin(\lambda x)^n \cos(\lambda x)^{-n-4} \_Y(x)\, c + \_Y(x)\,\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-lambda^2-c*sin(lambda*x)^n*c
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            -> Mathieu
               -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2+lambda^2+c*sin(lambda*x)^n*cos(lambda*x)^(-n-4),y(x), singsol=all
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+\[Lambda]^2+c*Sin[\[Lambda]*x]^n*Cos[\[Lambda]*x]^(-n-4),y[x],x,Include
```

Not solved

13.2 problem 48

Internal problem ID [10546]
Internal file name [`OUTPUT/9493_Monday_June_06_2022_02_55_37_PM_29498587/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-5. Equations containing combinations of trigonometric functions.
Problem number: 48.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 \sin(\lambda x)\, a = b \sin(\lambda x) \cos(\lambda x)^n}$$

13.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 \sin(\lambda x)\, a + b \sin(\lambda x) \cos(\lambda x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 \sin(\lambda x)\, a + b \sin(\lambda x) \cos(\lambda x)^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b \sin(\lambda x) \cos(\lambda x)^n$, $f_1(x) = 0$ and $f_2(x) = a \sin(\lambda x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{a \sin(\lambda x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= a\lambda \cos(\lambda x) \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= a^2 \sin(\lambda x)^3 b \cos(\lambda x)^n \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$a \sin(\lambda x) u''(x) - a\lambda \cos(\lambda x) u'(x) + a^2 \sin(\lambda x)^3 b \cos(\lambda x)^n u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sqrt{\cos(\lambda x)} \left(c_1 \operatorname{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}} \cos(\lambda x)^{1+\frac{n}{2}}}{2+n} \right) + c_2 \operatorname{BesselY}\left(\frac{1}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}} \cos(\lambda x)^{1+\frac{n}{2}}}{2+n} \right) \right)$$

The above shows that

$$u'(x) = \frac{\sin(\lambda x) \lambda \left(\operatorname{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}} \cos(\lambda x)^{1+\frac{n}{2}}}{2+n} \right) \cos(\lambda x)^{1+\frac{n}{2}} \sqrt{\frac{ab}{\lambda^2}}\, c_1 + \cos(\lambda x)^{1+\frac{n}{2}} \sqrt{\frac{ab}{\lambda^2}} \operatorname{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}}}{} \right.\right.}{\sqrt{\cos(\lambda x)}}$$

Using the above in (1) gives the solution

$$y = -\frac{\lambda \left(\operatorname{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}} \cos(\lambda x)^{1+\frac{n}{2}}}{2+n} \right) \cos(\lambda x)^{1+\frac{n}{2}} \sqrt{\frac{ab}{\lambda^2}}\, c_1 + \cos(\lambda x)^{1+\frac{n}{2}} \sqrt{\frac{ab}{\lambda^2}} \operatorname{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}} \cos(\lambda x)^{1}}{2+n} \right.\right.}{\cos(\lambda x)\, a \left(c_1 \operatorname{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}} \cos(\lambda x)^{1+\frac{n}{2}}}{2+n} \right) + c_2 \mathrm{B} \right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-\operatorname{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}}\, \cos(\lambda x)^{1+\frac{n}{2}}}{2+n}\right) \cos\left(\lambda x\right)^{1+\frac{n}{2}} \sqrt{\frac{ab}{\lambda^2}}\, c_3 - \operatorname{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}}\, \cos(\lambda x)^{1+\frac{n}{2}}}{2+n}\right) \sqrt{\frac{ab}{\lambda^2}}\, \cos\left(\lambda x\right)^{}}{\left(c_3 \operatorname{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}}\, \cos(\lambda x)^{1+\frac{n}{2}}}{2+n}\right) + \operatorname{BesselY}\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-\operatorname{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}}\, \cos(\lambda x)^{1+\frac{n}{2}}}{2+n}\right) \cos\left(\lambda x\right)^{1+\frac{n}{2}} \sqrt{\frac{ab}{\lambda^2}}\, c_3 - \operatorname{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}}\, \cos(\lambda x)^{1+\frac{n}{2}}}{2+n}\right) \sqrt{\frac{ab}{\lambda^2}}\, \cos\left(\lambda x\right)^{}}{\left(c_3 \operatorname{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}}\, \cos(\lambda x)^{1+\frac{n}{2}}}{2+n}\right) + \operatorname{BesselY}\right.} \quad (1)$$

Verification of solutions

$$y = \frac{\left(-\operatorname{BesselJ}\left(\frac{n+3}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}}\, \cos(\lambda x)^{1+\frac{n}{2}}}{2+n}\right) \cos\left(\lambda x\right)^{1+\frac{n}{2}} \sqrt{\frac{ab}{\lambda^2}}\, c_3 - \operatorname{BesselY}\left(\frac{n+3}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}}\, \cos(\lambda x)^{1+\frac{n}{2}}}{2+n}\right) \sqrt{\frac{ab}{\lambda^2}}\, \cos\left(\lambda x\right)^{}}{\left(c_3 \operatorname{BesselJ}\left(\frac{1}{2+n}, \frac{2\sqrt{\frac{ab}{\lambda^2}}\, \cos(\lambda x)^{1+\frac{n}{2}}}{2+n}\right) + \operatorname{BesselY}\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = lambda*cos(lambda*x)*(diff(y(x
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying an equivalence, under non-integer power transformations,
            to LODEs admitting Liouvillian solutions.
            -> Trying a Liouvillian solution using Kovacics algorithm
            <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            <- Bessel successful
         <- special function solution successful
         Change of variables used:
            [x = arccos(t)/lambda]
         Linear ODE actually solved:
            4*a*b*t^n*(-t^2+1)^(3/2)*u(t)+4*(-t^2+1)^(3/2)*lambda^2*diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 256

```
dsolve(diff(y(x),x)=a*sin(lambda*x)*y(x)^2+b*sin(lambda*x)*cos(lambda*x)^n,y(x), singsol=all
```

$$y(x) = \frac{\left(-\sqrt{\frac{ab}{\lambda^2}}\, \text{BesselY}\left(\frac{3+n}{n+2}, \frac{2\sqrt{\frac{ab}{\lambda^2}}\, \cos(x\lambda)^{\frac{n}{2}+1}}{n+2}\right) \cos(x\lambda)^{\frac{n}{2}+1} c_1 - \text{BesselJ}\left(\frac{3+n}{n+2}, \frac{2\sqrt{\frac{ab}{\lambda^2}}\, \cos(x\lambda)^{\frac{n}{2}+1}}{n+2}\right) \sqrt{\frac{ab}{\lambda^2}}\, \cos(x\lambda)}{\left(\text{BesselY}\left(\frac{1}{n+2}, \frac{2\sqrt{\frac{ab}{\lambda^2}}\, \cos(x\lambda)^{\frac{n}{2}+1}}{n+2}\right) c_1 + \text{Bessel}}$$

✓ Solution by Mathematica

Time used: 1.409 (sec). Leaf size: 695

```
DSolve[y'[x]==a*Sin[\[Lambda]*x]*y[x]^2+b*Sin[\[Lambda]*x]*Cos[\[Lambda]*x]^n,y[x],x,Include
```

$$y(x) \to \frac{\sqrt{a}\sqrt{b}\,\text{Gamma}\left(1+\frac{1}{n+2}\right)\cos^{\frac{n}{2}}(\lambda x)\,\text{BesselJ}\left(\frac{1}{n+2}-1, \frac{2\sqrt{a}\sqrt{b}\cos^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right) - \sqrt{a}\sqrt{b}\,\text{Gamma}\left(1+\frac{1}{n+2}\right)\cos^{\frac{n}{2}}}{}$$

$$y(x) \to \frac{\frac{\sqrt{a}\sqrt{b}\cos^{\frac{n}{2}}(\lambda x)\left(\text{BesselJ}\left(-\frac{n+3}{n+2}, \frac{2\sqrt{a}\sqrt{b}\cos^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right) - \text{BesselJ}\left(\frac{n+1}{n+2}, \frac{2\sqrt{a}\sqrt{b}\cos^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right)\right)}{\text{BesselJ}\left(-\frac{1}{n+2}, \frac{2\sqrt{a}\sqrt{b}\cos^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right)} + \lambda\sec(\lambda x)}{2a}$$

13.3 problem 49

Internal problem ID [10547]
Internal file name [`OUTPUT/9494_Monday_June_06_2022_02_55_40_PM_10899333/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-5. Equations containing combinations of trigonometric functions.
Problem number: 49.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \sin(\lambda x) y^2 - a \cos(\lambda x)^n y = -a \cos(\lambda x)^{n-1}}$$

13.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \lambda \sin(\lambda x) y^2 + a \cos(\lambda x)^n y - a \cos(\lambda x)^{n-1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \lambda \sin(\lambda x) y^2 + a \cos(\lambda x)^n y - \frac{a \cos(\lambda x)^n}{\cos(\lambda x)}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a \cos(\lambda x)^{n-1}$, $f_1(x) = a \cos(\lambda x)^n$ and $f_2(x) = \lambda \sin(\lambda x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\lambda \sin(\lambda x) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \lambda^2 \cos(\lambda x) \\ f_1 f_2 &= a \cos(\lambda x)^n \lambda \sin(\lambda x) \\ f_2^2 f_0 &= -\lambda^2 \sin(\lambda x)^2 a \cos(\lambda x)^{n-1} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\lambda \sin(\lambda x)\, u''(x) - \left(\lambda^2 \cos(\lambda x) + a \cos(\lambda x)^n \lambda \sin(\lambda x)\right) u'(x) - \lambda^2 \sin(\lambda x)^2 a \cos(\lambda x)^{n-1} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = \mathrm{DESol}\left(\left\{-\sin(\lambda x)\cos(\lambda x)^{n-1} a\lambda \_Y(x) - \cos(\lambda x)^n \_Y'(x)\, a \right.\right. \\ \left.\left. - \cot(\lambda x) \_Y'(x)\, \lambda + \_Y''(x)\right\}, \{\_Y(x)\}\right) \end{aligned}$$

The above shows that

$$\begin{aligned} u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{-\sin(\lambda x)\cos(\lambda x)^{n-1} a\lambda \_Y(x) - \cos(\lambda x)^n \_Y'(x)\, a \right.\right. \\ \left.\left. - \cot(\lambda x) \_Y'(x)\, \lambda + \_Y''(x)\right\}, \{\_Y(x)\}\right) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{-\sin(\lambda x)\cos(\lambda x)^{n-1} a\lambda \_Y(x) - \cos(\lambda x)^n \_Y'(x)\, a - \cot(\lambda x) \_Y'(x)\, \lambda + \_Y''(x)\right.\right.}{\lambda \sin(\lambda x)\, \mathrm{DESol}\left(\left\{-\sin(\lambda x)\cos(\lambda x)^{n-1} a\lambda \_Y(x) - \cos(\lambda x)^n \_Y'(x)\, a - \cot(\lambda x) \_Y'(x)\, \lambda + \_Y\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\csc(\lambda x)\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{-\sin(\lambda x)\cos(\lambda x)^{n-1} a\lambda \_Y(x) - \cos(\lambda x)^n \_Y'(x)\, a - \cot(\lambda x) \_Y'(x)\, \lambda + \_\right.\right.\right.}{\mathrm{DESol}\left(\left\{-\sin(\lambda x)\cos(\lambda x)^{n-1} a\lambda \_Y(x) - \cos(\lambda x)^n \_Y'(x)\, a - \cot(\lambda x) \_Y'(x)\, \lambda + \_Y''(x)\right\}\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\csc(\lambda x)\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{-\sin(\lambda x)\cos(\lambda x)^{n-1} a\lambda\_Y(x) - \cos(\lambda x)^{n}\_Y'(x)\, a - \cot(\lambda x)\_Y'(x)\,\lambda + \_\right.\right.\right.}{\mathrm{DESol}\left(\left\{-\sin(\lambda x)\cos(\lambda x)^{n-1} a\lambda\_Y(x) - \cos(\lambda x)^{n}\_Y'(x)\, a - \cot(\lambda x)\_Y'(x)\,\lambda + \_Y''(x)\right\}\right.} \quad (1)$$

Verification of solutions

$$y = -\frac{\csc(\lambda x)\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{-\sin(\lambda x)\cos(\lambda x)^{n-1} a\lambda\_Y(x) - \cos(\lambda x)^{n}\_Y'(x)\, a - \cot(\lambda x)\_Y'(x)\,\lambda + \_\right.\right.\right.}{\mathrm{DESol}\left(\left\{-\sin(\lambda x)\cos(\lambda x)^{n-1} a\lambda\_Y(x) - \cos(\lambda x)^{n}\_Y'(x)\, a - \cot(\lambda x)\_Y'(x)\,\lambda + \_Y''(x)\right\}\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (cos(lambda*x)^n*sin(lambda*x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            -> Mathieu
               -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=lambda*sin(lambda*x)*y(x)^2+a*cos(lambda*x)^n*y(x)-a*cos(lambda*x)^(n-1)
```

No solution found

✓ Solution by Mathematica

Time used: 150.623 (sec). Leaf size: 467

```
DSolve[y'[x]==\[Lambda]*Sin[\[Lambda]*x]*y[x]^2+a*Cos[\[Lambda]*x]^n*y[x]-a*Cos[\[Lambda]*x]
```

$$\text{Solve}\left[\int_1^x -\frac{\exp\left(-\frac{a\cos^{n+1}(\lambda K[1])\csc(\lambda K[1])\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{n+1}{2},\frac{n+3}{2},\cos^2(\lambda K[1])\right)\sqrt{\sin^2(\lambda K[1])}}{(n+1)\lambda}\right)\tan(\lambda K[1])\left(-a\csc(\lambda K[1])\,\text{co}\right.}{(\cos(\lambda K[1])y(x)-1)^2}\right.$$

$$+\int_1^{y(x)}\left(\frac{\exp\left(-\frac{a\cos^{n+1}(x\lambda)\csc(x\lambda)\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{n+1}{2},\frac{n+3}{2},\cos^2(x\lambda)\right)\sqrt{\sin^2(x\lambda)}}{(n+1)\lambda}\right)}{(\cos(x\lambda)K[2]-1)^2}\right.$$

$$-\int_1^x\left(\frac{2\exp\left(-\frac{a\cos^{n+1}(\lambda K[1])\csc(\lambda K[1])\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{n+1}{2},\frac{n+3}{2},\cos^2(\lambda K[1])\right)\sqrt{\sin^2(\lambda K[1])}}{(n+1)\lambda}\right)\left(-a\csc(\lambda K[1])\cos^n(\lambda\right.}{(\cos(\lambda K[1])K[2]-1)^3}\right.$$

13.4 problem 50

Internal problem ID [10548]
Internal file name [`OUTPUT/9495_Monday_June_06_2022_02_56_33_PM_78479147/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-5. Equations containing combinations of trigonometric functions.
Problem number: 50.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 \cos(\lambda x)\, a = b \cos(\lambda x) \sin(\lambda x)^n}$$

13.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 \cos(\lambda x)\, a + b \cos(\lambda x) \sin(\lambda x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 \cos(\lambda x)\, a + b \cos(\lambda x) \sin(\lambda x)^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b \cos(\lambda x) \sin(\lambda x)^n$, $f_1(x) = 0$ and $f_2(x) = \cos(\lambda x)\, a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\cos(\lambda x)\, a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -a\lambda \sin(\lambda x) \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \cos(\lambda x)^3 a^2 b \sin(\lambda x)^n \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\cos(\lambda x)\, a u''(x) + a\lambda \sin(\lambda x)\, u'(x) + \cos(\lambda x)^3 a^2 b \sin(\lambda x)^n u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{-\csc\left(\frac{\pi(n+3)}{2+n}\right) c_1 \operatorname{BesselI}\left(-\frac{1}{2+n}, 2\sqrt{-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) \pi \left(-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{1}{4+2n}} + c_2 \sin(\lambda x) \operatorname{BesselI}\left(\frac{1}{2+n}, 2\sqrt{-\cdots}\right.}{(2+n)\,\Gamma\left(\frac{n+3}{2+n}\right)}$$

The above shows that

$$u'(x) = \frac{\left(\Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} c_2 \cos(\lambda x)(2+n) \operatorname{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) + \cos(\lambda x)\, c_2 \operatorname{BesselI}\left(\cdots\right.\right.}{\cdots}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} c_2 \cos(\lambda x)(2+n) \operatorname{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) + \cos(\lambda x)\, c_2 \operatorname{BesselI}\cdots\right.}{\cos(\lambda x)\, a \left(-\csc\left(\frac{\pi(n+3)}{2+n}\right) c_1 \operatorname{BesselI}\left(-\frac{1}{2+n}, 2\sqrt{-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right)\cdots\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$y =$

$$-\frac{\left(\Gamma\left(\frac{n+3}{2+n}\right)^{2}\left(-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^{2}(2+n)^{2}}\right)^{\frac{n+1}{4+2n}}(2+n)\,\mathrm{BesselI}\left(\frac{n+3}{2+n},2\sqrt{-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^{2}(2+n)^{2}}}\right)-\csc(\lambda x)\,\pi c_{3}\csc\left(\frac{\pi(n+3)}{2+n}\right)\left(-\right.\right.}{\left(-\csc\left(\frac{\pi(n+3)}{2+n}\right)c_{3}\,\mathrm{BesselI}\left(-\frac{1}{2+n},2\sqrt{-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^{2}(2+n)^{2}}}\right)\pi\left(-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^{2}(2+n)^{2}}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\Gamma\left(\frac{n+3}{2+n}\right)^{2}\left(-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^{2}(2+n)^{2}}\right)^{\frac{n+1}{4+2n}}(2+n)\,\mathrm{BesselI}\left(\frac{n+3}{2+n},2\sqrt{-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^{2}(2+n)^{2}}}\right)-\csc(\lambda x)\,\pi c_{3}\csc\left(\frac{\pi(n+3)}{2+n}\right)\left(-\right.\right.}{\left(-\csc\left(\frac{\pi(n+3)}{2+n}\right)c_{3}\,\mathrm{BesselI}\left(-\frac{1}{2+n},2\sqrt{-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^{2}(2+n)^{2}}}\right)\pi\left(-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^{2}(2+n)^{2}}\right.\right.} \tag{1}$$

Verification of solutions

$y =$

$$-\frac{\left(\Gamma\left(\frac{n+3}{2+n}\right)^{2}\left(-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^{2}(2+n)^{2}}\right)^{\frac{n+1}{4+2n}}(2+n)\,\mathrm{BesselI}\left(\frac{n+3}{2+n},2\sqrt{-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^{2}(2+n)^{2}}}\right)-\csc(\lambda x)\,\pi c_{3}\csc\left(\frac{\pi(n+3)}{2+n}\right)\left(-\right.\right.}{\left(-\csc\left(\frac{\pi(n+3)}{2+n}\right)c_{3}\,\mathrm{BesselI}\left(-\frac{1}{2+n},2\sqrt{-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^{2}(2+n)^{2}}}\right)\pi\left(-\frac{ab\sin(\lambda x)^{2+n}}{\lambda^{2}(2+n)^{2}}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -lambda*sin(lambda*x)*(diff(y(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying an equivalence, under non-integer power transformations,
            to LODEs admitting Liouvillian solutions.
            -> Trying a Liouvillian solution using Kovacics algorithm
            <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Kummer
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
               <- hyper3 successful: received ODE is equivalent to the 0F1 ODE
            <- Kummer successful
         <- special function solution successful
         Change of variables used:
            [x = arccos(t)/lambda]
         Linear ODE actually solved:
            4*a*b*(-t^2+1)^(1/2*n)*t^3*u(t)-4*lambda^2*diff(u(t),t)+(-4*lambda^2*t^3+4*lambd
      <- change of variables successful
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 949

```
dsolve(diff(y(x),x)=a*cos(lambda*x)*y(x)^2+b*cos(lambda*x)*sin(lambda*x)^n,y(x), singsol=all
```

Expression too large to display

✓ Solution by Mathematica

Time used: 1.376 (sec). Leaf size: 633

```
DSolve[y'[x]==a*Cos[\[Lambda]*x]*y[x]^2+b*Cos[\[Lambda]*x]*Sin[\[Lambda]*x]^n,y[x],x,Include
```

$$y(x) \to \frac{\csc(\lambda x)\left(-\lambda \operatorname{Gamma}\left(1+\frac{1}{n+2}\right)\operatorname{BesselJ}\left(\frac{1}{n+2},\frac{2\sqrt{a}\sqrt{b}\sin^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right)+\sqrt{a}\sqrt{b}\sin^{\frac{n}{2}+1}(\lambda x)\left(\operatorname{Gamma}\left(1+\frac{1}{n+2}\right)\right.\right.}{}$$

$$y(x) \to \frac{\frac{\sqrt{a}\sqrt{b}\sin^{\frac{n}{2}}(\lambda x)\left(\operatorname{BesselJ}\left(\frac{n+1}{n+2},\frac{2\sqrt{a}\sqrt{b}\sin^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right)-\operatorname{BesselJ}\left(-\frac{n+3}{n+2},\frac{2\sqrt{a}\sqrt{b}\sin^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right)\right)}{\operatorname{BesselJ}\left(-\frac{1}{n+2},\frac{2\sqrt{a}\sqrt{b}\sin^{\frac{n}{2}+1}(x\lambda)}{n\lambda+2\lambda}\right)}-\lambda\csc(\lambda x)}{2a}$$

13.5 problem 51

Internal problem ID [10549]
Internal file name [`OUTPUT/9496_Monday_June_06_2022_02_57_09_PM_54105406/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-5. Equations containing combinations of trigonometric functions.
Problem number: 51.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \sin(\lambda x)\, y^2 - a\, x^n \cos(\lambda x)\, y = -x^n a}$$

13.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \lambda \sin(\lambda x)\, y^2 + a\, x^n \cos(\lambda x)\, y - x^n a \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \lambda \sin(\lambda x)\, y^2 + a\, x^n \cos(\lambda x)\, y - x^n a$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -x^n a$, $f_1(x) = x^n \cos(\lambda x)\, a$ and $f_2(x) = \lambda \sin(\lambda x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\lambda \sin(\lambda x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \lambda^2 \cos(\lambda x) \\ f_1 f_2 &= x^n \cos(\lambda x)\, a\lambda \sin(\lambda x) \\ f_2^2 f_0 &= -x^n a\, \lambda^2 \sin(\lambda x)^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\lambda \sin(\lambda x)\, u''(x) - \left(x^n \cos(\lambda x)\, a\lambda \sin(\lambda x) + \lambda^2 \cos(\lambda x)\right) u'(x) - x^n a\, \lambda^2 \sin(\lambda x)^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = -\cos(\lambda x) \left(c_2 \lambda \left(\int \mathrm{e}^{\int (x^n \cos(\lambda x) a + 2\tan(\lambda x)\lambda) dx} \sin(\lambda x)\, dx\right) - c_1\right)$$

The above shows that

$$\begin{aligned} u'(x) = -\lambda \sin(\lambda x) \Bigg(&\cos(\lambda x)\, c_2 \mathrm{e}^{\int (x^n \cos(\lambda x) a + 2\tan(\lambda x)\lambda) dx} \\ &- c_2 \lambda \left(\int \mathrm{e}^{\int (x^n \cos(\lambda x) a + 2\tan(\lambda x)\lambda) dx} \sin(\lambda x)\, dx\right) + c_1\Bigg) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\cos(\lambda x)\, c_2 \mathrm{e}^{\int (x^n \cos(\lambda x) a + 2\tan(\lambda x)\lambda) dx} - c_2 \lambda \left(\int \mathrm{e}^{\int (x^n \cos(\lambda x) a + 2\tan(\lambda x)\lambda) dx} \sin(\lambda x)\, dx\right) + c_1}{\cos(\lambda x) \left(c_2 \lambda \left(\int \mathrm{e}^{\int (x^n \cos(\lambda x) a + 2\tan(\lambda x)\lambda) dx} \sin(\lambda x)\, dx\right) - c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\sec(\lambda x)\, \lambda \left(\int \mathrm{e}^{\int (x^n \cos(\lambda x) a + 2\tan(\lambda x)\lambda) dx} \sin(\lambda x)\, dx\right) - \sec(\lambda x)\, c_3 - \mathrm{e}^{\int (x^n \cos(\lambda x) a + 2\tan(\lambda x)\lambda) dx}}{\lambda \left(\int \mathrm{e}^{\int (x^n \cos(\lambda x) a + 2\tan(\lambda x)\lambda) dx} \sin(\lambda x)\, dx\right) - c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{\sec \left(\lambda x \right) \lambda \left(\int {\mathrm e}^{\int \left(x^{n} \cos \left(\lambda x \right) a +2 \tan \left(\lambda x \right) \lambda \right)d x} \sin \left(\lambda x \right)d x \right)-\sec \left(\lambda x \right) c_{3} -{\mathrm e}^{\int \left(x^{n} \cos \left(\lambda x \right) a +2 \tan \left(\lambda x \right) \lambda \right)d x}}{\lambda \left(\int {\mathrm e}^{\int \left(x^{n} \cos \left(\lambda x \right) a +2 \tan \left(\lambda x \right) \lambda \right)d x} \sin \left(\lambda x \right)d x \right)-c_{3}} \tag{1}$$

Verification of solutions

$$y = \frac{\sec \left(\lambda x \right) \lambda \left(\int {\mathrm e}^{\int \left(x^{n} \cos \left(\lambda x \right) a +2 \tan \left(\lambda x \right) \lambda \right)d x} \sin \left(\lambda x \right)d x \right)-\sec \left(\lambda x \right) c_{3} -{\mathrm e}^{\int \left(x^{n} \cos \left(\lambda x \right) a +2 \tan \left(\lambda x \right) \lambda \right)d x}}{\lambda \left(\int {\mathrm e}^{\int \left(x^{n} \cos \left(\lambda x \right) a +2 \tan \left(\lambda x \right) \lambda \right)d x} \sin \left(\lambda x \right)d x \right)-c_{3}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = cos(lambda*x)*(a*x^n*sin(lambd
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            <- linear symmetries successful
         Change of variables used:
            [x = arccos(t)/lambda]
         Linear ODE actually solved:
            (2*(-t^2+1)^(1/2)*a*(arccos(t)/lambda)^n*t^2-2*(-t^2+1)^(1/2)*a*(arccos(t)/lambd
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 103

```
dsolve(diff(y(x),x)=lambda*sin(lambda*x)*y(x)^2+a*x^n*cos(lambda*x)*y(x)-a*x^n,y(x), singsol
```

$$y(x) = \frac{-c_1 \mathrm{e}^{\int (x^n \cos(x\lambda) a + 2\tan(x\lambda)\lambda)dx} + \sec(x\lambda)\,\lambda\left(\int \mathrm{e}^{\int (x^n \cos(x\lambda) a + 2\tan(x\lambda)\lambda)dx} \sin(x\lambda)\,dx\right) c_1 - \sec(x\lambda)}{\lambda\left(\int \mathrm{e}^{\int (x^n \cos(x\lambda) a + 2\tan(x\lambda)\lambda)dx} \sin(x\lambda)\,dx\right) c_1 - 1}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==\[Lambda]*Sin[\[Lambda]*x]*y[x]^2+a*x^n*Cos[\[Lambda]*x]*y[x]-a*x^n,y[x],x,Inc
```

Not solved

13.6 problem 52

Internal problem ID [10550]
Internal file name [`OUTPUT/9497_Monday_June_06_2022_02_57_18_PM_24345774/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-5. Equations containing combinations of trigonometric functions.
Problem number: 52.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{\sin(2x)^{n+1} y' - a y^2 \sin(x)^{2n} = b\cos(x)^{2n}}$$

13.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \left(a\,y^2 \sin(x)^{2n} + b\cos(x)^{2n}\right)\sin(2x)^{-n-1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a\,y^2 \sin(x)^{2n} \left(\frac{\sin(2x)}{2}\right)^{-n} 2^{-n}}{2\cos(x)\sin(x)} + \frac{b\cos(x)^{2n} \left(\frac{\sin(2x)}{2}\right)^{-n} 2^{-n}}{2\cos(x)\sin(x)}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \sin(2x)^{-n-1}\cos(x)^{2n} b$, $f_1(x) = 0$ and $f_2(x) = \sin(2x)^{-n-1}\sin(x)^{2n} a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\sin(x)^{2n} a \sin(2x)^{-n-1} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$f_2' = \frac{2\sin(2x)^{-n-1}\sin(x)^{2n}\,n\cos(x)\,a}{\sin(x)} - \frac{2\sin(x)^{2n}\,a\sin(2x)^{-n-1}\,(n+1)\cos(2x)}{\sin(2x)}$$
$$f_1f_2 = 0$$
$$f_2^2f_0 = \sin(x)^{4n}\,a^2\sin(2x)^{-3-3n}\,b\cos(x)^{2n}$$

Substituting the above terms back in equation (2) gives

$$\sin(x)^{2n}\,a\sin(2x)^{-n-1}\,u''(x) - \left(\frac{2\sin(2x)^{-n-1}\sin(x)^{2n}\,n\cos(x)\,a}{\sin(x)} - \frac{2\sin(x)^{2n}\,a\sin(2x)^{-n-1}\,(n+1)\cos$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \cot(x)^{-\frac{n}{2}}\left(c_1\cot(x)^{\frac{\sqrt{-b4^{-n}a+n^2}}{2}} + c_2\cot(x)^{-\frac{\sqrt{-b4^{-n}a+n^2}}{2}}\right)$$

The above shows that

$$u'(x) = \frac{(\cot(x)+\tan(x))\left(c_2\left(n+\sqrt{-b4^{-n}a+n^2}\right)\cot(x)^{-\frac{\sqrt{-b4^{-n}a+n^2}}{2}} - \cot(x)^{\frac{\sqrt{-b4^{-n}a+n^2}}{2}}\,c_1\left(-n+\sqrt{-b4^{-n}a\right.\right.}{2}$$

Using the above in (1) gives the solution

$$y = -\frac{(\cot(x)+\tan(x))\left(c_2\left(n+\sqrt{-b4^{-n}a+n^2}\right)\cot(x)^{-\frac{\sqrt{-b4^{-n}a+n^2}}{2}} - \cot(x)^{\frac{\sqrt{-b4^{-n}a+n^2}}{2}}\,c_1\left(-n+\sqrt{-b4^{-n}\right.\right.}{2a\left(c_1\cot(x)^{\frac{\sqrt{-b4^{-n}a+n^2}}{2}} + c_2\cot(x)^{-\frac{\sqrt{-b4^{-n}a+n^2}}{2}}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\sin(x)^{-2n}\sec(x)\csc(x)\sin(2x)^{n+1}\left(\left(n+\sqrt{-b4^{-n}a+n^2}\right)\cot(x)^{-\frac{\sqrt{-b4^{-n}a+n^2}}{2}} - \cot(x)^{\frac{\sqrt{-b4^{-n}a+n^2}}{2}} c_3 (\right.}{2\left(c_3\cot(x)^{\frac{\sqrt{-b4^{-n}a+n^2}}{2}} + \cot(x)^{-\frac{\sqrt{-b4^{-n}a+n^2}}{2}}\right)a}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sin(x)^{-2n}\sec(x)\csc(x)\sin(2x)^{n+1}\left(\left(n+\sqrt{-b4^{-n}a+n^2}\right)\cot(x)^{-\frac{\sqrt{-b4^{-n}a+n^2}}{2}} - \cot(x)^{\frac{\sqrt{-b4^{-n}a+n^2}}{2}} c_3 (\right.}{2\left(c_3\cot(x)^{\frac{\sqrt{-b4^{-n}a+n^2}}{2}} + \cot(x)^{-\frac{\sqrt{-b4^{-n}a+n^2}}{2}}\right)a} \tag{1}$$

Verification of solutions

$$y = -\frac{\sin(x)^{-2n}\sec(x)\csc(x)\sin(2x)^{n+1}\left(\left(n+\sqrt{-b4^{-n}a+n^2}\right)\cot(x)^{-\frac{\sqrt{-b4^{-n}a+n^2}}{2}} - \cot(x)^{\frac{\sqrt{-b4^{-n}a+n^2}}{2}} c_3 (\right.}{2\left(c_3\cot(x)^{\frac{\sqrt{-b4^{-n}a+n^2}}{2}} + \cot(x)^{-\frac{\sqrt{-b4^{-n}a+n^2}}{2}}\right)a}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (sin(x)^2*n+cos(x)^2*n+sin(x)^
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Group is reducible or imprimitive
         <- Kovacics algorithm successful
         Change of variables used:
            [x = arccos(t)]
         Linear ODE actually solved:
            b*a*(-t^2+1)^n*t^(2*n)*u(t)+2^(2*n+2)*t^(2*n+1)*(-t^2+1)^(n+1)*(-3*t^2+n+1)*diff
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 232

```
dsolve(sin(2*x)^(n+1)*diff(y(x),x)=a*y(x)^2*sin(x)^(2*n)+b*cos(x)^(2*n),y(x), singsol=all)
```

$$y(x) = -\frac{\csc(x)\sin(2x)^n\left(\sin(x)^{\frac{\sqrt{n^2-4^{-n}ab}}{2}}\left(n+\sqrt{n^2-4^{-n}ab}\right)\cos(x)^{-\frac{\sqrt{n^2-4^{-n}ab}}{2}}-\cos(x)^{\frac{\sqrt{n^2-4^{-n}ab}}{2}}\sin(x)^{-\sqrt{n}}\right.}{a\left(\cos(x)^{\frac{\sqrt{n^2-4^{-n}ab}}{2}}\sin(x)^{-\frac{\sqrt{n^2-4^{-n}ab}}{2}}c_1+\cos(x)^{-\frac{\sqrt{n^2-4^{-n}ab}}{2}}\sin(x)\right.}$$

✓ Solution by Mathematica

Time used: 33.745 (sec). Leaf size: 132

```
DSolve[Sin[2*x]^(n+1)*y'[x]==a*y[x]^2*Sin[x]^(2*n)+b*Cos[x]^(2*n),y[x],x,IncludeSingularSolu
```

$$\text{Solve}\left[\int_1^{\sqrt{\frac{a\cos^{-2n}(x)\sin^{2n}(x)}{b}}y(x)}\frac{1}{K[1]^2-\sqrt{\frac{2^{2n+2}n^2}{ab}}K[1]+1}dK[1]=\frac{1}{2}b\sin^{-n}(2x)\cos^{2n}(x)\left(\log\left(\tan\left(\frac{x}{2}\right)\right)\right.\right.$$
$$\left.\left.-\log\left(\cos(x)\sec^2\left(\frac{x}{2}\right)\right)\right)\sqrt{\frac{a\sin^{2n}(x)\cos^{-2n}(x)}{b}}+c_1,y(x)\right]$$

13.7 problem 53

Internal problem ID [10551]
Internal file name [`OUTPUT/9498_Monday_June_06_2022_02_58_18_PM_50440844/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-5. Equations containing combinations of trigonometric functions.
Problem number: 53.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 + \tan(x)\, y = a(1-a)\cot(x)^2}$$

13.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -a^2\cot(x)^2 + \cot(x)^2 a - \tan(x)\, y + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a^2\cot(x)^2 + \cot(x)^2 a - \tan(x)\, y + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2\cot(x)^2 + \cot(x)^2 a$, $f_1(x) = -\tan(x)$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= -\tan(x) \\ f_2^2f_0 &= -a^2\cot(x)^2 + \cot(x)^2 a \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \tan(x)\,u'(x) + \left(-a^2\cot(x)^2 + \cot(x)^2 a\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1\sin(x)^a + c_2\sin(x)^{1-a}$$

The above shows that

$$u'(x) = \cot(x)\left(-c_2(-1+a)\sin(x)^{1-a} + c_1\sin(x)^a a\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\cot(x)\left(-c_2(-1+a)\sin(x)^{1-a} + c_1\sin(x)^a a\right)}{c_1\sin(x)^a + c_2\sin(x)^{1-a}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\cot(x)\,c_3\sin(x)^{2a}a + \cos(x)(-1+a)}{c_3\sin(x)^{2a} + \sin(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\cot(x)\,c_3\sin(x)^{2a}a + \cos(x)(-1+a)}{c_3\sin(x)^{2a} + \sin(x)} \tag{1}$$

Verification of solutions

$$y = \frac{-\cot(x) c_3 \sin(x)^{2a} a + \cos(x)(-1+a)}{c_3 \sin(x)^{2a} + \sin(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -tan(x)*(diff(y(x), x))+(a^2*c
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         <- LODE of Euler type successful
         Change of variables used:
            [x = arcsin(t)]
         Linear ODE actually solved:
            (a^2-a)*u(t)-t^2*diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x)=y(x)^2-y(x)*tan(x)+a*(1-a)*cot(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{-\cot(x)\sin(x)^{2a} a + c_1 \cos(x)(a-1)}{c_1 \sin(x) + \sin(x)^{2a}}$$

✓ Solution by Mathematica

Time used: 7.444 (sec). Leaf size: 230

```
DSolve[y'[x]==y[x]^2-y[x]*Tan[x]+a*(1-a)*Cot[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{i\cot(x)\left(\left(\sqrt{a-1}\sqrt{a}\sqrt{-\frac{(2a-1)^2}{(a-1)a}}-i\right)\left(-\sin^2(x)\right)^{\frac{1}{2}i\sqrt{a-1}\sqrt{a}\sqrt{-\frac{(2a-1)^2}{(a-1)a}}}-\left(\sqrt{a-1}\sqrt{a}\sqrt{-\frac{(2a-1)^2}{(a-1)a}}+i\right)c_1\right)}{2\left(\left(-\sin^2(x)\right)^{\frac{1}{2}i\sqrt{a-1}\sqrt{a}\sqrt{\frac{1}{a-a^2}-4}}+c_1\right)}$$

$$y(x) \to \frac{1}{2}i\left(\sqrt{a-1}\sqrt{a}\sqrt{\frac{1}{a-a^2}-4}+i\right)\cot(x)$$

13.8 problem 54

Internal problem ID [10552]
Internal file name [`OUTPUT/9499_Monday_June_06_2022_02_58_20_PM_18999849/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-5. Equations containing combinations of trigonometric functions.
Problem number: 54.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 + my\tan(x) = b^2\cos(x)^{2m}}$$

13.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 - my\tan(x) + b^2\cos(x)^{2m} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - my\tan(x) + b^2\cos(x)^{2m}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b^2\cos(x)^{2m}$, $f_1(x) = -m\tan(x)$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= -m\tan(x) \\ f_2^2 f_0 &= b^2\cos(x)^{2m} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + m\tan(x)\,u'(x) + b^2\cos(x)^{2m}\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = {} & c_1\sin\left(b\sqrt{\cos(x)^{2m}}\,\cos(x)^{-m}\sin(x)\,\mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{1}{2}-\frac{m}{2}\right],\left[\frac{3}{2}\right],\sin(x)^2\right)\right) \\ & + c_2\cos\left(b\sqrt{\cos(x)^{2m-2}}\,\cos(x)^{-m+1}\sin(x)\,\mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{1}{2}-\frac{m}{2}\right],\left[\frac{3}{2}\right],\sin(x)^2\right)\right) \end{aligned}$$

The above shows that

$$\begin{aligned} u'(x) = {} & b\cos(x)^{-m+1}\left(-\frac{\sin(x)^2(m-1)\,\mathrm{hypergeom}\left(\left[\frac{3}{2},\frac{3}{2}-\frac{m}{2}\right],\left[\frac{5}{2}\right],\sin(x)^2\right)}{3}\right. \\ & \left.+\mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{1}{2}-\frac{m}{2}\right],\left[\frac{3}{2}\right],\sin(x)^2\right)\right)\left(-\cos(x)\sqrt{\cos(x)^{2m-2}}\,\sin\left(b\sqrt{\cos(x)^{2m-2}}\,\cos(x)^{-m+1}\right.\right. \\ & \left.+\cos\left(b\sqrt{\cos(x)^{2m}}\,\cos(x)^{-m}\sin(x)\,\mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{1}{2}-\frac{m}{2}\right],\left[\frac{3}{2}\right],\sin(x)^2\right)\right)\sqrt{\cos(x)^{2m}}\,c_1\right) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{b\cos(x)^{-m+1}\left(-\frac{\sin(x)^2(m-1)\,\mathrm{hypergeom}\left(\left[\frac{3}{2},\frac{3}{2}-\frac{m}{2}\right],\left[\frac{5}{2}\right],\sin(x)^2\right)}{3}+\mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{1}{2}-\frac{m}{2}\right],\left[\frac{3}{2}\right],\sin(x)^2\right)\right)\left(-\right.}{c_1\sin\left(b\sqrt{\cos(x)^{2m}}\,\cos(x)^{-m}\,\mathrm{si}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\cos(x)\sqrt{\cos(x)^{2m-2}}\,\sin\left(b\sqrt{\cos(x)^{2m-2}}\,\cos(x)^{-m+1}\sin(x)\,\text{hypergeom}\left(\left[\frac{1}{2}, \frac{1}{2}-\frac{m}{2}\right], \left[\frac{3}{2}\right], \sin(x)^2\right)\right)}{c_3 \sin\left(b\sqrt{\cos(x)^{2m}}\,\cos(x)^{-m}\sin(x\right.}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\cos(x)\sqrt{\cos(x)^{2m-2}}\,\sin\left(b\sqrt{\cos(x)^{2m-2}}\,\cos(x)^{-m+1}\sin(x)\,\text{hypergeom}\left(\left[\frac{1}{2}, \frac{1}{2}-\frac{m}{2}\right], \left[\frac{3}{2}\right], \sin(x)^2\right)\right)}{c_3 \sin\left(b\sqrt{\cos(x)^{2m}}\,\cos(x)^{-m}\sin(x\right.} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\cos(x)\sqrt{\cos(x)^{2m-2}}\,\sin\left(b\sqrt{\cos(x)^{2m-2}}\,\cos(x)^{-m+1}\sin(x)\,\text{hypergeom}\left(\left[\frac{1}{2}, \frac{1}{2}-\frac{m}{2}\right], \left[\frac{3}{2}\right], \sin(x)^2\right)\right)}{c_3 \sin\left(b\sqrt{\cos(x)^{2m}}\,\cos(x)^{-m}\sin(x\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -m*tan(x)*(diff(y(x), x))-b^2*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         <- linear_1 successful
         Change of variables used:
            [x = arcsin(t)]
         Linear ODE actually solved:
            b^2*(-t^2+1)^m*u(t)+(m*t-t)*diff(u(t),t)+(-t^2+1)*diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 222

```
dsolve(diff(y(x),x)=y(x)^2-m*y(x)*tan(x)+b^2*cos(x)^(2*m),y(x), singsol=all)
```

$$y(x) = \frac{\left(-\frac{\sin(x)^2(m-1)\,\mathrm{hypergeom}\left(\left[\frac{3}{2},-\frac{m}{2}+\frac{3}{2}\right],\left[\frac{5}{2}\right],\sin(x)^2\right)}{3} + \mathrm{hypergeom}\left(\left[\frac{1}{2},-\frac{m}{2}+\frac{1}{2}\right],\left[\frac{3}{2}\right],\sin(x)^2\right)\right) b\cos(x)^{-m+1}\left(s\right.}{c_1\cos\left(b\sqrt{\cos(x)^{2m-2}}\,\cos(x)^{-m+}\right.}$$

✓ Solution by Mathematica

Time used: 4.179 (sec). Leaf size: 73

```
DSolve[y'[x]==y[x]^2-m*y[x]*Tan[x]+b^2*Cos[x]^(2*m),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{b^2}\cos^m(x)\tan\left(-\frac{\sqrt{b^2}\sqrt{\sin^2(x)}\csc(x)\cos^{m+1}(x)\,\mathrm{Hypergeometric2F1}\left(\frac{1}{2},\frac{m+1}{2},\frac{m+3}{2},\cos^2(x)\right)}{m+1} + c_1\right)$$

13.9 problem 55

Internal problem ID [10553]
Internal file name [`OUTPUT/9500_Monday_June_06_2022_02_58_37_PM_44320937/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-5. Equations containing combinations of trigonometric functions.
Problem number: 55.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - my\cot(x) = b^2 \sin(x)^{2m}}$$

13.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + my\cot(x) + b^2 \sin(x)^{2m} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + my\cot(x) + b^2 \sin(x)^{2m}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b^2 \sin(x)^{2m}$, $f_1(x) = \cot(x)\, m$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= 0\\
f_1f_2 &= \cot(x)\,m\\
f_2^2f_0 &= b^2\sin(x)^{2m}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \cot(x)\,mu'(x) + b^2\sin(x)^{2m}\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
u(x) &= c_1\sin\left(\sqrt{\left(\csc(x)^2\right)^{-m}\sin(x)^4}\,\csc(x)^2\left(\csc(x)^2\right)^{\frac{m}{2}}\cot(x)\operatorname{hypergeom}\left(\left[\frac{1}{2}, 1+\frac{m}{2}\right], \left[\frac{3}{2}\right], -\cot(x)^2\right)b\right)\\
&\quad + c_2\cos\left(\sqrt{\left(\csc(x)^2\right)^{-m}\sin(x)^4}\,\csc(x)^2\left(\csc(x)^2\right)^{\frac{m}{2}}\cot(x)\operatorname{hypergeom}\left(\left[\frac{1}{2}, 1+\frac{m}{2}\right], \left[\frac{3}{2}\right], -\cot(x)^2\right)b\right)
\end{aligned}$$

The above shows that

$$u'(x) = \frac{b\left(\csc(x)^2\right)^{-\frac{m}{2}}\left(-\frac{\cot(x)^2(2+m)\operatorname{hypergeom}\left(\left[\frac{3}{2}, 2+\frac{m}{2}\right], \left[\frac{5}{2}\right], -\cot(x)^2\right)}{3} + \operatorname{hypergeom}\left(\left[\frac{1}{2}, 1+\frac{m}{2}\right], \left[\frac{3}{2}\right], -\cot(x)^2\right)\right)\left(\right.}{}$$

Using the above in (1) gives the solution

$$y = -\frac{b\left(\csc(x)^2\right)^{-\frac{m}{2}}\left(-\frac{\cot(x)^2(2+m)\operatorname{hypergeom}\left(\left[\frac{3}{2}, 2+\frac{m}{2}\right], \left[\frac{5}{2}\right], -\cot(x)^2\right)}{3} + \operatorname{hypergeom}\left(\left[\frac{1}{2}, 1+\frac{m}{2}\right], \left[\frac{3}{2}\right], -\cot(x)^2\right)\right.}{\sqrt{\left(\csc(x)^2\right)^{-m}\sin(x)^4}\left(c_1\sin\left(\sqrt{\left(\csc(x)^2\right)^{-m}\sin(x)^4}\,\csc(\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-\sin\left(\sqrt{\left(\csc\left(x\right)^{2}\right)^{-m}\sin\left(x\right)^{4}}\,\csc\left(x\right)^{2}\left(\csc\left(x\right)^{2}\right)^{\frac{m}{2}}\cot\left(x\right)\mathrm{hypergeom}\left(\left[\frac{1}{2}, 1+\frac{m}{2}\right], \left[\frac{3}{2}\right], -\cot\left(x\right)^{2}\right)b\right)\right.}{\left(c_3\sin\left(\sqrt{\left(\csc\left(x\right)^{2}\right)^{-m}\sin\left(x\right)^{4}}\,\csc\left(x\right)^{2}\left(\csc\left(x\right)^{2}\right)^{\frac{m}{2}}\cot\left(x\right)\mathrm{hy}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-\sin\left(\sqrt{\left(\csc\left(x\right)^{2}\right)^{-m}\sin\left(x\right)^{4}}\,\csc\left(x\right)^{2}\left(\csc\left(x\right)^{2}\right)^{\frac{m}{2}}\cot\left(x\right)\mathrm{hypergeom}\left(\left[\frac{1}{2}, 1+\frac{m}{2}\right], \left[\frac{3}{2}\right], -\cot\left(x\right)^{2}\right)b\right)\right.}{\left(c_3\sin\left(\sqrt{\left(\csc\left(x\right)^{2}\right)^{-m}\sin\left(x\right)^{4}}\,\csc\left(x\right)^{2}\left(\csc\left(x\right)^{2}\right)^{\frac{m}{2}}\cot\left(x\right)\mathrm{hy}\right.\right.} \tag{1}$$

Verification of solutions

$$y = \frac{\left(-\sin\left(\sqrt{\left(\csc\left(x\right)^{2}\right)^{-m}\sin\left(x\right)^{4}}\,\csc\left(x\right)^{2}\left(\csc\left(x\right)^{2}\right)^{\frac{m}{2}}\cot\left(x\right)\mathrm{hypergeom}\left(\left[\frac{1}{2}, 1+\frac{m}{2}\right], \left[\frac{3}{2}\right], -\cot\left(x\right)^{2}\right)b\right)\right.}{\left(c_3\sin\left(\sqrt{\left(\csc\left(x\right)^{2}\right)^{-m}\sin\left(x\right)^{4}}\,\csc\left(x\right)^{2}\left(\csc\left(x\right)^{2}\right)^{\frac{m}{2}}\cot\left(x\right)\mathrm{hy}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = m*cot(x)*(diff(y(x), x))-b^2*s
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         <- linear_1 successful
         Change of variables used:
            [x = arccot(t)]
         Linear ODE actually solved:
            b^2*u(t)+t*(m*t^2+2*t^2+m+2)*(t^2+1)^m*diff(u(t),t)+(t^4+2*t^2+1)*(t^2+1)^m*diff
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 281

```
dsolve(diff(y(x),x)=y(x)^2+m*y(x)*cot(x)+b^2*sin(x)^(2*m),y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{\left(\csc(x)^2\right)^{-m}\sin(x)^4}\left(-c_1\sin\left(\sqrt{\left(\csc(x)^2\right)^{-m}\sin(x)^4}\,\csc(x)^2\left(\csc(x)^2\right)^{\frac{m}{2}}\cot(x)\,\mathrm{hypergeom}\left(\left[\frac{1}{2}, 1 \right.\right.\right.\right.}{c_1\cos\left(\sqrt{\left(\csc(x)^2\right)^{-m}\sin(x)^4}\right.}$$

✓ Solution by Mathematica

Time used: 5.352 (sec). Leaf size: 72

```
DSolve[y'[x]==y[x]^2+m*y[x]*Cot[x]+b^2*Sin[x]^(2*m),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{b^2}\sin^m(x)\tan\left(\frac{\sqrt{b^2}\sqrt{\cos^2(x)}\sec(x)\sin^{m+1}(x)\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{m+1}{2},\frac{m+3}{2},\sin^2(x)\right)}{m+1} + c_1\right)$$

13.10 problem 56

Internal problem ID [10554]
Internal file name [`OUTPUT/9501_Monday_June_06_2022_02_59_13_PM_29078961/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-5. Equations containing combinations of trigonometric functions.
Problem number: 56.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$\boxed{y' - y^2 = -2\lambda^2 \tan(x)^2 - 2\lambda^2 \cot(\lambda x)^2}$$

13.10.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 - 2\lambda^2 \tan(x)^2 - 2\lambda^2 \cot(\lambda x)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - 2\lambda^2 \tan(x)^2 - 2\lambda^2 \cot(\lambda x)^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -2\lambda^2 \tan(x)^2 - 2\lambda^2 \cot(\lambda x)^2$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= -2\lambda^2\tan(x)^2 - 2\lambda^2\cot(\lambda x)^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-2\lambda^2\tan(x)^2 - 2\lambda^2\cot(\lambda x)^2\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2-2*lambda^2*tan(x)^2-2*lambda^2*cot(lambda*x)^2,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2-2*\[Lambda]^2*Tan[x]^2-2*\[Lambda]^2*Cot[\[Lambda]*x]^2,y[x],x,IncludeS
```

Not solved

13.11 problem 57

Internal problem ID [10555]
Internal file name [`OUTPUT/9502_Monday_June_06_2022_02_59_50_PM_45559588/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-5. Equations containing combinations of trigonometric functions.
Problem number: 57.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = 2ab + \lambda a + b\lambda + a(\lambda - a)\tan(\lambda x)^2 + b(\lambda - b)\cot(\lambda x)^2}$$

13.11.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -b^2\cot(\lambda x)^2 + b\cot(\lambda x)^2\lambda - a^2\tan(\lambda x)^2 + a\tan(\lambda x)^2\lambda + 2ab + \lambda a + b\lambda + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -b^2\cot(\lambda x)^2 + b\cot(\lambda x)^2\lambda - a^2\tan(\lambda x)^2 + a\tan(\lambda x)^2\lambda + 2ab + \lambda a + b\lambda + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -b^2\cot(\lambda x)^2 + b\cot(\lambda x)^2\lambda - a^2\tan(\lambda x)^2 + a\tan(\lambda x)^2\lambda + 2ab + \lambda a + b\lambda$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -b^2 \cot(\lambda x)^2 + b\cot(\lambda x)^2 \lambda - a^2 \tan(\lambda x)^2 + a\tan(\lambda x)^2 \lambda + 2ab + \lambda a + b\lambda \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-b^2 \cot(\lambda x)^2 + b\cot(\lambda x)^2 \lambda - a^2 \tan(\lambda x)^2 + a\tan(\lambda x)^2 \lambda + 2ab + \lambda a + b\lambda\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) &= c_1 \cos(\lambda x)^{\frac{a}{\lambda}} \sin(\lambda x)^{\frac{b}{\lambda}} \\ &\quad + c_2 \cos(\lambda x)^{\frac{\lambda-a}{\lambda}} \sin(\lambda x)^{\frac{\lambda-b}{\lambda}} \operatorname{hypergeom}\left(\left[1, \frac{-b+\lambda-a}{\lambda}\right], \left[-\frac{-3\lambda+2a}{2\lambda}\right], \cos(\lambda x)^2\right) \end{aligned}$$

The above shows that

$$u'(x) = \frac{-4\cos(\lambda x)^{\frac{2\lambda-a}{\lambda}} \sin(\lambda x)^{\frac{2\lambda-b}{\lambda}} c_2 \lambda(a+b-\lambda) \operatorname{hypergeom}\left(\left[2, \frac{2\lambda-a-b}{\lambda}\right], \left[-\frac{-5\lambda+2a}{2\lambda}\right], \cos(\lambda x)^2\right) - 2\left(-\frac{3\lambda}{2} + \cdots\right.}{\cdots}$$

Using the above in (1) gives the solution

$$y = -\frac{-4\cos(\lambda x)^{\frac{2\lambda-a}{\lambda}} \sin(\lambda x)^{\frac{2\lambda-b}{\lambda}} c_2 \lambda(a+b-\lambda) \operatorname{hypergeom}\left(\left[2, \frac{2\lambda-a-b}{\lambda}\right], \left[-\frac{-5\lambda+2a}{2\lambda}\right], \cos(\lambda x)^2\right) - 2\left(-\frac{3\lambda}{2} \cdots\right.}{(-3\lambda+2a)\left(c_1 \cos(\lambda \cdots\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{4\lambda\cos(\lambda x)^2 \sin(\lambda x)^2 (a+b-\lambda) \operatorname{hypergeom}\left(\left[2, \frac{2\lambda-a-b}{\lambda}\right], \left[-\frac{-5\lambda+2a}{2\lambda}\right], \cos(\lambda x)^2\right) + \left((6\lambda^2 + (-7a-3b)\cdots\right.}{(-3\lambda+2a)\left(c_3 \cos(\lambda x)\cdots\right.}$$

Summary
The solution(s) found are the following

$$y = \frac{4\lambda \cos(\lambda x)^2 \sin(\lambda x)^2 (a+b-\lambda) \operatorname{hypergeom}\left(\left[2, \frac{2\lambda-a-b}{\lambda}\right], \left[-\frac{-5\lambda+2a}{2\lambda}\right], \cos(\lambda x)^2\right) + \left((6\lambda^2 + (-7a-3b)\right.}{(-3\lambda+2a)\left(c_3 \cos(\lambda x)\right.} \tag{1}$$

Verification of solutions

$$y = \frac{4\lambda \cos(\lambda x)^2 \sin(\lambda x)^2 (a+b-\lambda) \operatorname{hypergeom}\left(\left[2, \frac{2\lambda-a-b}{\lambda}\right], \left[-\frac{-5\lambda+2a}{2\lambda}\right], \cos(\lambda x)^2\right) + \left((6\lambda^2 + (-7a-3b)\right.}{(-3\lambda+2a)\left(c_3 \cos(\lambda x)\right.}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a^2*tan(lambda*x)^2-a*tan(lam
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               -> Whittaker
                  -> hyper3: Equivalence to 1F1 under a power @ Moebius
               -> hypergeometric
                  -> heuristic approach
                  <- heuristic approach successful
               <- hypergeometric successful
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 268

```
dsolve(diff(y(x),x)=y(x)^2+lambda*a+lambda*b+2*a*b+a*(lambda-a)*tan(lambda*x)^2+b*(lambda-b)
```

$$y(x) = \frac{4c_1\lambda\cos(x\lambda)^2\sin(x\lambda)^2(b-\lambda+a)\,\text{hypergeom}\left(\left[2, \frac{2\lambda-b-a}{\lambda}\right], \left[-\frac{2a-5\lambda}{2\lambda}\right], \cos(x\lambda)^2\right) - 2c_1\left(\left(-3\lambda^2+\left(\frac{7a}{2}+\right.\right.\right.}{(2a-3\lambda)\left(c_1\cos(x\lambda)\sin\right.}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+\[Lambda]*a+\[Lambda]*b+2*a*b+a*(\[Lambda]-a)*Tan[\[Lambda]*x]^2+b*(\[L
```

Not solved

13.12 problem 58

Internal problem ID [10556]
Internal file name [`OUTPUT/9503_Monday_June_06_2022_02_59_53_PM_14938955/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-5. Equations containing combinations of trigonometric functions.
Problem number: 58.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = -\frac{\lambda^2}{2} - \frac{3\lambda^2 \tan(\lambda x)^2}{4} + a\cos(\lambda x)^2 \sin(\lambda x)^n}$$

13.12.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 - \frac{\lambda^2}{2} - \frac{3\lambda^2 \tan(\lambda x)^2}{4} + a\cos(\lambda x)^2 \sin(\lambda x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - \frac{\lambda^2}{2} - \frac{3\lambda^2 \tan(\lambda x)^2}{4} + a\cos(\lambda x)^2 \sin(\lambda x)^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{\lambda^2}{2} - \frac{3\lambda^2 \tan(\lambda x)^2}{4} + a\cos(\lambda x)^2 \sin(\lambda x)^n$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -\frac{\lambda^2}{2} - \frac{3\lambda^2 \tan(\lambda x)^2}{4} + a\cos(\lambda x)^2 \sin(\lambda x)^n \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-\frac{\lambda^2}{2} - \frac{3\lambda^2 \tan(\lambda x)^2}{4} + a\cos(\lambda x)^2 \sin(\lambda x)^n\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{-\csc\left(\frac{\pi(n+3)}{2+n}\right) c_1 \operatorname{BesselI}\left(-\frac{1}{2+n}, 2\sqrt{-\frac{a\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) \pi \left(-\frac{a\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{1}{4+2n}} + c_2 \sin(\lambda x) \operatorname{BesselI}\left(\frac{1}{2+n}, 2\sqrt{-\right.}{\sqrt{\cos(\lambda x)}\,(2+n)\,\Gamma\left(\frac{n+3}{2+n}\right)}$$

The above shows that

$$u'(x) = \frac{\left(2\Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{a\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} c_2 \cos(\lambda x)^2 (2+n)^2 \operatorname{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{a\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) + c_2 \operatorname{BesselI}\left(\frac{1}{2+n}, 2\sqrt{\right.\right.}{}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(2\Gamma\left(\frac{n+3}{2+n}\right)^2 \left(-\frac{a\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} c_2 \cos(\lambda x)^2 (2+n)^2 \operatorname{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{a\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) + c_2 \operatorname{BesselI}\left(\frac{1}{2+n},\right.\right.}{2\cos(\lambda x)\left(-\csc\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-2\Gamma\left(\frac{n+3}{2+n}\right)^2 \sin\left(\frac{\pi(n+3)}{2+n}\right) \left(-\frac{a\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} \cos(\lambda x)(2+n)^2 \operatorname{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{a\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) + 2\left(-\frac{a \sin}{\lambda^2}\right.\right.}{-2c_3 \operatorname{Bes}}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-2\Gamma\left(\frac{n+3}{2+n}\right)^2 \sin\left(\frac{\pi(n+3)}{2+n}\right) \left(-\frac{a\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} \cos(\lambda x)(2+n)^2 \operatorname{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{a\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) + 2\left(-\frac{a \sin}{\lambda^2}\right.\right.}{-2c_3 \operatorname{Bes}} \tag{1}$$

Verification of solutions

$$y = \frac{\left(-2\Gamma\left(\frac{n+3}{2+n}\right)^2 \sin\left(\frac{\pi(n+3)}{2+n}\right) \left(-\frac{a\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}\right)^{\frac{n+1}{4+2n}} \cos(\lambda x)(2+n)^2 \operatorname{BesselI}\left(\frac{n+3}{2+n}, 2\sqrt{-\frac{a\sin(\lambda x)^{2+n}}{\lambda^2(2+n)^2}}\right) + 2\left(-\frac{a \sin}{\lambda^2}\right.\right.}{-2c_3 \operatorname{Bes}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = ((1/2)*lambda^2+(3/4)*lambda^2
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying an equivalence, under non-integer power transformations,
            to LODEs admitting Liouvillian solutions.
            -> Trying a Liouvillian solution using Kovacics algorithm
            <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
               <- hyper3 successful: received ODE is equivalent to the 0F1 ODE
            <- Whittaker successful
         <- special function solution successful
         Change of variables used:
            [x = arccos(t)/lambda]
         Linear ODE actually solved:
            (4*a*(-t^2+1)^(1/2*n)*t^4+lambda^2*t^2-3*lambda^2)*u(t)-4*t^3*lambda^2*diff(u(t)
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 1194

```
dsolve(diff(y(x),x)=y(x)^2-1/2*lambda^2-3/4*lambda^2*tan(lambda*x)^2+a*cos(lambda*x)^2*sin(l
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2-1/2*\[Lambda]^2-3/4*\[Lambda]^2*Tan[\[Lambda]*x]^2+a*Cos[\[Lambda]*x]^2
```

Not solved

13.13 problem 59

Internal problem ID [10557]
Internal file name [`OUTPUT/9504_Monday_June_06_2022_03_00_31_PM_30962163/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.6-5. Equations containing combinations of trigonometric functions.
Problem number: 59.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \sin(\lambda x)\, y^2 - a \sin(\lambda x)\, y = -a \tan(\lambda x)}$$

13.13.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \lambda \sin(\lambda x)\, y^2 + a \sin(\lambda x)\, y - a \tan(\lambda x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \lambda \sin(\lambda x)\, y^2 + a \sin(\lambda x)\, y - a \tan(\lambda x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a \tan(\lambda x)$, $f_1(x) = a \sin(\lambda x)$ and $f_2(x) = \lambda \sin(\lambda x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\lambda \sin(\lambda x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \lambda^2 \cos(\lambda x) \\ f_1 f_2 &= a \sin(\lambda x)^2 \lambda \\ f_2^2 f_0 &= -a \tan(\lambda x)\, \lambda^2 \sin(\lambda x)^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\lambda \sin(\lambda x)\, u''(x) - \left(a \sin(\lambda x)^2 \lambda + \lambda^2 \cos(\lambda x)\right) u'(x) - a \tan(\lambda x)\, \lambda^2 \sin(\lambda x)^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = -\mathrm{e}^{-\frac{a\cos(\lambda x)}{\lambda}} c_2 \lambda + \cos(\lambda x) \left(a\ \mathrm{expIntegral}_1\left(\frac{a\cos(\lambda x)}{\lambda}\right) c_2 + c_1\right)$$

The above shows that

$$u'(x) = -\lambda \sin(\lambda x) \left(a\ \mathrm{expIntegral}_1\left(\frac{a\cos(\lambda x)}{\lambda}\right) c_2 + c_1\right)$$

Using the above in (1) gives the solution

$$y = \frac{a\ \mathrm{expIntegral}_1\left(\frac{a\cos(\lambda x)}{\lambda}\right) c_2 + c_1}{-\mathrm{e}^{-\frac{a\cos(\lambda x)}{\lambda}} c_2 \lambda + \cos(\lambda x) \left(a\ \mathrm{expIntegral}_1\left(\frac{a\cos(\lambda x)}{\lambda}\right) c_2 + c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\mathrm{expIntegral}_1\left(\frac{a\cos(\lambda x)}{\lambda}\right) a + c_3}{-\mathrm{e}^{-\frac{a\cos(\lambda x)}{\lambda}} \lambda + \cos(\lambda x) \left(\mathrm{expIntegral}_1\left(\frac{a\cos(\lambda x)}{\lambda}\right) a + c_3\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{\operatorname{expIntegral}_1\left(\frac{a \cos(\lambda x)}{\lambda}\right) a + c_3}{-\mathrm{e}^{-\frac{a \cos(\lambda x)}{\lambda}} \lambda + \cos(\lambda x) \left(\operatorname{expIntegral}_1\left(\frac{a \cos(\lambda x)}{\lambda}\right) a + c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\operatorname{expIntegral}_1\left(\frac{a \cos(\lambda x)}{\lambda}\right) a + c_3}{-\mathrm{e}^{-\frac{a \cos(\lambda x)}{\lambda}} \lambda + \cos(\lambda x) \left(\operatorname{expIntegral}_1\left(\frac{a \cos(\lambda x)}{\lambda}\right) a + c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (sin(lambda*x)^2*a+lambda*cos(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         <- linear_1 successful
         Change of variables used:
            [x = arccos(t)/lambda]
         Linear ODE actually solved:
            (-2*a*t^4+4*a*t^2-2*a)*u(t)+(2*a*t^5-4*a*t^3+2*a*t)*diff(u(t),t)+(2*lambda*t^5-4
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 61

```
dsolve(diff(y(x),x)=lambda*sin(lambda*x)*y(x)^2+a*sin(lambda*x)*y(x)-a*tan(lambda*x),y(x), s
```

$$y(x) = \frac{\operatorname{expIntegral}_1\left(\frac{a\cos(x\lambda)}{\lambda}\right) c_1 a + 1}{\cos(x\lambda)\operatorname{expIntegral}_1\left(\frac{a\cos(x\lambda)}{\lambda}\right) c_1 a - \mathrm{e}^{-\frac{a\cos(x\lambda)}{\lambda}} c_1 \lambda + \cos(x\lambda)}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==\[Lambda]*Sin[\[Lambda]*x]*y[x]^2+a*Sin[\[Lambda]*x]*y[x]-a*Tan[\[Lambda]*x],y
```

Not solved

14 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-1. Equations containing arcsine.

14.1 problem 1

Internal problem ID [10558]
Internal file name [`OUTPUT/9505_Monday_June_06_2022_03_00_33_PM_57882991/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-1. Equations containing arcsine.
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 - \lambda \arcsin(x)^n y = -a^2 + a\lambda \arcsin(x)^n}$$

14.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + \lambda \arcsin(x)^n y - a^2 + a\lambda \arcsin(x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \lambda \arcsin(x)^n y - a^2 + a\lambda \arcsin(x)^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2 + a\lambda \arcsin(x)^n$, $f_1(x) = \arcsin(x)^n \lambda$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \arcsin(x)^n \lambda \\ f_2^2 f_0 &= -a^2 + a\lambda \arcsin(x)^n \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \arcsin(x)^n \lambda u'(x) + \left(-a^2 + a\lambda \arcsin(x)^n\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\int \frac{a\left(\int \mathrm{e}^{-\left(\int \left(-\arcsin(x)^n\lambda+2a\right)dx\right)}dx\right)-c_1 a+\mathrm{e}^{-\left(\int \left(-\arcsin(x)^n\lambda+2a\right)dx\right)}}{-c_1+\int \mathrm{e}^{-\left(\int \left(-\arcsin(x)^n\lambda+2a\right)dx\right)}dx}dx} c_2$$

The above shows that

$$u'(x) = \frac{\left(a\left(\int \mathrm{e}^{-\left(\int \left(-\arcsin(x)^n\lambda+2a\right)dx\right)}dx\right) - c_1 a + \mathrm{e}^{-\left(\int \left(-\arcsin(x)^n\lambda+2a\right)dx\right)}\right) \mathrm{e}^{\int \frac{a\left(\int \mathrm{e}^{-\left(\int \left(-\arcsin(x)^n\lambda+2a\right)dx\right)}dx\right)-c_1 a+\mathrm{e}^{-(\int (-\mathrm{arc}}}{-c_1+\int \mathrm{e}^{-\left(\int \left(-\arcsin(x)^n\lambda+2a\right)dx\right)}dx}}}{-c_1+\int \mathrm{e}^{-\left(\int \left(-\arcsin(x)^n\lambda+2a\right)dx\right)}dx}$$

Using the above in (1) gives the solution

$$y = -\frac{a\left(\int \mathrm{e}^{-\left(\int \left(-\arcsin(x)^n\lambda+2a\right)dx\right)}dx\right) - c_1 a + \mathrm{e}^{-\left(\int \left(-\arcsin(x)^n\lambda+2a\right)dx\right)}}{-c_1+\int \mathrm{e}^{-\left(\int \left(-\arcsin(x)^n\lambda+2a\right)dx\right)}dx}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-a\left(\int \mathrm{e}^{-\left(\int \left(-\arcsin(x)^n\lambda+2a\right)dx\right)}dx\right) + c_3 a - \mathrm{e}^{-\left(\int \left(-\arcsin(x)^n\lambda+2a\right)dx\right)}}{-c_3+\int \mathrm{e}^{-\left(\int \left(-\arcsin(x)^n\lambda+2a\right)dx\right)}dx}$$

Summary
The solution(s) found are the following

$$y = \frac{-a\left(\int e^{-\left(\int \left(-\arcsin(x)^n \lambda+2a\right)dx\right)}dx\right)+c_3 a-e^{-\left(\int \left(-\arcsin(x)^n \lambda+2a\right)dx\right)}}{-c_3+\int e^{-\left(\int \left(-\arcsin(x)^n \lambda+2a\right)dx\right)}dx} \tag{1}$$

Verification of solutions

$$y = \frac{-a\left(\int e^{-\left(\int \left(-\arcsin(x)^n \lambda+2a\right)dx\right)}dx\right)+c_3 a-e^{-\left(\int \left(-\arcsin(x)^n \lambda+2a\right)dx\right)}}{-c_3+\int e^{-\left(\int \left(-\arcsin(x)^n \lambda+2a\right)dx\right)}dx}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = lambda*arcsin(x)^n*(diff(y(x),
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(y(x)^2+y(x)+lambda*arcsin(x)^n*y(x)*x+x^
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 71

```
dsolve(diff(y(x),x)=y(x)^2+lambda*arcsin(x)^n*y(x)-a^2+a*lambda*arcsin(x)^n,y(x), singsol=al
```

$$y(x) = \frac{-c_1 a - a\left(\int \mathrm{e}^{-\left(\int (-\lambda \arcsin(x)^n + 2a)dx\right)}dx\right) - \mathrm{e}^{-\left(\int (-\lambda \arcsin(x)^n + 2a)dx\right)}}{c_1 + \int \mathrm{e}^{-\left(\int (-\lambda \arcsin(x)^n + 2a)dx\right)}dx}$$

✓ Solution by Mathematica

Time used: 6.556 (sec). Leaf size: 398

```
DSolve[y'[x]==y[x]^2+\[Lambda]*ArcSin[x]^n*y[x]-a^2+a*\[Lambda]*ArcSin[x]^n,y[x],x,IncludeSi
```

$$\text{Solve}\Bigg[\int_1^x$$

$$-\frac{\exp\left(\frac{1}{2}i\lambda \arcsin(K[1])^n \left(\arcsin(K[1])^2\right)^{-n}\left((-i\arcsin(K[1]))^n\Gamma(n+1, i\arcsin(K[1])) - (i\arcsin(K[1]))^n\right.\right.}{n\lambda(a+y(x))}$$

$$+\int_1^{y(x)}\left(\frac{\exp\left(\frac{1}{2}i\lambda \arcsin(x)^n \left(\arcsin(x)^2\right)^{-n}\left((-i\arcsin(x))^n\Gamma(n+1, i\arcsin(x)) - (i\arcsin(x))^n\Gamma(n+1,\right.\right.}{n\lambda(a+K[2])^2}\right.$$

$$-\int_1^x\left(\frac{\exp\left(\frac{1}{2}i\lambda \arcsin(K[1])^n \left(\arcsin(K[1])^2\right)^{-n}\left((-i\arcsin(K[1]))^n\Gamma(n+1, i\arcsin(K[1])) - (i\arcsin(\right.\right.}{n\lambda(a+K[2])^2}\right.$$

14.2 problem 2

Internal problem ID [10559]
Internal file name [`OUTPUT/9506_Monday_June_06_2022_03_00_35_PM_80242153/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-1. Equations containing arcsine.
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - \lambda x \arcsin(x)^n y = \arcsin(x)^n \lambda}$$

14.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + \arcsin(x)^n \lambda x y + \arcsin(x)^n \lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \arcsin(x)^n \lambda x y + \arcsin(x)^n \lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \arcsin(x)^n \lambda$, $f_1(x) = \arcsin(x)^n \lambda x$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= \arcsin(x)^n \lambda x \\ f_2^2 f_0 &= \arcsin(x)^n \lambda \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \arcsin(x)^n \lambda x u'(x) + \arcsin(x)^n \lambda u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x\left(c_1\left(\int e^{\int \frac{\arcsin(x)^n \lambda x^2 - 2}{x}dx}dx\right) + c_2\right)$$

The above shows that

$$u'(x) = c_1\left(\int e^{\int \frac{\arcsin(x)^n \lambda x^2 - 2}{x}dx}dx\right) + c_2 + xc_1 e^{\int \frac{\arcsin(x)^n \lambda x^2 - 2}{x}dx}$$

Using the above in (1) gives the solution

$$y = -\frac{c_1\left(\int e^{\int \frac{\arcsin(x)^n \lambda x^2 - 2}{x}dx}dx\right) + c_2 + xc_1 e^{\int \frac{\arcsin(x)^n \lambda x^2 - 2}{x}dx}}{x\left(c_1\left(\int e^{\int \frac{\arcsin(x)^n \lambda x^2 - 2}{x}dx}dx\right) + c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3\left(\int e^{\int \frac{\arcsin(x)^n \lambda x^2 - 2}{x}dx}dx\right) - 1 - xc_3 e^{\int \frac{\arcsin(x)^n \lambda x^2 - 2}{x}dx}}{x\left(c_3\left(\int e^{\int \frac{\arcsin(x)^n \lambda x^2 - 2}{x}dx}dx\right) + 1\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_3\left(\int e^{\int \frac{\arcsin(x)^n \lambda\, x^2-2}{x}dx}dx\right)-1-x c_3 e^{\int \frac{\arcsin(x)^n \lambda\, x^2-2}{x}dx}}{x\left(c_3\left(\int e^{\int \frac{\arcsin(x)^n \lambda\, x^2-2}{x}dx}dx\right)+1\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-c_3\left(\int e^{\int \frac{\arcsin(x)^n \lambda\, x^2-2}{x}dx}dx\right)-1-x c_3 e^{\int \frac{\arcsin(x)^n \lambda\, x^2-2}{x}dx}}{x\left(c_3\left(\int e^{\int \frac{\arcsin(x)^n \lambda\, x^2-2}{x}dx}dx\right)+1\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 78

```
dsolve(diff(y(x),x)=y(x)^2+lambda*x*arcsin(x)^n*y(x)+lambda*arcsin(x)^n,y(x), singsol=all)
```

$$y(x) = \frac{e^{\int \frac{\lambda \arcsin(x)^n x^2-2}{x}dx} x+\int e^{\int \frac{\lambda \arcsin(x)^n x^2-2}{x}dx}dx-c_1}{\left(c_1-\left(\int e^{\int \frac{\lambda \arcsin(x)^n x^2-2}{x}dx}dx\right)\right)x}$$

✓ Solution by Mathematica

Time used: 4.147 (sec). Leaf size: 256

```
DSolve[y'[x]==y[x]^2+\[Lambda]*x*ArcSin[x]^n*y[x]+\[Lambda]*ArcSin[x]^n,y[x],x,IncludeSingul
```

$$y(x) \to$$

$$-\frac{\int_1^x \frac{\exp\left(-2^{-n-3}\lambda \arcsin(K[1])^n \left(\arcsin(K[1])^2\right)^{-n} (\Gamma(n+1,2i\arcsin(K[1]))(-i\arcsin(K[1]))^n+(i\arcsin(K[1]))^n\Gamma(n+1,-2i\arcsin(K[1]))}{K[1]^2}}{x\left(\int_1^x \frac{\exp\left(-2^{-n-3}\lambda \arcsin(K[1])^n (\arcsin(K[1])^2)^{-n} (\Gamma(n+1,2i\arcsin(K[1])\right)}{K[1]^2}\right.}$$

$$y(x) \to -\frac{1}{x}$$

14.3 problem 3

Internal problem ID [10560]
Internal file name [`OUTPUT/9507_Monday_June_06_2022_03_00_41_PM_35430875/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-1. Equations containing arcsine.
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' + (k+1)\, x^k y^2 - \lambda \arcsin(x)^n \left(x^{k+1} y - 1\right) = 0}$$

14.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x^{k+1} \arcsin(x)^n \lambda y - x^k y^2 k - x^k y^2 - \arcsin(x)^n \lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^k x \arcsin(x)^n \lambda y - x^k y^2 k - x^k y^2 - \arcsin(x)^n \lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\arcsin(x)^n \lambda$, $f_1(x) = x^{k+1} \arcsin(x)^n \lambda$ and $f_2(x) = -x^k k - x^k$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(-x^k k - x^k)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{k^2 x^k}{x} - \frac{k\, x^k}{x} \\ f_1 f_2 &= x^{k+1} \arcsin(x)^n \lambda\left(-x^k k - x^k\right) \\ f_2^2 f_0 &= -\left(-x^k k - x^k\right)^2 \arcsin(x)^n \lambda \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\left(-x^k k - x^k\right) u''(x) - \left(-\frac{k^2 x^k}{x} - \frac{k\, x^k}{x} + x^{k+1} \arcsin(x)^n \lambda\left(-x^k k - x^k\right)\right) u'(x) - \left(-x^k k - x^k\right)^2 \arcsin(x)$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{k+1}\left(\left(\int x^{-2k-2} \mathrm{e}^{\int \left(x^{k+1} \arcsin(x)^n \lambda + \frac{k}{x}\right) dx} dx\right) c_2 + c_1\right)$$

The above shows that

$$u'(x) = c_2 x^{-k-1} \mathrm{e}^{\int \frac{x^{k+2} \lambda \arcsin(x)^n + k}{x} dx} + (k+1)\left(\left(\int \mathrm{e}^{\int \frac{x^{k+2} \lambda \arcsin(x)^n + k}{x} dx} x^{-2k-2} dx\right) c_2 + c_1\right) x^k$$

Using the above in (1) gives the solution

$$y = -\frac{\left(c_2 x^{-k-1} \mathrm{e}^{\int \frac{x^{k+2} \lambda \arcsin(x)^n + k}{x} dx} + (k+1)\left(\left(\int \mathrm{e}^{\int \frac{x^{k+2} \lambda \arcsin(x)^n + k}{x} dx} x^{-2k-2} dx\right) c_2 + c_1\right) x^k\right) x^{-k-1}}{\left(-x^k k - x^k\right)\left(\left(\int x^{-2k-2} \mathrm{e}^{\int \left(x^{k+1} \arcsin(x)^n \lambda + \frac{k}{x}\right) dx} dx\right) c_2 + c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(k+1)\left(\int x^{-2k-2} \mathrm{e}^{\int \left(x^{k+1} \arcsin(x)^n \lambda + \frac{k}{x}\right) dx} dx + c_3\right) x^{-k-1} + x^{-3k-2} \mathrm{e}^{\int \left(x^{k+1} \arcsin(x)^n \lambda + \frac{k}{x}\right) dx}}{(k+1)\left(\int \mathrm{e}^{\int \frac{x^{k+2} \lambda \arcsin(x)^n + k}{x} dx} x^{-2k-2} dx + c_3\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{(k+1)\left(\int x^{-2k-2} e^{\int \left(x^{k+1} \arcsin(x)^{n} \lambda+\frac{k}{x}\right)dx} dx + c_3\right) x^{-k-1} + x^{-3k-2} e^{\int \left(x^{k+1} \arcsin(x)^{n} \lambda+\frac{k}{x}\right)dx}}{(k+1)\left(\int e^{\int \frac{x^{k+2} \lambda \arcsin(x)^{n}+k}{x}dx} x^{-2k-2} dx + c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{(k+1)\left(\int x^{-2k-2} e^{\int \left(x^{k+1} \arcsin(x)^{n} \lambda+\frac{k}{x}\right)dx} dx + c_3\right) x^{-k-1} + x^{-3k-2} e^{\int \left(x^{k+1} \arcsin(x)^{n} \lambda+\frac{k}{x}\right)dx}}{(k+1)\left(\int e^{\int \frac{x^{k+2} \lambda \arcsin(x)^{n}+k}{x}dx} x^{-2k-2} dx + c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (arcsin(x)^n*x^(1+k)*lambda*x+
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-((-x^k*k-x^k)*y(x)^2+y(x)+arcsin(x)^n*x^(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 180

```
dsolve(diff(y(x),x)=-(k+1)*x^k*y(x)^2+lambda*arcsin(x)^n*(x^(k+1)*y(x)-1),y(x), singsol=all)
```

$$y(x) = \frac{x^{-1-k}\left(x^{1+k}e^{\int \frac{\arcsin(x)^n x^{1+k}\lambda x-2k-2}{x}dx} + \left(\int x^k e^{\lambda\left(\int \arcsin(x)^n x^{1+k}dx\right)-2\left(\int \frac{1}{x}dx\right)(1+k)}dx\right)k + \int x^k e^{\lambda\left(\int \arcsin(x)^n x^{1+k}d\right.}\right.}{\left(\int x^k e^{\lambda\left(\int \arcsin(x)^n x^{1+k}dx\right)-2\left(\int \frac{1}{x}dx\right)(1+k)}dx\right)k + \int x^k e^{\lambda\left(\int \arcsin(x)^n x^{1+k}dx\right)-2\left(\int \frac{1}{x}dx\right)(1+k)}dx}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==-(k+1)*x^k*y[x]^2+\[Lambda]*ArcSin[x]^n*(x^(k+1)*y[x]-1),y[x],x,IncludeSingula
```

Not solved

14.4 problem 4

Internal problem ID [10561]
Internal file name [`OUTPUT/9508_Monday_June_06_2022_03_00_49_PM_11041830/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-1. Equations containing arcsine.
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \arcsin(x)^n y^2 - ay = ab - b^2\lambda \arcsin(x)^n}$$

14.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \lambda \arcsin(x)^n y^2 + ya + ab - b^2\lambda \arcsin(x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \lambda \arcsin(x)^n y^2 + ya + ab - b^2\lambda \arcsin(x)^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = ab - b^2\lambda \arcsin(x)^n$, $f_1(x) = a$ and $f_2(x) = \arcsin(x)^n \lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\arcsin(x)^n \lambda u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{\arcsin(x)^n\, n\lambda}{\sqrt{-x^2+1}\,\arcsin(x)} \\
f_1f_2 &= a\lambda\arcsin(x)^n \\
f_2^2f_0 &= \arcsin(x)^{2n}\,\lambda^2\left(ab - b^2\lambda\arcsin(x)^n\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\arcsin(x)^n\,\lambda u''(x) - \left(\frac{\arcsin(x)^n\, n\lambda}{\sqrt{-x^2+1}\,\arcsin(x)} + a\lambda\arcsin(x)^n\right)u'(x) + \arcsin(x)^{2n}\,\lambda^2\left(ab - b^2\lambda\arcsin(x)^n\right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
u(x) = \mathrm{DESol}\Bigg(\Bigg\{_Y''(x) - \frac{n_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)} - a_Y'(x) - \arcsin(x)^{2n}\,b^2\lambda^2_Y(x) \\
+ \arcsin(x)^n\,ab\lambda_Y(x)\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

The above shows that

$$\begin{aligned}
u'(x) = \frac{\partial}{\partial x}\,\mathrm{DESol}\Bigg(\Bigg\{_Y''(x) - \frac{n_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)} - a_Y'(x) \\
- \arcsin(x)^{2n}\,b^2\lambda^2_Y(x) + \arcsin(x)^n\,ab\lambda_Y(x)\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)} - a\_Y'(x) - \arcsin(x)^{2n}\,b^2\lambda^2\_Y(x) + \arcsin(x)^n\,ab\lambda\_Y(x)\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)} - a\_Y'(x) - \arcsin(x)^{2n}\,b^2\lambda^2\_Y(x) + \arcsin(x)^n\,ab\lambda\_\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\left(-\arcsin(x)^{1+2n}\_\_Y(x)b^2\lambda^2+\arcsin(x)^{n+1}\_\_Y(x)ab\lambda-\arcsin(x)\left(a\_\_Y'(x)-\_\_Y''(x)\right)\right)\sqrt{-x^2+1}-n\_\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)}\right\}\cdots\right)\right)}{\lambda\operatorname{DESol}\left(\left\{\frac{\left(-\arcsin(x)^{1+2n}\_\_Y(x)b^2\lambda^2+\arcsin(x)^{n+1}\_\_Y(x)ab\lambda-\arcsin(x)\left(a\_\_Y'(x)-\_\_Y''(x)\right)\right)\sqrt{-x^2+1}-n\_\_Y\cdots}{\sqrt{-x^2+1}\,\arcsin(x)}\right.\right.}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\left(-\arcsin(x)^{1+2n}\_\_Y(x)b^2\lambda^2+\arcsin(x)^{n+1}\_\_Y(x)ab\lambda-\arcsin(x)\left(a\_\_Y'(x)-\_\_Y''(x)\right)\right)\sqrt{-x^2+1}-n\_\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)}\right\}\cdots\right)\right)}{\lambda\operatorname{DESol}\left(\left\{\frac{\left(-\arcsin(x)^{1+2n}\_\_Y(x)b^2\lambda^2+\arcsin(x)^{n+1}\_\_Y(x)ab\lambda-\arcsin(x)\left(a\_\_Y'(x)-\_\_Y''(x)\right)\right)\sqrt{-x^2+1}-n\_\_Y\cdots}{\sqrt{-x^2+1}\,\arcsin(x)}\right.\right.} \tag{1}
$$

Verification of solutions

$$
y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\left(-\arcsin(x)^{1+2n}\_\_Y(x)b^2\lambda^2+\arcsin(x)^{n+1}\_\_Y(x)ab\lambda-\arcsin(x)\left(a\_\_Y'(x)-\_\_Y''(x)\right)\right)\sqrt{-x^2+1}-n\_\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)}\right\}\cdots\right)\right)}{\lambda\operatorname{DESol}\left(\left\{\frac{\left(-\arcsin(x)^{1+2n}\_\_Y(x)b^2\lambda^2+\arcsin(x)^{n+1}\_\_Y(x)ab\lambda-\arcsin(x)\left(a\_\_Y'(x)-\_\_Y''(x)\right)\right)\sqrt{-x^2+1}-n\_\_Y\cdots}{\sqrt{-x^2+1}\,\arcsin(x)}\right.\right.}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a*(-x^2+1)^(1/2)*arcsin(x)+n)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(lambda*arcsin(x)^n*y(x)^2+y(x)+y(x)*a*x+
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 87

```
dsolve(diff(y(x),x)=lambda*arcsin(x)^n*y(x)^2+a*y(x)+a*b-b^2*lambda*arcsin(x)^n,y(x), singso
```

$$y(x) = \frac{-b\lambda\left(\int \arcsin\left(x\right)^{n} \mathrm{e}^{-\left(\int (2\arcsin(x)^{n}\lambda b-a)dx\right)}dx\right) - c_1 b - \mathrm{e}^{-\left(\int (2\arcsin(x)^{n}\lambda b-a)dx\right)}}{c_1 + \lambda\left(\int \arcsin\left(x\right)^{n} \mathrm{e}^{-\left(\int (2\arcsin(x)^{n}\lambda b-a)dx\right)}dx\right)}$$

✓ Solution by Mathematica

Time used: 7.093 (sec). Leaf size: 428

```
DSolve[y'[x]==\[Lambda]*ArcSin[x]^n*y[x]^2+a*y[x]+a*b-b^2*\[Lambda]*ArcSin[x]^n,y[x],x,Inclu
```

$$\text{Solve}\left[\int_1^x \frac{i\exp\left(aK[1]-ib\lambda\arcsin(K[1])^n\left(\arcsin(K[1])^2\right)^{-n}\left((-i\arcsin(K[1]))^n\Gamma(n+1,i\arcsin(K[1]))\right.\right.}{an\lambda(b\ -}\right.$$

$$+\int_1^{y(x)}\left(-\int_1^x\left(\frac{i\exp\left(aK[1]-ib\lambda\arcsin(K[1])^n\left(\arcsin(K[1])^2\right)^{-n}\left((-i\arcsin(K[1]))^n\Gamma(n+1,i\arcsin(\right.\right.}{an(b+K[2])}\right.\right.$$

$$-\frac{i\exp\left(ax-ib\lambda\arcsin(x)^n\left(\arcsin(x)^2\right)^{-n}\left((-i\arcsin(x))^n\Gamma(n+1,i\arcsin(x))-(i\arcsin(x))^n\Gamma(n+1,-\right.\right.}{an\lambda(b+K[2])^2}$$

14.5 problem 5

Internal problem ID [10562]
Internal file name [`OUTPUT/9509_Monday_June_06_2022_03_00_54_PM_61372346/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-1. Equations containing arcsine.
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \arcsin(x)^n y^2 + b\lambda\, x^m \arcsin(x)^n y = bm\, x^{m-1}}$$

14.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \lambda \arcsin(x)^n y^2 - b\lambda\, x^m \arcsin(x)^n y + bm\, x^{m-1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \lambda \arcsin(x)^n y^2 - b\lambda\, x^m \arcsin(x)^n y + \frac{b\, x^m m}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = bm\, x^{m-1}$, $f_1(x) = -b\lambda\, x^m \arcsin(x)^n$ and $f_2(x) = \arcsin(x)^n \lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\arcsin(x)^n \lambda u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{\arcsin(x)^n n\lambda}{\sqrt{-x^2+1}\,\arcsin(x)} \\ f_1 f_2 &= -b\,\lambda^2 x^m \arcsin(x)^{2n} \\ f_2^2 f_0 &= \arcsin(x)^{2n} \lambda^2 bm\,x^{m-1} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\arcsin(x)^n \lambda u''(x) - \left(-b\,\lambda^2 x^m \arcsin(x)^{2n} + \frac{\arcsin(x)^n n\lambda}{\sqrt{-x^2+1}\,\arcsin(x)}\right) u'(x) + \arcsin(x)^{2n} \lambda^2 bm\,x^{m-1} u(x)$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) + b\,x^m \lambda \arcsin(x)^n \_Y'(x) - \frac{n\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)} + bm\,x^{m-1}\lambda\_Y(x)\arcsin(x)^n\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + b\,x^m \lambda \arcsin(x)^n \_Y'(x) - \frac{n\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)} + bm\,x^{m-1}\lambda\_Y(x)\arcsin(x)^n\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + b\,x^m \lambda \arcsin(x)^n \_Y'(x) - \frac{n\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)} + bm\,x^{m-1}\lambda\_Y(x)\arcsin(x)^n\right\},\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\_Y''(x) + b\,x^m \lambda \arcsin(x)^n \_Y'(x) - \frac{n\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)} + bm\,x^{m-1}\lambda\_Y(x)\arcsin(\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(b\lambda\left(m\_\_Y(x)x^{m-1}+\_\_Y'(x)x^{m}\right)\arcsin(x)^{n+1}+\arcsin(x)\_\_Y''(x)\right)\sqrt{-x^2+1}-n\_\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)}\right\},\{\_\_Y(x)\}\right)\right)\mathrm{ar}}{\lambda\,\mathrm{DESol}\left(\left\{\frac{\left(b\lambda\left(m\,x^{m}\_\_Y(x)+\_\_Y'(x)x^{m+1}\right)\arcsin(x)^{n+1}+\arcsin(x)\_\_Y''(x)x\right)\sqrt{-x^2+1}-\_\_Y'(x)xn}{\sqrt{-x^2+1}\,x\arcsin(x)}\right\},\{\_\_Y(x)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(b\lambda\left(m\_\_Y(x)x^{m-1}+\_\_Y'(x)x^{m}\right)\arcsin(x)^{n+1}+\arcsin(x)\_\_Y''(x)\right)\sqrt{-x^2+1}-n\_\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)}\right\},\{\_\_Y(x)\}\right)\right)\mathrm{ar}}{\lambda\,\mathrm{DESol}\left(\left\{\frac{\left(b\lambda\left(m\,x^{m}\_\_Y(x)+\_\_Y'(x)x^{m+1}\right)\arcsin(x)^{n+1}+\arcsin(x)\_\_Y''(x)x\right)\sqrt{-x^2+1}-\_\_Y'(x)xn}{\sqrt{-x^2+1}\,x\arcsin(x)}\right\},\{\_\_Y(x)} \quad (1)$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(b\lambda\left(m\_\_Y(x)x^{m-1}+\_\_Y'(x)x^{m}\right)\arcsin(x)^{n+1}+\arcsin(x)\_\_Y''(x)\right)\sqrt{-x^2+1}-n\_\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)}\right\},\{\_\_Y(x)\}\right)\right)\mathrm{ar}}{\lambda\,\mathrm{DESol}\left(\left\{\frac{\left(b\lambda\left(m\,x^{m}\_\_Y(x)+\_\_Y'(x)x^{m+1}\right)\arcsin(x)^{n+1}+\arcsin(x)\_\_Y''(x)x\right)\sqrt{-x^2+1}-\_\_Y'(x)xn}{\sqrt{-x^2+1}\,x\arcsin(x)}\right\},\{\_\_Y(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(x^m*(-x^2+1)^(1/2)*arcsin(x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(lambda*arcsin(x)^n*y(x)^2+y(x)-b*lambda*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=lambda*arcsin(x)^n*y(x)^2-b*lambda*x^m*arcsin(x)^n*y(x)+b*m*x^(m-1),y(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==\[Lambda]*ArcSin[x]^n*y[x]^2-b*\[Lambda]*x^m*ArcSin[x]^n*y[x]+b*m*x^(m-1),y[x]
```

Not solved

14.6 problem 6

Internal problem ID [10563]
Internal file name [`OUTPUT/9510_Monday_June_06_2022_03_01_01_PM_44105884/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-1. Equations containing arcsine.
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$y' - \lambda \arcsin(x)^n y^2 = \beta m x^{m-1} - \lambda \beta^2 x^{2m} \arcsin(x)^n$$

14.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \lambda \arcsin(x)^n y^2 + \beta m x^{m-1} - \lambda \beta^2 x^{2m} \arcsin(x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\lambda \beta^2 x^{2m} \arcsin(x)^n + \lambda \arcsin(x)^n y^2 + \frac{\beta m x^m}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \beta m x^{m-1} - \lambda \beta^2 x^{2m} \arcsin(x)^n$, $f_1(x) = 0$ and $f_2(x) = \arcsin(x)^n \lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\arcsin(x)^n \lambda u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{\arcsin(x)^n \, n\lambda}{\sqrt{-x^2+1}\, \arcsin(x)} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= \arcsin(x)^{2n} \lambda^2 \left(\beta m \, x^{m-1} - \lambda\, \beta^2 x^{2m} \arcsin(x)^n\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\arcsin(x)^n \, \lambda u''(x) - \frac{\arcsin(x)^n \, n\lambda u'(x)}{\sqrt{-x^2+1}\, \arcsin(x)} + \arcsin(x)^{2n} \lambda^2 \left(\beta m \, x^{m-1} - \lambda\, \beta^2 x^{2m} \arcsin(x)^n\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
u(x) = \mathrm{DESol}\Bigg(\Bigg\{_Y''(x) &- \frac{n_Y'(x)}{\sqrt{-x^2+1}\, \arcsin(x)} - x^{2m}\beta^2_Y(x)\, \lambda^2 \arcsin(x)^{2n} \\
&+ m\beta\, x^{m-1}\lambda_Y(x) \arcsin(x)^n\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

The above shows that

$$\begin{aligned}
u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\Bigg(\Bigg\{_Y''(x) &- \frac{n_Y'(x)}{\sqrt{-x^2+1}\, \arcsin(x)} - x^{2m}\beta^2_Y(x)\, \lambda^2 \arcsin(x)^{2n} \\
&+ m\beta\, x^{m-1}\lambda_Y(x) \arcsin(x)^n\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{\sqrt{-x^2+1}\, \arcsin(x)} - x^{2m}\beta^2\_Y(x)\, \lambda^2 \arcsin(x)^{2n} + m\beta\, x^{m-1}\lambda\_Y(x) \arcsin(x)^n\right.\right.\right.}{\lambda\, \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{\sqrt{-x^2+1}\, \arcsin(x)} - x^{2m}\beta^2\_Y(x)\, \lambda^2 \arcsin(x)^{2n} + m\beta\, x^{m-1}\lambda\_Y(x)\, \mathrm{arcsi}\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\left(-x^{2m}\arcsin(x)^{1+2n}\_\_Y(x)\beta^2\lambda^2+x^{m-1}\arcsin(x)^{n+1}\_\_Y(x)\beta\lambda m+\arcsin(x)\_\_Y''(x)\right)\sqrt{-x^2+1}-n\_\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)}\right\},\{\right.\right.}{\lambda\operatorname{DESol}\left(\left\{\frac{\left(-x^{1+2m}\beta^2\lambda^2\arcsin(x)^{1+2n}\_\_Y(x)+m\beta x^m\lambda\arcsin(x)^{n+1}\_\_Y(x)+\arcsin(x)\_\_Y''(x)x\right)\sqrt{-x^2+1}-\_\_Y'(x}{\sqrt{-x^2+1}\,x\arcsin(x)}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\left(-x^{2m}\arcsin(x)^{1+2n}\_\_Y(x)\beta^2\lambda^2+x^{m-1}\arcsin(x)^{n+1}\_\_Y(x)\beta\lambda m+\arcsin(x)\_\_Y''(x)\right)\sqrt{-x^2+1}-n\_\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)}\right\},\{\right.\right.}{\lambda\operatorname{DESol}\left(\left\{\frac{\left(-x^{1+2m}\beta^2\lambda^2\arcsin(x)^{1+2n}\_\_Y(x)+m\beta x^m\lambda\arcsin(x)^{n+1}\_\_Y(x)+\arcsin(x)\_\_Y''(x)x\right)\sqrt{-x^2+1}-\_\_Y'(x}{\sqrt{-x^2+1}\,x\arcsin(x)}\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\left(-x^{2m}\arcsin(x)^{1+2n}\_\_Y(x)\beta^2\lambda^2+x^{m-1}\arcsin(x)^{n+1}\_\_Y(x)\beta\lambda m+\arcsin(x)\_\_Y''(x)\right)\sqrt{-x^2+1}-n\_\_Y'(x)}{\sqrt{-x^2+1}\,\arcsin(x)}\right\},\{\right.\right.}{\lambda\operatorname{DESol}\left(\left\{\frac{\left(-x^{1+2m}\beta^2\lambda^2\arcsin(x)^{1+2n}\_\_Y(x)+m\beta x^m\lambda\arcsin(x)^{n+1}\_\_Y(x)+\arcsin(x)\_\_Y''(x)x\right)\sqrt{-x^2+1}-\_\_Y'(x}{\sqrt{-x^2+1}\,x\arcsin(x)}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = n*(diff(y(x), x))/((-x^2+1)^(1
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
         -> trying with_periodic_functions in the coefficients
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(lambda*arcsin(x)^n*y(x)^2+y(x)+x^2*(beta
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=lambda*arcsin(x)^n*y(x)^2+beta*m*x^(m-1)-lambda*beta^2*x^(2*m)*arcsin(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==\[Lambda]*ArcSin[x]^n*y[x]^2+\[Beta]*m*x^(m-1)-\[Lambda]*\[Beta]^2*x^(2*m)*Arc
```

Not solved

14.7 problem 7

Internal problem ID [10564]
Internal file name [`OUTPUT/9511_Monday_June_06_2022_03_01_08_PM_99759205/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-1. Equations containing arcsine.
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$\boxed{y' - \lambda \arcsin(x)^n (y - a\,x^m - b)^2 = am\,x^{m-1}}$$

14.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= x^{2m}\arcsin(x)^n a^2\lambda + 2x^m \arcsin(x)^n ab\lambda - 2x^m \arcsin(x)^n a\lambda y + b^2\lambda \arcsin(x)^n - 2\arcsin(x)^n b\lambda y\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^{2m}\arcsin(x)^n a^2\lambda+2x^m \arcsin(x)^n ab\lambda-2x^m \arcsin(x)^n a\lambda y+b^2\lambda \arcsin(x)^n-2\arcsin(x)^n b\lambda y+\lambda\,\mathrm{a}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^{2m}\arcsin(x)^n a^2\lambda+2x^m \arcsin(x)^n ab\lambda+b^2\lambda \arcsin(x)^n+am\,x^{m-1}$, $f_1(x) = -2a\lambda\,x^m \arcsin(x)^n - 2\arcsin(x)^n \lambda b$ and $f_2(x) = \arcsin(x)^n \lambda$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{\arcsin(x)^n \lambda u}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = \frac{\arcsin(x)^n n\lambda}{\sqrt{-x^2+1}\arcsin(x)}$$
$$f_1f_2 = (-2a\lambda\, x^m \arcsin(x)^n - 2\arcsin(x)^n \lambda b)\arcsin(x)^n \lambda$$
$$f_2^2 f_0 = \arcsin(x)^{2n} \lambda^2 \left(x^{2m} \arcsin(x)^n a^2\lambda + 2x^m \arcsin(x)^n ab\lambda + b^2\lambda \arcsin(x)^n + am\, x^{m-1}\right)$$

Substituting the above terms back in equation (2) gives

$$\arcsin(x)^n \lambda u''(x) - \left(\frac{\arcsin(x)^n n\lambda}{\sqrt{-x^2+1}\arcsin(x)} + (-2a\lambda\, x^m \arcsin(x)^n - 2\arcsin(x)^n \lambda b)\arcsin(x)^n \lambda\right) u'(x$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{-\frac{n\_Y'(x)}{\sqrt{-x^2+1}} + \arcsin(x)\left(\lambda^2\_Y(x)\left(a^2x^{2m} + 2ab\, x^m + b^2\right)\arcsin(x)^{2n} + \_Y''(x) + \arcsin(x)^n\right.}{\arcsin(x)}\right.\right.$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-\frac{n\_Y'(x)}{\sqrt{-x^2+1}} + \lambda^2\_Y(x)\left(a^2x^{2m} + 2ab\, x^m + b^2\right)\arcsin(x)^{1+2n} + \left(a\, x^{m-1}m\_Y(x) + 2\_Y'(x\right.}{\arcsin(x)}\right.\right.$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-\frac{n\_Y'(x)}{\sqrt{-x^2+1}}+\lambda^2\_Y(x)(a^2x^{2m}+2ab\,x^m+b^2)\arcsin(x)^{1+2n}+\left(a\,x^{m-1}m\_Y(x)+2\_Y'(x)(a\,x^m+b)\right)\lambda\arcsin(x)^n}{\arcsin(x)}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{-\frac{n\_Y'(x)}{\sqrt{-x^2+1}}+\arcsin(x)\left(\lambda^2\_Y(x)(a^2x^{2m}+2ab\,x^m+b^2)\arcsin(x)^{2n}+\_Y''(x)+\arcsin(x)^n am\,x^{m-1}\lambda\_Y(x)+\right.}{\arcsin(x)}\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\left(\lambda^2 \_\_Y(x)(a^2x^{2m}+2ab\,x^m+b^2)\arcsin(x)^{1+2n}+\left(a\,x^{m-1}m\_\_Y(x)+2\_\_Y'(x)(a\,x^m+b)\right)\lambda\arcsin(x)^{n+1}+\arcsin(x)\_\right.}{\sqrt{-x^2+1}\,\arcsin(x)}\right.\right.\right.}{\lambda\operatorname{DESol}\left(\left\{\frac{\left(\lambda^2 \_\_Y(x)(a^2x^{2m}+2ab\,x^m+b^2)\arcsin(x)^{1+2n}+\left(a\,x^{m-1}m\_\_Y(x)+2\_\_Y'(x)(a\,x^m+b)\right)\lambda\arcsin(x)^{n+1}+\mathrm{ar}\right.}{\sqrt{-x^2+1}\,\arcsin(x)}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\left(\lambda^2 \_\_Y(x)(a^2x^{2m}+2ab\,x^m+b^2)\arcsin(x)^{1+2n}+\left(a\,x^{m-1}m\_\_Y(x)+2\_\_Y'(x)(a\,x^m+b)\right)\lambda\arcsin(x)^{n+1}+\arcsin(x)\_\right.}{\sqrt{-x^2+1}\,\arcsin(x)}\right.\right.\right.}{\lambda\operatorname{DESol}\left(\left\{\frac{\left(\lambda^2 \_\_Y(x)(a^2x^{2m}+2ab\,x^m+b^2)\arcsin(x)^{1+2n}+\left(a\,x^{m-1}m\_\_Y(x)+2\_\_Y'(x)(a\,x^m+b)\right)\lambda\arcsin(x)^{n+1}+\mathrm{ar}\right.}{\sqrt{-x^2+1}\,\arcsin(x)}\right.\right.} \quad (1)$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\left(\lambda^2 \_\_Y(x)(a^2x^{2m}+2ab\,x^m+b^2)\arcsin(x)^{1+2n}+\left(a\,x^{m-1}m\_\_Y(x)+2\_\_Y'(x)(a\,x^m+b)\right)\lambda\arcsin(x)^{n+1}+\arcsin(x)\_\right.}{\sqrt{-x^2+1}\,\arcsin(x)}\right.\right.\right.}{\lambda\operatorname{DESol}\left(\left\{\frac{\left(\lambda^2 \_\_Y(x)(a^2x^{2m}+2ab\,x^m+b^2)\arcsin(x)^{1+2n}+\left(a\,x^{m-1}m\_\_Y(x)+2\_\_Y'(x)(a\,x^m+b)\right)\lambda\arcsin(x)^{n+1}+\mathrm{ar}\right.}{\sqrt{-x^2+1}\,\arcsin(x)}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (d) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)=lambda*arcsin(x)^n*(y(x)-a*x^m-b)^2+a*m*x^(m-1),y(x), singsol=all)
```

$$y(x) = a\,x^{m} + b + \frac{1}{c_1 - \lambda\left(\int \arcsin\left(x\right)^{n} dx\right)}$$

✓ Solution by Mathematica

Time used: 4.054 (sec). Leaf size: 87

```
DSolve[y'[x]==\[Lambda]*ArcSin[x]^n*(y[x]-a*x^m-b)^2+a*m*x^(m-1),y[x],x,IncludeSingularSolut
```

$$y(x) \to ax^{m} + \frac{1}{\frac{1}{2}i\lambda \arcsin(x)^{n}\left(\arcsin(x)^{2}\right)^{-n}\left((i\arcsin(x))^{n}\Gamma(n+1,-i\arcsin(x)) - (-i\arcsin(x))^{n}\Gamma(n+1,i\arcsin(\right.}$$
$$+ b$$

$$y(x) \to ax^{m} + b$$

14.8 problem 8

Internal problem ID [10565]
Internal file name [`OUTPUT/9512_Monday_June_06_2022_03_01_23_PM_37025169/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-1. Equations containing arcsine.
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - \lambda \arcsin(x)^n y^2 - ky = \lambda b^2 x^{2k} \arcsin(x)^n}$$

14.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\lambda \arcsin(x)^n y^2 + ky + \lambda b^2 x^{2k} \arcsin(x)^n}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{\lambda b^2 x^{2k} \arcsin(x)^n}{x} + \frac{\lambda \arcsin(x)^n y^2}{x} + \frac{ky}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\lambda b^2 x^{2k} \arcsin(x)^n}{x}$, $f_1(x) = \frac{k}{x}$ and $f_2(x) = \frac{\lambda \arcsin(x)^n}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{\lambda \arcsin(x)^n u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{\lambda \arcsin(x)^n}{x^2} + \frac{\lambda \arcsin(x)^n n}{x\sqrt{-x^2+1}\,\arcsin(x)} \\
f_1f_2 &= \frac{k\lambda \arcsin(x)^n}{x^2} \\
f_2^2f_0 &= \frac{\lambda^3 \arcsin(x)^{3n} b^2x^{2k}}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\lambda \arcsin(x)^n u''(x)}{x} - \left(-\frac{\lambda \arcsin(x)^n}{x^2} + \frac{\lambda \arcsin(x)^n n}{x\sqrt{-x^2+1}\,\arcsin(x)} + \frac{k\lambda \arcsin(x)^n}{x^2}\right)u'(x) + \frac{\lambda^3 \arcsin(x)^{3n}}{x^3}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1\mathrm{e}^{ib\lambda\left(\int x^{k-1}\arcsin(x)^n dx\right)} + c_2\mathrm{e}^{-ib\lambda\left(\int x^{k-1}\arcsin(x)^n dx\right)}$$

The above shows that

$$u'(x) = ib\,x^{k-1}\lambda \arcsin(x)^n\,\mathrm{e}^{-ib\lambda\left(\int x^{k-1}\arcsin(x)^n dx\right)}\left(c_1\mathrm{e}^{2ib\lambda\left(\int x^{k-1}\arcsin(x)^n dx\right)} - c_2\right)$$

Using the above in (1) gives the solution

$$y = -\frac{ib\,x^{k-1}\mathrm{e}^{-ib\lambda\left(\int x^{k-1}\arcsin(x)^n dx\right)}\left(c_1\mathrm{e}^{2ib\lambda\left(\int x^{k-1}\arcsin(x)^n dx\right)} - c_2\right)x}{c_1\mathrm{e}^{ib\lambda\left(\int x^{k-1}\arcsin(x)^n dx\right)} + c_2\mathrm{e}^{-ib\lambda\left(\int x^{k-1}\arcsin(x)^n dx\right)}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{ib\,x^k\left(c_3\mathrm{e}^{2ib\lambda\left(\int x^{k-1}\arcsin(x)^n dx\right)} - 1\right)}{c_3\mathrm{e}^{2ib\lambda\left(\int x^{k-1}\arcsin(x)^n dx\right)} + 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{ib\,x^{k}\left(c_3 e^{2ib\lambda\left(\int x^{k-1}\arcsin(x)^{n}dx\right)}-1\right)}{c_3 e^{2ib\lambda\left(\int x^{k-1}\arcsin(x)^{n}dx\right)}+1} \tag{1}$$

Verification of solutions

$$y = -\frac{ib\,x^{k}\left(c_3 e^{2ib\lambda\left(\int x^{k-1}\arcsin(x)^{n}dx\right)}-1\right)}{c_3 e^{2ib\lambda\left(\int x^{k-1}\arcsin(x)^{n}dx\right)}+1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 29

```
dsolve(x*diff(y(x),x)=lambda*arcsin(x)^n*y(x)^2+k*y(x)+lambda*b^2*x^(2*k)*arcsin(x)^n,y(x),
```

$$y(x) = -\tan\left(-\lambda b\left(\int x^{-1+k}\arcsin(x)^{n}\,dx\right)+c_1\right)b\,x^{k}$$

✓ Solution by Mathematica

Time used: 1.716 (sec). Leaf size: 48

```
DSolve[x*y'[x]==\[Lambda]*ArcSin[x]^n*y[x]^2+k*y[x]+\[Lambda]*b^2*x^(2*k)*ArcSin[x]^n,y[x],x
```

$$y(x) \to \sqrt{b^2}x^{k}\tan\left(\sqrt{b^2}\int_1^x \lambda\arcsin(K[1])^{n}K[1]^{k-1}dK[1]+c_1\right)$$

14.9 problem 9

Internal problem ID [10566]
Internal file name [`OUTPUT/9513_Monday_June_06_2022_03_01_26_PM_22323186/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-1. Equations containing arcsine.
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x-\left(a\,x^{2m}y^2+yx^nb+c\right)\arcsin\left(x\right)^m+yn=0}$$

14.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y'&=F(x,y)\\&=\frac{\arcsin\left(x\right)^m x^{2m}a\,y^2+\arcsin\left(x\right)^m x^n by+\arcsin\left(x\right)^m c-ny}{x}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=\frac{\arcsin\left(x\right)^m x^{2m}a\,y^2}{x}+\frac{\arcsin\left(x\right)^m x^n by}{x}+\frac{\arcsin\left(x\right)^m c}{x}-\frac{ny}{x}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=\frac{\arcsin(x)^m c}{x}$, $f_1(x)=\frac{\arcsin(x)^m x^n b-n}{x}$ and $f_2(x)=\frac{\arcsin(x)^m x^{2m}a}{x}$. Let

$$\begin{aligned}y&=\frac{-u'}{f_2u}\\&=\frac{-u'}{\frac{\arcsin(x)^m x^{2m}au}{x}}\end{aligned}\tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{\arcsin(x)^m\, m\, x^{2m}a}{\sqrt{-x^2+1}\,\arcsin(x)\, x} + \frac{2\arcsin(x)^m\, x^{2m}ma}{x^2} - \frac{\arcsin(x)^m\, x^{2m}a}{x^2}\\
f_1f_2 &= \frac{(\arcsin(x)^m\, x^n b - n)\arcsin(x)^m\, x^{2m}a}{x^2}\\
f_2^2f_0 &= \frac{\arcsin(x)^{3m}\, x^{4m}a^2c}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\arcsin(x)^m\, x^{2m}au''(x)}{x} - \left(\frac{\arcsin(x)^m\, m\, x^{2m}a}{\sqrt{-x^2+1}\,\arcsin(x)\, x} + \frac{2\arcsin(x)^m\, x^{2m}ma}{x^2} - \frac{\arcsin(x)^m\, x^{2m}a}{x^2} + \frac{(\arcsin(}{}\right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
u(x) = \mathrm{DESol}\Bigg(\Bigg\{&_Y''(x) - \frac{2m_Y'(x)}{x} + \frac{n_Y'(x)}{x} - b\,x^{n-1}\arcsin(x)^m\,_Y'(x) + \frac{_Y'(x)}{x}\\
&- \frac{m_Y'(x)}{\arcsin(x)\sqrt{-x^2+1}} + ac\,x^{2m-2}_Y(x)\arcsin(x)^{2m}\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

The above shows that

$$\begin{aligned}
u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\Bigg(\Bigg\{&_Y''(x) - \frac{2m_Y'(x)}{x} + \frac{n_Y'(x)}{x} - b\,x^{n-1}\arcsin(x)^m\,_Y'(x)\\
&+ \frac{_Y'(x)}{x} - \frac{m_Y'(x)}{\arcsin(x)\sqrt{-x^2+1}} + ac\,x^{2m-2}_Y(x)\arcsin(x)^{2m}\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{2m\_Y'(x)}{x} + \frac{n\_Y'(x)}{x} - b\,x^{n-1}\arcsin(x)^m\,\_Y'(x) + \frac{\_Y'(x)}{x} - \frac{m\_Y'(x)}{\arcsin(x)\sqrt{-x^2+1}} -\right.\right.\right.}{a\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{2m\_Y'(x)}{x} + \frac{n\_Y'(x)}{x} - b\,x^{n-1}\arcsin(x)^m\,\_Y'(x) + \frac{\_Y'(x)}{x} - \frac{m}{\arcsin(}\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x^{-2m+1} \arcsin(x)^{-m} \left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{\left(\arcsin(x)^{1+2m} x^{2m-1} \_\_Y(x) a c - x^{n} \arcsin(x)^{m+1} \_\_Y'(x) b - 2\left(-\frac{\_\_Y''(x) x}{2} + \_\_Y'(\cdots\right.\right.}{\sqrt{-x^2+1}\, \arcsin(x) x}\cdots\right.\right.}{a\, \mathrm{DESol}\left(\left\{\frac{\left(a\, x^{2m} c \arcsin(x)^{1+2m} \_\_Y(x) - b\, x^{n+1} \arcsin(x)^{m+1} \_\_Y'(x) - 2\left(-\frac{\_\_Y''(x) x}{2} + \_\_Y'(x)\left(m - \frac{n}{2} - \frac{1}{2}\right)\right) \cdots\right.}{\sqrt{-x^2+1}\, x^2 \arcsin(x)}\cdots\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^{-2m+1} \arcsin(x)^{-m} \left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{\left(\arcsin(x)^{1+2m} x^{2m-1} \_\_Y(x) a c - x^{n} \arcsin(x)^{m+1} \_\_Y'(x) b - 2\left(-\frac{\_\_Y''(x) x}{2} + \_\_Y'(\cdots\right.\right.}{\sqrt{-x^2+1}\, \arcsin(x) x}\cdots\right.\right.}{a\, \mathrm{DESol}\left(\left\{\frac{\left(a\, x^{2m} c \arcsin(x)^{1+2m} \_\_Y(x) - b\, x^{n+1} \arcsin(x)^{m+1} \_\_Y'(x) - 2\left(-\frac{\_\_Y''(x) x}{2} + \_\_Y'(x)\left(m - \frac{n}{2} - \frac{1}{2}\right)\right) \cdots\right.}{\sqrt{-x^2+1}\, x^2 \arcsin(x)}\cdots\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{x^{-2m+1} \arcsin(x)^{-m} \left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{\left(\arcsin(x)^{1+2m} x^{2m-1} \_\_Y(x) a c - x^{n} \arcsin(x)^{m+1} \_\_Y'(x) b - 2\left(-\frac{\_\_Y''(x) x}{2} + \_\_Y'(\cdots\right.\right.}{\sqrt{-x^2+1}\, \arcsin(x) x}\cdots\right.\right.}{a\, \mathrm{DESol}\left(\left\{\frac{\left(a\, x^{2m} c \arcsin(x)^{1+2m} \_\_Y(x) - b\, x^{n+1} \arcsin(x)^{m+1} \_\_Y'(x) - 2\left(-\frac{\_\_Y''(x) x}{2} + \_\_Y'(x)\left(m - \frac{n}{2} - \frac{1}{2}\right)\right) \cdots\right.}{\sqrt{-x^2+1}\, x^2 \arcsin(x)}\cdots\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (b*x^(n-1)*arcsin(x)^m*x*(-x^2
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(x^(-1+2*m)*arcsin(x)^m*a*y(x)^2+y(x)+(b*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x)=(a*x^(2*m)*y(x)^2+b*x^n*y(x)+c)*arcsin(x)^m-n*y(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y'[x]==(a*x^(2*m)*y[x]^2+b*x^n*y[x]+c)*ArcSin[x]^m-n*y[x],y[x],x,IncludeSingularSol
```

Not solved

15 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-2. Equations containing arccosine.

15.1 problem 10

Internal problem ID [10567]
Internal file name [`OUTPUT/9514_Monday_June_06_2022_03_01_38_PM_72944279/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-2. Equations containing arccosine.
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - \lambda \arccos(x)^n y = -a^2 + a\lambda \arccos(x)^n}$$

15.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + \lambda \arccos(x)^n y - a^2 + a\lambda \arccos(x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \lambda \arccos(x)^n y - a^2 + a\lambda \arccos(x)^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2 + a\lambda \arccos(x)^n$, $f_1(x) = \arccos(x)^n \lambda$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \arccos(x)^n \lambda \\ f_2^2 f_0 &= -a^2 + a\lambda \arccos(x)^n \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \arccos(x)^n \lambda u'(x) + \left(-a^2 + a\lambda \arccos(x)^n\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

Expression too large to display

The above shows that

Expression too large to display

Using the above in (1) gives the solution

Expression too large to display

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

Expression too large to display

Summary

The solution(s) found are the following

Expression too large to display (1)

Verification of solutions

Expression too large to display

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = arccos(x)^n*lambda*(diff(y(x),
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(y(x)^2+y(x)+arccos(x)^n*lambda*y(x)*x+x^
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 386

```
dsolve(diff(y(x),x)=y(x)^2+lambda*arccos(x)^n*y(x)-a^2+a*lambda*arccos(x)^n,y(x), singsol=al
```

Expression too large to display

✓ Solution by Mathematica

Time used: 8.046 (sec). Leaf size: 404

```
DSolve[y'[x]==y[x]^2+\[Lambda]*ArcCos[x]^n*y[x]-a^2+a*\[Lambda]*ArcCos[x]^n,y[x],x,IncludeSi
```

$$\text{Solve}\left[\int_1^x \frac{i \exp\left(\frac{1}{2}\lambda \arccos(K[1])^n \Gamma(n+1, -i\arccos(K[1]))(-i\arccos(K[1]))^{-n} + \frac{1}{2}\lambda(i\arccos(K[1]))^{-n} \text{a}}{n\lambda(a+y(x))}\right.$$

$$+\int_1^{y(x)}\left(-\int_1^x\left(\frac{i\exp\left(\frac{1}{2}\lambda\arccos(K[1])^n\Gamma(n+1,-i\arccos(K[1]))(-i\arccos(K[1]))^{-n}+\frac{1}{2}\lambda(i\arccos(K[1}{n\lambda(a+K[2])}\right.\right.$$

$$-\frac{i\exp\left(\frac{1}{2}\lambda\arccos(x)^n\Gamma(n+1,-i\arccos(x))(-i\arccos(x))^{-n}-2ax+\frac{1}{2}\lambda(i\arccos(x))^{-n}\arccos(x)^n\Gamma(n+}{n\lambda(a+K[2])^2}$$

15.2 problem 11

Internal problem ID [10568]
Internal file name [`OUTPUT/9515_Monday_June_06_2022_03_02_52_PM_27537007/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-2. Equations containing arccosine.
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - \lambda x \arccos(x)^n y = \arccos(x)^n \lambda}$$

15.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + \arccos(x)^n \lambda x y + \arccos(x)^n \lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \arccos(x)^n \lambda x y + \arccos(x)^n \lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \arccos(x)^n \lambda$, $f_1(x) = \arccos(x)^n \lambda x$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \arccos(x)^n \lambda x \\ f_2^2 f_0 &= \arccos(x)^n \lambda \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \arccos(x)^n \lambda x u'(x) + \arccos(x)^n \lambda u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x\left(\left(\int e^{\int \frac{\arccos(x)^n \lambda x^2 - 2}{x} dx} dx\right) c_1 + c_2\right)$$

The above shows that

$$u'(x) = \left(\int e^{\int \frac{\arccos(x)^n \lambda x^2 - 2}{x} dx} dx\right) c_1 + c_2 + x\, e^{\int \frac{\arccos(x)^n \lambda x^2 - 2}{x} dx} c_1$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\int e^{\int \frac{\arccos(x)^n \lambda x^2 - 2}{x} dx} dx\right) c_1 + c_2 + x\, e^{\int \frac{\arccos(x)^n \lambda x^2 - 2}{x} dx} c_1}{x\left(\left(\int e^{\int \frac{\arccos(x)^n \lambda x^2 - 2}{x} dx} dx\right) c_1 + c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\left(\int e^{\int \frac{\arccos(x)^n \lambda x^2 - 2}{x} dx} dx\right) c_3 - 1 - x\, e^{\int \frac{\arccos(x)^n \lambda x^2 - 2}{x} dx} c_3}{x\left(\left(\int e^{\int \frac{\arccos(x)^n \lambda x^2 - 2}{x} dx} dx\right) c_3 + 1\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\left(\int e^{\int \frac{\arccos(x)^n \lambda x^2-2}{x}dx}dx\right) c_3 - 1 - x\, e^{\int \frac{\arccos(x)^n \lambda x^2-2}{x}dx} c_3}{x\left(\left(\int e^{\int \frac{\arccos(x)^n \lambda x^2-2}{x}dx}dx\right) c_3 + 1\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-\left(\int e^{\int \frac{\arccos(x)^n \lambda x^2-2}{x}dx}dx\right) c_3 - 1 - x\, e^{\int \frac{\arccos(x)^n \lambda x^2-2}{x}dx} c_3}{x\left(\left(\int e^{\int \frac{\arccos(x)^n \lambda x^2-2}{x}dx}dx\right) c_3 + 1\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 78

```
dsolve(diff(y(x),x)=y(x)^2+lambda*x*arccos(x)^n*y(x)+lambda*arccos(x)^n,y(x), singsol=all)
```

$$y(x) = \frac{e^{\int \frac{\arccos(x)^n \lambda x^2-2}{x}dx} x + \int e^{\int \frac{\arccos(x)^n \lambda x^2-2}{x}dx}dx - c_1}{\left(c_1 - \left(\int e^{\int \frac{\arccos(x)^n \lambda x^2-2}{x}dx}dx\right)\right) x}$$

✓ Solution by Mathematica

Time used: 5.617 (sec). Leaf size: 253

```
DSolve[y'[x]==y[x]^2+\[Lambda]*x*ArcCos[x]^n*y[x]+\[Lambda]*ArcCos[x]^n,y[x],x,IncludeSingul
```

$$y(x) \to$$

$$-\frac{x \int_1^x \frac{\exp\left(2^{-n-3}\lambda \arccos(K[1])^n \left(\arccos(K[1])^2\right)^{-n}(\Gamma(n+1,2i\arccos(K[1]))(-i\arccos(K[1]))^n+(i\arccos(K[1]))^n\Gamma(n+1,-2i\arccos(K[1}{K[1]^2}}{x^2\left(\int_1^x \frac{\exp\left(2^{-n-3}\lambda \arccos(K[1])^n(\arccos(K[1])^2)^{-n}}{}}$$

$$y(x) \to -\frac{1}{x}$$

15.3 problem 12

Internal problem ID [10569]
Internal file name [`OUTPUT/9516_Monday_June_06_2022_03_02_55_PM_9309576/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-2. Equations containing arccosine.
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + (k+1)\, x^k y^2 - \lambda \arccos(x)^n \left(x^{k+1} y - 1\right) = 0}$$

15.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x^{k+1} \arccos(x)^n \lambda y - x^k y^2 k - x^k y^2 - \arccos(x)^n \lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^k x \arccos(x)^n \lambda y - x^k y^2 k - x^k y^2 - \arccos(x)^n \lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = -\arccos(x)^n \lambda$, $f_1(x) = x^{k+1} \arccos(x)^n \lambda$ and $f_2(x) = -x^k k - x^k$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(-x^k k - x^k)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{k^2 x^k}{x} - \frac{k\, x^k}{x} \\ f_1 f_2 &= x^{k+1} \arccos(x)^n \lambda(-x^k k - x^k) \\ f_2^2 f_0 &= -(-x^k k - x^k)^2 \arccos(x)^n \lambda \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(-x^k k - x^k)\, u''(x) - \left(-\frac{k^2 x^k}{x} - \frac{k\, x^k}{x} + x^{k+1} \arccos(x)^n \lambda(-x^k k - x^k)\right) u'(x) - (-x^k k - x^k)^2 \arccos(x$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{k+1} \left(\left(\int x^{-2k-2} e^{\int \left(x^{k+1} \arccos(x)^n \lambda + \frac{k}{x}\right) dx} dx\right) c_2 + c_1\right)$$

The above shows that

$$u'(x) = c_2 x^{-k-1} e^{\int \frac{x^{k+2} \lambda \arccos(x)^n + k}{x} dx} + (k+1) \left(\left(\int e^{\int \frac{x^{k+2} \lambda \arccos(x)^n + k}{x} dx} x^{-2k-2} dx\right) c_2 + c_1\right) x^k$$

Using the above in (1) gives the solution

$$y = -\frac{\left(c_2 x^{-k-1} e^{\int \frac{x^{k+2} \lambda \arccos(x)^n + k}{x} dx} + (k+1) \left(\left(\int e^{\int \frac{x^{k+2} \lambda \arccos(x)^n + k}{x} dx} x^{-2k-2} dx\right) c_2 + c_1\right) x^k\right) x^{-k-1}}{(-x^k k - x^k) \left(\left(\int x^{-2k-2} e^{\int \left(x^{k+1} \arccos(x)^n \lambda + \frac{k}{x}\right) dx} dx\right) c_2 + c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(k+1) \left(\int x^{-2k-2} e^{\int \left(x^{k+1} \arccos(x)^n \lambda + \frac{k}{x}\right) dx} dx + c_3\right) x^{-k-1} + x^{-3k-2} e^{\int \left(x^{k+1} \arccos(x)^n \lambda + \frac{k}{x}\right) dx}}{(k+1) \left(\int e^{\int \frac{x^{k+2} \lambda \arccos(x)^n + k}{x} dx} x^{-2k-2} dx + c_3\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{(k+1)\left(\int x^{-2k-2} e^{\int \left(x^{k+1} \arccos(x)^{n} \lambda+\frac{k}{x}\right)dx} dx + c_3\right) x^{-k-1} + x^{-3k-2} e^{\int \left(x^{k+1} \arccos(x)^{n} \lambda+\frac{k}{x}\right)dx}}{(k+1)\left(\int e^{\int \frac{x^{k+2} \lambda \arccos(x)^{n}+k}{x} dx} x^{-2k-2} dx + c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{(k+1)\left(\int x^{-2k-2} e^{\int \left(x^{k+1} \arccos(x)^{n} \lambda+\frac{k}{x}\right)dx} dx + c_3\right) x^{-k-1} + x^{-3k-2} e^{\int \left(x^{k+1} \arccos(x)^{n} \lambda+\frac{k}{x}\right)dx}}{(k+1)\left(\int e^{\int \frac{x^{k+2} \lambda \arccos(x)^{n}+k}{x} dx} x^{-2k-2} dx + c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (arccos(x)^n*x^(1+k)*lambda*x+
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-((-x^k*k-x^k)*y(x)^2+y(x)+arccos(x)^n*x^(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 180

```
dsolve(diff(y(x),x)=-(k+1)*x^k*y(x)^2+lambda*arccos(x)^n*(x^(k+1)*y(x)-1),y(x), singsol=all)
```

$$y(x) = \frac{x^{-1-k}\left(x^{1+k}\mathrm{e}^{\int \frac{\arccos(x)^n x^{1+k}\lambda x-2k-2}{x}dx} + \left(\int x^k \mathrm{e}^{\lambda\left(\int \arccos(x)^n x^{1+k}dx\right)-2\left(\int \frac{1}{x}dx\right)(1+k)}dx\right)k + \int x^k \mathrm{e}^{\lambda\left(\int \arccos(x)^n x^{1+k}d\right.}\right.}{\left(\int x^k \mathrm{e}^{\lambda\left(\int \arccos(x)^n x^{1+k}dx\right)-2\left(\int \frac{1}{x}dx\right)(1+k)}dx\right)k + \int x^k \mathrm{e}^{\lambda\left(\int \arccos(x)^n x^{1+k}dx\right)-2\left(\int \frac{1}{x}dx\right)(1+k)}dx}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==-(k+1)*x^k*y[x]^2+\[Lambda]*ArcCos[x]^n*(x^(k+1)*y[x]-1),y[x],x,IncludeSingula
```

Not solved

15.4 problem 13

Internal problem ID [10570]
Internal file name [`OUTPUT/9517_Monday_June_06_2022_03_03_02_PM_51106550/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-2. Equations containing arccosine.
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - \lambda \arccos(x)^n y^2 - ay = ab - b^2\lambda \arccos(x)^n}$$

15.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \arccos(x)^n \lambda y^2 + ya + ab - b^2\lambda \arccos(x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \arccos(x)^n \lambda y^2 + ya + ab - b^2\lambda \arccos(x)^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = ab - b^2\lambda \arccos(x)^n$, $f_1(x) = a$ and $f_2(x) = \arccos(x)^n \lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\arccos(x)^n \lambda u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{\arccos(x)^n n\lambda}{\sqrt{-x^2+1}\,\arccos(x)} \\ f_1 f_2 &= a\lambda \arccos(x)^n \\ f_2^2 f_0 &= \arccos(x)^{2n} \lambda^2 \left(ab - b^2\lambda \arccos(x)^n\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\arccos(x)^n \lambda u''(x) - \left(-\frac{\arccos(x)^n n\lambda}{\sqrt{-x^2+1}\,\arccos(x)} + a\lambda \arccos(x)^n\right) u'(x) + \arccos(x)^{2n} \lambda^2 \left(ab - b^2\lambda \arccos(\right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} - a\_Y'(x) - \arccos(x)^{2n} b^2\lambda^2\_Y(x) + \arccos(x)^n ab\lambda\_Y(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} - a\_Y'(x) - \arccos(x)^{2n} b^2\lambda^2\_Y(x) + \arccos(x)^n ab\lambda\_Y(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} - a\_Y'(x) - \arccos(x)^{2n} b^2\lambda^2\_Y(x) + \arccos(x)^n ab\lambda\_Y(x)\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} - a\_Y'(x) - \arccos(x)^{2n} b^2\lambda^2\_Y(x) + \arccos(x)^n ab\lambda\_\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(-\arccos(x)^{1+2n}\_\_Y(x)b^2\lambda^2+\arccos(x)^{n+1}\_\_Y(x)ab\lambda-\arccos(x)\left(a\_\_Y'(x)-\_\_Y''(x)\right)\right)\sqrt{-x^2+1}+n\_\_Y'(x)}{\sqrt{-x^2+1}\,\arccos(x)}\right\}\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{\left(-\arccos(x)^{1+2n}\_\_Y(x)b^2\lambda^2+\arccos(x)^{n+1}\_\_Y(x)ab\lambda-\arccos(x)\left(a\_\_Y'(x)-\_\_Y''(x)\right)\right)\sqrt{-x^2+1}+n\_\_}{\sqrt{-x^2+1}\,\arccos(x)}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(-\arccos(x)^{1+2n}\_\_Y(x)b^2\lambda^2+\arccos(x)^{n+1}\_\_Y(x)ab\lambda-\arccos(x)\left(a\_\_Y'(x)-\_\_Y''(x)\right)\right)\sqrt{-x^2+1}+n\_\_Y'(x)}{\sqrt{-x^2+1}\,\arccos(x)}\right\}\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{\left(-\arccos(x)^{1+2n}\_\_Y(x)b^2\lambda^2+\arccos(x)^{n+1}\_\_Y(x)ab\lambda-\arccos(x)\left(a\_\_Y'(x)-\_\_Y''(x)\right)\right)\sqrt{-x^2+1}+n\_\_}{\sqrt{-x^2+1}\,\arccos(x)}\right.\right.} \quad (1)$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(-\arccos(x)^{1+2n}\_\_Y(x)b^2\lambda^2+\arccos(x)^{n+1}\_\_Y(x)ab\lambda-\arccos(x)\left(a\_\_Y'(x)-\_\_Y''(x)\right)\right)\sqrt{-x^2+1}+n\_\_Y'(x)}{\sqrt{-x^2+1}\,\arccos(x)}\right\}\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{\left(-\arccos(x)^{1+2n}\_\_Y(x)b^2\lambda^2+\arccos(x)^{n+1}\_\_Y(x)ab\lambda-\arccos(x)\left(a\_\_Y'(x)-\_\_Y''(x)\right)\right)\sqrt{-x^2+1}+n\_\_}{\sqrt{-x^2+1}\,\arccos(x)}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a*(-x^2+1)^(1/2)*arccos(x)-n)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(arccos(x)^n*lambda*y(x)^2+y(x)+y(x)*a*x+
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 382

```
dsolve(diff(y(x),x)=lambda*arccos(x)^n*y(x)^2+a*y(x)+a*b-b^2*lambda*arccos(x)^n,y(x), singso
```

Expression too large to display

✓ Solution by Mathematica

Time used: 9.288 (sec). Leaf size: 420

```
DSolve[y'[x]==\[Lambda]*ArcCos[x]^n*y[x]^2+a*y[x]+a*b-b^2*\[Lambda]*ArcCos[x]^n,y[x],x,Inclu
```

$$\text{Solve}\Bigg[\int_1^x$$

$$-\frac{i\exp\left(-b\lambda\arccos(K[1])^n\Gamma(n+1,-i\arccos(K[1]))(-i\arccos(K[1]))^{-n}-b\lambda(i\arccos(K[1]))^{-n}\arccos(K}{an\lambda(b+y(x))}$$

$$+\int_1^{y(x)}\left(\frac{i\exp\left(-b\lambda\arccos(x)^n\Gamma(n+1,-i\arccos(x))(-i\arccos(x))^{-n}+ax-b\lambda(i\arccos(x))^{-n}\arccos(x)}{an\lambda(b+K[2])^2}\right.$$

$$-\int_1^x\left(\frac{i\exp\left(-b\lambda\arccos(K[1])^n\Gamma(n+1,-i\arccos(K[1]))(-i\arccos(K[1]))^{-n}-b\lambda(i\arccos(K[1]))^{-n}\text{arc}}{an\lambda(b+K}\right.$$

15.5 problem 14

Internal problem ID [10571]
Internal file name [`OUTPUT/9518_Monday_June_06_2022_03_03_13_PM_87171803/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-2. Equations containing arccosine.
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \arccos(x)^n y^2 + b\lambda x^m \arccos(x)^n y = bm x^{m-1}}$$

15.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \arccos(x)^n \lambda y^2 - b\lambda x^m \arccos(x)^n y + bm x^{m-1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \arccos(x)^n \lambda y^2 - b\lambda x^m \arccos(x)^n y + \frac{b x^m m}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = bm x^{m-1}$, $f_1(x) = -b\lambda x^m \arccos(x)^n$ and $f_2(x) = \arccos(x)^n \lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\arccos(x)^n \lambda u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{\arccos(x)^n n\lambda}{\sqrt{-x^2+1}\arccos(x)} \\ f_1 f_2 &= -b\lambda^2 x^m \arccos(x)^{2n} \\ f_2^2 f_0 &= \arccos(x)^{2n}\lambda^2 bm\, x^{m-1} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\arccos(x)^n \lambda u''(x) - \left(-b\lambda^2 x^m \arccos(x)^{2n} - \frac{\arccos(x)^n n\lambda}{\sqrt{-x^2+1}\arccos(x)}\right) u'(x) + \arccos(x)^{2n}\lambda^2 bm\, x^{m-1} u(x)$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \text{DESol}\left(\left\{\_Y''(x) + b\,x^m\lambda\arccos(x)^n\_Y'(x) + \frac{n\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} + bm\,x^{m-1}\lambda\_Y(x)\arccos(x)^n\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\text{DESol}\left(\left\{\_Y''(x) + b\,x^m\lambda\arccos(x)^n\_Y'(x) + \frac{n\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} + bm\,x^{m-1}\lambda\_Y(x)\arccos(x)^n\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\text{DESol}\left(\left\{\_Y''(x) + b\,x^m\lambda\arccos(x)^n\_Y'(x) + \frac{n\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} + bm\,x^{m-1}\lambda\_Y(x)\arccos(x)^n\right\},\right.\right.}{\lambda\,\text{DESol}\left(\left\{\_Y''(x) + b\,x^m\lambda\arccos(x)^n\_Y'(x) + \frac{n\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} + bm\,x^{m-1}\lambda\_Y(x)\arccos(\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(b\lambda\left(m\_\_Y(x)x^{m-1}+\_\_Y'(x)x^{m}\right)\arccos(x)^{n+1}+\arccos(x)\_\_Y''(x)\right)\sqrt{-x^{2}+1}+n\_\_Y'(x)}{\sqrt{-x^{2}+1}\,\arccos(x)}\right\},\{\_\_Y(x)\}\right)\right)\mathrm{a}}{\lambda\,\mathrm{DESol}\left(\left\{\frac{\left(b\lambda\left(m\,x^{m}\_\_Y(x)+\_\_Y'(x)x^{m+1}\right)\arccos(x)^{n+1}+\arccos(x)\_\_Y''(x)x\right)\sqrt{-x^{2}+1}+\_\_Y'(x)xn}{\sqrt{-x^{2}+1}\,x\arccos(x)}\right\},\{\_\_Y(x}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(b\lambda\left(m\_\_Y(x)x^{m-1}+\_\_Y'(x)x^{m}\right)\arccos(x)^{n+1}+\arccos(x)\_\_Y''(x)\right)\sqrt{-x^{2}+1}+n\_\_Y'(x)}{\sqrt{-x^{2}+1}\,\arccos(x)}\right\},\{\_\_Y(x)\}\right)\right)\mathrm{a}}{\lambda\,\mathrm{DESol}\left(\left\{\frac{\left(b\lambda\left(m\,x^{m}\_\_Y(x)+\_\_Y'(x)x^{m+1}\right)\arccos(x)^{n+1}+\arccos(x)\_\_Y''(x)x\right)\sqrt{-x^{2}+1}+\_\_Y'(x)xn}{\sqrt{-x^{2}+1}\,x\arccos(x)}\right\},\{\_\_Y(x}\quad(1)$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(b\lambda\left(m\_\_Y(x)x^{m-1}+\_\_Y'(x)x^{m}\right)\arccos(x)^{n+1}+\arccos(x)\_\_Y''(x)\right)\sqrt{-x^{2}+1}+n\_\_Y'(x)}{\sqrt{-x^{2}+1}\,\arccos(x)}\right\},\{\_\_Y(x)\}\right)\right)\mathrm{a}}{\lambda\,\mathrm{DESol}\left(\left\{\frac{\left(b\lambda\left(m\,x^{m}\_\_Y(x)+\_\_Y'(x)x^{m+1}\right)\arccos(x)^{n+1}+\arccos(x)\_\_Y''(x)x\right)\sqrt{-x^{2}+1}+\_\_Y'(x)xn}{\sqrt{-x^{2}+1}\,x\arccos(x)}\right\},\{\_\_Y(x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -((-x^2+1)^(1/2)*x^m*arccos(x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(arccos(x)^n*lambda*y(x)^2+y(x)-b*lambda*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=lambda*arccos(x)^n*y(x)^2-b*lambda*x^m*arccos(x)^n*y(x)+b*m*x^(m-1),y(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==\[Lambda]*ArcCos[x]^n*y[x]^2-b*\[Lambda]*x^m*ArcCos[x]^n*y[x]+b*m*x^(m-1),y[x]
```

Not solved

15.6 problem 15

Internal problem ID [10572]
Internal file name [`OUTPUT/9519_Monday_June_06_2022_03_03_20_PM_88076586/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-2. Equations containing arccosine.
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \arccos\left(x\right)^{n} y^{2} = \beta m\, x^{m-1} - \lambda\, \beta^{2} x^{2m} \arccos\left(x\right)^{n}}$$

15.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \arccos\left(x\right)^{n} \lambda\, y^{2} + \beta m\, x^{m-1} - \lambda\, \beta^{2} x^{2m} \arccos\left(x\right)^{n} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\lambda\, \beta^{2} x^{2m} \arccos\left(x\right)^{n} + \arccos\left(x\right)^{n} \lambda\, y^{2} + \frac{\beta m\, x^{m}}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \beta m\, x^{m-1} - \lambda\, \beta^{2} x^{2m} \arccos\left(x\right)^{n}$, $f_1(x) = 0$ and $f_2(x) = \arccos\left(x\right)^{n} \lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\arccos\left(x\right)^{n} \lambda u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{\arccos(x)^n n\lambda}{\sqrt{-x^2+1}\,\arccos(x)} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= \arccos(x)^{2n} \lambda^2 \left(\beta m\, x^{m-1} - \lambda\, \beta^2 x^{2m} \arccos(x)^n\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\arccos(x)^n \lambda u''(x) + \frac{\arccos(x)^n n\lambda u'(x)}{\sqrt{-x^2+1}\,\arccos(x)} + \arccos(x)^{2n} \lambda^2 \left(\beta m\, x^{m-1} - \lambda\, \beta^2 x^{2m} \arccos(x)^n\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} - x^{2m}\beta^2\_Y(x)\,\lambda^2 \arccos(x)^{2n} + m\beta\, x^{m-1}\lambda\_Y(x) \arccos(x)^n\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} - x^{2m}\beta^2\_Y(x)\,\lambda^2 \arccos(x)^{2n} + m\beta\, x^{m-1}\lambda\_Y(x) \arccos(x)^n\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} - x^{2m}\beta^2\_Y(x)\,\lambda^2 \arccos(x)^{2n} + m\beta\, x^{m-1}\lambda\_Y(x) \arccos(x)^n \cdots\right.\right.\right.}{\lambda\, \mathrm{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} - x^{2m}\beta^2\_Y(x)\,\lambda^2 \arccos(x)^{2n} + m\beta\, x^{m-1}\lambda\_Y(x) \operatorname{arcc}\cdots\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\left(-x^{2m}\arccos(x)^{1+2n}\_\_Y(x)\beta^{2}\lambda^{2}+x^{m-1}\arccos(x)^{n+1}\_\_Y(x)\beta\lambda m+\arccos(x)\_\_Y''(x)\right)\sqrt{-x^{2}+1}+n\_\_Y'(x)}{\sqrt{-x^{2}+1}\arccos(x)}\right\},\right.\right.}{\lambda\operatorname{DESol}\left(\left\{\frac{\left(-\arccos(x)^{1+2n}\lambda^{2}\beta^{2}x^{1+2m}\_\_Y(x)+\arccos(x)^{n+1}\beta\lambda m\,x^{m}\_\_Y(x)+\arccos(x)\_\_Y''(x)x\right)\sqrt{-x^{2}+1}+\_\_Y'(x)}{\sqrt{-x^{2}+1}\,x\arccos(x)}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\left(-x^{2m}\arccos(x)^{1+2n}\_\_Y(x)\beta^{2}\lambda^{2}+x^{m-1}\arccos(x)^{n+1}\_\_Y(x)\beta\lambda m+\arccos(x)\_\_Y''(x)\right)\sqrt{-x^{2}+1}+n\_\_Y'(x)}{\sqrt{-x^{2}+1}\arccos(x)}\right\},\right.\right.}{\lambda\operatorname{DESol}\left(\left\{\frac{\left(-\arccos(x)^{1+2n}\lambda^{2}\beta^{2}x^{1+2m}\_\_Y(x)+\arccos(x)^{n+1}\beta\lambda m\,x^{m}\_\_Y(x)+\arccos(x)\_\_Y''(x)x\right)\sqrt{-x^{2}+1}+\_\_Y'(x)}{\sqrt{-x^{2}+1}\,x\arccos(x)}\right.\right.} \quad (1)$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\left(-x^{2m}\arccos(x)^{1+2n}\_\_Y(x)\beta^{2}\lambda^{2}+x^{m-1}\arccos(x)^{n+1}\_\_Y(x)\beta\lambda m+\arccos(x)\_\_Y''(x)\right)\sqrt{-x^{2}+1}+n\_\_Y'(x)}{\sqrt{-x^{2}+1}\arccos(x)}\right\},\right.\right.}{\lambda\operatorname{DESol}\left(\left\{\frac{\left(-\arccos(x)^{1+2n}\lambda^{2}\beta^{2}x^{1+2m}\_\_Y(x)+\arccos(x)^{n+1}\beta\lambda m\,x^{m}\_\_Y(x)+\arccos(x)\_\_Y''(x)x\right)\sqrt{-x^{2}+1}+\_\_Y'(x)}{\sqrt{-x^{2}+1}\,x\arccos(x)}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -n*(diff(y(x), x))/((-x^2+1)^(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
         -> trying with_periodic_functions in the coefficients
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(arccos(x)^n*lambda*y(x)^2+y(x)+x^2*(beta
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=lambda*arccos(x)^n*y(x)^2+beta*m*x^(m-1)-lambda*beta^2*x^(2*m)*arccos(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==\[Lambda]*ArcCos[x]^n*y[x]^2+\[Beta]*m*x^(m-1)-\[Lambda]*\[Beta]^2*x^(2*m)*Arc
```

Not solved

15.7 problem 16

Internal problem ID [10573]
Internal file name [`OUTPUT/9520_Monday_June_06_2022_03_03_27_PM_85306196/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-2. Equations containing arccosine.
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$\boxed{y' - \lambda \arccos(x)^n (y - a\,x^m - b)^2 = am\,x^{m-1}}$$

15.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= x^{2m}\arccos(x)^n a^2\lambda + 2x^m \arccos(x)^n ab\lambda - 2x^m\arccos(x)^n a\lambda y + b^2\lambda\arccos(x)^n - 2\arccos(x)^n b\lambda\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^{2m}\arccos(x)^n a^2\lambda+2x^m\arccos(x)^n ab\lambda-2x^m\arccos(x)^n a\lambda y+b^2\lambda\arccos(x)^n-2\arccos(x)^n b\lambda y+a$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^{2m}\arccos(x)^n a^2\lambda+2x^m\arccos(x)^n ab\lambda+b^2\lambda\arccos(x)^n+am\,x^{m-1}$, $f_1(x) = -2a\lambda\,x^m\arccos(x)^n - 2\arccos(x)^n\lambda b$ and $f_2(x) = \arccos(x)^n\lambda$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{\arccos(x)^n\lambda u}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{\arccos(x)^n\,n\lambda}{\sqrt{-x^2+1}\,\arccos(x)} \\ f_1f_2 &= (-2a\lambda\,x^m\arccos(x)^n - 2\arccos(x)^n\,\lambda b)\arccos(x)^n\,\lambda \\ f_2^2f_0 &= \arccos(x)^{2n}\,\lambda^2\left(x^{2m}\arccos(x)^n\,a^2\lambda + 2x^m\arccos(x)^n\,ab\lambda + b^2\lambda\arccos(x)^n + am\,x^{m-1}\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\arccos(x)^n\,\lambda u''(x) - \left(-\frac{\arccos(x)^n\,n\lambda}{\sqrt{-x^2+1}\,\arccos(x)} + (-2a\lambda\,x^m\arccos(x)^n - 2\arccos(x)^n\,\lambda b)\arccos(x)^n\,\lambda\right) u$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{\frac{n\_Y'(x)}{\sqrt{-x^2+1}} + \arccos(x)\left(\lambda^2\_Y(x)\,(a^2x^{2m} + 2ab\,x^m + b^2)\arccos(x)^{2n} + \_Y''(x) + x^{m-1}\arccos\right.}{\arccos(x)}\right.\right.$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{\frac{n\_Y'(x)}{\sqrt{-x^2+1}} + \lambda^2\_Y(x)\,(a^2x^{2m} + 2ab\,x^m + b^2)\arccos(x)^{1+2n} + (a\,x^{m-1}m\_Y(x) + 2\_Y'(x)\,(}{\arccos(x)}\right.\right.$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{\frac{n\_Y'(x)}{\sqrt{-x^2+1}} + \lambda^2\_Y(x)(a^2x^{2m}+2ab\,x^m+b^2)\arccos(x)^{1+2n} + \left(a\,x^{m-1}m\_Y(x) + 2\_Y'(x)(a\,x^m+b)\right)\lambda\arccos(x)^{n+}}{\arccos(x)}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{\frac{n\_Y'(x)}{\sqrt{-x^2+1}} + \arccos(x)\left(\lambda^2\_Y(x)(a^2x^{2m}+2ab\,x^m+b^2)\arccos(x)^{2n} + \_Y''(x) + x^{m-1}\arccos(x)^n am\lambda\_Y(x)+\right.}{\arccos(x)}\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{\left(\lambda^2 \_\_Y(x)\left(a^2x^{2m}+2ab\,x^m+b^2\right)\arccos(x)^{1+2n}+\left(a\,x^{m-1}m\_\_Y(x)+2\_\_Y'(x)(a\,x^m+b)\right)\lambda\arccos(x)^{n+1}+\arccos(x\right.}{\sqrt{-x^2+1}\,\arccos(x)}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{\left(\lambda^2 \_\_Y(x)\left(a^2x^{2m}+2ab\,x^m+b^2\right)\arccos(x)^{1+2n}+\left(a\,x^{m-1}m\_\_Y(x)+2\_\_Y'(x)(a\,x^m+b)\right)\lambda\arccos(x)^{n+1}+a\right.}{\sqrt{-x^2+1}\,\arccos(x)}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{\left(\lambda^2 \_\_Y(x)\left(a^2x^{2m}+2ab\,x^m+b^2\right)\arccos(x)^{1+2n}+\left(a\,x^{m-1}m\_\_Y(x)+2\_\_Y'(x)(a\,x^m+b)\right)\lambda\arccos(x)^{n+1}+\arccos(x\right.}{\sqrt{-x^2+1}\,\arccos(x)}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{\left(\lambda^2 \_\_Y(x)\left(a^2x^{2m}+2ab\,x^m+b^2\right)\arccos(x)^{1+2n}+\left(a\,x^{m-1}m\_\_Y(x)+2\_\_Y'(x)(a\,x^m+b)\right)\lambda\arccos(x)^{n+1}+a\right.}{\sqrt{-x^2+1}\,\arccos(x)}\right.\right.} \quad (1)$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{\left(\lambda^2 \_\_Y(x)\left(a^2x^{2m}+2ab\,x^m+b^2\right)\arccos(x)^{1+2n}+\left(a\,x^{m-1}m\_\_Y(x)+2\_\_Y'(x)(a\,x^m+b)\right)\lambda\arccos(x)^{n+1}+\arccos(x\right.}{\sqrt{-x^2+1}\,\arccos(x)}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{\left(\lambda^2 \_\_Y(x)\left(a^2x^{2m}+2ab\,x^m+b^2\right)\arccos(x)^{1+2n}+\left(a\,x^{m-1}m\_\_Y(x)+2\_\_Y'(x)(a\,x^m+b)\right)\lambda\arccos(x)^{n+1}+a\right.}{\sqrt{-x^2+1}\,\arccos(x)}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (d) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 165

```
dsolve(diff(y(x),x)=lambda*arccos(x)^n*(y(x)-a*x^m-b)^2+a*m*x^(m-1),y(x), singsol=all)
```

$$y(x) = \frac{\lambda(a\,x^m+b)\left((n+2)\,\mathrm{LommelS1}\left(n+\frac{1}{2},\frac{1}{2},\arccos(x)\right)-\mathrm{LommelS1}\left(n+\frac{3}{2},\frac{3}{2},\arccos(x)\right)\arccos(x)+a}{\lambda\left((n+2)\,\mathrm{LommelS1}\left(n+\frac{1}{2},\frac{1}{2},\arccos(x)\right)-\mathrm{LommelS1}\left(n+\frac{3}{2},\frac{3}{2},\arccos(x)\right)\mathrm{arccos}}$$

✓ Solution by Mathematica

Time used: 4.776 (sec). Leaf size: 86

```
DSolve[y'[x]==\[Lambda]*ArcCos[x]^n*(y[x]-a*x^m-b)^2+a*m*x^(m-1),y[x],x,IncludeSingularSolut
```

$$y(x) \to ax^m + \frac{1}{-\frac{1}{2}\lambda\arccos(x)^n(-i\arccos(x))^{-n}\Gamma(n+1,-i\arccos(x))-\frac{1}{2}\lambda(i\arccos(x))^{-n}\arccos(x)^n\Gamma(n+1,i\,\mathrm{arcc}} + b$$

$$y(x) \to ax^m + b$$

15.8 problem 17

Internal problem ID [10574]
Internal file name [`OUTPUT/9521_Monday_June_06_2022_03_03_44_PM_79556931/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-2. Equations containing arccosine.
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - \lambda \arccos(x)^n y^2 - ky = \lambda b^2 x^{2k} \arccos(x)^n}$$

15.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\arccos(x)^n \lambda y^2 + ky + \lambda b^2 x^{2k} \arccos(x)^n}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{\lambda b^2 x^{2k} \arccos(x)^n}{x} + \frac{\arccos(x)^n \lambda y^2}{x} + \frac{ky}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\lambda b^2 x^{2k} \arccos(x)^n}{x}$, $f_1(x) = \frac{k}{x}$ and $f_2(x) = \frac{\arccos(x)^n \lambda}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{\arccos(x)^n \lambda u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{\arccos(x)^n n\lambda}{\sqrt{-x^2+1}\,\arccos(x)\,x} - \frac{\arccos(x)^n \lambda}{x^2} \\
f_1 f_2 &= \frac{k \arccos(x)^n \lambda}{x^2} \\
f_2^2 f_0 &= \frac{\arccos(x)^{3n} \lambda^3 b^2 x^{2k}}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\arccos(x)^n \lambda u''(x)}{x} - \left(-\frac{\arccos(x)^n n\lambda}{\sqrt{-x^2+1}\,\arccos(x)\,x} - \frac{\arccos(x)^n \lambda}{x^2} + \frac{k \arccos(x)^n \lambda}{x^2}\right) u'(x) + \frac{\arccos(x)^{3n} \lambda}{x^3}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \mathrm{e}^{ib\lambda\left(\int x^{k-1} \arccos(x)^n dx\right)} + c_2 \mathrm{e}^{-ib\lambda\left(\int x^{k-1} \arccos(x)^n dx\right)}$$

The above shows that

$$u'(x) = ib\, x^{k-1} \lambda \arccos(x)^n \mathrm{e}^{-ib\lambda\left(\int x^{k-1} \arccos(x)^n dx\right)} \left(\mathrm{e}^{2ib\lambda\left(\int x^{k-1} \arccos(x)^n dx\right)} c_1 - c_2\right)$$

Using the above in (1) gives the solution

$$y = -\frac{ib\, x^{k-1} \mathrm{e}^{-ib\lambda\left(\int x^{k-1} \arccos(x)^n dx\right)} \left(\mathrm{e}^{2ib\lambda\left(\int x^{k-1} \arccos(x)^n dx\right)} c_1 - c_2\right) x}{c_1 \mathrm{e}^{ib\lambda\left(\int x^{k-1} \arccos(x)^n dx\right)} + c_2 \mathrm{e}^{-ib\lambda\left(\int x^{k-1} \arccos(x)^n dx\right)}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{ib\, x^k \left(\mathrm{e}^{2ib\lambda\left(\int x^{k-1} \arccos(x)^n dx\right)} c_3 - 1\right)}{\mathrm{e}^{2ib\lambda\left(\int x^{k-1} \arccos(x)^n dx\right)} c_3 + 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{i b\, x^{k} \left({\mathrm e}^{2 i b \lambda \left(\int x^{k -1} \arccos \left(x \right)^{n}d x \right)} c_{3}-1\right)}{{\mathrm e}^{2 i b \lambda \left(\int x^{k -1} \arccos \left(x \right)^{n}d x \right)} c_{3}+1} \tag{1}$$

Verification of solutions

$$y = -\frac{i b\, x^{k} \left({\mathrm e}^{2 i b \lambda \left(\int x^{k -1} \arccos \left(x \right)^{n}d x \right)} c_{3}-1\right)}{{\mathrm e}^{2 i b \lambda \left(\int x^{k -1} \arccos \left(x \right)^{n}d x \right)} c_{3}+1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 29

```
dsolve(x*diff(y(x),x)=lambda*arccos(x)^n*y(x)^2+k*y(x)+lambda*b^2*x^(2*k)*arccos(x)^n,y(x),
```

$$y(x) = -\tan\left(-\lambda b\left(\int x^{-1+k} \arccos\left(x\right)^{n} dx\right)+c_1\right) b\, x^{k}$$

✓ Solution by Mathematica

Time used: 2.128 (sec). Leaf size: 48

```
DSolve[x*y'[x]==\[Lambda]*ArcCos[x]^n*y[x]^2+k*y[x]+\[Lambda]*b^2*x^(2*k)*ArcCos[x]^n,y[x],x
```

$$y(x) \to \sqrt{b^2} x^k \tan\left(\sqrt{b^2} \int_1^x \lambda \arccos(K[1])^n K[1]^{k-1} dK[1] + c_1\right)$$

15.9 problem 18

Internal problem ID [10575]
Internal file name [OUTPUT/9522_Monday_June_06_2022_03_03_46_PM_39589339/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-2. Equations containing arccosine.
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y'x-\left(a\,x^{2m}y^2+yx^nb+c\right)\arccos\left(x\right)^m+yn=0}$$

15.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= \frac{\arccos\left(x\right)^m x^{2m}a\,y^2+\arccos\left(x\right)^m x^n by+\arccos\left(x\right)^m c-ny}{x}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=\frac{\arccos\left(x\right)^m x^{2m}a\,y^2}{x}+\frac{\arccos\left(x\right)^m x^n by}{x}+\frac{\arccos\left(x\right)^m c}{x}-\frac{ny}{x}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=\frac{\arccos(x)^m c}{x}$, $f_1(x)=\frac{\arccos(x)^m x^n b-n}{x}$ and $f_2(x)=\frac{\arccos(x)^m x^{2m}a}{x}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2u}\\ &= \frac{-u'}{\frac{\arccos(x)^m x^{2m}au}{x}}\end{aligned}\tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{\arccos(x)^m\, m\, x^{2m}a}{\sqrt{-x^2+1}\,\arccos(x)\,x} + \frac{2\arccos(x)^m\, x^{2m}ma}{x^2} - \frac{\arccos(x)^m\, x^{2m}a}{x^2} \\
f_1f_2 &= \frac{(\arccos(x)^m\, x^nb - n)\arccos(x)^m\, x^{2m}a}{x^2} \\
f_2^2f_0 &= \frac{\arccos(x)^{3m}\, x^{4m}a^2c}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\arccos(x)^m\, x^{2m}au''(x)}{x} - \left(-\frac{\arccos(x)^m\, m\, x^{2m}a}{\sqrt{-x^2+1}\,\arccos(x)\,x} + \frac{2\arccos(x)^m\, x^{2m}ma}{x^2} - \frac{\arccos(x)^m\, x^{2m}a}{x^2} + \frac{(\arccos$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) + \frac{m\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} - b\,x^{n-1}\arccos(x)^m\_Y'(x) + ac\,x^{2m-2}\_Y(x)\arccos(x)^{2m} + \frac{n\_Y'(x)}{x} - \frac{2m\_Y'(x)}{x} + \frac{\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{m\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} - b\,x^{n-1}\arccos(x)^m\_Y'(x) + ac\,x^{2m-2}\_Y(x)\arccos(x)^{2m} + \frac{n\_Y'(x)}{x} - \frac{2m\_Y'(x)}{x} + \frac{\_Y'(x)}{x}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{m\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} - b\,x^{n-1}\arccos(x)^m\_Y'(x) + ac\,x^{2m-2}\_Y(x)\arccos(x)^{2m} + \right.\right.\right.}{a\,\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{m\_Y'(x)}{\arccos(x)\sqrt{-x^2+1}} - b\,x^{n-1}\arccos(x)^m\_Y'(x) + ac\,x^{2m-2}\_Y(x)\arccos\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x^{-2m+1}\arccos(x)^{-m}\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(ac\,x^{2m-1}\_\_Y(x)\arccos(x)^{1+2m}-b\,x^{n}\arccos(x)^{m+1}\_\_Y'(x)-2\left(-\frac{\_\_Y''(x)x}{2}+\_\_Y\cdots\right)\right)}{\sqrt{-x^{2}+1}\,\arccos(x)x}\right.\right.\right)}{a\,\mathrm{DESol}\left(\left\{\frac{\left(x^{2m}\arccos(x)^{1+2m}\_\_Y(x)ac-\arccos(x)^{m+1}x^{n+1}\_\_Y'(x)b-2\left(-\frac{\_\_Y''(x)x}{2}+\_\_Y'(x)\left(m-\frac{n}{2}-\frac{1}{2}\right)\right)\cdots\right)}{\sqrt{-x^{2}+1}\,\arccos(x)x^{2}}\right.\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^{-2m+1}\arccos(x)^{-m}\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(ac\,x^{2m-1}\_\_Y(x)\arccos(x)^{1+2m}-b\,x^{n}\arccos(x)^{m+1}\_\_Y'(x)-2\left(-\frac{\_\_Y''(x)x}{2}+\_\_Y\cdots\right)\right)}{\sqrt{-x^{2}+1}\,\arccos(x)x}\right.\right.\right)}{a\,\mathrm{DESol}\left(\left\{\frac{\left(x^{2m}\arccos(x)^{1+2m}\_\_Y(x)ac-\arccos(x)^{m+1}x^{n+1}\_\_Y'(x)b-2\left(-\frac{\_\_Y''(x)x}{2}+\_\_Y'(x)\left(m-\frac{n}{2}-\frac{1}{2}\right)\right)\cdots\right)}{\sqrt{-x^{2}+1}\,\arccos(x)x^{2}}\right.\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{x^{-2m+1}\arccos(x)^{-m}\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\left(ac\,x^{2m-1}\_\_Y(x)\arccos(x)^{1+2m}-b\,x^{n}\arccos(x)^{m+1}\_\_Y'(x)-2\left(-\frac{\_\_Y''(x)x}{2}+\_\_Y\cdots\right)\right)}{\sqrt{-x^{2}+1}\,\arccos(x)x}\right.\right.\right)}{a\,\mathrm{DESol}\left(\left\{\frac{\left(x^{2m}\arccos(x)^{1+2m}\_\_Y(x)ac-\arccos(x)^{m+1}x^{n+1}\_\_Y'(x)b-2\left(-\frac{\_\_Y''(x)x}{2}+\_\_Y'(x)\left(m-\frac{n}{2}-\frac{1}{2}\right)\right)\cdots\right)}{\sqrt{-x^{2}+1}\,\arccos(x)x^{2}}\right.\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x^(n-1)*(-x^2+1)^(1/2)*arccos
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(x^(-1+2*m)*a*arccos(x)^m*y(x)^2+y(x)+(x^
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x)=(a*x^(2*m)*y(x)^2+b*x^n*y(x)+c)*arccos(x)^m-n*y(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y'[x]==(a*x^(2*m)*y[x]^2+b*x^n*y[x]+c)*ArcCos[x]^m-n*y[x],y[x],x,IncludeSingularSol
```

Not solved

16 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.

16.1 problem 19

Internal problem ID [10576]
Internal file name [`OUTPUT/9523_Monday_June_06_2022_03_04_01_PM_31783764/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - \lambda \arctan(x)^n y = -a^2 + a\lambda \arctan(x)^n}$$

16.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + \lambda \arctan(x)^n y - a^2 + a\lambda \arctan(x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \lambda \arctan(x)^n y - a^2 + a\lambda \arctan(x)^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2 + a\lambda \arctan(x)^n$, $f_1(x) = \arctan(x)^n \lambda$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \arctan(x)^n \lambda \\ f_2^2 f_0 &= -a^2 + a\lambda \arctan(x)^n \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \arctan(x)^n \lambda u'(x) + \left(-a^2 + a\lambda \arctan(x)^n\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\int \frac{a\left(\int \mathrm{e}^{-\left(\int(-\arctan(x)^n\lambda+2a)dx\right)}dx\right) - c_1 a + \mathrm{e}^{-\left(\int(-\arctan(x)^n\lambda+2a)dx\right)}}{-c_1 + \int \mathrm{e}^{-\left(\int(-\arctan(x)^n\lambda+2a)dx\right)}dx} dx} c_2$$

The above shows that

$$u'(x) = \frac{\left(a\left(\int \mathrm{e}^{-\left(\int(-\arctan(x)^n\lambda+2a)dx\right)}dx\right) - c_1 a + \mathrm{e}^{-\left(\int(-\arctan(x)^n\lambda+2a)dx\right)}\right) \mathrm{e}^{\int \frac{a\left(\int \mathrm{e}^{-\left(\int(-\arctan(x)^n\lambda+2a)dx\right)}dx\right) - c_1 a + \mathrm{e}^{-(\int(-a \cdots}}{-c_1 + \int \mathrm{e}^{-\left(\int(-\arctan(x)^n\lambda+2a)dx\right)} \cdots}}}{-c_1 + \int \mathrm{e}^{-\left(\int(-\arctan(x)^n\lambda+2a)dx\right)}dx}$$

Using the above in (1) gives the solution

$$y = -\frac{a\left(\int \mathrm{e}^{-\left(\int(-\arctan(x)^n\lambda+2a)dx\right)}dx\right) - c_1 a + \mathrm{e}^{-\left(\int(-\arctan(x)^n\lambda+2a)dx\right)}}{-c_1 + \int \mathrm{e}^{-\left(\int(-\arctan(x)^n\lambda+2a)dx\right)}dx}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-a\left(\int \mathrm{e}^{-\left(\int(-\arctan(x)^n\lambda+2a)dx\right)}dx\right) + c_3 a - \mathrm{e}^{-\left(\int(-\arctan(x)^n\lambda+2a)dx\right)}}{-c_3 + \int \mathrm{e}^{-\left(\int(-\arctan(x)^n\lambda+2a)dx\right)}dx}$$

Summary
The solution(s) found are the following

$$y = \frac{-a\left(\int e^{-\left(\int (-\arctan(x)^n \lambda+2a)dx\right)}dx\right) + c_3 a - e^{-\left(\int (-\arctan(x)^n \lambda+2a)dx\right)}}{-c_3 + \int e^{-\left(\int (-\arctan(x)^n \lambda+2a)dx\right)}dx} \tag{1}$$

Verification of solutions

$$y = \frac{-a\left(\int e^{-\left(\int (-\arctan(x)^n \lambda+2a)dx\right)}dx\right) + c_3 a - e^{-\left(\int (-\arctan(x)^n \lambda+2a)dx\right)}}{-c_3 + \int e^{-\left(\int (-\arctan(x)^n \lambda+2a)dx\right)}dx}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = arctan(x)^n*lambda*(diff(y(x),
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 71

```
dsolve(diff(y(x),x)=y(x)^2+lambda*arctan(x)^n*y(x)-a^2+a*lambda*arctan(x)^n,y(x), singsol=al
```

$$y(x) = \frac{-c_1 a - a\left(\int e^{-\left(\int (-\arctan(x)^n \lambda + 2a)dx\right)} dx\right) - e^{-\left(\int (-\arctan(x)^n \lambda + 2a)dx\right)}}{c_1 + \int e^{-\left(\int (-\arctan(x)^n \lambda + 2a)dx\right)} dx}$$

✓ Solution by Mathematica

Time used: 7.862 (sec). Leaf size: 210

```
DSolve[y'[x]==y[x]^2+\[Lambda]*ArcTan[x]^n*y[x]-a^2+a*\[Lambda]*ArcTan[x]^n,y[x],x,IncludeSi
```

$$\text{Solve}\left[\int_1^x \frac{\exp\left(-\int_1^{K[2]} (2a - \lambda \arctan(K[1])^n)\, dK[1]\right) (-\lambda \arctan(K[2])^n + a - y(x))}{n\lambda(a + y(x))} dK[2]\right.$$

$$+ \int_1^{y(x)} \left(\frac{\exp\left(-\int_1^x (2a - \lambda \arctan(K[1])^n)\, dK[1]\right)}{n\lambda(a + K[3])^2}\right.$$

$$- \int_1^x \left(-\frac{\exp\left(-\int_1^{K[2]} (2a - \lambda \arctan(K[1])^n)\, dK[1]\right) (-\lambda \arctan(K[2])^n + a - K[3])}{n\lambda(a + K[3])^2} - \frac{\exp\left(-\int_1^{K[2]} (2a\right.}{n\lambda}\right.$$

16.2 problem 20

Internal problem ID [10577]
Internal file name [`OUTPUT/9524_Monday_June_06_2022_03_04_08_PM_65810070/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - \lambda x \arctan(x)^n y = \arctan(x)^n \lambda}$$

16.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + \arctan(x)^n \lambda x y + \arctan(x)^n \lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \arctan(x)^n \lambda x y + \arctan(x)^n \lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \arctan(x)^n \lambda$, $f_1(x) = \arctan(x)^n \lambda x$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \arctan(x)^n \lambda x \\ f_2^2 f_0 &= \arctan(x)^n \lambda \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \arctan(x)^n \lambda x u'(x) + \arctan(x)^n \lambda u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x \left(c_1 + c_2 \left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n \lambda\right)}{x^2+1} dx}}{x^2\left(x^2+1\right)} dx\right)\right)$$

The above shows that

$$u'(x) = \frac{\left(x^2+1\right) x c_2 \left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n \lambda\right)}{x^2+1} dx}}{x^2\left(x^2+1\right)} dx\right) + c_2 e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n \lambda\right)}{x^2+1} dx} + c_1 x(x^2+1)}{x\left(x^2+1\right)}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(x^2+1\right) x c_2 \left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n \lambda\right)}{x^2+1} dx}}{x^2\left(x^2+1\right)} dx\right) + c_2 e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n \lambda\right)}{x^2+1} dx} + c_1 x(x^2+1)}{x^2\left(x^2+1\right)\left(c_1 + c_2 \left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n \lambda\right)}{x^2+1} dx}}{x^2\left(x^2+1\right)} dx\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-x^3 - x)\left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n\lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right) - c_3x^3 - c_3x - e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n\lambda\right)}{x^2+1}dx}}{x^2\left(x^2+1\right)\left(c_3 + \int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n\lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{(-x^3 - x)\left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n\lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right) - c_3x^3 - c_3x - e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n\lambda\right)}{x^2+1}dx}}{x^2\left(x^2+1\right)\left(c_3 + \int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n\lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right)} \tag{1}$$

Verification of solutions

$$y = \frac{(-x^3 - x)\left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n\lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right) - c_3x^3 - c_3x - e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n\lambda\right)}{x^2+1}dx}}{x^2\left(x^2+1\right)\left(c_3 + \int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\arctan(x)^n\lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 78

```
dsolve(diff(y(x),x)=y(x)^2+lambda*x*arctan(x)^n*y(x)+lambda*arctan(x)^n,y(x), singsol=all)
```

$$y(x) = \frac{e^{\int \frac{\arctan(x)^n \lambda x^2 - 2}{x} dx} x + \int e^{\int \frac{\arctan(x)^n \lambda x^2 - 2}{x} dx} dx - c_1}{\left(c_1 - \left(\int e^{\int \frac{\arctan(x)^n \lambda x^2 - 2}{x} dx} dx\right)\right) x}$$

✓ Solution by Mathematica

Time used: 7.063 (sec). Leaf size: 120

```
DSolve[y'[x]==y[x]^2+\[Lambda]*x*ArcTan[x]^n*y[x]+\[Lambda]*ArcTan[x]^n,y[x],x,IncludeSingul
```

$$y(x) \to -\frac{\exp\left(-\int_1^x -\lambda \arctan(K[1])^n K[1] dK[1]\right) + x \int_1^x \frac{\exp\left(-\int_1^{K[2]} -\lambda \arctan(K[1])^n K[1] dK[1]\right)}{K[2]^2} dK[2] + c_1 x}{x^2 \left(\int_1^x \frac{\exp\left(-\int_1^{K[2]} -\lambda \arctan(K[1])^n K[1] dK[1]\right)}{K[2]^2} dK[2] + c_1\right)}$$

$$y(x) \to -\frac{1}{x}$$

16.3 problem 21

Internal problem ID [10578]
Internal file name [`OUTPUT/9525_Monday_June_06_2022_03_04_13_PM_78552728/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + (k+1)\, x^k y^2 - \lambda \arctan(x)^n \left(x^{k+1} y - 1\right) = 0}$$

16.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x^{k+1} \arctan(x)^n \lambda y - x^k y^2 k - x^k y^2 - \arctan(x)^n \lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^k x \arctan(x)^n \lambda y - x^k y^2 k - x^k y^2 - \arctan(x)^n \lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\arctan(x)^n \lambda$, $f_1(x) = x^{k+1} \arctan(x)^n \lambda$ and $f_2(x) = -x^k k - x^k$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(-x^k k - x^k)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{k^2 x^k}{x} - \frac{k\,x^k}{x} \\ f_1 f_2 &= x^{k+1} \arctan(x)^n \lambda(-x^k k - x^k) \\ f_2^2 f_0 &= -(-x^k k - x^k)^2 \arctan(x)^n \lambda \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(-x^k k - x^k) u''(x) - \left(-\frac{k^2 x^k}{x} - \frac{k\,x^k}{x} + x^{k+1} \arctan(x)^n \lambda(-x^k k - x^k)\right) u'(x) - (-x^k k - x^k)^2 \arctan(x$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{k+1}\left(\left(\int x^{-2k-2} e^{\int \left(x^{k+1} \arctan(x)^n \lambda + \frac{k}{x}\right) dx} dx\right) c_2 + c_1\right)$$

The above shows that

$$\begin{aligned} u'(x) = {} & c_2 x^{-k-1} e^{\int \frac{x^{k+2} \lambda \arctan(x)^n + k}{x} dx} \\ & + (k+1)\left(\left(\int e^{\int \frac{x^{k+2} \lambda \arctan(x)^n + k}{x} dx} x^{-2k-2} dx\right) c_2 + c_1\right) x^k \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(c_2 x^{-k-1} e^{\int \frac{x^{k+2} \lambda \arctan(x)^n + k}{x} dx} + (k+1)\left(\left(\int e^{\int \frac{x^{k+2} \lambda \arctan(x)^n + k}{x} dx} x^{-2k-2} dx\right) c_2 + c_1\right) x^k\right) x^{-k-1}}{(-x^k k - x^k)\left(\left(\int x^{-2k-2} e^{\int \left(x^{k+1} \arctan(x)^n \lambda + \frac{k}{x}\right) dx} dx\right) c_2 + c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(k+1)\left(\int x^{-2k-2} e^{\int \left(x^{k+1} \arctan(x)^n \lambda + \frac{k}{x}\right) dx} dx + c_3\right) x^{-k-1} + x^{-3k-2} e^{\int \left(x^{k+1} \arctan(x)^n \lambda + \frac{k}{x}\right) dx}}{(k+1)\left(\int e^{\int \frac{x^{k+2} \lambda \arctan(x)^n + k}{x} dx} x^{-2k-2} dx + c_3\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{(k+1)\left(\int x^{-2k-2} \mathrm{e}^{\int \left(x^{k+1} \arctan(x)^{n} \lambda+\frac{k}{x}\right)dx} dx+c_{3}\right) x^{-k-1}+x^{-3k-2} \mathrm{e}^{\int \left(x^{k+1} \arctan(x)^{n} \lambda+\frac{k}{x}\right)dx}}{(k+1)\left(\int \mathrm{e}^{\int \frac{x^{k+2} \lambda \arctan(x)^{n}+k}{x}dx} x^{-2k-2}dx+c_{3}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{(k+1)\left(\int x^{-2k-2} \mathrm{e}^{\int \left(x^{k+1} \arctan(x)^{n} \lambda+\frac{k}{x}\right)dx} dx+c_{3}\right) x^{-k-1}+x^{-3k-2} \mathrm{e}^{\int \left(x^{k+1} \arctan(x)^{n} \lambda+\frac{k}{x}\right)dx}}{(k+1)\left(\int \mathrm{e}^{\int \frac{x^{k+2} \lambda \arctan(x)^{n}+k}{x}dx} x^{-2k-2}dx+c_{3}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x^(1+k)*arctan(x)^n*x*lambda+
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 180

```
dsolve(diff(y(x),x)=-(k+1)*x^k*y(x)^2+lambda*arctan(x)^n*(x^(k+1)*y(x)-1),y(x), singsol=all)
```

$$y(x) = \frac{x^{-1-k}\left(\left(\left(\int x^k \mathrm{e}^{\lambda\left(\int \arctan(x)^n x^{1+k} dx\right)-2\left(\int \frac{1}{x} dx\right)(1+k)} dx\right) k + x^{1+k} \mathrm{e}^{\int \frac{x^{1+k} \arctan(x)^n x\lambda-2k-2}{x} dx} + \int x^k \mathrm{e}^{\lambda\left(\int \arctan(x)^n x^{1+k} \right.}\right.\right.}{\left(\int x^k \mathrm{e}^{\lambda\left(\int \arctan(x)^n x^{1+k} dx\right)-2\left(\int \frac{1}{x} dx\right)(1+k)} dx\right) k + \int x^k \mathrm{e}^{\lambda\left(\int \arctan(x)^n x^{1+k} dx\right)-2\left(\int \frac{1}{x} dx\right)(1+k)} dx}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==-(k+1)*x^k*y[x]^2+\[Lambda]*ArcTan[x]^n*(x^(k+1)*y[x]-1),y[x],x,IncludeSingula
```

Not solved

16.4 problem 22

Internal problem ID [10579]
Internal file name [`OUTPUT/9526_Monday_June_06_2022_03_04_23_PM_74571615/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \arctan(x)^n y^2 - ay = ab - b^2 \lambda \arctan(x)^n}$$

16.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \lambda \arctan(x)^n y^2 + ya + ab - b^2 \lambda \arctan(x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \lambda \arctan(x)^n y^2 + ya + ab - b^2 \lambda \arctan(x)^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = ab - b^2 \lambda \arctan(x)^n$, $f_1(x) = a$ and $f_2(x) = \arctan(x)^n \lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\arctan(x)^n \lambda u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{\arctan(x)^n n\lambda}{(x^2+1)\arctan(x)} \\ f_1 f_2 &= a\lambda \arctan(x)^n \\ f_2^2 f_0 &= \arctan(x)^{2n} \lambda^2 \left(ab - b^2\lambda \arctan(x)^n\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\arctan(x)^n \lambda u''(x) - \left(\frac{\arctan(x)^n n\lambda}{(x^2+1)\arctan(x)} + a\lambda \arctan(x)^n\right) u'(x) + \arctan(x)^{2n} \lambda^2 \left(ab - b^2\lambda \arctan(x)^n\right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{(x^2+1)\arctan(x)} - a\_Y'(x) - \arctan(x)^{2n} b^2\lambda^2\_Y(x) + \arctan(x)^n ab\lambda\_Y(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{(x^2+1)\arctan(x)} - a\_Y'(x) - \arctan(x)^{2n} b^2\lambda^2\_Y(x) + \arctan(x)^n ab\lambda\_Y(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{(x^2+1)\arctan(x)} - a\_Y'(x) - \arctan(x)^{2n} b^2\lambda^2\_Y(x) + \arctan(x)^n ab\lambda\_Y(x)\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{(x^2+1)\arctan(x)} - a\_Y'(x) - \arctan(x)^{2n} b^2\lambda^2\_Y(x) + \arctan(x)^n ab\lambda\_\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-b^2\_\_Y(x)\lambda^2 \arctan(x)^{1+2n}(x^2+1)+ab\lambda\_\_Y(x)\arctan(x)^{n+1}(x^2+1)+\_\_Y''(x)(x^2+1)\arctan(x)-\_\_Y'(x)(a(x^2}{(x^2+1)\arctan(x)}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{-b^2\_\_Y(x)\lambda^2\arctan(x)^{1+2n}(x^2+1)+ab\lambda\_\_Y(x)\arctan(x)^{n+1}(x^2+1)+\_\_Y''(x)(x^2+1)\arctan(x)-\_\_Y'(x}{(x^2+1)\arctan(x)}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-b^2\_\_Y(x)\lambda^2 \arctan(x)^{1+2n}(x^2+1)+ab\lambda\_\_Y(x)\arctan(x)^{n+1}(x^2+1)+\_\_Y''(x)(x^2+1)\arctan(x)-\_\_Y'(x)(a(x^2}{(x^2+1)\arctan(x)}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{-b^2\_\_Y(x)\lambda^2\arctan(x)^{1+2n}(x^2+1)+ab\lambda\_\_Y(x)\arctan(x)^{n+1}(x^2+1)+\_\_Y''(x)(x^2+1)\arctan(x)-\_\_Y'(x}{(x^2+1)\arctan(x)}\right.\right.} \quad (1)$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-b^2\_\_Y(x)\lambda^2 \arctan(x)^{1+2n}(x^2+1)+ab\lambda\_\_Y(x)\arctan(x)^{n+1}(x^2+1)+\_\_Y''(x)(x^2+1)\arctan(x)-\_\_Y'(x)(a(x^2}{(x^2+1)\arctan(x)}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{-b^2\_\_Y(x)\lambda^2\arctan(x)^{1+2n}(x^2+1)+ab\lambda\_\_Y(x)\arctan(x)^{n+1}(x^2+1)+\_\_Y''(x)(x^2+1)\arctan(x)-\_\_Y'(x}{(x^2+1)\arctan(x)}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (arctan(x)*a*x^2+a*arctan(x)+n
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 87

```
dsolve(diff(y(x),x)=lambda*arctan(x)^n*y(x)^2+a*y(x)+a*b-b^2*lambda*arctan(x)^n,y(x), singso
```

$$y(x) = \frac{-b\lambda\left(\int \arctan(x)^n \mathrm{e}^{-\left(\int (2\arctan(x)^n \lambda b-a)dx\right)}dx\right) - c_1 b - \mathrm{e}^{-\left(\int (2\arctan(x)^n \lambda b-a)dx\right)}}{c_1 + \lambda\left(\int \arctan(x)^n \mathrm{e}^{-\left(\int (2\arctan(x)^n \lambda b-a)dx\right)}dx\right)}$$

✓ Solution by Mathematica

Time used: 10.998 (sec). Leaf size: 240

```
DSolve[y'[x]==\[Lambda]*ArcTan[x]^n*y[x]^2+a*y[x]+a*b-b^2*\[Lambda]*ArcTan[x]^n,y[x],x,Inclu
```

$$\text{Solve}\left[\int_1^x \frac{\exp\left(-\int_1^{K[2]}(2b\lambda\arctan(K[1])^n-a)\,dK[1]\right)(-b\lambda\arctan(K[2])^n+\lambda y(x)\arctan(K[2])^n+a)}{an\lambda(b+y(x))} d\right.$$

$$+\int_1^{y(x)}\left(-\int_1^x\left(\frac{\exp\left(-\int_1^{K[2]}(2b\lambda\arctan(K[1])^n-a)\,dK[1]\right)\arctan(K[2])^n}{an(b+K[3])}-\frac{\exp\left(-\int_1^{K[2]}(2b\lambda\arctan(\right.}{}\right.\right.$$

$$\left.\left.-\frac{\exp\left(-\int_1^x(2b\lambda\arctan(K[1])^n-a)\,dK[1]\right)}{an\lambda(b+K[3])^2}\right)dK[3]=c_1,y(x)\right]$$

16.5 problem 23

Internal problem ID [10580]
Internal file name [`OUTPUT/9527_Monday_June_06_2022_03_04_28_PM_46831198/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \arctan(x)^n y^2 + b\lambda x^m \arctan(x)^n y = bm x^{m-1}}$$

16.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \lambda \arctan(x)^n y^2 - b\lambda x^m \arctan(x)^n y + bm x^{m-1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \lambda \arctan(x)^n y^2 - b\lambda x^m \arctan(x)^n y + \frac{b x^m m}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = bm x^{m-1}$, $f_1(x) = -b\lambda x^m \arctan(x)^n$ and $f_2(x) = \arctan(x)^n \lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\arctan(x)^n \lambda u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{\arctan(x)^n n\lambda}{(x^2+1)\arctan(x)} \\ f_1f_2 &= -b\,\lambda^2 x^m \arctan(x)^{2n} \\ f_2^2 f_0 &= \arctan(x)^{2n} \lambda^2 bm\, x^{m-1} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\arctan(x)^n \lambda u''(x) - \left(\frac{\arctan(x)^n n\lambda}{(x^2+1)\arctan(x)} - b\,\lambda^2 x^m \arctan(x)^{2n}\right) u'(x) + \arctan(x)^{2n} \lambda^2 bm\, x^{m-1} u(x) =$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{(x^2+1)\arctan(x)} + \arctan(x)^n b\lambda\, x^m \_Y'(x) + \arctan(x)^n \lambda bm\, x^{m-1} \_Y(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{(x^2+1)\arctan(x)} + \arctan(x)^n b\lambda\, x^m \_Y'(x) + \arctan(x)^n \lambda bm\, x^{m-1} \_Y(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{(x^2+1)\arctan(x)} + \arctan(x)^n b\lambda\, x^m \_Y'(x) + \arctan(x)^n \lambda bm\, x^{m-1} \_Y(x)\right\}, \{\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{(x^2+1)\arctan(x)} + \arctan(x)^n b\lambda\, x^m \_Y'(x) + \arctan(x)^n \lambda bm\, x^{m-1} \_Y(\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{b\lambda(x^2+1)\left(m\_\_Y(x)x^{m-1}+\_\_Y'(x)x^m\right)\arctan(x)^{n+1}+\_\_Y''(x)(x^2+1)\arctan(x)-n\_\_Y'(x)}{(x^2+1)\arctan(x)}\right\},\{\_\_Y(x)\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{b\lambda\left(m\,x^m\_\_Y(x)+m\_\_Y(x)x^{2+m}+\_\_Y'(x)x^{3+m}+\_\_Y'(x)x^{m+1}\right)\arctan(x)^{n+1}-x\left((-x^2-1)\arctan(x)\_\_Y''(x)+n\_\_\right.}{(x^2+1)\arctan(x)x}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{b\lambda(x^2+1)\left(m\_\_Y(x)x^{m-1}+\_\_Y'(x)x^m\right)\arctan(x)^{n+1}+\_\_Y''(x)(x^2+1)\arctan(x)-n\_\_Y'(x)}{(x^2+1)\arctan(x)}\right\},\{\_\_Y(x)\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{b\lambda\left(m\,x^m\_\_Y(x)+m\_\_Y(x)x^{2+m}+\_\_Y'(x)x^{3+m}+\_\_Y'(x)x^{m+1}\right)\arctan(x)^{n+1}-x\left((-x^2-1)\arctan(x)\_\_Y''(x)+n\_\_\right.}{(x^2+1)\arctan(x)x}\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{b\lambda(x^2+1)\left(m\_\_Y(x)x^{m-1}+\_\_Y'(x)x^m\right)\arctan(x)^{n+1}+\_\_Y''(x)(x^2+1)\arctan(x)-n\_\_Y'(x)}{(x^2+1)\arctan(x)}\right\},\{\_\_Y(x)\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{b\lambda\left(m\,x^m\_\_Y(x)+m\_\_Y(x)x^{2+m}+\_\_Y'(x)x^{3+m}+\_\_Y'(x)x^{m+1}\right)\arctan(x)^{n+1}-x\left((-x^2-1)\arctan(x)\_\_Y''(x)+n\_\_\right.}{(x^2+1)\arctan(x)x}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(x^m*arctan(x)^n*arctan(x)*b*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=lambda*arctan(x)^n*y(x)^2-b*lambda*x^m*arctan(x)^n*y(x)+b*m*x^(m-1),y(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==\[Lambda]*ArcTan[x]^n*y[x]^2-b*\[Lambda]*x^m*ArcTan[x]^n*y[x]+b*m*x^(m-1),y[x]
```

Not solved

16.6 problem 24

Internal problem ID [10581]
Internal file name [`OUTPUT/9528_Monday_June_06_2022_03_04_39_PM_60608499/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \arctan(x)^n y^2 = \beta m\, x^{m-1} - \lambda\, \beta^2 x^{2m} \arctan(x)^n}$$

16.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \lambda \arctan(x)^n y^2 + \beta m\, x^{m-1} - \lambda\, \beta^2 x^{2m} \arctan(x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\lambda\, \beta^2 x^{2m} \arctan(x)^n + \lambda \arctan(x)^n y^2 + \frac{\beta m\, x^m}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = \beta m\, x^{m-1} - \lambda\, \beta^2 x^{2m} \arctan(x)^n$, $f_1(x) = 0$ and $f_2(x) = \arctan(x)^n \lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\arctan(x)^n \lambda u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{\arctan(x)^n n\lambda}{(x^2+1)\arctan(x)} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \arctan(x)^{2n} \lambda^2 \left(\beta m\, x^{m-1} - \lambda\, \beta^2 x^{2m} \arctan(x)^n\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\arctan(x)^n \lambda u''(x) - \frac{\arctan(x)^n n\lambda u'(x)}{(x^2+1)\arctan(x)} + \arctan(x)^{2n} \lambda^2 \left(\beta m\, x^{m-1} - \lambda\, \beta^2 x^{2m} \arctan(x)^n\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{(x^2+1)\arctan(x)} - \arctan(x)^{2n} x^{2m}\beta^2\lambda^2\_Y(x) + \arctan(x)^n \beta m\, x^{m-1}\lambda\_Y(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{(x^2+1)\arctan(x)} - \arctan(x)^{2n} x^{2m}\beta^2\lambda^2\_Y(x) + \arctan(x)^n \beta m\, x^{m-1}\lambda\_Y(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{(x^2+1)\arctan(x)} - \arctan(x)^{2n} x^{2m}\beta^2\lambda^2\_Y(x) + \arctan(x)^n \beta m\, x^{m-1}\lambda\_Y(x)\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{(x^2+1)\arctan(x)} - \arctan(x)^{2n} x^{2m}\beta^2\lambda^2\_Y(x) + \arctan(x)^n \beta m\, x^{m-1}\lambda\_\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{\_\_Y''(x)\,(x^2+1)\arctan(x) - n\_\_Y'(x) - \beta^2\_\_Y(x)x^{2m}\lambda^2\arctan(x)^{1+2n}(x^2+1) + m\beta\lambda\_\_Y(x)\arctan(x)^{n+1}x^{m-1}(x}{(x^2+1)\arctan(x)}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{-\beta^2\lambda^2\_\_Y(x)(x^{3+2m}+x^{1+2m})\arctan(x)^{1+2n} + m\beta\lambda\_\_Y(x)(x^m+x^{2+m})\arctan(x)^{n+1} - x\left((-x^2-1)\arctan(x)\_\_\right.}{x(x^2+1)\arctan(x)}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{\_\_Y''(x)\,(x^2+1)\arctan(x) - n\_\_Y'(x) - \beta^2\_\_Y(x)x^{2m}\lambda^2\arctan(x)^{1+2n}(x^2+1) + m\beta\lambda\_\_Y(x)\arctan(x)^{n+1}x^{m-1}(x}{(x^2+1)\arctan(x)}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{-\beta^2\lambda^2\_\_Y(x)(x^{3+2m}+x^{1+2m})\arctan(x)^{1+2n} + m\beta\lambda\_\_Y(x)(x^m+x^{2+m})\arctan(x)^{n+1} - x\left((-x^2-1)\arctan(x)\_\_\right.}{x(x^2+1)\arctan(x)}\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{\_\_Y''(x)\,(x^2+1)\arctan(x) - n\_\_Y'(x) - \beta^2\_\_Y(x)x^{2m}\lambda^2\arctan(x)^{1+2n}(x^2+1) + m\beta\lambda\_\_Y(x)\arctan(x)^{n+1}x^{m-1}(x}{(x^2+1)\arctan(x)}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{-\beta^2\lambda^2\_\_Y(x)(x^{3+2m}+x^{1+2m})\arctan(x)^{1+2n} + m\beta\lambda\_\_Y(x)(x^m+x^{2+m})\arctan(x)^{n+1} - x\left((-x^2-1)\arctan(x)\_\_\right.}{x(x^2+1)\arctan(x)}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = n*(diff(y(x), x))/((x^2+1)*arc
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=lambda*arctan(x)^n*y(x)^2+beta*m*x^(m-1)-lambda*beta^2*x^(2*m)*arctan(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==\[Lambda]*ArcTan[x]^n*y[x]^2+\[Beta]*m*x^(m-1)-\[Lambda]*\[Beta]^2*x^(2*m)*Arc
```

Not solved

16.7 problem 25

Internal problem ID [10582]
Internal file name [`OUTPUT/9529_Monday_June_06_2022_03_04_55_PM_64490874/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]`

$$\boxed{y' - \lambda \arctan(x)^n (y - a\,x^m - b)^2 = am\,x^{m-1}}$$

16.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= x^{2m}\arctan(x)^n a^2\lambda + 2x^m\arctan(x)^n ab\lambda - 2x^m\arctan(x)^n a\lambda y + b^2\lambda\arctan(x)^n - 2\arctan(x)^n b\lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^{2m}\arctan(x)^n a^2\lambda + 2x^m\arctan(x)^n ab\lambda - 2x^m\arctan(x)^n a\lambda y + b^2\lambda\arctan(x)^n - 2\arctan(x)^n b\lambda y + \lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^{2m}\arctan(x)^n a^2\lambda + 2x^m\arctan(x)^n ab\lambda + b^2\lambda\arctan(x)^n + am\,x^{m-1}$, $f_1(x) = -2a\lambda\,x^m\arctan(x)^n - 2\arctan(x)^n\lambda b$ and $f_2(x) = \arctan(x)^n\lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{\arctan(x)^n\lambda u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{\arctan(x)^n n\lambda}{(x^2+1)\arctan(x)} \\
f_1f_2 &= (-2a\lambda\,x^m \arctan(x)^n - 2\arctan(x)^n \lambda b)\arctan(x)^n \lambda \\
f_2^2f_0 &= \arctan(x)^{2n} \lambda^2 (x^{2m}\arctan(x)^n a^2\lambda + 2x^m \arctan(x)^n ab\lambda + b^2\lambda\arctan(x)^n + am\,x^{m-1})
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\arctan(x)^n \lambda u''(x) - \left(\frac{\arctan(x)^n n\lambda}{(x^2+1)\arctan(x)} + (-2a\lambda\,x^m\arctan(x)^n - 2\arctan(x)^n\lambda b)\arctan(x)^n\lambda\right)u'(x$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
u(x) = \mathrm{DESol}\Bigg(\Bigg\{&_Y''(x) - \frac{n_Y'(x)}{(x^2+1)\arctan(x)} + 2\arctan(x)^n \lambda\,x^m a_Y'(x) \\
&+ 2\arctan(x)^n b\lambda_Y'(x) + \arctan(x)^{2n} x^{2m}a^2\lambda^2_Y(x) \\
&+ 2\arctan(x)^{2n} x^m ab\,\lambda^2_Y(x) + \arctan(x)^{2n} b^2\lambda^2_Y(x) \\
&+ \arctan(x)^n am\,x^{m-1}\lambda_Y(x)\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

The above shows that

$$\begin{aligned}
u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\Bigg(\Bigg\{&_Y''(x) - \frac{n_Y'(x)}{(x^2+1)\arctan(x)} + 2\arctan(x)^n \lambda\,x^m a_Y'(x) \\
&+ 2\arctan(x)^n b\lambda_Y'(x) + \arctan(x)^{2n} x^{2m}a^2\lambda^2_Y(x) \\
&+ 2\arctan(x)^{2n} x^m ab\,\lambda^2_Y(x) + \arctan(x)^{2n} b^2\lambda^2_Y(x) \\
&+ \arctan(x)^n am\,x^{m-1}\lambda_Y(x)\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{(x^2+1)\arctan(x)} + 2\arctan(x)^n\lambda\,x^m a\_Y'(x) + 2\arctan(x)^n b\lambda\_Y'(x) + \mathrm{arcta}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{n\_Y'(x)}{(x^2+1)\arctan(x)} + 2\arctan(x)^n\lambda\,x^m a\_Y'(x) + 2\arctan(x)^n b\lambda\_Y'(x) + \right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\frac{\lambda^2 \_Y(x)\left(x^2+1\right)\left(a^2 x^{2m}+2ab\, x^m+b^2\right) \arctan(x)^{1+2n}+\left(a\, x^{m-1} m \_Y(x)+2 \_Y'(x)(\cdots\right.}{\left(x^2+1\right) \arctan(x \cdots}\right.\right.\right.}{\lambda \operatorname{DESol}\left(\left\{\frac{\_Y(x)\left(a^2 x^{1+2m}+a^2 x^{3+2m}+2a\, x^{3+m} b+2a\, x^{m+1} b+b^2 x\left(x^2+1\right)\right)\lambda^2 \arctan(x)^{1+2n}+\left(2a\, x^{3+m} \_Y'(x)+2a\, x^{m+1} \_Y'(\cdots\right.}{x\left(x^2+1\right) \cdots}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\frac{\lambda^2 \_Y(x)\left(x^2+1\right)\left(a^2 x^{2m}+2ab\, x^m+b^2\right) \arctan(x)^{1+2n}+\left(a\, x^{m-1} m \_Y(x)+2 \_Y'(x)(\cdots\right.}{\left(x^2+1\right) \arctan(x \cdots}\right.\right.\right.}{\lambda \operatorname{DESol}\left(\left\{\frac{\_Y(x)\left(a^2 x^{1+2m}+a^2 x^{3+2m}+2a\, x^{3+m} b+2a\, x^{m+1} b+b^2 x\left(x^2+1\right)\right)\lambda^2 \arctan(x)^{1+2n}+\left(2a\, x^{3+m} \_Y'(x)+2a\, x^{m+1} \_Y'(\cdots\right.}{x\left(x^2+1\right) \cdots}\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\frac{\lambda^2 \_Y(x)\left(x^2+1\right)\left(a^2 x^{2m}+2ab\, x^m+b^2\right) \arctan(x)^{1+2n}+\left(a\, x^{m-1} m \_Y(x)+2 \_Y'(x)(\cdots\right.}{\left(x^2+1\right) \arctan(x \cdots}\right.\right.\right.}{\lambda \operatorname{DESol}\left(\left\{\frac{\_Y(x)\left(a^2 x^{1+2m}+a^2 x^{3+2m}+2a\, x^{3+m} b+2a\, x^{m+1} b+b^2 x\left(x^2+1\right)\right)\lambda^2 \arctan(x)^{1+2n}+\left(2a\, x^{3+m} \_Y'(x)+2a\, x^{m+1} \_Y'(\cdots\right.}{x\left(x^2+1\right) \cdots}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (d) successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)=lambda*arctan(x)^n*(y(x)-a*x^m-b)^2+a*m*x^(m-1),y(x), singsol=all)
```

$$y(x) = a\, x^{m} + b + \frac{1}{c_1 - \lambda \left(\int \arctan\left(x\right)^{n} dx\right)}$$

✓ Solution by Mathematica

Time used: 2.089 (sec). Leaf size: 44

```
DSolve[y'[x]==\[Lambda]*ArcTan[x]^n*(y[x]-a*x^m-b)^2+a*m*x^(m-1),y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{1}{-\int_1^x \lambda \arctan(K[2])^n dK[2] + c_1} + ax^m + b$$
$$y(x) \to ax^m + b$$

16.8 problem 26

Internal problem ID [10583]
Internal file name [`OUTPUT/9530_Monday_June_06_2022_03_05_10_PM_55650788/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - \lambda \arctan(x)^n y^2 - ky = \lambda b^2 x^{2k} \arctan(x)^n}$$

16.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\lambda \arctan(x)^n y^2 + ky + \lambda b^2 x^{2k} \arctan(x)^n}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{\lambda b^2 x^{2k} \arctan(x)^n}{x} + \frac{\lambda \arctan(x)^n y^2}{x} + \frac{ky}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\lambda b^2 x^{2k} \arctan(x)^n}{x}$, $f_1(x) = \frac{k}{x}$ and $f_2(x) = \frac{\arctan(x)^n \lambda}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{\arctan(x)^n \lambda u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{\arctan(x)^n n\lambda}{(x^2+1)\arctan(x)\, x} - \frac{\lambda \arctan(x)^n}{x^2} \\
f_1 f_2 &= \frac{k \arctan(x)^n \lambda}{x^2} \\
f_2^2 f_0 &= \frac{\arctan(x)^{3n} \lambda^3 b^2 x^{2k}}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\arctan(x)^n \lambda u''(x)}{x} - \left(\frac{\arctan(x)^n n\lambda}{(x^2+1)\arctan(x)\, x} - \frac{\lambda \arctan(x)^n}{x^2} + \frac{k \arctan(x)^n \lambda}{x^2}\right) u'(x) + \frac{\arctan(x)^{3n} \lambda^3 b^2}{x^3}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 e^{ib\lambda\left(\int x^{k-1} \arctan(x)^n dx\right)} + c_2 e^{-ib\lambda\left(\int x^{k-1} \arctan(x)^n dx\right)}$$

The above shows that

$$u'(x) = ib\, x^{k-1} \lambda \arctan(x)^n\, e^{-ib\lambda\left(\int x^{k-1} \arctan(x)^n dx\right)} \left(c_1 e^{2ib\lambda\left(\int x^{k-1} \arctan(x)^n dx\right)} - c_2\right)$$

Using the above in (1) gives the solution

$$y = -\frac{ib\, x^{k-1} e^{-ib\lambda\left(\int x^{k-1} \arctan(x)^n dx\right)} \left(c_1 e^{2ib\lambda\left(\int x^{k-1} \arctan(x)^n dx\right)} - c_2\right) x}{c_1 e^{ib\lambda\left(\int x^{k-1} \arctan(x)^n dx\right)} + c_2 e^{-ib\lambda\left(\int x^{k-1} \arctan(x)^n dx\right)}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{ib\, x^k \left(c_3 e^{2ib\lambda\left(\int x^{k-1} \arctan(x)^n dx\right)} - 1\right)}{c_3 e^{2ib\lambda\left(\int x^{k-1} \arctan(x)^n dx\right)} + 1}$$

Summary
The solution(s) found are the following

$$y = -\frac{ib\,x^k\left(c_3 \mathrm{e}^{2ib\lambda\left(\int x^{k-1}\arctan(x)^n dx\right)}-1\right)}{c_3 \mathrm{e}^{2ib\lambda\left(\int x^{k-1}\arctan(x)^n dx\right)}+1} \tag{1}$$

Verification of solutions

$$y = -\frac{ib\,x^k\left(c_3 \mathrm{e}^{2ib\lambda\left(\int x^{k-1}\arctan(x)^n dx\right)}-1\right)}{c_3 \mathrm{e}^{2ib\lambda\left(\int x^{k-1}\arctan(x)^n dx\right)}+1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 29

```
dsolve(x*diff(y(x),x)=lambda*arctan(x)^n*y(x)^2+k*y(x)+lambda*b^2*x^(2*k)*arctan(x)^n,y(x),
```

$$y(x) = -\tan\left(-\lambda b\left(\int x^{-1+k}\arctan(x)^n\,dx\right)+c_1\right)b\,x^k$$

✓ Solution by Mathematica
Time used: 1.992 (sec). Leaf size: 48

```
DSolve[x*y'[x]==\[Lambda]*ArcTan[x]^n*y[x]^2+k*y[x]+\[Lambda]*b^2*x^(2*k)*ArcTan[x]^n,y[x],x
```

$$y(x) \to \sqrt{b^2}x^k\tan\left(\sqrt{b^2}\int_1^x \lambda\arctan(K[1])^n K[1]^{k-1}dK[1]+c_1\right)$$

16.9 problem 27

Internal problem ID [10584]
Internal file name [OUTPUT/9531_Monday_June_06_2022_03_05_13_PM_92844768/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - \left(a\,x^{2m}y^2 + yx^nb + c\right)\arctan\left(x\right)^m + yn = 0}$$

16.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\arctan\left(x\right)^m x^{2m}a\,y^2 + \arctan\left(x\right)^m x^nby + \arctan\left(x\right)^m c - ny}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{\arctan\left(x\right)^m x^{2m}a\,y^2}{x} + \frac{\arctan\left(x\right)^m x^nby}{x} + \frac{\arctan\left(x\right)^m c}{x} - \frac{ny}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\arctan(x)^m c}{x}$, $f_1(x) = \frac{\arctan(x)^m x^n b - n}{x}$ and $f_2(x) = \frac{\arctan(x)^m x^{2m}a}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{\frac{\arctan(x)^m x^{2m}au}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{\arctan(x)^m\, m\, x^{2m}a}{(x^2+1)\arctan(x)\,x} + \frac{2\arctan(x)^m\, x^{2m}ma}{x^2} - \frac{\arctan(x)^m\, x^{2m}a}{x^2}\\
f_1f_2 &= \frac{(\arctan(x)^m\, x^n b - n)\arctan(x)^m\, x^{2m}a}{x^2}\\
f_2^2f_0 &= \frac{\arctan(x)^{3m}\, x^{4m}a^2c}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\arctan(x)^m\, x^{2m}au''(x)}{x} - \left(\frac{\arctan(x)^m\, m\, x^{2m}a}{(x^2+1)\arctan(x)\,x} + \frac{2\arctan(x)^m\, x^{2m}ma}{x^2} - \frac{\arctan(x)^m\, x^{2m}a}{x^2} + \frac{(\arctan(\cdots}{\cdots}\right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
u(x) = \mathrm{DESol}\Bigg(\Bigg\{&_Y''(x) - \frac{m_Y'(x)}{\arctan(x)(x^2+1)} - \frac{2m_Y'(x)}{x} + \frac{_Y'(x)}{x}\\
&- b\,x^{n-1}\arctan(x)^m\,_Y'(x) + \frac{n_Y'(x)}{x}\\
&+ ac\,x^{2m-2}_Y(x)\arctan(x)^{2m}\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

The above shows that

$$\begin{aligned}
u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\Bigg(\Bigg\{&_Y''(x) - \frac{m_Y'(x)}{\arctan(x)(x^2+1)} - \frac{2m_Y'(x)}{x} + \frac{_Y'(x)}{x}\\
&- b\,x^{n-1}\arctan(x)^m\,_Y'(x) + \frac{n_Y'(x)}{x}\\
&+ ac\,x^{2m-2}_Y(x)\arctan(x)^{2m}\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{m\_Y'(x)}{\arctan(x)(x^2+1)} - \frac{2m\_Y'(x)}{x} + \frac{\_Y'(x)}{x} - b\,x^{n-1}\arctan(x)^m\,\_Y'(x) + \frac{n\_Y'(x)}{x} + \cdots\right.\right.\right.}{a\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{m\_Y'(x)}{\arctan(x)(x^2+1)} - \frac{2m\_Y'(x)}{x} + \frac{\_Y'(x)}{x} - b\,x^{n-1}\arctan(x)^m\,\_Y'(x) + \cdots\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x^{-2m+1} \arctan(x)^{-m} \left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{a c\, x^{2m-1} \_\_Y(x) \arctan(x)^{1+2m} (x^2+1) - b\, x^n \arctan(x)^{m+1} (x^2+1) \_\_Y'(x) + \_\_Y''(x)}{x(x^2+1)\arctan(x)}\right.\right.\right.}{a\, \mathrm{DESol}\left(\left\{\frac{ac\_\_Y(x)(x^{2m}+x^{2+2m})\arctan(x)^{1+2m} - \_\_Y'(x) b (x^{n+1}+x^{n+3})\arctan(x)^{m+1} - 2\left(-\frac{\_\_Y''(x)\left(x^2+1\right)\operatorname{arcta}}{2}\right.}{(x^2+1)x^2\arctan(x)}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^{-2m+1} \arctan(x)^{-m} \left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{a c\, x^{2m-1} \_\_Y(x) \arctan(x)^{1+2m} (x^2+1) - b\, x^n \arctan(x)^{m+1} (x^2+1) \_\_Y'(x) + \_\_Y''(x)}{x(x^2+1)\arctan(x)}\right.\right.\right.}{a\, \mathrm{DESol}\left(\left\{\frac{ac\_\_Y(x)(x^{2m}+x^{2+2m})\arctan(x)^{1+2m} - \_\_Y'(x) b (x^{n+1}+x^{n+3})\arctan(x)^{m+1} - 2\left(-\frac{\_\_Y''(x)\left(x^2+1\right)\operatorname{arcta}}{2}\right.}{(x^2+1)x^2\arctan(x)}\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{x^{-2m+1} \arctan(x)^{-m} \left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{a c\, x^{2m-1} \_\_Y(x) \arctan(x)^{1+2m} (x^2+1) - b\, x^n \arctan(x)^{m+1} (x^2+1) \_\_Y'(x) + \_\_Y''(x)}{x(x^2+1)\arctan(x)}\right.\right.\right.}{a\, \mathrm{DESol}\left(\left\{\frac{ac\_\_Y(x)(x^{2m}+x^{2+2m})\arctan(x)^{1+2m} - \_\_Y'(x) b (x^{n+1}+x^{n+3})\arctan(x)^{m+1} - 2\left(-\frac{\_\_Y''(x)\left(x^2+1\right)\operatorname{arcta}}{2}\right.}{(x^2+1)x^2\arctan(x)}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (arctan(x)^m*arctan(x)*x^(n-1)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x)=(a*x^(2*m)*y(x)^2+b*x^n*y(x)+c)*arctan(x)^m-n*y(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y'[x]==(a*x^(2*m)*y[x]^2+b*x^n*y[x]+c)*ArcTan[x]^m-n*y[x],y[x],x,IncludeSingularSol
```

Not solved

17 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-4. Equations containing arccotangent.

17.1 problem 28

Internal problem ID [10585]
Internal file name [`OUTPUT/9532_Monday_June_06_2022_03_05_30_PM_38612860/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-4. Equations containing arccotangent.
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - \lambda \operatorname{arccot}(x)^n y = -a^2 + a\lambda \operatorname{arccot}(x)^n}$$

17.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + \lambda \operatorname{arccot}(x)^n y - a^2 + a\lambda \operatorname{arccot}(x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \lambda \operatorname{arccot}(x)^n y - a^2 + a\lambda \operatorname{arccot}(x)^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2 + a\lambda \operatorname{arccot}(x)^n$, $f_1(x) = \operatorname{arccot}(x)^n \lambda$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \operatorname{arccot}(x)^n \lambda \\ f_2^2 f_0 &= -a^2 + a\lambda \operatorname{arccot}(x)^n \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \operatorname{arccot}(x)^n \lambda u'(x) + \left(-a^2 + a\lambda \operatorname{arccot}(x)^n\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\int \frac{a\left(\int \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)} dx\right) - c_1 a + \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)}}{-c_1 + \int \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)} dx} dx} c_2$$

The above shows that

$$u'(x) = \frac{\left(a\left(\int \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)} dx\right) - c_1 a + \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)}\right) \mathrm{e}^{\int \frac{a\left(\int \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)} dx\right) - c_1 a + \mathrm{e}^{-(\int (- \operatorname{ar}}}{-c_1 + \int \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)} d}}}{-c_1 + \int \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)} dx}$$

Using the above in (1) gives the solution

$$y = -\frac{a\left(\int \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)} dx\right) - c_1 a + \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)}}{-c_1 + \int \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)} dx}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-a\left(\int \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)} dx\right) + c_3 a - \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)}}{-c_3 + \int \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)} dx}$$

Summary
The solution(s) found are the following

$$y = \frac{-a\left(\int \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^{n} \lambda+2a)dx\right)}dx\right) + c_3 a - \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^{n} \lambda+2a)dx\right)}}{-c_3 + \int \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^{n} \lambda+2a)dx\right)}dx} \tag{1}$$

Verification of solutions

$$y = \frac{-a\left(\int \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^{n} \lambda+2a)dx\right)}dx\right) + c_3 a - \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^{n} \lambda+2a)dx\right)}}{-c_3 + \int \mathrm{e}^{-\left(\int (-\operatorname{arccot}(x)^{n} \lambda+2a)dx\right)}dx}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = arccot(x)^n*lambda*(diff(y(x),
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 71

```
dsolve(diff(y(x),x)=y(x)^2+lambda*arccot(x)^n*y(x)-a^2+a*lambda*arccot(x)^n,y(x), singsol=al
```

$$y(x) = \frac{-c_1 a - a\left(\int e^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)} dx\right) - e^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)}}{c_1 + \int e^{-\left(\int (-\operatorname{arccot}(x)^n \lambda + 2a) dx\right)} dx}$$

✓ Solution by Mathematica

Time used: 8.548 (sec). Leaf size: 210

```
DSolve[y'[x]==y[x]^2+\[Lambda]*ArcCot[x]^n*y[x]-a^2+a*\[Lambda]*ArcCot[x]^n,y[x],x,IncludeSi
```

$$\text{Solve}\left[\int_1^x -\frac{\exp\left(-\int_1^{K[2]} (2a - \lambda \cot^{-1}(K[1])^n)\, dK[1]\right)(-\lambda \cot^{-1}(K[2])^n + a - y(x))}{n\lambda(a + y(x))} dK[2] + \int_1^{y(x)} \left(-\int_1^x \left(\frac{\exp\left(-\int_1^{K[2]} (2a - \lambda \cot^{-1}(K[1])^n)\, dK[1]\right)(-\lambda \cot^{-1}(K[2])^n + a - K[3])}{n\lambda(a + K[3])^2} + \frac{\exp\left(-\int_1^{K[\ldots}\right.}{\ldots}\right.\right. - \frac{\exp\left(-\int_1^x (2a - \lambda \cot^{-1}(K[1])^n)\, dK[1]\right)}{n\lambda(a + K[3])^2}\right) dK[3] = c_1, y(x)\right]$$

18 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.

18.1 problem 29

Internal problem ID [10586]
Internal file name [`OUTPUT/9533_Monday_June_06_2022_03_05_36_PM_11542127/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - \lambda x \operatorname{arccot}(x)^n y = \operatorname{arccot}(x)^n \lambda}$$

18.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + \operatorname{arccot}(x)^n \lambda x y + \operatorname{arccot}(x)^n \lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \operatorname{arccot}(x)^n \lambda x y + \operatorname{arccot}(x)^n \lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \operatorname{arccot}(x)^n \lambda$, $f_1(x) = \operatorname{arccot}(x)^n \lambda x$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \operatorname{arccot}(x)^n \lambda x \\ f_2^2 f_0 &= \operatorname{arccot}(x)^n \lambda \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \operatorname{arccot}(x)^n \lambda x u'(x) + \operatorname{arccot}(x)^n \lambda u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x\left(c_1 - c_2\left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n\lambda\right)}{x^2+1}dx}}{x^2\left(x^2+1\right)}dx\right)\right)$$

The above shows that

$$u'(x) = \frac{-\left(x^2+1\right)x c_2\left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n\lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right) - c_2 e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n\lambda\right)}{x^2+1}dx} + c_1 x(x^2+1)}{x\left(x^2+1\right)}$$

Using the above in (1) gives the solution

$$y = -\frac{-\left(x^2+1\right)x c_2\left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n\lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right) - c_2 e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n\lambda\right)}{x^2+1}dx} + c_1 x(x^2+1)}{x^2\left(x^2+1\right)\left(c_1 - c_2\left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n\lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(x^3+x)\left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n \lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right) - c_3x^3 - c_3x + e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n \lambda\right)}{x^2+1}dx}}{x^2\left(x^2+1\right)\left(c_3 - \left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n \lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right)\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{(x^3+x)\left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n \lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right) - c_3x^3 - c_3x + e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n \lambda\right)}{x^2+1}dx}}{x^2\left(x^2+1\right)\left(c_3 - \left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n \lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{(x^3+x)\left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n \lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right) - c_3x^3 - c_3x + e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n \lambda\right)}{x^2+1}dx}}{x^2\left(x^2+1\right)\left(c_3 - \left(\int \frac{e^{\int \frac{x\left(2+\left(x^2+1\right)\operatorname{arccot}(x)^n \lambda\right)}{x^2+1}dx}}{x^2(x^2+1)}dx\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 78

```
dsolve(diff(y(x),x)=y(x)^2+lambda*x*arccot(x)^n*y(x)+lambda*arccot(x)^n,y(x), singsol=all)
```

$$y(x) = \frac{e^{\int \frac{x^2 \operatorname{arccot}(x)^n \lambda - 2}{x} dx} x + \int e^{\int \frac{x^2 \operatorname{arccot}(x)^n \lambda - 2}{x} dx} dx - c_1}{\left(c_1 - \left(\int e^{\int \frac{x^2 \operatorname{arccot}(x)^n \lambda - 2}{x} dx} dx\right)\right) x}$$

✓ Solution by Mathematica

Time used: 7.258 (sec). Leaf size: 120

```
DSolve[y'[x]==y[x]^2+\[Lambda]*x*ArcCot[x]^n*y[x]+\[Lambda]*ArcCot[x]^n,y[x],x,IncludeSingul
```

$y(x) \to$

$$-\frac{\exp\left(-\int_1^x -\lambda \cot^{-1}(K[1])^n K[1] dK[1]\right) + x \int_1^x \frac{\exp\left(-\int_1^{K[2]} -\lambda \cot^{-1}(K[1])^n K[1] dK[1]\right)}{K[2]^2} dK[2] + c_1 x}{x^2 \left(\int_1^x \frac{\exp\left(-\int_1^{K[2]} -\lambda \cot^{-1}(K[1])^n K[1] dK[1]\right)}{K[2]^2} dK[2] + c_1\right)}$$

$$y(x) \to -\frac{1}{x}$$

18.2 problem 30

Internal problem ID [10587]
Internal file name [`OUTPUT/9534_Monday_June_06_2022_03_05_40_PM_62741443/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + (k+1)\, x^k y^2 - \lambda \operatorname{arccot}(x)^n \left(x^{k+1} y - 1\right) = 0}$$

18.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x^{k+1} \operatorname{arccot}(x)^n \lambda y - x^k y^2 k - x^k y^2 - \operatorname{arccot}(x)^n \lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^k x \operatorname{arccot}(x)^n \lambda y - x^k y^2 k - x^k y^2 - \operatorname{arccot}(x)^n \lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = -\operatorname{arccot}(x)^n \lambda$, $f_1(x) = x^{k+1} \operatorname{arccot}(x)^n \lambda$ and $f_2(x) = -x^k k - x^k$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(-x^k k - x^k)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{k^2 x^k}{x} - \frac{k\, x^k}{x} \\
f_1 f_2 &= x^{k+1} \operatorname{arccot}(x)^n \lambda(-x^k k - x^k) \\
f_2^2 f_0 &= -(-x^k k - x^k)^2 \operatorname{arccot}(x)^n \lambda
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(-x^k k - x^k)\, u''(x) - \left(-\frac{k^2 x^k}{x} - \frac{k\, x^k}{x} + x^{k+1} \operatorname{arccot}(x)^n \lambda(-x^k k - x^k)\right) u'(x) - (-x^k k - x^k)^2 \operatorname{arccot}(x$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{k+1}\left(\left(\int x^{-2k-2} e^{\int \left(x^{k+1} \operatorname{arccot}(x)^n \lambda + \frac{k}{x}\right) dx} dx\right) c_2 + c_1\right)$$

The above shows that

$$u'(x) = c_2 x^{-k-1} e^{\int \frac{x^{k+2} \lambda \operatorname{arccot}(x)^n + k}{x} dx} + (k+1)\left(\left(\int e^{\int \frac{x^{k+2} \lambda \operatorname{arccot}(x)^n + k}{x} dx} x^{-2k-2} dx\right) c_2 + c_1\right) x^k$$

Using the above in (1) gives the solution

$$y = -\frac{\left(c_2 x^{-k-1} e^{\int \frac{x^{k+2} \lambda \operatorname{arccot}(x)^n + k}{x} dx} + (k+1)\left(\left(\int e^{\int \frac{x^{k+2} \lambda \operatorname{arccot}(x)^n + k}{x} dx} x^{-2k-2} dx\right) c_2 + c_1\right) x^k\right) x^{-k-1}}{(-x^k k - x^k)\left(\left(\int x^{-2k-2} e^{\int \left(x^{k+1} \operatorname{arccot}(x)^n \lambda + \frac{k}{x}\right) dx} dx\right) c_2 + c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(k+1)\left(\int x^{-2k-2} e^{\int \left(x^{k+1} \operatorname{arccot}(x)^n \lambda + \frac{k}{x}\right) dx} dx + c_3\right) x^{-k-1} + x^{-3k-2} e^{\int \left(x^{k+1} \operatorname{arccot}(x)^n \lambda + \frac{k}{x}\right) dx}}{(k+1)\left(\int e^{\int \frac{x^{k+2} \lambda \operatorname{arccot}(x)^n + k}{x} dx} x^{-2k-2} dx + c_3\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{(k+1)\left(\int x^{-2k-2} \mathrm{e}^{\int \left(x^{k+1} \operatorname{arccot}(x)^{n} \lambda+\frac{k}{x}\right)dx} dx + c_3\right) x^{-k-1} + x^{-3k-2} \mathrm{e}^{\int \left(x^{k+1} \operatorname{arccot}(x)^{n} \lambda+\frac{k}{x}\right)dx}}{(k+1)\left(\int \mathrm{e}^{\int \frac{x^{k+2} \lambda \operatorname{arccot}(x)^{n}+k}{x} dx} x^{-2k-2} dx + c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{(k+1)\left(\int x^{-2k-2} \mathrm{e}^{\int \left(x^{k+1} \operatorname{arccot}(x)^{n} \lambda+\frac{k}{x}\right)dx} dx + c_3\right) x^{-k-1} + x^{-3k-2} \mathrm{e}^{\int \left(x^{k+1} \operatorname{arccot}(x)^{n} \lambda+\frac{k}{x}\right)dx}}{(k+1)\left(\int \mathrm{e}^{\int \frac{x^{k+2} \lambda \operatorname{arccot}(x)^{n}+k}{x} dx} x^{-2k-2} dx + c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x^(1+k)*arccot(x)^n*x*lambda+
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 184

```
dsolve(diff(y(x),x)=-(k+1)*x^k*y(x)^2+lambda*arccot(x)^n*(x^(k+1)*y(x)-1),y(x), singsol=all)
```

$$y(x) = \frac{x^{-1-k}\left(\left(\left(\int x^k \mathrm{e}^{\lambda\left(\int x^{1+k} \operatorname{arccot}(x)^n dx\right)-2\left(\int \frac{1}{x}dx\right)(1+k)}dx\right)k + x^{1+k}\mathrm{e}^{\int \frac{x^{1+k} \operatorname{arccot}(x)^n x\lambda-2k-2}{x}dx} + \int x^k \mathrm{e}^{\lambda\left(\int x^{1+k} \operatorname{arccot}(x)^n\right.}\right.\right.}{\left(\int x^k \mathrm{e}^{\lambda\left(\int x^{1+k} \operatorname{arccot}(x)^n dx\right)-2\left(\int \frac{1}{x}dx\right)(1+k)}dx\right)k + \int x^k \mathrm{e}^{\lambda\left(\int x^{1+k} \operatorname{arccot}(x)^n dx\right)-2\left(\int \frac{1}{x}dx\right)(1+k)}dx}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==-(k+1)*x^k*y[x]^2+\[Lambda]*ArcCot[x]^n*(x^(k+1)*y[x]-1),y[x],x,IncludeSingula
```

Not solved

18.3 problem 31

Internal problem ID [10588]
Internal file name [`OUTPUT/9535_Monday_June_06_2022_03_05_51_PM_82945462/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \operatorname{arccot}(x)^n y^2 - ay = ab - b^2\lambda \operatorname{arccot}(x)^n}$$

18.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \lambda \operatorname{arccot}(x)^n y^2 + ya + ab - b^2\lambda \operatorname{arccot}(x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \lambda \operatorname{arccot}(x)^n y^2 + ya + ab - b^2\lambda \operatorname{arccot}(x)^n$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = ab - b^2\lambda \operatorname{arccot}(x)^n$, $f_1(x) = a$ and $f_2(x) = \operatorname{arccot}(x)^n \lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\operatorname{arccot}(x)^n \lambda u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{\operatorname{arccot}(x)^n n\lambda}{(x^2+1)\operatorname{arccot}(x)} \\ f_1 f_2 &= a\lambda \operatorname{arccot}(x)^n \\ f_2^2 f_0 &= \operatorname{arccot}(x)^{2n} \lambda^2 \left(ab - b^2\lambda \operatorname{arccot}(x)^n\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\operatorname{arccot}(x)^n \lambda u''(x) - \left(-\frac{\operatorname{arccot}(x)^n n\lambda}{(x^2+1)\operatorname{arccot}(x)} + a\lambda \operatorname{arccot}(x)^n\right) u'(x) + \operatorname{arccot}(x)^{2n} \lambda^2 \left(ab - b^2\lambda \operatorname{arccot}(x)^n\right)$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \operatorname{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)} - a\_Y'(x) - \operatorname{arccot}(x)^{2n} b^2\lambda^2\_Y(x) + \operatorname{arccot}(x)^n ab\lambda\_Y(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)} - a\_Y'(x) - \operatorname{arccot}(x)^{2n} b^2\lambda^2\_Y(x) + \operatorname{arccot}(x)^n ab\lambda\_Y(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)} - a\_Y'(x) - \operatorname{arccot}(x)^{2n} b^2\lambda^2\_Y(x) + \operatorname{arccot}(x)^n ab\lambda\_Y(x)\right\}\right.\right.}{\lambda \operatorname{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)} - a\_Y'(x) - \operatorname{arccot}(x)^{2n} b^2\lambda^2\_Y(x) + \operatorname{arccot}(x)^n ab\lambda\_\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{-b^2\_\_Y(x)\lambda^2\operatorname{arccot}(x)^{1+2n}(x^2+1)+ab\lambda\_\_Y(x)\operatorname{arccot}(x)^{n+1}(x^2+1)+\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)-(a(x^2+1)\operatorname{arcco}}{\operatorname{arccot}(x)(x^2+1)}\right.\right.\right.}{\lambda\operatorname{DESol}\left(\left\{\frac{-b^2\_\_Y(x)\lambda^2\operatorname{arccot}(x)^{1+2n}(x^2+1)+ab\lambda\_\_Y(x)\operatorname{arccot}(x)^{n+1}(x^2+1)+\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)-(a(x^2+1}{\operatorname{arccot}(x)(x^2+1)}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{-b^2\_\_Y(x)\lambda^2\operatorname{arccot}(x)^{1+2n}(x^2+1)+ab\lambda\_\_Y(x)\operatorname{arccot}(x)^{n+1}(x^2+1)+\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)-(a(x^2+1)\operatorname{arcco}}{\operatorname{arccot}(x)(x^2+1)}\right.\right.\right.}{\lambda\operatorname{DESol}\left(\left\{\frac{-b^2\_\_Y(x)\lambda^2\operatorname{arccot}(x)^{1+2n}(x^2+1)+ab\lambda\_\_Y(x)\operatorname{arccot}(x)^{n+1}(x^2+1)+\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)-(a(x^2+1}{\operatorname{arccot}(x)(x^2+1)}\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{-b^2\_\_Y(x)\lambda^2\operatorname{arccot}(x)^{1+2n}(x^2+1)+ab\lambda\_\_Y(x)\operatorname{arccot}(x)^{n+1}(x^2+1)+\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)-(a(x^2+1)\operatorname{arcco}}{\operatorname{arccot}(x)(x^2+1)}\right.\right.\right.}{\lambda\operatorname{DESol}\left(\left\{\frac{-b^2\_\_Y(x)\lambda^2\operatorname{arccot}(x)^{1+2n}(x^2+1)+ab\lambda\_\_Y(x)\operatorname{arccot}(x)^{n+1}(x^2+1)+\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)-(a(x^2+1}{\operatorname{arccot}(x)(x^2+1)}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (arccot(x)*a*x^2+a*arccot(x)-n
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 87

```
dsolve(diff(y(x),x)=lambda*arccot(x)^n*y(x)^2+a*y(x)+a*b-b^2*lambda*arccot(x)^n,y(x), singso
```

$$y(x) = \frac{-b\lambda\left(\int \operatorname{arccot}(x)^n \mathrm{e}^{-\left(\int (2\operatorname{arccot}(x)^n \lambda b-a)dx\right)}dx\right) - c_1 b - \mathrm{e}^{-\left(\int (2\operatorname{arccot}(x)^n \lambda b-a)dx\right)}}{c_1 + \lambda\left(\int \operatorname{arccot}(x)^n \mathrm{e}^{-\left(\int (2\operatorname{arccot}(x)^n \lambda b-a)dx\right)}dx\right)}$$

✓ Solution by Mathematica

Time used: 11.807 (sec). Leaf size: 240

```
DSolve[y'[x]==\[Lambda]*ArcCot[x]^n*y[x]^2+a*y[x]+a*b-b^2*\[Lambda]*ArcCot[x]^n,y[x],x,Inclu
```

$$\mathrm{Solve}\left[\int_1^x -\frac{\exp\left(-\int_1^{K[2]}(2b\lambda\cot^{-1}(K[1])^n-a)\,dK[1]\right)(-b\lambda\cot^{-1}(K[2])^n+\lambda y(x)\cot^{-1}(K[2])^n+a)}{an\lambda(b+y(x))}dK[2]\right.$$
$$+\int_1^{y(x)}\left(\frac{\exp\left(-\int_1^x(2b\lambda\cot^{-1}(K[1])^n-a)\,dK[1]\right)}{an\lambda(b+K[3])^2}\right.$$
$$-\int_1^x\left(\frac{\exp\left(-\int_1^{K[2]}(2b\lambda\cot^{-1}(K[1])^n-a)\,dK[1]\right)(-b\lambda\cot^{-1}(K[2])^n+\lambda K[3]\cot^{-1}(K[2])^n+a)}{an\lambda(b+K[3])^2}-\frac{\exp(\cdots}{\cdots}\right.$$

18.4 problem 32

Internal problem ID [10589]
Internal file name [`OUTPUT/9536_Monday_June_06_2022_03_05_59_PM_97148518/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \operatorname{arccot}(x)^n y^2 + b\lambda x^m \operatorname{arccot}(x)^n y = bm x^{m-1}}$$

18.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \lambda \operatorname{arccot}(x)^n y^2 - b\lambda x^m \operatorname{arccot}(x)^n y + bm x^{m-1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \lambda \operatorname{arccot}(x)^n y^2 - b\lambda x^m \operatorname{arccot}(x)^n y + \frac{b x^m m}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = bm x^{m-1}$, $f_1(x) = -b\lambda x^m \operatorname{arccot}(x)^n$ and $f_2(x) = \operatorname{arccot}(x)^n \lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\operatorname{arccot}(x)^n \lambda u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{\operatorname{arccot}(x)^n n\lambda}{(x^2+1)\operatorname{arccot}(x)} \\ f_1f_2 &= -b\,\lambda^2x^m \operatorname{arccot}(x)^{2n} \\ f_2^2f_0 &= \operatorname{arccot}(x)^{2n}\,\lambda^2bm\,x^{m-1} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\operatorname{arccot}(x)^n\,\lambda u''(x) - \left(-\frac{\operatorname{arccot}(x)^n n\lambda}{(x^2+1)\operatorname{arccot}(x)} - b\,\lambda^2x^m \operatorname{arccot}(x)^{2n}\right)u'(x) + \operatorname{arccot}(x)^{2n}\,\lambda^2bm\,x^{m-1}u(x) =$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \operatorname{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)} + \operatorname{arccot}(x)^n b\lambda\,x^m\_Y'(x) + \operatorname{arccot}(x)^n\,\lambda bm\,x^{m-1}\_Y(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)} + \operatorname{arccot}(x)^n b\lambda\,x^m\_Y'(x) + \operatorname{arccot}(x)^n\,\lambda bm\,x^{m-1}\_Y(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)} + \operatorname{arccot}(x)^n b\lambda\,x^m\_Y'(x) + \operatorname{arccot}(x)^n\,\lambda bm\,x^{m-1}\_Y(x)\right\}, \{\_\right.\right.}{\lambda\operatorname{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)} + \operatorname{arccot}(x)^n b\lambda\,x^m\_Y'(x) + \operatorname{arccot}(x)^n\,\lambda bm\,x^{m-1}\_Y(x\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{b\lambda(x^2+1)\left(m\_\_Y(x)x^{m-1}+\_\_Y'(x)x^{m}\right)\operatorname{arccot}(x)^{n+1}+\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)+n\_\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)}\right\}, \{\_\_Y(x)\}\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{b\lambda\left(m\,x^{m}\_\_Y(x)+m\_\_Y(x)x^{2+m}+\_\_Y'(x)x^{3+m}+\_\_Y'(x)x^{m+1}\right)\operatorname{arccot}(x)^{n+1}+x\left(\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)+n\_\_Y\right.}{(x^2+1)\operatorname{arccot}(x)x}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{b\lambda(x^2+1)\left(m\_\_Y(x)x^{m-1}+\_\_Y'(x)x^{m}\right)\operatorname{arccot}(x)^{n+1}+\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)+n\_\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)}\right\}, \{\_\_Y(x)\}\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{b\lambda\left(m\,x^{m}\_\_Y(x)+m\_\_Y(x)x^{2+m}+\_\_Y'(x)x^{3+m}+\_\_Y'(x)x^{m+1}\right)\operatorname{arccot}(x)^{n+1}+x\left(\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)+n\_\_Y\right.}{(x^2+1)\operatorname{arccot}(x)x}\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\,\mathrm{DESol}\left(\left\{\frac{b\lambda(x^2+1)\left(m\_\_Y(x)x^{m-1}+\_\_Y'(x)x^{m}\right)\operatorname{arccot}(x)^{n+1}+\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)+n\_\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)}\right\}, \{\_\_Y(x)\}\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{b\lambda\left(m\,x^{m}\_\_Y(x)+m\_\_Y(x)x^{2+m}+\_\_Y'(x)x^{3+m}+\_\_Y'(x)x^{m+1}\right)\operatorname{arccot}(x)^{n+1}+x\left(\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)+n\_\_Y\right.}{(x^2+1)\operatorname{arccot}(x)x}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(x^m*arccot(x)*arccot(x)^n*b*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=lambda*arccot(x)^n*y(x)^2-b*lambda*x^m*arccot(x)^n*y(x)+b*m*x^(m-1),y(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==\[Lambda]*ArcCot[x]^n*y[x]^2-b*\[Lambda]*x^m*ArcCot[x]^n*y[x]+b*m*x^(m-1),y[x]
```

Not solved

18.5 problem 33

Internal problem ID [10590]
Internal file name [`OUTPUT/9537_Monday_June_06_2022_03_06_10_PM_51668749/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \operatorname{arccot}(x)^n y^2 = \beta m\, x^{m-1} - \lambda\, \beta^2 x^{2m} \operatorname{arccot}(x)^n}$$

18.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \lambda \operatorname{arccot}(x)^n y^2 + \beta m\, x^{m-1} - \lambda\, \beta^2 x^{2m} \operatorname{arccot}(x)^n \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\lambda\, \beta^2 x^{2m} \operatorname{arccot}(x)^n + \lambda \operatorname{arccot}(x)^n y^2 + \frac{\beta m\, x^m}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \beta m\, x^{m-1} - \lambda\, \beta^2 x^{2m} \operatorname{arccot}(x)^n$, $f_1(x) = 0$ and $f_2(x) = \operatorname{arccot}(x)^n \lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\operatorname{arccot}(x)^n \lambda u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{\operatorname{arccot}(x)^n n\lambda}{(x^2+1)\operatorname{arccot}(x)} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= \operatorname{arccot}(x)^{2n} \lambda^2 \left(\beta m\, x^{m-1} - \lambda\, \beta^2 x^{2m} \operatorname{arccot}(x)^n\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\operatorname{arccot}(x)^n \lambda u''(x) + \frac{\operatorname{arccot}(x)^n n\lambda u'(x)}{(x^2+1)\operatorname{arccot}(x)} + \operatorname{arccot}(x)^{2n} \lambda^2 \left(\beta m\, x^{m-1} - \lambda\, \beta^2 x^{2m} \operatorname{arccot}(x)^n\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)} - x^{2m} \operatorname{arccot}(x)^{2n} \beta^2 \lambda^2 \_Y(x) + x^{m-1} \operatorname{arccot}(x)^n \beta m \lambda \_Y(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)} - x^{2m} \operatorname{arccot}(x)^{2n} \beta^2 \lambda^2 \_Y(x) + x^{m-1} \operatorname{arccot}(x)^n \beta m \lambda \_Y(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)} - x^{2m} \operatorname{arccot}(x)^{2n} \beta^2 \lambda^2 \_Y(x) + x^{m-1} \operatorname{arccot}(x)^n \beta m \lambda \_Y(x)\right\}\right.\right.}{\lambda\, \mathrm{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)} - x^{2m} \operatorname{arccot}(x)^{2n} \beta^2 \lambda^2 \_Y(x) + x^{m-1} \operatorname{arccot}(x)^n \beta m \lambda\_\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)+n\_\_Y'(x)-\beta^2\_\_Y(x)x^{2m}\lambda^2\operatorname{arccot}(x)^{1+2n}(x^2+1)+m\beta\lambda\_\_Y(x)\operatorname{arccot}(x)^{n+1}x^{m-1}(x^2}{(x^2+1)\operatorname{arccot}(x)}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{-\beta^2\lambda^2\_\_Y(x)(x^{3+2m}+x^{1+2m})\operatorname{arccot}(x)^{1+2n}+m\beta\lambda\_\_Y(x)(x^m+x^{2+m})\operatorname{arccot}(x)^{n+1}+x\left(\_\_Y''(x)(x^2+1)\operatorname{arcco}\right.}{(x^2+1)\operatorname{arccot}(x)x}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)+n\_\_Y'(x)-\beta^2\_\_Y(x)x^{2m}\lambda^2\operatorname{arccot}(x)^{1+2n}(x^2+1)+m\beta\lambda\_\_Y(x)\operatorname{arccot}(x)^{n+1}x^{m-1}(x^2}{(x^2+1)\operatorname{arccot}(x)}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{-\beta^2\lambda^2\_\_Y(x)(x^{3+2m}+x^{1+2m})\operatorname{arccot}(x)^{1+2n}+m\beta\lambda\_\_Y(x)(x^m+x^{2+m})\operatorname{arccot}(x)^{n+1}+x\left(\_\_Y''(x)(x^2+1)\operatorname{arcco}\right.}{(x^2+1)\operatorname{arccot}(x)x}\right.\right.} \quad (1)$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\_\_Y''(x)(x^2+1)\operatorname{arccot}(x)+n\_\_Y'(x)-\beta^2\_\_Y(x)x^{2m}\lambda^2\operatorname{arccot}(x)^{1+2n}(x^2+1)+m\beta\lambda\_\_Y(x)\operatorname{arccot}(x)^{n+1}x^{m-1}(x^2}{(x^2+1)\operatorname{arccot}(x)}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\frac{-\beta^2\lambda^2\_\_Y(x)(x^{3+2m}+x^{1+2m})\operatorname{arccot}(x)^{1+2n}+m\beta\lambda\_\_Y(x)(x^m+x^{2+m})\operatorname{arccot}(x)^{n+1}+x\left(\_\_Y''(x)(x^2+1)\operatorname{arcco}\right.}{(x^2+1)\operatorname{arccot}(x)x}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -n*(diff(y(x), x))/((x^2+1)*ar
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=lambda*arccot(x)^n*y(x)^2+beta*m*x^(m-1)-lambda*beta^2*x^(2*m)*arccot(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==\[Lambda]*ArcCot[x]^n*y[x]^2+\[Beta]*m*x^(m-1)-\[Lambda]*\[Beta]^2*x^(2*m)*Arc
```

Not solved

18.6 problem 34

Internal problem ID [10591]
Internal file name [`OUTPUT/9538_Monday_June_06_2022_03_06_31_PM_7594206/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$\boxed{y' - \lambda \operatorname{arccot}(x)^n (y - a\,x^m - b)^2 = am\,x^{m-1}}$$

18.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x^{2m} \operatorname{arccot}(x)^n a^2\lambda + 2x^m \operatorname{arccot}(x)^n ab\lambda - 2x^m \operatorname{arccot}(x)^n a\lambda y + b^2\lambda \operatorname{arccot}(x)^n - 2\operatorname{arccot}(x)^n b\lambda y \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^{2m} \operatorname{arccot}(x)^n a^2\lambda + 2x^m \operatorname{arccot}(x)^n ab\lambda - 2x^m \operatorname{arccot}(x)^n a\lambda y + b^2\lambda \operatorname{arccot}(x)^n - 2\operatorname{arccot}(x)^n b\lambda y + \lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^{2m} \operatorname{arccot}(x)^n a^2\lambda + 2x^m \operatorname{arccot}(x)^n ab\lambda + b^2\lambda \operatorname{arccot}(x)^n + am\,x^{m-1}$, $f_1(x) = -2a\lambda\,x^m \operatorname{arccot}(x)^n - 2\operatorname{arccot}(x)^n \lambda b$ and $f_2(x) = \operatorname{arccot}(x)^n \lambda$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\operatorname{arccot}(x)^n \lambda u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{\operatorname{arccot}(x)^n\,n\lambda}{(x^2+1)\operatorname{arccot}(x)}\\
f_1f_2 &= \left(-2a\lambda\,x^m\operatorname{arccot}(x)^n - 2\operatorname{arccot}(x)^n\,\lambda b\right)\operatorname{arccot}(x)^n\,\lambda\\
f_2^2f_0 &= \operatorname{arccot}(x)^{2n}\,\lambda^2\left(x^{2m}\operatorname{arccot}(x)^n\,a^2\lambda + 2x^m\operatorname{arccot}(x)^n\,ab\lambda + b^2\lambda\operatorname{arccot}(x)^n + am\,x^{m-1}\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\operatorname{arccot}(x)^n\,\lambda u''(x) - \left(-\frac{\operatorname{arccot}(x)^n\,n\lambda}{(x^2+1)\operatorname{arccot}(x)} + \left(-2a\lambda\,x^m\operatorname{arccot}(x)^n - 2\operatorname{arccot}(x)^n\,\lambda b\right)\operatorname{arccot}(x)^n\,\lambda\right)u'(x$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
u(x) = \mathrm{DESol}\Bigg(\Bigg\{&_Y''(x) + \frac{n_Y'(x)}{\operatorname{arccot}(x)\,(x^2+1)} + 2\operatorname{arccot}(x)^n\,x^ma\lambda_Y'(x)\\
&+ 2\operatorname{arccot}(x)^n\,b\lambda_Y'(x) + x^{2m}\operatorname{arccot}(x)^{2n}\,a^2\lambda^2_Y(x)\\
&+ 2x^m\operatorname{arccot}(x)^{2n}\,ab\,\lambda^2_Y(x) + \operatorname{arccot}(x)^{2n}\,b^2\lambda^2_Y(x)\\
&+ x^{m-1}\operatorname{arccot}(x)^n\,am\lambda_Y(x)\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

The above shows that

$$\begin{aligned}
u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\Bigg(\Bigg\{&_Y''(x) + \frac{n_Y'(x)}{\operatorname{arccot}(x)\,(x^2+1)} + 2\operatorname{arccot}(x)^n\,x^ma\lambda_Y'(x)\\
&+ 2\operatorname{arccot}(x)^n\,b\lambda_Y'(x) + x^{2m}\operatorname{arccot}(x)^{2n}\,a^2\lambda^2_Y(x)\\
&+ 2x^m\operatorname{arccot}(x)^{2n}\,ab\,\lambda^2_Y(x) + \operatorname{arccot}(x)^{2n}\,b^2\lambda^2_Y(x)\\
&+ x^{m-1}\operatorname{arccot}(x)^n\,am\lambda_Y(x)\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)} + 2\operatorname{arccot}(x)^n\,x^ma\lambda\_Y'(x) + 2\operatorname{arccot}(x)^n\,b\lambda\_Y'(x) + x^{2m}\operatorname{ar}\right.\right.\right.}{\lambda\,\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{n\_Y'(x)}{\operatorname{arccot}(x)(x^2+1)} + 2\operatorname{arccot}(x)^n\,x^ma\lambda\_Y'(x) + 2\operatorname{arccot}(x)^n\,b\lambda\_Y'(x) + x\right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\frac{\lambda^2 \_\_Y(x)\left(x^2+1\right)\left(a^2 x^{2m}+2ab\, x^m+b^2\right) \operatorname{arccot}(x)^{1+2n}+\left(x^2+1\right)\left(a\, x^{m-1} m \_\_Y(x)+2\_\_\right.}{\operatorname{arccot}(x)\left(x^2+1\right)}\right.\right.\right.}{\lambda \operatorname{DESol}\left(\left\{\frac{\left(a^2 x^{1+2m}+a^2 x^{3+2m}+2a\, x^{3+m} b+2a\, x^{m+1} b+b^2 x\left(x^2+1\right)\right)\_\_Y(x)\lambda^2 \operatorname{arccot}(x)^{1+2n}+\left(2a\, x^{3+m}\_\_Y'(x)+2a\, x^{m+1}\_\_Y'(x\right.}{x\left(x^2+1\right) \mathrm{a}}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\frac{\lambda^2 \_\_Y(x)\left(x^2+1\right)\left(a^2 x^{2m}+2ab\, x^m+b^2\right) \operatorname{arccot}(x)^{1+2n}+\left(x^2+1\right)\left(a\, x^{m-1} m \_\_Y(x)+2\_\_\right.}{\operatorname{arccot}(x)\left(x^2+1\right)}\right.\right.\right.}{\lambda \operatorname{DESol}\left(\left\{\frac{\left(a^2 x^{1+2m}+a^2 x^{3+2m}+2a\, x^{3+m} b+2a\, x^{m+1} b+b^2 x\left(x^2+1\right)\right)\_\_Y(x)\lambda^2 \operatorname{arccot}(x)^{1+2n}+\left(2a\, x^{3+m}\_\_Y'(x)+2a\, x^{m+1}\_\_Y'(x\right.}{x\left(x^2+1\right) \mathrm{a}}\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\frac{\lambda^2 \_\_Y(x)\left(x^2+1\right)\left(a^2 x^{2m}+2ab\, x^m+b^2\right) \operatorname{arccot}(x)^{1+2n}+\left(x^2+1\right)\left(a\, x^{m-1} m \_\_Y(x)+2\_\_\right.}{\operatorname{arccot}(x)\left(x^2+1\right)}\right.\right.\right.}{\lambda \operatorname{DESol}\left(\left\{\frac{\left(a^2 x^{1+2m}+a^2 x^{3+2m}+2a\, x^{3+m} b+2a\, x^{m+1} b+b^2 x\left(x^2+1\right)\right)\_\_Y(x)\lambda^2 \operatorname{arccot}(x)^{1+2n}+\left(2a\, x^{3+m}\_\_Y'(x)+2a\, x^{m+1}\_\_Y'(x\right.}{x\left(x^2+1\right) \mathrm{a}}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (d) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)=lambda*arccot(x)^n*(y(x)-a*x^m-b)^2+a*m*x^(m-1),y(x), singsol=all)
```

$$y(x) = a\, x^m + b + \frac{1}{c_1 - \lambda \left(\int \operatorname{arccot}\left(x\right)^n dx\right)}$$

✓ Solution by Mathematica

Time used: 2.259 (sec). Leaf size: 44

```
DSolve[y'[x]==\[Lambda]*ArcCot[x]^n*(y[x]-a*x^m-b)^2+a*m*x^(m-1),y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{1}{-\int_1^x \lambda \cot^{-1}(K[2])^n dK[2] + c_1} + ax^m + b$$
$$y(x) \to ax^m + b$$

18.7 problem 35

Internal problem ID [10592]
Internal file name [`OUTPUT/9539_Monday_June_06_2022_03_06_51_PM_86672655/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - \lambda \operatorname{arccot}(x)^{n} y^{2} - ky = \lambda b^{2} x^{2k} \operatorname{arccot}(x)^{n}}$$

18.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{\lambda \operatorname{arccot}(x)^{n} y^{2} + ky + \lambda b^{2} x^{2k} \operatorname{arccot}(x)^{n}}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{\lambda b^{2} x^{2k} \operatorname{arccot}(x)^{n}}{x} + \frac{\lambda \operatorname{arccot}(x)^{n} y^{2}}{x} + \frac{ky}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = \frac{\lambda b^{2} x^{2k} \operatorname{arccot}(x)^{n}}{x}$, $f_1(x) = \frac{k}{x}$ and $f_2(x) = \frac{\operatorname{arccot}(x)^{n} \lambda}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{\operatorname{arccot}(x)^{n} \lambda u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{\operatorname{arccot}(x)^n n\lambda}{(x^2+1)\operatorname{arccot}(x)\, x} - \frac{\operatorname{arccot}(x)^n \lambda}{x^2} \\
f_1 f_2 &= \frac{k \operatorname{arccot}(x)^n \lambda}{x^2} \\
f_2^2 f_0 &= \frac{\operatorname{arccot}(x)^{3n} \lambda^3 b^2 x^{2k}}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\operatorname{arccot}(x)^n \lambda u''(x)}{x} - \left(-\frac{\operatorname{arccot}(x)^n n\lambda}{(x^2+1)\operatorname{arccot}(x)\, x} - \frac{\operatorname{arccot}(x)^n \lambda}{x^2} + \frac{k \operatorname{arccot}(x)^n \lambda}{x^2}\right) u'(x) + \frac{\operatorname{arccot}(x)^{3n} \lambda^3 b}{x^3}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \mathrm{e}^{ib\lambda\left(\int x^{k-1} \operatorname{arccot}(x)^n dx\right)} + c_2 \mathrm{e}^{-ib\lambda\left(\int x^{k-1} \operatorname{arccot}(x)^n dx\right)}$$

The above shows that

$$u'(x) = ib\, x^{k-1} \lambda \operatorname{arccot}(x)^n \mathrm{e}^{-ib\lambda\left(\int x^{k-1} \operatorname{arccot}(x)^n dx\right)} \left(\mathrm{e}^{2ib\lambda\left(\int x^{k-1} \operatorname{arccot}(x)^n dx\right)} c_1 - c_2\right)$$

Using the above in (1) gives the solution

$$y = -\frac{ib\, x^{k-1} \mathrm{e}^{-ib\lambda\left(\int x^{k-1} \operatorname{arccot}(x)^n dx\right)} \left(\mathrm{e}^{2ib\lambda\left(\int x^{k-1} \operatorname{arccot}(x)^n dx\right)} c_1 - c_2\right) x}{c_1 \mathrm{e}^{ib\lambda\left(\int x^{k-1} \operatorname{arccot}(x)^n dx\right)} + c_2 \mathrm{e}^{-ib\lambda\left(\int x^{k-1} \operatorname{arccot}(x)^n dx\right)}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{ib\, x^k \left(\mathrm{e}^{2ib\lambda\left(\int x^{k-1} \operatorname{arccot}(x)^n dx\right)} c_3 - 1\right)}{\mathrm{e}^{2ib\lambda\left(\int x^{k-1} \operatorname{arccot}(x)^n dx\right)} c_3 + 1}$$

Summary
The solution(s) found are the following

$$y = -\frac{ib\,x^k\left(\mathrm{e}^{2ib\lambda\left(\int x^{k-1}\operatorname{arccot}(x)^n dx\right)}c_3 - 1\right)}{\mathrm{e}^{2ib\lambda\left(\int x^{k-1}\operatorname{arccot}(x)^n dx\right)}c_3 + 1} \tag{1}$$

Verification of solutions

$$y = -\frac{ib\,x^k\left(\mathrm{e}^{2ib\lambda\left(\int x^{k-1}\operatorname{arccot}(x)^n dx\right)}c_3 - 1\right)}{\mathrm{e}^{2ib\lambda\left(\int x^{k-1}\operatorname{arccot}(x)^n dx\right)}c_3 + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 29

```
dsolve(x*diff(y(x),x)=lambda*arccot(x)^n*y(x)^2+k*y(x)+lambda*b^2*x^(2*k)*arccot(x)^n,y(x),
```

$$y(x) = -\tan\left(-\lambda b\left(\int x^{-1+k}\operatorname{arccot}(x)^n\,dx\right) + c_1\right) b\,x^k$$

✓ Solution by Mathematica
Time used: 2.591 (sec). Leaf size: 48

```
DSolve[x*y'[x]==\[Lambda]*ArcCot[x]^n*y[x]^2+k*y[x]+\[Lambda]*b^2*x^(2*k)*ArcCot[x]^n,y[x],x
```

$$y(x) \to \sqrt{b^2}x^k\tan\left(\sqrt{b^2}\int_1^x \lambda\cot^{-1}(K[1])^n K[1]^{k-1}dK[1] + c_1\right)$$

18.8 problem 36

Internal problem ID [10593]
Internal file name [OUTPUT/9540_Monday_June_06_2022_03_06_54_PM_87619199/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.7-3. Equations containing arctangent.
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y'x - \left(a\,x^{2m} y^2 + y x^n b + c\right) \operatorname{arccot}\left(x\right)^m + yn = 0}$$

18.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\operatorname{arccot}\left(x\right)^m x^{2m} a\, y^2 + \operatorname{arccot}\left(x\right)^m x^n b y + \operatorname{arccot}\left(x\right)^m c - n y}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{\operatorname{arccot}\left(x\right)^m x^{2m} a\, y^2}{x} + \frac{\operatorname{arccot}\left(x\right)^m x^n b y}{x} + \frac{\operatorname{arccot}\left(x\right)^m c}{x} - \frac{n y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = \frac{\operatorname{arccot}(x)^m c}{x}$, $f_1(x) = \frac{\operatorname{arccot}(x)^m x^n b - n}{x}$ and $f_2(x) = \frac{\operatorname{arccot}(x)^m x^{2m} a}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{\operatorname{arccot}(x)^m x^{2m} a u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{\operatorname{arccot}(x)^m\, m\, x^{2m}a}{(x^2+1)\operatorname{arccot}(x)\, x} + \frac{2\operatorname{arccot}(x)^m\, x^{2m} m a}{x^2} - \frac{\operatorname{arccot}(x)^m\, x^{2m} a}{x^2}\\
f_1f_2 &= \frac{(\operatorname{arccot}(x)^m\, x^n b - n)\operatorname{arccot}(x)^m\, x^{2m} a}{x^2}\\
f_2^2f_0 &= \frac{\operatorname{arccot}(x)^{3m}\, x^{4m} a^2 c}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\operatorname{arccot}(x)^m\, x^{2m} a u''(x)}{x} - \left(-\frac{\operatorname{arccot}(x)^m\, m\, x^{2m}a}{(x^2+1)\operatorname{arccot}(x)\, x} + \frac{2\operatorname{arccot}(x)^m\, x^{2m} m a}{x^2} - \frac{\operatorname{arccot}(x)^m\, x^{2m} a}{x^2} + \frac{(\operatorname{arccot}(\right.$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
u(x) = \mathrm{DESol}\Bigg(\Bigg\{&_Y''(x) + \frac{m_Y'(x)}{(x^2+1)\operatorname{arccot}(x)} - \frac{2m_Y'(x)}{x} + \frac{_Y'(x)}{x}\\
&- b\,x^{n-1}\operatorname{arccot}(x)^m\,_Y'(x) + \frac{n_Y'(x)}{x}\\
&+ ac\,x^{2m-2}_Y(x)\operatorname{arccot}(x)^{2m}\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

The above shows that

$$\begin{aligned}
u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\Bigg(\Bigg\{&_Y''(x) + \frac{m_Y'(x)}{(x^2+1)\operatorname{arccot}(x)} - \frac{2m_Y'(x)}{x} + \frac{_Y'(x)}{x}\\
&- b\,x^{n-1}\operatorname{arccot}(x)^m\,_Y'(x) + \frac{n_Y'(x)}{x}\\
&+ ac\,x^{2m-2}_Y(x)\operatorname{arccot}(x)^{2m}\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{m\_Y'(x)}{(x^2+1)\operatorname{arccot}(x)} - \frac{2m\_Y'(x)}{x} + \frac{\_Y'(x)}{x} - b\,x^{n-1}\operatorname{arccot}(x)^m\,\_Y'(x) + \frac{n\_Y'(x)}{x} + \right.\right.\right.}{a\,\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{m\_Y'(x)}{(x^2+1)\operatorname{arccot}(x)} - \frac{2m\_Y'(x)}{x} + \frac{\_Y'(x)}{x} - b\,x^{n-1}\operatorname{arccot}(x)^m\,\_Y'(x) + \right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x^{-2m+1} \operatorname{arccot}(x)^{-m} \Bigg(\frac{\partial}{\partial x} \mathrm{DESol}\Bigg(\Bigg\{\frac{a c\, x^{2m-1}\_\_Y(x) \operatorname{arccot}(x)^{1+2m}(x^2+1) - b\, x^n \operatorname{arccot}(x)^{m+1}(x^2+1)\_\_Y'(x) + \_\_Y''(x)(\cdots}{x(x^2+1)\operatorname{arccot}(x)}}{a\, \mathrm{DESol}\Bigg(\Bigg\{\frac{ac\_\_Y(x)(x^{2m}+x^{2+2m})\operatorname{arccot}(x)^{1+2m} - \_\_Y'(x) b(x^{n+1}+x^{n+3})\operatorname{arccot}(x)^{m+1} - 2\Big(-\frac{\_\_Y''(x)(x^2+1)\operatorname{arcco}\cdots}{2}}{(x^2+1)x^2\operatorname{arccot}(x)}}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^{-2m+1} \operatorname{arccot}(x)^{-m} \Bigg(\frac{\partial}{\partial x} \mathrm{DESol}\Bigg(\Bigg\{\frac{a c\, x^{2m-1}\_\_Y(x) \operatorname{arccot}(x)^{1+2m}(x^2+1) - b\, x^n \operatorname{arccot}(x)^{m+1}(x^2+1)\_\_Y'(x) + \_\_Y''(x)(\cdots}{x(x^2+1)\operatorname{arccot}(x)}}{a\, \mathrm{DESol}\Bigg(\Bigg\{\frac{ac\_\_Y(x)(x^{2m}+x^{2+2m})\operatorname{arccot}(x)^{1+2m} - \_\_Y'(x) b(x^{n+1}+x^{n+3})\operatorname{arccot}(x)^{m+1} - 2\Big(-\frac{\_\_Y''(x)(x^2+1)\operatorname{arcco}\cdots}{2}}{(x^2+1)x^2\operatorname{arccot}(x)}} \tag{1}$$

Verification of solutions

$$y = -\frac{x^{-2m+1} \operatorname{arccot}(x)^{-m} \Bigg(\frac{\partial}{\partial x} \mathrm{DESol}\Bigg(\Bigg\{\frac{a c\, x^{2m-1}\_\_Y(x) \operatorname{arccot}(x)^{1+2m}(x^2+1) - b\, x^n \operatorname{arccot}(x)^{m+1}(x^2+1)\_\_Y'(x) + \_\_Y''(x)(\cdots}{x(x^2+1)\operatorname{arccot}(x)}}{a\, \mathrm{DESol}\Bigg(\Bigg\{\frac{ac\_\_Y(x)(x^{2m}+x^{2+2m})\operatorname{arccot}(x)^{1+2m} - \_\_Y'(x) b(x^{n+1}+x^{n+3})\operatorname{arccot}(x)^{m+1} - 2\Big(-\frac{\_\_Y''(x)(x^2+1)\operatorname{arcco}\cdots}{2}}{(x^2+1)x^2\operatorname{arccot}(x)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (arccot(x)*x^(n-1)*arccot(x)^m
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x)=(a*x^(2*m)*y(x)^2+b*x^n*y(x)+c)*arccot(x)^m-n*y(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y'[x]==(a*x^(2*m)*y[x]^2+b*x^n*y[x]+c)*ArcCot[x]^m-n*y[x],y[x],x,IncludeSingularSol
```

Not solved

19 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).

19.1 problem 1

Internal problem ID [10594]
Internal file name [`OUTPUT/9541_Monday_June_06_2022_03_07_16_PM_10349265/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - f(x)\, y = -a^2 - f(x)\, a}$$

19.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + f(x)\, y - a^2 - f(x)\, a \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + f(x)\, y - a^2 - f(x)\, a$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2 - f(x)\, a$, $f_1(x) = f(x)$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= f(x) \\ f_2^2 f_0 &= -a^2 - f(x)\,a \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - f(x)\,u'(x) + \left(-a^2 - f(x)\,a\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\left(\int \mathrm{e}^{2xa+\int f(x)dx}dx\right) c_1 + c_2\right) \mathrm{e}^{-xa}$$

The above shows that

$$u'(x) = -\mathrm{e}^{-xa}\left(\int \mathrm{e}^{2xa+\int f(x)dx}dx\right) c_1a - \mathrm{e}^{-xa}c_2a + c_1\mathrm{e}^{xa+\int f(x)dx}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-\mathrm{e}^{-xa}\left(\int \mathrm{e}^{2xa+\int f(x)dx}dx\right) c_1a - \mathrm{e}^{-xa}c_2a + c_1\mathrm{e}^{xa+\int f(x)dx}\right) \mathrm{e}^{xa}}{\left(\int \mathrm{e}^{2xa+\int f(x)dx}dx\right) c_1 + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3\mathrm{e}^{2xa+\int f(x)dx} + a\left(\int \mathrm{e}^{2xa+\int f(x)dx}dx\right) c_3 + a}{\left(\int \mathrm{e}^{2xa+\int f(x)dx}dx\right) c_3 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_3\mathrm{e}^{2xa+\int f(x)dx} + a\left(\int \mathrm{e}^{2xa+\int f(x)dx}dx\right) c_3 + a}{\left(\int \mathrm{e}^{2xa+\int f(x)dx}dx\right) c_3 + 1} \tag{1}$$

Verification of solutions

$$y = \frac{-c_3 \mathrm{e}^{2xa+\int f(x)dx} + a\left(\int \mathrm{e}^{2xa+\int f(x)dx}dx\right) c_3 + a}{\left(\int \mathrm{e}^{2xa+\int f(x)dx}dx\right) c_3 + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 52

```
dsolve(diff(y(x),x)=y(x)^2+f(x)*y(x)-a^2-a*f(x),y(x), singsol=all)
```

$$y(x) = \frac{-a\left(\int \mathrm{e}^{\int f(x)dx+2ax}dx\right) + c_1 a + \mathrm{e}^{\int f(x)dx+2ax}}{-\left(\int \mathrm{e}^{\int f(x)dx+2ax}dx\right) + c_1}$$

✓ Solution by Mathematica

Time used: 0.719 (sec). Leaf size: 166

```
DSolve[y'[x]==y[x]^2+f[x]*y[x]-a^2-a*f[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\int_1^x \frac{\exp\left(-\int_1^{K[2]}(-2a-f(K[1]))dK[1]\right)(a+f(K[2])+y(x))}{a-y(x)}dK[2]\right.$$
$$+\int_1^{y(x)}\left(\frac{\exp\left(-\int_1^x(-2a-f(K[1]))dK[1]\right)}{(K[3]-a)^2}\right.$$
$$-\int_1^x\left(\frac{\exp\left(-\int_1^{K[2]}(-2a-f(K[1]))dK[1]\right)(a+f(K[2])+K[3])}{(a-K[3])^2}+\frac{\exp\left(-\int_1^{K[2]}(-2a-f(K[1]))dK[1]\right)}{a-K[3]}\right.$$

19.2 problem 2

Internal problem ID [10595]
Internal file name [`OUTPUT/9542_Monday_June_06_2022_03_07_17_PM_47552009/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) + ay = -ab - b^2 f(x)}$$

19.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)\, y^2 - ya - ab - b^2 f(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\, y^2 - ya - ab - b^2 f(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -ab - b^2 f(x)$, $f_1(x) = -a$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1 f_2 &= -f(x)\, a \\ f_2^2 f_0 &= f(x)^2 \left(-ab - b^2 f(x)\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\, u''(x) - (f'(x) - f(x)\, a)\, u'(x) + f(x)^2 \left(-ab - b^2 f(x)\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\int \frac{f(x)\left(b\left(\int f(x)\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx\right) - c_1 b + \mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}\right)}{-c_1 + \int f(x)\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx} dx} c_2$$

The above shows that

$$u'(x) = \frac{f(x)\left(b\left(\int f(x)\,\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx\right) - c_1 b + \mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}\right)\mathrm{e}^{\int \frac{f(x)\left(b\left(\int f(x)\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx\right) - c_1 b + \mathrm{e}^{-\left(\int (2f(x)b+a}{-c_1 + \int f(x)\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx}}}{-c_1 + \int f(x)\,\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx}$$

Using the above in (1) gives the solution

$$y = -\frac{b\left(\int f(x)\,\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx\right) - c_1 b + \mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}}{-c_1 + \int f(x)\,\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{b\left(\int f(x)\,\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx\right) - bc_3 + \mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}}{c_3 - \left(\int f(x)\,\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{b\left(\int f(x)\,\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx\right) - bc_3 + \mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}}{c_3 - \left(\int f(x)\,\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx\right)} \tag{1}$$

Verification of solutions

$$y = \frac{b\left(\int f(x)\,\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx\right) - bc_3 + \mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}}{c_3 - \left(\int f(x)\,\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(a*f(x)-(diff(f(x), x)))*(dif
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 65

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-a*y(x)-a*b-b^2*f(x),y(x), singsol=all)
```

$$y(x) = \frac{-c_1 b - b\left(\int f(x)\,\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx\right) - \mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}}{c_1 + \int f(x)\,\mathrm{e}^{-\left(\int (2f(x)b+a)dx\right)}dx}$$

✓ Solution by Mathematica

Time used: 0.955 (sec). Leaf size: 185

```
DSolve[y'[x]==f[x]*y[x]^2-a*y[x]-a*b-b^2*f[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\int_1^x \frac{\exp\left(-\int_1^{K[2]}(a+2bf(K[1]))dK[1]\right)(a+bf(K[2])-f(K[2])y(x))}{a(b+y(x))}dK[2]\right.$$
$$+\int_1^{y(x)}\left(\frac{\exp\left(-\int_1^x(a+2bf(K[1]))dK[1]\right)}{a(b+K[3])^2}\right.$$
$$-\int_1^x\left(-\frac{\exp\left(-\int_1^{K[2]}(a+2bf(K[1]))dK[1]\right)f(K[2])}{a(b+K[3])}-\frac{\exp\left(-\int_1^{K[2]}(a+2bf(K[1]))dK[1]\right)(a+bf(K[2]}{a(b+K[3])^2}\right.$$

19.3 problem 3

Internal problem ID [10596]
Internal file name [`OUTPUT/9543_Monday_June_06_2022_03_07_19_PM_13365483/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 - xf(x)\,y = f(x)}$$

19.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 + f(x)\,xy + f(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + f(x)\,xy + f(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = f(x)$, $f_1(x) = xf(x)$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= x f(x) \\ f_2^2 f_0 &= f(x) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - x f(x)\, u'(x) + f(x)\, u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x\left(\left(\int e^{\int \frac{f(x)x^2-2}{x}dx} dx\right) c_1 + c_2\right)$$

The above shows that

$$u'(x) = \left(\int e^{\int \frac{f(x)x^2-2}{x}dx} dx\right) c_1 + c_2 + x\, e^{\int \frac{f(x)x^2-2}{x}dx} c_1$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\int e^{\int \frac{f(x)x^2-2}{x}dx} dx\right) c_1 + c_2 + x\, e^{\int \frac{f(x)x^2-2}{x}dx} c_1}{x\left(\left(\int e^{\int \frac{f(x)x^2-2}{x}dx} dx\right) c_1 + c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\left(\int e^{\int \frac{f(x)x^2-2}{x}dx} dx\right) c_3 - 1 - x\, e^{\int \frac{f(x)x^2-2}{x}dx} c_3}{x\left(\left(\int e^{\int \frac{f(x)x^2-2}{x}dx} dx\right) c_3 + 1\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{-\left(\int e^{\int \frac{f(x)x^2-2}{x}dx}dx\right)c_3 - 1 - x\,e^{\int \frac{f(x)x^2-2}{x}dx}c_3}{x\left(\left(\int e^{\int \frac{f(x)x^2-2}{x}dx}dx\right)c_3 + 1\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-\left(\int e^{\int \frac{f(x)x^2-2}{x}dx}dx\right)c_3 - 1 - x\,e^{\int \frac{f(x)x^2-2}{x}dx}c_3}{x\left(\left(\int e^{\int \frac{f(x)x^2-2}{x}dx}dx\right)c_3 + 1\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 69

```
dsolve(diff(y(x),x)=y(x)^2+x*f(x)*y(x)+f(x),y(x), singsol=all)
```

$$y(x) = \frac{e^{\int \frac{f(x)x^2-2}{x}dx}x + \int e^{\int \frac{f(x)x^2-2}{x}dx}dx - c_1}{\left(c_1 - \left(\int e^{\int \frac{f(x)x^2-2}{x}dx}dx\right)\right)x}$$

✓ Solution by Mathematica

Time used: 1.074 (sec). Leaf size: 111

```
DSolve[y'[x]==y[x]^2+x*f[x]*y[x]+f[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\exp\left(-\int_1^x -f(K[1])K[1]dK[1]\right) + x\int_1^x \frac{\exp\left(-\int_1^{K[2]} -f(K[1])K[1]dK[1]\right)}{K[2]^2}dK[2] + c_1 x}{x^2\left(\int_1^x \frac{\exp\left(-\int_1^{K[2]} -f(K[1])K[1]dK[1]\right)}{K[2]^2}dK[2] + c_1\right)}$$

$$y(x) \to -\frac{1}{x}$$

19.4 problem 4

Internal problem ID [10597]
Internal file name [`OUTPUT/9544_Monday_June_06_2022_03_07_20_PM_89866051/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) + a\,x^n f(x)\,y = an\,x^{n-1}}$$

19.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)\,y^2 - a\,x^n f(x)\,y + an\,x^{n-1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\,y^2 - a\,x^n f(x)\,y + \frac{x^n na}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = an\,x^{n-1}$, $f_1(x) = -f(x)\,a\,x^n$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1f_2 &= -f(x)^2\,a\,x^n \\ f_2^2f_0 &= f(x)^2\,an\,x^{n-1} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\,u''(x) - \left(-f(x)^2\,a\,x^n + f'(x)\right)u'(x) + f(x)^2\,an\,x^{n-1}u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{-a\left(\int f(x)x^n dx\right)}\left(c_1 + \left(\int f(x)\,\mathrm{e}^{a\left(\int f(x)x^n dx\right)}dx\right)c_2\right)$$

The above shows that

$$u'(x) = f(x)\left(-x^n\mathrm{e}^{-a\left(\int f(x)x^n dx\right)}\left(\int f(x)\,\mathrm{e}^{a\left(\int f(x)x^n dx\right)}dx\right)c_2a - x^n\mathrm{e}^{-a\left(\int f(x)x^n dx\right)}c_1a + c_2\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-x^n\mathrm{e}^{-a\left(\int f(x)x^n dx\right)}\left(\int f(x)\,\mathrm{e}^{a\left(\int f(x)x^n dx\right)}dx\right)c_2a - x^n\mathrm{e}^{-a\left(\int f(x)x^n dx\right)}c_1a + c_2\right)\mathrm{e}^{\int f(x)a\,x^n dx}}{c_1 + \left(\int f(x)\,\mathrm{e}^{a\left(\int f(x)x^n dx\right)}dx\right)c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{ac_3x^n + a\left(\int f(x)\,\mathrm{e}^{a\left(\int f(x)x^n dx\right)}dx\right)x^n - \mathrm{e}^{a\left(\int f(x)x^n dx\right)}}{c_3 + \int f(x)\,\mathrm{e}^{a\left(\int f(x)x^n dx\right)}dx}$$

Summary

The solution(s) found are the following

$$y = \frac{ac_3x^n + a\left(\int f(x)\,\mathrm{e}^{a\left(\int f(x)x^n dx\right)}dx\right)x^n - \mathrm{e}^{a\left(\int f(x)x^n dx\right)}}{c_3 + \int f(x)\,\mathrm{e}^{a\left(\int f(x)x^n dx\right)}dx} \tag{1}$$

Verification of solutions

$$y = \frac{ac_3x^n + a\left(\int f(x)\,\mathrm{e}^{a\left(\int f(x)x^n dx\right)}dx\right)x^n - \mathrm{e}^{a\left(\int f(x)x^n dx\right)}}{c_3 + \int f(x)\,\mathrm{e}^{a\left(\int f(x)x^n dx\right)}dx}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(a*x^n*f(x)^2-(diff(f(x), x))
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(f(x)*y(x)^2+y(x)-a*x^n*f(x)*y(x)*x+x^2*a
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-a*x^n*f(x)*y(x)+a*n*x^(n-1),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2-a*x^n*f[x]*y[x]+a*n*x^(n-1),y[x],x,IncludeSingularSolutions -> Tru
```

Not solved

19.5 problem 5

Internal problem ID [10598]
Internal file name [`OUTPUT/9545_Monday_June_06_2022_03_07_22_PM_12930354/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) = an\, x^{n-1} - a^2 x^{2n} f(x)}$$

19.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)\, y^2 + an\, x^{n-1} - a^2 x^{2n} f(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a^2 x^{2n} f(x) + f(x)\, y^2 + \frac{x^n n a}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = an\, x^{n-1} - a^2 x^{2n} f(x)$, $f_1(x) = 0$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= f(x)^2 \left(an\, x^{n-1} - a^2x^{2n} f(x)\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\, u''(x) - f'(x)\, u'(x) + f(x)^2 \left(an\, x^{n-1} - a^2x^{2n} f(x)\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} + f(x)\left(an\, x^{n-1} - a^2x^{2n} f(x)\right)\_Y(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} + f(x)\left(an\, x^{n-1} - a^2x^{2n} f(x)\right)\_Y(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} + f(x)\left(an\, x^{n-1} - a^2x^{2n} f(x)\right)\_Y(x)\right\}, \{\_Y(x)\}\right)}{f(x)\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} + f(x)\left(an\, x^{n-1} - a^2x^{2n} f(x)\right)\_Y(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} + f(x)\left(an\, x^{n-1} - a^2x^{2n} f(x)\right)\_Y(x)\right\}, \{\_Y(x)\}\right)}{f(x)\,\mathrm{DESol}\left(\left\{\frac{-f(x)^3x^{1+2n}\_Y(x)a^2 + f(x)^2x^n\_Y(x)an + \_Y''(x)xf(x) - f'(x)\_Y'(x)x}{xf(x)}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} + f(x)\left(an\,x^{n-1} - a^2x^{2n}f(x)\right)\_Y(x)\right\}, \{\_Y(x)\}\right)}{f(x)\operatorname{DESol}\left(\left\{\frac{-f(x)^3x^{1+2n}\_Y(x)a^2 + f(x)^2x^n\_Y(x)an + \_Y''(x)xf(x) - f'(x)\_Y'(x)x}{xf(x)}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} + f(x)\left(an\,x^{n-1} - a^2x^{2n}f(x)\right)\_Y(x)\right\}, \{\_Y(x)\}\right)}{f(x)\operatorname{DESol}\left(\left\{\frac{-f(x)^3x^{1+2n}\_Y(x)a^2 + f(x)^2x^n\_Y(x)an + \_Y''(x)xf(x) - f'(x)\_Y'(x)x}{xf(x)}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(2*a*x^(2*n)*n*f(x)+a*x^(2*n)*(diff(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2+a*n*x^(n-1)-a^2*x^(2*n)*f(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2+a*n*x^(n-1)-a^2*x^(2*n)*f[x],y[x],x,IncludeSingularSolutions -> Tr
```

Not solved

19.6 problem 6

Internal problem ID [10599]
Internal file name [`OUTPUT/9546_Monday_June_06_2022_03_07_28_PM_86558589/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + (n+1)\, x^n y^2 - x^{n+1} f(x)\, y = -f(x)}$$

19.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -x^n y^2 n + x^{n+1} f(x)\, y - x^n y^2 - f(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -x^n y^2 n + x^n x f(x)\, y - x^n y^2 - f(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = -f(x)$, $f_1(x) = f(x)\, x^{n+1}$ and $f_2(x) = -n\, x^n - x^n$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(-n\, x^n - x^n)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{n^2 x^n}{x} - \frac{x^n n}{x} \\ f_1 f_2 &= f(x)\, x^{n+1}(-n\, x^n - x^n) \\ f_2^2 f_0 &= -(-n\, x^n - x^n)^2\, f(x) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(-n\, x^n - x^n)\, u''(x) - \left(-\frac{n^2 x^n}{x} - \frac{x^n n}{x} + f(x)\, x^{n+1}(-n\, x^n - x^n)\right) u'(x) - (-n\, x^n - x^n)^2\, f(x)\, u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^{n+1}\left(\left(\int x^{-2n-2} \mathrm{e}^{\int \left(f(x)x^{n+1} + \frac{n}{x}\right)dx} dx\right) c_2 + c_1\right)$$

The above shows that

$$u'(x) = x^n(n+1)\left(\left(\int \mathrm{e}^{\int \frac{f(x)x^{2+n}+n}{x} dx} x^{-2n-2} dx\right) c_2 + c_1\right) + c_2 x^{-n-1} \mathrm{e}^{\int \frac{f(x)x^{2+n}+n}{x} dx}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(x^n(n+1)\left(\left(\int \mathrm{e}^{\int \frac{f(x)x^{2+n}+n}{x} dx} x^{-2n-2} dx\right) c_2 + c_1\right) + c_2 x^{-n-1} \mathrm{e}^{\int \frac{f(x)x^{2+n}+n}{x} dx}\right) x^{-n-1}}{(-n\, x^n - x^n)\left(\left(\int x^{-2n-2} \mathrm{e}^{\int \left(f(x)x^{n+1} + \frac{n}{x}\right)dx} dx\right) c_2 + c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(n+1)\left(\int x^{-2n-2} \mathrm{e}^{\int \left(f(x)x^{n+1} + \frac{n}{x}\right)dx} dx + c_3\right) x^{-n-1} + x^{-2-3n} \mathrm{e}^{\int \left(f(x)x^{n+1} + \frac{n}{x}\right)dx}}{(n+1)\left(\int \mathrm{e}^{\int \frac{f(x)x^{2+n}+n}{x} dx} x^{-2n-2} dx + c_3\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{(n+1)\left(\int x^{-2n-2} e^{\int \left(f(x)x^{n+1}+\frac{n}{x}\right)dx} dx + c_3\right) x^{-n-1} + x^{-2-3n} e^{\int \left(f(x)x^{n+1}+\frac{n}{x}\right)dx}}{(n+1)\left(\int e^{\int \frac{f(x)x^{2+n}+n}{x} dx} x^{-2n-2} dx + c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{(n+1)\left(\int x^{-2n-2} e^{\int \left(f(x)x^{n+1}+\frac{n}{x}\right)dx} dx + c_3\right) x^{-n-1} + x^{-2-3n} e^{\int \left(f(x)x^{n+1}+\frac{n}{x}\right)dx}}{(n+1)\left(\int e^{\int \frac{f(x)x^{2+n}+n}{x} dx} x^{-2n-2} dx + c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x^(n+1)*f(x)*x+n)*(diff(y(x),
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-((-x^n*n-x^n)*y(x)^2+y(x)+x^(n+1)*f(x)*y(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 169

```
dsolve(diff(y(x),x)=-(n+1)*x^n*y(x)^2+x^(n+1)*f(x)*y(x)-f(x),y(x), singsol=all)
```

$$y(x) = \frac{x^{-n-1}\left(x^{n+1}\mathrm{e}^{\int \frac{x^{n+1}f(x)x-2n-2}{x}dx} + \left(\int x^{n}\mathrm{e}^{\int x^{n+1}f(x)dx+(-2n-2)\left(\int \frac{1}{x}dx\right)}dx\right)n + \int x^{n}\mathrm{e}^{\int x^{n+1}f(x)dx+(-2n-2)\left(\int \frac{1}{x}dx\right)}d}{\left(\int x^{n}\mathrm{e}^{\int x^{n+1}f(x)dx+(-2n-2)\left(\int \frac{1}{x}dx\right)}dx\right)n + \int x^{n}\mathrm{e}^{\int x^{n+1}f(x)dx+(-2n-2)\left(\int \frac{1}{x}dx\right)}dx - c_1}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==-(n+1)*x^n*y[x]^2+x^(n+1)*f[x]*y[x]-f[x],y[x],x,IncludeSingularSolutions -> Tr
```

Not solved

19.7 problem 7

Internal problem ID [10600]
Internal file name [`OUTPUT/9547_Monday_June_06_2022_03_07_31_PM_8322341/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - y^2 f(x) - yn = f(x)\, x^{2n} a}$$

19.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{f(x)\, y^2 + ny + f(x)\, x^{2n} a}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{f(x)\, a\, x^{2n}}{x} + \frac{f(x)\, y^2}{x} + \frac{ny}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{f(x) a\, x^{2n}}{x}$, $f_1(x) = \frac{n}{x}$ and $f_2(x) = \frac{f(x)}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{f(x)u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{f'(x)}{x} - \frac{f(x)}{x^2} \\ f_1f_2 &= \frac{nf(x)}{x^2} \\ f_2^2 f_0 &= \frac{f(x)^3\, a\, x^{2n}}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{f(x)\,u''(x)}{x} - \left(\frac{f'(x)}{x} - \frac{f(x)}{x^2} + \frac{nf(x)}{x^2}\right)u'(x) + \frac{f(x)^3\, a\, x^{2n} u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \mathrm{e}^{i\sqrt{a}\left(\int f(x)x^{n-1}dx\right)} + c_2 \mathrm{e}^{-i\sqrt{a}\left(\int f(x)x^{n-1}dx\right)}$$

The above shows that

$$u'(x) = ix^{n-1} f(x)\,\sqrt{a}\,\mathrm{e}^{-i\sqrt{a}\left(\int f(x)x^{n-1}dx\right)}\left(c_1 \mathrm{e}^{2i\sqrt{a}\left(\int f(x)x^{n-1}dx\right)} - c_2\right)$$

Using the above in (1) gives the solution

$$y = -\frac{ix^{n-1}\sqrt{a}\,\mathrm{e}^{-i\sqrt{a}\left(\int f(x)x^{n-1}dx\right)}\left(c_1 \mathrm{e}^{2i\sqrt{a}\left(\int f(x)x^{n-1}dx\right)} - c_2\right)x}{c_1 \mathrm{e}^{i\sqrt{a}\left(\int f(x)x^{n-1}dx\right)} + c_2 \mathrm{e}^{-i\sqrt{a}\left(\int f(x)x^{n-1}dx\right)}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{ix^n\sqrt{a}\left(c_3 \mathrm{e}^{2i\sqrt{a}\left(\int f(x)x^{n-1}dx\right)} - 1\right)}{c_3 \mathrm{e}^{2i\sqrt{a}\left(\int f(x)x^{n-1}dx\right)} + 1}$$

Summary
The solution(s) found are the following

$$y = -\frac{i x^{n} \sqrt{a}\, \left(c_{3} {\mathrm e}^{2 i \sqrt{a}\, \left(\int f \left(x \right) x^{n -1}d x \right)}-1\right)}{c_{3} {\mathrm e}^{2 i \sqrt{a}\, \left(\int f \left(x \right) x^{n -1}d x \right)}+1} \tag{1}$$

Verification of solutions

$$y = -\frac{i x^{n} \sqrt{a}\, \left(c_{3} {\mathrm e}^{2 i \sqrt{a}\, \left(\int f \left(x \right) x^{n -1}d x \right)}-1\right)}{c_{3} {\mathrm e}^{2 i \sqrt{a}\, \left(\int f \left(x \right) x^{n -1}d x \right)}+1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 30

```
dsolve(x*diff(y(x),x)=f(x)*y(x)^2+n*y(x)+a*x^(2*n)*f(x),y(x), singsol=all)
```

$$y(x) = -\tan\left(-\sqrt{a}\left(\int f(x)\, x^{n-1}dx\right)+c_1\right)\sqrt{a}\, x^n$$

✓ Solution by Mathematica
Time used: 0.577 (sec). Leaf size: 41

```
DSolve[x*y'[x]==f[x]*y[x]^2+n*y[x]+a*x^(2*n)*f[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{a}x^n \tan\left(\sqrt{a}\int_1^x f(K[1])K[1]^{n-1}dK[1]+c_1\right)$$

19.8 problem 8

Internal problem ID [10601]
Internal file name [`OUTPUT/9548_Monday_June_06_2022_03_07_32_PM_96364581/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - x^{2n} f(x)\, y^2 - (f(x)\, a\, x^n - n)\, y = f(x)\, b}$$

19.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{a\, x^n f(x)\, y + x^{2n} f(x)\, y^2 + f(x)\, b - ny}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x^n f(x)\, ay}{x} + \frac{x^{2n} f(x)\, y^2}{x} + \frac{f(x)\, b}{x} - \frac{ny}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{f(x)b}{x}$, $f_1(x) = \frac{f(x)a\, x^n - n}{x}$ and $f_2(x) = \frac{f(x)x^{2n}}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{f(x)x^{2n}u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{f'(x)\, x^{2n}}{x} + \frac{2 f(x)\, x^{2n} n}{x^2} - \frac{f(x)\, x^{2n}}{x^2} \\
f_1 f_2 &= \frac{(f(x)\, a\, x^n - n)\, f(x)\, x^{2n}}{x^2} \\
f_2^2 f_0 &= \frac{f(x)^3\, x^{4n} b}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{f(x)\, x^{2n} u''(x)}{x} - \left(\frac{f'(x)\, x^{2n}}{x} + \frac{2 f(x)\, x^{2n} n}{x^2} - \frac{f(x)\, x^{2n}}{x^2} + \frac{(f(x)\, a\, x^n - n)\, f(x)\, x^{2n}}{x^2}\right) u'(x) + \frac{f(x)^3\, x^{4n} b u(x}{x^3}$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(c_1 \operatorname{BesselJ}\left(\frac{\sqrt{3}\,\sqrt{-b}}{8a}, \frac{\sqrt{3}\,\sqrt{2}\,\sqrt{b\, x^{2n}}\, x^{-n}}{8a}\right) + c_2 \operatorname{BesselY}\left(\frac{\sqrt{3}\,\sqrt{-b}}{8a}, \frac{\sqrt{3}\,\sqrt{2}\,\sqrt{b\, x^{2n}}\, x^{-n}}{8a}\right)\right) \mathrm{e}^{\frac{\left(\int \left(\frac{a\, x^n f(x)}{x} + \frac{f'(x)}{f(x)} + \frac{3n}{x}\right) dx\right)}{2}}$$

The above shows that

$$u'(x) = \frac{\left(c_1 \operatorname{BesselJ}\left(\frac{\sqrt{3}\,\sqrt{-b}}{8a}, \frac{\sqrt{3}\,\sqrt{2}\,\sqrt{b\, x^{2n}}\, x^{-n}}{8a}\right) + c_2 \operatorname{BesselY}\left(\frac{\sqrt{3}\,\sqrt{-b}}{8a}, \frac{\sqrt{3}\,\sqrt{2}\,\sqrt{b\, x^{2n}}\, x^{-n}}{8a}\right)\right) \left(f(x)^2\, a\, x^n + f'(x)\, x + 3n\right.}{2 x f(x)}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(f(x)^2\, a\, x^n + f'(x)\, x + 3n f(x)\right) \mathrm{e}^{\frac{\left(\int \frac{f(x)^2 a\, x^n + f'(x) x + 3n f(x)}{f(x) x} dx\right)}{2}} x^{-2n} \mathrm{e}^{\int \left(-\frac{a\, x^{n-1} f(x)}{2} - \frac{f'(x)}{2 f(x)} - \frac{3n}{2x}\right) dx}}{2 f(x)^2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(f(x)^2 a\, x^n + f'(x)\, x + 3n f(x)\right) x^{-2n}}{2 f\left(x\right)^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(f(x)^2 a\, x^n + f'(x)\, x + 3n f(x)\right) x^{-2n}}{2 f\left(x\right)^2} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(f(x)^2 a\, x^n + f'(x)\, x + 3n f(x)\right) x^{-2n}}{2 f\left(x\right)^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 65

```
dsolve(x*diff(y(x),x)=x^(2*n)*f(x)*y(x)^2+(a*x^n*f(x)-n)*y(x)+b*f(x),y(x), singsol=all)
```

$$y(x) = -\frac{\left(a^2 + \tanh\left(\frac{\sqrt{a^2(a^2-4b)}\,\left(a\left(\int f(x)x^{n-1}dx\right)+c_1\right)}{2a^2}\right)\sqrt{a^2\left(a^2-4b\right)}\right)x^{-n}}{2a}$$

✓ Solution by Mathematica

Time used: 2.272 (sec). Leaf size: 82

```
DSolve[x*y'[x]==x^(2*n)*f[x]*y[x]^2+(a*x^n*f[x]-n)*y[x]+b*f[x],y[x],x,IncludeSingularSolutio
```

$$\text{Solve}\left[\int_1^{\sqrt{\frac{x^{2n}}{b}}y(x)}\frac{1}{K[1]^2-\sqrt{\frac{a^2}{b}}K[1]+1}dK[1]=\int_1^x\frac{bf(K[2])\sqrt{\frac{K[2]^{2n}}{b}}}{K[2]}dK[2]+c_1,y(x)\right]$$

19.9 problem 9

Internal problem ID [10602]
Internal file name [`OUTPUT/9549_Monday_June_06_2022_03_07_35_PM_92126063/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) - g(x)\, y = -f(x)\, a^2 - a g(x)}$$

19.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)\, y^2 + g(x)\, y - f(x)\, a^2 - a g(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\, y^2 + g(x)\, y - f(x)\, a^2 - a g(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = -f(x)\, a^2 - a g(x)$, $f_1(x) = g(x)$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1 f_2 &= g(x)\, f(x) \\ f_2^2 f_0 &= f(x)^2 \left(-f(x)\, a^2 - a g(x)\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\, u''(x) - (f'(x) + g(x)\, f(x))\, u'(x) + f(x)^2 \left(-f(x)\, a^2 - a g(x)\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = -\left(\int e^{\int (2f(x)a + g(x))dx} f(x)\, dx + c_1\right) e^{-a\left(\int f(x)dx\right)} c_2$$

The above shows that

$$u'(x) = c_2 f(x) \left(-e^{\int (2f(x)a+g(x))dx - a\left(\int f(x)dx\right)} + a\, e^{-a\left(\int f(x)dx\right)} \left(\int e^{\int (2f(x)a+g(x))dx} f(x)\, dx + c_1\right)\right)$$

Using the above in (1) gives the solution

$$y = \frac{\left(-e^{\int (2f(x)a+g(x))dx - a\left(\int f(x)dx\right)} + a\, e^{-a\left(\int f(x)dx\right)} \left(\int e^{\int (2f(x)a+g(x))dx} f(x)\, dx + c_1\right)\right) e^{\int f(x) a dx}}{\int e^{\int (2f(x)a+g(x))dx} f(x)\, dx + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-e^{\int (2f(x)a+g(x))dx} + c_3 a + \left(\int e^{\int (2f(x)a+g(x))dx} f(x)\, dx\right) a}{\int e^{\int (2f(x)a+g(x))dx} f(x)\, dx + c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{-e^{\int (2f(x)a+g(x))dx} + c_3 a + \left(\int e^{\int (2f(x)a+g(x))dx} f(x)\, dx\right) a}{\int e^{\int (2f(x)a+g(x))dx} f(x)\, dx + c_3} \tag{1}$$

Verification of solutions

$$y = \frac{-e^{\int (2f(x)a+g(x))dx} + c_3 a + \left(\int e^{\int (2f(x)a+g(x))dx} f(x)\, dx\right) a}{\int e^{\int (2f(x)a+g(x))dx} f(x)\, dx + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 67

```
dsolve(diff(y(x),x)=f(x)*y(x)^2+g(x)*y(x)-a^2*f(x)-a*g(x),y(x), singsol=all)
```

$$y(x) = \frac{-a\left(\int e^{\int g(x)dx+2a\left(\int f(x)dx\right)} f(x)\, dx\right) + c_1 a + e^{\int g(x)dx+2a\left(\int f(x)dx\right)}}{-\left(\int e^{\int g(x)dx+2a\left(\int f(x)dx\right)} f(x)\, dx\right) + c_1}$$

✓ Solution by Mathematica

Time used: 1.122 (sec). Leaf size: 201

```
DSolve[y'[x]==f[x]*y[x]^2+g[x]*y[x]-a^2*f[x]-a*g[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\int_1^x -\frac{\exp\left(-\int_1^{K[2]}(-2af(K[1])-g(K[1]))dK[1]\right)(af(K[2])+y(x)f(K[2])+g(K[2]))}{a-y(x)}dK[2] + \int_1^{y(x)}\left(-\int_1^x\left(-\frac{\exp\left(-\int_1^{K[2]}(-2af(K[1])-g(K[1]))dK[1]\right)f(K[2])}{a-K[3]} - \frac{\exp\left(-\int_1^{K[2]}(-2af(K[1])-g\right.}{}\right.\right.\right.$$

$$\left.\left. - \frac{\exp\left(-\int_1^x(-2af(K[1])-g(K[1]))dK[1]\right)}{(K[3]-a)^2}\right)dK[3]=c_1,y(x)\right]$$

19.10 problem 10

Internal problem ID [10603]
Internal file name [`OUTPUT/9550_Monday_June_06_2022_03_07_36_PM_81058003/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) - g(x)\, y = an\, x^{n-1} - a\, x^n g(x) - a^2 x^{2n} f(x)}$$

19.10.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)\, y^2 + g(x)\, y + an\, x^{n-1} - a\, x^n g(x) - a^2 x^{2n} f(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a^2 x^{2n} f(x) - a\, x^n g(x) + f(x)\, y^2 + \frac{x^n na}{x} + g(x)\, y$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = an\, x^{n-1} - a\, x^n g(x) - a^2 x^{2n} f(x)$, $f_1(x) = g(x)$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= f'(x) \\
f_1 f_2 &= g(x)\, f(x) \\
f_2^2 f_0 &= f(x)^2 \left(an\, x^{n-1} - a\, x^n g(x) - a^2 x^{2n} f(x)\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\, u''(x) - (f'(x) + g(x)\, f(x))\, u'(x) + f(x)^2 \left(an\, x^{n-1} - a\, x^n g(x) - a^2 x^{2n} f(x)\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(f'(x) + g(x)\, f(x))\_Y'(x)}{f(x)} + f(x) \left(an\, x^{n-1} - a\, x^n g(x) - a^2 x^{2n} f(x)\right)\_Y(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(f'(x) + g(x)\, f(x))\_Y'(x)}{f(x)} + f(x) \left(an\, x^{n-1} - a\, x^n g(x) - a^2 x^{2n} f(x)\right)\_Y(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(f'(x)+g(x)f(x))\_Y'(x)}{f(x)} + f(x) \left(an\, x^{n-1} - a\, x^n g(x) - a^2 x^{2n} f(x)\right)\_Y(x)\right\}, \{\_ \right.}{f(x)\, \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(f'(x)+g(x)f(x))\_Y'(x)}{f(x)} + f(x) \left(an\, x^{n-1} - a\, x^n g(x) - a^2 x^{2n} f(x)\right)\_Y(x)\right\}, \{ \right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(f'(x)+g(x)f(x))\_Y'(x)}{f(x)} + f(x) \left(an\, x^{n-1} - a\, x^n g(x) - a^2 x^{2n} f(x)\right)\_Y(x)\right\}, \{ \right.}{f(x)\, \mathrm{DESol}\left(\left\{\frac{-f(x)^3 x^{1+2n}\_Y(x) a^2 + \_Y''(x) x f(x) - f(x)^2 g(x) x^{n+1}\_Y(x) a - (f'(x)+g(x)f(x))\_Y'(x) x + f(x)^2 x^n \_Y(x) an}{x f(x)} \right.\right.}$$

Summary
The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\_Y''(x) - \frac{(f'(x)+g(x)f(x))\_Y'(x)}{f(x)} + f(x)\left(an\,x^{n-1} - a\,x^{n}g(x) - a^{2}x^{2n}f(x)\right)\_Y(x)\right\}, \{}{f(x)\operatorname{DESol}\left(\left\{\frac{-f(x)^{3}x^{1+2n}\_Y(x)a^{2}+\_Y''(x)xf(x)-f(x)^{2}g(x)x^{n+1}\_Y(x)a-(f'(x)+g(x)f(x))\_Y'(x)x+f(x)^{2}x^{n}\_Y(x)an}{xf(x)}\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\_Y''(x) - \frac{(f'(x)+g(x)f(x))\_Y'(x)}{f(x)} + f(x)\left(an\,x^{n-1} - a\,x^{n}g(x) - a^{2}x^{2n}f(x)\right)\_Y(x)\right\}, \{}{f(x)\operatorname{DESol}\left(\left\{\frac{-f(x)^{3}x^{1+2n}\_Y(x)a^{2}+\_Y''(x)xf(x)-f(x)^{2}g(x)x^{n+1}\_Y(x)a-(f'(x)+g(x)f(x))\_Y'(x)x+f(x)^{2}x^{n}\_Y(x)an}{xf(x)}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (g(x)*f(x)+diff(f(x), x))*(dif
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(f(x)*y(x)^2+y(x)+g(x)*y(x)*x+x^2*(a*n*x^
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2+g(x)*y(x)+a*n*x^(n-1)-a*x^n*g(x)-a^2*f(x)*x^(2*n),y(x), sing
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2+g[x]*y[x]+a*n*x^(n-1)-a*x^n*g[x]-a^2*f[x]*x^(2*n),y[x],x,IncludeSi
```

Not solved

19.11 problem 11

Internal problem ID [10604]
Internal file name [`OUTPUT/9551_Monday_June_06_2022_03_08_07_PM_96640301/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) + a\,x^n g(x)\,y = an\,x^{n-1} + a^2 x^{2n}(g(x) - f(x))}$$

19.11.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a^2 x^{2n} g(x) - a^2 x^{2n} f(x) - x^n g(x)\,ay + an\,x^{n-1} + f(x)\,y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a^2 x^{2n} g(x) - a^2 x^{2n} f(x) - x^n g(x)\,ay + \frac{x^n na}{x} + f(x)\,y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a^2 x^{2n} g(x) - a^2 x^{2n} f(x) + an\,x^{n-1}$, $f_1(x) = -a\,x^n g(x)$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f\,(x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1f_2 &= -x^n f(x)\, g(x)\, a \\ f_2^2 f_0 &= f(x)^2 \left(a^2x^{2n}g(x) - a^2x^{2n}f(x) + an\, x^{n-1}\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\, u''(x) - (-x^n f(x)\, g(x)\, a + f'(x))\, u'(x) + f(x)^2 \left(a^2x^{2n}g(x) - a^2x^{2n}f(x) + an\, x^{n-1}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(-x^n f(x)\, g(x)\, a + f'(x))\_Y'(x)}{f(x)} + f(x)\left(a^2x^{2n}g(x) - a^2x^{2n}f(x) + an\, x^{n-1}\right)\_Y(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(-x^n f(x)\, g(x)\, a + f'(x))\_Y'(x)}{f(x)} + f(x)\left(a^2x^{2n}g(x) - a^2x^{2n}f(x) + an\, x^{n-1}\right)\_Y(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(-x^n f(x)g(x)a+f'(x))\_Y'(x)}{f(x)} + f(x)\left(a^2x^{2n}g(x) - a^2x^{2n}f(x) + an\, x^{n-1}\right)\_Y(x)\right.}{f(x)\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(-x^n f(x)g(x)a+f'(x))\_Y'(x)}{f(x)} + f(x)\left(a^2x^{2n}g(x) - a^2x^{2n}f(x) + an\, x^{n-1}\right)\_Y(x\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(-x^n f(x)g(x)a+f'(x))\_Y'(x)}{f(x)} + f(x)\left(a^2x^{2n}g(x) - a^2x^{2n}f(x) + an\, x^{n-1}\right)\_Y(x)\right\}\right.}{f(x)\,\mathrm{DESol}\left(\left\{\frac{-f(x)^2\_Y(x)a^2(f(x)-g(x))x^{1+2n}+f(x)^2x^n\_Y(x)an+a\,x^{n+1}f(x)g(x)\_Y'(x)-f'(x)\_Y'(x)x+\_Y''(x)xf(x}{f(x)x}\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(-x^n f(x) g(x) a + f'(x))\_Y'(x)}{f(x)} + f(x)\left(a^2 x^{2n} g(x) - a^2 x^{2n} f(x) + a n\, x^{n-1}\right)\_Y(x)\right\}}{f(x)\,\mathrm{DESol}\left(\left\{\frac{-f(x)^2\_Y(x) a^2 (f(x)-g(x)) x^{1+2n} + f(x)^2 x^n\_Y(x) a n + a\, x^{n+1} f(x) g(x)\_Y'(x) - f'(x)\_Y'(x) x + \_Y''(x) x f(x}{f(x) x}\right.\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(-x^n f(x) g(x) a + f'(x))\_Y'(x)}{f(x)} + f(x)\left(a^2 x^{2n} g(x) - a^2 x^{2n} f(x) + a n\, x^{n-1}\right)\_Y(x)\right\}}{f(x)\,\mathrm{DESol}\left(\left\{\frac{-f(x)^2\_Y(x) a^2 (f(x)-g(x)) x^{1+2n} + f(x)^2 x^n\_Y(x) a n + a\, x^{n+1} f(x) g(x)\_Y'(x) - f'(x)\_Y'(x) x + \_Y''(x) x f(x}{f(x) x}\right.\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(f(x)*x^n*g(x)*a-(diff(f(x),
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(f(x)*y(x)^2+y(x)-a*x^n*g(x)*y(x)*x+x^2*(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-a*x^n*g(x)*y(x)+a*n*x^(n-1)+a^2*x^(2*n)*(g(x)-f(x)),y(x), si
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2-a*x^n*g[x]*y[x]+a*n*x^(n-1)+a^2*x^(2*n)*(g[x]-f[x]),y[x],x,Include
```

Not solved

19.12 problem 12

Internal problem ID [10605]
Internal file name [`OUTPUT/9552_Monday_June_06_2022_03_08_37_PM_49442143/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - a\,\mathrm{e}^{\lambda x} y^2 - a\,\mathrm{e}^{\lambda x} f(x)\,y = \lambda f(x)}$$

19.12.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \mathrm{e}^{\lambda x} a\,y^2 + a\,\mathrm{e}^{\lambda x} f(x)\,y + \lambda f(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \mathrm{e}^{\lambda x} a\,y^2 + a\,\mathrm{e}^{\lambda x} f(x)\,y + \lambda f(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \lambda f(x)$, $f_1(x) = a\,\mathrm{e}^{\lambda x} f(x)$ and $f_2(x) = \mathrm{e}^{\lambda x} a$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\mathrm{e}^{\lambda x} a u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= a\lambda\,\mathrm{e}^{\lambda x} \\ f_1f_2 &= a^2\mathrm{e}^{2\lambda x}f(x) \\ f_2^2f_0 &= \mathrm{e}^{2\lambda x}f(x)\,a^2\lambda \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\mathrm{e}^{\lambda x}au''(x) - \left(a^2\mathrm{e}^{2\lambda x}f(x) + a\lambda\,\mathrm{e}^{\lambda x}\right)u'(x) + \mathrm{e}^{2\lambda x}f(x)\,a^2\lambda u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{\mathrm{e}^{\lambda x}\left(\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)}dx\right)c_2 + c_1\lambda\right)}{\lambda}$$

The above shows that

$$u'(x) = \frac{\mathrm{e}^{\lambda x}\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)}dx\right)c_2\lambda + \mathrm{e}^{\lambda x}c_1\lambda^2 + c_2\mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)}}{\lambda}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\mathrm{e}^{\lambda x}\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)}dx\right)c_2\lambda + \mathrm{e}^{\lambda x}c_1\lambda^2 + c_2\mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)}\right)\mathrm{e}^{-2\lambda x}}{a\left(\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)}dx\right)c_2 + c_1\lambda\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\mathrm{e}^{\lambda x}\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)}dx\right)\lambda + \mathrm{e}^{\lambda x}c_3\lambda^2 + \mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)}\right)\mathrm{e}^{-2\lambda x}}{a\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)}dx + \lambda c_3\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\mathrm{e}^{\lambda x}\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)}dx\right)\lambda + \mathrm{e}^{\lambda x}c_3\lambda^2 + \mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)}\right)\mathrm{e}^{-2\lambda x}}{a\left(\int \mathrm{e}^{-\lambda x + a\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)}dx + \lambda c_3\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(e^{\lambda x}\left(\int e^{-\lambda x+a\left(\int e^{\lambda x} f(x)dx\right)}dx\right)\lambda + e^{\lambda x} c_3 \lambda^2 + e^{a\left(\int e^{\lambda x} f(x)dx\right)}\right) e^{-2\lambda x}}{a\left(\int e^{-\lambda x+a\left(\int e^{\lambda x} f(x)dx\right)}dx + \lambda c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a*exp(lambda*x)*f(x)+lambda)*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         <- linear_1 successful
         Change of variables used:
            [x = ln(t)/lambda]
         Linear ODE actually solved:
            a*f(ln(t)/lambda)*u(t)-a*t*f(ln(t)/lambda)*diff(u(t),t)+t*lambda*diff(diff(u(t),
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 92

```
dsolve(diff(y(x),x)=a*exp(lambda*x)*y(x)^2+a*exp(lambda*x)*f(x)*y(x)+lambda*f(x),y(x), sings
```

$$y(x) = \frac{-c_1 e^{-2x\lambda + a\left(\int e^{x\lambda} f(x)dx\right)} - e^{-x\lambda}\left(\int e^{-x\lambda + a\left(\int e^{x\lambda} f(x)dx\right)}dx\right)c_1\lambda - \lambda^2 e^{-x\lambda}}{a\left(\left(\int e^{-x\lambda + a\left(\int e^{x\lambda} f(x)dx\right)}dx\right)c_1 + \lambda\right)}$$

✓ Solution by Mathematica

Time used: 4.45 (sec). Leaf size: 166

```
DSolve[y'[x]==a*Exp[\[Lambda]*x]*y[x]^2+a*Exp[\[Lambda]*x]*f[x]*y[x]+\[Lambda]*f[x],y[x],x,I
```

$$y(x) \to -\frac{\lambda e^{-2\lambda x}\left(\exp\left(-\int_1^{e^{x\lambda}} -\frac{af\left(\frac{\log(K[1])}{\lambda}\right)}{\lambda}dK[1]\right) + e^{\lambda x}\int_1^{e^{x\lambda}} \frac{\exp\left(-\int_1^{K[2]} -\frac{af\left(\frac{\log(K[1])}{\lambda}\right)}{\lambda}dK[1]\right)}{K[2]^2}dK[2] + c_1 e^{\lambda x}\right)}{a\left(\int_1^{e^{x\lambda}} \frac{\exp\left(-\int_1^{K[2]} -\frac{af\left(\frac{\log(K[1])}{\lambda}\right)}{\lambda}dK[1]\right)}{K[2]^2}dK[2] + c_1\right)}$$

$$y(x) \to -\frac{\lambda e^{\lambda(-x)}}{a}$$

19.13 problem 13

Internal problem ID [10606]
Internal file name [`OUTPUT/9553_Monday_June_06_2022_03_08_39_PM_4505664/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) + a\,\mathrm{e}^{\lambda x} f(x)\,y = a\lambda\,\mathrm{e}^{\lambda x}}$$

19.13.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)\,y^2 - a\,\mathrm{e}^{\lambda x} f(x)\,y + a\lambda\,\mathrm{e}^{\lambda x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\,y^2 - a\,\mathrm{e}^{\lambda x} f(x)\,y + a\lambda\,\mathrm{e}^{\lambda x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\lambda\,\mathrm{e}^{\lambda x}$, $f_1(x) = -a\,\mathrm{e}^{\lambda x} f(x)$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1 f_2 &= -f(x)^2 \mathrm{e}^{\lambda x} a \\ f_2^2 f_0 &= \lambda f(x)^2 a \, \mathrm{e}^{\lambda x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x) u''(x) - \left(-f(x)^2 \mathrm{e}^{\lambda x} a + f'(x)\right) u'(x) + \lambda f(x)^2 a \, \mathrm{e}^{\lambda x} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{-a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)} \left(c_1 + \left(\int f(x) \, \mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)} dx\right) c_2\right)$$

The above shows that

$$u'(x) = -\left(a\left(c_1 + \left(\int f(x) \, \mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)} dx\right) c_2\right) \mathrm{e}^{\lambda x - a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)} - c_2\right) f(x)$$

Using the above in (1) gives the solution

$$y = \frac{\left(a\left(c_1 + \left(\int f(x) \, \mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)} dx\right) c_2\right) \mathrm{e}^{\lambda x - a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)} - c_2\right) \mathrm{e}^{\int a \, \mathrm{e}^{\lambda x} f(x) dx}}{c_1 + \left(\int f(x) \, \mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)} dx\right) c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(a\left(c_3 + \int f(x) \, \mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)} dx\right) \mathrm{e}^{\lambda x - a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)} - 1\right) \mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)}}{c_3 + \int f(x) \, \mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)} dx}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(a\left(c_3 + \int f(x) \, \mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)} dx\right) \mathrm{e}^{\lambda x - a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)} - 1\right) \mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)}}{c_3 + \int f(x) \, \mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x} f(x) dx\right)} dx} \tag{1}$$

Verification of solutions

$$y = \frac{\left(a\left(c_3 + \int f(x)\,\mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x} f(x)dx\right)}dx\right)\mathrm{e}^{\lambda x - a\left(\int \mathrm{e}^{\lambda x} f(x)dx\right)} - 1\right)\mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x} f(x)dx\right)}}{c_3 + \int f\left(x\right)\mathrm{e}^{a\left(\int \mathrm{e}^{\lambda x} f(x)dx\right)}dx}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(exp(lambda*x)*f(x)^2*a-(diff
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-a*exp(lambda*x)*f(x)*y(x)+a*lambda*exp(lambda*x),y(x), sings
```

No solution found

✓ Solution by Mathematica

Time used: 48.456 (sec). Leaf size: 207

```
DSolve[y'[x]==f[x]*y[x]^2-a*Exp[\[Lambda]*x]*f[x]*y[x]+a*\[Lambda]*Exp[\[Lambda]*x],y[x],x,I
```

$$y(x) \rightarrow \frac{a \exp\left(\int_1^{e^{x\lambda}} -\frac{af\left(\frac{\log(K[1])}{\lambda}\right)}{\lambda} dK[1] + 2\lambda x\right) \left(\int_1^{e^{x\lambda}} \frac{\exp\left(-\int_1^{K[2]} -\frac{af\left(\frac{\log(K[1])}{\lambda}\right)}{\lambda} dK[1]\right)}{K[2]^2} dK[2] + c_1\right.}{\exp\left(\int_1^{e^{x\lambda}} -\frac{af\left(\frac{\log(K[1])}{\lambda}\right)}{\lambda} dK[1] + \lambda x\right) \int_1^{e^{x\lambda}} \frac{\exp\left(-\int_1^{K[2]} -\frac{af\left(\frac{\log(K[1])}{\lambda}\right)}{\lambda} dK[1]\right)}{K[2]^2} dK[2] + c_1 \exp\left(\int_1^{e^{x\lambda}} -\frac{af\left(\frac{\log(K[1])}{\lambda}\right)}{\lambda}\right.}$$

$$y(x) \rightarrow ae^{\lambda x}$$

19.14 problem 14

Internal problem ID [10607]
Internal file name [`OUTPUT/9554_Monday_June_06_2022_03_08_42_PM_50840373/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) = a\lambda\, e^{\lambda x} - a^2 e^{2\lambda x} f(x)}$$

19.14.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)\, y^2 + a\lambda\, e^{\lambda x} - a^2 e^{2\lambda x} f(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\, y^2 + a\lambda\, e^{\lambda x} - a^2 e^{2\lambda x} f(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\lambda\, e^{\lambda x} - a^2 e^{2\lambda x} f(x)$, $f_1(x) = 0$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1f_2 &= 0 \\ f_2^2f_0 &= f(x)^2\left(a\lambda\,\mathrm{e}^{\lambda x} - a^2\mathrm{e}^{2\lambda x}f(x)\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\,u''(x) - f'(x)\,u'(x) + f(x)^2\left(a\lambda\,\mathrm{e}^{\lambda x} - a^2\mathrm{e}^{2\lambda x}f(x)\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{-f(x)^3\,\mathrm{e}^{2\lambda x}\_Y(x)\,a^2 + f(x)^2\,\mathrm{e}^{\lambda x}\_Y(x)\,a\lambda + \_Y''(x)\,f(x) - f'(x)\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-f(x)^3\,\mathrm{e}^{2\lambda x}\_Y(x)\,a^2 + f(x)^2\,\mathrm{e}^{\lambda x}\_Y(x)\,a\lambda + \_Y''(x)\,f(x) - f'(x)\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-f(x)^3\mathrm{e}^{2\lambda x}\_Y(x)a^2+f(x)^2\mathrm{e}^{\lambda x}\_Y(x)a\lambda+\_Y''(x)f(x)-f'(x)\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x)\,\mathrm{DESol}\left(\left\{\frac{-f(x)^3\mathrm{e}^{2\lambda x}\_Y(x)a^2+f(x)^2\mathrm{e}^{\lambda x}\_Y(x)a\lambda+\_Y''(x)f(x)-f'(x)\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-f(x)^3\mathrm{e}^{2\lambda x}\_Y(x)a^2+f(x)^2\mathrm{e}^{\lambda x}\_Y(x)a\lambda+\_Y''(x)f(x)-f'(x)\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x)\,\mathrm{DESol}\left(\left\{\frac{-f(x)^3\mathrm{e}^{2\lambda x}\_Y(x)a^2+f(x)^2\mathrm{e}^{\lambda x}\_Y(x)a\lambda+\_Y''(x)f(x)-f'(x)\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{-f(x)^{3} e^{2\lambda x} \_\_Y(x) a^{2}+f(x)^{2} e^{\lambda x} \_\_Y(x) a\lambda+\_\_Y''(x) f(x)-f'(x) \_\_Y'(x)}{f(x)}\right\}, \left\{\_\_Y(x)\right\}\right)}{f(x) \operatorname{DESol}\left(\left\{\frac{-f(x)^{3} e^{2\lambda x} \_\_Y(x) a^{2}+f(x)^{2} e^{\lambda x} \_\_Y(x) a\lambda+\_\_Y''(x) f(x)-f'(x) \_\_Y'(x)}{f(x)}\right\}, \left\{\_\_Y(x)\right\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{-f(x)^{3} e^{2\lambda x} \_\_Y(x) a^{2}+f(x)^{2} e^{\lambda x} \_\_Y(x) a\lambda+\_\_Y''(x) f(x)-f'(x) \_\_Y'(x)}{f(x)}\right\}, \left\{\_\_Y(x)\right\}\right)}{f(x) \operatorname{DESol}\left(\left\{\frac{-f(x)^{3} e^{2\lambda x} \_\_Y(x) a^{2}+f(x)^{2} e^{\lambda x} \_\_Y(x) a\lambda+\_\_Y''(x) f(x)-f'(x) \_\_Y'(x)}{f(x)}\right\}, \left\{\_\_Y(x)\right\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2+a*lambda*exp(lambda*x)-a^2*exp(2*lambda*x)*f(x),y(x), singso
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2+a*\[Lambda]*Exp[\[Lambda]*x]-a^2*Exp[2*\[Lambda]*x]*f[x],y[x],x,In
```

Not solved

19.15 problem 15

Internal problem ID [10608]
Internal file name [`OUTPUT/9555_Monday_June_06_2022_03_08_45_PM_80987473/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) - \lambda y = a^2 e^{2\lambda x} f(x)}$$

19.15.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)\, y^2 + \lambda y + a^2 e^{2\lambda x} f(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\, y^2 + \lambda y + a^2 e^{2\lambda x} f(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a^2 e^{2\lambda x} f(x)$, $f_1(x) = \lambda$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1 f_2 &= \lambda f(x) \\ f_2^2 f_0 &= f(x)^3 e^{2\lambda x} a^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x) u''(x) - (f'(x) + \lambda f(x)) u'(x) + f(x)^3 e^{2\lambda x} a^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 e^{ia\left(\int e^{\lambda x} f(x) dx\right)} + c_2 e^{-ia\left(\int e^{\lambda x} f(x) dx\right)}$$

The above shows that

$$u'(x) = ia\, e^{\lambda x} f(x) \left(c_1 e^{ia\left(\int e^{\lambda x} f(x) dx\right)} - c_2 e^{-ia\left(\int e^{\lambda x} f(x) dx\right)}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{ia\, e^{\lambda x} \left(c_1 e^{ia\left(\int e^{\lambda x} f(x) dx\right)} - c_2 e^{-ia\left(\int e^{\lambda x} f(x) dx\right)}\right)}{c_1 e^{ia\left(\int e^{\lambda x} f(x) dx\right)} + c_2 e^{-ia\left(\int e^{\lambda x} f(x) dx\right)}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{ia\, e^{\lambda x} \left(c_3 e^{ia\left(\int e^{\lambda x} f(x) dx\right)} - e^{-ia\left(\int e^{\lambda x} f(x) dx\right)}\right)}{c_3 e^{ia\left(\int e^{\lambda x} f(x) dx\right)} + e^{-ia\left(\int e^{\lambda x} f(x) dx\right)}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{ia\, e^{\lambda x} \left(c_3 e^{ia\left(\int e^{\lambda x} f(x) dx\right)} - e^{-ia\left(\int e^{\lambda x} f(x) dx\right)}\right)}{c_3 e^{ia\left(\int e^{\lambda x} f(x) dx\right)} + e^{-ia\left(\int e^{\lambda x} f(x) dx\right)}} \tag{1}$$

Verification of solutions

$$y = -\frac{ia\,\mathrm{e}^{\lambda x}\left(c_3\mathrm{e}^{ia\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)} - \mathrm{e}^{-ia\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)}\right)}{c_3\mathrm{e}^{ia\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)} + \mathrm{e}^{-ia\left(\int \mathrm{e}^{\lambda x}f(x)dx\right)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 26

```
dsolve(diff(y(x),x)=f(x)*y(x)^2+lambda*y(x)+a^2*exp(2*lambda*x)*f(x),y(x), singsol=all)
```

$$y(x) = -\tan\left(-a\left(\int \mathrm{e}^{x\lambda}f(x)\,dx\right) + c_1\right)a\,\mathrm{e}^{x\lambda}$$

✓ Solution by Mathematica

Time used: 0.615 (sec). Leaf size: 47

```
DSolve[y'[x]==f[x]*y[x]^2+\[Lambda]*y[x]+a^2*Exp[2*\[Lambda]*x]*f[x],y[x],x,IncludeSingularS
```

$$y(x) \to \sqrt{a^2}e^{\lambda x}\tan\left(\sqrt{a^2}\int_1^x e^{\lambda K[1]}f(K[1])dK[1] + c_1\right)$$

19.16 problem 16

Internal problem ID [10609]
Internal file name [`OUTPUT/9556_Monday_June_06_2022_03_08_46_PM_10138442/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) + f(x) \left(e^{\lambda x} a + b\right) y = a\lambda\, e^{\lambda x}}$$

19.16.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -a\, e^{\lambda x} f(x)\, y + a\lambda\, e^{\lambda x} - f(x)\, by + f(x)\, y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -a\, e^{\lambda x} f(x)\, y + a\lambda\, e^{\lambda x} - f(x)\, by + f(x)\, y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\lambda\, e^{\lambda x}$, $f_1(x) = -a\, e^{\lambda x} f(x) - f(x)\, b$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1 f_2 &= \left(-a\,\mathrm{e}^{\lambda x} f(x) - f(x)\, b\right) f(x) \\ f_2^2 f_0 &= \lambda f(x)^2\, a\,\mathrm{e}^{\lambda x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\, u''(x) - \left(\left(-a\,\mathrm{e}^{\lambda x} f(x) - f(x)\, b\right) f(x) + f'(x)\right) u'(x) + \lambda f(x)^2\, a\,\mathrm{e}^{\lambda x} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{-\left(\int f(x)\left(\mathrm{e}^{\lambda x} a + b\right) dx\right)} \left(c_1 + \left(\int f(x)\, \mathrm{e}^{\int f(x)\left(\mathrm{e}^{\lambda x} a+b\right) dx} dx\right) c_2\right)$$

The above shows that

$$\begin{aligned} u'(x) = -\Bigg(a\left(c_1 + \left(\int f(x)\, \mathrm{e}^{\int f(x)\left(\mathrm{e}^{\lambda x} a+b\right) dx} dx\right) c_2\right) \mathrm{e}^{\lambda x - \left(\int f(x)\left(\mathrm{e}^{\lambda x} a+b\right) dx\right)} \\ + \mathrm{e}^{-\left(\int f(x)\left(\mathrm{e}^{\lambda x} a+b\right) dx\right)} \left(\int f(x)\, \mathrm{e}^{\int f(x)\left(\mathrm{e}^{\lambda x} a+b\right) dx} dx\right) c_2 b + \mathrm{e}^{-\left(\int f(x)\left(\mathrm{e}^{\lambda x} a+b\right) dx\right)} c_1 b \\ - c_2\Bigg) f(x) \end{aligned}$$

Using the above in (1) gives the solution

$$y = \frac{\left(a\left(c_1 + \left(\int f(x)\, \mathrm{e}^{\int f(x)\left(\mathrm{e}^{\lambda x} a+b\right) dx} dx\right) c_2\right) \mathrm{e}^{\lambda x - \left(\int f(x)\left(\mathrm{e}^{\lambda x} a+b\right) dx\right)} + \mathrm{e}^{-\left(\int f(x)\left(\mathrm{e}^{\lambda x} a+b\right) dx\right)} \left(\int f(x)\, \mathrm{e}^{\int f(x)\left(\mathrm{e}^{\lambda x} a+b\right) da}\right.\right.}{c_1 + \left(\int f(x)\, \mathrm{e}^{\int f(x)\left(\mathrm{e}^{\lambda x} a+b\right) dx} dx\right) c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\mathrm{e}^{\lambda x} a + b\right) \left(\int f(x)\, \mathrm{e}^{\int f(x)\left(\mathrm{e}^{\lambda x} a+b\right) dx} dx\right) + \mathrm{e}^{\lambda x} c_3 a + b c_3 - \mathrm{e}^{\int f(x)\left(\mathrm{e}^{\lambda x} a+b\right) dx}}{c_3 + \int f(x)\, \mathrm{e}^{\int f(x)\left(\mathrm{e}^{\lambda x} a+b\right) dx} dx}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(e^{\lambda x} a + b\right) \left(\int f(x)\, e^{\int f(x)\left(e^{\lambda x} a + b\right) dx} dx\right) + e^{\lambda x} c_3 a + b c_3 - e^{\int f(x)\left(e^{\lambda x} a + b\right) dx}}{c_3 + \int f\left(x\right) e^{\int f(x)\left(e^{\lambda x} a + b\right) dx} dx} \tag{1}$$

Verification of solutions

$$y = \frac{\left(e^{\lambda x} a + b\right) \left(\int f(x)\, e^{\int f(x)\left(e^{\lambda x} a + b\right) dx} dx\right) + e^{\lambda x} c_3 a + b c_3 - e^{\int f(x)\left(e^{\lambda x} a + b\right) dx}}{c_3 + \int f\left(x\right) e^{\int f(x)\left(e^{\lambda x} a + b\right) dx} dx}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-exp(lambda*x)*f(x)^2*a-f(x)^
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-f(x)*(a*exp(lambda*x)+b)*y(x)+a*lambda*exp(lambda*x),y(x), s
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2-f[x]*(a*Exp[\[Lambda]*x]+b)*y[x]+a*\[Lambda]*Exp[\[Lambda]*x],y[x]
```

Not solved

19.17 problem 17

Internal problem ID [10610]
Internal file name [`OUTPUT/9557_Monday_June_06_2022_03_08_52_PM_9131604/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - e^{\lambda x} f(x)\, y^2 - (f(x)\, a - \lambda)\, y = b\, e^{-\lambda x} f(x)}$$

19.17.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= e^{\lambda x} f(x)\, y^2 + b\, e^{-\lambda x} f(x) + f(x)\, ay - \lambda y \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = e^{\lambda x} f(x)\, y^2 + b\, e^{-\lambda x} f(x) + f(x)\, ay - \lambda y$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = b\, e^{-\lambda x} f(x)$, $f_1(x) = f(x)\, a - \lambda$ and $f_2(x) = e^{\lambda x} f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{e^{\lambda x} f(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f(x) \mathrm{e}^{\lambda x} \lambda + \mathrm{e}^{\lambda x} f'(x) \\ f_1 f_2 &= (f(x) a - \lambda) \mathrm{e}^{\lambda x} f(x) \\ f_2^2 f_0 &= \mathrm{e}^{2\lambda x} f(x)^3 b \,\mathrm{e}^{-\lambda x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\mathrm{e}^{\lambda x} f(x) u''(x) - \left(f(x) \mathrm{e}^{\lambda x} \lambda + \mathrm{e}^{\lambda x} f'(x) + (f(x) a - \lambda) \mathrm{e}^{\lambda x} f(x)\right) u'(x) + \mathrm{e}^{2\lambda x} f(x)^3 b \,\mathrm{e}^{-\lambda x} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\frac{a(\int f(x)dx)}{2}} \cosh\left(\frac{\sqrt{a^2 (a^2 - 4b)}\,(a(\int f(x)\,dx) + c_1)}{2a^2}\right) c_2$$

The above shows that

$$u'(x) = \frac{c_2 \mathrm{e}^{\frac{a(\int f(x)dx)}{2}} f(x) \left(a^2 \cosh\left(\frac{\sqrt{a^2(a^2-4b)}\,(a(\int f(x)dx)+c_1)}{2a^2}\right) + \sqrt{a^2 (a^2 - 4b)} \sinh\left(\frac{\sqrt{a^2(a^2-4b)}\,(a(\int f(x)dx)+c_1)}{2a^2}\right)\right)}{2a}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(a^2 \cosh\left(\frac{\sqrt{a^2(a^2-4b)}\,(a(\int f(x)dx)+c_1)}{2a^2}\right) + \sqrt{a^2 (a^2 - 4b)} \sinh\left(\frac{\sqrt{a^2(a^2-4b)}\,(a(\int f(x)dx)+c_1)}{2a^2}\right)\right) \mathrm{e}^{-\lambda x}}{2a \cosh\left(\frac{\sqrt{a^2(a^2-4b)}\,(a(\int f(x)dx)+c_1)}{2a^2}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(a^2 + \tanh\left(\frac{\sqrt{a^2(a^2-4b)}\,(a(\int f(x)dx)+c_3)}{2a^2}\right) \sqrt{a^2 (a^2 - 4b)}\right) \mathrm{e}^{-\lambda x}}{2a}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(a^{2}+\tanh \left(\frac{\sqrt{a^{2} \left(a^{2}-4 b \right)}\, \left(a \left(\int f \left(x \right)d x \right)+c_{3}\right)}{2 a^{2}}\right) \sqrt{a^{2} \left(a^{2}-4 b \right)}\right) {\mathrm e}^{-\lambda x}}{2 a} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(a^{2}+\tanh \left(\frac{\sqrt{a^{2} \left(a^{2}-4 b \right)}\, \left(a \left(\int f \left(x \right)d x \right)+c_{3}\right)}{2 a^{2}}\right) \sqrt{a^{2} \left(a^{2}-4 b \right)}\right) {\mathrm e}^{-\lambda x}}{2 a}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 59

```
dsolve(diff(y(x),x)=exp(lambda*x)*f(x)*y(x)^2+(a*f(x)-lambda)*y(x)+b*exp(-lambda*x)*f(x),y(x
```

$$y(x) = -\frac{\left(a^{2}+\tanh \left(\frac{\sqrt{a^{2} \left(a^{2}-4 b \right)}\, \left(a \left(\int f \left(x \right)d x \right)+c_{1}\right)}{2 a^{2}}\right) \sqrt{a^{2} \left(a^{2}-4 b \right)}\right) {\mathrm e}^{-x \lambda}}{2 a}$$

✓ Solution by Mathematica

Time used: 1.392 (sec). Leaf size: 87

```
DSolve[y'[x]==Exp[\[Lambda]*x]*f[x]*y[x]^2+(a*f[x]-\[Lambda])*y[x]+b*Exp[-\[Lambda]*x]*f[x],
```

$$\text{Solve}\left[\int_1^{\sqrt{\frac{e^{2x\lambda}}{b}}y(x)}\frac{1}{K[1]^2-\sqrt{\frac{a^2}{b}}K[1]+1}dK[1]=\int_1^x be^{-\lambda K[2]}\sqrt{\frac{e^{2\lambda K[2]}}{b}}f(K[2])dK[2]\right.$$
$$\left.+c_1,y(x)\right]$$

19.18 problem 18

Internal problem ID [10611]
Internal file name [`OUTPUT/9558_Monday_June_06_2022_03_08_54_PM_11539278/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) - g(x)\,y = a\lambda\,\mathrm{e}^{\lambda x} - a\,\mathrm{e}^{\lambda x} g(x) - a^2 \mathrm{e}^{2\lambda x} f(x)}$$

19.18.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)\,y^2 + g(x)\,y + a\lambda\,\mathrm{e}^{\lambda x} - a\,\mathrm{e}^{\lambda x} g(x) - a^2 \mathrm{e}^{2\lambda x} f(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\,y^2 + g(x)\,y + a\lambda\,\mathrm{e}^{\lambda x} - a\,\mathrm{e}^{\lambda x} g(x) - a^2 \mathrm{e}^{2\lambda x} f(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\lambda\,\mathrm{e}^{\lambda x} - a\,\mathrm{e}^{\lambda x} g(x) - a^2 \mathrm{e}^{2\lambda x} f(x)$, $f_1(x) = g(x)$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1f_2 &= g(x)\,f(x) \\ f_2^2f_0 &= f(x)^2\left(a\lambda\,\mathrm{e}^{\lambda x} - a\,\mathrm{e}^{\lambda x}g(x) - a^2\mathrm{e}^{2\lambda x}f(x)\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\,u''(x) - (f'(x) + g(x)\,f(x))\,u'(x) + f(x)^2\left(a\lambda\,\mathrm{e}^{\lambda x} - a\,\mathrm{e}^{\lambda x}g(x) - a^2\mathrm{e}^{2\lambda x}f(x)\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{-f(x)^3\,\mathrm{e}^{2\lambda x}\_Y(x)\,a^2 + \_Y''(x)\,f(x) + af(x)^2\_Y(x)\,(\lambda - g(x))\,\mathrm{e}^{\lambda x} - (f'(x) + g(x)\,f(x))\_}{f(x)}\right.\right.$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-f(x)^3\,\mathrm{e}^{2\lambda x}\_Y(x)\,a^2 + \_Y''(x)\,f(x) + af(x)^2\_Y(x)\,(\lambda - g(x))\,\mathrm{e}^{\lambda x} - (f'(x) + g(x)\,f(x)}{f(x)}\right.\right.$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-f(x)^3\mathrm{e}^{2\lambda x}\_Y(x)a^2 + \_Y''(x)f(x) + af(x)^2\_Y(x)(\lambda - g(x))\mathrm{e}^{\lambda x} - (f'(x)+g(x)f(x))\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x)\,\mathrm{DESol}\left(\left\{\frac{-f(x)^3\mathrm{e}^{2\lambda x}\_Y(x)a^2 + \_Y''(x)f(x) + af(x)^2\_Y(x)(\lambda - g(x))\mathrm{e}^{\lambda x} - (f'(x)+g(x)f(x))\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-f(x)^3\mathrm{e}^{2\lambda x}\_Y(x)a^2 + \_Y''(x)f(x) + af(x)^2\_Y(x)(\lambda - g(x))\mathrm{e}^{\lambda x} - (f'(x)+g(x)f(x))\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x)\,\mathrm{DESol}\left(\left\{\frac{-f(x)^3\mathrm{e}^{2\lambda x}\_Y(x)a^2 + \_Y''(x)f(x) + af(x)^2\_Y(x)(\lambda - g(x))\mathrm{e}^{\lambda x} - (f'(x)+g(x)f(x))\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-f(x)^3 \mathrm{e}^{2\lambda x} \_Y(x) a^2 + \_Y''(x) f(x) + a f(x)^2 \_Y(x) (\lambda - g(x)) \mathrm{e}^{\lambda x} - (f'(x) + g(x) f(x)) \_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x) \mathrm{DESol}\left(\left\{\frac{-f(x)^3 \mathrm{e}^{2\lambda x} \_Y(x) a^2 + \_Y''(x) f(x) + a f(x)^2 \_Y(x) (\lambda - g(x)) \mathrm{e}^{\lambda x} - (f'(x) + g(x) f(x)) \_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-f(x)^3 \mathrm{e}^{2\lambda x} \_Y(x) a^2 + \_Y''(x) f(x) + a f(x)^2 \_Y(x) (\lambda - g(x)) \mathrm{e}^{\lambda x} - (f'(x) + g(x) f(x)) \_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x) \mathrm{DESol}\left(\left\{\frac{-f(x)^3 \mathrm{e}^{2\lambda x} \_Y(x) a^2 + \_Y''(x) f(x) + a f(x)^2 \_Y(x) (\lambda - g(x)) \mathrm{e}^{\lambda x} - (f'(x) + g(x) f(x)) \_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (g(x)*f(x)+diff(f(x), x))*(dif
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2+g(x)*y(x)+a*lambda*exp(lambda*x)-a*exp(lambda*x)*g(x)-a^2*ex
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2+g[x]*y[x]+a*\[Lambda]*Exp[\[Lambda]*x]-a*Exp[\[Lambda]*x]*g[x]-a^2
```

Not solved

19.19 problem 19

Internal problem ID [10612]
Internal file name [`OUTPUT/9559_Monday_June_06_2022_03_08_58_PM_23846366/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) + a\,\mathrm{e}^{\lambda x} g(x)\,y = a\lambda\,\mathrm{e}^{\lambda x} + a^2 \mathrm{e}^{2\lambda x}(g(x) - f(x))}$$

19.19.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= a^2\mathrm{e}^{2\lambda x} g(x) - a^2\mathrm{e}^{2\lambda x} f(x) - g(x)\,\mathrm{e}^{\lambda x} a y + a\lambda\,\mathrm{e}^{\lambda x} + f(x)\,y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = a^2\mathrm{e}^{2\lambda x} g(x) - a^2\mathrm{e}^{2\lambda x} f(x) - g(x)\,\mathrm{e}^{\lambda x} a y + a\lambda\,\mathrm{e}^{\lambda x} + f(x)\,y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a^2\mathrm{e}^{2\lambda x} g(x) - a^2\mathrm{e}^{2\lambda x} f(x) + a\lambda\,\mathrm{e}^{\lambda x}$, $f_1(x) = -a\,\mathrm{e}^{\lambda x} g(x)$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1 f_2 &= -f(x)\, g(x)\, \mathrm{e}^{\lambda x} a \\ f_2^2 f_0 &= f(x)^2 \left(a^2 \mathrm{e}^{2\lambda x} g(x) - a^2 \mathrm{e}^{2\lambda x} f(x) + a\lambda\, \mathrm{e}^{\lambda x}\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\, u''(x) - \left(-f(x)\, g(x)\, \mathrm{e}^{\lambda x} a + f'(x)\right) u'(x) + f(x)^2 \left(a^2 \mathrm{e}^{2\lambda x} g(x) - a^2 \mathrm{e}^{2\lambda x} f(x) + a\lambda\, \mathrm{e}^{\lambda x}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{-a^2 f(x)^2 \_Y(x)\,(f(x) - g(x))\, \mathrm{e}^{2\lambda x} + \_Y''(x)\, f(x) + f(x)\, a(\lambda f(x) \_Y(x) + g(x) \_Y'(x))\, \mathrm{e}^{\lambda x}}{f(x)}\right.\right.$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-a^2 f(x)^2 \_Y(x)\,(f(x) - g(x))\, \mathrm{e}^{2\lambda x} + \_Y''(x)\, f(x) + f(x)\, a(\lambda f(x) \_Y(x) + g(x) \_Y'(x)}{f(x)}\right.\right.$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-a^2 f(x)^2 \_Y(x)(f(x)-g(x))\mathrm{e}^{2\lambda x} + \_Y''(x) f(x) + f(x) a\left(\lambda f(x) \_Y(x) + g(x) \_Y'(x)\right)\mathrm{e}^{\lambda x} - f'(x) \_Y'(x)}{f(x)}\right\}, \{\_\right.}{f(x)\, \mathrm{DESol}\left(\left\{\frac{-a^2 f(x)^2 \_Y(x)(f(x)-g(x))\mathrm{e}^{2\lambda x} + \_Y''(x) f(x) + f(x) a\left(\lambda f(x) \_Y(x) + g(x) \_Y'(x)\right)\mathrm{e}^{\lambda x} - f'(x) \_Y'(x)}{f(x)}\right\}, \{\_\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-a^2 f(x)^2 \_Y(x)(f(x)-g(x))\mathrm{e}^{2\lambda x} + \_Y''(x) f(x) + f(x) a\left(\lambda f(x) \_Y(x) + g(x) \_Y'(x)\right)\mathrm{e}^{\lambda x} - f'(x) \_Y'(x)}{f(x)}\right\}, \{\_\right.}{f(x)\, \mathrm{DESol}\left(\left\{\frac{-a^2 f(x)^2 \_Y(x)(f(x)-g(x))\mathrm{e}^{2\lambda x} + \_Y''(x) f(x) + f(x) a\left(\lambda f(x) \_Y(x) + g(x) \_Y'(x)\right)\mathrm{e}^{\lambda x} - f'(x) \_Y'(x)}{f(x)}\right\}, \{\_\right.}$$

Summary
The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-a^2 f(x)^2 \_\_Y(x)(f(x)-g(x))\mathrm{e}^{2\lambda x}+\_\_Y''(x) f(x)+f(x) a\left(\lambda f(x) \_\_Y(x)+g(x) \_\_Y'(x)\right)\mathrm{e}^{\lambda x}-f'(x) \_\_Y'(x)}{f(x)}\right\}, \{\_}{f(x)\,\mathrm{DESol}\left(\left\{\frac{-a^2 f(x)^2 \_\_Y(x)(f(x)-g(x))\mathrm{e}^{2\lambda x}+\_\_Y''(x) f(x)+f(x) a\left(\lambda f(x) \_\_Y(x)+g(x) \_\_Y'(x)\right)\mathrm{e}^{\lambda x}-f'(x) \_\_Y'(x)}{f(x)}\right\}, \{\_} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-a^2 f(x)^2 \_\_Y(x)(f(x)-g(x))\mathrm{e}^{2\lambda x}+\_\_Y''(x) f(x)+f(x) a\left(\lambda f(x) \_\_Y(x)+g(x) \_\_Y'(x)\right)\mathrm{e}^{\lambda x}-f'(x) \_\_Y'(x)}{f(x)}\right\}, \{\_}{f(x)\,\mathrm{DESol}\left(\left\{\frac{-a^2 f(x)^2 \_\_Y(x)(f(x)-g(x))\mathrm{e}^{2\lambda x}+\_\_Y''(x) f(x)+f(x) a\left(\lambda f(x) \_\_Y(x)+g(x) \_\_Y'(x)\right)\mathrm{e}^{\lambda x}-f'(x) \_\_Y'(x)}{f(x)}\right\}, \{\_}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-exp(lambda*x)*f(x)*g(x)*a+di
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-a*exp(lambda*x)*g(x)*y(x)+a*lambda*exp(lambda*x)+a^2*exp(2*l
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2-a*Exp[\[Lambda]*x]*g[x]*y[x]+a*\[Lambda]*Exp[\[Lambda]*x]+a^2*Exp[
```

Not solved

19.20 problem 20

Internal problem ID [10613]
Internal file name [`OUTPUT/9560_Monday_June_06_2022_03_09_05_PM_90585740/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) = 2a\lambda x\,\mathrm{e}^{\lambda x^2} - a^2 f(x)\,\mathrm{e}^{2\lambda x^2}}$$

19.20.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)\,y^2 + 2a\lambda x\,\mathrm{e}^{\lambda x^2} - a^2 f(x)\,\mathrm{e}^{2\lambda x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\,y^2 + 2a\lambda x\,\mathrm{e}^{\lambda x^2} - a^2 f(x)\,\mathrm{e}^{2\lambda x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 2a\lambda x\,\mathrm{e}^{\lambda x^2} - a^2 f(x)\,\mathrm{e}^{2\lambda x^2}$, $f_1(x) = 0$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= f(x)^2 \left(2a\lambda x\, \mathrm{e}^{\lambda x^2} - a^2 f(x)\, \mathrm{e}^{2\lambda x^2}\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\, u''(x) - f'(x)\, u'(x) + f(x)^2 \left(2a\lambda x\, \mathrm{e}^{\lambda x^2} - a^2 f(x)\, \mathrm{e}^{2\lambda x^2}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{-f(x)^3\, \mathrm{e}^{2\lambda x^2} \_Y(x)\, a^2 + 2f(x)^2\, \mathrm{e}^{\lambda x^2} \_Y(x)\, a\lambda x - f'(x) \_Y'(x) + \_Y''(x)\, f(x)}{f(x)}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-f(x)^3\, \mathrm{e}^{2\lambda x^2} \_Y(x)\, a^2 + 2f(x)^2\, \mathrm{e}^{\lambda x^2} \_Y(x)\, a\lambda x - f'(x) \_Y'(x) + \_Y''(x)\, f(x)}{f(x)}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-f(x)^3 \mathrm{e}^{2\lambda x^2} \_Y(x) a^2 + 2f(x)^2 \mathrm{e}^{\lambda x^2} \_Y(x) a\lambda x - f'(x) \_Y'(x) + \_Y''(x) f(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x)\, \mathrm{DESol}\left(\left\{\frac{-f(x)^3 \mathrm{e}^{2\lambda x^2} \_Y(x) a^2 + 2f(x)^2 \mathrm{e}^{\lambda x^2} \_Y(x) a\lambda x - f'(x) \_Y'(x) + \_Y''(x) f(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-f(x)^3 \mathrm{e}^{2\lambda x^2} \_Y(x) a^2 + 2f(x)^2 \mathrm{e}^{\lambda x^2} \_Y(x) a\lambda x - f'(x) \_Y'(x) + \_Y''(x) f(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x)\, \mathrm{DESol}\left(\left\{\frac{-f(x)^3 \mathrm{e}^{2\lambda x^2} \_Y(x) a^2 + 2f(x)^2 \mathrm{e}^{\lambda x^2} \_Y(x) a\lambda x - f'(x) \_Y'(x) + \_Y''(x) f(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-f(x)^3 \mathrm{e}^{2\lambda\,x^2}\_Y(x)a^2+2f(x)^2\mathrm{e}^{\lambda\,x^2}\_Y(x)a\lambda x-f'(x)\_Y'(x)+\_Y''(x)f(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x)\,\mathrm{DESol}\left(\left\{\frac{-f(x)^3 \mathrm{e}^{2\lambda\,x^2}\_Y(x)a^2+2f(x)^2\mathrm{e}^{\lambda\,x^2}\_Y(x)a\lambda x-f'(x)\_Y'(x)+\_Y''(x)f(x)}{f(x)}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{-f(x)^3 \mathrm{e}^{2\lambda\,x^2}\_Y(x)a^2+2f(x)^2\mathrm{e}^{\lambda\,x^2}\_Y(x)a\lambda x-f'(x)\_Y'(x)+\_Y''(x)f(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x)\,\mathrm{DESol}\left(\left\{\frac{-f(x)^3 \mathrm{e}^{2\lambda\,x^2}\_Y(x)a^2+2f(x)^2\mathrm{e}^{\lambda\,x^2}\_Y(x)a\lambda x-f'(x)\_Y'(x)+\_Y''(x)f(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*((diff(f(x), x))*exp(2*x^2*lambda)*a
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2+2*a*lambda*x*exp(lambda*x^2)-a^2*f(x)*exp(2*lambda*x^2),y(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2+2*a*\[Lambda]*x*Exp[\[Lambda]*x^2]-a^2*f[x]*Exp[2*\[Lambda]*x^2],y
```

Not solved

19.21 problem 21

Internal problem ID [10614]
Internal file name [`OUTPUT/9561_Monday_June_06_2022_03_09_08_PM_29123275/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) - y\lambda x = a\,\mathrm{e}^{\lambda x} f(x)}$$

19.21.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)\,y^2 + \lambda x y + a\,\mathrm{e}^{\lambda x} f(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\,y^2 + \lambda x y + a\,\mathrm{e}^{\lambda x} f(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\,\mathrm{e}^{\lambda x} f(x)$, $f_1(x) = \lambda x$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1 f_2 &= f(x) \lambda x \\ f_2^2 f_0 &= f(x)^3 e^{\lambda x} a \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x) u''(x) - (f(x) \lambda x + f'(x)) u'(x) + f(x)^3 e^{\lambda x} a u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{\_Y''(x) f(x) + f(x)^3 e^{\lambda x} a \_Y(x) - (f(x) \lambda x + f'(x)) \_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{\_Y''(x) f(x) + f(x)^3 e^{\lambda x} a \_Y(x) - (f(x) \lambda x + f'(x)) \_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{\_Y''(x) f(x) + f(x)^3 e^{\lambda x} a \_Y(x) - (f(x) \lambda x + f'(x)) \_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x) \mathrm{DESol}\left(\left\{\frac{\_Y''(x) f(x) + f(x)^3 e^{\lambda x} a \_Y(x) - (f(x) \lambda x + f'(x)) \_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{\_Y''(x) f(x) + f(x)^3 e^{\lambda x} a \_Y(x) - (f(x) \lambda x + f'(x)) \_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x) \mathrm{DESol}\left(\left\{\frac{\_Y''(x) f(x) + f(x)^3 e^{\lambda x} a \_Y(x) - (f(x) \lambda x + f'(x)) \_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\_Y''(x) f(x)+f(x)^{3} \mathrm{e}^{\lambda x} a \_Y(x)-(f(x) \lambda x+f'(x)) \_Y'(x)}{f(x)}\right\}, \left\{\_Y(x)\right\}\right)}{f(x) \operatorname{DESol}\left(\left\{\frac{\_Y''(x) f(x)+f(x)^{3} \mathrm{e}^{\lambda x} a \_Y(x)-(f(x) \lambda x+f'(x)) \_Y'(x)}{f(x)}\right\}, \left\{\_Y(x)\right\}\right)} \quad (1)$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\_Y''(x) f(x)+f(x)^{3} \mathrm{e}^{\lambda x} a \_Y(x)-(f(x) \lambda x+f'(x)) \_Y'(x)}{f(x)}\right\}, \left\{\_Y(x)\right\}\right)}{f(x) \operatorname{DESol}\left(\left\{\frac{\_Y''(x) f(x)+f(x)^{3} \mathrm{e}^{\lambda x} a \_Y(x)-(f(x) \lambda x+f'(x)) \_Y'(x)}{f(x)}\right\}, \left\{\_Y(x)\right\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (f(x)*lambda*x+diff(f(x), x))*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2+lambda*x*y(x)+a*f(x)*exp(lambda*x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2+\[Lambda]*x*y[x]+a*f[x]*Exp[\[Lambda]*x],y[x],x,IncludeSingularSol
```

Not solved

19.22 problem 22

Internal problem ID [10615]
Internal file name [`OUTPUT/9562_Monday_June_06_2022_03_09_10_PM_16193780/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$\boxed{y' - y^2 f(x) = -a \tanh(\lambda x)^2 (f(x) a + \lambda) + \lambda a}$$

19.22.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\tanh(\lambda x)^2 f(x) a^2 - a \tanh(\lambda x)^2 \lambda + f(x) y^2 + \lambda a \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\tanh(\lambda x)^2 f(x) a^2 - a \tanh(\lambda x)^2 \lambda + f(x) y^2 + \lambda a$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\tanh(\lambda x)^2 f(x)\, a^2 - a \tanh(\lambda x)^2 \lambda + \lambda a$, $f_1(x) = 0$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= f(x)^2 \left(-\tanh(\lambda x)^2 f(x)\, a^2 - a \tanh(\lambda x)^2 \lambda + \lambda a\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\, u''(x) - f'(x)\, u'(x) + f(x)^2 \left(-\tanh(\lambda x)^2 f(x)\, a^2 - a \tanh(\lambda x)^2 \lambda + \lambda a\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*tanh(lambda*x)*(-2*f(x)*tanh(lambda*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-a*tanh(lambda*x)^2*(a*f(x)+lambda)+a*lambda,y(x), singsol=al
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2-a*Tanh[\[Lambda]*x]^2*(a*f[x]+\[Lambda])+a*\[Lambda],y[x],x,Includ
```

Not solved

19.23 problem 23

Internal problem ID [10616]
Internal file name [`OUTPUT/9563_Monday_June_06_2022_03_09_18_PM_28124635/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$\boxed{y' - y^2 f(x) = -a\coth(\lambda x)^2 (f(x)\, a + \lambda) + \lambda a}$$

19.23.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\coth(\lambda x)^2 f(x)\, a^2 - a\coth(\lambda x)^2 \lambda + f(x)\, y^2 + \lambda a \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\coth(\lambda x)^2 f(x)\, a^2 - a\coth(\lambda x)^2 \lambda + f(x)\, y^2 + \lambda a$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\coth(\lambda x)^2 f(x) a^2 - a\coth(\lambda x)^2 \lambda + \lambda a$, $f_1(x) = 0$ and $f_2(x) = f(x)$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{f(x) u}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= f'(x) \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= f(x)^2 \left(-\coth(\lambda x)^2 f(x) a^2 - a\coth(\lambda x)^2 \lambda + \lambda a\right)
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
f(x) u''(x) - f'(x) u'(x) + f(x)^2 \left(-\coth(\lambda x)^2 f(x) a^2 - a\coth(\lambda x)^2 \lambda + \lambda a\right) u(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+coth(lambda*x)*y(x)*(-2*f(x)*coth(lambda*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-a*coth(lambda*x)^2*(a*f(x)+lambda)+a*lambda,y(x), singsol=al
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2-a*Coth[\[Lambda]*x]^2*(a*f[x]+\[Lambda])+a*\[Lambda],y[x],x,Includ
```

Not solved

19.24 problem 24

Internal problem ID [10617]
Internal file name [`OUTPUT/9564_Monday_June_06_2022_03_09_30_PM_15801037/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) = -f(x)\, a^2 + a\lambda \sinh(\lambda x) - \sinh(\lambda x)^2 f(x)\, a^2}$$

19.24.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)\, y^2 - f(x)\, a^2 + a\lambda \sinh(\lambda x) - \sinh(\lambda x)^2 f(x)\, a^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\, y^2 - f(x)\, a^2 + a\lambda \sinh(\lambda x) - \sinh(\lambda x)^2 f(x)\, a^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -f(x)\, a^2 + a\lambda \sinh(\lambda x) - \sinh(\lambda x)^2 f(x)\, a^2$, $f_1(x) = 0$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= f'(x) \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= f(x)^2 \left(-f(x) a^2 + a\lambda \sinh(\lambda x) - \sinh(\lambda x)^2 f(x) a^2\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x) u''(x) - f'(x) u'(x) + f(x)^2 \left(-f(x) a^2 + a\lambda \sinh(\lambda x) - \sinh(\lambda x)^2 f(x) a^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned}
u(x) = \mathrm{DESol}\Bigg(\Bigg\{&_Y''(x) - \frac{f'(x)_Y'(x)}{f(x)} \\
&+ \left(-f(x) a^2 + a\lambda \sinh(\lambda x) - \sinh(\lambda x)^2 f(x) a^2\right) f(x)_Y(x)\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

The above shows that

$$\begin{aligned}
u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\Bigg(\Bigg\{&_Y''(x) - \frac{f'(x)_Y'(x)}{f(x)} \\
&+ \left(-f(x) a^2 + a\lambda \sinh(\lambda x) - \sinh(\lambda x)^2 f(x) a^2\right) f(x)_Y(x)\Bigg\}, \{_Y(x)\}\Bigg)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} + \left(-f(x) a^2 + a\lambda \sinh(\lambda x) - \sinh(\lambda x)^2 f(x) a^2\right) f(x)\_Y(x)\right\}, \right.}{f(x)\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} + \left(-f(x) a^2 + a\lambda \sinh(\lambda x) - \sinh(\lambda x)^2 f(x) a^2\right) f(x)\_Y(x) \right.\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-f(x)^3 \cosh(\lambda x)^2 \_Y(x) a^2 + f(x)^2 \sinh(\lambda x) \_Y(x) a\lambda + \_Y''(x) f(x) - f'(x)\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{-f(x)^3 \cosh(\lambda x)^2 \_Y(x) a^2 + f(x)^2 \sinh(\lambda x) \_Y(x) a\lambda + \_Y''(x) f(x) - f'(x)\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right) f(x)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-f(x)^3 \cosh(\lambda x)^2 \_\_Y(x) a^2 + f(x)^2 \sinh(\lambda x) \_\_Y(x) a \lambda + \_\_Y''(x) f(x) - f'(x) \_\_Y'(x)}{f(x)}\right\}, \{\_\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{-f(x)^3 \cosh(\lambda x)^2 \_\_Y(x) a^2 + f(x)^2 \sinh(\lambda x) \_\_Y(x) a \lambda + \_\_Y''(x) f(x) - f'(x) \_\_Y'(x)}{f(x)}\right\}, \{\_\_Y(x)\}\right) f(x)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{-f(x)^3 \cosh(\lambda x)^2 \_\_Y(x) a^2 + f(x)^2 \sinh(\lambda x) \_\_Y(x) a \lambda + \_\_Y''(x) f(x) - f'(x) \_\_Y'(x)}{f(x)}\right\}, \{\_\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{-f(x)^3 \cosh(\lambda x)^2 \_\_Y(x) a^2 + f(x)^2 \sinh(\lambda x) \_\_Y(x) a \lambda + \_\_Y''(x) f(x) - f'(x) \_\_Y'(x)}{f(x)}\right\}, \{\_\_Y(x)\}\right) f(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = -4*y(x)*x/(a^2-2*x^2), y(x)`      *** S
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(2*sinh(2*lambda*x)*f(x)*a*lambda+co
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+4*y(x)*x/(a^2-2*x^2), y(x)`      *** Subl
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-a^2*f(x)+a*lambda*sinh(lambda*x)-a^2*f(x)*sinh(lambda*x)^2,y
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2-a^2*f[x]+a*\[Lambda]*Sinh[\[Lambda]*x]-a^2*f[x]*Sinh[\[Lambda]*x]^
```

Not solved

19.25 problem 25

Internal problem ID [10618]
Internal file name [`OUTPUT/9565_Monday_June_06_2022_03_09_39_PM_54305745/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y'x - y^2 f(x) = a - a^2 f(x) \ln(x)^2}$$

19.25.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{a^2 f(x) \ln(x)^2 - f(x)\, y^2 - a}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{a^2 f(x) \ln(x)^2}{x} + \frac{f(x)\, y^2}{x} + \frac{a}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{-a+a^2 f(x)\ln(x)^2}{x}$, $f_1(x) = 0$ and $f_2(x) = \frac{f(x)}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{f(x)u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{f'(x)}{x} - \frac{f(x)}{x^2} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= -\frac{f(x)^2 \left(-a + a^2 f(x) \ln(x)^2\right)}{x^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{f(x)\, u''(x)}{x} - \left(\frac{f'(x)}{x} - \frac{f(x)}{x^2}\right) u'(x) - \frac{f(x)^2 \left(-a + a^2 f(x) \ln(x)^2\right) u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(\frac{f'(x)}{x} - \frac{f(x)}{x^2}\right) x\_Y'(x)}{f(x)} - \frac{f(x)\left(-a + a^2 f(x) \ln(x)^2\right)\_Y(x)}{x^2}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(\frac{f'(x)}{x} - \frac{f(x)}{x^2}\right) x\_Y'(x)}{f(x)} - \frac{f(x)\left(-a + a^2 f(x) \ln(x)^2\right)\_Y(x)}{x^2}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(\frac{f'(x)}{x} - \frac{f(x)}{x^2}\right) x\_Y'(x)}{f(x)} - \frac{f(x)\left(-a+a^2 f(x) \ln(x)^2\right)\_Y(x)}{x^2}\right\}, \{\_Y(x)\}\right)\right) x}{f(x)\, \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(\frac{f'(x)}{x} - \frac{f(x)}{x^2}\right) x\_Y'(x)}{f(x)} - \frac{f(x)\left(-a+a^2 f(x) \ln(x)^2\right)\_Y(x)}{x^2}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{-\frac{f(x)^3 \ln(x)^2 \_Y(x) a^2 - f(x)^2 a \_Y(x) - \_Y''(x) x^2 f(x) + f'(x) \_Y'(x) x^2 - f(x) \_Y'(x) x}{x^2 f(x)}\right\}, \{\_Y(x)\}\right)\right) x}{f(x) \mathrm{DESol}\left(\left\{-\frac{f(x)^3 \ln(x)^2 \_Y(x) a^2 - f(x)^2 a \_Y(x) - \_Y''(x) x^2 f(x) + f'(x) \_Y'(x) x^2 - f(x) \_Y'(x) x}{x^2 f(x)}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{-\frac{f(x)^3 \ln(x)^2 \_Y(x) a^2 - f(x)^2 a \_Y(x) - \_Y''(x) x^2 f(x) + f'(x) \_Y'(x) x^2 - f(x) \_Y'(x) x}{x^2 f(x)}\right\}, \{\_Y(x)\}\right)\right) x}{f(x) \mathrm{DESol}\left(\left\{-\frac{f(x)^3 \ln(x)^2 \_Y(x) a^2 - f(x)^2 a \_Y(x) - \_Y''(x) x^2 f(x) + f'(x) \_Y'(x) x^2 - f(x) \_Y'(x) x}{x^2 f(x)}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{-\frac{f(x)^3 \ln(x)^2 \_Y(x) a^2 - f(x)^2 a \_Y(x) - \_Y''(x) x^2 f(x) + f'(x) \_Y'(x) x^2 - f(x) \_Y'(x) x}{x^2 f(x)}\right\}, \{\_Y(x)\}\right)\right) x}{f(x) \mathrm{DESol}\left(\left\{-\frac{f(x)^3 \ln(x)^2 \_Y(x) a^2 - f(x)^2 a \_Y(x) - \_Y''(x) x^2 f(x) + f'(x) \_Y'(x) x^2 - f(x) \_Y'(x) x}{x^2 f(x)}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x)=f(x)*y(x)^2+a-a^2*f(x)*(ln(x))^2,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y'[x]==f[x]*y[x]^2+a-a^2*f[x]*(Log[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

19.26 problem 26

Internal problem ID [10619]
Internal file name [`OUTPUT/9566_Monday_June_06_2022_03_09_41_PM_20497896/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$\boxed{y'x - f(x)\left(y + a\ln\left(x\right)\right)^2 = -a}$$

19.26.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= \frac{a^2 f(x)\ln\left(x\right)^2 + 2\ln\left(x\right)f(x)\,ay + f(x)\,y^2 - a}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a^2 f(x)\ln\left(x\right)^2}{x} + \frac{2\ln\left(x\right)f(x)\,ay}{x} + \frac{f(x)\,y^2}{x} - \frac{a}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{-a+a^2 f(x)\ln(x)^2}{x}$, $f_1(x) = \frac{2af(x)\ln(x)}{x}$ and $f_2(x) = \frac{f(x)}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{\frac{f(x)u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{f'(x)}{x} - \frac{f(x)}{x^2} \\ f_1 f_2 &= \frac{2a f(x)^2 \ln(x)}{x^2} \\ f_2^2 f_0 &= \frac{f(x)^2 \left(-a + a^2 f(x) \ln(x)^2\right)}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{f(x)\, u''(x)}{x} - \left(\frac{f'(x)}{x} - \frac{f(x)}{x^2} + \frac{2a f(x)^2 \ln(x)}{x^2}\right) u'(x) + \frac{f(x)^2 \left(-a + a^2 f(x) \ln(x)^2\right) u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(\frac{f'(x)}{x} - \frac{f(x)}{x^2} + \frac{2a f(x)^2 \ln(x)}{x^2}\right) x\_Y'(x)}{f(x)} + \frac{f(x)\left(-a + a^2 f(x) \ln(x)^2\right)\_Y(x)}{x^2}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(\frac{f'(x)}{x} - \frac{f(x)}{x^2} + \frac{2a f(x)^2 \ln(x)}{x^2}\right) x\_Y'(x)}{f(x)} + \frac{f(x)\left(-a + a^2 f(x) \ln(x)^2\right)\_Y(x)}{x^2}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(\frac{f'(x)}{x} - \frac{f(x)}{x^2} + \frac{2a f(x)^2 \ln(x)}{x^2}\right) x\_Y'(x)}{f(x)} + \frac{f(x)\left(-a + a^2 f(x) \ln(x)^2\right)\_Y(x)}{x^2}\right\}, \{\_Y(x)\}\right)\right) x}{f(x)\, \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(\frac{f'(x)}{x} - \frac{f(x)}{x^2} + \frac{2a f(x)^2 \ln(x)}{x^2}\right) x\_Y'(x)}{f(x)} + \frac{f(x)\left(-a + a^2 f(x) \ln(x)^2\right)\_Y(x)}{x^2}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{\_Y''(x) x^2 f(x) - 2\left(f(x)^2 \ln(x) a + \frac{f'(x) x}{2} - \frac{f(x)}{2}\right) x \_Y'(x) + f(x)^2 a \left(f(x) \ln(x)^2 a - 1\right) \_Y(x)}{x^2 f(x)}\right\}, \{\_Y(x)\}\right)\right) x}{f(x) \mathrm{DESol}\left(\left\{\frac{\_Y''(x) x^2 f(x) - 2\left(f(x)^2 \ln(x) a + \frac{f'(x) x}{2} - \frac{f(x)}{2}\right) x \_Y'(x) + f(x)^2 a \left(f(x) \ln(x)^2 a - 1\right) \_Y(x)}{x^2 f(x)}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{\_Y''(x) x^2 f(x) - 2\left(f(x)^2 \ln(x) a + \frac{f'(x) x}{2} - \frac{f(x)}{2}\right) x \_Y'(x) + f(x)^2 a \left(f(x) \ln(x)^2 a - 1\right) \_Y(x)}{x^2 f(x)}\right\}, \{\_Y(x)\}\right)\right) x}{f(x) \mathrm{DESol}\left(\left\{\frac{\_Y''(x) x^2 f(x) - 2\left(f(x)^2 \ln(x) a + \frac{f'(x) x}{2} - \frac{f(x)}{2}\right) x \_Y'(x) + f(x)^2 a \left(f(x) \ln(x)^2 a - 1\right) \_Y(x)}{x^2 f(x)}\right\}, \{\_Y(x)\}\right)} \quad (1)$$

Verification of solutions

$$y = -\frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{\_Y''(x) x^2 f(x) - 2\left(f(x)^2 \ln(x) a + \frac{f'(x) x}{2} - \frac{f(x)}{2}\right) x \_Y'(x) + f(x)^2 a \left(f(x) \ln(x)^2 a - 1\right) \_Y(x)}{x^2 f(x)}\right\}, \{\_Y(x)\}\right)\right) x}{f(x) \mathrm{DESol}\left(\left\{\frac{\_Y''(x) x^2 f(x) - 2\left(f(x)^2 \ln(x) a + \frac{f'(x) x}{2} - \frac{f(x)}{2}\right) x \_Y'(x) + f(x)^2 a \left(f(x) \ln(x)^2 a - 1\right) \_Y(x)}{x^2 f(x)}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (d) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(x*diff(y(x),x)=f(x)*(y(x)+a*ln(x))^2-a,y(x), singsol=all)
```

$$y(x) = -a\ln(x) + \frac{1}{c_1 - \left(\int \frac{f(x)}{x}dx\right)}$$

✓ Solution by Mathematica

Time used: 0.48 (sec). Leaf size: 42

```
DSolve[x*y'[x]==f[x]*(y[x]+a*Log[x])^2-a,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -a\log(x) + \frac{1}{-\int_1^x \frac{f(K[2])}{K[2]}dK[2] + c_1}$$
$$y(x) \to -a\log(x)$$

19.27 problem 27

Internal problem ID [10620]
Internal file name [`OUTPUT/9567_Monday_June_06_2022_03_09_43_PM_63722438/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) + ax \ln(x) f(x) y = a \ln(x) + a}$$

19.27.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x) y^2 - ax \ln(x) f(x) y + a \ln(x) + a \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x) y^2 - ax \ln(x) f(x) y + a \ln(x) + a$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a \ln(x) + a$, $f_1(x) = -f(x) \ln(x) ax$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$f_2' = f'(x)$$
$$f_1f_2 = -xf(x)^2 \ln(x)\,a$$
$$f_2^2f_0 = f(x)^2\,(a\ln(x) + a)$$

Substituting the above terms back in equation (2) gives

$$f(x)\,u''(x) - \left(-xf(x)^2 \ln(x)\,a + f'(x)\right) u'(x) + f(x)^2\,(a\ln(x) + a)\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \text{DESol}\left(\left\{\_Y''(x) - \frac{\left(-xf(x)^2 \ln(x)\,a + f'(x)\right)\_Y'(x)}{f(x)} + a(1+\ln(x))\,f(x)\_Y(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\text{DESol}\left(\left\{\_Y''(x) - \frac{\left(-xf(x)^2 \ln(x)\,a + f'(x)\right)\_Y'(x)}{f(x)} + a(1+\ln(x))\,f(x)\_Y(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\text{DESol}\left(\left\{\_Y''(x) - \frac{\left(-xf(x)^2 \ln(x)a + f'(x)\right)\_Y'(x)}{f(x)} + a(1+\ln(x))\,f(x)\_Y(x)\right\}, \{\_Y(x)\}\right)}{f(x)\,\text{DESol}\left(\left\{\_Y''(x) - \frac{\left(-xf(x)^2 \ln(x)a + f'(x)\right)\_Y'(x)}{f(x)} + a(1+\ln(x))\,f(x)\_Y(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\_Y''(x) f(x)+\left(x f(x)^{2} \ln(x) a-f'(x)\right)\_Y'(x)+a(1+\ln(x)) f(x)^{2}\_Y(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x) \operatorname{DESol}\left(\left\{\frac{\_Y''(x) f(x)+\left(x f(x)^{2} \ln(x) a-f'(x)\right)\_Y'(x)+a(1+\ln(x)) f(x)^{2}\_Y(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\_Y''(x) f(x)+\left(x f(x)^{2} \ln(x) a-f'(x)\right)\_Y'(x)+a(1+\ln(x)) f(x)^{2}\_Y(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x) \operatorname{DESol}\left(\left\{\frac{\_Y''(x) f(x)+\left(x f(x)^{2} \ln(x) a-f'(x)\right)\_Y'(x)+a(1+\ln(x)) f(x)^{2}\_Y(x)}{f(x)}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{\_Y''(x) f(x)+\left(x f(x)^{2} \ln(x) a-f'(x)\right)\_Y'(x)+a(1+\ln(x)) f(x)^{2}\_Y(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x) \operatorname{DESol}\left(\left\{\frac{\_Y''(x) f(x)+\left(x f(x)^{2} \ln(x) a-f'(x)\right)\_Y'(x)+a(1+\ln(x)) f(x)^{2}\_Y(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-f(x)^2*ln(x)*a*x+diff(f(x),
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-a*x*ln(x)*f(x)*y(x)+a*ln(x)+a,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2-a*x*Log[x]*f[x]*y[x]+a*Log[x]+a,y[x],x,IncludeSingularSolutions ->
```

Not solved

19.28 problem 28

Internal problem ID [10621]
Internal file name [`OUTPUT/9568_Monday_June_06_2022_03_09_45_PM_32821835/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + a\ln(x)\,y^2 - af(x)\,(\ln(x)\,x - x)\,y = -f(x)}$$

19.28.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= ax\ln(x)\,f(x)\,y - f(x)\,axy - a\ln(x)\,y^2 - f(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = ax\ln(x)\,f(x)\,y - f(x)\,axy - a\ln(x)\,y^2 - f(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -f(x)$, $f_1(x) = f(x)\ln(x)\,ax - axf(x)$ and $f_2(x) = -a\ln(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-a\ln(x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{a}{x} \\ f_1 f_2 &= -(f(x)\ln(x) a x - a x f(x)) a \ln(x) \\ f_2^2 f_0 &= -a^2 f(x) \ln(x)^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-a\ln(x) u''(x) - \left(-(f(x)\ln(x) a x - a x f(x)) a \ln(x) - \frac{a}{x}\right) u'(x) - a^2 f(x) \ln(x)^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x(\ln(x) - 1)\left(c_2\left(\int \frac{e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x) a x^2 + 1}{x\ln(x)}dx}}{x^2(\ln(x)-1)^2}dx\right) + c_1\right)$$

The above shows that

$$u'(x) = \frac{x c_2 \ln(x)(\ln(x)-1)\left(\int \frac{e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x) a x^2 + 1}{x\ln(x)}dx}}{x^2(\ln(x)-1)^2}dx\right) + c_2 e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x) a x^2 + 1}{x\ln(x)}dx} + c_1 x \ln(x)(\ln}{x(\ln(x)-1)}$$

Using the above in (1) gives the solution

$$y = \frac{x c_2 \ln(x)(\ln(x)-1)\left(\int \frac{e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x) a x^2 + 1}{x\ln(x)}dx}}{x^2(\ln(x)-1)^2}dx\right) + c_2 e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x) a x^2 + 1}{x\ln(x)}dx} + c_1 x \ln(x)(\ln}{x^2(\ln(x)-1)^2 a\ln(x)\left(c_2\left(\int \frac{e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x) a x^2 + 1}{x\ln(x)}dx}}{x^2(\ln(x)-1)^2}dx\right) + c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x \ln(x)(\ln(x)-1)\left(\int \frac{e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x) a x^2 + 1}{x\ln(x)}dx}}{x^2(\ln(x)-1)^2}dx\right) + c_3 x \ln(x)^2 - \ln(x) c_3 x + e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x)a}{x\ln(x)}}}{x^2(\ln(x)-1)^2 a \ln(x)\left(\int \frac{e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x) a x^2 + 1}{x\ln(x)}dx}}{x^2(\ln(x)-1)^2}dx + c_3\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{x \ln(x)(\ln(x)-1)\left(\int \frac{e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x) a x^2 + 1}{x\ln(x)}dx}}{x^2(\ln(x)-1)^2}dx\right) + c_3 x \ln(x)^2 - \ln(x) c_3 x + e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x)a}{x\ln(x)}}}{x^2(\ln(x)-1)^2 a \ln(x)\left(\int \frac{e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x) a x^2 + 1}{x\ln(x)}dx}}{x^2(\ln(x)-1)^2}dx + c_3\right)} \tag{1}$$

Verification of solutions

$$y = \frac{x \ln(x)(\ln(x)-1)\left(\int \frac{e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x) a x^2 + 1}{x\ln(x)}dx}}{x^2(\ln(x)-1)^2}dx\right) + c_3 x \ln(x)^2 - \ln(x) c_3 x + e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x)a}{x\ln(x)}}}{x^2(\ln(x)-1)^2 a \ln(x)\left(\int \frac{e^{\int \frac{f(x)\ln(x)^2 a x^2 - f(x)\ln(x) a x^2 + 1}{x\ln(x)}dx}}{x^2(\ln(x)-1)^2}dx + c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (ln(x)^2*f(x)*a*x^2-ln(x)*f(x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 227

```
dsolve(diff(y(x),x)=-a*ln(x)*y(x)^2+a*f(x)*(x*ln(x)-x)*y(x)-f(x),y(x), singsol=all)
```

$$y(x) = \frac{-x(\ln(x)-1)\,\mathrm{e}^{\int \frac{f(x)\ln(x)^2 a x^2+\left(-2x^2 a f(x)-2\right)\ln(x)+x^2 a f(x)}{x(\ln(x)-1)}dx} + c_1 a - \left(\int \ln(x)\,\mathrm{e}^{a\left(\int \frac{x f(x)\ln(x)^2}{\ln(x)-1}dx\right)-2a\left(\int \frac{x f(x)\ln(x)}{\ln(x)-1}dx\right)+\cdots}\right.}{a x(\ln(x)-1)\left(c_1 a - \left(\int \ln(x)\,\mathrm{e}^{a\left(\int \frac{x f(x)\ln(x)^2}{\ln(x)-1}dx\right)-2a\left(\int \frac{x f(x)\ln(x)}{\ln(x)-1}dx\right)+a\left(\int \frac{x f(x)}{\ln(x)-1}dx\right)-2\left(\int \frac{\ln\cdots}{x(\ln\cdots}\right.}\right.\right.}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==-a*Log[x]*y[x]^2+a*f[x]*(x*Log[x]-x)*y[x]-f[x],y[x],x,IncludeSingularSolutions
```

Not solved

19.29 problem 29

Internal problem ID [10622]
Internal file name [`OUTPUT/9569_Monday_June_06_2022_03_09_51_PM_66941966/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \lambda \sin(\lambda x)\, y^2 - f(x) \cos(\lambda x)\, y = -f(x)}$$

19.29.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \lambda \sin(\lambda x)\, y^2 + f(x) \cos(\lambda x)\, y - f(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \lambda \sin(\lambda x)\, y^2 + f(x) \cos(\lambda x)\, y - f(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -f(x)$, $f_1(x) = \cos(\lambda x)\, f(x)$ and $f_2(x) = \lambda \sin(\lambda x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\lambda \sin(\lambda x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \lambda^2 \cos(\lambda x) \\
f_1 f_2 &= \cos(\lambda x) f(x) \lambda \sin(\lambda x) \\
f_2^2 f_0 &= -\lambda^2 \sin(\lambda x)^2 f(x)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\lambda \sin(\lambda x) u''(x) - \left(\cos(\lambda x) f(x) \lambda \sin(\lambda x) + \lambda^2 \cos(\lambda x)\right) u'(x) - \lambda^2 \sin(\lambda x)^2 f(x) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = -\cos(\lambda x)\left(c_2 \lambda \left(\int e^{\int (\cos(\lambda x) f(x) + 2\tan(\lambda x)\lambda) dx} \sin(\lambda x)\, dx\right) - c_1\right)$$

The above shows that

$$\begin{aligned}
u'(x) = -\lambda \sin(\lambda x) \Bigg(&\cos(\lambda x)\, c_2 e^{\int (\cos(\lambda x) f(x) + 2\tan(\lambda x)\lambda) dx} \\
&- c_2 \lambda \left(\int e^{\int (\cos(\lambda x) f(x) + 2\tan(\lambda x)\lambda) dx} \sin(\lambda x)\, dx\right) + c_1\Bigg)
\end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\cos(\lambda x)\, c_2 e^{\int (\cos(\lambda x) f(x) + 2\tan(\lambda x)\lambda) dx} - c_2 \lambda \left(\int e^{\int (\cos(\lambda x) f(x) + 2\tan(\lambda x)\lambda) dx} \sin(\lambda x)\, dx\right) + c_1}{\cos(\lambda x)\left(c_2 \lambda \left(\int e^{\int (\cos(\lambda x) f(x) + 2\tan(\lambda x)\lambda) dx} \sin(\lambda x)\, dx\right) - c_1\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\sec(\lambda x) \lambda \left(\int e^{\int (\cos(\lambda x) f(x) + 2\tan(\lambda x)\lambda) dx} \sin(\lambda x)\, dx\right) - \sec(\lambda x) c_3 - e^{\int (\cos(\lambda x) f(x) + 2\tan(\lambda x)\lambda) dx}}{\lambda \left(\int e^{\int (\cos(\lambda x) f(x) + 2\tan(\lambda x)\lambda) dx} \sin(\lambda x)\, dx\right) - c_3}$$

Summary
The solution(s) found are the following

$$y = \frac{\sec(\lambda x)\,\lambda\left(\int e^{\int (\cos(\lambda x) f(x)+2\tan(\lambda x)\lambda)dx}\sin(\lambda x)\,dx\right) - \sec(\lambda x)\,c_3 - e^{\int (\cos(\lambda x) f(x)+2\tan(\lambda x)\lambda)dx}}{\lambda\left(\int e^{\int (\cos(\lambda x) f(x)+2\tan(\lambda x)\lambda)dx}\sin(\lambda x)\,dx\right) - c_3} \tag{1}$$

Verification of solutions

$$y = \frac{\sec(\lambda x)\,\lambda\left(\int e^{\int (\cos(\lambda x) f(x)+2\tan(\lambda x)\lambda)dx}\sin(\lambda x)\,dx\right) - \sec(\lambda x)\,c_3 - e^{\int (\cos(\lambda x) f(x)+2\tan(\lambda x)\lambda)dx}}{\lambda\left(\int e^{\int (\cos(\lambda x) f(x)+2\tan(\lambda x)\lambda)dx}\sin(\lambda x)\,dx\right) - c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = cos(lambda*x)*(sin(lambda*x)*f
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            <- linear symmetries successful
         Change of variables used:
            [x = arccos(t)/lambda]
         Linear ODE actually solved:
            (2*(-t^2+1)^(1/2)*f(arccos(t)/lambda)*t^2-2*(-t^2+1)^(1/2)*f(arccos(t)/lambda))*
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 97

```
dsolve(diff(y(x),x)=lambda*sin(lambda*x)*y(x)^2+f(x)*cos(lambda*x)*y(x)-f(x),y(x), singsol=a
```

$$y(x) = \frac{\sec(x\lambda)\,\lambda\left(\int \mathrm{e}^{\int (f(x)\cos(x\lambda)+2\tan(x\lambda)\lambda)dx}\sin(x\lambda)\,dx\right)c_1 - c_1\mathrm{e}^{\int (f(x)\cos(x\lambda)+2\tan(x\lambda)\lambda)dx} - \sec(x\lambda)}{\lambda\left(\int \mathrm{e}^{\int (f(x)\cos(x\lambda)+2\tan(x\lambda)\lambda)dx}\sin(x\lambda)\,dx\right)c_1 - 1}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==\[Lambda]*Sin[\[Lambda]*x]*y[x]^2+f[x]*Cos[\[Lambda]*x]*y[x]-f[x],y[x],x,Inclu
```

Not solved

19.30 problem 30

Internal problem ID [10623]
Internal file name [`OUTPUT/9570_Monday_June_06_2022_03_09_54_PM_68250168/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 f(x) = -f(x)\, a^2 + a\lambda \sin(\lambda x) + a^2 f(x) \sin(\lambda x)^2}$$

19.30.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)\, y^2 - f(x)\, a^2 + a\lambda \sin(\lambda x) + a^2 f(x) \sin(\lambda x)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\, y^2 - f(x)\, a^2 + a\lambda \sin(\lambda x) + a^2 f(x) \sin(\lambda x)^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -f(x)\, a^2 + a\lambda \sin(\lambda x) + a^2 f(x) \sin(\lambda x)^2$, $f_1(x) = 0$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= f(x)^2 \left(-f(x) a^2 + a\lambda \sin(\lambda x) + a^2 f(x) \sin(\lambda x)^2\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x) u''(x) - f'(x) u'(x) + f(x)^2 \left(-f(x) a^2 + a\lambda \sin(\lambda x) + a^2 f(x) \sin(\lambda x)^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = \mathrm{DESol}\Bigg(\bigg\{ & \_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} \\ & + f(x) \left(-f(x) a^2 + a\lambda \sin(\lambda x) + a^2 f(x) \sin(\lambda x)^2\right) \_Y(x)\bigg\}, \{\_Y(x)\}\Bigg) \end{aligned}$$

The above shows that

$$\begin{aligned} u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\Bigg(\bigg\{ & \_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} \\ & + f(x) \left(-f(x) a^2 + a\lambda \sin(\lambda x) + a^2 f(x) \sin(\lambda x)^2\right) \_Y(x)\bigg\}, \{\_Y(x)\}\Bigg) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} + f(x) \left(-f(x) a^2 + a\lambda \sin(\lambda x) + a^2 f(x) \sin(\lambda x)^2\right) \_Y(x)\right\}, \{\_\right.}{f(x) \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} + f(x) \left(-f(x) a^2 + a\lambda \sin(\lambda x) + a^2 f(x) \sin(\lambda x)^2\right) \_Y(x)\right\}, \right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{f(x)^2 \sin(\lambda x)\_Y(x) a\lambda - f(x)^3 \_Y(x) a^2 \cos(\lambda x)^2 + \_Y''(x) f(x) - f'(x)\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{f(x)^2 \sin(\lambda x)\_Y(x) a\lambda - f(x)^3 \_Y(x) a^2 \cos(\lambda x)^2 + \_Y''(x) f(x) - f'(x)\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right) f(x)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{f(x)^{2}\sin(\lambda x)\_Y(x)a\lambda - f(x)^{3}\_Y(x)a^{2}\cos(\lambda x)^{2} + \_Y''(x)f(x) - f'(x)\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{f(x)^{2}\sin(\lambda x)\_Y(x)a\lambda - f(x)^{3}\_Y(x)a^{2}\cos(\lambda x)^{2} + \_Y''(x)f(x) - f'(x)\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)f(x)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{f(x)^{2}\sin(\lambda x)\_Y(x)a\lambda - f(x)^{3}\_Y(x)a^{2}\cos(\lambda x)^{2} + \_Y''(x)f(x) - f'(x)\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{f(x)^{2}\sin(\lambda x)\_Y(x)a\lambda - f(x)^{3}\_Y(x)a^{2}\cos(\lambda x)^{2} + \_Y''(x)f(x) - f'(x)\_Y'(x)}{f(x)}\right\}, \{\_Y(x)\}\right)f(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(-2*sin(2*lambda*x)*f(x)*a*lambda+co
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-a^2*f(x)+a*lambda*sin(lambda*x)+a^2*f(x)*sin(lambda*x)^2,y(x
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2-a^2*f[x]+a*\[Lambda]*Sin[\[Lambda]*x]+a^2*f[x]*Sin[\[Lambda]*x]^2,
```

Not solved

19.31 problem 31

Internal problem ID [10624]
Internal file name [`OUTPUT/9571_Monday_June_06_2022_03_10_03_PM_24124929/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - f(x)\, y^2 = -a^2 f(x) + a\lambda \cos(\lambda x) + a^2 f(x) \cos(\lambda x)^2}$$

19.31.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)\, y^2 - a^2 f(x) + a\lambda \cos(\lambda x) + a^2 f(x) \cos(\lambda x)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\, y^2 - a^2 f(x) + a\lambda \cos(\lambda x) + a^2 f(x) \cos(\lambda x)^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -a^2 f(x) + a\lambda \cos(\lambda x) + a^2 f(x) \cos(\lambda x)^2$, $f_1(x) = 0$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= f(x)^2 \left(-a^2 f(x) + a\lambda \cos(\lambda x) + a^2 f(x) \cos(\lambda x)^2\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\, u''(x) - f'(x)\, u'(x) + f(x)^2 \left(-a^2 f(x) + a\lambda \cos(\lambda x) + a^2 f(x) \cos(\lambda x)^2\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = \mathrm{DESol}\Bigg(\Bigg\{&\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} \\ &+ f(x)\left(-a^2 f(x) + a\lambda \cos(\lambda x) + a^2 f(x) \cos(\lambda x)^2\right)\_Y(x)\Bigg\}, \{\_Y(x)\}\Bigg) \end{aligned}$$

The above shows that

$$\begin{aligned} u'(x) = \frac{d}{dx} \mathrm{DESol}\Bigg(\Bigg\{&\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} \\ &+ f(x)\left(-a^2 f(x) + a\lambda \cos(\lambda x) + a^2 f(x) \cos(\lambda x)^2\right)\_Y(x)\Bigg\}, \{\_Y(x)\}\Bigg) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{d}{dx} \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} + f(x)\left(-a^2 f(x) + a\lambda \cos(\lambda x) + a^2 f(x) \cos(\lambda x)^2\right)\_Y(x)\right\}, \{\_\right.}{f(x)\, \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{f'(x)\_Y'(x)}{f(x)} + f(x)\left(-a^2 f(x) + a\lambda \cos(\lambda x) + a^2 f(x) \cos(\lambda x)^2\right)\_Y(x)\right\},\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{d}{dx} \mathrm{DESol}\left(\left\{\frac{-f(x)^3 \sin(\lambda x)^2 \_Y(x) a^2 + \cos(\lambda x) f(x)^2 \_Y(x) a\lambda - f'(x)\_Y'(x) + \_Y''(x) f(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}{f(x)\, \mathrm{DESol}\left(\left\{\frac{-f(x)^3 \sin(\lambda x)^2 \_Y(x) a^2 + \cos(\lambda x) f(x)^2 \_Y(x) a\lambda - f'(x)\_Y'(x) + \_Y''(x) f(x)}{f(x)}\right\}, \{\_Y(x)\}\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\frac{d}{dx}\operatorname{DESol}\left(\left\{\frac{-f(x)^3\sin(\lambda x)^2\_\_Y(x)a^2+\cos(\lambda x)f(x)^2\_\_Y(x)a\lambda-f'(x)\_\_Y'(x)+\_\_Y''(x)f(x)}{f(x)}\right\},\{\_\_Y(x)\}\right)}{f(x)\operatorname{DESol}\left(\left\{\frac{-f(x)^3\sin(\lambda x)^2\_\_Y(x)a^2+\cos(\lambda x)f(x)^2\_\_Y(x)a\lambda-f'(x)\_\_Y'(x)+\_\_Y''(x)f(x)}{f(x)}\right\},\{\_\_Y(x)\}\right)} \quad (1)$$

Verification of solutions

$$y = -\frac{\frac{d}{dx}\operatorname{DESol}\left(\left\{\frac{-f(x)^3\sin(\lambda x)^2\_\_Y(x)a^2+\cos(\lambda x)f(x)^2\_\_Y(x)a\lambda-f'(x)\_\_Y'(x)+\_\_Y''(x)f(x)}{f(x)}\right\},\{\_\_Y(x)\}\right)}{f(x)\operatorname{DESol}\left(\left\{\frac{-f(x)^3\sin(\lambda x)^2\_\_Y(x)a^2+\cos(\lambda x)f(x)^2\_\_Y(x)a\lambda-f'(x)\_\_Y'(x)+\_\_Y''(x)f(x)}{f(x)}\right\},\{\_\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(2*sin(2*lambda*x)*f(x)*a*lambda+2*l
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-a^2*f(x)+a*lambda*cos(lambda*x)+a^2*f(x)*cos(lambda*x)^2,y(x
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2-a^2*f[x]+a*\[Lambda]*cos[\[Lambda]*x]+a^2*f[x]*Cos[\[Lambda]*x]^2,
```

Not solved

19.32 problem 32

Internal problem ID [10625]
Internal file name [`OUTPUT/9572_Monday_June_06_2022_03_10_12_PM_82219677/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

Unable to solve or complete the solution.

$$\boxed{y' - f(x)\, y^2 = -a \tan(\lambda x)^2 \,(a f(x) - \lambda) + a\lambda}$$

19.32.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -f(x) \tan(\lambda x)^2 \, a^2 + \tan(\lambda x)^2 \, a\lambda + f(x)\, y^2 + a\lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -f(x) \tan(\lambda x)^2 \, a^2 + \tan(\lambda x)^2 \, a\lambda + f(x)\, y^2 + a\lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -f(x)\tan(\lambda x)^2 a^2 + \tan(\lambda x)^2 a\lambda + a\lambda$, $f_1(x) = 0$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= f(x)^2 \left(-f(x)\tan(\lambda x)^2 a^2 + \tan(\lambda x)^2 a\lambda + a\lambda\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\, u''(x) - f'(x)\, u'(x) + f(x)^2 \left(-f(x)\tan(\lambda x)^2 a^2 + \tan(\lambda x)^2 a\lambda + a\lambda\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+tan(lambda*x)*y(x)*(2*f(x)*tan(lambda*x)^
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-a*tan(lambda*x)^2*(a*f(x)-lambda)+a*lambda,y(x), singsol=all
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2-a*Tan[\[Lambda]*x]^2*(a*f[x]-\[Lambda])+a*\[Lambda],y[x],x,Include
```

Not solved

19.33 problem 33

Internal problem ID [10626]
Internal file name [`OUTPUT/9573_Monday_June_06_2022_03_10_25_PM_94623270/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-1. Equations containing arbitrary functions (but not containing their derivatives).
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$\boxed{y' - f(x)\,y^2 = -a\cot(\lambda x)^2\,(af(x) - \lambda) + a\lambda}$$

19.33.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -f(x)\cot(\lambda x)^2\,a^2 + \cot(\lambda x)^2\,a\lambda + f(x)\,y^2 + a\lambda \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -f(x)\cot(\lambda x)^2\,a^2 + \cot(\lambda x)^2\,a\lambda + f(x)\,y^2 + a\lambda$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -f(x)\cot(\lambda x)^2 a^2 + \cot(\lambda x)^2 a\lambda + a\lambda$, $f_1(x) = 0$ and $f_2(x) = f(x)$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{f(x)\, u}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= f'(x) \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= f(x)^2 \left(-f(x)\cot(\lambda x)^2 a^2 + \cot(\lambda x)^2 a\lambda + a\lambda\right)
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
f(x)\, u''(x) - f'(x)\, u'(x) + f(x)^2 \left(-f(x)\cot(\lambda x)^2 a^2 + \cot(\lambda x)^2 a\lambda + a\lambda\right) u(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-cot(lambda*x)*y(x)*(2*cot(lambda*x)^2*f(x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-a*cot(lambda*x)^2*(a*f(x)-lambda)+a*lambda,y(x), singsol=all
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2-a*Cot[\[Lambda]*x]^2*(a*f[x]-\[Lambda])+a*\[Lambda],y[x],x,Include
```

Not solved

20 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-2. Equations containing arbitrary functions and their derivatives.

20.1 problem 34

Internal problem ID [10627]
Internal file name [OUTPUT/9574_Monday_June_06_2022_03_10_37_PM_73313790/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-2. Equations containing arbitrary functions and their derivatives.
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 = -f(x)^2 + f'(x)}$$

20.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 - f(x)^2 + f'(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - f(x)^2 + f'(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -f(x)^2 + f'(x)$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= 0 \\
f_1f_2 &= 0 \\
f_2^2 f_0 &= -f(x)^2 + f'(x)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(-f(x)^2 + f'(x)\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\int \mathrm{e}^{2\left(\int f(x)dx\right)}dx + c_1\right) \mathrm{e}^{-\left(\int f(x)dx\right)} c_2$$

The above shows that

$$u'(x) = c_2\left(\mathrm{e}^{\int f(x)dx} - f(x)\, c_1\mathrm{e}^{-\left(\int f(x)dx\right)} - f(x)\left(\int \mathrm{e}^{2\left(\int f(x)dx\right)}dx\right) \mathrm{e}^{-\left(\int f(x)dx\right)}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\mathrm{e}^{\int f(x)dx} - f(x)\, c_1\mathrm{e}^{-\left(\int f(x)dx\right)} - f(x)\left(\int \mathrm{e}^{2\left(\int f(x)dx\right)}dx\right) \mathrm{e}^{-\left(\int f(x)dx\right)}\right) \mathrm{e}^{\int f(x)dx}}{\int \mathrm{e}^{2\left(\int f(x)dx\right)}dx + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\mathrm{e}^{2\left(\int f(x)dx\right)} + c_3f(x) + \left(\int \mathrm{e}^{2\left(\int f(x)dx\right)}dx\right) f(x)}{\int \mathrm{e}^{2\left(\int f(x)dx\right)}dx + c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{-\mathrm{e}^{2\left(\int f(x)dx\right)} + c_3f(x) + \left(\int \mathrm{e}^{2\left(\int f(x)dx\right)}dx\right) f(x)}{\int \mathrm{e}^{2\left(\int f(x)dx\right)}dx + c_3} \tag{1}$$

Verification of solutions

$$y = \frac{-e^{2\left(\int f(x)dx\right)} + c_3 f(x) + \left(\int e^{2\left(\int f(x)dx\right)}dx\right) f(x)}{\int e^{2\left(\int f(x)dx\right)}dx + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (f(x)^2-(diff(f(x), x)))*y(x),
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(y(x)^2+y(x)+x^2*(-f(x)^2+diff(f(x), x)))
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(diff(y(x),x)=y(x)^2-f(x)^2+diff(f(x),x),y(x), singsol=all)
```

$$y(x) = \frac{-f(x) \left(\int \mathrm{e}^{2\left(\int f(x)dx\right)}dx\right) + f(x) c_1 + \mathrm{e}^{2\left(\int f(x)dx\right)}}{c_1 - \left(\int \mathrm{e}^{2\left(\int f(x)dx\right)}dx\right)}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2-f[x]^2+f'[x],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

20.2 problem 35

Internal problem ID [10628]
Internal file name [OUTPUT/9575_Monday_June_06_2022_03_10_38_PM_90443355/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-2. Equations containing arbitrary functions and their derivatives.
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - f(x)\,y^2 + f(x)\,g(x)\,y = g'(x)}$$

20.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)\,y^2 - f(x)\,g(x)\,y + g'(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\,y^2 - f(x)\,g(x)\,y + g'(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = g'(x)$, $f_1(x) = -f(x)\,g(x)$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1 f_2 &= -f(x)^2 g(x) \\ f_2^2 f_0 &= f(x)^2 g'(x) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x) u''(x) - \left(f'(x) - f(x)^2 g(x)\right) u'(x) + f(x)^2 g'(x) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{-\left(\int f(x)g(x)dx\right)} \left(c_1 + \left(\int f(x)\, \mathrm{e}^{\int f(x)g(x)dx} dx\right) c_2\right)$$

The above shows that

$$u'(x) \\ = f(x) \left(-g(x)\, \mathrm{e}^{-\left(\int f(x)g(x)dx\right)} \left(\int f(x)\, \mathrm{e}^{\int f(x)g(x)dx} dx\right) c_2 - g(x)\, \mathrm{e}^{-\left(\int f(x)g(x)dx\right)} c_1 + c_2\right)$$

Using the above in (1) gives the solution

$$y = \\ -\frac{\left(-g(x)\, \mathrm{e}^{-\left(\int f(x)g(x)dx\right)} \left(\int f(x)\, \mathrm{e}^{\int f(x)g(x)dx} dx\right) c_2 - g(x)\, \mathrm{e}^{-\left(\int f(x)g(x)dx\right)} c_1 + c_2\right) \mathrm{e}^{\int f(x)g(x)dx}}{c_1 + \left(\int f(x)\, \mathrm{e}^{\int f(x)g(x)dx} dx\right) c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\int f(x)\, \mathrm{e}^{\int f(x)g(x)dx} dx\right) g(x) + c_3 g(x) - \mathrm{e}^{\int f(x)g(x)dx}}{c_3 + \int f(x)\, \mathrm{e}^{\int f(x)g(x)dx} dx}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\int f(x)\, \mathrm{e}^{\int f(x)g(x)dx}dx\right) g(x) + c_3 g(x) - \mathrm{e}^{\int f(x)g(x)dx}}{c_3 + \int f(x)\, \mathrm{e}^{\int f(x)g(x)dx}dx} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\int f(x)\, \mathrm{e}^{\int f(x)g(x)dx}dx\right) g(x) + c_3 g(x) - \mathrm{e}^{\int f(x)g(x)dx}}{c_3 + \int f(x)\, \mathrm{e}^{\int f(x)g(x)dx}dx}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(f(x)^2*g(x)-(diff(f(x), x)))
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(f(x)*y(x)^2+y(x)-g(x)*f(x)*y(x)*x+(diff(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2-f(x)*g(x)*y(x)+diff(g(x),x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2-f[x]*g[x]*y[x]+g'[x],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

20.3 problem 36

Internal problem ID [10629]
Internal file name [`OUTPUT/9576_Monday_June_06_2022_03_10_40_PM_92412809/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-2. Equations containing arbitrary functions and their derivatives.
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + f'(x)\, y^2 - f(x)\, g(x)\, y = -g(x)}$$

20.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -f'(x)\, y^2 + f(x)\, g(x)\, y - g(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -f'(x)\, y^2 + f(x)\, g(x)\, y - g(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -g(x)$, $f_1(x) = f(x)\, g(x)$ and $f_2(x) = -f'(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-f'(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -f''(x) \\ f_1f_2 &= -f(x)\,g(x)\,f'(x) \\ f_2^2 f_0 &= -f'(x)^2 g(x) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-f'(x)\,u''(x) - (-f''(x) - f(x)\,g(x)\,f'(x))\,u'(x) - f'(x)^2 g(x)\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = f(x)\left(c_1 + c_2\left(\int \frac{e^{\int \frac{f(x)g(x)f'(x)+f''(x)}{f'(x)}dx}}{f(x)^2}dx\right)\right)$$

The above shows that

$$u'(x) = \frac{f'(x)\,f(x)\left(\int \frac{e^{\int \frac{f(x)g(x)f'(x)+f''(x)}{f'(x)}dx}}{f(x)^2}dx\right)c_2 + f'(x)\,f(x)\,c_1 + c_2 e^{\int \frac{f(x)g(x)f'(x)+f''(x)}{f'(x)}dx}}{f(x)}$$

Using the above in (1) gives the solution

$$y = \frac{f'(x)\,f(x)\left(\int \frac{e^{\int \frac{f(x)g(x)f'(x)+f''(x)}{f'(x)}dx}}{f(x)^2}dx\right)c_2 + f'(x)\,f(x)\,c_1 + c_2 e^{\int \frac{f(x)g(x)f'(x)+f''(x)}{f'(x)}dx}}{f(x)^2 f'(x)\left(c_1 + c_2\left(\int \frac{e^{\int \frac{f(x)g(x)f'(x)+f''(x)}{f'(x)}dx}}{f(x)^2}dx\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\int \frac{e^{\int \frac{f(x)g(x)f'(x)+f''(x)}{f'(x)}dx}}{f(x)^2}dx\right)f(x)\,f'(x) + f'(x)\,f(x)\,c_3 + e^{\int \frac{f(x)g(x)f'(x)+f''(x)}{f'(x)}dx}}{f(x)^2 f'(x)\left(c_3 + \int \frac{e^{\int \frac{f(x)g(x)f'(x)+f''(x)}{f'(x)}dx}}{f(x)^2}dx\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\int \frac{e^{\int \frac{f(x) g(x) f'(x)+f''(x)}{f'(x)}dx}}{f(x)^{2}}dx\right) f(x) f'(x) + f'(x) f(x) c_{3} + e^{\int \frac{f(x) g(x) f'(x)+f''(x)}{f'(x)}dx}}{f\left(x\right)^{2} f'\left(x\right) \left(c_{3} + \int \frac{e^{\int \frac{f(x) g(x) f'(x)+f''(x)}{f'(x)}dx}}{f(x)^{2}}dx\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\int \frac{e^{\int \frac{f(x) g(x) f'(x)+f''(x)}{f'(x)}dx}}{f(x)^{2}}dx\right) f(x) f'(x) + f'(x) f(x) c_{3} + e^{\int \frac{f(x) g(x) f'(x)+f''(x)}{f'(x)}dx}}{f\left(x\right)^{2} f'\left(x\right) \left(c_{3} + \int \frac{e^{\int \frac{f(x) g(x) f'(x)+f''(x)}{f'(x)}dx}}{f(x)^{2}}dx\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (g(x)*f(x)*(diff(f(x), x))+dif
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(-(diff(f(x), x))*y(x)^2+y(x)+g(x)*f(x)*y
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 102

```
dsolve(diff(y(x),x)=-diff(f(x),x)*y(x)^2+f(x)*g(x)*y(x)-g(x),y(x), singsol=all)
```

$$y(x) = \frac{f(x)\,\mathrm{e}^{\int \frac{g(x)f(x)^2-2\frac{d}{dx}f(x)}{f(x)}dx} + \int \left(\frac{d}{dx}f(x)\right)\mathrm{e}^{\int g(x)f(x)dx-2\left(\int \frac{\frac{d}{dx}f(x)}{f(x)}dx\right)}dx - c_1}{f(x)\left(\int \left(\frac{d}{dx}f(x)\right)\mathrm{e}^{\int g(x)f(x)dx-2\left(\int \frac{\frac{d}{dx}f(x)}{f(x)}dx\right)}dx - c_1\right)}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==-f'[x]*y[x]^2+f[x]*g[x]*y[x]-g[x],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

20.4 problem 37

Internal problem ID [10630]
Internal file name [`OUTPUT/9577_Monday_June_06_2022_03_10_45_PM_76170662/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-2. Equations containing arbitrary functions and their derivatives.
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$\boxed{y' - g(x)\,(y - f(x))^2 = f'(x)}$$

20.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)^2\, g(x) - 2f(x)\, g(x)\, y + g(x)\, y^2 + f'(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)^2\, g(x) - 2f(x)\, g(x)\, y + g(x)\, y^2 + f'(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = f(x)^2\, g(x) + f'(x)$, $f_1(x) = -2f(x)\, g(x)$ and $f_2(x) = g(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{g(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= g'(x) \\
f_1 f_2 &= -2f(x)\, g(x)^2 \\
f_2^2 f_0 &= g(x)^2 \left(f(x)^2 g(x) + f'(x)\right)
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$g(x)\, u''(x) - \left(g'(x) - 2f(x)\, g(x)^2\right) u'(x) + g(x)^2 \left(f(x)^2 g(x) + f'(x)\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(c_1 + \int g(x)\, dx\right) \mathrm{e}^{-\left(\int f(x)g(x)dx\right)} c_2$$

The above shows that

$$u'(x) = g(x)\, \mathrm{e}^{-\left(\int f(x)g(x)dx\right)} c_2 \left(1 - f(x) \left(\int g(x)\, dx\right) - c_1 f(x)\right)$$

Using the above in (1) gives the solution

$$y = -\frac{1 - f(x) \left(\int g(x)\, dx\right) - c_1 f(x)}{c_1 + \int g\,(x)\, dx}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-1 + f(x) \left(\int g(x)\, dx\right) + c_3 f(x)}{c_3 + \int g\,(x)\, dx}$$

Summary
The solution(s) found are the following

$$y = \frac{-1 + f(x) \left(\int g(x)\, dx\right) + c_3 f(x)}{c_3 + \int g\,(x)\, dx} \tag{1}$$

Verification of solutions

$$y = \frac{-1 + f(x) \left(\int g(x) \, dx\right) + c_3 f(x)}{c_3 + \int g(x) \, dx}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (d) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)=g(x)*(y(x)-f(x))^2+diff(f(x),x),y(x), singsol=all)
```

$$y(x) = f(x) + \frac{1}{c_1 - \left(\int g(x) \, dx\right)}$$

✓ Solution by Mathematica

Time used: 0.35 (sec). Leaf size: 31

```
DSolve[y'[x]==g[x]*(y[x]-f[x])^2+f'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to f(x) + \frac{1}{-\int_1^x g(K[2])dK[2] + c_1}$$

$$y(x) \to f(x)$$

20.5 problem 38

Internal problem ID [10631]
Internal file name [`OUTPUT/9578_Monday_June_06_2022_03_10_47_PM_6587525/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-2. Equations containing arbitrary functions and their derivatives.
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \frac{f'(x)\, y^2}{g\left(x\right)} = -\frac{g'(x)}{f\left(x\right)}}$$

20.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{f'(x)\, y^2 f(x) - g(x)\, g'(x)}{g\left(x\right) f\left(x\right)} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{f'(x)\, y^2}{g\left(x\right)} - \frac{g'(x)}{f\left(x\right)}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{g'(x)}{f(x)}$, $f_1(x) = 0$ and $f_2(x) = \frac{f'(x)}{g(x)}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{f'(x)u}{g(x)}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{f'(x)\, g'(x)}{g(x)^2} + \frac{f''(x)}{g(x)} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= -\frac{f'(x)^2 g'(x)}{g(x)^2 f(x)}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{f'(x)\, u''(x)}{g(x)} - \left(-\frac{f'(x)\, g'(x)}{g(x)^2} + \frac{f''(x)}{g(x)}\right) u'(x) - \frac{f'(x)^2 g'(x)\, u(x)}{g(x)^2 f(x)} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(-\frac{f'(x)g'(x)}{g(x)^2} + \frac{f''(x)}{g(x)}\right) g(x)\_Y'(x)}{f'(x)} - \frac{f'(x)\, g'(x)\_Y(x)}{g(x)\, f(x)}\right\}, \{\_Y(x)\}\right)
$$

The above shows that

$$
u'(x) = \frac{d}{dx}\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{\left(-\frac{f'(x)g'(x)}{g(x)^2} + \frac{f''(x)}{g(x)}\right) g(x)\_Y'(x)}{f'(x)} - \frac{f'(x)\, g'(x)\_Y(x)}{g(x)\, f(x)}\right\}, \{\_Y(x)\}\right)
$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{d}{dx}\operatorname{DESol}\left(\left\{\_Y''(x) - \frac{\left(-\frac{f'(x)g'(x)}{g(x)^2} + \frac{f''(x)}{g(x)}\right)g(x)\_Y'(x)}{f'(x)} - \frac{f'(x)g'(x)\_Y(x)}{g(x)f(x)}\right\}, \{\_Y(x)\}\right)\right)g(x)}{f'(x)\operatorname{DESol}\left(\left\{\_Y''(x) - \frac{\left(-\frac{f'(x)g'(x)}{g(x)^2} + \frac{f''(x)}{g(x)}\right)g(x)\_Y'(x)}{f'(x)} - \frac{f'(x)g'(x)\_Y(x)}{g(x)f(x)}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{d}{dx}\operatorname{DESol}\left(\left\{\frac{\_Y''(x)g(x)f(x)f'(x) - f(x)g(x)f''(x)\_Y'(x) + f'(x)g'(x)\left(\_Y'(x)f(x) - \_Y(x)f'(x)\right)}{g(x)f'(x)f(x)}\right\}, \{\_Y(x)\}\right)\right)g(x)}{f'(x)\operatorname{DESol}\left(\left\{\frac{\_Y''(x)g(x)f(x)f'(x) - f(x)g(x)f''(x)\_Y'(x) + f'(x)g'(x)\left(\_Y'(x)f(x) - \_Y(x)f'(x)\right)}{g(x)f'(x)f(x)}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{d}{dx}\operatorname{DESol}\left(\left\{\frac{\_Y''(x)g(x)f(x)f'(x) - f(x)g(x)f''(x)\_Y'(x) + f'(x)g'(x)\left(\_Y'(x)f(x) - \_Y(x)f'(x)\right)}{g(x)f'(x)f(x)}\right\}, \{\_Y(x)\}\right)\right)g(x)}{f'(x)\operatorname{DESol}\left(\left\{\frac{\_Y''(x)g(x)f(x)f'(x) - f(x)g(x)f''(x)\_Y'(x) + f'(x)g'(x)\left(\_Y'(x)f(x) - \_Y(x)f'(x)\right)}{g(x)f'(x)f(x)}\right\}, \{\_Y(x)\}\right)} \quad (1)$$

Verification of solutions

$$y = -\frac{\left(\frac{d}{dx}\operatorname{DESol}\left(\left\{\frac{\_Y''(x)g(x)f(x)f'(x) - f(x)g(x)f''(x)\_Y'(x) + f'(x)g'(x)\left(\_Y'(x)f(x) - \_Y(x)f'(x)\right)}{g(x)f'(x)f(x)}\right\}, \{\_Y(x)\}\right)\right)g(x)}{f'(x)\operatorname{DESol}\left(\left\{\frac{\_Y''(x)g(x)f(x)f'(x) - f(x)g(x)f''(x)\_Y'(x) + f'(x)g'(x)\left(\_Y'(x)f(x) - \_Y(x)f'(x)\right)}{g(x)f'(x)f(x)}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = ((diff(diff(f(x), x), x))*g(x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-((diff(f(x), x))*y(x)^2/g(x)+y(x)-x^2*(di
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 58

```
dsolve(diff(y(x),x)=diff(f(x),x)/g(x)*y(x)^2-diff(g(x),x)/f(x),y(x), singsol=all)
```

$$y(x) = \frac{-\left(\int \frac{\frac{d}{dx}f(x)}{g(x)f(x)^2}dx\right) g(x)\, f(x) - c_1 f(x)\, g(x) - 1}{f\,(x)^2 \left(\int \frac{\frac{d}{dx}f(x)}{g(x)f(x)^2}dx + c_1\right)}$$

✓ Solution by Mathematica

Time used: 0.347 (sec). Leaf size: 160

```
DSolve[y'[x]==f'[x]/g[x]*y[x]^2-g'[x]/f[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\int_1^{y(x)} \left(\frac{1}{(g(x)+f(x)K[2])^2} - \int_1^x \left(\frac{2(f(K[1])K[2]^2 f'(K[1]) - g(K[1])g'(K[1]))}{g(K[1])(g(K[1])+f(K[1])K[2])^3} - \frac{2K[2]f'(K[1])}{g(K[1])(g(K[1])+f(K[1])K[2])^2}\right) dK[1]\right) dK[2] + \int_1^x -\frac{f(K[1])y(x)^2 f'(K[1]) - g(K[1])g'(K[1])}{f(K[1])g(K[1])(g(K[1])+f(K[1])y(x))^2} dK[1] = c_1, y(x)\right]$$

20.6 problem 39

Internal problem ID [10632]
Internal file name [`OUTPUT/9579_Monday_June_06_2022_03_10_49_PM_7500533/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-2. Equations containing arbitrary functions and their derivatives.
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{f(x)^2 y' - f'(x) y^2 + g(x) (y - f(x)) = 0}$$

20.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{f'(x) y^2 + f(x) g(x) - y g(x)}{f(x)^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{f'(x) y^2}{f(x)^2} + \frac{g(x)}{f(x)} - \frac{g(x) y}{f(x)^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{g(x)}{f(x)}$, $f_1(x) = -\frac{g(x)}{f(x)^2}$ and $f_2(x) = \frac{f'(x)}{f(x)^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{f'(x)u}{f(x)^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2f'(x)^2}{f(x)^3} + \frac{f''(x)}{f(x)^2} \\ f_1 f_2 &= -\frac{g(x)\, f'(x)}{f(x)^4} \\ f_2^2 f_0 &= \frac{f'(x)^2 g(x)}{f(x)^5} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{f'(x)\, u''(x)}{f(x)^2} - \left(-\frac{2f'(x)^2}{f(x)^3} + \frac{f''(x)}{f(x)^2} - \frac{g(x)\, f'(x)}{f(x)^4}\right) u'(x) + \frac{f'(x)^2 g(x)\, u(x)}{f(x)^5} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{2f'(x)^2 f(x)^2 \_Y'(x) + \_Y''(x)\, f(x)^3 f'(x) - f(x)^3 f''(x) \_Y'(x) + f'(x)^2 g(x) \_Y(x) + f'(x}{f(x)^3 f'(x)}\right.\right.$$

The above shows that

$$u'(x) = \frac{d}{dx} \mathrm{DESol}\left(\left\{\frac{2f'(x)^2 f(x)^2 \_Y'(x) + \_Y''(x)\, f(x)^3 f'(x) - f(x)^3 f''(x) \_Y'(x) + f'(x)^2 g(x) \_Y(x) + f}{f(x)^3 f'(x)}\right.\right.$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{d}{dx} \mathrm{DESol}\left(\left\{\frac{2f'(x)^2 f(x)^2 \_Y'(x) + \_Y''(x) f(x)^3 f'(x) - f(x)^3 f''(x) \_Y'(x) + f'(x)^2 g(x) \_Y(x) + f'(x) f(x) g(x) \_Y'(x)}{f(x)^3 f'(x)}\right\}, \{\_\right.\right.}{f'(x)\, \mathrm{DESol}\left(\left\{\frac{2f'(x)^2 f(x)^2 \_Y'(x) + \_Y''(x) f(x)^3 f'(x) - f(x)^3 f''(x) \_Y'(x) + f'(x)^2 g(x) \_Y(x) + f'(x) f(x) g(x) \_Y'(x)}{f(x)^3 f'(x)}\right\}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{d}{dx}\operatorname{DESol}\left(\left\{\frac{2f'(x)^2 f(x)^2 \_\_Y'(x)+\_\_Y''(x) f(x)^3 f'(x)-f(x)^3 f''(x) \_\_Y'(x)+f'(x)^2 g(x) \_\_Y(x)+f'(x) f(x) g(x) \_\_Y'(x)}{f(x)^3 f'(x)}\right\}, \{\_\right.\right.}{f'(x)\operatorname{DESol}\left(\left\{\frac{2f'(x)^2 f(x)^2 \_\_Y'(x)+\_\_Y''(x) f(x)^3 f'(x)-f(x)^3 f''(x) \_\_Y'(x)+f'(x)^2 g(x) \_\_Y(x)+f'(x) f(x) g(x) \_\_Y'(x)}{f(x)^3 f'(x)}\right\}\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{d}{dx}\operatorname{DESol}\left(\left\{\frac{2f'(x)^2 f(x)^2 \_\_Y'(x)+\_\_Y''(x) f(x)^3 f'(x)-f(x)^3 f''(x) \_\_Y'(x)+f'(x)^2 g(x) \_\_Y(x)+f'(x) f(x) g(x) \_\_Y'(x)}{f(x)^3 f'(x)}\right\}, \{\_\right.\right.}{f'(x)\operatorname{DESol}\left(\left\{\frac{2f'(x)^2 f(x)^2 \_\_Y'(x)+\_\_Y''(x) f(x)^3 f'(x)-f(x)^3 f''(x) \_\_Y'(x)+f'(x)^2 g(x) \_\_Y(x)+f'(x) f(x) g(x) \_\_Y'(x)}{f(x)^3 f'(x)}\right\}\right.} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\frac{d}{dx}\operatorname{DESol}\left(\left\{\frac{2f'(x)^2 f(x)^2 \_\_Y'(x)+\_\_Y''(x) f(x)^3 f'(x)-f(x)^3 f''(x) \_\_Y'(x)+f'(x)^2 g(x) \_\_Y(x)+f'(x) f(x) g(x) \_\_Y'(x)}{f(x)^3 f'(x)}\right\}, \{\_\right.\right.}{f'(x)\operatorname{DESol}\left(\left\{\frac{2f'(x)^2 f(x)^2 \_\_Y'(x)+\_\_Y''(x) f(x)^3 f'(x)-f(x)^3 f''(x) \_\_Y'(x)+f'(x)^2 g(x) \_\_Y(x)+f'(x) f(x) g(x) \_\_Y'(x)}{f(x)^3 f'(x)}\right\}\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(-(diff(diff(f(x), x), x))*f(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-((diff(f(x), x))*y(x)^2/f(x)^2+y(x)-g(x)*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(f(x)^2*diff(y(x),x)-diff(f(x),x)*y(x)^2+g(x)*(y(x)-f(x))=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[f[x]^2*y'[x]-f'[x]*y[x]^2+g[x]*(y[x]-f[x])==0,y[x],x,IncludeSingularSolutions -> True
```

Not solved

20.7 problem 40

Internal problem ID [10633]
Internal file name [OUTPUT/9580_Monday_June_06_2022_03_10_52_PM_112628/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-2. Equations containing arbitrary functions and their derivatives.
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - f'(x)\, y^2 - a\, e^{\lambda x} f(x)\, y = a\, e^{\lambda x}}$$

20.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f'(x)\, y^2 + a\, e^{\lambda x} f(x)\, y + a\, e^{\lambda x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f'(x)\, y^2 + a\, e^{\lambda x} f(x)\, y + a\, e^{\lambda x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a\, e^{\lambda x}$, $f_1(x) = a\, e^{\lambda x} f(x)$ and $f_2(x) = f'(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f'(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f''(x) \\ f_1 f_2 &= a\,e^{\lambda x} f(x) f'(x) \\ f_2^2 f_0 &= f'(x)^2 a\,e^{\lambda x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f'(x) u''(x) - \left(f''(x) + a\,e^{\lambda x} f(x) f'(x)\right) u'(x) + f'(x)^2 a\,e^{\lambda x} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = f(x)\left(c_1 + c_2\left(\int \frac{e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f(x)^2}dx\right)\right)$$

The above shows that

$$u'(x) = \frac{f'(x) f(x)\left(\int \frac{e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f(x)^2}dx\right) c_2 + f'(x) f(x) c_1 + c_2 e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f(x)}$$

Using the above in (1) gives the solution

$$y = -\frac{f'(x) f(x)\left(\int \frac{e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f(x)^2}dx\right) c_2 + f'(x) f(x) c_1 + c_2 e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f(x)^2 f'(x)\left(c_1 + c_2\left(\int \frac{e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f(x)^2}dx\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\left(\int \frac{e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f(x)^2}dx\right) f(x) f'(x) - f'(x) f(x) c_3 - e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f(x)^2 f'(x)\left(c_3 + \int \frac{e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f(x)^2}dx\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\left(\int \frac{e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f(x)^2}dx\right) f(x)\, f'(x) - f'(x)\, f(x)\, c_3 - e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f\left(x\right)^2 f'\left(x\right) \left(c_3 + \int \frac{e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f(x)^2}dx\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-\left(\int \frac{e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f(x)^2}dx\right) f(x)\, f'(x) - f'(x)\, f(x)\, c_3 - e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f\left(x\right)^2 f'\left(x\right) \left(c_3 + \int \frac{e^{\int \frac{f''(x)+a\,e^{\lambda x} f(x) f'(x)}{f'(x)}dx}}{f(x)^2}dx\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (exp(lambda*x)*f(x)*a*(diff(f(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
            checking if the LODE is missing y
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 114

```
dsolve(diff(y(x),x)=diff(f(x),x)*y(x)^2+a*exp(lambda*x)*f(x)*y(x)+a*exp(lambda*x),y(x), sing
```

$$y(x) = \frac{-f(x)\,e^{\int \frac{e^{x\lambda} f(x)^2 a - 2\frac{d}{dx}f(x)}{f(x)}dx} - \left(\int \left(\frac{d}{dx}f(x)\right) e^{a\left(\int e^{x\lambda} f(x)dx\right) - 2\left(\int \frac{\frac{d}{dx}f(x)}{f(x)}dx\right)}dx\right) - c_1}{f(x)\left(c_1 + \int \left(\frac{d}{dx}f(x)\right) e^{a\left(\int e^{x\lambda} f(x)dx\right) - 2\left(\int \frac{\frac{d}{dx}f(x)}{f(x)}dx\right)}dx\right)}$$

✓ Solution by Mathematica

Time used: 84.356 (sec). Leaf size: 167

```
DSolve[y'[x]==f'[x]*y[x]^2+a*Exp[\[Lambda]*x]*f[x]*y[x]+a*Exp[\[Lambda]*x],y[x],x,IncludeSin
```

$$y(x) \to -\frac{a\exp\left(\int_1^{e^{x\lambda}} -\frac{af\left(\frac{\log(K[1])}{\lambda}\right)}{\lambda}dK[1]\right)\left(1 + c_1\int_1^{e^{x\lambda}}\exp\left(-\int_1^{K[2]} -\frac{af\left(\frac{\log(K[1])}{\lambda}\right)}{\lambda}dK[1]\right)dK[2]\right)}{af\left(\frac{\log(e^{\lambda x})}{\lambda}\right)\exp\left(\int_1^{e^{x\lambda}} -\frac{af\left(\frac{\log(K[1])}{\lambda}\right)}{\lambda}dK[1]\right)\left(1 + c_1\int_1^{e^{x\lambda}}\exp\left(-\int_1^{K[2]} -\frac{af\left(\frac{\log(K[1])}{\lambda}\right)}{\lambda}dK[1]\right)dK[2]\right) -}$$

20.8 problem 41

Internal problem ID [10634]
Internal file name [`OUTPUT/9581_Monday_June_06_2022_03_10_54_PM_90496217/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-2. Equations containing arbitrary functions and their derivatives.
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - f(x)\, y^2 - g'(x)\, y = a f(x)\, \mathrm{e}^{2g(x)}}$$

20.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)\, y^2 + g'(x)\, y + a f(x)\, \mathrm{e}^{2g(x)} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\, y^2 + g'(x)\, y + a f(x)\, \mathrm{e}^{2g(x)}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a f(x)\, \mathrm{e}^{2g(x)}$, $f_1(x) = g'(x)$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1f_2 &= f(x)\, g'(x) \\ f_2^2 f_0 &= f(x)^3\, a\, \mathrm{e}^{2g(x)} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\, u''(x) - (f'(x) + f(x)\, g'(x))\, u'(x) + f(x)^3\, a\, \mathrm{e}^{2g(x)} u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \mathrm{e}^{i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)} + c_2 \mathrm{e}^{-i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)}$$

The above shows that

$$u'(x) = i\sqrt{a}\, f(x)\, \mathrm{e}^{g(x)} \left(c_1 \mathrm{e}^{i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)} - c_2 \mathrm{e}^{-i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{i\sqrt{a}\, \mathrm{e}^{g(x)} \left(c_1 \mathrm{e}^{i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)} - c_2 \mathrm{e}^{-i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)}\right)}{c_1 \mathrm{e}^{i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)} + c_2 \mathrm{e}^{-i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{i\sqrt{a}\, \mathrm{e}^{g(x)} \left(c_3 \mathrm{e}^{i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)} - \mathrm{e}^{-i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)}\right)}{c_3 \mathrm{e}^{i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)} + \mathrm{e}^{-i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)}}$$

Summary

The solution(s) found are the following

$$y = -\frac{i\sqrt{a}\, \mathrm{e}^{g(x)} \left(c_3 \mathrm{e}^{i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)} - \mathrm{e}^{-i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)}\right)}{c_3 \mathrm{e}^{i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)} + \mathrm{e}^{-i\sqrt{a}\,\left(\int f(x)\mathrm{e}^{g(x)}dx\right)}} \tag{1}$$

Verification of solutions

$$y = -\frac{i\sqrt{a}\, e^{g(x)} \left(c_3 e^{i\sqrt{a}\,\left(\int f(x) e^{g(x)} dx\right)} - e^{-i\sqrt{a}\,\left(\int f(x) e^{g(x)} dx\right)}\right)}{c_3 e^{i\sqrt{a}\,\left(\int f(x) e^{g(x)} dx\right)} + e^{-i\sqrt{a}\,\left(\int f(x) e^{g(x)} dx\right)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 28

```
dsolve(diff(y(x),x)=f(x)*y(x)^2+diff(g(x),x)*y(x)+a*f(x)*exp(2*g(x)),y(x), singsol=all)
```

$$y(x) = -\tan\left(-\sqrt{a}\left(\int f(x)\, e^{g(x)} dx\right) + c_1\right)\sqrt{a}\, e^{g(x)}$$

✓ Solution by Mathematica

Time used: 0.635 (sec). Leaf size: 41

```
DSolve[y'[x]==f[x]*y[x]^2+g'[x]*y[x]+a*f[x]*Exp[2*g[x]],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \sqrt{a} e^{g(x)} \tan\left(\sqrt{a}\int_1^x e^{g(K[1])} f(K[1]) dK[1] + c_1\right)$$

20.9 problem 42

Internal problem ID [10635]
Internal file name [`OUTPUT/9582_Monday_June_06_2022_03_10_56_PM_41579351/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.8-2. Equations containing arbitrary functions and their derivatives.
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_Riccati]`

Unable to solve or complete the solution.

$$y' - y^2 = -\frac{f''(x)}{f(x)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (diff(diff(f(x), x), x))*y(x)/
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      <- linear_1 successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(diff(y(x),x)=y(x)^2-diff(f(x),x$2)/f(x),y(x), singsol=all)
```

$$y(x)=\frac{-\left(\int \frac{1}{f(x)^{2}}dx\right)\left(\frac{d}{dx}f(x)\right)f(x)-\left(\frac{d}{dx}f(x)\right)c_{1}f(x)-1}{\left(\int \frac{1}{f(x)^{2}}dx+c_{1}\right)f(x)^{2}}$$

✓ Solution by Mathematica

Time used: 0.365 (sec). Leaf size: 132

```
DSolve[y'[x]==y[x]^2-f''[x]/f[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\int_1^{y(x)}\left(\frac{1}{(f(x)K[2]+f'(x))^2}-\int_1^x\left(\frac{2(f(K[1])K[2]^2-f''(K[1]))}{(f(K[1])K[2]+f'(K[1]))^3}-\frac{2K[2]}{(f(K[1])K[2]+f'(K[1]))^2}\right)dK[1]\right)dK[2]+\int_1^x-\frac{f(K[1])y(x)^2-f''(K[1])}{f(K[1])\left(f(K[1])y(x)+f'(K[1])\right)^2}dK[1]=c_1,y(x)\right]$$

21 Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations

21.1 problem 1

Internal problem ID [10636]
Internal file name [`OUTPUT/9583_Monday_June_06_2022_03_10_56_PM_78843346/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 = a^2 f(ax + b)}$$

21.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + a^2 f(ax + b) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + a^2 f(ax + b)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = a^2 f(ax + b)$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = 0$$
$$f_1 f_2 = 0$$
$$f_2^2 f_0 = a^2 f(ax + b)$$

Substituting the above terms back in equation (2) gives

$$u''(x) + a^2 f(ax + b) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{a^2 f(ax+b)\_Y(x) + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{a^2 f(ax+b)\_Y(x) + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\{a^2 f(ax+b)\_Y(x) + \_Y''(x)\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\{a^2 f(ax+b)\_Y(x) + \_Y''(x)\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\{a^2 f(ax+b)\_Y(x) + \_Y''(x)\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\{a^2 f(ax+b)\_Y(x) + \_Y''(x)\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\{a^2 f(ax+b)\_Y(x) + \_Y''(x)\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\{a^2 f(ax+b)\_Y(x) + \_Y''(x)\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{a^2 f(ax+b)\_Y(x) + \_Y''(x)\right\}, \left\{\_Y(x)\right\}\right)}{\operatorname{DESol}\left(\left\{a^2 f(ax+b)\_Y(x) + \_Y''(x)\right\}, \left\{\_Y(x)\right\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2+a^2*f(a*x+b),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+a^2*f[a*x+b],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

21.2 problem 2

Internal problem ID [10637]
Internal file name [`OUTPUT/9584_Monday_June_06_2022_03_10_58_PM_85596946/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = \frac{f\left(\frac{1}{x}\right)}{x^4}}$$

21.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y^2x^4 + f\left(\frac{1}{x}\right)}{x^4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \frac{f\left(\frac{1}{x}\right)}{x^4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{f\left(\frac{1}{x}\right)}{x^4}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= \frac{f\left(\frac{1}{x}\right)}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{f\left(\frac{1}{x}\right)u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{f\left(\frac{1}{x}\right)\_Y(x)}{x^4} + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{d}{dx}\,\mathrm{DESol}\left(\left\{\frac{f\left(\frac{1}{x}\right)\_Y(x)}{x^4} + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{d}{dx}\,\mathrm{DESol}\left(\left\{\frac{f\left(\frac{1}{x}\right)\_Y(x)}{x^4} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{f\left(\frac{1}{x}\right)\_Y(x)}{x^4} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{d}{dx}\,\mathrm{DESol}\left(\left\{\frac{\_Y''(x)x^4 + f\left(\frac{1}{x}\right)\_Y(x)}{x^4}\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{\_Y''(x)x^4 + f\left(\frac{1}{x}\right)\_Y(x)}{x^4}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{d}{dx}\operatorname{DESol}\left(\left\{\frac{\_Y''(x)\,x^4 + f\left(\frac{1}{x}\right)\_Y(x)}{x^4}\right\}, \{\_Y(x)\}\right)}{\operatorname{DESol}\left(\left\{\frac{\_Y''(x)\,x^4 + f\left(\frac{1}{x}\right)\_Y(x)}{x^4}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{d}{dx}\operatorname{DESol}\left(\left\{\frac{\_Y''(x)\,x^4 + f\left(\frac{1}{x}\right)\_Y(x)}{x^4}\right\}, \{\_Y(x)\}\right)}{\operatorname{DESol}\left(\left\{\frac{\_Y''(x)\,x^4 + f\left(\frac{1}{x}\right)\_Y(x)}{x^4}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2+1/x^4*f(1/x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+1/x^4*f[1/x],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

21.3 problem 3

Internal problem ID [10638]
Internal file name [`OUTPUT/9585_Monday_June_06_2022_03_10_59_PM_98380475/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = \frac{f\left(\frac{ax+b}{cx+d}\right)}{(cx+d)^4}}$$

21.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{c^4x^4y^2 + 4c^3d\,x^3y^2 + 6c^2d^2x^2y^2 + 4c\,d^3x\,y^2 + d^4y^2 + f\left(\frac{ax+b}{cx+d}\right)}{(cx+d)^4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{c^4x^4y^2}{(cx+d)^4} + \frac{4c^3d\,x^3y^2}{(cx+d)^4} + \frac{6c^2d^2x^2y^2}{(cx+d)^4} + \frac{4c\,d^3x\,y^2}{(cx+d)^4} + \frac{d^4y^2}{(cx+d)^4} + \frac{f\left(\frac{ax+b}{cx+d}\right)}{(cx+d)^4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{f\left(\frac{ax+b}{cx+d}\right)}{(cx+d)^4}$, $f_1(x) = 0$ and $f_2(x) = \frac{c^4x^4+4c^3d\,x^3+6c^2d^2x^2+4c\,d^3x+d^4}{(cx+d)^4}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{(c^4x^4+4c^3d\,x^3+6c^2d^2x^2+4c\,d^3x+d^4)u}{(cx+d)^4}}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= \frac{4c^4x^3+12c^3d\,x^2+12c^2d^2x+4c\,d^3}{(cx+d)^4} - \frac{4(c^4x^4+4c^3d\,x^3+6c^2d^2x^2+4c\,d^3x+d^4)\,c}{(cx+d)^5} \\
f_1f_2 &= 0 \\
f_2^2f_0 &= \frac{(c^4x^4+4c^3d\,x^3+6c^2d^2x^2+4c\,d^3x+d^4)^2 f\left(\frac{ax+b}{cx+d}\right)}{(cx+d)^{12}}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{(c^4x^4+4c^3d\,x^3+6c^2d^2x^2+4c\,d^3x+d^4)\,u''(x)}{(cx+d)^4} - \left(\frac{4c^4x^3+12c^3d\,x^2+12c^2d^2x+4c\,d^3}{(cx+d)^4} - \frac{4(c^4x^4+4c^3d\,x
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = \mathrm{DESol}\left(\left\{\frac{f\left(\frac{ax+b}{cx+d}\right)\_Y(x) + \_Y''(x)\,(cx+d)^4}{(cx+d)^4}\right\}, \{\_Y(x)\}\right)
$$

The above shows that

$$
u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{f\left(\frac{ax+b}{cx+d}\right)\_Y(x) + \_Y''(x)\,(cx+d)^4}{(cx+d)^4}\right\}, \{\_Y(x)\}\right)
$$

Using the above in (1) gives the solution

$$
y = -\frac{\left(\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{f\left(\frac{ax+b}{cx+d}\right)\_Y(x)+\_Y''(x)(cx+d)^4}{(cx+d)^4}\right\}, \{\_Y(x)\}\right)\right)(cx+d)^4}{(c^4x^4+4c^3d\,x^3+6c^2d^2x^2+4c\,d^3x+d^4)\,\mathrm{DESol}\left(\left\{\frac{f\left(\frac{ax+b}{cx+d}\right)\_Y(x)+\_Y''(x)(cx+d)^4}{(cx+d)^4}\right\}, \{\_Y(x)\}\right)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\frac{f\left(\frac{ax+b}{cx+d}\right)\_Y(x)+\_Y''(x)(cx+d)^4}{(cx+d)^4}\right\}, \{\_Y(x)\}\right)}{\operatorname{DESol}\left(\left\{\frac{f\left(\frac{ax+b}{cx+d}\right)\_Y(x)+\_Y''(x)(cx+d)^4}{(cx+d)^4}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\frac{f\left(\frac{ax+b}{cx+d}\right)\_Y(x)+\_Y''(x)(cx+d)^4}{(cx+d)^4}\right\}, \{\_Y(x)\}\right)}{\operatorname{DESol}\left(\left\{\frac{f\left(\frac{ax+b}{cx+d}\right)\_Y(x)+\_Y''(x)(cx+d)^4}{(cx+d)^4}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\frac{f\left(\frac{ax+b}{cx+d}\right)\_Y(x)+\_Y''(x)(cx+d)^4}{(cx+d)^4}\right\}, \{\_Y(x)\}\right)}{\operatorname{DESol}\left(\left\{\frac{f\left(\frac{ax+b}{cx+d}\right)\_Y(x)+\_Y''(x)(cx+d)^4}{(cx+d)^4}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-2*(2*c*d^3*x+y(x))/x, y(x)`      *** Subl
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-2*(70*c*d^3*x+y(x))/x, y(x)`      *** Sub
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-(35*c*d^3*x+2*y(x))/x, y(x)`      *** Sub
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-(7*c*d^3*x+2*y(x))/x, y(x)`      *** Subl
      Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2+1/(c*x+d)^4*f((a*x+b)/(c*x+d)),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+1/(c*x+d)^4*f[(a*x+b)/(c*x+d)],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

21.4 problem 4

Internal problem ID [10639]
Internal file name [`OUTPUT/9586_Monday_June_06_2022_03_11_01_PM_74540310/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{x^2y' - x^4f(x)\,y^2 = 1}$$

21.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^4f(x)\,y^2+1}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2f(x)\,y^2 + \frac{1}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{1}{x^2}$, $f_1(x) = 0$ and $f_2(x) = x^2f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{x^2f\,(x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= 2f(x)\,x + x^2 f'(x)\\
f_1f_2 &= 0\\
f_2^2 f_0 &= x^2 f(x)^2
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^2 f(x)\,u''(x) - \left(2f(x)\,x + x^2 f'(x)\right) u'(x) + x^2 f(x)^2\, u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(2f(x)\,x + x^2 f'(x))\_Y'(x)}{x^2 f(x)} + f(x)\_Y(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{d}{dx}\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(2f(x)\,x + x^2 f'(x))\_Y'(x)}{x^2 f(x)} + f(x)\_Y(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{d}{dx}\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(2f(x)x + x^2 f'(x))\_Y'(x)}{x^2 f(x)} + f(x)\_Y(x)\right\}, \{\_Y(x)\}\right)}{x^2 f(x)\,\mathrm{DESol}\left(\left\{\_Y''(x) - \frac{(2f(x)x + x^2 f'(x))\_Y'(x)}{x^2 f(x)} + f(x)\_Y(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{d}{dx}\,\mathrm{DESol}\left(\left\{\frac{\_Y''(x)xf(x) + (-xf'(x) - 2f(x))\_Y'(x) + f(x)^2\_Y(x)x}{xf(x)}\right\}, \{\_Y(x)\}\right)}{x^2 f(x)\,\mathrm{DESol}\left(\left\{\frac{\_Y''(x)xf(x) + (-xf'(x) - 2f(x))\_Y'(x) + f(x)^2\_Y(x)x}{xf(x)}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{d}{dx}\operatorname{DESol}\left(\left\{\frac{\_Y''(x) x f(x)+(-x f'(x)-2 f(x))\_Y'(x)+f(x)^{2}\_Y(x) x}{x f(x)}\right\}, \{\_Y(x)\}\right)}{x^{2} f(x) \operatorname{DESol}\left(\left\{\frac{\_Y''(x) x f(x)+(-x f'(x)-2 f(x))\_Y'(x)+f(x)^{2}\_Y(x) x}{x f(x)}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{d}{dx}\operatorname{DESol}\left(\left\{\frac{\_Y''(x) x f(x)+(-x f'(x)-2 f(x))\_Y'(x)+f(x)^{2}\_Y(x) x}{x f(x)}\right\}, \{\_Y(x)\}\right)}{x^{2} f(x) \operatorname{DESol}\left(\left\{\frac{\_Y''(x) x f(x)+(-x f'(x)-2 f(x))\_Y'(x)+f(x)^{2}\_Y(x) x}{x f(x)}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(x^2*diff(y(x),x)=x^4*f(x)*y(x)^2+1,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*y'[x]==x^4*f[x]*y[x]^2+1,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

21.5 problem 5

Internal problem ID [10640]
Internal file name [`OUTPUT/9587_Monday_June_06_2022_03_11_03_PM_91047867/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{x^2 y' - y^2 x^4 = x^{2n} f(a\,x^n + b) - \frac{n^2}{4} + \frac{1}{4}}$$

21.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{4y^2x^4 + 4x^{2n} f(a\,x^n + b) - n^2 + 1}{4x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 y^2 + \frac{x^{2n} f(a\,x^n + b)}{x^2} - \frac{n^2}{4x^2} + \frac{1}{4x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{4x^{2n} f(a\,x^n+b) - n^2+1}{4x^2}$, $f_1(x) = 0$ and $f_2(x) = x^2$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^2 u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2x \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{x^2(4x^{2n} f(a\,x^n + b) - n^2 + 1)}{4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^2 u''(x) - 2xu'(x) + \frac{x^2(4x^{2n} f(a\,x^n + b) - n^2 + 1)\, u(x)}{4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\left(x^{2n} f(a\,x^n + b) - \frac{n^2}{4} + \frac{1}{4}\right)\_Y(x) - \frac{2\_Y'(x)}{x} + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\left(x^{2n} f(a\,x^n + b) - \frac{n^2}{4} + \frac{1}{4}\right)\_Y(x) - \frac{2\_Y'(x)}{x} + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\left(x^{2n} f(a\,x^n + b) - \frac{n^2}{4} + \frac{1}{4}\right)\_Y(x) - \frac{2\_Y'(x)}{x} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{x^2\,\mathrm{DESol}\left(\left\{\left(x^{2n} f(a\,x^n + b) - \frac{n^2}{4} + \frac{1}{4}\right)\_Y(x) - \frac{2\_Y'(x)}{x} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\left(x^{2n} f(a\,x^n + b) - \frac{n^2}{4} + \frac{1}{4}\right)\_Y(x) - \frac{2\_Y'(x)}{x} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{x^2\,\mathrm{DESol}\left(\left\{\frac{4x^{2n+1}\_Y(x) f(a\,x^n+b) + 4\_Y''(x)x - 8\_Y'(x) + x(-n^2+1)\_Y(x)}{4x}\right\}, \{\_Y(x)\}\right)}$$

Summary
The solution(s) found are the following

$$
y = -\frac{\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\left(x^{2n} f(a\,x^{n}+b) - \frac{n^{2}}{4} + \frac{1}{4}\right)\_Y(x) - \frac{2\_Y'(x)}{x} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{x^{2} \operatorname{DESol}\left(\left\{\frac{4x^{2n+1}\_Y(x) f(a\,x^{n}+b)+4\_Y''(x)x-8\_Y'(x)+x(-n^{2}+1)\_Y(x)}{4x}\right\}, \{\_Y(x)\}\right)} \tag{1}
$$

Verification of solutions

$$
y = -\frac{\frac{\partial}{\partial x} \operatorname{DESol}\left(\left\{\left(x^{2n} f(a\,x^{n}+b) - \frac{n^{2}}{4} + \frac{1}{4}\right)\_Y(x) - \frac{2\_Y'(x)}{x} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{x^{2} \operatorname{DESol}\left(\left\{\frac{4x^{2n+1}\_Y(x) f(a\,x^{n}+b)+4\_Y''(x)x-8\_Y'(x)+x(-n^{2}+1)\_Y(x)}{4x}\right\}, \{\_Y(x)\}\right)}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(x^2*diff(y(x),x)=x^4*y(x)^2+x^(2*n)*f(a*x^n+b)+1/4*(1-n^2),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*y'[x]==x^4*y[x]^2+x^(2*n)*f[a*x^n+b]+1/4*(1-n^2),y[x],x,IncludeSingularSolutions
```

Not solved

21.6 problem 6

Internal problem ID [10641]
Internal file name [`OUTPUT/9588_Monday_June_06_2022_03_11_05_PM_78843544/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - f(x)\,y^2 - g(x)\,y = h(x)}$$

21.6.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)\,y^2 + yg(x) + h(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = f(x)\,y^2 + yg(x) + h(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = h(x)$, $f_1(x) = g(x)$ and $f_2(x) = f(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{f(x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= f'(x) \\ f_1f_2 &= f(x)\,g(x) \\ f_2^2 f_0 &= h(x)\,f(x)^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$f(x)\,u''(x) - (f(x)\,g(x) + f'(x))\,u'(x) + h(x)\,f(x)^2\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{h(x)\,f(x)\_Y(x) - \frac{(f(x)\,g(x) + f'(x))\_Y'(x)}{f(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{d}{dx}\mathrm{DESol}\left(\left\{h(x)\,f(x)\_Y(x) - \frac{(f(x)\,g(x) + f'(x))\_Y'(x)}{f(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{d}{dx}\mathrm{DESol}\left(\left\{h(x)\,f(x)\_Y(x) - \frac{(f(x)g(x)+f'(x))\_Y'(x)}{f(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{f(x)\,\mathrm{DESol}\left(\left\{h(x)\,f(x)\_Y(x) - \frac{(f(x)g(x)+f'(x))\_Y'(x)}{f(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{d}{dx}\mathrm{DESol}\left(\left\{h(x)\,f(x)\_Y(x) - \frac{(f(x)g(x)+f'(x))\_Y'(x)}{f(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{f(x)\,\mathrm{DESol}\left(\left\{h(x)\,f(x)\_Y(x) - \frac{(f(x)g(x)+f'(x))\_Y'(x)}{f(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{d}{dx}\operatorname{DESol}\left(\left\{h(x)\, f(x)\_Y(x) - \frac{(f(x)g(x)+f'(x))\_Y'(x)}{f(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{f(x)\operatorname{DESol}\left(\left\{h(x)\, f(x)\_Y(x) - \frac{(f(x)g(x)+f'(x))\_Y'(x)}{f(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{d}{dx}\operatorname{DESol}\left(\left\{h(x)\, f(x)\_Y(x) - \frac{(f(x)g(x)+f'(x))\_Y'(x)}{f(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{f(x)\operatorname{DESol}\left(\left\{h(x)\, f(x)\_Y(x) - \frac{(f(x)g(x)+f'(x))\_Y'(x)}{f(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (g(x)*f(x)+diff(f(x), x))*(dif
      Methods for second order ODEs:
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(f(x)*y(x)^2+y(x)+g(x)*y(x)*x+x^2*h(x))/x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
      trying separable
      trying inverse linear
      trying homogeneous types:
      trying Chini
      differential order: 1; looking for linear symmetries
      trying exact
      Looking for potential symmetries
      trying Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
      trying inverse_Riccati
      trying 1st order ODE linearizable_by_differentiation
   -> trying a symmetry pattern of the form [F(x)*G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)*G(y)]
   -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=f(x)*y(x)^2+g(x)*y(x)+h(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==f[x]*y[x]^2+g[x]*y[x]+h[x],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

21.7 problem 7

Internal problem ID [10642]
Internal file name [`OUTPUT/9589_Monday_June_06_2022_03_11_07_PM_128411/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = e^{2\lambda x} f\left(e^{\lambda x}\right) - \frac{\lambda^2}{4}}$$

21.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 + e^{2\lambda x} f\left(e^{\lambda x}\right) - \frac{\lambda^2}{4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + e^{2\lambda x} f\left(e^{\lambda x}\right) - \frac{\lambda^2}{4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = e^{2\lambda x} f\left(e^{\lambda x}\right) - \frac{\lambda^2}{4}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= e^{2\lambda x} f(e^{\lambda x}) - \frac{\lambda^2}{4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(e^{2\lambda x} f(e^{\lambda x}) - \frac{\lambda^2}{4}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y(x)\, e^{2\lambda x} f(e^{\lambda x}) - \frac{\_Y(x)\, \lambda^2}{4} + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y(x)\, e^{2\lambda x} f(e^{\lambda x}) - \frac{\_Y(x)\, \lambda^2}{4} + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y(x)\, e^{2\lambda x} f(e^{\lambda x}) - \frac{\_Y(x)\lambda^2}{4} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\_Y(x)\, e^{2\lambda x} f(e^{\lambda x}) - \frac{\_Y(x)\lambda^2}{4} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y(x)\, e^{2\lambda x} f(e^{\lambda x}) - \frac{\_Y(x)\lambda^2}{4} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\_Y(x)\, e^{2\lambda x} f(e^{\lambda x}) - \frac{\_Y(x)\lambda^2}{4} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\_Y(x)\,\mathrm{e}^{2\lambda x} f\left(\mathrm{e}^{\lambda x}\right) - \frac{\_Y(x)\lambda^2}{4} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\operatorname{DESol}\left(\left\{\_Y(x)\,\mathrm{e}^{2\lambda x} f\left(\mathrm{e}^{\lambda x}\right) - \frac{\_Y(x)\lambda^2}{4} + \_Y''(x)\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\_Y(x)\,\mathrm{e}^{2\lambda x} f\left(\mathrm{e}^{\lambda x}\right) - \frac{\_Y(x)\lambda^2}{4} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\operatorname{DESol}\left(\left\{\_Y(x)\,\mathrm{e}^{2\lambda x} f\left(\mathrm{e}^{\lambda x}\right) - \frac{\_Y(x)\lambda^2}{4} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = -8*y(x)*x/((lambda-2*x)*(2*x+lambda)), 
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+4*y(x)*lambda*(2*exp(2*lambda*x)*f(exp(la
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-2*lambda*K[1], y(x)`      *** Sublevel 2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+8*y(x)*x/((lambda-2*x)*(2*x+lambda)), y(x
      Methods for first order ODEs:
      --- Trying classification methods ---
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2+exp(2*lambda*x)*f(exp(lambda*x))-1/4*lambda^2,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+Exp[2*\[Lambda]*x]*f[Exp[\[Lambda]*x]]-1/4*\[Lambda]^2,y[x],x,IncludeSi
```

Not solved

21.8 problem 8

Internal problem ID [10643]
Internal file name [`OUTPUT/9590_Monday_June_06_2022_03_11_12_PM_76019185/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$y' - y^2 = -\frac{\lambda^2}{4} + \frac{e^{2\lambda x} f\left(\frac{a\,e^{\lambda x}+b}{c\,e^{\lambda x}+d}\right)}{\left(c\,e^{\lambda x}+d\right)^4}$$

21.8.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-e^{4\lambda x}c^4\lambda^2 + 4\,e^{4\lambda x}c^4y^2 - 4\,e^{3\lambda x}c^3d\,\lambda^2 + 16\,e^{3\lambda x}c^3d\,y^2 - 6\,e^{2\lambda x}c^2d^2\lambda^2 + 24\,e^{2\lambda x}c^2d^2y^2 - 4\,e^{\lambda x}c\,d^3\lambda^2 + 1}{4\left(c\,e^{\lambda x}+d\right)^4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{3\,e^{2\lambda x}c^2d^2\lambda^2}{2\left(c\,e^{\lambda x}+d\right)^4} + \frac{6\,e^{2\lambda x}c^2d^2y^2}{\left(c\,e^{\lambda x}+d\right)^4} - \frac{e^{\lambda x}c\,d^3\lambda^2}{\left(c\,e^{\lambda x}+d\right)^4} + \frac{4\,e^{\lambda x}c\,d^3y^2}{\left(c\,e^{\lambda x}+d\right)^4} - \frac{e^{4\lambda x}c^4\lambda^2}{4\left(c\,e^{\lambda x}+d\right)^4} + \frac{e^{4\lambda x}c^4y^2}{\left(c\,e^{\lambda x}+d\right)^4} - \frac{e^{3\lambda x}c^3d\,\lambda^2}{\left(c\,e^{\lambda x}+d\right)^4} +$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{-\mathrm{e}^{4\lambda x} c^4 \lambda^2 - 4\,\mathrm{e}^{3\lambda x} c^3 d\,\lambda^2 - 6\,\mathrm{e}^{2\lambda x} c^2 d^2 \lambda^2 - 4\,\mathrm{e}^{\lambda x} c\, d^3 \lambda^2 - d^4 \lambda^2 + 4\,\mathrm{e}^{2\lambda x} f\left(\frac{a\,\mathrm{e}^{\lambda x}+b}{c\,\mathrm{e}^{\lambda x}+d}\right)}{4\left(c\,\mathrm{e}^{\lambda x}+d\right)^4}$, $f_1(x) = 0$ and $f_2(x) = \frac{16c\,\mathrm{e}^{\lambda x} d^3 + 4\,\mathrm{e}^{4\lambda x} c^4 + 16\,\mathrm{e}^{3\lambda x} c^3 d + 24c^2 d^2 \mathrm{e}^{2\lambda x} + 4d^4}{4\left(c\,\mathrm{e}^{\lambda x}+d\right)^4}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{\left(16c\,\mathrm{e}^{\lambda x} d^3 + 4\,\mathrm{e}^{4\lambda x} c^4 + 16\,\mathrm{e}^{3\lambda x} c^3 d + 24c^2 d^2 \mathrm{e}^{2\lambda x} + 4d^4\right) u}{4\left(c\,\mathrm{e}^{\lambda x}+d\right)^4}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - \left(f_2' + f_1 f_2\right) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
f_2' = \frac{16\,\mathrm{e}^{4\lambda x} c^4 \lambda + 48\,\mathrm{e}^{3\lambda x} c^3 d\lambda + 48\,\mathrm{e}^{2\lambda x} c^2 d^2 \lambda + 16c\,\mathrm{e}^{\lambda x} d^3 \lambda}{4\left(c\,\mathrm{e}^{\lambda x}+d\right)^4} - \frac{\left(16c\,\mathrm{e}^{\lambda x} d^3 + 4\,\mathrm{e}^{4\lambda x} c^4 + 16\,\mathrm{e}^{3\lambda x} c^3 d + 24c^2 d\right.}{\left(c\,\mathrm{e}^{\lambda x}+d\right)^5}
$$

$$
f_1 f_2 = 0
$$

$$
f_2^2 f_0 = \frac{\left(16c\,\mathrm{e}^{\lambda x} d^3 + 4\,\mathrm{e}^{4\lambda x} c^4 + 16\,\mathrm{e}^{3\lambda x} c^3 d + 24c^2 d^2 \mathrm{e}^{2\lambda x} + 4d^4\right)^2 \left(-\mathrm{e}^{4\lambda x} c^4 \lambda^2 - 4\,\mathrm{e}^{3\lambda x} c^3 d\,\lambda^2 - 6\,\mathrm{e}^{2\lambda x} c^2 d^2 \lambda^2 -\right.}{64\left(c\,\mathrm{e}^{\lambda x}+d\right)^{12}}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{\left(16c\,\mathrm{e}^{\lambda x} d^3 + 4\,\mathrm{e}^{4\lambda x} c^4 + 16\,\mathrm{e}^{3\lambda x} c^3 d + 24c^2 d^2 \mathrm{e}^{2\lambda x} + 4d^4\right) u''(x)}{4\left(c\,\mathrm{e}^{\lambda x}+d\right)^4} - \left(\frac{16\,\mathrm{e}^{4\lambda x} c^4 \lambda + 48\,\mathrm{e}^{3\lambda x} c^3 d\lambda + 48\,\mathrm{e}^{2\lambda x} c^2 d^2 \lambda}{4\left(c\,\mathrm{e}^{\lambda x}+d\right)^4}\right.
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = \mathrm{DESol}\left(\left\{\frac{\mathrm{e}^{2\lambda x} f\left(\frac{a\,\mathrm{e}^{\lambda x}+b}{c\,\mathrm{e}^{\lambda x}+d}\right) \_Y(x) + 6\left(c^2 d^2 \mathrm{e}^{2\lambda x} + \frac{2\,\mathrm{e}^{3\lambda x} c^3 d}{3} + \frac{\mathrm{e}^{4\lambda x} c^4}{6} + \frac{2d^3\left(c\,\mathrm{e}^{\lambda x}+\frac{d}{4}\right)}{3}\right)\left(-\frac{\_Y(x)\lambda^2}{4} + \_Y''(x)\right.}{\mathrm{e}^{4\lambda x} c^4 + 4\,\mathrm{e}^{3\lambda x} c^3 d + 6c^2 d^2 \mathrm{e}^{2\lambda x} + 4c\,\mathrm{e}^{\lambda x} d^3 + d^4}\right.\right.
$$

The above shows that

$$
u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{\mathrm{e}^{2\lambda x} f\left(\frac{a\,\mathrm{e}^{\lambda x}+b}{c\,\mathrm{e}^{\lambda x}+d}\right) \_Y(x) + 6\left(c^2 d^2 \mathrm{e}^{2\lambda x} + \frac{2\,\mathrm{e}^{3\lambda x} c^3 d}{3} + \frac{\mathrm{e}^{4\lambda x} c^4}{6} + \frac{2d^3\left(c\,\mathrm{e}^{\lambda x}+\frac{d}{4}\right)}{3}\right)\left(-\frac{\_Y(x)\lambda^2}{4} + \_Y\right.}{\mathrm{e}^{4\lambda x} c^4 + 4\,\mathrm{e}^{3\lambda x} c^3 d + 6c^2 d^2 \mathrm{e}^{2\lambda x} + 4c\,\mathrm{e}^{\lambda x} d^3 + d^4}\right.\right.
$$

Using the above in (1) gives the solution

$$y = -\frac{4\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left(\left\{\frac{e^{2\lambda x} f\left(\frac{a\,e^{\lambda x}+b}{c\,e^{\lambda x}+d}\right)\_Y(x)+6\left(c^2d^2e^{2\lambda x}+\frac{2\,e^{3\lambda x}c^3d}{3}+\frac{e^{4\lambda x}c^4}{6}+\frac{2d^3\left(c\,e^{\lambda x}+\frac{d}{4}\right)}{3}\right)\left(-\frac{\_Y(x)\lambda^2}{4}\cdots\right.}{e^{4\lambda x}c^4+4\,e^{3\lambda x}c^3d+6c^2d^2e^{2\lambda x}+4c\,e^{\lambda x}d^3+d^4}\right.\right.\right.\cdots\right.}{\left(16c\,e^{\lambda x}d^3+4\,e^{4\lambda x}c^4+16\,e^{3\lambda x}c^3d+24c^2d^2e^{2\lambda x}+4d^4\right)\operatorname{DESol}\left(\left(\left\{\frac{e^{2\lambda x} f\left(\frac{a\,e^{\lambda x}+b}{c\,e^{\lambda x}+d}\right)\_Y(x)+6\left(c^2d^2e^{2\lambda x}+\frac{2\,e^{3\lambda x}c^3}{3}\cdots\right.}{e^{4\lambda x}c^4+4\,e^{3\lambda x}c^3d+\cdots}\right.\right.\right.\cdots}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1}=c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{e^{2\lambda x} f\left(\frac{a\,e^{\lambda x}+b}{c\,e^{\lambda x}+d}\right)\_Y(x)+6\left(c^2d^2e^{2\lambda x}+\frac{2\,e^{3\lambda x}c^3d}{3}+\frac{e^{4\lambda x}c^4}{6}+\frac{2d^3\left(c\,e^{\lambda x}+\frac{d}{4}\right)}{3}\right)\left(-\frac{\_Y(x)\lambda^2}{4}+\_Y''(x)\right)}{e^{4\lambda x}c^4+4\,e^{3\lambda x}c^3d+6c^2d^2e^{2\lambda x}+4c\,e^{\lambda x}d^3+d^4}\right\},\{\_Y(x\cdots\right.}{\operatorname{DESol}\left(\left\{\frac{e^{2\lambda x} f\left(\frac{a\,e^{\lambda x}+b}{c\,e^{\lambda x}+d}\right)\_Y(x)+6\left(c^2d^2e^{2\lambda x}+\frac{2\,e^{3\lambda x}c^3d}{3}+\frac{e^{4\lambda x}c^4}{6}+\frac{2d^3\left(c\,e^{\lambda x}+\frac{d}{4}\right)}{3}\right)\left(-\frac{\_Y(x)\lambda^2}{4}+\_Y''(x)\right)}{e^{4\lambda x}c^4+4\,e^{3\lambda x}c^3d+6c^2d^2e^{2\lambda x}+4c\,e^{\lambda x}d^3+d^4}\right\},\{\_Y(x)\cdots\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{e^{2\lambda x} f\left(\frac{a\,e^{\lambda x}+b}{c\,e^{\lambda x}+d}\right)\_Y(x)+6\left(c^2d^2e^{2\lambda x}+\frac{2\,e^{3\lambda x}c^3d}{3}+\frac{e^{4\lambda x}c^4}{6}+\frac{2d^3\left(c\,e^{\lambda x}+\frac{d}{4}\right)}{3}\right)\left(-\frac{\_Y(x)\lambda^2}{4}+\_Y''(x)\right)}{e^{4\lambda x}c^4+4\,e^{3\lambda x}c^3d+6c^2d^2e^{2\lambda x}+4c\,e^{\lambda x}d^3+d^4}\right\},\{\_Y(x\cdots\right.}{\operatorname{DESol}\left(\left\{\frac{e^{2\lambda x} f\left(\frac{a\,e^{\lambda x}+b}{c\,e^{\lambda x}+d}\right)\_Y(x)+6\left(c^2d^2e^{2\lambda x}+\frac{2\,e^{3\lambda x}c^3d}{3}+\frac{e^{4\lambda x}c^4}{6}+\frac{2d^3\left(c\,e^{\lambda x}+\frac{d}{4}\right)}{3}\right)\left(-\frac{\_Y(x)\lambda^2}{4}+\_Y''(x)\right)}{e^{4\lambda x}c^4+4\,e^{3\lambda x}c^3d+6c^2d^2e^{2\lambda x}+4c\,e^{\lambda x}d^3+d^4}\right\},\{\_Y(x)\cdots\right.} \quad (1)$$

Verification of solutions

$y =$

$$-\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\frac{e^{2\lambda x} f\left(\frac{a\,e^{\lambda x}+b}{c\,e^{\lambda x}+d}\right)\_Y(x)+6\left(c^2d^2e^{2\lambda x}+\frac{2\,e^{3\lambda x}c^3d}{3}+\frac{e^{4\lambda x}c^4}{6}+\frac{2d^3\left(c\,e^{\lambda x}+\frac{d}{4}\right)}{3}\right)\left(-\frac{\_Y(x)\lambda^2}{4}+\_Y''(x)\right)}{e^{4\lambda x}c^4+4\,e^{3\lambda x}c^3d+6c^2d^2e^{2\lambda x}+4c\,e^{\lambda x}d^3+d^4}\right\}, \{\_Y(x\right.}{\operatorname{DESol}\left(\left\{\frac{e^{2\lambda x} f\left(\frac{a\,e^{\lambda x}+b}{c\,e^{\lambda x}+d}\right)\_Y(x)+6\left(c^2d^2e^{2\lambda x}+\frac{2\,e^{3\lambda x}c^3d}{3}+\frac{e^{4\lambda x}c^4}{6}+\frac{2d^3\left(c\,e^{\lambda x}+\frac{d}{4}\right)}{3}\right)\left(-\frac{\_Y(x)\lambda^2}{4}+\_Y''(x)\right)}{e^{4\lambda x}c^4+4\,e^{3\lambda x}c^3d+6c^2d^2e^{2\lambda x}+4c\,e^{\lambda x}d^3+d^4}\right\}, \{\_Y(x)\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-2*lambda*K[1], y(x)`      *** Sublevel 2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+2*lambda*K[1], y(x)`      *** Sublevel 2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+4*y(x)*lambda*(2*f((exp(lambda*x)*a+b)/(e
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-2*lambda*d^4, y(x)`      *** Sublevel 2 *
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2-lambda^2/4+exp(2*lambda*x)/(c*exp(lambda*x)+d)^4*f((a*exp(lambda*
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2-\[Lambda]^2/4+Exp[2*\[Lambda]*x]/(c*Exp[\[Lambda]*x]+d)^4*f[(a*Exp[\[La
```

Not solved

21.9 problem 9

Internal problem ID [10644]
Internal file name [`OUTPUT/9591_Monday_June_06_2022_03_11_19_PM_91442332/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = -\lambda^2 + \frac{f(\coth(\lambda x))}{\sinh(\lambda x)^4}}$$

21.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-\lambda^2 \sinh(\lambda x)^4 + y^2 \sinh(\lambda x)^4 + f(\coth(\lambda x))}{\sinh(\lambda x)^4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - \lambda^2 + \frac{f(\coth(\lambda x))}{\sinh(\lambda x)^4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{-\lambda^2 \sinh(\lambda x)^4 + f(\coth(\lambda x))}{\sinh(\lambda x)^4}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = 0$$
$$f_1 f_2 = 0$$
$$f_2^2 f_0 = \frac{-\lambda^2 \sinh(\lambda x)^4 + f(\coth(\lambda x))}{\sinh(\lambda x)^4}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{\left(-\lambda^2 \sinh(\lambda x)^4 + f(\coth(\lambda x))\right) u(x)}{\sinh(\lambda x)^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \text{DESol}\left(\left\{f(\coth(\lambda x)) \operatorname{csch}(\lambda x)^4 \_Y(x) - \_Y(x)\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \text{DESol}\left(\left\{f(\coth(\lambda x)) \operatorname{csch}(\lambda x)^4 \_Y(x) - \_Y(x)\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \text{DESol}\left(\left\{f(\coth(\lambda x)) \operatorname{csch}(\lambda x)^4 \_Y(x) - \_Y(x)\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\text{DESol}\left(\left\{f(\coth(\lambda x)) \operatorname{csch}(\lambda x)^4 \_Y(x) - \_Y(x)\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \text{DESol}\left(\left\{f(\coth(\lambda x)) \operatorname{csch}(\lambda x)^4 \_Y(x) - \_Y(x)\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\text{DESol}\left(\left\{f(\coth(\lambda x)) \operatorname{csch}(\lambda x)^4 \_Y(x) - \_Y(x)\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x} \text{DESol}\left(\left\{f(\coth(\lambda x)) \operatorname{csch}(\lambda x)^4 \_Y(x) - \_Y(x)\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\text{DESol}\left(\left\{f(\coth(\lambda x)) \operatorname{csch}(\lambda x)^4 \_Y(x) - \_Y(x)\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{f(\coth(\lambda x))\,\mathrm{csch}(\lambda x)^4\,\_Y(x) - \_Y(x)\,\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{f(\coth(\lambda x))\,\mathrm{csch}(\lambda x)^4\,\_Y(x) - \_Y(x)\,\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+2*y(x)*x/((lambda-x)*(lambda+x)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x) = -2*y(x)*x/((lambda-x)*(lambda+x)), y(x)
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*lambda*(sinh(lambda*x)*(D(f))(coth(l
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2-lambda^2+sinh(lambda*x)^(-4)*f(coth(lambda*x)),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2-\[Lambda]^2+Sinh[\[Lambda]*x]^(-4)*f[Coth[\[Lambda]*x]],y[x],x,IncludeS
```

Not solved

21.10 problem 10

Internal problem ID [10645]
Internal file name [`OUTPUT/9592_Monday_June_06_2022_03_11_25_PM_41454593/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = -\lambda^2 + \frac{f(\tanh(\lambda x))}{\cosh(\lambda x)^4}}$$

21.10.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-\cosh(\lambda x)^4 \lambda^2 + \cosh(\lambda x)^4 y^2 + f(\tanh(\lambda x))}{\cosh(\lambda x)^4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - \lambda^2 + \frac{f(\tanh(\lambda x))}{\cosh(\lambda x)^4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{-\cosh(\lambda x)^4\lambda^2 + f(\tanh(\lambda x))}{\cosh(\lambda x)^4}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= \frac{-\cosh(\lambda x)^4\,\lambda^2 + f(\tanh(\lambda x))}{\cosh(\lambda x)^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{(-\cosh(\lambda x)^4\,\lambda^2 + f(\tanh(\lambda x)))\,u(x)}{\cosh(\lambda x)^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \text{DESol}\left(\left\{f(\tanh(\lambda x))\,\text{sech}(\lambda x)^4\,\_Y(x) - \_Y(x)\,\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\text{DESol}\left(\left\{f(\tanh(\lambda x))\,\text{sech}(\lambda x)^4\,\_Y(x) - \_Y(x)\,\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\text{DESol}\left(\left\{f(\tanh(\lambda x))\,\text{sech}(\lambda x)^4\,\_Y(x) - \_Y(x)\,\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\text{DESol}\left(\left\{f(\tanh(\lambda x))\,\text{sech}(\lambda x)^4\,\_Y(x) - \_Y(x)\,\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\text{DESol}\left(\left\{f(\tanh(\lambda x))\,\text{sech}(\lambda x)^4\,\_Y(x) - \_Y(x)\,\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\text{DESol}\left(\left\{f(\tanh(\lambda x))\,\text{sech}(\lambda x)^4\,\_Y(x) - \_Y(x)\,\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x}\text{DESol}\left(\left\{f(\tanh(\lambda x))\,\text{sech}(\lambda x)^4\,\_Y(x) - \_Y(x)\,\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\text{DESol}\left(\left\{f(\tanh(\lambda x))\,\text{sech}(\lambda x)^4\,\_Y(x) - \_Y(x)\,\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{f(\tanh(\lambda x)) \operatorname{sech}(\lambda x)^4 \_Y(x) - \_Y(x)\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{f(\tanh(\lambda x)) \operatorname{sech}(\lambda x)^4 \_Y(x) - \_Y(x)\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*lambda*(cosh(lambda*x)*(D(f))(tanh(l
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2-lambda^2+cosh(lambda*x)^(-4)*f(tanh(lambda*x)),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2-\[Lambda]^2+Cosh[\[Lambda]*x]^(-4)*f[Tanh[\[Lambda]*x]],y[x],x,IncludeS
```

Not solved

21.11 problem 11

Internal problem ID [10646]
Internal file name [`OUTPUT/9593_Monday_June_06_2022_03_11_31_PM_11209688/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{x^2y' - x^2y^2 = f(a\ln(x) + b) + \frac{1}{4}}$$

21.11.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{4x^2y^2 + 4f(a\ln(x) + b) + 1}{4x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \frac{f(a\ln(x) + b)}{x^2} + \frac{1}{4x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{4f(a\ln(x)+b)+1}{4x^2}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= \frac{4f(a\ln(x)+b)+1}{4x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{(4f(a\ln(x)+b)+1)\,u(x)}{4x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\frac{(4f(a\ln(x)+b)+1)\_Y(x)}{4x^2} + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{(4f(a\ln(x)+b)+1)\_Y(x)}{4x^2} + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{(4f(a\ln(x)+b)+1)\_Y(x)}{4x^2} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{(4f(a\ln(x)+b)+1)\_Y(x)}{4x^2} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\left\{\frac{4\_Y''(x)x^2+4\_Y(x)f(a\ln(x)+b)+\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{4\_Y''(x)x^2+4\_Y(x)f(a\ln(x)+b)+\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{4\_Y''(x)\,x^2+4\_Y(x)\,f(a\ln(x)+b)+\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{4\_Y''(x)\,x^2+4\_Y(x)\,f(a\ln(x)+b)+\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\frac{4\_Y''(x)\,x^2+4\_Y(x)\,f(a\ln(x)+b)+\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\frac{4\_Y''(x)\,x^2+4\_Y(x)\,f(a\ln(x)+b)+\_Y(x)}{4x^2}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(x^2*diff(y(x),x)=x^2*y(x)^2+f(a*ln(x)+b)+1/4,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*y'[x]==x^2*y[x]^2+f[a*Log[x]+b]+1/4,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

21.12 problem 12

Internal problem ID [10647]
Internal file name [`OUTPUT/9594_Monday_June_06_2022_03_11_32_PM_45494961/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = \lambda^2 + \frac{f(\cot(\lambda x))}{\sin(\lambda x)^4}}$$

21.12.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\lambda^2 \sin(\lambda x)^4 + y^2 \sin(\lambda x)^4 + f(\cot(\lambda x))}{\sin(\lambda x)^4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \lambda^2 + \frac{f(\cot(\lambda x))}{\sin(\lambda x)^4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\lambda^2 \sin(\lambda x)^4 + f(\cot(\lambda x))}{\sin(\lambda x)^4}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= \frac{\lambda^2 \sin(\lambda x)^4 + f(\cot(\lambda x))}{\sin(\lambda x)^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{(\lambda^2 \sin(\lambda x)^4 + f(\cot(\lambda x)))\, u(x)}{\sin(\lambda x)^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc(\lambda x)^4 f(\cot(\lambda x))\_Y(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc(\lambda x)^4 f(\cot(\lambda x))\_Y(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc(\lambda x)^4 f(\cot(\lambda x))\_Y(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc(\lambda x)^4 f(\cot(\lambda x))\_Y(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc(\lambda x)^4 f(\cot(\lambda x))\_Y(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc(\lambda x)^4 f(\cot(\lambda x))\_Y(x)\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc(\lambda x)^4 f(\cot(\lambda x))\_Y(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc(\lambda x)^4 f(\cot(\lambda x))\_Y(x)\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\_Y(x)\,\lambda^2+\_Y''(x)+\csc(\lambda x)^4 f(\cot(\lambda x))\,\_Y(x)\right\},\left\{\_Y(x)\right\}\right)}{\operatorname{DESol}\left(\left\{\_Y(x)\,\lambda^2+\_Y''(x)+\csc(\lambda x)^4 f(\cot(\lambda x))\,\_Y(x)\right\},\left\{\_Y(x)\right\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-2*y(x)*x/(lambda^2+x^2), y(x)`      *** S
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x) = 2*y(x)*x/(lambda^2+x^2), y(x)`      ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*lambda*((D(f))(cot(lambda*x))*cot(la
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2+lambda^2+sin(lambda*x)^(-4)*f(cot(lambda*x)),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+\[Lambda]^2+Sin[\[Lambda]*x]^(-4)*f[Cot[\[Lambda]*x]],y[x],x,IncludeSin
```

Not solved

21.13 problem 13

Internal problem ID [10648]
Internal file name [`OUTPUT/9595_Monday_June_06_2022_03_11_40_PM_48937354/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = \lambda^2 + \frac{f(\tan(\lambda x))}{\cos(\lambda x)^4}}$$

21.13.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{\cos(\lambda x)^4 \lambda^2 + \cos(\lambda x)^4 y^2 + f(\tan(\lambda x))}{\cos(\lambda x)^4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \lambda^2 + \frac{f(\tan(\lambda x))}{\cos(\lambda x)^4}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\cos(\lambda x)^4 \lambda^2 + f(\tan(\lambda x))}{\cos(\lambda x)^4}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{\cos(\lambda x)^4 \lambda^2 + f(\tan(\lambda x))}{\cos(\lambda x)^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{(\cos(\lambda x)^4 \lambda^2 + f(\tan(\lambda x)))\, u(x)}{\cos(\lambda x)^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{f(\tan(\lambda x)) \sec(\lambda x)^4 \_Y(x) + \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{f(\tan(\lambda x)) \sec(\lambda x)^4 \_Y(x) + \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{f(\tan(\lambda x)) \sec(\lambda x)^4 \_Y(x) + \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{f(\tan(\lambda x)) \sec(\lambda x)^4 \_Y(x) + \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{f(\tan(\lambda x)) \sec(\lambda x)^4 \_Y(x) + \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{f(\tan(\lambda x)) \sec(\lambda x)^4 \_Y(x) + \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{f(\tan(\lambda x)) \sec(\lambda x)^4 \_Y(x) + \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\left\{f(\tan(\lambda x)) \sec(\lambda x)^4 \_Y(x) + \_Y(x)\, \lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{f(\tan(\lambda x))\sec(\lambda x)^4 \_Y(x) + \_Y(x)\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{\operatorname{DESol}\left(\left\{f(\tan(\lambda x))\sec(\lambda x)^4 \_Y(x) + \_Y(x)\lambda^2 + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = 0, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*lambda*(cos(lambda*x)*(D(f))(tan(lam
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2+lambda^2+cos(lambda*x)^(-4)*f(tan(lambda*x)),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+\[Lambda]^2+Cos[\[Lambda]*x]^(-4)*f[Tan[\[Lambda]*x]],y[x],x,IncludeSin
```

Not solved

21.14 problem 14

Internal problem ID [10649]
Internal file name [`OUTPUT/9596_Monday_June_06_2022_03_11_50_PM_77761706/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.2. Riccati Equation. subsection 1.2.9. Some Transformations
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = \lambda^2 + \frac{f\left(\frac{\sin(\lambda x+a)}{\sin(\lambda x+b)}\right)}{\sin(\lambda x+b)^4}}$$

21.14.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\lambda^2 \sin(\lambda x+b)^4 + y^2 \sin(\lambda x+b)^4 + f\left(\frac{\sin(\lambda x+a)}{\sin(\lambda x+b)}\right)}{\sin(\lambda x+b)^4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{\lambda^2 \sin(\lambda x)^4 \cos(b)^4}{(\sin(\lambda x)\cos(b)+\cos(\lambda x)\sin(b))^4} + \frac{4\lambda^2 \sin(\lambda x)^3 \cos(b)^3 \cos(\lambda x)\sin(b)}{(\sin(\lambda x)\cos(b)+\cos(\lambda x)\sin(b))^4} + \frac{6\lambda^2 \sin(\lambda x)^2 \cos(b)^2 \cos(}{(\sin(\lambda x)\cos(b)+\cos(\lambda}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\lambda^2 \sin(\lambda x+b)^4 + f\left(\frac{\sin(\lambda x+a)}{\sin(\lambda x+b)}\right)}{\sin(\lambda x+b)^4}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{u} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= \frac{\lambda^2 \sin(\lambda x+b)^4 + f\left(\frac{\sin(\lambda x+a)}{\sin(\lambda x+b)}\right)}{\sin(\lambda x+b)^4}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
u''(x) + \frac{\left(\lambda^2 \sin(\lambda x+b)^4 + f\left(\frac{\sin(\lambda x+a)}{\sin(\lambda x+b)}\right)\right) u(x)}{\sin(\lambda x+b)^4} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
\begin{aligned}
u(x) = \mathrm{DESol}\Big(\{&\_Y(x)\,\lambda^2 + \_Y''(x) \\
&+ \csc(\lambda x+b)^4 f(\csc(\lambda x+b)\sin(\lambda x+a))\,\_Y(x)\}, \{\_Y(x)\}\Big)
\end{aligned}
$$

The above shows that

$$
\begin{aligned}
u'(x) = \frac{\partial}{\partial x}\mathrm{DESol}\Big(\{&\_Y(x)\,\lambda^2 + \_Y''(x) \\
&+ \csc(\lambda x+b)^4 f(\csc(\lambda x+b)\sin(\lambda x+a))\,\_Y(x)\}, \{\_Y(x)\}\Big)
\end{aligned}
$$

Using the above in (1) gives the solution

$$
y = -\frac{\frac{\partial}{\partial x}\mathrm{DESol}\left(\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc(\lambda x+b)^4 f(\csc(\lambda x+b)\sin(\lambda x+a))\,\_Y(x)\}, \{\_Y(x)\}\right)}{\mathrm{DESol}\left(\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc(\lambda x+b)^4 f(\csc(\lambda x+b)\sin(\lambda x+a))\,\_Y(x)\}, \{\_Y(x)\}\right)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc\left(\lambda x + b\right)^4 f\left(\csc\left(\lambda x + b\right)\sin\left(\lambda x + a\right)\right)\_Y(x)\right\}, \left\{\_Y(x)\right\}\right)}{\mathrm{DESol}\left(\left\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc\left(\lambda x + b\right)^4 f\left(\csc\left(\lambda x + b\right)\sin\left(\lambda x + a\right)\right)\_Y(x)\right\}, \left\{\_Y(x)\right\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc\left(\lambda x + b\right)^4 f\left(\csc\left(\lambda x + b\right)\sin\left(\lambda x + a\right)\right)\_Y(x)\right\}, \left\{\_Y(x)\right\}\right)}{\mathrm{DESol}\left(\left\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc\left(\lambda x + b\right)^4 f\left(\csc\left(\lambda x + b\right)\sin\left(\lambda x + a\right)\right)\_Y(x)\right\}, \left\{\_Y(x)\right\}\right)} \quad (1)$$

Verification of solutions

$$y = -\frac{\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc\left(\lambda x + b\right)^4 f\left(\csc\left(\lambda x + b\right)\sin\left(\lambda x + a\right)\right)\_Y(x)\right\}, \left\{\_Y(x)\right\}\right)}{\mathrm{DESol}\left(\left\{\_Y(x)\,\lambda^2 + \_Y''(x) + \csc\left(\lambda x + b\right)^4 f\left(\csc\left(\lambda x + b\right)\sin\left(\lambda x + a\right)\right)\_Y(x)\right\}, \left\{\_Y(x)\right\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*lambda*(sin(lambda*x+a)*cos(lambda*x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=y(x)^2+lambda^2+sin(lambda*x+b)^(-4)*f(sin(lambda*x+a)/sin(lambda*x+b)),
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==y[x]^2+\[Lambda]^2+Sin[\[Lambda]*x+b]^(-4)*f[Sin[\[Lambda]*x+a]/Sin[\[Lambda]*
```

Not solved

22 Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions

22.1 problem 1

Internal problem ID [10650]
Internal file name [`OUTPUT/9597_Monday_June_06_2022_03_13_08_PM_16682338/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{yy' - y = A}$$

22.1.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{y}{y+A} dy = x + c_1$$
$$y - A\ln(y + A) = x + c_1$$

Solving for y gives these solutions

$$y_1 = -A\left(\text{LambertW}\left(-\frac{e^{-\frac{A+c_1+x}{A}}}{A}\right) + 1\right)$$
$$= -A\left(\text{LambertW}\left(-\frac{e^{\frac{-A-x}{A}}}{c_1 A}\right) + 1\right)$$

Summary
The solution(s) found are the following

$$y = -A\left(\text{LambertW}\left(-\frac{e^{\frac{-A-x}{A}}}{c_1 A}\right) + 1\right) \tag{1}$$

Verification of solutions

$$y = -A\left(\mathrm{LambertW}\left(-\frac{e^{\frac{-A-x}{A}}}{c_1 A}\right)+1\right)$$

Verified OK.

22.1.2 Maple step by step solution

Let's solve

$yy' - y = A$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{y+A} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'y}{y+A}dx = \int 1dx + c_1$

- Evaluate integral

  $y - A\ln(y+A) = x + c_1$

- Solve for y

  $y = -A\left(\mathit{LambertW}\left(-\frac{e^{-\frac{A+c_1+x}{A}}}{A}\right)+1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 30

```
dsolve(y(x)*diff(y(x),x)-y(x)=A,y(x), singsol=all)
```

$$y(x) = -A\left(\operatorname{LambertW}\left(-\frac{e^{\frac{-A-c_1-x}{A}}}{A}\right)+1\right)$$

✓ Solution by Mathematica

Time used: 60.032 (sec). Leaf size: 28

```
DSolve[y[x]*y'[x]-y[x]==A,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -A\left(1+W\left(-\frac{e^{-\frac{A+x+c_1}{A}}}{A}\right)\right)$$

22.2 problem 2

Internal problem ID [10651]
Internal file name [`OUTPUT/9598_Monday_June_06_2022_03_13_09_PM_4007698/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{yy' - y = Ax + B}$$

22.2.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{Ax + B + y}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(Ax+B+y)(b_3-a_2)}{y} - \frac{(Ax+B+y)^2 a_3}{y^2} - \frac{A(xa_2+ya_3+a_1)}{y} - \left(\frac{1}{y} - \frac{Ax+B+y}{y^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{A^2x^2a_3 + 2ABxa_3 - A\,x^2b_2 + 2Axya_2 + 2Axya_3 - 2Axyb_3 + A\,y^2a_3 - Axb_1 + Aya_1 + B^2a_3 - Bxb_2 +}{y^2}$$
$$= 0$$

Setting the numerator to zero gives

$$-A^2x^2a_3 - 2ABxa_3 + A\,x^2b_2 - 2Axya_2 - 2Axya_3 + 2Axyb_3 - A\,y^2a_3 + Axb_1 - Aya_1 - B^2a_3 + Bxb_2 - Bya_2 - 2Bya_3 + 2Byb_3 - y^2a_2 - y^2a_3 + b_2y^2 + y^2b_3 + Bb_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-A^2a_3v_1^2 - 2ABa_3v_1 - 2Aa_2v_1v_2 - 2Aa_3v_1v_2 - Aa_3v_2^2 + Ab_2v_1^2 + 2Ab_3v_1v_2 - Aa_1v_2 + Ab_1v_1 - B^2a_3 - Ba_2v_2 - 2Ba_3v_2 + Bb_2v_1 + 2Bb_3v_2 - a_2v_2^2 - a_3v_2^2 + b_2v_2^2 + b_3v_2^2 + Bb_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-A^2a_3 + Ab_2)\, v_1^2 + (-2Aa_2 - 2Aa_3 + 2Ab_3)\, v_1v_2 \\
&+ (-2ABa_3 + Ab_1 + Bb_2)\, v_1 + (-Aa_3 - a_2 - a_3 + b_2 + b_3)\, v_2^2 \\
&+ (-Aa_1 - Ba_2 - 2Ba_3 + 2Bb_3)\, v_2 - B^2a_3 + Bb_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-A^2a_3 + Ab_2 &= 0 \\
-B^2a_3 + Bb_1 &= 0 \\
-2Aa_2 - 2Aa_3 + 2Ab_3 &= 0 \\
-2ABa_3 + Ab_1 + Bb_2 &= 0 \\
-Aa_1 - Ba_2 - 2Ba_3 + 2Bb_3 &= 0 \\
-Aa_3 - a_2 - a_3 + b_2 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= a_1 \\
a_2 &= \frac{Aa_1}{B} \\
a_3 &= a_3 \\
b_1 &= Ba_3 \\
b_2 &= Aa_3 \\
b_3 &= \frac{Aa_1 + Ba_3}{B}
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= \frac{Ax + B}{B} \\
\eta &= \frac{Ay}{B}
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= \frac{Ay}{B} - \left(\frac{Ax+B+y}{y}\right)\left(\frac{Ax+B}{B}\right) \\
&= \frac{-A^2x^2 - 2ABx - Axy + A\,y^2 - B^2 - By}{By} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-A^2x^2-2ABx-Axy+A\,y^2-B^2-By}{By}} dy
\end{aligned}
$$

Which results in

$$
S = B\left(\frac{\ln\left(A^2x^2 + 2ABx + Axy - A\,y^2 + B^2 + By\right)}{2A} + \frac{(Ax+B)\operatorname{arctanh}\left(\frac{Ax-2Ay+B}{\sqrt{4A^3x^2+8A^2Bx+A^2x^2+4A\,B^2+2AB}}\right.}{A\sqrt{4A^3x^2 + 8A^2Bx + A^2x^2 + 4A\,B^2 + 2ABx}}\right.
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{Ax + B + y}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{B(Ax + B + y)}{A^2x^2 + (2Bx + y(x-y))A + B(B+y)} \\
S_y &= -\frac{By}{A^2x^2 + (2Bx + y(x-y))A + B(B+y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{B\left(\ln\left(A^2x^2 + (2Bx + y(x-y))A + B(B+y)\right)\sqrt{4A+1} + 2\,\mathrm{arctanh}\left(\frac{(x-2y)A+B}{\sqrt{4A+1}\,(Ax+B)}\right)\right)}{2\sqrt{4A+1}\,A} = c_1$$

Which simplifies to

$$\frac{B\left(\ln\left(A^2x^2 + (2Bx + y(x-y))A + B(B+y)\right)\sqrt{4A+1} + 2\,\mathrm{arctanh}\left(\frac{(x-2y)A+B}{\sqrt{4A+1}\,(Ax+B)}\right)\right)}{2\sqrt{4A+1}\,A} = c_1$$

Summary

The solution(s) found are the following

$$
\frac{B\left(\ln\left(A^2x^2+(2Bx+y(x-y))A+B(B+y)\right)\sqrt{4A+1}+2\operatorname{arctanh}\left(\frac{(x-2y)A+B}{\sqrt{4A+1}\,(Ax+B)}\right)\right)}{2\sqrt{4A+1}\,A} = c_1 \tag{1}
$$

Verification of solutions

$$
\frac{B\left(\ln\left(A^2x^2+(2Bx+y(x-y))A+B(B+y)\right)\sqrt{4A+1}+2\operatorname{arctanh}\left(\frac{(x-2y)A+B}{\sqrt{4A+1}\,(Ax+B)}\right)\right)}{2\sqrt{4A+1}\,A} = c_1
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 4.266 (sec). Leaf size: 68

```
dsolve(y(x)*diff(y(x),x)-y(x)=A*x+B,y(x), singsol=all)
```

$$
y(x) = -\frac{(xA+B)\operatorname{RootOf}\left(\_Z^2-A+\_Z+\mathrm{e}^{\operatorname{RootOf}\left((xA+B)^2\left(-2\,\mathrm{e}^{-\_Z}\cosh\left((\_Z+2\ln(xA+B)+2c_1)\sqrt{4A+1}\right)+4A-2\,\mathrm{e}^{-\_Z}+1\right)\right)}\right)}{A}
$$

✓ Solution by Mathematica

Time used: 0.184 (sec). Leaf size: 88

```
DSolve[y[x]*y'[x]-y[x]==A*x+B,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{\frac{2\arctan\left(\frac{\frac{2Ay(x)}{Ax+B}-1}{\sqrt{-4A-1}}\right)}{\sqrt{-4A-1}}+\log\left(-\frac{Ay(x)^2}{(Ax+B)^2}+\frac{y(x)}{Ax+B}+1\right)}{2A}=\frac{\log(Ax+B)}{A}+c_1,y(x)\right]$$

22.3 problem 3

Internal problem ID [10652]
Internal file name [`OUTPUT/9599_Monday_June_06_2022_03_13_14_PM_3277166/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = -\frac{2x}{9} + A + \frac{B}{\sqrt{x}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 90

```
dsolve(y(x)*diff(y(x),x)-y(x)=-2/9*x+A+B*x^(-1/2),y(x), singsol=all)
```

$$y(x) = -\frac{9\left(A\sqrt{x}+B-\frac{2x^{\frac{3}{2}}}{9}\right)A}{3A\sqrt{x}+3\,\mathrm{RootOf}\left(18A^3\left(\int^{\_Z}\frac{1}{-2\_a^3B^2+9\_aA^3-9A^3}d\_a\right)-9A\left(\int\frac{1}{9xA-2x^2+9B\sqrt{x}}dx\right)+2c_1\right)B}$$

✓ Solution by Mathematica

Time used: 8.154 (sec). Leaf size: 415

```
DSolve[y[x]*y'[x]-y[x]==-2/9*x+A+B*x^(-1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\Bigg[6\mathrm{RootSum}\Bigg[8\#1^6-72\#1^4A-36\#1^4y(x)-72\#1^3B+162\#1^2A^2$$
$$+162\#1^2Ay(x)+54\#1^2y(x)^2+324\#1AB+162\#1By(x)-81Ay(x)^2+162B^2$$
$$-27y(x)^3\&,\frac{-2\#1^3\log\left(\sqrt{x}-\#1\right)+9\#1A\log\left(\sqrt{x}-\#1\right)+9B\log\left(\sqrt{x}-\#1\right)+9\#1y(x)\log\left(\sqrt{x}-\right.}{8\#1^5-48\#1^3A-24\#1^3y(x)-36\#1^2B+54\#1A^2+54\#1Ay(x)+18\#1y(x)^2+54AB+}$$
$$+\int_1^{y(x)}\left(\frac{162K[1]}{8x^3-72Ax^2-36K[1]x^2-72Bx^{3/2}+162A^2x+54K[1]^2x+162AK[1]x+324AB\sqrt{x}+162BK}\right.$$
$$+\frac{162K[1]}{-8x^3+72Ax^2+36K[1]x^2+72Bx^{3/2}-162A^2x-54K[1]^2x-162AK[1]x-324AB\sqrt{x}-162BK[1]\sqrt{}}$$

22.4 problem 4

Internal problem ID [10653]
Internal file name [`OUTPUT/9600_Monday_June_06_2022_03_13_15_PM_16949097/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = 2A\left(\sqrt{x} + 4A + \frac{3A^2}{\sqrt{x}}\right)}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 120

```
dsolve(y(x)*diff(y(x),x)-y(x)=2*A*(x^(1/2)+4*A+3*A^2*x^(-1/2)),y(x), singsol=all)
```

$$\frac{-\sqrt{\frac{-6A^2-8A\sqrt{x}-2x+2y(x)}{y(x)}}\,\sqrt{2}+4\sqrt{-\frac{A^2}{y(x)}}\,\operatorname{arctanh}\left(\frac{\sqrt{-\frac{A^2}{y(x)}}\,(3A+\sqrt{x})}{\sqrt{\frac{-3A^2-4A\sqrt{x}-x+y(x)}{y(x)}}\,A}\right)+c_1\sqrt{-\frac{A^2}{y(x)}}}{\sqrt{-\frac{A^2}{y(x)}}}$$
$$= 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==2*A*(x^(1/2)+4*A+3*A^2*x^(-1/2)),y[x],x,IncludeSingularSolutions ->
```

Not solved

22.5 problem 5

Internal problem ID [10654]
Internal file name [`OUTPUT/9601_Monday_June_06_2022_03_13_16_PM_16781380/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = Ax + \frac{B}{x} - \frac{B^2}{x^3}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 179

```
dsolve(y(x)*diff(y(x),x)-y(x)=A*x+B/x-B^2*x^(-3),y(x), singsol=all)
```

$$\frac{\left(-y(x)\,x^2B-B^2x\right)\left(\int^{-\frac{x^2}{2y(x)x+2B}}\frac{e^{\frac{2\,\mathrm{arctanh}\left(\frac{4A\_a-1}{\sqrt{4A+1}}\right)}{\sqrt{4A+1}}}\left(4A\_a^2-2\_a-1\right)}{\_a^2}d\_a\right)+2y(x)\left(-y(x)^2x^2+\left(x^3-2Bx\right)\right.}{x\left(y\left(x\right)x+B\right)}$$
$$=0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==A*x+B/x-B^2*x^(-3),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.6 problem 6

Internal problem ID [10655]
Internal file name [`OUTPUT/9602_Monday_June_06_2022_03_13_18_PM_90018615/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = A\,x^{k-1} - kB\,x^k + k\,B^2 x^{2k-1}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(2*k^2*B^2*x^(-1+2*k)-k*B^2*x^(-1+2*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = 0, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x) = -y(x)/x, y(x)`      *** Sublevel 2 ***
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=A*x^(k-1)-k*B*x^k+k*B^2*x^(2*k-1),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==A*x^(k-1)-k*B*x^k+k*B^2*x^(2*k-1),y[x],x,IncludeSingularSolutions ->
```

Not solved

22.7 problem 7

Internal problem ID [10656]
Internal file name [OUTPUT/9603_Monday_June_06_2022_03_13_21_PM_1762640/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = \frac{A}{x} - \frac{A^2}{x^3}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 118

```
dsolve(y(x)*diff(y(x),x)-y(x)=A*x^(-1)-A^2*x^(-3),y(x), singsol=all)
```

$$y(x) = -\frac{\left(-c_1 x^2 + A\,\mathrm{e}^{\mathrm{RootOf}\left(2\_Z A\,\mathrm{e}^{2\_Z} - x^2 \mathrm{e}^{2\_Z} + 2 c_1 x^2 \mathrm{e}^{\_Z} - c_1^2 x^2 - 2A\,\mathrm{e}^{2\_Z} + 2 A c_1 \mathrm{e}^{\_Z}\right)}\right) \mathrm{e}^{-\mathrm{RootOf}\left(2\_Z A\,\mathrm{e}^{2\_Z} - x^2 \mathrm{e}^{2\_Z} + 2 c_1 x^2 \mathrm{e}^{\_Z} - c_1^2}}{x}$$

✓ Solution by Mathematica

Time used: 0.569 (sec). Leaf size: 63

```
DSolve[y[x]*y'[x]-y[x]==A*x^(-1)-A^2*x^(-3),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x^2\left(-\frac{1}{A} + \frac{2x^2 \log\left(\frac{x^2}{A+xy(x)}\right) + 2A - c_1 x^2 + 2xy(x)}{(A - x^2 + xy(x))^2}\right) = 0, y(x)\right]$$

22.8 problem 8

Internal problem ID [10657]
Internal file name [`OUTPUT/9604_Monday_June_06_2022_03_13_22_PM_9151881/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_Abel, `2nd type`, `class A`]]`

Unable to solve or complete the solution.

$$\boxed{yy' - y = A + B\,\mathrm{e}^{-\frac{2x}{A}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 76

```
dsolve(y(x)*diff(y(x),x)-y(x)=A+B*exp(-2*x/A),y(x), singsol=all)
```

$$c_1 - 2\arctan\left(\frac{y(x)+A}{y\left(x\right)\sqrt{\frac{-AB\,\mathrm{e}^{-\frac{2x}{A}}-\left(y(x)+A\right)^2}{y(x)^2}}}\right)A - 2\sqrt{\frac{-AB\,\mathrm{e}^{-\frac{2x}{A}}-\left(y\left(x\right)+A\right)^2}{y\left(x\right)^2}}\,y(x) = 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==A+B*Exp[-2*x/A],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.9 problem 9

Internal problem ID [10658]
Internal file name [`OUTPUT/9605_Monday_June_06_2022_03_13_23_PM_32679015/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = A\left(e^{\frac{2x}{A}} - 1\right)}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 82

```
dsolve(y(x)*diff(y(x),x)-y(x)=A*(exp(2*x/A)-1),y(x), singsol=all)
```

$$c_1 + 2A \arctan\left(\frac{A - y(x)}{y(x)\sqrt{\frac{e^{\frac{2x}{A}}A^2-(A-y(x))^2}{y(x)^2}}}\right) + 2y(x)\sqrt{\frac{e^{\frac{2x}{A}}A^2 - (A - y(x))^2}{y(x)^2}} = 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==A*(Exp[2*x/A]-1),y[x],x,IncludeSingularSolutions -> True]
```

$$\{\}$$

22.10 problem 10

Internal problem ID [10659]
Internal file name [`OUTPUT/9606_Monday_June_06_2022_03_13_24_PM_76864051/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = -\frac{2(m+1)}{(m+3)^2} + A\,x^m}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*A*x^m*m*(m+3)^2/(x*(A*x^m*m^2+6*A*x^
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = 2*(1+m)*y(x)/((m^2*x+6*m*x-2*m+9*x-2)*x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x) = -y(x)*A*x^m*m*(m+3)^2/(x*(A*x^m*m^2+6*A
      Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-(2*(m+1))/(m+3)^2+A*x^m,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-(2*(m+1))/(m+3)^2+A*x^m,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.11 problem 11

Internal problem ID [10660]
Internal file name [`OUTPUT/9607_Monday_June_06_2022_03_13_26_PM_78895175/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{2x}{9} + 6A^2\left(1 + \frac{2A}{\sqrt{x}}\right)$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 354

```
dsolve(y(x)*diff(y(x),x)-y(x)=-2/9*x+6*A^2*(1+2*A*x^(-1/2)),y(x), singsol=all)
```

$$y(x) = \frac{\cdots}{3\,\mathrm{e}^{\mathrm{RootOf}\left(36A^2\mathrm{e}^{\_Z}\ln(2)+18A^2\mathrm{e}^{\_Z}\ln\left(\frac{(3A-\sqrt{x})(6A-\sqrt{x})(36A^2-x)}{(9A^2-x)(6A+\sqrt{x})(3A+\sqrt{x})(\mathrm{e}^{\_Z}+9)^2}\right)+108A^2c_1\mathrm{e}^{\_Z}+36A^2\mathrm{e}^{\_Z}\_Z+6A\sqrt{x}\,\mathrm{e}^{\_Z}\ln(2)+3A\sqrt{x}\,\mathrm{e}^{\_Z}\ln\left(\cdots\right.\right.}}$$

✓ Solution by Mathematica

Time used: 12.331 (sec). Leaf size: 488

```
DSolve[y[x]*y'[x]-y[x]==-2/9*x+6*A^2*(1+2*A*x^(-1/2)),y[x],x,IncludeSingularSolutions -> Tru
```

$$\mathrm{Solve}\left[\frac{2^{2/3}\left(\frac{-\frac{6(6A-\sqrt{x})(3A+\sqrt{x})^2}{y(x)}-9\sqrt{x}}{\sqrt[3]{A^3}}+54\right)\left(\frac{6(6A-\sqrt{x})(3A+\sqrt{x})^2+9\sqrt{x}y(x)}{\sqrt[3]{A^3}y(x)}+27\right)\left(-\frac{\left(3\left(3\sqrt[3]{A^3}+\sqrt{x}\right)y(x)+2(6A-\sqrt{x}\cdots\right.}{\cdots}\right.}{6561\left(\frac{\left(2(6A-\cdots\right.}{\cdots}\right.}\right.$$

$$\left.+c_1, y(x)\right]$$

22.12 problem 12

Internal problem ID [10661]
Internal file name [`OUTPUT/9608_Monday_June_06_2022_03_13_31_PM_8547601/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = \frac{2m-2}{(m-3)^2} + \frac{2A\left(m(m+3)\sqrt{x} + (4m^2+3m+9)A + \frac{3m(m+3)A^2}{\sqrt{x}}\right)}{(m-3)^2}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/2)*A*y(x)*m*(3*A^2-x)*(m+3)/(4*A^2*x^(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/12)*(12*A^3*y(x)*m^2+36*A^3*y(x)*m-8*A
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=2*(m-1)/(m-3)^2+2*A/(m-3)^2*(m*(m+3)*x^(1/2)+(4*m^2+3*m+9)*A+3
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==2*(m-1)/(m-3)^2+2*A/(m-3)^2*(m*(m+3)*x^(1/2)+(4*m^2+3*m+9)*A+3*m*(m+
```

Not solved

22.13 problem 13

Internal problem ID [10662]
Internal file name [`OUTPUT/9609_Monday_June_06_2022_03_13_33_PM_8713158/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = \frac{(2m+1)\,x}{4m^2} + \frac{A}{x} - \frac{A^2}{x^3}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 166

```
dsolve(y(x)*diff(y(x),x)-y(x)=(2*m+1)/(4*m^2)*x+A*1/x-A^2*1/(x^3),y(x), singsol=all)
```

$$\frac{2^{-\frac{m}{1+m}} y(x) \left(\frac{-2y(x)mx-2Am-x^2}{2y(x)x+2A}\right)^{\frac{1}{1+m}} (y(x)\,x+A) \left(\frac{(-1-2m)x^2+2y(x)mx+2Am}{y(x)x+A}\right)^{\frac{1+2m}{1+m}} - x\left(A\left(\int^{-\frac{x^2}{2y(x)x+2A}} \frac{(-m+\_\,}{}\right.\right.}{x}$$

$= 0$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==(2*m+1)/(4*m^2)*x+A*1/x-A^2*1/(x^3),y[x],x,IncludeSingularSolutions
```

Not solved

22.14 problem 14

Internal problem ID [10663]
Internal file name [`OUTPUT/9610_Monday_June_06_2022_03_13_35_PM_29052279/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - y = \frac{4}{9}x + 2A\,x^2 + 2A^2x^3$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 177

```
dsolve(y(x)*diff(y(x),x)-y(x)=4/9*x+2*A*x^2+2*A^2*x^3,y(x), singsol=all)
```

$$-\frac{27\left(Ay(x)\,3^{\frac{1}{4}}\left(xA+\frac{1}{3}\right)\left(\frac{A(9A^2x^2-9y(x)A+9xA+2)(3A\,x^2+3y(x)+x)}{(-9y(x)A+3xA+1)^2}\right)^{\frac{1}{4}}-\frac{\left(\int^{\frac{(3xA+1)^2}{-9y(x)A+3xA+1}}\frac{(\_a^2-1)^{\frac{1}{4}}}{\sqrt{\_a}}d\_a+c_1\right)\left(\frac{1}{3}-3y(\right.}{9}\right.}{\sqrt{\frac{(3xA+1)^2}{-9y(x)A+3xA+1}}\,(-9y(x)\,A+3xA+1)}$$

$= 0$

✓ Solution by Mathematica

Time used: 3.439 (sec). Leaf size: 170

```
DSolve[y[x]*y'[x]-y[x]==4/9*x+2*A*x^2+2*A^2*x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\sqrt[4]{\frac{(-9Ay(x)+3Ax+1)^2}{(3Ax+1)^4}-1}\left(\frac{(-9Ay(x)+3Ax+1)\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{3}{4},\frac{3}{2},\frac{(3Ax}{}\right.}{2\sqrt[4]{3}(3Ax+1)\sqrt{(3Ax+1)^2}\sqrt[4]{\frac{A\,(6(3Ax+1)y(x)-27Ay(x)^2+x}{(3Ax+1)^4}}}\right.\right.$$

$$\left.\left.+\sqrt{(3Ax+1)^2}\right)+c_1=0,y(x)\right]$$

22.15 problem 15

Internal problem ID [10664]
Internal file name [`OUTPUT/9611_Monday_June_06_2022_03_13_36_PM_76543715/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{3x}{16} + \frac{5A}{x^{\frac{1}{3}}} - \frac{12A^2}{x^{\frac{5}{3}}}$$

Unable to determine ODE type.

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-3/16*x+5*A*x^(-1/3)-12*A^2*x^(-5/3),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-3/16*x+5*A*x^(-1/3)-12*A^2*x^(-5/3),y[x],x,IncludeSingularSolutions
```

Not solved

22.16 problem 16

Internal problem ID [10665]
Internal file name [`OUTPUT/9612_Monday_June_06_2022_03_15_48_PM_62995207/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = \frac{A}{x}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 57

```
dsolve(y(x)*diff(y(x),x)-y(x)=A*1/x,y(x), singsol=all)
```

$$\frac{\operatorname{erf}\left(\frac{(y(x)-x)\sqrt{2}}{2\sqrt{-A}}\right)\sqrt{2}\,\sqrt{\pi}\,x-2\,\mathrm{e}^{\frac{(y(x)-x)^2}{2A}}\sqrt{-A}+c_1x}{x}=0$$

✓ Solution by Mathematica

Time used: 0.827 (sec). Leaf size: 64

```
DSolve[y[x]*y'[x]-y[x]==A*1/x,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{x}{\sqrt{A}}=\frac{2e^{\frac{(x-y(x))^2}{2A}}}{\sqrt{2\pi}\text{erfi}\left(\frac{y(x)-x}{\sqrt{2}\sqrt{A}}\right)+2c_1},y(x)\right]$$

22.17 problem 17

Internal problem ID [10666]
Internal file name [`OUTPUT/9613_Monday_June_06_2022_03_15_49_PM_99424666/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational, [_Abel, `2nd type`, `class B`]]`

Unable to solve or complete the solution.

$$yy' - y = -\frac{x}{4} + \frac{A\left(\sqrt{x} + 5A + \frac{3A^2}{\sqrt{x}}\right)}{4}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   found: 2 potential symmetries. Proceeding with integration step
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 257

```
dsolve(y(x)*diff(y(x),x)-y(x)=-1/4*x+1/4*A*(x^(1/2)+5*A+3*A^2*x^(-1/2)),y(x), singsol=all)
```

$$-\frac{3\left(-4\left(\left(\int^{\frac{6A\sqrt{x}-2x+3y(x)}{12A^2-4A\sqrt{x}+2y(x)}}\frac{\mathrm{e}^{-\frac{2}{2\_a+1}}\sqrt{2\_a+1}}{\sqrt{2\_a-3}}d\_a\right)A+\frac{c_1}{2}\right)\left(A^2-\frac{A\sqrt{x}}{3}+\frac{y(x)}{6}\right)\sqrt{-\frac{(3A-\sqrt{x})^2}{6A^2-2A\sqrt{x}+y(x)}}+\frac{\sqrt{\frac{3A^2+2A\ldots}{6A^2-\ldots}}}{\ldots}\ldots\right)}{\sqrt{-\frac{(3A-\sqrt{x})^2}{6A^2-2A\sqrt{x}+y(x)}}\left(6A^2-2A\sqrt{x}+y(x)\right)}$$
$$=0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-1/4*x+1/4*A*(x^(1/2)+5*A+3*A^2*x^(-1/2)),y[x],x,IncludeSingularSolu
```

Not solved

22.18 problem 18

Internal problem ID [10667]
Internal file name [`OUTPUT/9614_Monday_June_06_2022_03_15_50_PM_41390442/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = \frac{2a^2}{\sqrt{8a^2 + x^2}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 715

```
dsolve(y(x)*diff(y(x),x)-y(x)=2*a^2/sqrt(x^2+8*a^2),y(x), singsol=all)
```

$$512\left(-\frac{33\left(a^4+\frac{23}{66}a^2x^2+\frac{1}{66}x^4\right)x\sqrt{8a^2+x^2}}{64}+a^6+\frac{75a^4x^2}{64}+\frac{27a^2x^4}{128}+\frac{x^6}{128}\right)\mathrm{e}^{-\frac{(-y(x)+x)^2\left(-64\sqrt{8a^2+x^2}\,a^6-108\sqrt{8a^2+x^2}\,a^4x^2-25\sqrt{8a^2+\right.}{2\left(128a^6+150a^4x^2-66\sqrt{8a^2+x^2}\,a^4x+27a^2x^4-23\sqrt{8a^2+x^2\right.}}$$

$$= 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==2*a^2/Sqrt[x^2+8*a^2],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.19 problem 19

Internal problem ID [10668]
Internal file name [`OUTPUT/9615_Monday_June_06_2022_03_15_54_PM_62818123/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational, [_Abel, `2nd type`, `class B`]]`

Unable to solve or complete the solution.

$$\boxed{yy' - y = 2x + \frac{A}{x^2}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 170

```
dsolve(y(x)*diff(y(x),x)-y(x)=2*x+A*x^(-2),y(x), singsol=all)
```

$$\frac{6\left(\sqrt{3}\,\operatorname{arctanh}\left(\frac{\sqrt{\frac{x\left(A^2\right)^{\frac{1}{3}}}{A}}\,(-2x+y(x))}{\sqrt{\frac{\left(4x^3-4y(x)x^2+y(x)^2x+2A\right)\left(A^2\right)^{\frac{1}{3}}}{y(x)^2A}}\,y(x)}\right)A+\frac{c_1}{6}\right)x\sqrt{\frac{x(A^2)^{\frac{1}{3}}}{A}}+2\sqrt{3}\,y(x)\left(-x^3-\frac{y(x)x^2}{2}+\frac{y(x)^2x}{2}-\right.}{\sqrt{\frac{x(A^2)^{\frac{1}{3}}}{A}}\,x}$$

$= 0$

✓ Solution by Mathematica

Time used: 2.08 (sec). Leaf size: 233

```
DSolve[y[x]*y'[x]-y[x]==2*x+A*x^(-2),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[c_1=\right.$$

$$-\frac{i\sqrt{-\frac{2A+4x^3-4x^2y(x)+xy(x)^2}{A}}\left(-6\sqrt{A}x^{3/2}\text{arcsinh}\left(\frac{\sqrt{x}(2x-y(x))}{\sqrt{2}\sqrt{A}}\right)+x^2(-y(x))\sqrt{\frac{2A+4x^3-4x^2y(x)+xy(x)^2}{A}}+xy(x)^2\right.}{4\sqrt{A}x^{3/2}\sqrt{\frac{2A+4x^3-4x^2y(x)+xy(x)^2}{A}}}$$

22.20 problem 20

Internal problem ID [10669]
Internal file name [`OUTPUT/9616_Monday_June_06_2022_03_15_55_PM_70549492/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{6X}{25} + \frac{2A\left(2\sqrt{x} + 19A + \frac{6A^2}{\sqrt{x}}\right)}{25}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-A*y(x)*(3*A^2-x)/(19*A^2*x^(3/2)+6*A^3*x-
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/24)*(24*A^3*y(x)-38*A^2*x+6*X*x-25*x^2
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-6/25*X+2/25*A*(2*x^(1/2)+19*A+6*A^2*x^(-1/2)),y(x), singsol=a
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-6/25*X+2/25*A*(2*x^(1/2)+19*A+6*A^2*x^(-1/2)),y[x],x,IncludeSingula
```

Not solved

22.21 problem 21

Internal problem ID [10670]
Internal file name [OUTPUT/9617_Monday_June_06_2022_03_16_03_PM_37437657/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = \frac{3x}{8} + \frac{3\sqrt{a^2 + x^2}}{8} - \frac{a^2}{16\sqrt{a^2 + x^2}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(6*(a^2+x^2)^(1/2)*a^2+6*(a^2+x^2)^(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(y(x)-2*x)/x, y(x)`      *** Sublevel 2 *
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/5)*(5*y(x)-6*x)/x, y(x)`      *** Subl
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=3/8*x+3/8*sqrt(x^2+a^2)-a^2/(16*sqrt(x^2+a^2)),y(x), singsol=a
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==3/8*x+3/8*Sqrt[x^2+a^2]-a^2/(16*Sqrt[x^2+a^2]),y[x],x,IncludeSingula
```

Not solved

22.22 problem 22

Internal problem ID [10671]
Internal file name [`OUTPUT/9618_Monday_June_06_2022_03_16_06_PM_10022587/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = -\frac{4x}{25} + \frac{A}{\sqrt{x}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 270

```
dsolve(y(x)*diff(y(x),x)-y(x)=-4/25*x+A*x^(-1/2),y(x), singsol=all)
```

$$\frac{625\sqrt{A\,x^{\frac{3}{2}}}\,c_1\left(-\frac{Ay(x)^2\sqrt{x}}{2}+\frac{16x^4}{625}-\frac{16x^3y(x)}{125}+\frac{6y(x)^2x^2}{25}-\frac{xy(x)^3}{5}+\frac{y(x)^4}{16}+A^2x+\frac{4Ay(x)x^{\frac{3}{2}}}{5}-\frac{8A\,x^{\frac{5}{2}}}{25}\right)\sqrt{\frac{A\sqrt{x}-\frac{4(}{}}{\sqrt{}}}}{}$$

$= 0$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-4/25*x+A*x^(-1/2),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.23 problem 23

Internal problem ID [10672]
Internal file name [`OUTPUT/9619_Monday_June_06_2022_03_16_07_PM_474942/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational, [_Abel, `2nd type`, `class B`]]`

Unable to solve or complete the solution.

$$\boxed{yy' - y = -\frac{9x}{100} + \frac{A}{x^{\frac{5}{3}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 581

```
dsolve(y(x)*diff(y(x),x)-y(x)=-9/100*x+A*x^(-5/3),y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 60.566 (sec). Leaf size: 7909

```
DSolve[y[x]*y'[x]-y[x]==-9/100*x+A*x^(-5/3),y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

22.24 problem 24

Internal problem ID [10673]
Internal file name [`OUTPUT/9620_Monday_June_06_2022_03_16_08_PM_55707895/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{12x}{49} + \frac{2A\left(5\sqrt{x} + 34A + \frac{15A^2}{\sqrt{x}}\right)}{49}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   found: 2 potential symmetries. Proceeding with integration step
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 270

```
dsolve(y(x)*diff(y(x),x)-y(x)=-12/49*x+2/49*A*(5*x^(1/2)+34*A+15*A^2*x^(-1/2)),y(x), singsol
```

$$\frac{\left(3A-\sqrt{x}\right)\left(36A^4+120A^3\sqrt{x}-80A\,x^{\frac{3}{2}}+52A^2x+84A^2y(x)+140A\sqrt{x}\,y(x)+16x^2-56y(x)\,x+49y(\right.}{8\sqrt{-\frac{\left(3A-\sqrt{x}\right)^2}{6A^2-2A\sqrt{x}+y(x)}}\left(\frac{15A^2+4A\sqrt{x}-3x+7y(x)}{6A^2-2A\sqrt{x}+y(x)}\right)^{\frac{3}{2}}\left(6A^2-2A\sqrt{x}+y\left(x\right)\right)^3A}$$

$$+\frac{\left(-54A^2-6A\sqrt{x}+8x-21y(x)\right)\sqrt{-\frac{\left(3A-\sqrt{x}\right)^2}{6A^2-2A\sqrt{x}+y(x)}}}{\sqrt{\frac{15A^2+4A\sqrt{x}-3x+7y(x)}{6A^2-2A\sqrt{x}+y(x)}}\left(36A^2-12A\sqrt{x}+6y\left(x\right)\right)}+c_1=0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-12/49*x+2/49*A*(5*x^(1/2)+34*A+15*A^2*x^(-1/2)),y[x],x,IncludeSingu
```

Not solved

22.25 problem 25

Internal problem ID [10674]
Internal file name [`OUTPUT/9621_Monday_June_06_2022_03_16_10_PM_91589618/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = -\frac{12x}{49} + \frac{A\left(25\sqrt{x} + 41A + \frac{10A^2}{\sqrt{x}}\right)}{98}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 1531

```
dsolve(y(x)*diff(y(x),x)-y(x)=-12/49*x+1/98*A*(25*x^(1/2)+41*A+10*A^2*x^(-1/2)),y(x), singso
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-12/49*x+1/98*A*(25*x^(1/2)+41*A+10*A^2*x^(-1/2)),y[x],x,IncludeSing
```

Not solved

22.26 problem 26

Internal problem ID [10675]
Internal file name [OUTPUT/9622_Monday_June_06_2022_03_16_12_PM_56059788/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{2x}{9} + \frac{A}{\sqrt{x}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 134

```
dsolve(y(x)*diff(y(x),x)-y(x)=-2/9*x+A*x^(-1/2),y(x), singsol=all)
```

$$y(x) = \frac{2\,3^{\frac{5}{6}}2^{\frac{1}{3}}\left(-2x^{\frac{3}{2}}\right.}{\sqrt{x}\left(\left(\left(27\tan\left(\mathrm{RootOf}\left(18\,3^{\frac{5}{6}}2^{\frac{1}{3}}\left(\int\frac{\left(\frac{A}{x^{\frac{3}{2}}}\right)^{\frac{2}{3}}\sqrt{x}}{-2x^{\frac{3}{2}}+9A}dx\right)+\ln\left(-8\sqrt{3}\,\sin\left(\_Z\right)\cos\left(\_Z\right)^{3}-8\cos\left(\_Z\right)^{4}-4\right.\right.\right.\right.\right.\right.}$$

✓ Solution by Mathematica

Time used: 1.355 (sec). Leaf size: 282

```
DSolve[y[x]*y'[x]-y[x]==-2/9*x+A*x^(-1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\log\left(9A^{2/3}+3\sqrt[3]{6}\sqrt[3]{A}\sqrt{x}\right.\right.$$

$$\left.+6^{2/3}x\right)+2\sqrt{3}\arctan\left(\frac{-\frac{6\sqrt[3]{6}\left(9A-2x^{3/2}+3\sqrt{x}y(x)\right)}{\sqrt[3]{A}y(x)}-27}{27\sqrt{3}}\right)+2\sqrt{3}\arctan\left(\frac{\frac{2\sqrt[3]{6}\sqrt{x}}{\sqrt[3]{A}}+3}{3\sqrt{3}}\right)+2\log\left(\frac{1}{27}\left(27-\frac{3\sqrt[3]{}}{}\right.\right.$$

22.27 problem 27

Internal problem ID [10676]
Internal file name [`OUTPUT/9623_Monday_June_06_2022_03_16_17_PM_43567883/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = -\frac{5x}{36} + \frac{A}{x^{\frac{7}{5}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/5)*y(x)*(25*x^(12/5)+252*A)/(-5*x^(17/
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-5/36*x+A*x^(-7/5),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-5/36*x+A*x^(-7/5),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.28 problem 28

Internal problem ID [10677]
Internal file name [`OUTPUT/9624_Monday_June_06_2022_03_16_20_PM_40767146/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{12x}{49} + \frac{6A\left(-3\sqrt{x} + 23A + \frac{12A^2}{\sqrt{x}}\right)}{49}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/2)*y(x)*(12*A^3+3*A*x+4*x^(3/2))/(23*A
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(A*y(x)-x)/(A*x), y(x)`      *** Sublevel
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/144)*(144*A^3*y(x)-138*A^2*x-49*x^2)/(
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-12/49*x+6/49*A*(-3*x^(1/2)+23*A+12*A^2*x^(-1/2)),y(x), singso
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-12/49*x+6/49*A*(-3*x^(1/2)+23*A+12*A^2*x^(-1/2)),y[x],x,IncludeSing
```

Not solved

22.29 problem 29

Internal problem ID [10678]
Internal file name [`OUTPUT/9625_Monday_June_06_2022_03_16_24_PM_71158636/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{30x}{121} + \frac{3A\left(21\sqrt{x} + 35A + \frac{6A^2}{\sqrt{x}}\right)}{242}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/2)*y(x)*(6*A^3-21*A*x+40*x^(3/2))/(35*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/7)*(7*A*y(x)+10*x)/(A*x), y(x)`      *
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/36)*(36*A^3*y(x)-105*A^2*x-242*x^2)/(A
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-30/121*x+3/242*A*(21*x^(1/2)+35*A+6*A^2*x^(-1/2)),y(x), sings
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-30/121*x+3/242*A*(21*x^(1/2)+35*A+6*A^2*x^(-1/2)),y[x],x,IncludeSin
```

Not solved

22.30 problem 30

Internal problem ID [10679]
Internal file name [`OUTPUT/9626_Monday_June_06_2022_03_16_31_PM_19939040/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{3x}{16} + \frac{A}{x^{\frac{5}{3}}}$$

Unable to determine ODE type.

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-3/16*x+A*x^(-5/3),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-3/16*x+A*x^(-5/3),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.31 problem 31

Internal problem ID [10680]
Internal file name [`OUTPUT/9627_Monday_June_06_2022_03_16_41_PM_51445858/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{12x}{49} + \frac{4A\left(-10\sqrt{x} + 27A + \frac{10A^2}{\sqrt{x}}\right)}{49}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(5*A^3+5*A*x+3*x^(3/2))/(10*A^3*x+27
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/20)*(20*A*y(x)-9*x)/(A*x), y(x)`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-12/49*x+4/49*A*(-10*x^(1/2)+27*A+10*A^2*x^(-1/2)),y(x), sings
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-12/49*x+4/49*A*(-10*x^(1/2)+27*A+10*A^2*x^(-1/2)),y[x],x,IncludeSin
```

Not solved

22.32 problem 32

Internal problem ID [10681]
Internal file name [`OUTPUT/9628_Monday_June_06_2022_03_16_45_PM_69846556/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = \frac{A}{\sqrt{x}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 222

```
dsolve(y(x)*diff(y(x),x)-y(x)=A*x^(-1/2),y(x), singsol=all)
```

$$\frac{\left(\operatorname{AiryBi}\left(-\frac{2^{\frac{1}{3}}\left(-A^{2}x^{\frac{3}{2}}\right)^{\frac{2}{3}}(y(x)-x)}{2A^{2}x}\right)c_{1}-\operatorname{AiryAi}\left(-\frac{2^{\frac{1}{3}}\left(-A^{2}x^{\frac{3}{2}}\right)^{\frac{2}{3}}(y(x)-x)}{2A^{2}x}\right)\right)2^{\frac{2}{3}}\left(-A^{2}x^{\frac{3}{2}}\right)^{\frac{1}{3}}-2A\left(-\operatorname{AiryBi}\left(\right.\right.}{2^{\frac{2}{3}}\left(-A^{2}x^{\frac{3}{2}}\right)^{\frac{1}{3}}\operatorname{AiryBi}\left(-\frac{2^{\frac{1}{3}}\left(-A^{2}x^{\frac{3}{2}}\right)^{\frac{2}{3}}(y(x)-x)}{2A^{2}x}\right)+2\operatorname{AiryBi}\left(1,-\right.}$$

$$=0$$

✓ Solution by Mathematica

Time used: 0.566 (sec). Leaf size: 139

```
DSolve[y[x]*y'[x]-y[x]==A*x^(-1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$\operatorname{Solve}\left[\frac{\sqrt[3]{-1}2^{2/3}\sqrt{x}\operatorname{AiryAi}\left(\frac{\left(-\frac{1}{2}\right)^{2/3}(x-y(x))}{A^{2/3}}\right)+2\sqrt[3]{A}\operatorname{AiryAiPrime}\left(\frac{\left(-\frac{1}{2}\right)^{2/3}(x-y(x))}{A^{2/3}}\right)}{\sqrt[3]{-1}2^{2/3}\sqrt{x}\operatorname{AiryBi}\left(\frac{\left(-\frac{1}{2}\right)^{2/3}(x-y(x))}{A^{2/3}}\right)+2\sqrt[3]{A}\operatorname{AiryBiPrime}\left(\frac{\left(-\frac{1}{2}\right)^{2/3}(x-y(x))}{A^{2/3}}\right)}+c_{1}=0,y(x)\right]$$

22.33 problem 33

Internal problem ID [10682]
Internal file name [`OUTPUT/9629_Monday_June_06_2022_03_16_46_PM_8383073/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational, [_Abel, `2nd type`, `class B`]]`

Unable to solve or complete the solution.

$$\boxed{yy' - y = \frac{A}{x^2}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 279

```
dsolve(y(x)*diff(y(x),x)-y(x)=A*x^(-2),y(x), singsol=all)
```

$$-\frac{\left(\operatorname{AiryBi}\left(-\frac{\left(x^3-2y(x)x^2+y(x)^2x+2A\right)2^{\frac{2}{3}}}{4(-A^2)^{\frac{1}{3}}x}\right)c_1-\operatorname{AiryAi}\left(-\frac{\left(x^3-2y(x)x^2+y(x)^2x+2A\right)2^{\frac{2}{3}}}{4(-A^2)^{\frac{1}{3}}x}\right)\right)A(-y(x)+x)\,2^{\frac{1}{3}}+2(}{-A2^{\frac{1}{3}}\left(-y\left(x\right)+x\right)\operatorname{AiryBi}\left(-\frac{\left(x^3-2y(x)x^2+y(x)^2x+2A\right)2^{\frac{2}{3}}}{4(-A^2)^{\frac{1}{3}}x}\right)+2}$$

$= 0$

✓ Solution by Mathematica

Time used: 1.053 (sec). Leaf size: 201

```
DSolve[y[x]*y'[x]-y[x]==A*x^(-2),y[x],x,IncludeSingularSolutions -> True]
```

$$\operatorname{Solve}\left[\frac{\operatorname{AiryAiPrime}\left(\frac{x^3-2y(x)x^2+y(x)^2x+2A}{2\sqrt[3]{2}A^{2/3}x}\right)-\frac{(x-y(x))\operatorname{AiryAi}\left(\frac{x^3-2y(x)x^2+y(x)^2x+2A}{2\sqrt[3]{2}A^{2/3}x}\right)}{2^{2/3}\sqrt[3]{A}}}{\operatorname{AiryBiPrime}\left(\frac{x^3-2y(x)x^2+y(x)^2x+2A}{2\sqrt[3]{2}A^{2/3}x}\right)-\frac{(x-y(x))\operatorname{AiryBi}\left(\frac{x^3-2y(x)x^2+y(x)^2x+2A}{2\sqrt[3]{2}A^{2/3}x}\right)}{2^{2/3}\sqrt[3]{A}}}+c_1=0,y(x)\right]$$

22.34 problem 34

Internal problem ID [10683]
Internal file name [`OUTPUT/9630_Monday_June_06_2022_03_16_47_PM_24109604/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = A(2+n)\left(\sqrt{x} + 2(2+n)\,A + \frac{(1+n)\,(n+3)\,A^2}{\sqrt{x}}\right)$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 410

```
dsolve(y(x)*diff(y(x),x)-y(x)=A*(n+2)*(x^(1/2)+2*(n+2)*A+(n+1)*(n+3)*A^2*x^(-1/2)),y(x), sin
```

$$\frac{(n+2)\left(\operatorname{BesselI}\left(\frac{3+n}{n+2}, -\sqrt{\frac{2(n+2)A\sqrt{x}+A^2(n^2+4n+3)+x-y(x)}{(n+2)^2A^2}}\right)c_1 + \operatorname{BesselK}\left(\frac{3+n}{n+2}, -\sqrt{\frac{2(n+2)A\sqrt{x}+A^2(n^2+4n+3)+x-}{(n+2)^2A^2}}\right.\right.}{A\sqrt{\frac{2(n+2)A\sqrt{x}+A^2(n^2+4n+3)+x-y(x)}{(n+2)^2A^2}}\,(n+2)\operatorname{BesselI}}$$

$= 0$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==A*(n+2)*(x^(1/2)+2*(n+2)*A+(n+1)*(n+3)*A^2*x^(-1/2)),y[x],x,IncludeS
```

Not solved

22.35 problem 35

Internal problem ID [10684]
Internal file name [`OUTPUT/9631_Monday_June_06_2022_03_16_49_PM_20236254/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = A(2+n)\left(\sqrt{x} + 2(2+n)\,A + \frac{(3+2n)\,A^2}{\sqrt{x}}\right)$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 474

```
dsolve(y(x)*diff(y(x),x)-y(x)=A*(n+2)*(x^(1/2)+2*(n+2)*A+(2*n+3)*A^2*x^(-1/2)),y(x), singsol
```

$$\frac{-(n+2)\left(\mathrm{BesselI}\left(\sqrt{\frac{(n+1)^2}{(n+2)^2}}+1,-\sqrt{\frac{2(n+2)A\sqrt{x}+(2n+3)A^2+x-y(x)}{(n+2)^2A^2}}\right)c_1+\mathrm{BesselK}\left(\sqrt{\frac{(n+1)^2}{(n+2)^2}}+1,-\sqrt{\frac{2(n+2)A\sqrt{}}{}}\right.\right.}{-A\sqrt{\frac{2(n+2)A\sqrt{x}+(2n+3)A^2+x-y(x)}{(n+2)^2A^2}}\,(n+2)\,\mathrm{Bessel}}$$

$$= 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==A*(n+2)*(x^(1/2)+2*(n+2)*A+(2*n+3)*A^2*x^(-1/2)),y[x],x,IncludeSingu
```

Not solved

22.36 problem 36

Internal problem ID [10685]
Internal file name [`OUTPUT/9632_Monday_June_06_2022_03_16_55_PM_1338498/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = A\sqrt{x} + 2A^2 + \frac{B}{\sqrt{x}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 407

```
dsolve(y(x)*diff(y(x),x)-y(x)=A*x^(1/2)+2*A^2+B*x^(-1/2),y(x), singsol=all)
```

$$\frac{-c_1\left(\sqrt{\frac{A^3-B}{A^3}}\,A-A-\sqrt{x}\right)\mathrm{BesselI}\left(\sqrt{\frac{A^3-B}{A^3}},-\sqrt{\frac{2A^2\sqrt{x}-y(x)A+xA+B}{A^3}}\right)+A\sqrt{\frac{2A^2\sqrt{x}-y(x)A+xA+B}{A^3}}\,\mathrm{BesselI}\left(\right.}{A\sqrt{\frac{2A^2\sqrt{x}-y(x)A+xA+B}{A^3}}\,\mathrm{BesselI}\left(\right.}$$

$$=0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==A*x^(1/2)+2*A^2+B*x^(-1/2),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.37 problem 37

Internal problem ID [10686]
Internal file name [`OUTPUT/9633_Monday_June_06_2022_03_16_58_PM_33792140/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = 2A^2 - A\sqrt{x}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 228

```
dsolve(y(x)*diff(y(x),x)-y(x)=2*A^2-A*x^(1/2),y(x), singsol=all)
```

$$\frac{\left(-2A+\sqrt{x}\right)\operatorname{BesselK}\left(1,-\sqrt{-\frac{2A\sqrt{x}-x+y(x)}{A^2}}\right)+\operatorname{BesselK}\left(0,-\sqrt{-\frac{2A\sqrt{x}-x+y(x)}{A^2}}\right)\sqrt{-\frac{2A\sqrt{x}-x+y(x)}{A^2}}\,A+c_1\left(\right.}{A\operatorname{BesselI}\left(0,\sqrt{-\frac{2A\sqrt{x}-x+y(x)}{A^2}}\right)\sqrt{-\frac{2A\sqrt{x}-x+y(x)}{A^2}}+\left(\right.}$$

$= 0$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==2*A^2-A*x^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.38 problem 38

Internal problem ID [10687]
Internal file name [OUTPUT/9634_Monday_June_06_2022_03_16_59_PM_96412727/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{x}{4} + \frac{6A\left(\sqrt{x} + 8A + \frac{5A^2}{\sqrt{x}}\right)}{49}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(60*A^3-12*A*x+49*x^(3/2))/(120*A^3*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/16)*(16*A*y(x)+49*x)/(A*x), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/60)*(60*A^3*y(x)-48*A^2*x-49*x^2)/(x*A
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-12/48*x+6/49*A*(x^(1/2)+8*A+5*A^2*x^(-1/2)),y(x), singsol=all
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-12/48*x+6/49*A*(x^(1/2)+8*A+5*A^2*x^(-1/2)),y[x],x,IncludeSingularS
```

Not solved

22.39 problem 39

Internal problem ID [10688]
Internal file name [`OUTPUT/9635_Monday_June_06_2022_03_17_03_PM_29636547/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{6x}{25} + \frac{6A\left(2\sqrt{x} + 7A + \frac{4A^2}{\sqrt{x}}\right)}{25}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(2*A^3-A*x+x^(3/2))/(7*A^2*x^(3/2)-x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/4)*(4*A*y(x)+3*x)/(A*x), y(x)`      **
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-6/25*x+6/25*A*(2*x^(1/2)+7*A+4*A^2*x^(-1/2)),y(x), singsol=al
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-6/25*x+6/25*A*(2*x^(1/2)+7*A+4*A^2*x^(-1/2)),y[x],x,IncludeSingular
```

Not solved

22.40 problem 40

Internal problem ID [10689]
Internal file name [`OUTPUT/9636_Monday_June_06_2022_03_17_06_PM_3102030/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = -\frac{3x}{16} + \frac{3A}{x^{\frac{1}{3}}} - \frac{12A^2}{x^{\frac{5}{3}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
trying Riccati to 2nd Order
-> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(4*x-3)*(diff(y(x), x))/(x*(x-1)
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- Riccati to 2nd Order successful
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 1488

```
dsolve(y(x)*diff(y(x),x)-y(x)=-3/16*x+3*A*x^(-1/3)-12*A^2*x^(-5/3),y(x), singsol=all)
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-3/16*x+3*A*x^(-1/3)-12*A^2*x^(-5/3),y[x],x,IncludeSingularSolutions
```

Not solved

22.41 problem 41

Internal problem ID [10690]
Internal file name [`OUTPUT/9637_Monday_June_06_2022_03_17_13_PM_36887324/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = \frac{3x}{8} + \frac{3\sqrt{b^2 + x^2}}{8} + \frac{3b^2}{16\sqrt{b^2 + x^2}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(2*(b^2+x^2)^(1/2)*b^2+2*(b^2+x^2)^(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/3)*(3*y(x)-2*x)/x, y(x)`      *** Subl
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=3/8*x+3/8*sqrt(x^2+b^2)+3*b^2/(16*sqrt(x^2+b^2)),y(x), singsol
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==3/8*x+3/8*Sqrt[x^2+b^2]+3*b^2/(16*Sqrt[x^2+b^2]),y[x],x,IncludeSingu
```

Not solved

22.42 problem 42

Internal problem ID [10691]
Internal file name [`OUTPUT/9638_Monday_June_06_2022_03_17_16_PM_88301991/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = \frac{9x}{32} + \frac{15\sqrt{b^2 + x^2}}{32} + \frac{3b^2}{64\sqrt{b^2 + x^2}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(6*(b^2+x^2)^(1/2)*b^2+6*(b^2+x^2)^(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/3)*(3*y(x)-10*x)/x, y(x)`       *** Sub
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/11)*(11*y(x)-6*x)/x, y(x)`       *** Su
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=9/32*x+15/32*sqrt(x^2+b^2)+3*b^2/(64*sqrt(x^2+b^2)),y(x), sing
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==9/32*x+15/32*Sqrt[x^2+b^2]+3*b^2/(64*Sqrt[x^2+b^2]),y[x],x,IncludeSi
```

Not solved

22.43 problem 43

Internal problem ID [10692]
Internal file name [`OUTPUT/9639_Monday_June_06_2022_03_17_22_PM_54791991/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{3x}{32} - \frac{3\sqrt{a^2 + x^2}}{32} + \frac{15a^2}{64\sqrt{a^2 + x^2}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(2*(a^2+x^2)^(1/2)*a^2+2*(a^2+x^2)^(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/3)*(3*y(x)+2*x)/x, y(x)`      *** Subl
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-3/32*x-3/32*sqrt(x^2+a^2)+15*a^2/(64*sqrt(x^2+a^2)),y(x), sin
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-3/32*x-3/32*Sqrt[x^2+a^2]+15*a^2/(64*Sqrt[x^2+a^2]),y[x],x,IncludeS
```

Not solved

22.44 problem 44

Internal problem ID [10693]
Internal file name [`OUTPUT/9640_Monday_June_06_2022_03_17_25_PM_32058918/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational, [_Abel, `2nd type`, `class A`]]`

Unable to solve or complete the solution.

$$\boxed{yy' - y = A\,x^2 - \frac{9}{625A}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 186

```
dsolve(y(x)*diff(y(x),x)-y(x)=A*x^2-9/625*A^(-1),y(x), singsol=all)
```

$$\frac{-\frac{125\,2^{\frac{5}{6}}\left(\frac{-46875A^2y(x)^2+\left(37500A^2x+4500A\right)y(x)+31250\left(xA-\frac{3}{25}\right)\left(xA+\frac{3}{25}\right)^2}{(50xA-125y(x)A+6)^2}\right)^{\frac{1}{6}}Ay(x)\sqrt{25xA+3}}{2}+50\left(xA-\frac{5y(x)A}{2}+\frac{3}{25}\right)\left(\frac{(25xA\cdots}{50xA-125\cdots}\right.}{\left(\frac{(25xA+3)^{\frac{3}{2}}}{50xA-125y(x)A+6}\right)^{\frac{1}{3}}\left(50xA-125y\left(x\right)A+6\right)}$$

$$=0$$

✓ Solution by Mathematica

Time used: 2.438 (sec). Leaf size: 198

```
DSolve[y[x]*y'[x]-y[x]==A*x^2-9/625*A^(-1),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{\sqrt[6]{\frac{46875A^2y(x)^2-1500A(25Ax+3)y(x)-2(25Ax-3)(25Ax+3)^2}{(25Ax+3)^3}}\left(\frac{(-125Ay(x)+\cdots}{\sqrt[3]{2}\sqrt{3}(25Ax+3)^{3/2}\sqrt[6]{\frac{-46875\cdots}{\cdots}}}\right.}{\sqrt[6]{2}}\right.$$

$$\left.+c_1=0,y(x)\right]$$

22.45 problem 45

Internal problem ID [10694]
Internal file name [`OUTPUT/9641_Monday_June_06_2022_03_17_26_PM_14824341/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{6}{25}x - A\,x^2$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 132

```
dsolve(y(x)*diff(y(x),x)-y(x)=-6/25*x-A*x^2,y(x), singsol=all)
```

$$c_1 + \frac{(2x-5y(x))\left(\int^{-\frac{10\sqrt{-xA}\,x}{-2x+5y(x)}}\frac{(\_a^2-6)^{\frac{1}{6}}}{\_a^{\frac{1}{3}}}d\_a\right)-\frac{5\,2^{\frac{5}{6}}\left(\frac{-50A\,x^3-12x^2+60y(x)x-75y(x)^2}{(-2x+5y(x))^2}\right)^{\frac{1}{6}}5^{\frac{2}{3}}\sqrt{-xA}\,y(x)}{2\left(-\frac{\sqrt{-xA}\,x}{-2x+5y(x)}\right)^{\frac{1}{3}}}}{2x-5y(x)} = 0$$

✓ Solution by Mathematica

Time used: 2.139 (sec). Leaf size: 162

```
DSolve[y[x]*y'[x]-y[x]==-6/25*x-A*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[c_1 = \frac{i\sqrt[6]{\frac{-2x^2(25Ax+6)+60xy(x)-75y(x)^2}{Ax^3}}\left(25Ax^2-\frac{\sqrt[6]{2}\sqrt[3]{5}(2x-5y(x))\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{5}{6},\frac{3}{2},-\frac{3(2}{}\right.}{\sqrt[6]{\frac{2x^2(25Ax+6)-60xy(x)+75y(}{Ax^3}}}\right.}{5\;2^{2/3}\sqrt{3}\sqrt[3]{5}\sqrt{A}x^{3/2}}\right.$$

22.46 problem 46

Internal problem ID [10695]
Internal file name [`OUTPUT/9642_Monday_June_06_2022_03_17_27_PM_31935674/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = \frac{6}{25}x - A\,x^2}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 180

```
dsolve(y(x)*diff(y(x),x)-y(x)=6/25*x-A*x^2,y(x), singsol=all)
```

$$-\frac{125\left(5^{\frac{1}{3}}2^{\frac{5}{6}}\left(-\frac{1250\left(\frac{3y(x)^2A}{2}+\left(-\frac{6xA}{5}+\frac{36}{125}\right)y(x)+\left(xA-\frac{6}{25}\right)^2x\right)A}{(50xA-125y(x)A-12)^2}\right)^{\frac{1}{6}}Ay(x)\sqrt{-25xA+6}-\frac{4\left(xA-\frac{5y(x)A}{2}-\frac{6}{25}\right)\left(\int\frac{2(-25}{-50xA+1}\right.}{}\right.}{\left(\frac{(-25xA+6)^{\frac{3}{2}}}{-50xA+125y(x)A+12}\right)^{\frac{1}{3}}(100xA-250y(x)A-24)}$$

$= 0$

✓ Solution by Mathematica

Time used: 3.324 (sec). Leaf size: 189

```
DSolve[y[x]*y'[x]-y[x]==6/25*x-A*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{\sqrt[3]{5}\sqrt[6]{-\frac{A\left(1875Ay(x)^2-60(25Ax-6)y(x)+2x(6-25Ax)^2\right)}{(25Ax-6)^3}}\left(\frac{(-125Ay(x)+50Ax-12)\text{ Hyperge}}{\sqrt[3]{10}\sqrt{18-75Ax}(25Ax-6)\sqrt[6]{\frac{A(1875Ay}{}}}\right.}{\sqrt[6]{2}}\right.$$

$$\left.+c_1=0,y(x)\right]$$

22.47 problem 47

Internal problem ID [10696]
Internal file name [`OUTPUT/9643_Monday_June_06_2022_03_17_27_PM_31172118/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 47.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = 12x + \frac{A}{x^{\frac{5}{2}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 110

```
dsolve(y(x)*diff(y(x),x)-y(x)=12*x+A*x^(-5/2),y(x), singsol=all)
```

$$c_1 + \frac{12\sqrt{3}\left(2^{\frac{2}{3}}\left(\frac{3y(x)^2 x^{\frac{3}{2}}}{4} - 6y(x)\,x^{\frac{5}{2}} + A + 12x^{\frac{7}{2}}\right)\left(\frac{48x^{\frac{7}{2}} - 24y(x)x^{\frac{5}{2}} + 3y(x)^2 x^{\frac{3}{2}} + 4A}{A}\right)^{\frac{1}{6}} - 56\,\mathrm{hypergeom}\left(\left[-\frac{1}{6}, \frac{1}{2}\right], \right.\right.}{\sqrt{-A\,x^{\frac{7}{2}}}} = 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==12*x+A*x^(-5/2),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.48 problem 48

Internal problem ID [10697]
Internal file name [OUTPUT/9644_Monday_June_06_2022_03_17_29_PM_17392824/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 48.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = \frac{63x}{4} + \frac{A}{x^{\frac{5}{3}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/3)*y(x)*(-189*x^(8/3)+20*A)/(63*x^(11/
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=63/4*x+A*x^(-5/3),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==63/4*x+A*x^(-5/3),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.49 problem 49

Internal problem ID [10698]
Internal file name [`OUTPUT/9645_Monday_June_06_2022_03_17_32_PM_34499080/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 49.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = 2x + 2A\left(10\sqrt{x} + 31A + \frac{30A^2}{\sqrt{x}}\right)}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   found: 2 potential symmetries. Proceeding with integration step
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 196

```
dsolve(y(x)*diff(y(x),x)-y(x)=2*x+2*A*(10*x^(1/2)+31*A+30*A^2*x^(-1/2)),y(x), singsol=all)
```

$$c_1 - \frac{\left(3A+\sqrt{x}\right) 2^{\frac{1}{3}} \left(\frac{12A^2+10A\sqrt{x}+2x-y(x)}{6A^2+2A\sqrt{x}+y(x)}\right)^{\frac{1}{3}} \left(\frac{15A^2+8A\sqrt{x}+x+y(x)}{6A^2+2A\sqrt{x}+y(x)}\right)^{\frac{1}{6}} y(x)}{4\sqrt{\frac{\left(3A+\sqrt{x}\right)^2}{6A^2+2A\sqrt{x}+y(x)}}\left(6A^2+2A\sqrt{x}+y\left(x\right)\right) A} - \left(\int^{\frac{6A\sqrt{x}+2x-3y(x)}{12A^2+4A\sqrt{x}+2y(x)}} \frac{\left(\_a+1\right)^{\frac{1}{3}} \left(2\_a+5\right)^{\frac{1}{6}}}{\sqrt{2\_a+3}} d\_a\right) = 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==2*x+2*A*(10*x^(1/2)+31*A+30*A^2*x^(-1/2)),y[x],x,IncludeSingularSolu
```

Not solved

22.50 problem 50

Internal problem ID [10699]
Internal file name [`OUTPUT/9646_Monday_June_06_2022_03_17_33_PM_58193520/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 50.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = 2x + 2A\left(-10\sqrt{x} + 19A + \frac{30A^2}{\sqrt{x}}\right)}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(15*A^3+5*A*x-x^(3/2))/(30*A^3*x+19*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/20)*(20*A*y(x)+3*x)/(A*x), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/120)*(120*A^3*y(x)-38*A^2*x-x^2)/(A^3*
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=2*x+2*A*(-10*x^(1/2)+19*A+30*A^2*x^(-1/2)),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==2*x+2*A*(-10*x^(1/2)+19*A+30*A^2*x^(-1/2)),y[x],x,IncludeSingularSol
```

Not solved

22.51 problem 51

Internal problem ID [10700]
Internal file name [`OUTPUT/9647_Monday_June_06_2022_03_17_37_PM_70045049/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 51.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{28x}{121} + \frac{2A\left(5\sqrt{x} + 106A + \frac{65A^2}{\sqrt{x}}\right)}{121}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/2)*y(x)*(65*A^3-5*A*x+28*x^(3/2))/(106
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/5)*(5*A*y(x)+21*x)/(A*x), y(x)`      *
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-28/121*x+2/121*A*(5*x^(1/2)+106*A+65*A^2*x^(-1/2)),y(x), sing
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-28/121*x+2/121*A*(5*x^(1/2)+106*A+65*A^2*x^(-1/2)),y[x],x,IncludeSi
```

Not solved

22.52 problem 52

Internal problem ID [10701]
Internal file name [`OUTPUT/9648_Monday_June_06_2022_03_17_41_PM_60740551/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 52.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$
\boxed{yy' - y = -\frac{12x}{49} + \frac{A\left(5\sqrt{x} + 262A + \frac{65A^2}{\sqrt{x}}\right)}{49}}
$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 4129

```
dsolve(y(x)*diff(y(x),x)-y(x)=-12/49*x+1/49*A*(5*x^(1/2)+262*A+65*A^2*x^(-1/2)),y(x), singso
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-28/121*x+2/121*A*(5*x^(1/2)+262*A+65*A^2*x^(-1/2)),y[x],x,IncludeSi
```

Not solved

22.53 problem 53

Internal problem ID [10702]
Internal file name [OUTPUT/9649_Monday_June_06_2022_03_17_48_PM_70219559/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 53.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = -\frac{12x}{49} + A\sqrt{x}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 133

```
dsolve(y(x)*diff(y(x),x)-y(x)=-12/49*x+A*x^(1/2),y(x), singsol=all)
```

$$\frac{196^{\frac{5}{6}}\left(\left(\frac{4\left(x-\frac{7y(x)}{4}\right)\sqrt{3}\,\mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{7}{6}\right],\left[\frac{3}{2}\right],\frac{3(-4x+7y(x))^{2}}{196x^{\frac{3}{2}}A}\right)}{7}+\sqrt{x}\,\sqrt{A\sqrt{x}}\,c_{1}\right)196^{\frac{1}{6}}\left(\frac{A\,x^{\frac{3}{2}}-\frac{12\left(x-\frac{7y(x)}{4}\right)^{2}}{49}}{x^{\frac{3}{2}}A}\right)^{\frac{1}{6}}-7\,14^{\frac{1}{3}}\,A\cdots\right)}{196\left(\frac{A\,x^{\frac{3}{2}}-\frac{12\left(x-\frac{7y(x)}{4}\right)^{2}}{49}}{x^{\frac{3}{2}}A}\right)^{\frac{1}{6}}\sqrt{A\sqrt{x}}\,\sqrt{x}}$$

$$=0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-12/49*x+A*x^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.54 problem 54

Internal problem ID [10703]
Internal file name [OUTPUT/9650_Monday_June_06_2022_03_17_50_PM_13823638/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 54.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = 6x + \frac{A}{x^4}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 217

```
dsolve(y(x)*diff(y(x),x)-y(x)=6*x+A*x^(-4),y(x), singsol=all)
```

$$c_1 + \frac{5\,5^{\frac{2}{3}}\left(x+\frac{y(x)}{2}\right)\left(xA3^{\frac{5}{6}}\operatorname{hypergeom}\left(\left[\frac{1}{6},\frac{2}{3}\right],\left[\frac{5}{3}\right],-\frac{2A}{3x^{3}(2x+y(x))^{2}}\right)\left(\frac{12x^{5}+12y(x)x^{4}+3y(x)^{2}x^{3}+2A}{x^{9}(2x+y(x))^{6}}\right)^{\frac{1}{6}}+\frac{24\left(\left(-\frac{}{x^{\frac{3}{2}}(2\cdots}\right.\right.}{\cdots}\cdots\right.}{2\left(-\frac{1}{x^{\frac{3}{2}}(2x+y(x))}\right)^{\frac{7}{3}}x^{\frac{11}{2}}\left(\frac{12x^{5}+12y(x)x^{4}+3y(x)^{2}x^{3}+2A}{x^{3}(2x+y(x))^{2}}\right)^{\frac{1}{6}}(2\cdots}$$

$$= 0$$

✓ Solution by Mathematica

Time used: 2.079 (sec). Leaf size: 213

```
DSolve[y[x]*y'[x]-y[x]==6*x+A*x^(-4),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[c_1=\frac{i\left(-\frac{2A+12x^5+12x^4y(x)+3x^3y(x)^2}{A}\right)^{5/6}\left(-10\ 2^{5/6}x^5\,\text{Hypergeometric2F1}\left(\frac{1}{6},\frac{1}{2},\frac{3}{2},-\frac{3x^3(2x+y(x))^2}{2A}\right)-5\right.}{2\sqrt[3]{2}\sqrt{3}\sqrt{A}x^{5/2}\left(\frac{2A+12x^5}{}\right.}\right.$$

22.55 problem 55

Internal problem ID [10704]
Internal file name [`OUTPUT/9651_Monday_June_06_2022_03_17_51_PM_68427508/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 55.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = 20x + \frac{A}{\sqrt{x}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/2)*y(x)*(-40*x^(3/2)+A)/(20*x^(5/2)+A*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/2)*(2*A*y(x)-x^2)/(A*x), y(x)`      **
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=20*x+A*x^(-1/2),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==20*x+A*x^(-1/2),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.56 problem 56

Internal problem ID [10705]
Internal file name [`OUTPUT/9652_Monday_June_06_2022_03_17_53_PM_60286823/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 56.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = \frac{15x}{4} + \frac{A}{x^7}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=15/4*x+A*x^(-7),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==15/4*x+A*x^(-7),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.57 problem 57

Internal problem ID [10706]
Internal file name [`OUTPUT/9653_Monday_June_06_2022_03_17_55_PM_19715917/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 57.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{10x}{49} + \frac{2A\left(4\sqrt{x} + 61A + \frac{12A^2}{\sqrt{x}}\right)}{49}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   found: 2 potential symmetries. Proceeding with integration step
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 200

```
dsolve(y(x)*diff(y(x),x)-y(x)=-10/49*x+2/49*A*(4*x^(1/2)+61*A+12*A^2*x^(-1/2)),y(x), singsol
```

$$c_1 - \frac{\left(3A+\sqrt{x}\right) 2^{\frac{2}{3}} \left(\frac{3A^2+16A\sqrt{x}+5x-7y(x)}{6A^2+2A\sqrt{x}+y(x)}\right)^{\frac{5}{6}} y(x)}{2\sqrt{\frac{\left(3A+\sqrt{x}\right)^2}{6A^2+2A\sqrt{x}+y(x)}}\left(\frac{-24A^2-2A\sqrt{x}+2x-7y(x)}{6A^2+2A\sqrt{x}+y(x)}\right)^{\frac{1}{3}} \left(6A^2+2A\sqrt{x}+y\left(x\right)\right) A} - \left(\int^{\frac{6A\sqrt{x}+2x-3y(x)}{12A^2+4A\sqrt{x}+2y(x)}} \frac{\left(10\_a+1\right)^{\frac{5}{6}}}{\sqrt{2\_a+3}\,\left(\_a-2\right)^{\frac{1}{3}}} d\_a\right) = 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-10/49*x+2/49*A*(4*x^(1/2)+61*A+12*A^2*x^(-1/2)),y[x],x,IncludeSingu
```

Not solved

22.58 problem 58

Internal problem ID [10707]
Internal file name [`OUTPUT/9654_Monday_June_06_2022_03_17_57_PM_56089828/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 58.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{12x}{49} + \frac{2A\left(\sqrt{x} + 166A + \frac{55A^2}{\sqrt{x}}\right)}{49}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 686

```
dsolve(y(x)*diff(y(x),x)-y(x)=-12/49*x+2/49*A*(x^(1/2)+166*A+55*A^2*x^(-1/2)),y(x), singsol=
```

$$c_1 + \frac{3\sqrt{6}\,4^{\frac{2}{3}}\left(\left(\sqrt{-35A^2-7A\sqrt{x}}\left(A\left(3+\right.\right.\right.\right.}{4\left(\left(\left(2\left(-18i-7\sqrt{6}\right)A\sqrt{x}-70A^2\sqrt{6}+120iA^2-12ix+21iy\left(x\right)\right)\sqrt{-35A^2-7A\sqrt{x}}+350\left(-\frac{\sqrt{6}\,x}{25}+\right.\right.\right.}$$
$$= 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-12/49*x+2/49*A*(x^(1/2)+166*A+55*A^2*x^(-1/2)),y[x],x,IncludeSingul
```

Not solved

22.59 problem 59

Internal problem ID [10708]
Internal file name [`OUTPUT/9655_Monday_June_06_2022_03_18_01_PM_51997936/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 59.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{4x}{25} + \frac{A\left(7\sqrt{x} + 49A + \frac{6A^2}{\sqrt{x}}\right)}{50}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/2)*y(x)*(6*A^3-7*A*x+16*x^(3/2))/(49*A
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/7)*(7*A*y(x)+12*x)/(A*x), y(x)`      *
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-4/25*x+1/50*A*(7*x^(1/2)+49*A+6*A^2*x^(-1/2)),y(x), singsol=a
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-4/25*x+1/50*A*(7*x^(1/2)+49*A+6*A^2*x^(-1/2)),y[x],x,IncludeSingula
```

Not solved

22.60 problem 60

Internal problem ID [10709]
Internal file name [OUTPUT/9656_Monday_June_06_2022_03_18_05_PM_56043275/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 60.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = \frac{15x}{4} + \frac{6A}{x^{\frac{1}{3}}} - \frac{3A^2}{x^{\frac{5}{3}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/3)*y(x)*(15*x^(8/3)-8*A*x^(4/3)+20*A^2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=15/4*x+6*A*x^(-1/3)-3*A^2*x^(-5/3),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==15/4*x+6*A*x^(-1/3)-3*A^2*x^(-5/3),y[x],x,IncludeSingularSolutions -
```

Not solved

22.61 problem 61

Internal problem ID [10710]
Internal file name [`OUTPUT/9657_Monday_June_06_2022_03_18_13_PM_97476693/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{3x}{16} + \frac{A}{x^{\frac{1}{3}}} + \frac{B}{x^{\frac{5}{3}}}$$

Unable to determine ODE type.

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-3/16*x+A*x^(-1/3)+B*x^(-5/3),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-3/16*x+A*x^(-1/3)+B*x^(-5/3),y[x],x,IncludeSingularSolutions -> Tru
```

Not solved

22.62 problem 62

Internal problem ID [10711]
Internal file name [`OUTPUT/9658_Monday_June_06_2022_03_20_15_PM_11874362/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 62.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{5x}{36} + \frac{A}{x^{\frac{3}{5}}} - \frac{B}{x^{\frac{7}{5}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/5)*y(x)*(25*x^(12/5)+108*A*x^(4/5)-252
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-5/36*x+A*x^(-3/5)-B*x^(-7/5),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-5/36*x+A*x^(-3/5)-B*x^(-7/5),y[x],x,IncludeSingularSolutions -> Tru
```

Timed out

22.63 problem 63

Internal problem ID [10712]
Internal file name [`OUTPUT/9659_Monday_June_06_2022_03_20_24_PM_10202307/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 63.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = \frac{k}{\sqrt{A\,x^2 + Bx + c}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/2)*y(x)*(2*A*x+B)/(A*x^2+B*x+c), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/2)*(B*x^2-2*k*y(x))/(k*x), y(x)`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=k*(A*x^2+B*x+c)^(-1/2),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==k*(A*x^2+B*x+c)^(-1/2),y[x],x,IncludeSingularSolutions -> True]
```

Timed out

22.64 problem 64

Internal problem ID [10713]
Internal file name [`OUTPUT/9660_Monday_June_06_2022_03_20_26_PM_79963363/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 64.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{12x}{49} + 3A\left(\frac{1}{49} + B\right)\sqrt{x} + 3A^2\left(\frac{4}{49} - \frac{5B}{2}\right) + \frac{15A^3\left(\frac{1}{49} - \frac{5B}{4}\right)}{4\sqrt{x}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/2)*y(x)*(1225*A^3*B+784*x*B*A-20*A^3+1
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(49*A*B*y(x)+A*y(x)+6*x)/(A*(1+49*B)*x),
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/15)*(3675*A^3*B*y(x)-60*A^3*y(x)-2940*
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-12/49*x+3*A*(1/49+B)*x^(1/2)+3*A^2*(4/49-5/2*B)+15/4*A^3*(1/4
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-12/49*x+3*A*(1/49+B)*x^(1/2)+3*A^2*(4/49-5/2*B)+15/4*A^3*(1/49-5/4*
```

Not solved

22.65 problem 65

Internal problem ID [10714]
Internal file name [OUTPUT/9661_Monday_June_06_2022_03_20_32_PM_46669422/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 65.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{6x}{25} + \frac{4B^2\left((2-A)\,x^{\frac{1}{3}} - \frac{3B(2A+1)}{2} + \frac{B^2(1-3A)}{x^{\frac{1}{3}}} - \frac{A\,B^3}{x^{\frac{2}{3}}}\right)}{75}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 3187

```
dsolve(y(x)*diff(y(x),x)-y(x)=-6/25*x+4/75*B^2*((2-A)*x^(1/3)-3/2*B*(2*A+1)+B^2*(1-3*A)*x^(-
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-6/25*x+4/75*B^2*((2-A)*x^(1/3)-3/2*B*(2*A+1)+B^2*(1-3*A)*x^(-1/3)-A
```

Not solved

22.66 problem 66

Internal problem ID [10715]
Internal file name [`OUTPUT/9662_Monday_June_06_2022_03_20_40_PM_45829850/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 66.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational, [_Abel, `2nd type`, `class B`]]`

Unable to solve or complete the solution.

$$\boxed{yy' - y = \frac{3x}{4} - \frac{3A\, x^{\frac{1}{3}}}{2} + \frac{3A^2}{4x^{\frac{1}{3}}} - \frac{27A^4}{625x^{\frac{5}{3}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(5/3)*y(x)*(375*x^(8/3)-250*x^2*A-125*A^2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(4*A*x+y(x))/x, y(x)`      *** Sublevel 2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/3)*(2*A*x+3*y(x))/x, y(x)`      *** Su
      Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=3/4*x-3/2*A*x^(1/3)+3/4*A^2*x^(-1/3)-27/625*A^4*x^(-5/3),y(x),
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==3/4*x-3/2*A*x^(1/3)+3/4*A^2*x^(-1/3)-27/625*A^4*x^(-5/3),y[x],x,Incl
```

Not solved

22.67 problem 67

Internal problem ID [10716]
Internal file name [`OUTPUT/9663_Monday_June_06_2022_03_20_46_PM_36629386/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 67.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{6x}{25} + \frac{7A\,x^{\frac{1}{3}}}{5} + \frac{31A^2}{3x^{\frac{1}{3}}} - \frac{100A^4}{3x^{\frac{5}{3}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/3)*y(x)*(-54*x^(8/3)+105*x^2*A-775*A^2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/3)*(35*A*x+3*y(x))/x, y(x)`      *** S
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-6/25*x+7/5*A*x^(1/3)+31/3*A^2*x^(-1/3)-100/3*A^4*x^(-5/3),y(x
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-6/25*x+7/5*A*x^(1/3)+31/3*A^2*x^(-1/3)-100/3*A^4*x^(-5/3),y[x],x,In
```

Not solved

22.68 problem 68

Internal problem ID [10717]
Internal file name [`OUTPUT/9664_Monday_June_06_2022_03_20_56_PM_3960468/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 68.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = -\frac{10x}{49} + \frac{13A^2}{5x^{\frac{1}{5}}} - \frac{7A^3}{20x^{\frac{4}{5}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(4/5)*y(x)*(-250*x^(9/5)-637*A^2*x^(3/5)+
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/13)*(13*A^2*y(x)-4*x^2)/(A^2*x), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-10/49*x+13/5*A^2*x^(-1/5)-7/20*A^3*x^(-4/5),y(x), singsol=all
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-10/49*x+13/5*A^2*x^(-1/5)-7/20*A^3*x^(-4/5),y[x],x,IncludeSingularS
```

Not solved

22.69 problem 69

Internal problem ID [10718]
Internal file name [`OUTPUT/9665_Monday_June_06_2022_03_21_00_PM_67032086/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{33x}{169} + \frac{286A^2}{3x^{\frac{5}{11}}} - \frac{770A^3}{9x^{\frac{13}{11}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/11)*y(x)*(-297*x^(24/11)-65910*A^2*x^(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-33/169*x+286/3*A^2*x^(-5/11)-770/9*A^3*x^(-13/11),y(x), sings
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-33/169*x+286/3*A^2*x^(-5/11)-770/9*A^3*x^(-13/11),y[x],x,IncludeSin
```

Timed out

22.70 problem 70

Internal problem ID [10719]
Internal file name [`OUTPUT/9666_Monday_June_06_2022_03_21_05_PM_82138169/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 70.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{21x}{100} + \frac{7A^2\left(\frac{123}{x^{\frac{1}{7}}} + \frac{280A}{x^{\frac{5}{7}}} - \frac{400A^2}{x^{\frac{9}{7}}}\right)}{9}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/7)*y(x)*(-189*x^(16/7)-12300*A^2*x^(8/
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-21/100*x+7/9*A^2*(123*x^(-1/7)+280*A*x^(-5/7)-400*A^2*x^(-9/7
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-21/100*x+7/9*A^2*(123*x^(-1/7)+280*A*x^(-5/7)-400*A^2*x^(-9/7)),y[x
```

Not solved

22.71 problem 71

Internal problem ID [10720]
Internal file name [`OUTPUT/9667_Monday_June_06_2022_03_21_15_PM_67997614/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 71.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = ax + b\,x^m}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(b*m*x^m+a*x)/(x*(a*x+b*x^m)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = -y(x)*(b*m*x^m+a*x)/(x*(a*x+b*x^m)), y(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-a*K[1]/x, y(x)`      *** Sublevel 2 ***
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=a*x+b*x^m,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==a*x+b*x^m,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

22.72 problem 72

Internal problem ID [10721]
Internal file name [`OUTPUT/9668_Monday_June_06_2022_03_21_16_PM_27444372/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 72.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - y = -\frac{(m+1)\,x}{(m+2)^2} + A\,x^{2m+1} + B\,x^{3m+1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(2*A*x^(1+2*m)*m^3+3*B*x^(1+3*m)*m^3
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+K[1]*(1+m)/(x*(m+2)^2), y(x)`      *** Su
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      <- quadrature successful
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=-(m+1)/(m+2)^2*x+A*x^(2*m+1)+B*x^(3*m+1),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==-(m+1)/(m+2)^2*x+A*x^(2*m+1)+B*x^(3*m+1),y[x],x,IncludeSingularSolut
```

Not solved

22.73 problem 73

Internal problem ID [10722]
Internal file name [`OUTPUT/9669_Monday_June_06_2022_03_21_23_PM_99331213/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 73.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = a^2\lambda\,\mathrm{e}^{2\lambda x} - a(b\lambda + 1)\,\mathrm{e}^{\lambda x} + b}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*a*lambda*(2*exp(2*lambda*x)*a*lambda
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = -y(x)*b/(x*(b+x)), y(x)`      *** Suble
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*b/(x*(b+x)), y(x)`      *** Sublevel
      Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=a^2*lambda*exp(2*lambda*x)-a*(b*lambda+1)*exp(lambda*x)+b,y(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==a^2*\[Lambda]*Exp[2*\[Lambda]*x]-a*(b*\[Lambda]+1)*Exp[\[Lambda]*x]+
```

Not solved

22.74 problem 74

Internal problem ID [10723]
Internal file name [`OUTPUT/9670_Monday_June_06_2022_03_21_25_PM_61614408/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 74.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = a^2 \lambda \,\mathrm{e}^{2\lambda x} + a\lambda x \,\mathrm{e}^{\lambda x} + b \,\mathrm{e}^{\lambda x}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*lambda*(2*a^2*lambda*exp(2*lambda*x)
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(-lambda*K[1]*x+y(x))/x, y(x)`      *** S
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(-a*lambda*x*K[1]-b*lambda*x*K[1]+y(x)*b)
      Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=a^2*lambda*exp(2*lambda*x)+a*lambda*x*exp(lambda*x)+b*exp(lamb
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==a^2*\[Lambda]*Exp[2*\[Lambda]*x]+a*\[Lambda]*x*Exp[\[Lambda]*x]+b*Ex
```

Not solved

22.75 problem 75

Internal problem ID [10724]
Internal file name [OUTPUT/9671_Monday_June_06_2022_03_21_27_PM_94139464/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 75.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - y = 2a^2\lambda \sin(2\lambda x) + 2a \sin(\lambda x)}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*lambda*(2*a*lambda*cos(2*lambda*x)+c
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = -y(x)*lambda*(2*a*lambda*cos(2*lambda*x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=2*a^2*lambda*sin(2*lambda*x)+2*a*sin(lambda*x),y(x), singsol=a
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==2*a^2*\[Lambda]*Sin[2*\[Lambda]*x]+2*a*Sin[\[Lambda]*x],y[x],x,Inclu
```

Not solved

22.76 problem 76

Internal problem ID [10725]
Internal file name [`OUTPUT/9672_Monday_June_06_2022_03_21_31_PM_37296922/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. Form $yy' - y = f(x)$. subsection 1.3.1-2. Solvable equations and their solutions
Problem number: 76.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - y = a^2 f'(x) f''(x) - \frac{(f(x) + b)^2 f''(x)}{f'(x)^3}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(a^2*(diff(f(x), x))^5*(diff(diff(di
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(y(x)*a^2+2*x*b)/(a^2*x), y(x)`       ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-y(x)=a^2*diff(f(x),x)*diff(f(x),x$2)-(f(x)+b)^2/( diff(f(x),x)^3)*d
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-y[x]==a^2*f'[x]*f''[x]-(f[x]+b)^2/( (f'[x])^3)*f''[x],y[x],x,IncludeSingul
```

Timed out

23 Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.2. Equations of the form $yy' = f(x)y + 1$

23.1 problem 1

Internal problem ID [10726]
Internal file name [`OUTPUT/9673_Monday_June_06_2022_03_21_32_PM_48237524/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.2. Equations of the form $yy' = f(x)y + 1$
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational, [_Abel, `2nd type`, `class A`]]`

Unable to solve or complete the solution.

$$\boxed{yy' - (ax + b)\, y = 1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 215

```
dsolve(y(x)*diff(y(x),x)=(a*x+b)*y(x)+1,y(x), singsol=all)
```

$$\frac{(ax+b)\left(\operatorname{AiryBi}\left(-\frac{\left(-2ay(x)+(ax+b)^2\right)2^{\frac{2}{3}}}{4(-a^2)^{\frac{1}{3}}}\right)c_1-\operatorname{AiryAi}\left(-\frac{\left(-2ay(x)+(ax+b)^2\right)2^{\frac{2}{3}}}{4(-a^2)^{\frac{1}{3}}}\right)\right)(-a^2)^{\frac{1}{3}}2^{\frac{1}{3}}+2\Big(\operatorname{AiryBi}\Big(}{2^{\frac{1}{3}}(-a^2)^{\frac{1}{3}}(ax+b)\operatorname{AiryBi}\left(-\frac{\left(-2ay(x)+(ax+b)^2\right)2^{\frac{2}{3}}}{4(-a^2)^{\frac{1}{3}}}\right)+2\operatorname{AiryBi}\Big(}$$
$$=0$$

✓ Solution by Mathematica

Time used: 0.901 (sec). Leaf size: 161

```
DSolve[y[x]*y'[x]==(a*x+b)*y[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{\sqrt[3]{2}(ax+b)\operatorname{AiryAi}\left(\frac{(b+ax)^2-2ay(x)}{2\sqrt[3]{2}a^{2/3}}\right)-2\sqrt[3]{a}\operatorname{AiryAiPrime}\left(\frac{(b+ax)^2-2ay(x)}{2\sqrt[3]{2}a^{2/3}}\right)}{\sqrt[3]{2}(ax+b)\operatorname{AiryBi}\left(\frac{(b+ax)^2-2ay(x)}{2\sqrt[3]{2}a^{2/3}}\right)-2\sqrt[3]{a}\operatorname{AiryBiPrime}\left(\frac{(b+ax)^2-2ay(x)}{2\sqrt[3]{2}a^{2/3}}\right)}+c_1=0,y(x)\right]$$

23.2 problem 2

Internal problem ID [10727]
Internal file name [`OUTPUT/9674_Monday_June_06_2022_03_21_33_PM_94051857/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.2. Equations of the form $yy' = f(x)y + 1$
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{y}{(ax+b)^2} = 1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 557

```
dsolve(y(x)*diff(y(x),x)=(a*x+b)^(-2)*y(x)+1,y(x), singsol=all)
```

$$\frac{-a\left(-\operatorname{AiryBi}\left(-\frac{2^{\frac{2}{3}}\left(-\frac{a^2(ax+b)^2y(x)^2}{2}+(-a^2x-ab)y(x)+a^4x^3+3a^3b\,x^2+3a^2b^2x+a\,b^3-\frac{1}{2}\right)}{2(a^2)^{\frac{1}{3}}(ax+b)^2}\right)c_1+\operatorname{AiryAi}\left(-\frac{2^{\frac{2}{3}}\left(-\frac{a^2(ax+b)^2y(\cdots}{2}\right.}{\cdots}\right.\cdots\right.}{a2^{\frac{1}{3}}\left(1+a\left(ax+b\right)y\left(x\right)\cdots\right.}$$

$$= 0$$

✓ Solution by Mathematica

Time used: 2.233 (sec). Leaf size: 561

```
DSolve[y[x]*y'[x]==(a*x+b)^(-2)*y[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{ay(x)(ax+b)\operatorname{AiryAi}\left(\frac{-2x^3a^4-6bx^2a^3+(b+ax)^2y(x)^2a^2-6b^2xa^2-2b^3a+2(b+ax)y(x)a+1}{2\sqrt[3]{2}(a(b+ax)^3)^{2/3}}\right)+\operatorname{AiryAi}\left(\frac{-2x^3a^4-6bx^2\cdots}{\cdots}\right.\cdots}{ay(x)(ax+b)\operatorname{AiryBi}\left(\frac{-2x^3a^4-6bx^2a^3+(b+ax)^2y(x)^2a^2-6b^2xa^2-2b^3a+2(b+ax)y(x)a+1}{2\sqrt[3]{2}(a(b+ax)^3)^{2/3}}\right)+\operatorname{AiryBi}\left(\frac{-2x^3a^4-6bx^2\cdots}{\cdots}\right.\cdots}\right.$$

$$\left.+c_1=0,y(x)\right]$$

23.3 problem 3

Internal problem ID [10728]
Internal file name [`OUTPUT/9675_Monday_June_06_2022_03_21_35_PM_22910527/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.2. Equations of the form $yy' = f(x)y + 1$
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \left(a - \frac{1}{ax}\right)y = 1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(y(x)*diff(y(x),x)=(a-1/(a*x))*y(x)+1,y(x), singsol=all)
```

$$y(x) = \frac{a^2 x - \mathrm{RootOf}\left(-\mathrm{e}^{\_Z} - \mathrm{expIntegral}_1\left(-\_Z\right) a^2 x + c_1 a^2 x\right)}{a}$$

✓ Solution by Mathematica

Time used: 0.268 (sec). Leaf size: 37

```
DSolve[y[x]*y'[x]==(a-1/(a*x))*y[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\mathrm{ExpIntegralEi}(a(ax - y(x))) + c_1 = \frac{e^{a(ax-y(x))}}{a^2 x}, y(x)\right]$$

23.4 problem 4

Internal problem ID [10729]
Internal file name [`OUTPUT/9676_Monday_June_06_2022_03_21_36_PM_48846722/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.2. Equations of the form $yy' = f(x)y + 1$
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{yy' - \frac{y}{\sqrt{ax+b}} = 1}$$

23.4.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y + \sqrt{ax+b}}{\sqrt{ax+b}\, y} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(y+\sqrt{ax+b}\right)(b_3-a_2)}{\sqrt{ax+b}\,y} - \frac{\left(y+\sqrt{ax+b}\right)^2 a_3}{(ax+b)\,y^2} - \left(\frac{a}{2\,(ax+b)\,y} - \frac{\left(y+\sqrt{ax+b}\right)a}{2\,(ax+b)^{\frac{3}{2}}\,y}\right)(xa_2+ya_3+a_1) - \left(\frac{1}{y\sqrt{ax+b}} - \frac{y+\sqrt{ax+b}}{\sqrt{ax+b}\,y^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{a^2x^2y^2a_2 - a^2x\,y^3a_3 - a^2x\,y^2a_1 - ab\,y^3a_3 - ab\,y^2a_1 - 2(ax+b)^{\frac{3}{2}}\,a\,x^2b_2 + (ax+b)^{\frac{3}{2}}\,a\,y^2a_3 - 2(ax+b)^{\frac{3}{2}} \cdots}{\cdots}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-a^2x^2y^2a_2+a^2x\,y^3a_3+a^2x\,y^2a_1+ab\,y^3a_3+ab\,y^2a_1+2(ax+b)^{\frac{3}{2}}\,a\,x^2b_2 \\
&-(ax+b)^{\frac{3}{2}}\,a\,y^2a_3+2(ax+b)^{\frac{3}{2}}\,b\,y^2b_2+2(ax+b)^{\frac{3}{2}}\,axb_1-(ax+b)^{\frac{3}{2}}\,aya_1 \\
&+2(ax+b)^{\frac{3}{2}}\,bxb_2-2(ax+b)^{\frac{3}{2}}\,bya_2+4(ax+b)^{\frac{3}{2}}\,byb_3+2a^2x^2y^2b_3 \\
&-4a^2x^2ya_3+\sqrt{ax+b}\,abxya_2-2(ax+b)^{\frac{3}{2}}\,y^2a_3+2(ax+b)^{\frac{3}{2}}\,bb_1 \\
&-2b^2y^2a_2+2b^2y^2b_3-4b^2ya_3+4abx\,y^2b_3-8abxya_3-3abx\,y^2a_2 \\
&+2(ax+b)^{\frac{3}{2}}\,ax\,y^2b_2-3(ax+b)^{\frac{3}{2}}\,axya_2+4(ax+b)^{\frac{3}{2}}\,axyb_3 \\
&+\sqrt{ax+b}\,a^2x^2ya_2+\sqrt{ax+b}\,a^2x\,y^2a_3+\sqrt{ax+b}\,a^2xya_1 \\
&+\sqrt{ax+b}\,ab\,y^2a_3+\sqrt{ax+b}\,abya_1-2(ax+b)^{\frac{5}{2}}\,a_3=0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&a^{2}x^{2}y^{2}a_{2}+a^{2}x\,y^{3}a_{3}+a^{2}x\,y^{2}a_{1}+ab\,y^{3}a_{3}+ab\,y^{2}a_{1}+2(ax+b)^{\frac{3}{2}}\,a\,x^{2}b_{2}\\
&-(ax+b)^{\frac{3}{2}}\,a\,y^{2}a_{3}+2(ax+b)^{\frac{3}{2}}\,b\,y^{2}b_{2}+2(ax+b)^{\frac{3}{2}}\,axb_{1}-(ax+b)^{\frac{3}{2}}\,aya_{1}\\
&+2(ax+b)^{\frac{3}{2}}\,bxb_{2}-2(ax+b)^{\frac{3}{2}}\,bya_{2}+4(ax+b)^{\frac{3}{2}}\,byb_{3}-2(ax+b)\,b\,y^{2}a_{2}\\
&+2(ax+b)\,b\,y^{2}b_{3}+\sqrt{ax+b}\,abxya_{2}-4(ax+b)^{2}\,ya_{3}-2(ax+b)^{\frac{3}{2}}\,y^{2}a_{3}\\
&+2(ax+b)^{\frac{3}{2}}\,bb_{1}+abx\,y^{2}a_{2}+2(ax+b)^{\frac{3}{2}}\,ax\,y^{2}b_{2}-3(ax+b)^{\frac{3}{2}}\,axya_{2}\\
&+4(ax+b)^{\frac{3}{2}}\,axyb_{3}-2(ax+b)\,ax\,y^{2}a_{2}+2(ax+b)\,ax\,y^{2}b_{3}\\
&+\sqrt{ax+b}\,a^{2}x^{2}ya_{2}+\sqrt{ax+b}\,a^{2}x\,y^{2}a_{3}+\sqrt{ax+b}\,a^{2}xya_{1}\\
&+\sqrt{ax+b}\,ab\,y^{2}a_{3}+\sqrt{ax+b}\,abya_{1}-2(ax+b)^{\frac{5}{2}}\,a_{3}=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-2b^{2}\sqrt{ax+b}\,ya_{2}+4b^{2}\sqrt{ax+b}\,yb_{3}-2b\sqrt{ax+b}\,y^{2}a_{3}+2a^{2}x^{3}\sqrt{ax+b}\,b_{2}\\
&-2a^{2}x^{2}\sqrt{ax+b}\,a_{3}+2a^{2}x^{2}\sqrt{ax+b}\,b_{1}+2b^{2}\sqrt{ax+b}\,y^{2}b_{2}\\
&+2b^{2}x\sqrt{ax+b}\,b_{2}-a^{2}x^{2}y^{2}a_{2}+a^{2}x\,y^{3}a_{3}+a^{2}x\,y^{2}a_{1}+ab\,y^{3}a_{3}+ab\,y^{2}a_{1}\\
&+2a^{2}x^{2}y^{2}b_{3}-4a^{2}x^{2}ya_{3}+4abx\sqrt{ax+b}\,y^{2}b_{2}+8abx\sqrt{ax+b}\,yb_{3}\\
&-4\sqrt{ax+b}\,abxya_{2}-2b^{2}y^{2}a_{2}+2b^{2}y^{2}b_{3}-4b^{2}ya_{3}+2b^{2}\sqrt{ax+b}\,b_{1}\\
&-2b^{2}\sqrt{ax+b}\,a_{3}+4abx\,y^{2}b_{3}-8abxya_{3}-3abx\,y^{2}a_{2}-2\sqrt{ax+b}\,a^{2}x^{2}ya_{2}\\
&+2a^{2}x^{2}\sqrt{ax+b}\,y^{2}b_{2}+4a^{2}x^{2}\sqrt{ax+b}\,yb_{3}+4ab\,x^{2}\sqrt{ax+b}\,b_{2}\\
&-2ax\sqrt{ax+b}\,y^{2}a_{3}-4abx\sqrt{ax+b}\,a_{3}+4abx\sqrt{ax+b}\,b_{1}=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{ax+b}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_{1}, y=v_{2}, \sqrt{ax+b}=v_{3}\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2a^{2}v_{1}^{2}v_{3}v_{2}^{2}b_{2}-a^{2}v_{1}^{2}v_{2}^{2}a_{2}-2v_{3}a^{2}v_{1}^{2}v_{2}a_{2}+a^{2}v_{1}v_{2}^{3}a_{3}+2a^{2}v_{1}^{3}v_{3}b_{2}\\
&+2a^{2}v_{1}^{2}v_{2}^{2}b_{3}+4a^{2}v_{1}^{2}v_{3}v_{2}b_{3}+4abv_{1}v_{3}v_{2}^{2}b_{2}+a^{2}v_{1}v_{2}^{2}a_{1}-4a^{2}v_{1}^{2}v_{2}a_{3}\\
&-2a^{2}v_{1}^{2}v_{3}a_{3}+2a^{2}v_{1}^{2}v_{3}b_{1}-3abv_{1}v_{2}^{2}a_{2}-4v_{3}abv_{1}v_{2}a_{2}+abv_{2}^{3}a_{3}\\
&+4abv_{1}^{2}v_{3}b_{2}+4abv_{1}v_{2}^{2}b_{3}+8abv_{1}v_{3}v_{2}b_{3}-2av_{1}v_{3}v_{2}^{2}a_{3}+2b^{2}v_{3}v_{2}^{2}b_{2}+abv_{2}^{2}a_{1}\\
&-8abv_{1}v_{2}a_{3}-4abv_{1}v_{3}a_{3}+4abv_{1}v_{3}b_{1}-2b^{2}v_{2}^{2}a_{2}-2b^{2}v_{3}v_{2}a_{2}+2b^{2}v_{1}v_{3}b_{2}\\
&+2b^{2}v_{2}^{2}b_{3}+4b^{2}v_{3}v_{2}b_{3}-2bv_{3}v_{2}^{2}a_{3}-4b^{2}v_{2}a_{3}-2b^{2}v_{3}a_{3}+2b^{2}v_{3}b_{1}=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&2a^2v_1^3v_3b_2 + 2a^2v_1^2v_3v_2^2b_2 + \left(-a^2a_2 + 2a^2b_3\right)v_1^2v_2^2 \\
&+ \left(-2a^2a_2 + 4a^2b_3\right)v_1^2v_2v_3 - 4a^2v_1^2v_2a_3 + \left(-2a^2a_3 + 2a^2b_1 + 4abb_2\right)v_1^2v_3 \\
&+ a^2v_1v_2^3a_3 + (4abb_2 - 2aa_3)v_1v_2^2v_3 + \left(a^2a_1 - 3aba_2 + 4abb_3\right)v_1v_2^2 \\
&+ (-4aba_2 + 8abb_3)v_1v_2v_3 - 8abv_1v_2a_3 + \left(-4aba_3 + 4abb_1 + 2b^2b_2\right)v_1v_3 \\
&+ abv_2^3a_3 + \left(2b^2b_2 - 2ba_3\right)v_2^2v_3 + \left(aba_1 - 2b^2a_2 + 2b^2b_3\right)v_2^2 \\
&+ \left(-2b^2a_2 + 4b^2b_3\right)v_2v_3 - 4b^2v_2a_3 + \left(-2b^2a_3 + 2b^2b_1\right)v_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a^2a_3 &= 0 \\
aba_3 &= 0 \\
-4a^2a_3 &= 0 \\
2a^2b_2 &= 0 \\
-4b^2a_3 &= 0 \\
-8aba_3 &= 0 \\
-2a^2a_2 + 4a^2b_3 &= 0 \\
-a^2a_2 + 2a^2b_3 &= 0 \\
-2b^2a_2 + 4b^2b_3 &= 0 \\
-2b^2a_3 + 2b^2b_1 &= 0 \\
2b^2b_2 - 2ba_3 &= 0 \\
-4aba_2 + 8abb_3 &= 0 \\
4abb_2 - 2aa_3 &= 0 \\
aba_1 - 2b^2a_2 + 2b^2b_3 &= 0 \\
a^2a_1 - 3aba_2 + 4abb_3 &= 0 \\
-4aba_3 + 4abb_1 + 2b^2b_2 &= 0 \\
-2a^2a_3 + 2a^2b_1 + 4abb_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= \frac{2bb_3}{a} \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{2ax + 2b}{a} \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y + \sqrt{ax + b}}{\sqrt{ax + b}\,y}\right)\left(\frac{2ax + 2b}{a}\right) \\
&= \frac{a\,y^2 - 2ax - 2b - 2\sqrt{ax + b}\,y}{ay} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{a\,y^2-2ax-2b-2\sqrt{ax+b}\,y}{ay}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln\left(-a\,y^2+2ax+2b+2\sqrt{ax+b}\,y\right)}{2} + \frac{\sqrt{ax+b}\,\operatorname{arctanh}\left(\frac{-2ay+2\sqrt{ax+b}}{2\sqrt{2a^2x+2ab+ax+b}}\right)}{\sqrt{2a^2x+2ab+ax+b}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y+\sqrt{ax+b}}{\sqrt{ax+b}\,y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{4a\left(\frac{\left(\left(-\frac{y^2}{2}+x\right)a+y^2+b\right)\sqrt{ax+b}}{2} + \left(\left(-\frac{y^2}{4}+x\right)a+b\right)y\right)}{\sqrt{ax+b}\,\left(-a\,y^2+2ax+2b+2\sqrt{ax+b}\,y\right)^2} \\ S_y &= -\frac{2\left(\left(\left(-\frac{y^2}{2}+x\right)a+b\right)\sqrt{ax+b}+(ax+b)\,y\right)ay}{\sqrt{ax+b}\,\left(-a\,y^2+2ax+2b+2\sqrt{ax+b}\,y\right)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{ax+b}\, \operatorname{arctanh}\left(\frac{-ay+\sqrt{ax+b}}{\sqrt{(2a+1)(ax+b)}}\right)}{\sqrt{(2a+1)\,(ax+b)}} + \frac{\ln\left(2y\sqrt{ax+b}+(-y^2+2x)\,a+2b\right)}{2} = c_1$$

Which simplifies to

$$\frac{\sqrt{ax+b}\, \operatorname{arctanh}\left(\frac{-ay+\sqrt{ax+b}}{\sqrt{(2a+1)(ax+b)}}\right)}{\sqrt{(2a+1)\,(ax+b)}} + \frac{\ln\left(2y\sqrt{ax+b}+(-y^2+2x)\,a+2b\right)}{2} = c_1$$

Summary

The solution(s) found are the following

$$\frac{\sqrt{ax+b}\, \operatorname{arctanh}\left(\frac{-ay+\sqrt{ax+b}}{\sqrt{(2a+1)(ax+b)}}\right)}{\sqrt{(2a+1)\,(ax+b)}} + \frac{\ln\left(2y\sqrt{ax+b}+(-y^2+2x)\,a+2b\right)}{2} = c_1 \tag{1}$$

Verification of solutions

$$\frac{\sqrt{ax+b}\, \operatorname{arctanh}\left(\frac{-ay+\sqrt{ax+b}}{\sqrt{(2a+1)(ax+b)}}\right)}{\sqrt{(2a+1)\,(ax+b)}} + \frac{\ln\left(2y\sqrt{ax+b}+(-y^2+2x)\,a+2b\right)}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = y(x)*a/(2*a*x+2*b), y(x)`      *** Subl
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 153

```
dsolve(y(x)*diff(y(x),x)=(a*x+b)^(-1/2)*y(x)+1,y(x), singsol=all)
```

$$\frac{2\,\mathrm{arctanh}\left(\frac{-\sqrt{ax+b}\,y(x)a+ax+b}{\sqrt{(1+2a)(ax+b)^2}}\right)ax}{\sqrt{(1+2a)\,(ax+b)^2}} + \ln\left(\left(ay(x)^2-2ax-2b\right)\sqrt{ax+b}-2(ax+b)\,y(x)\right) + \frac{2\,\mathrm{arctanh}\left(\frac{-\sqrt{ax+b}\,y(x)a+ax+b}{\sqrt{(1+2a)(ax+b)^2}}\right)b}{\sqrt{(1+2a)\,(ax+b)^2}} - \frac{\ln(ax+b)}{2} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.256 (sec). Leaf size: 90

```
DSolve[y[x]*y'[x]==(a*x+b)^(-1/2)*y[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{\frac{2\arctan\left(\frac{\frac{ay(x)}{\sqrt{ax+b}}-1}{\sqrt{-2a-1}}\right)}{\sqrt{-2a-1}}+\log\left(-\frac{ay(x)^2}{ax+b}+\frac{2y(x)}{\sqrt{ax+b}}+2\right)}{a}=\frac{\log(ax+b)}{a}+c_1,y(x)\right]$$

23.5 problem 5

Internal problem ID [10730]
Internal file name [`OUTPUT/9677_Monday_June_06_2022_03_21_38_PM_47417038/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.2. Equations of the form $yy' = f(x)y + 1$
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_Abel, `2nd type`, `class B`]]`

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{3y}{\sqrt{a\,x^{\frac{3}{2}} + 8x}} = 1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 293

```
dsolve(y(x)*diff(y(x),x)=3*(a*x^(3/2)+8*x)^(-1/2)*y(x)+1,y(x), singsol=all)
```

$$\frac{\left(-\frac{a\sqrt{x}\left(-2a\,x^{\frac{3}{2}}+\sqrt{x}\,a y(x)^{2}-8\sqrt{x\left(8+\sqrt{x}\,a\right)}\,y(x)-16x\right)}{\left(\sqrt{x}\,a y(x)-4\sqrt{x\left(8+\sqrt{x}\,a\right)}\right)^{2}}\right)^{\frac{1}{4}}\sqrt{2\sqrt{x}\,a+16}\,a\sqrt{x}\,y(x)+4\sqrt{-\frac{\sqrt{2\sqrt{x}\,a+16}\,\sqrt{x\left(8+\sqrt{x}\,a\right)}}{\sqrt{x}\,a y(x)-4\sqrt{x\left(8+\sqrt{x}\,a\right)}}}\left(\sqrt{x}\,a\right.}{\sqrt{-\frac{\sqrt{2\sqrt{x}\,a+16}\,\sqrt{x\left(8+\sqrt{x}\,a\right)}}{\sqrt{x}\,a y(x)-4\sqrt{x\left(8+\sqrt{x}\,a\right)}}}\left(\sqrt{x}\,a y\left(x\right)-4\sqrt{x\left(8\right.}\right.}$$

$= 0$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==3*(a*x^(3/2)+8*x)^(-1/2)*y[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

23.6 problem 6

Internal problem ID [10731]
Internal file name [`OUTPUT/9678_Monday_June_06_2022_03_21_43_PM_99671830/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.2. Equations of the form $yy' = f(x)y + 1$
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \left(\frac{a}{x^{\frac{2}{3}}} - \frac{2}{3a\,x^{\frac{1}{3}}}\right)y = 1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 187

```
dsolve(y(x)*diff(y(x),x)=(a*x^(-2/3)-2/3*a^(-1)*x^(-1/3))*y(x)+1,y(x), singsol=all)
```

$$\frac{-\sqrt{\frac{x^{\frac{2}{3}}+ay(x)}{a^4}}\,\mathrm{BesselI}\left(1,\frac{2\sqrt{\frac{x^{\frac{2}{3}}+ay(x)}{a^4}}}{3}\right)c_1a^2+\mathrm{BesselK}\left(1,-\frac{2\sqrt{\frac{x^{\frac{2}{3}}+ay(x)}{a^4}}}{3}\right)\sqrt{\frac{x^{\frac{2}{3}}+ay(x)}{a^4}}\,a^2+x^{\frac{1}{3}}\,\mathrm{BesselI}\left(0,\frac{2\sqrt{\ldots}}{\ldots}\right.}{-\mathrm{BesselI}\left(1,\frac{2\sqrt{\frac{x^{\frac{2}{3}}+ay(x)}{a^4}}}{3}\right)\sqrt{\frac{x^{\frac{2}{3}}+ay(x)}{a^4}}\,a^2+x^{\frac{1}{3}}\,\mathrm{BesselI}\left(0,\frac{2\sqrt{\frac{x^{\frac{2}{3}}+a\ldots}{a^4}}}{3}\right.}$$

$= 0$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==(a*x^(-2/3)-2/3*a^(-1)*x^(-1/3))*y[x]+1,y[x],x,IncludeSingularSolutions -
```

Not solved

23.7 problem 7

Internal problem ID [10732]
Internal file name [`OUTPUT/9679_Monday_June_06_2022_03_21_45_PM_42964674/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.2. Equations of the form $yy' = f(x)y + 1$
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - e^{\lambda x} y a = 1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 83

```
dsolve(y(x)*diff(y(x),x)=a*exp(lambda*x)*y(x)+1,y(x), singsol=all)
```

$$c_1 + a\ \mathrm{erf}\left(\frac{(\lambda y(x) - \mathrm{e}^{x\lambda} a)\sqrt{2}}{2\sqrt{-\lambda}}\right)\sqrt{2}\,\sqrt{\pi} - 2\sqrt{-\lambda}\,\mathrm{e}^{\frac{y(x)^2\lambda^2 - 2y(x)\mathrm{e}^{x\lambda}a\lambda + a^2\mathrm{e}^{2x\lambda} - 2x\,\lambda^2}{2\lambda}} = 0$$

✓ Solution by Mathematica

Time used: 1.687 (sec). Leaf size: 83

```
DSolve[y[x]*y'[x]==a*Exp[\[Lambda]*x]*y[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[-\frac{ae^{\lambda x}}{\sqrt{\lambda}} = \frac{2e^{\frac{(ae^{\lambda x} - \lambda y(x))^2}{2\lambda}}}{\sqrt{2\pi}\mathrm{erfi}\left(\frac{\lambda y(x) - ae^{\lambda x}}{\sqrt{2}\sqrt{\lambda}}\right) + 2c_1}, y(x)\right]$$

23.8 problem 8

Internal problem ID [10733]
Internal file name [`OUTPUT/9680_Monday_June_06_2022_03_21_47_PM_71577788/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.2. Equations of the form $yy' = f(x)y + 1$
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \left(a\,\mathrm{e}^{\lambda x} + b\,\mathrm{e}^{-\lambda x}\right) y = 1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
   -> Calling odsolve with the ODE`, diff(y(x), x) = (x*exp(lambda*y(x))*lambda+(exp(lambda*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
      trying separable
      trying inverse linear
      trying homogeneous types:
      trying Chini
      differential order: 1; looking for linear symmetries
      trying exact
      Looking for potential symmetries
      trying inverse_Riccati
      trying an equivalence to an Abel ODE
      differential order: 1; trying a linearization to 2nd order
      --- trying a change of variables {x -> y(x), y(x) -> x}
      differential order: 1; trying a linearization to 2nd order
      trying 1st order ODE linearizable_by_differentiation
      --- Trying Lie symmetry methods, 1st order ---
      `, `-> Computing symmetries using: way = 3
      `, `-> Computing symmetries using: way = 4
      `, `-> Computing symmetries using: way = 5
      trying symmetry patterns for 1st order ODEs
      -> trying a symmetry pattern of the form [F(x)*G(y), 0]
      -> trying a symmetry pattern of the form [0, F(x)*G(y)]
      -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
      `, `-> Computing symmetries using: way = HINT
         -> Calling odsolve with the ODE`, diff(f__1(x), x), f__1(x)`          *** Subleve
            Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=(a*exp(lambda*x)+b*exp(-lambda*x))*y(x)+1,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==(a*Exp[\[Lambda]*x]+b*Exp[-\[Lambda]*x])*y[x]+1,y[x],x,IncludeSingularSol
```

Not solved

23.9 problem 9

Internal problem ID [10734]
Internal file name [`OUTPUT/9681_Monday_June_06_2022_03_21_49_PM_97771181/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.2. Equations of the form $yy' = f(x)y + 1$
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - ay\cosh(x) = 1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
   -> Calling odsolve with the ODE`, diff(y(x), x) = a*sinh(y(x))+x, y(x), implicit`      **
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
      trying separable
      trying inverse linear
      trying homogeneous types:
      trying Chini
      differential order: 1; looking for linear symmetries
      trying exact
      Looking for potential symmetries
      trying inverse_Riccati
      trying an equivalence to an Abel ODE
      differential order: 1; trying a linearization to 2nd order
      --- trying a change of variables {x -> y(x), y(x) -> x}
      differential order: 1; trying a linearization to 2nd order
      trying 1st order ODE linearizable_by_differentiation
      --- Trying Lie symmetry methods, 1st order ---
      `, `-> Computing symmetries using: way = 3
      `, `-> Computing symmetries using: way = 4
      `, `-> Computing symmetries using: way = 5
      trying symmetry patterns for 1st order ODEs
      -> trying a symmetry pattern of the form [F(x)*G(y), 0]
      -> trying a symmetry pattern of the form [0, F(x)*G(y)]
      -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
      -> trying a symmetry pattern of the form [F(x),G(x)]
      -> trying a symmetry pattern of the form [F(y),G(y)]
      -> trying a symmetry pattern of the form [F(x)+G(y), 0]
      -> trying a symmetry pattern of the form [0, F(x)+G(y)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=a*y(x)*cosh(x)+1,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==a*y[x]*Cosh[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

23.10 problem 10

Internal problem ID [10735]
Internal file name [`OUTPUT/9682_Monday_June_06_2022_03_21_52_PM_3984291/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.2. Equations of the form $yy' = f(x)y + 1$
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - ay\sinh(x) = 1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
   -> Calling odsolve with the ODE`, diff(y(x), x) = a*cosh(y(x))+x, y(x), implicit`      **
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
      trying separable
      trying inverse linear
      trying homogeneous types:
      trying Chini
      differential order: 1; looking for linear symmetries
      trying exact
      Looking for potential symmetries
      trying inverse_Riccati
      trying an equivalence to an Abel ODE
      differential order: 1; trying a linearization to 2nd order
      --- trying a change of variables {x -> y(x), y(x) -> x}
      differential order: 1; trying a linearization to 2nd order
      trying 1st order ODE linearizable_by_differentiation
      --- Trying Lie symmetry methods, 1st order ---
      `, `-> Computing symmetries using: way = 3
      `, `-> Computing symmetries using: way = 4
      `, `-> Computing symmetries using: way = 5
      trying symmetry patterns for 1st order ODEs
      -> trying a symmetry pattern of the form [F(x)*G(y), 0]
      -> trying a symmetry pattern of the form [0, F(x)*G(y)]
      -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
      -> trying a symmetry pattern of the form [F(x),G(x)]
      -> trying a symmetry pattern of the form [F(y),G(y)]
      -> trying a symmetry pattern of the form [F(x)+G(y), 0]
      -> trying a symmetry pattern of the form [0, F(x)+G(y)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=a*y(x)*sinh(x)+1,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==a*y[x]*Sinh[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

23.11 problem 11

Internal problem ID [10736]
Internal file name [`OUTPUT/9683_Monday_June_06_2022_03_21_55_PM_71241704/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.2. Equations of the form $yy' = f(x)y + 1$
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - a\cos(\lambda x)\, y = 1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
   -> Calling odsolve with the ODE`, diff(y(x), x) = (lambda*x+a*sin(lambda*y(x)))/lambda, y
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
      trying separable
      trying inverse linear
      trying homogeneous types:
      trying Chini
      differential order: 1; looking for linear symmetries
      trying exact
      Looking for potential symmetries
      trying inverse_Riccati
      trying an equivalence to an Abel ODE
      differential order: 1; trying a linearization to 2nd order
      --- trying a change of variables {x -> y(x), y(x) -> x}
      differential order: 1; trying a linearization to 2nd order
      trying 1st order ODE linearizable_by_differentiation
      --- Trying Lie symmetry methods, 1st order ---
      `, `-> Computing symmetries using: way = 3
      `, `-> Computing symmetries using: way = 4
      `, `-> Computing symmetries using: way = 5
      trying symmetry patterns for 1st order ODEs
      -> trying a symmetry pattern of the form [F(x)*G(y), 0]
      -> trying a symmetry pattern of the form [0, F(x)*G(y)]
      -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
      `, `-> Computing symmetries using: way = HINT
         -> Calling odsolve with the ODE`, diff(f__1(y), y) = cos(lambda*y)*f__1(y)*lambda/s
            Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=a*cos(lambda*x)*y(x)+1,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==a*Cos[\[Lambda]*x]*y[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

23.12 problem 12

Internal problem ID [10737]
Internal file name [`OUTPUT/9684_Monday_June_06_2022_03_21_58_PM_79559776/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.2. Equations of the form $yy' = f(x)y + 1$
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \sin(\lambda x)\, ya = 1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
   -> Calling odsolve with the ODE`, diff(y(x), x) = -(a*cos(lambda*y(x))-lambda*x)/lambda,
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
      trying separable
      trying inverse linear
      trying homogeneous types:
      trying Chini
      differential order: 1; looking for linear symmetries
      trying exact
      Looking for potential symmetries
      trying inverse_Riccati
      trying an equivalence to an Abel ODE
      differential order: 1; trying a linearization to 2nd order
      --- trying a change of variables {x -> y(x), y(x) -> x}
      differential order: 1; trying a linearization to 2nd order
      trying 1st order ODE linearizable_by_differentiation
      --- Trying Lie symmetry methods, 1st order ---
      `, `-> Computing symmetries using: way = 3
      `, `-> Computing symmetries using: way = 4
      `, `-> Computing symmetries using: way = 5
      trying symmetry patterns for 1st order ODEs
      -> trying a symmetry pattern of the form [F(x)*G(y), 0]
      -> trying a symmetry pattern of the form [0, F(x)*G(y)]
      -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
      `, `-> Computing symmetries using: way = HINT
         -> Calling odsolve with the ODE`, diff(f__1(y), y) = -f__1(y)*lambda*sin(lambda*y)/
            Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=a*sin(lambda*x)*y(x)+1,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==a*Sin[\[Lambda]*x]*y[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

24 Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$

24.1 problem 1

Internal problem ID [10738]
Internal file name [`OUTPUT/9685_Monday_June_06_2022_03_22_01_PM_52889186/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - (ax + 3b)\, y = -ab\, x^2 + c\, x^3 - 2b^2 x}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 224

```
dsolve(y(x)*diff(y(x),x)=(a*x+3*b)*y(x)+c*x^3-a*b*x^2-2*b^2*x,y(x), singsol=all)
```

$$\frac{x\left(\frac{-2y(x)^2+x(ax+4b)y(x)-ab\,x^3+c\,x^4-2b^2x^2}{(bx-y(x))^2}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{a\,\operatorname{arctanh}\left(\frac{-abx+2c\,x^2+ay(x)}{(-bx+y(x))\sqrt{a^2+8c}}\right)}{2\sqrt{a^2+8c}}} y(x)+\sqrt{\frac{x^2}{-bx+y(x)}}\,(bx-y(x))\left(\left(\int \frac{x^2}{-bx+y(x)} \cdots\right.\right.}{\sqrt{\frac{x^2}{-bx+y(x)}}\,(bx-y(x))}$$

$= 0$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==(a*x+3*b)*y[x]+c*x^3-a*b*x^2-2*b^2*x,y[x],x,IncludeSingularSolutions -> T
```

Not solved

24.2 problem 2

Internal problem ID [10739]
Internal file name [`OUTPUT/9686_Monday_June_06_2022_03_22_02_PM_81500461/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

$$\boxed{yy' - (3ax + b)\, y = -x^3a^2 - ab\, x^2 + cx}$$

24.2.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-x^3a^2 - ab\, x^2 + 3axy + by + cx}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 3 to use as anstaz gives

$$\xi = x^3a_7 + x^2ya_8 + x\, y^2a_9 + y^3a_{10} + x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = x^3b_7 + x^2yb_8 + x\, y^2b_9 + y^3b_{10} + x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&3x^2b_7 + 2xyb_8 + y^2b_9 + 2xb_4 + yb_5 + b_2 \\
&+ \frac{(-x^3a^2 - ab\,x^2 + 3axy + by + cx)\,(-3x^2a_7 + x^2b_8 - 2xya_8 + 2xyb_9 - y^2a_9 + 3y^2b_{10} - 2xa_4 + xb_5 - ya_5}{y} \\
&- \frac{(-x^3a^2 - ab\,x^2 + 3axy + by + cx)^2\,(x^2a_8 + 2xya_9 + 3y^2a_{10} + xa_5 + 2ya_6 + a_3)}{y^2} \\
&- \frac{(-3a^2x^2 - 2abx + 3ay + c)\,(x^3a_7 + x^2ya_8 + x\,y^2a_9 + y^3a_{10} + x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1)}{y} \\
&- \left(\frac{3ax + b}{y} - \frac{-x^3a^2 - ab\,x^2 + 3axy + by + cx}{y^2}\right)(x^3b_7 + x^2yb_8 \\
&+ x\,y^2b_9 + y^3b_{10} + x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\text{Expression too large to display} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-a^4a_5$$

$$-6a^3ba_{10}$$

$$-2a^4a_6 - 4a^3ba$$

$$-16aba_{10} - 6aa$$

$$24a^2ba_{10} - 14a^2a_9$$

$$-abb_1 - c$$

$$2aba_1 - 2bca_3 - 2c$$

$$-a^4a_3 - 2a^3ba_5 - a^2b^2a_8 + 2a^2c$$

$$2abca_3 - a^2b_1 - abb_2 - c$$

$$-b^2a_3 - 3aa_1 - ba_2 + bb_3 -$$

$$-2b^2a_6 - 3aa_3 - ba_5 + 2bb_6 -$$

$$-2a^3ba_3 - a^2b^2a_5 + 2a^2ca_5 + 2abca_8 - a^2$$

$$-3a^2b^2a_{10} + 12a^3a_6 + 16a^2ba_9 + 6a^2ca_{10} - 4a^2a$$

$$-3b^2a_{10} - 3aa_6 - ba_9 + 3bb_{10} -$$

$$-a^2b^2a_3 + 2a^2ca_3 + 2abca_5 - a^2b_2 - abb_4 - c$$

$$-4a^3ba_6 - 2a^2b^2a_9 + 6a^3a_5 + 8a^2ba_8 + 4a^2ca_9 + 6a^2a$$

$$6a\,b^2a_{10} - 15a^2a_6 - 9aba_9 - 4abb_{10} - 18aca_{10} - 9a$$

$$2a\,b^2a_3 + 3a^2a_1 + 3aba_2 - 2abb_3 - 6aca_3 - 2bca_5 - 2c^2a_6 - 3c$$

$$-4aba_3 - b^2a_5 - 4bca_6 - 6aa_2 + 3ab_3 - 2ba_4 + bb_5 - 2ca_5 + 3$$

$$16a^2ba_6 + 4a\,b^2a_9 + 6abca_{10} - 5a^2a_5 - 3a^2b_6 - 2aba_8 - 3abb_9 - 12aca_9 - 12a$$

$$-10aba_6 - 2b^2a_9 - 6bca_{10} - 6aa_5 + 6ab_6 - 2ba_8 + 2bb_9 - 2ca_9 + 4c$$

$$-2a^2b^2a_6 + 6a^3a_3 + 8a^2ba_5 + 4a^2ca_6 + 2a\,b^2a_8 + 4abca_9 + 5a^2a_4 - 2a^2b_5 + 5aba_7 - 2abb$$

$$8a^2ba_3 + 2a\,b^2a_5 + 4abca_6 + 4a^2a_2 - 2a^2b_3 + 4aba_4 - 2abb_5 - 6aca_5 - 2bca_8 - 2c^2a_9 - 4c$$

$$4a\,b^2a_6 - 6a^2a_3 - 3aba_5 - 3abb_6 - 12aca_6 - b^2a_8 - 4bca_9 - 3c^2a_{10} - 9aa_4 + 3ab_5 - 3ba_7 + bb_8 - 3ca_8 + 3$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{cb_6}{a} \\
a_3 &= a_3 \\
a_4 &= -bb_6 \\
a_5 &= 0 \\
a_6 &= 0 \\
a_7 &= -ab_6 \\
a_8 &= 0 \\
a_9 &= 0 \\
a_{10} &= 0 \\
b_1 &= 0 \\
b_2 &= ca_3 \\
b_3 &= \frac{aba_3 + cb_6}{a} \\
b_4 &= -aba_3 \\
b_5 &= 3aa_3 - 2bb_6 \\
b_6 &= b_6 \\
b_7 &= -a^2a_3 \\
b_8 &= -3ab_6 \\
b_9 &= 0 \\
b_{10} &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{x(a^2x^2 + abx - c)}{a} \\
\eta &= -\frac{y(3a^2x^2 + 2abx - ay - c)}{a}
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -\frac{y(3a^2x^2+2abx-ay-c)}{a} - \left(\frac{-x^3a^2-ab\,x^2+3axy+by+cx}{y}\right)\left(-\frac{x(a^2x^2+abx-c)}{a}\right) \\
&= \frac{-a^4x^6-2a^3b\,x^5+3a^3x^4y-a^2b^2x^4+4a^2b\,x^3y+2a^2c\,x^4-3a^2x^2y^2+a\,b^2x^2y+2abc\,x^3-2abx\,y^2-}{ay} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta}dy \\
&= \int \frac{1}{\frac{-a^4x^6-2a^3b\,x^5+3a^3x^4y-a^2b^2x^4+4a^2b\,x^3y+2a^2c\,x^4-3a^2x^2y^2+a\,b^2x^2y+2abc\,x^3-2abx\,y^2-3ac\,x^2y+a\,y^3-bcxy-c^2x^2+c\,y^2}{ay}}dy
\end{aligned}
$$

Which results in

$$
S = a\left(-\frac{\ln\left(-a^2x^2-abx+ay+c\right)}{c} + \frac{\frac{\ln\left(a^2x^4+ab\,x^3-2a\,x^2y-bxy-c\,x^2+y^2\right)}{2} - \frac{bx\,\operatorname{arctanh}\left(\frac{-2a\,x^2-bx+2y}{\sqrt{b^2x^2+4c\,x^2}}\right)}{\sqrt{b^2x^2+4c\,x^2}}}{c}\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-x^3a^2 - ab\,x^2 + 3axy + by + cx}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= -\frac{a(x^3a^2 + ab\,x^2 + (-3ay - c)\,x - by)}{(a^2x^4 + ab\,x^3 + (-2ay - c)\,x^2 - bxy + y^2)\,(a^2x^2 + abx - ay - c)}\\
S_y &= -\frac{ay}{(a^2x^4 + ab\,x^3 + (-2ay - c)\,x^2 - bxy + y^2)\,(a^2x^2 + abx - ay - c)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{a\left(2b \operatorname{arctanh}\left(\frac{2a\,x^2+bx-2y}{x\sqrt{b^2+4c}}\right) - 2\ln\left(-a^2x^2 + (-bx+y)\,a + c\right)\sqrt{b^2+4c} + \ln\left(a^2x^4 + ab\,x^3 + (-2ay - c)\,x^2\right.}{2\sqrt{b^2+4c}\,c}$$

Which simplifies to

$$\frac{a\left(2b \operatorname{arctanh}\left(\frac{2a\,x^2+bx-2y}{x\sqrt{b^2+4c}}\right) - 2\ln\left(-a^2x^2 + (-bx+y)\,a + c\right)\sqrt{b^2+4c} + \ln\left(a^2x^4 + ab\,x^3 + (-2ay - c)\,x^2\right.}{2\sqrt{b^2+4c}\,c}$$

Summary
The solution(s) found are the following

$$\frac{a\left(2b \operatorname{arctanh}\left(\frac{2a\,x^2+bx-2y}{x\sqrt{b^2+4c}}\right)-2\ln\left(-a^2x^2+(-bx+y)\,a+c\right)\sqrt{b^2+4c}+\ln\left(a^2x^4+ab\,x^3+(-2ay-c)\,x^2\right.\right.}{2\sqrt{b^2+4c}\,c} \tag{1}$$

$= c_1$

Verification of solutions

$$\frac{a\left(2b \operatorname{arctanh}\left(\frac{2a\,x^2+bx-2y}{x\sqrt{b^2+4c}}\right)-2\ln\left(-a^2x^2+(-bx+y)\,a+c\right)\sqrt{b^2+4c}+\ln\left(a^2x^4+ab\,x^3+(-2ay-c)\,x^2\right.\right.}{2\sqrt{b^2+4c}\,c}$$

$= c_1$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 826

```
dsolve(y(x)*diff(y(x),x)=(3*a*x+b)*y(x)-a^2*x^3-a*b*x^2+c*x,y(x), singsol=all)
```

$y(x)$

$$= \frac{}{-b\,\mathrm{e}^{\operatorname{RootOf}\left(2a\,b^8 \operatorname{arctanh}\left(\frac{b^2\left(9b^2+27c+2\,\mathrm{e}^{\_Z}\right)}{9\sqrt{b^2\left(b^2+4c\right)\left(b^2+3c\right)^2}}\right)+20a\,b^6c \operatorname{arctanh}\left(\frac{b^2\left(9b^2+27c+2\,\mathrm{e}^{\_Z}\right)}{9\sqrt{b^2\left(b^2+4c\right)\left(b^2+3c\right)^2}}\right)+66a\,b^4c^2 \operatorname{arctanh}\left(\frac{b^2\left(9b^2+27c+2\,\mathrm{e}^{\_Z}\right)}{9\sqrt{b^2\left(b^2+4c\right)\left(b^2+3c\right)^2}}\right.\right.}}$$

✓ Solution by Mathematica

Time used: 6.592 (sec). Leaf size: 194

```
DSolve[y[x]*y'[x]==(3*a*x+b)*y[x]-a^2*x^3-a*b*x^2+c*x,y[x],x,IncludeSingularSolutions -> Tru
```

$$\text{Solve}\left[\frac{2ab\left(\text{RootSum}\left[\#1^4a^2+\#1^3ab-2\#1^2ay(x)-\#1^2c-\#1by(x)+y(x)^2\&,\frac{-2\#1^3a^2\log(x-\#1)-\#1^2}{}\right.\right.}{c(3a+b-}\right.$$

24.3 problem 3

Internal problem ID [10740]
Internal file name [`OUTPUT/9687_Monday_June_06_2022_03_22_05_PM_72048347/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{2yy' - (7ax + 5b)\, y = -3x^3a^2 - 3b^2x - 2c\, x^2}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 4589

```
dsolve(2*y(x)*diff(y(x),x)=(7*a*x+5*b)*y(x)-3*a^2*x^3-2*c*x^2-3*b^2*x,y(x), singsol=all)
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[2*y[x]*y'[x]==(7*a*x+5*b)*y[x]-3*a^2*x^3-2*c*x^2-3*b^2*x,y[x],x,IncludeSingularSoluti
```

Not solved

24.4 problem 4

Internal problem ID [10741]
Internal file name [`OUTPUT/9688_Monday_June_06_2022_03_22_13_PM_18411353/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - ((3-m)\,x-1)\,y = -(m-1)\,ax}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=((3-m)*x-1)*y(x)+(m-1)*(x^2-x^2-a*x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==((3-m)*x-1)*y[x]+(m-1)*(x^2-x^2-a*x),y[x],x,IncludeSingularSolutions -> T
```

Not solved

24.5 problem 5

Internal problem ID [10742]
Internal file name [`OUTPUT/9689_Monday_June_06_2022_03_22_16_PM_26947723/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' + x(a x^2 + b) y = -x}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 179

```
dsolve(y(x)*diff(y(x),x)+x*(a*x^2+b)*y(x)+x=0,y(x), singsol=all)
```

$$\frac{2\,\mathrm{AiryBi}\left(1, \frac{4ay(x)+(a\,x^2+b)^2}{4a^{\frac{2}{3}}}\right) a^{\frac{1}{3}} c_1 + c_1(a\,x^2+b)\,\mathrm{AiryBi}\left(\frac{4ay(x)+(a\,x^2+b)^2}{4a^{\frac{2}{3}}}\right) - 2\,\mathrm{AiryAi}\left(1, \frac{4ay(x)+(a\,x^2+b)^2}{4a^{\frac{2}{3}}}\right) a \cdots}{2\,\mathrm{AiryBi}\left(1, \frac{4ay(x)+(a\,x^2+b)^2}{4a^{\frac{2}{3}}}\right) a^{\frac{1}{3}} + \mathrm{AiryBi}\left(\frac{4ay(x)+(a\,x^2+b)^2}{4a^{\frac{2}{3}}}\right)(a\,x^2+b \cdots}$$
$$= 0$$

✓ Solution by Mathematica

Time used: 0.492 (sec). Leaf size: 143

```
DSolve[y[x]*y'[x]+x*(a*x^2+b)*y[x]+x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\frac{(ax^2+b)\,\mathrm{AiryAi}\left(\frac{(ax^2+b)^2+4ay(x)}{4a^{2/3}}\right)+2\sqrt[3]{a}\,\mathrm{AiryAiPrime}\left(\frac{(ax^2+b)^2+4ay(x)}{4a^{2/3}}\right)}{(ax^2+b)\,\mathrm{AiryBi}\left(\frac{(ax^2+b)^2+4ay(x)}{4a^{2/3}}\right)+2\sqrt[3]{a}\,\mathrm{AiryBiPrime}\left(\frac{(ax^2+b)^2+4ay(x)}{4a^{2/3}}\right)} + c_1 = 0, y(x)\right]$$

24.6 problem 6

Internal problem ID [10743]
Internal file name [`OUTPUT/9690_Monday_June_06_2022_03_22_17_PM_23437655/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' + a\left(1 - \frac{1}{x}\right)y = a^2$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x)+a*(1-x^(-1))*y(x)=a^2,y(x), singsol=all)
```

$$y(x) = a\left(-x + \text{RootOf}\left(-e^{-\_Z} - \text{expIntegral}_1\left(-\_Z\right)x + c_1 x\right)\right)$$

✓ Solution by Mathematica

Time used: 0.208 (sec). Leaf size: 30

```
DSolve[y[x]*y'[x]+a*(1-x^(-1))*y[x]==a^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\text{ExpIntegralEi}\left(x + \frac{y(x)}{a}\right) + c_1 = \frac{e^{\frac{y(x)}{a}+x}}{x}, y(x)\right]$$

24.7 problem 7

Internal problem ID [10744]
Internal file name [`OUTPUT/9691_Monday_June_06_2022_03_22_18_PM_56791468/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - a\left(1 - \frac{b}{x}\right)y = a^2 b}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(y(x)*diff(y(x),x)-a*(1-b*x^(-1))*y(x)=a^2*b,y(x), singsol=all)
```

$$y(x) = a\left(-\operatorname{RootOf}\left(-\mathrm{e}^{-Z} b-\operatorname{expIntegral}_{1}\left(-\_Z\right) x+c_{1} x\right) b+x\right)$$

✓ Solution by Mathematica

Time used: 0.293 (sec). Leaf size: 45

```
DSolve[y[x]*y'[x]-a*(1-b*x^(-1))*y[x]==a^2*b,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\text{ExpIntegralEi}\left(\frac{ax-y(x)}{ab}\right)+c_1=\frac{be^{\frac{ax-y(x)}{ab}}}{x},y(x)\right]$$

24.8 problem 8

Internal problem ID [10745]
Internal file name [`OUTPUT/9692_Monday_June_06_2022_03_22_19_PM_7903388/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - x^{n-1}((2n+1)\,x + an)\,y = -n\,x^{2n}(x+a)}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 153

```
dsolve(y(x)*diff(y(x),x)=x^(n-1)*((1+2*n)*x+a*n)*y(x)-n*x^(2*n)*(x+a),y(x), singsol=all)
```

$$y(x) = -\frac{2\left(-\frac{\sqrt{-n^2}\, x \tan\left(\frac{\mathrm{RootOf}\left(-2an\,\mathrm{e}^{\_Z+\_a}-nx\,\mathrm{e}^{\_Z+\_a}-\tan\left(\frac{\_a\sqrt{-n^2}}{2}\right)\_Z\sqrt{-n^2}\,x+2c_1 x\,\mathrm{e}^{\_a}\right)\sqrt{-n^2}}{2}\right)}{2}+n\left(a+\frac{x}{2}\right)\right)x^n}{\tan\left(\frac{\mathrm{RootOf}\left(-2an\,\mathrm{e}^{\_Z+\_a}-nx\,\mathrm{e}^{\_Z+\_a}-\tan\left(\frac{\_a\sqrt{-n^2}}{2}\right)\_Z\sqrt{-n^2}\,x+2c_1 x\,\mathrm{e}^{\_a}\right)\sqrt{-n^2}}{2}\right)\sqrt{-n^2}-n}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==x^(n-1)*((1+2*n)*x+a*n)*y[x]-n*x^(2*n)*(x+a),y[x],x,IncludeSingularSoluti
```

Not solved

24.9 problem 9

Internal problem ID [10746]
Internal file name [`OUTPUT/9693_Monday_June_06_2022_03_22_23_PM_14690200/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - a(-nb + x)\, x^{n-1} y = c\left(x^2 - (2n+1)\, bx + n(1+n)\, b^2\right) x^{-1+2n}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 1972

```
dsolve(y(x)*diff(y(x),x)=a*(x-n*b)*x^(n-1)*y(x)+c*(x^2-(2*n+1)*b*x+n*(n+1)*b^2)*x^(2*n-1),y(
```

Expression too large to display

✓ Solution by Mathematica

Time used: 0.744 (sec). Leaf size: 200

```
DSolve[y[x]*y'[x]==a*(x-n*b)*x^(n-1)*y[x]+c*(x^2-(2*n+1)*b*x+n*(n+1)*b^2)*x^(2*n-1),y[x],x,I
```

$$\text{Solve}\left[\frac{a^2\left(-\frac{2a\,\text{arctanh}\left(\frac{a^2-\frac{2ac(n+1)y(x)}{-bcx^n-bcnx^n+cx^{n+1}}}{a\sqrt{a^2+4c(n+1)}}\right)}{\sqrt{a^2+4c(n+1)}}-\log\left(a^2\left(\frac{ay(x)}{-bcx^n-bcnx^n+cx^{n+1}}+1\right)-\frac{a^2c(n+1)y(x)^2}{(-bcx^n-bcnx^n+cx^{n+1})^2}\right)\right)}{2c(n+1)}\right.$$

$$\left.+c_1,y(x)\right]$$

24.10 problem 10

Internal problem ID [10747]
Internal file name [`OUTPUT/9694_Monday_June_06_2022_03_22_25_PM_92796696/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \left(a(2n+k)\,x^{k} + b\right) x^{n-1} y = \left(-a^{2} n\, x^{2k} - ab\, x^{k} + c\right) x^{-1+2n}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(x^k*k^2*a+3*x^k*k*a*n+2*x^k*a*n^2-a
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(2*a^2*n*x^(2*k)*k+2*x^(2*k)*a^2*n^2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=(a*(2*n+k)*x^k+b)*x^(n-1)*y(x)+(-a^2*n*x^(2*k)-a*b*x^k+c)*x^(2*n-1)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==(a*(2*n+k)*x^k+b)*x^(n-1)*y[x]+(-a^2*n*x^(2*k)-a*b*x^k+c)*x^(2*n-1),y[x],
```

Not solved

24.11 problem 11

Internal problem ID [10748]
Internal file name [`OUTPUT/9695_Monday_June_06_2022_03_22_44_PM_84902151/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - \left(a(2n+k)\,x^{2k} + b(2m-k)\right)x^{m-k-1}y = -\frac{a^2m\,x^{4k} + c\,x^{2k} + b^2m}{x}$$

Unable to determine ODE type.

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=(a*(2*n+k)*x^(2*k)+b*(2*m-k))*x^(m-k-1)*y(x)-(a^2*m*x^(4*k)+c*x^(2*
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==(a*(2*n+k)*x^(2*k)+b*(2*m-k))*x^(m-k-1)*y[x]-(a^2*m*x^(4*k)+c*x^(2*k)+b^2
```

Timed out

24.12 problem 12

Internal problem ID [10749]
Internal file name [`OUTPUT/9696_Monday_June_06_2022_03_24_54_PM_95434992/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - \frac{((m+2L-3)x+n-2L+3)y}{x} = ((m-L-1)x^2+(n-m-2L+3)x-n+L-2)x^{1-2L}$$

Unable to determine ODE type.

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=((m+2*L-3)*x+n-2*L+3)*1/x*y(x)+((m-L-1)*x^2+(n-m-2*L+3)*x-n+L-2)*x^
```

No solution found

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==((m+2*L-3)*x+n-2*L+3)*1/x*y[x]+((m-L-1)*x^2+(n-m-2*L+3)*x-n+L-2)*x^(1-2*L
```

Timed out

24.13 problem 13

Internal problem ID [10750]
Internal file name [`OUTPUT/9697_Monday_June_06_2022_03_26_59_PM_64549683/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - \left(a(2n+1)\,x^2 + cx + b(-1+2n)\right)x^{n-2}y = -\left(n\,a^2x^4 + ac\,x^3 + n\,b^2 + bcx + d\,x^2\right)x^{2n-3}$$

Unable to determine ODE type.

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=(a*(2*n+1)*x^2+c*x+b*(2*n-1))*x^(n-2)*y(x)-(n*a^2*x^4+a*c*x^3+d*x^2
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==(a*(2*n+1)*x^2+c*x+b*(2*n-1))*x^(n-2)*y[x]-(n*a^2*x^4+a*c*x^3+d*x^2+b*c*x
```

Timed out

24.14 problem 14

Internal problem ID [10751]
Internal file name [`OUTPUT/9698_Monday_June_06_2022_03_28_49_PM_67334612/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - (a(n-1)\,x + b(2\lambda + n))\,x^{\lambda-1}(ax+b)^{-\lambda-2}\,y = -(anx + b(\lambda + n))\,x^{2\lambda-1}(ax+b)^{-2\lambda-3}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(3*a^2*n*x^2-2*a*b*lambda*n*x+4*a*b*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(2*a^2*n*x^2-a*b*lambda*n*x-2*a^2*x^
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=(a*(n-1)*x+b*(2*lambda+n))*x^(lambda-1)*(a*x+b)^(-lambda-2)*y(x)-(a
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==(a*(n-1)*x+b*(2*\[Lambda]+n))*x^(\[Lambda]-1)*(a*x+b)^(-\[Lambda]-2)*y[x]
```

Not solved

24.15 problem 15

Internal problem ID [10752]
Internal file name [`OUTPUT/9699_Monday_June_06_2022_03_29_10_PM_72950326/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{a((m-1)\,x+1)\,y}{x} = \frac{a^2(mx+1)\,(x-1)}{x}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 279

```
dsolve(y(x)*diff(y(x),x)-a*((m-1)*x+1)*1/x*y(x)=a^2*1/x*(m*x+1)*(x-1),y(x), singsol=all)
```

$$-\frac{27(m-1)\left(-54m^4xa(m+2)\left(m+\frac{1}{2}\right)\left(\int^{\frac{9m\left((m-1)y(x)+3\left(\frac{1}{3}+\left(x-\frac{1}{3}\right)m\right)a\right)}{(m-1)(1+2m)(m+2)(-y(x)+a)}}\frac{\_a\left((m^2+m-2)\_a-9m\right)^{\frac{1}{1+m}}\left((2m^2-\cdots\right.}{8\left(\left(m^2-\frac{1}{2}m-\frac{1}{2}\right)\_a+\frac{9m}{2}\right)\left((m^2+m-2)\_a-9m\cdots\right.}\cdots\right.\right.}{m\,(2\cdots}$$
$$=0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-a*((m-1)*x+1)*1/x*y[x]==a^2*1/x*(m*x+1)*(x-1),y[x],x,IncludeSingularSoluti
```

Not solved

24.16 problem 16

Internal problem ID [10753]
Internal file name [`OUTPUT/9700_Monday_June_06_2022_03_29_14_PM_62636486/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$
\boxed{yy' - a\left(1 - \frac{b}{\sqrt{x}}\right)y = \frac{a^2 b}{\sqrt{x}}}
$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 269

```
dsolve(y(x)*diff(y(x),x)-a*(1-b*x^(-1/2))*y(x)=a^2*b*x^(-1/2),y(x), singsol=all)
```

$$\frac{(b^2)^{\frac{1}{3}} c_1 2^{\frac{2}{3}} \left(-\sqrt{x}+b\right) \operatorname{AiryBi}\left(\frac{2^{\frac{1}{3}}\left(-2\sqrt{x}\,ab+(b^2+x)a-y(x)\right)}{2(b^2)^{\frac{1}{3}}a}\right)+2\operatorname{AiryBi}\left(1, \frac{2^{\frac{1}{3}}\left(-2\sqrt{x}\,ab+(b^2+x)a-y(x)\right)}{2(b^2)^{\frac{1}{3}}a}\right) c_1 b - 2\,\mathrm{Ai}}{(b^2)^{\frac{1}{3}} 2^{\frac{2}{3}} \left(-\sqrt{x}+b\right) \operatorname{AiryBi}\left(\frac{2^{\frac{1}{3}}\left(-2\sqrt{x}\,ab+(b^2+x)a-y(x)\right)}{2(b^2)^{\frac{1}{3}}a}\right)}$$

$= 0$

✓ Solution by Mathematica

Time used: 1.905 (sec). Leaf size: 323

```
DSolve[y[x]*y'[x]-a*(1-b*x^(-1/2))*y[x]==a^2*b*x^(-1/2),y[x],x,IncludeSingularSolutions -> T
```

$$\text{Solve}\left[\frac{\sqrt[3]{-1}2^{2/3}\sqrt[3]{\left(b-\sqrt{x}\right)^3}\,\text{AiryAi}\left(\frac{\left(-\frac{1}{2}\right)^{2/3}\left(\left(b-\sqrt{x}\right)^3\right)^{2/3}\left(a\left(b-\sqrt{x}\right)^2-y(x)\right)}{ab^{2/3}\left(b-\sqrt{x}\right)^2}\right)-2\sqrt[3]{b}\,\text{AiryAiPrime}\left(\frac{\left(-\frac{1}{2}\right)^{2/3}\left(\left(b-\right.\right.}{}\right.}{\sqrt[3]{-1}2^{2/3}\sqrt[3]{\left(b-\sqrt{x}\right)^3}\,\text{AiryBi}\left(\frac{\left(-\frac{1}{2}\right)^{2/3}\left(\left(b-\sqrt{x}\right)^3\right)^{2/3}\left(a\left(b-\sqrt{x}\right)^2-y(x)\right)}{ab^{2/3}\left(b-\sqrt{x}\right)^2}\right)-2\sqrt[3]{b}\,\text{AiryBiPrime}\left(\frac{\left(-\frac{1}{2}\right)^{2/3}\left(\left(b-\right.\right.}{}\right.}\right.$$

$$\left.+c_1=0,y(x)\right]$$

24.17 problem 17

Internal problem ID [10754]
Internal file name [`OUTPUT/9701_Monday_June_06_2022_03_29_16_PM_5596839/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{3y}{(ax+b)^{\frac{1}{3}} x^{\frac{5}{3}}} = \frac{3}{(ax+b)^{\frac{2}{3}} x^{\frac{7}{3}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 143

```
dsolve(y(x)*diff(y(x),x)=3*(a*x+b)^(-1/3)*x^(-5/3)*y(x)+3*(a*x+b)^(-2/3)*x^(-7/3),y(x), sing
```

$y(x) =$

$$-\frac{6\sqrt{3}}{(ax+b)^{\frac{1}{3}}\, x^{\frac{2}{3}} \left(\left(\left(ax+b\right)^{\frac{1}{3}} x^{\frac{5}{3}} \left(\frac{a}{(ax+b)^2 x^4}\right)^{\frac{1}{3}}+2\right)\sqrt{3}+3x^{\frac{5}{3}}\,(ax+b)^{\frac{1}{3}} \left(\frac{a}{(ax+b)^2 x^4}\right)^{\frac{1}{3}} \tan\left(\operatorname{RootOf}\left(\sqrt{3}\right.\right.\right.}$$

✓ Solution by Mathematica

Time used: 1.769 (sec). Leaf size: 312

```
DSolve[y[x]*y'[x]==3*(a*x+b)^(-1/3)*x^(-5/3)*y[x]+3*(a*x+b)^(-2/3)*x^(-7/3),y[x],x,IncludeSi
```

$$\text{Solve}\left[\frac{1}{6}\left(2\sqrt{3}\arctan\left(\frac{-\frac{2\left(x^{2/3}y(x)\sqrt[3]{ax+b}+3\right)}{\sqrt[3]{ax^3}y(x)}-1}{\sqrt{3}}\right)+2\log\left(\frac{-x^{2/3}y(x)\sqrt[3]{ax+b}-3}{\sqrt[3]{ax^3}y(x)}+1\right)-\log\left(\frac{\left(x^{2/3}\right.}{(}\right.\right.\right.$$

24.18 problem 18

Internal problem ID [10755]
Internal file name [OUTPUT/9702_Monday_June_06_2022_03_29_21_PM_54352336/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$3yy' - \frac{(-7\lambda s(3s+4\lambda)x + 6s - 2\lambda)y}{x^{\frac{1}{3}}} = \frac{6\lambda sx - 6}{x^{\frac{2}{3}}} + 2(\lambda s(3s+4\lambda)x + 5\lambda)(-\lambda s(3s+4\lambda)x + 3s + 4\lambda$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(2/3)*y(x)*(28*lambda^2*s*x+21*lambda*s^2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/3)*y(x)*(112*lambda^4*s^2*x^3+168*lamb
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/9)*(-x^2*lambda+3*s*x^2+9*y(x))/x, y(x
```

✗ Solution by Maple

```
dsolve(3*y(x)*diff(y(x),x)=(-7*lambda*s*(3*s+4*lambda)*x+6*s-2*lambda)*x^(-1/3)*y(x)+6*(lamb
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[3*y[x]*y'[x]==(-7*\[Lambda]*s*(3*s+4*\[Lambda])*x+6*s-2*\[Lambda])*x^(-1/3)*y[x]+6*(\
```

Timed out

24.19 problem 19

Internal problem ID [10756]
Internal file name [`OUTPUT/9703_Monday_June_06_2022_03_29_28_PM_56850454/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' + \frac{a(6x-1)\,y}{2x} = -\frac{a^2(x-1)\,(4x-1)}{2x}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 364

```
dsolve(y(x)*diff(y(x),x)+1/2*a*(6*x-1)*1/x*y(x)=-1/2*a^2*(x-1)*(4*x-1)*1/x,y(x), singsol=all
```

$$c_1 + \frac{\sqrt{2}\left(\frac{i(i\sqrt{-x}\,a+2ax+y(x)-a)\sqrt{-x}}{xa}\right)^{\frac{3}{2}}\left(-\frac{i(i\sqrt{-x}\,a+2ax+y(x)-a)\sqrt{-x}\ \text{hypergeom}\left(\left[\frac{1}{2},\frac{3}{2}\right],\left[\frac{7}{2}\right],\frac{i(i\sqrt{-x}\,a+2ax+y(x)-a)\sqrt{-x}}{2xa}\right)}{8xa} + \frac{5\left(4i\sqrt{2}\right.}{}\right.}{2\left(\frac{3}{2}-\frac{4i\sqrt{2}\,x-i\sqrt{2}-6i\sqrt{-x}+4x+2}{2\left(4i\sqrt{2}\,x+i\sqrt{2}-2i\sqrt{-x}-4\sqrt{2}\,\sqrt{-x}+4x-2\right)}\right)\text{hypergeom}\left(\left[-2,-1\right],\left[-\frac{1}{2}\right],\frac{i(i\sqrt{-x}\,a+2ax}{2a}\right.}$$
$$= 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+1/2*a*(6*x-1)*1/x*y[x]==-1/2*a^2*(x-1)*(4*x-1)*1/x,y[x],x,IncludeSingularS
```

Not solved

24.20 problem 20

Internal problem ID [10757]
Internal file name [`OUTPUT/9704_Monday_June_06_2022_03_29_35_PM_3440477/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{a\left(1 + \frac{2b}{x^2}\right) y}{2} = \frac{a^2\left(3x + \frac{4b}{x}\right)}{16}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-1/2*a*(1+2*b*x^(-2))*y(x)=1/16*a^2*(3*x+4*b/x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-1/2*a*(1+2*b*x^(-2))*y[x]==1/16*a^2*(3*x+4*b/x),y[x],x,IncludeSingularSolu
```

Not solved

24.21 problem 21

Internal problem ID [10758]
Internal file name [`OUTPUT/9705_Monday_June_06_2022_03_29_39_PM_44919740/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' + \frac{a(13x - 20)\,y}{14x^{\frac{9}{7}}} = -\frac{3a^2(x-1)\,(x-8)}{14x^{\frac{11}{17}}}$$

Unable to determine ODE type.

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+1/14*a*(13*x-20)*x^(-9/7)*y(x)=-3/14*a^2*(x-1)*(x-8)*x^(-11/17),y(x
```

No solution found

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+1/14*a*(13*x-20)*x^(-9/7)*y[x]==-3/14*a^2*(x-1)*(x-8)*x^(-11/17),y[x],x,In
```

Timed out

24.22 problem 22

Internal problem ID [10759]
Internal file name [`OUTPUT/9706_Monday_June_06_2022_03_31_43_PM_93423319/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' + \frac{5a(23x-16)\,y}{56x^{\frac{9}{7}}} = -\frac{3a^2(x-1)\,(25x-32)}{56x^{\frac{11}{17}}}$$

Unable to determine ODE type.

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+5/56*a*(23*x-16)*x^(-9/7)*y(x)=-3/56*a^2*(x-1)*(25*x-32)*x^(-11/17)
```

No solution found

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+5/56*a*(23*x-16)*x^(-9/7)*y[x]==-3/56*a^2*(x-1)*(25*x-32)*x^(-11/17),y[x],
```

Timed out

24.23 problem 23

Internal problem ID [10760]
Internal file name [`OUTPUT/9707_Monday_June_06_2022_03_33_51_PM_26459903/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' + \frac{a(19x + 85)\, y}{26x^{\frac{18}{13}}} = -\frac{3a^2(x-1)\,(x+25)}{26x^{\frac{23}{13}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(5/13)*y(x)*(19*x+306)/(x*(19*x+85)), y(x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/13)*y(x)*(3*x^2-240*x+575)/((x-1)*x*(x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+1/26*a*(19*x+85)*x^(-18/13)*y(x)=-3/26*a^2*(x-1)*(x+25)*x^(-23/13),
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+1/26*a*(19*x+85)*x^(-18/13)*y[x]==-3/26*a^2*(x-1)*(x+25)*x^(-23/13),y[x],x
```

Timed out

24.24 problem 24

Internal problem ID [10761]
Internal file name [`OUTPUT/9708_Monday_June_06_2022_03_33_56_PM_57206178/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' + \frac{a(13x-18)\,y}{15x^{\frac{7}{5}}} = -\frac{4a^2(x-1)\,(x-6)}{15x^{\frac{9}{5}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(2/5)*y(x)*(13*x-63)/(x*(13*x-18)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/5)*y(x)*(x^2+28*x-54)/((x-1)*x*(x-6)),
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+1/15*a*(13*x-18)*x^(-7/5)*y(x)=-4/15*a^2*(x-1)*(x-6)*x^(-9/5),y(x),
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+1/15*a*(13*x-18)*x^(-7/5)*y[x]==-4/15*a^2*(x-1)*(x-6)*x^(-9/5),y[x],x,Incl
```

Timed out

24.25 problem 25

Internal problem ID [10762]
Internal file name [`OUTPUT/9709_Monday_June_06_2022_03_34_00_PM_79407301/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' + \frac{a(5x+1)\,y}{2\sqrt{x}} = a^2(-x^2+1)$$

Unable to determine ODE type.

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+1/2*a*(5*x+1)*x^(-1/2)*y(x)=a^2*(1-x^2),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+1/2*a*(5*x+1)*x^(-1/2)*y[x]==a^2*(1-x^2),y[x],x,IncludeSingularSolutions -
```

Not solved

24.26 problem 26

Internal problem ID [10763]
Internal file name [`OUTPUT/9710_Monday_June_06_2022_04_47_16_PM_49734204/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' + \frac{3a(19x-14)\,x^{\frac{7}{5}}y}{35} = -\frac{4a^2(x-1)\,(9x-14)\,x^{\frac{9}{5}}}{35}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(2/5)*y(x)*(114*x-49)/(x*(19*x-14)), y(x)
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/5)*y(x)*(171*x^2-322*x+126)/(x*(9*x-14
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(72/35)*a/x, y(x)`      *** Sublevel 2 **
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+3/35*a*(19*x-14)*x^(7/5)*y(x)=-4/35*a^2*(x-1)*(9*x-14)*x^(9/5),y(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+3/35*a*(19*x-14)*x^(7/5)*y[x]==-4/35*a^2*(x-1)*(9*x-14)*x^(9/5),y[x],x,Inc
```

Timed out

24.27 problem 27

Internal problem ID [10764]
Internal file name [`OUTPUT/9711_Monday_June_06_2022_04_47_19_PM_1175050/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' + \frac{3a(3x+7)\,y}{10x^{\frac{13}{10}}} = -\frac{a^2(x-1)\,(x+9)}{5x^{\frac{8}{5}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/10)*y(x)*(9*x+91)/(x*(3*x+7)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(2/5)*y(x)*(x-6)^2/(x*(x^(1/10)-1)*(x^(1/
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+3/10*a*(3*x+7)*x^(-13/10)*y(x)=-1/5*a^2*(x-1)*(x+9)*x^(-8/5),y(x),
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+3/10*a*(3*x+7)*x^(-13/10)*y[x]==-1/5*a^2*(x-1)*(x+9)*x^(-8/5),y[x],x,Inclu
```

Timed out

24.28 problem 28

Internal problem ID [10765]
Internal file name [OUTPUT/9712_Monday_June_06_2022_04_47_22_PM_62025219/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' + \frac{a(7x-12)\,y}{10x^{\frac{7}{5}}} = -\frac{a^2(x-1)\,(x-16)}{10x^{\frac{9}{5}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 758

```
dsolve(y(x)*diff(y(x),x)+1/10*a*(7*x-12)*x^(-7/5)*y(x)=-1/10*a^2*(x-1)*(x-16)*x^(-9/5),y(x),
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+1/10*a*(7*x-12)*x^(-7/5)*y[x]==-1/10*a^2*(x-1)*(x-16)*x^(-9/5),y[x],x,Incl
```

Timed out

24.29 problem 29

Internal problem ID [10766]
Internal file name [`OUTPUT/9713_Monday_June_06_2022_04_47_24_PM_53594480/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' + \frac{3a(13x-8)\,y}{20x^{\frac{7}{5}}} = -\frac{a^2(x-1)\,(27x-32)}{20x^{\frac{9}{5}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(2/5)*y(x)*(13*x-28)/(x*(13*x-8)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/5)*y(x)*(27*x^2+236*x-288)/((x-1)*x*(2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+3/20*a*(13*x-8)*x^(-7/5)*y(x)=-1/20*a^2*(x-1)*(27*x-32)*x^(-9/5),y(
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+3/20*a*(13*x-8)*x^(-7/5)*y[x]==-1/20*a^2*(x-1)*(27*x-32)*x^(-9/5),y[x],x,I
```

Timed out

24.30 problem 30

Internal problem ID [10767]
Internal file name [`OUTPUT/9714_Monday_June_06_2022_04_47_27_PM_70978277/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$
\boxed{yy' + \frac{3a(3x+11)\,y}{14x^{\frac{10}{7}}} = -\frac{a^2(x-1)\,(x-27)}{14x^{\frac{13}{7}}}}
$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/7)*y(x)*(9*x+110)/(x*(3*x+11)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/7)*y(x)*(x^2+168*x-351)/((x-1)*x*(x-27
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+3/14*a*(3*x+11)*x^(-10/7)*y(x)=-1/14*a^2*(x-1)*(x-27)*x^(-13/7),y(x
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+3/14*a*(3*x+11)*x^(-10/7)*y[x]==-1/14*a^2*(x-1)*(x-27)*x^(-13/7),y[x],x,In
```

Timed out

24.31 problem 31

Internal problem ID [10768]
Internal file name [`OUTPUT/9715_Monday_June_06_2022_04_47_30_PM_61373009/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{a(1+x)\,y}{2x^{\frac{7}{4}}} = \frac{a^2(x-1)\,(3x+5)}{4x^{\frac{5}{2}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 187

```
dsolve(y(x)*diff(y(x),x)-1/2*a*(x+1)*x^(-7/4)*y(x)=1/4*a^2*(x-1)*(3*x+5)*x^(-5/2),y(x), sing
```

$$\frac{\frac{36\sqrt{-\frac{(-1+x)a+x^{\frac{3}{4}}y(x)}{x^{\frac{3}{4}}\left(y(x)+x^{\frac{1}{4}}a\right)}}\,\sqrt{13}\,55^{\frac{1}{6}}\left(x-\frac{15}{2}\right)\left(\frac{(3x+5)a+3x^{\frac{3}{4}}y(x)}{x^{\frac{3}{4}}\left(y(x)+x^{\frac{1}{4}}a\right)}\right)^{\frac{5}{6}}}{20449}+1458000\left(\int^{-\frac{90\left(2x^{\frac{3}{4}}y(x)+2ax-15a\right)}{143\left(x^{\frac{3}{4}}y(x)+ax\right)}}\frac{\_\_a\sqrt{11\_\_a-90}\,(13\_\_}{(143\_\_a+180)^{\frac{4}{3}}\,(20449\_\_a^{3}-119}\right.}{\left(\frac{a}{x^{\frac{3}{4}}\left(y(x)+x^{\frac{1}{4}}a\right)}\right)^{\frac{4}{3}}x}$$

$$=0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-1/2*a*(x+1)*x^(-7/4)*y[x]==1/4*a^2*(x-1)*(3*x+5)*x^(-5/2),y[x],x,IncludeSi
```

Timed out

24.32 problem 32

Internal problem ID [10769]
Internal file name [`OUTPUT/9716_Monday_June_06_2022_04_47_32_PM_87061287/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - \frac{a(1+x)\,y}{2x^{\frac{7}{4}}} = \frac{a^2(x-1)\,(x+5)}{4x^{\frac{5}{2}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/2)*y(x)*(x^2+12*x-25)/(x*(x+5)*(x^(1/4
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-1/2*a*(x+1)*x^(-7/4)*y(x)=1/4*a^2*(x-1)*(x+5)*x^(-5/2),y(x), singso
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-1/2*a*(x+1)*x^(-7/4)*y[x]==1/4*a^2*(x-1)*(x+5)*x^(-5/2),y[x],x,IncludeSing
```

Timed out

24.33 problem 33

Internal problem ID [10770]
Internal file name [`OUTPUT/9717_Monday_June_06_2022_04_47_35_PM_8570371/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - \frac{a(4x+3)\,y}{14x^{\frac{8}{7}}} = -\frac{a^2(x-1)\,(16x+5)}{14x^{\frac{9}{7}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(4/7)*y(x)*(x+6)/(x*(4*x+3)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/7)*y(x)*(80*x^2+22*x+45)/((x-1)*x*(16*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-1/14*a*(4*x+3)*x^(-8/7)*y(x)=-1/14*a^2*(x-1)*(16*x+5)*x^(-9/7),y(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-1/14*a*(4*x+3)*x^(-8/7)*y[x]==-1/14*a^2*(x-1)*(16*x+5)*x^(-9/7),y[x],x,Inc
```

Timed out

24.34 problem 34

Internal problem ID [10771]
Internal file name [`OUTPUT/9718_Monday_June_06_2022_04_47_38_PM_68775627/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' + \frac{a(13x-3)\,y}{6x^{\frac{2}{3}}} = -\frac{a^2(x-1)\,(5x-1)}{6x^{\frac{1}{3}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/3)*y(x)*(13*x+6)/(x*(13*x-3)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/3)*y(x)*(25*x^2-12*x-1)/(x*(x^(1/3)-1)
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(8/13)*a/x, y(x)`      *** Sublevel 2 ***
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+1/6*a*(13*x-3)*x^(-2/3)*y(x)=-1/6*a^2*(x-1)*(5*x-1)*x^(-1/3),y(x),
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+1/6*a*(13*x-3)*x^(-2/3)*y[x]==-1/6*a^2*(x-1)*(5*x-1)*x^(-1/3),y[x],x,Inclu
```

Not solved

24.35 problem 35

Internal problem ID [10772]
Internal file name [`OUTPUT/9719_Monday_June_06_2022_04_47_41_PM_49926942/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{a(8x-1)\,y}{28x^{\frac{8}{7}}} = \frac{a^2(x-1)\,(32x+3)}{28x^{\frac{9}{7}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(8/7)*y(x)*(x^(1/7)-1)*(x^(6/7)+x^(5/7)+x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/7)*y(x)*(160*x^2+58*x+27)/((x-1)*x*(32
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-1/28*a*(8*x-1)*x^(-8/7)*y(x)=1/28*a^2*(x-1)*(32*x+3)*x^(-9/7),y(x),
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-1/28*a*(8*x-1)*x^(-8/7)*y[x]==1/28*a^2*(x-1)*(32*x+3)*x^(-9/7),y[x],x,Incl
```

Timed out

24.36 problem 36

Internal problem ID [10773]
Internal file name [`OUTPUT/9720_Monday_June_06_2022_04_47_44_PM_87836436/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{a(5x-4)\,y}{x^4} = \frac{a^2(x-1)\,(3x-1)}{x^7}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 167

```
dsolve(y(x)*diff(y(x),x)-a*(5*x-4)*x^(-4)*y(x)=a^2*(x-1)*(3*x-1)*x^(-7),y(x), singsol=all)
```

$$c_1 - \frac{9\, 2^{\frac{2}{3}} \sqrt{\frac{x^3 y(x)+a x-a}{(y(x) x^2+a) x}}\, \left(x-\frac{3}{4}\right) 5^{\frac{1}{6}}}{5x \left(-\frac{a}{(y(x)x^2+a)x}\right)^{\frac{1}{3}} \left(\frac{3x^3 y(x)+3ax-a}{(y(x)x^2+a)x}\right)^{\frac{1}{6}}} - 729 \left(\int^{\frac{\frac{9x^3 y(x)}{5}+\frac{9ax}{5}-\frac{27a}{20}}{(y(x)x^2+a)x}} \frac{\_a\sqrt{20\_a-9}}{(5\_a-9)^{\frac{1}{3}} (9+4\_a)^{\frac{1}{6}} (400\_a^3-1701\_a+729)} d\_a\right) = 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-a*(5*x-4)*x^(-4)*y[x]==a^2*(x-1)*(3*x-1)*x^(-7),y[x],x,IncludeSingularSolu
```

Not solved

24.37 problem 37

Internal problem ID [10774]
Internal file name [`OUTPUT/9721_Monday_June_06_2022_04_47_46_PM_92742251/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{2a(3x-10)\,y}{5x^4} = \frac{a^2(x-1)\,(8x-5)}{5x^7}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-2/5*a*(3*x-10)*x^(-4)*y(x)=1/5*a^2*(x-1)*(8*x-5)*x^(-7),y(x), sings
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-2/5*a*(3*x-10)*x^(-4)*y[x]==1/5*a^2*(x-1)*(8*x-5)*x^(-7),y[x],x,IncludeSin
```

Not solved

24.38 problem 38

Internal problem ID [10775]
Internal file name [`OUTPUT/9722_Monday_June_06_2022_04_47_48_PM_48890699/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' + \frac{a(39x-4)\,y}{42x^{\frac{9}{7}}} = -\frac{a^2(x-1)\,(9x-1)}{42x^{\frac{11}{7}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(6/7)*y(x)*(13*x-6)/(x*(39*x-4)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/7)*y(x)*(27*x^2+40*x-11)/((x-1)*x*(9*x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+1/42*a*(39*x-4)*x^(-9/7)*y(x)=-1/42*a^2*(x-1)*(9*x-1)*x^(-11/7),y(x
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+1/42*a*(39*x-4)*x^(-9/7)*y[x]==-1/42*a^2*(x-1)*(9*x-1)*x^(-11/7),y[x],x,In
```

Timed out

24.39 problem 39

Internal problem ID [10776]
Internal file name [`OUTPUT/9723_Monday_June_06_2022_04_47_51_PM_99545307/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' + \frac{a(x-2)\,y}{x} = \frac{2a^2(x-1)}{x}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 116

```
dsolve(y(x)*diff(y(x),x)+a*(x-2)*x^(-1)*y(x)=2*a^2*(x-1)*x^(-1),y(x), singsol=all)
```

$$\frac{\sqrt{\frac{(1-x)a-y(x)}{ax+y(x)}}\, e^{\frac{ax+y(x)}{2a}} y(x)+x\left(\int^{\frac{a}{ax+y(x)}}\frac{\sqrt{\_a-1}\, e^{\frac{1}{2\_a}}}{\sqrt{\_a}}d\_a+c_1\right)\sqrt{\frac{a}{ax+y(x)}}\,(ax+y(x))}{\sqrt{\frac{a}{ax+y(x)}}\, x\,(ax+y\,(x))}=0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+a*(x-2)*x^(-1)*y[x]==2*a^2*(x-1)*x^(-1),y[x],x,IncludeSingularSolutions ->
```

Not solved

24.40 problem 40

Internal problem ID [10777]
Internal file name [OUTPUT/9724_Monday_June_06_2022_04_47_52_PM_55668297/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' + \frac{a(3x-2)\,y}{x} = -\frac{2a^2(x-1)^2}{x}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+a*(3*x-2)*x^(-1)*y(x)=-2*a^2*(x-1)^2*x^(-1),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+a*(3*x-2)*x^(-1)*y[x]==-2*a^2*(x-1)^2*x^(-1),y[x],x,IncludeSingularSolutio
```

Not solved

24.41 problem 41

Internal problem ID [10778]
Internal file name [`OUTPUT/9725_Monday_June_06_2022_04_47_55_PM_34886484/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' + \frac{a\left(1 - \frac{b}{x^2}\right) y}{x} = \frac{a^2 b}{x}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+a*(1-b*x^(-2))*x^(-1)*y(x)=a^2*b*x^(-1),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+a*(1-b*x^(-2))*x^(-1)*y[x]==a^2*b*x^(-1),y[x],x,IncludeSingularSolutions -
```

Not solved

24.42 problem 42

Internal problem ID [10779]
Internal file name [`OUTPUT/9726_Monday_June_06_2022_04_47_57_PM_92274431/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{a(3x-4)\,y}{4x^{\frac{5}{2}}} = \frac{a^2(x-1)\,(x+2)}{4x^4}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/2)*y(x)*(9*x-20)/(x*(3*x-4)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(2*x^2+3*x-8)/(x*(x+2)*(x^(1/2)-1)*(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-1/4*a*(3*x-4)*x^(-5/2)*y(x)=1/4*a^2*(x-1)*(x+2)*x^(-4),y(x), singso
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-1/4*a*(3*x-4)*x^(-5/2)*y[x]==1/4*a^2*(x-1)*(x+2)*x^(-4),y[x],x,IncludeSing
```

Not solved

24.43 problem 43

Internal problem ID [10780]
Internal file name [OUTPUT/9727_Monday_June_06_2022_04_48_00_PM_3906511/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' + \frac{a(33x+2)\,y}{30x^{\frac{6}{5}}} = -\frac{a^2(x-1)\,(9x-4)}{30x^{\frac{7}{5}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 4330

```
dsolve(y(x)*diff(y(x),x)+1/30*a*(33*x+2)*x^(-6/5)*y(x)=-1/30*a^2*(x-1)*(9*x-4)*x^(-7/5),y(x)
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+1/30*a*(33*x+2)*x^(-6/5)*y[x]==-1/30*a^2*(x-1)*(9*x-4)*x^(-7/5),y[x],x,Inc
```

Timed out

24.44 problem 44

Internal problem ID [10781]
Internal file name [`OUTPUT/9728_Monday_June_06_2022_04_48_03_PM_71952739/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - \frac{a(x-8)\,y}{8x^{\frac{5}{2}}} = -\frac{a^2(x-1)\,(3x-4)}{8x^4}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/2)*y(x)*(3*x-40)/(x*(x-8)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(6*x^2-21*x+16)/(x*(3*x-4)*(x^(1/2)-
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-1/8*a*(x-8)*x^(-5/2)*y(x)=-1/8*a^2*(x-1)*(3*x-4)*x^(-4),y(x), sings
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-1/8*a*(x-8)*x^(-5/2)*y[x]==-1/8*a^2*(x-1)*(3*x-4)*x^(-4),y[x],x,IncludeSin
```

Not solved

24.45 problem 45

Internal problem ID [10782]
Internal file name [`OUTPUT/9729_Monday_June_06_2022_04_48_06_PM_22405435/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' + \frac{a(17x+18)\,y}{30x^{\frac{22}{15}}} = -\frac{a^2(x-1)\,(x+4)}{30x^{\frac{29}{15}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/15)*y(x)*(119*x+396)/(x*(17*x+18)), y(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/15)*y(x)*(x^2-42*x+116)/((x-1)*x*(x+4)
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+1/30*a*(17*x+18)*x^(-22/15)*y(x)=-1/30*a^2*(x-1)*(x+4)*x^(-29/15),y
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+1/30*a*(17*x+18)*x^(-22/15)*y[x]==-1/30*a^2*(x-1)*(x+4)*x^(-29/15),y[x],x,
```

Timed out

24.46 problem 46

Internal problem ID [10783]
Internal file name [`OUTPUT/9730_Monday_June_06_2022_04_48_09_PM_74919152/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{a(6x-13)\,y}{13x^{\frac{5}{2}}} = -\frac{a^2(x-1)\,(x-13)}{26x^4}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/2)*y(x)*(18*x-65)/(x*(6*x-13)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-2*y(x)*(x^2-21*x+26)/(x*(x^(1/2)-1)*(x^(1
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-1/13*a*(6*x-13)*x^(-5/2)*y(x)=-1/26*a^2*(x-1)*(x-13)*x^(-4),y(x), s
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-1/13*a*(6*x-13)*x^(-5/2)*y[x]==-1/26*a^2*(x-1)*(x-13)*x^(-4),y[x],x,Includ
```

Not solved

24.47 problem 47

Internal problem ID [10784]
Internal file name [`OUTPUT/9731_Monday_June_06_2022_04_48_13_PM_60336236/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 47.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' + \frac{a(24x + 11)\, x^{\frac{27}{20}} y}{30} = -\frac{a^2(x - 1)\,(9x + 1)}{60x^{\frac{17}{10}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(3/20)*y(x)*(376*x+99)/(x*(24*x+11)), y(x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/10)*y(x)*(27*x^2+56*x+17)/(x*(9*x+1)*(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+1/30*a*(24*x+11)*x^(27/20)*y(x)=-1/60*a^2*(x-1)*(9*x+1)*x^(-17/10),
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+1/30*a*(24*x+11)*x^(27/20)*y[x]==-1/60*a^2*(x-1)*(9*x+1)*x^(-17/10),y[x],x
```

Timed out

24.48 problem 48

Internal problem ID [10785]
Internal file name [OUTPUT/9732_Monday_June_06_2022_04_48_19_PM_49754049/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 48.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{2a(3x+2)\,y}{5x^{\frac{8}{5}}} = \frac{a^2(x-1)\,(8x+1)}{5x^{\frac{11}{5}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/5)*y(x)*(9*x+16)/(x*(3*x+2)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/5)*y(x)*(4*x+1)*(2*x-11)/((x-1)*x*(8*x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-2/5*a*(3*x+2)*x^(-8/5)*y(x)=1/5*a^2*(x-1)*(8*x+1)*x^(-11/5),y(x), s
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-2/5*a*(3*x+2)*x^(-8/5)*y[x]==1/5*a^2*(x-1)*(8*x+1)*x^(-11/5),y[x],x,Includ
```

Timed out

24.49 problem 49

Internal problem ID [10786]
Internal file name [`OUTPUT/9733_Monday_June_06_2022_04_48_22_PM_38261084/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 49.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{6a(4x+1)\,y}{5x^{\frac{7}{5}}} = \frac{a^2(x-1)\,(27x+8)}{5x^{\frac{9}{5}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/5)*y(x)*(8*x+7)/(x*(4*x+1)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/5)*y(x)*(27*x^2+76*x+72)/((x-1)*x*(27*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-6/5*a*(4*x+1)*x^(-7/5)*y(x)=1/5*a^2*(x-1)*(27*x+8)*x^(-9/5),y(x), s
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-6/5*a*(4*x+1)*x^(-7/5)*y[x]==1/5*a^2*(x-1)*(27*x+8)*x^(-9/5),y[x],x,Includ
```

Timed out

24.50 problem 50

Internal problem ID [10787]
Internal file name [`OUTPUT/9734_Monday_June_06_2022_04_48_25_PM_64664074/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 50.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{a(x+4)\,y}{5x^{\frac{8}{5}}} = \frac{a^2(x-1)\,(3x+7)}{5x^{\frac{3}{5}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/5)*y(x)*(3*x+32)/(x*(x+4)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/5)*y(x)*(21*x^2+8*x+21)/(x*(3*x+7)*(x-
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+7*y(x)*a/(x*(7*a-x)), y(x)`       *** Subl
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-1/5*a*(x+4)*x^(-8/5)*y(x)=1/5*a^2*(x-1)*(3*x+7)*x^(-3/5),y(x), sing
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-1/5*a*(x+4)*x^(-8/5)*y[x]==1/5*a^2*(x-1)*(3*x+7)*x^(-3/5),y[x],x,IncludeSi
```

Not solved

24.51 problem 51

Internal problem ID [10788]
Internal file name [`OUTPUT/9735_Monday_June_06_2022_04_48_29_PM_62088419/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 51.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - \frac{a(x+4)\,y}{5x^{\frac{8}{5}}} = \frac{a^2(x-1)\,(3x+7)}{5x^{\frac{11}{5}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 190

```
dsolve(y(x)*diff(y(x),x)-1/5*a*(x+4)*x^(-8/5)*y(x)=1/5*a^2*(x-1)*(3*x+7)*x^(-11/5),y(x), sin
```

$$\frac{\frac{360\, 2^{\frac{1}{3}} \sqrt{17}\, \sqrt{-\frac{(-1+x)a+y(x)x^{\frac{3}{5}}}{x^{\frac{3}{5}}\left(y(x)+a\,x^{\frac{2}{5}}\right)}}\; 91^{\frac{5}{6}} \left(x-\frac{21}{4}\right) \left(\frac{(3x+7)a+3y(x)x^{\frac{3}{5}}}{x^{\frac{3}{5}}\left(y(x)+a\,x^{\frac{2}{5}}\right)}\right)^{\frac{7}{6}}}{4444531} + 31255875x \left(\int^{-\frac{315\left(4y(x)x^{\frac{3}{5}}+4ax-21a\right)}{884\left(y(x)x^{\frac{3}{5}}+ax\right)}} \frac{\sqrt{52\_\_a-315}\,(68}{(11492\_\_a^2-53235\_\_a-}\right.}{x\left(\frac{a}{x^{\frac{3}{5}}\left(y(x)+a\,x^{\frac{2}{5}}\right)}\right)^{\frac{5}{3}}}$$

$= 0$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-1/5*a*(x+4)*x^(-8/5)*y[x]==1/5*a^2*(x-1)*(3*x+7)*x^(-11/5),y[x],x,IncludeS
```

Timed out

24.52 problem 52

Internal problem ID [10789]
Internal file name [`OUTPUT/9736_Monday_June_06_2022_04_48_31_PM_33126839/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 52.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{a(2x-1)\,y}{x^{\frac{5}{2}}} = \frac{a^2(x-1)\,(3x+1)}{2x^4}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 189

```
dsolve(y(x)*diff(y(x),x)-a*(2*x-1)*x^(-5/2)*y(x)=1/2*a^2*(x-1)*(3*x+1)*x^(-4),y(x), singsol=
```

$$\frac{\frac{18\sqrt{\frac{(-1+x)a+y(x)x^{\frac{3}{2}}}{x(y(x)\sqrt{x}+a)}}\,\sqrt{5}\,7^{\frac{5}{6}}\left(x+\frac{3}{2}\right)\left(\frac{(-3x-1)a-3y(x)x^{\frac{3}{2}}}{x(y(x)\sqrt{x}+a)}\right)^{\frac{1}{6}}}{1225}+1458\left(\int^{\frac{-\frac{18y(x)x^{\frac{3}{2}}}{35}+\frac{9(-2x-3)a}{35}}{x(y(x)\sqrt{x}+a)}}\frac{\_a(5\_a-9)^{\frac{1}{6}}\sqrt{7\_a+9}}{(35\_a+18)^{\frac{2}{3}}(1225\_a^3-3159\_a-1458)}\right.}{x\left(-\frac{a}{x(y(x)\sqrt{x}+a)}\right)^{\frac{2}{3}}}$$

$= 0$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-a*(2*x-1)*x^(-5/2)*y[x]==1/2*a^2*(x-1)*(3*x+1)*x^(-4),y[x],x,IncludeSingul
```

Not solved

24.53 problem 53

Internal problem ID [10790]
Internal file name [`OUTPUT/9737_Monday_June_06_2022_04_48_34_PM_47478066/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 53.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' + \frac{a(x-6)\,y}{5x^{\frac{7}{5}}} = \frac{2a^2(x-1)\,(x+4)}{5x^{\frac{9}{5}}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 156

```
dsolve(y(x)*diff(y(x),x)+1/5*a*(x-6)*x^(-7/5)*y(x)=2/5*a^2*(x-1)*(x+4)*x^(-9/5),y(x), singso
```

$$c_1 - \frac{80\sqrt{3}\left(ay(x)\,x^{\frac{2}{5}}+\frac{x^{\frac{4}{5}}y(x)^2}{8}+\frac{\left(y(x)x^{\frac{7}{5}}-2a(x+24)(-1+x)\right)a}{24}\right)\left(ay(x)\,x^{\frac{2}{5}}+\frac{x^{\frac{4}{5}}y(x)^2}{8}+\frac{a\left(y(x)x^{\frac{7}{5}}+\frac{a(x+4)^2}{2}\right)}{4}\right)\sqrt{\frac{-y(x)}{x^{\frac{2}{5}}\left(y\right.}}}{9\left(\frac{a}{x^{\frac{2}{5}}\left(y(x)+x^{\frac{3}{5}}a\right)}\right)^{\frac{5}{2}}\left(a\,(x+4)+y\,(x)\,x^{\frac{2}{5}}\right)^2\left(y\,(x)\,x^{\frac{2}{5}}+ax\right)^2x} = 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+1/5*a*(x-6)*x^(-7/5)*y[x]==2/5*a^2*(x-1)*(x+4)*x^(-9/5),y[x],x,IncludeSing
```

Timed out

24.54 problem 54

Internal problem ID [10791]
Internal file name [`OUTPUT/9738_Monday_June_06_2022_04_48_36_PM_24600607/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 54.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' + \frac{a(21x+19)\,y}{5x^{\frac{7}{5}}} = -\frac{2a^2(x-1)\,(9x-4)}{5x^{\frac{9}{5}}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(7/5)*y(x)*(6*x+19)/(x*(21*x+19)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/5)*y(x)*(9*x^2+52*x-36)/((x-1)*x*(9*x-
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+1/5*a*(21*x+19)*x^(-7/5)*y(x)=-2/5*a^2*(x-1)*(9*x-4)*x^(-9/5),y(x),
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+1/5*a*(21*x+19)*x^(-7/5)*y[x]==-2/5*a^2*(x-1)*(9*x-4)*x^(-9/5),y[x],x,Incl
```

Timed out

24.55 problem 55

Internal problem ID [10792]
Internal file name [`OUTPUT/9739_Monday_June_06_2022_04_48_39_PM_54450107/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 55.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$yy' - \frac{3ay}{x^{\frac{7}{4}}} = \frac{a^2(x-1)(x-9)}{4x^{\frac{5}{2}}}$$

Unable to determine ODE type.

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-3*a*x^(-7/4)*y(x)=1/4*a^2*(x-1)*(x-9)*x^(-5/2),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-3*a*x^(-7/4)*y[x]==1/4*a^2*(x-1)*(x-9)*x^(-5/2),y[x],x,IncludeSingularSolu
```

Not solved

24.56 problem 56

Internal problem ID [10793]
Internal file name [`OUTPUT/9740_Monday_June_06_2022_04_51_00_PM_3336046/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 56.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \frac{a((1+k)\,x-1)\,y}{x^2} = \frac{a^2(1+k)\,(x-1)}{x^2}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 143

```
dsolve(y(x)*diff(y(x),x)-a*((k+1)*x-1)*x^(-2)*y(x)=a^2*(k+1)*(x-1)*x^(-2),y(x), singsol=all)
```

$$\frac{\left(\frac{ax}{-y(x)x+a}\right)^{-\frac{1}{1+k}} x^2 \left(\frac{(-1+x)a+y(x)x}{-y(x)x+a}\right)^{\frac{1}{1+k}} e^{\frac{-y(x)x+a}{a(1+k)x}} y(x) - \left(\int^{\frac{ax}{-y(x)x+a}} (\_a-1)^{\frac{1}{1+k}} e^{\frac{1}{(1+k)\_a}} \_a^{-\frac{1}{1+k}} d\_a - c_1\right) (-}{-y(x)x+a}$$
$= 0$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-a*((k+1)*x-1)*x^(-2)*y[x]==a^2*(k+1)*(x-1)*x^(-2),y[x],x,IncludeSingularSo
```

Not solved

24.57 problem 57

Internal problem ID [10794]
Internal file name [`OUTPUT/9741_Monday_June_06_2022_09_30_54_PM_9550685/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 57.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - a((k-2)\,x + 2k - 3)\,x^{-k}y = a^2(k-2)\,(x-1)^2\,x^{1-2k}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(k^2*x+2*k^2-3*k*x-3*k+2*x)/(x*(k*x+
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-(2*k*x-2*k-3*x+1)*y(x)/(x*(x-1)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = (2*k*x-2*k-3*x+1)*y(x)/(x*(x-1)), y(x)`
      Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-a*( (k-2)*x + 2*k - 3)*x^(-k)*y(x)=a^2*(k-2)*(x-1)^2*x^(1-2*k),y(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-a*( (k-2)*x + 2*k - 3)*x^(-k)*y[x]==a^2*(k-2)*(x-1)^2*x^(1-2*k),y[x],x,Inc
```

Not solved

24.58 problem 58

Internal problem ID [10795]
Internal file name [`OUTPUT/9742_Monday_June_06_2022_09_31_03_PM_18647736/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 58.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - \frac{a((4k-7)\,x - 4k + 5)\,x^{-k}y}{2} = \frac{a^2(-3+2k)\,(x-1)^2\,x^{1-2k}}{2}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(4*k^2*x-4*k^2-11*k*x+5*k+7*x)/(x*(4
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-1/2*a*( (4*k-7)*x - 4*k + 5)*x^(-k)*y(x)=1/2*a^2*(2*k-3)*(x-1)^2*x^
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-1/2*a*( (4*k-7)*x - 4*k + 5)*x^(-k)*y[x]==1/2*a^2*(2*k-3)*(x-1)^2*x^(1-2*k
```

Not solved

24.59 problem 59

Internal problem ID [10796]
Internal file name [`OUTPUT/9743_Monday_June_06_2022_09_31_13_PM_94900719/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 59.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - ((-1+2n)\, x - an)\, x^{-n-1}y = n(x-a)\, x^{-2n}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 151

```
dsolve(y(x)*diff(y(x),x)-((2*n-1)*x-a*n)*x^(-n-1)*y(x)=n*(x-a)*x^(-2*n),y(x), singsol=all)
```

$$y(x) = \frac{2\left(-\frac{\sqrt{-n^2}\, x \tan\left(\frac{\operatorname{RootOf}\left(-2an\, e^{\_Z+\_a}+nx\, e^{\_Z+\_a}-\tan\left(\frac{\_a\sqrt{-n^2}}{2}\right)\_Z\sqrt{-n^2}\, x+2c_1 x\, e^{\_a}\right)\sqrt{-n^2}}{2}\right)}{2}+\left(a-\frac{x}{2}\right)n\right)x^{-n}}{\tan\left(\frac{\operatorname{RootOf}\left(-2an\, e^{\_Z+\_a}+nx\, e^{\_Z+\_a}-\tan\left(\frac{\_a\sqrt{-n^2}}{2}\right)\_Z\sqrt{-n^2}\, x+2c_1 x\, e^{\_a}\right)\sqrt{-n^2}}{2}\right)\sqrt{-n^2}+n}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-((2*n-1)*x-a*n)*x^(-n-1)*y[x]==n*(x-a)*x^(-2*n),y[x],x,IncludeSingularSolu
```

Not solved

24.60 problem 60

Internal problem ID [10797]
Internal file name [`OUTPUT/9744_Monday_June_06_2022_09_31_16_PM_78988747/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 60.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - ((1+n)\,x - an)\,x^{n-1}(x-a)^{-2-n}\,y = n\,x^{2n}(x-a)^{-2n-3}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(2*a*n+3*x)/(x*(a-x)), y(x)`      **
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(a^2*n^2-a*n^2*x-a^2*n+2*a*n*x-2*n*x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = -y(x)*(2*a*n+3*x)/(x*(a-x)), y(x)`
      Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-((n+1)*x-a*n)*x^(n-1)*(x-a)^(-n-2)*y(x)=n*x^(2*n)*(x-a)^(-2*n-3),y(
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-((n+1)*x-a*n)*x^(n-1)*(x-a)^(-n-2)*y[x]==n*x^(2*n)*(x-a)^(-2*n-3),y[x],x,I
```

Not solved

24.61 problem 61

Internal problem ID [10798]
Internal file name [`OUTPUT/9745_Monday_June_06_2022_09_31_26_PM_14092097/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - a((-3+2k)\,x+1)\,x^{-k}y = a^2(k-2)\,((k-1)\,x+1)\,x^{2-2k}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(2*k^2*x-5*k*x+k+3*x)/(x*(2*k*x-3*x+
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-(-1+k)*(2*k*x-3*x+2)*y(x)/(x*(k*x-x+1)),
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = (-1+k)*(2*k*x-3*x+2)*y(x)/(x*(k*x-x+1))
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-a*((2*k-3)*x+1)*x^(-k)*y(x)=a^2*(k-2)*((k-1)*x+1)*x^(2*(1-k)),y(x),
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-a*((2*k-3)*x+1)*x^(-k)*y[x]==a^2*(k-2)*((k-1)*x+1)*x^(2*(1-k)),y[x],x,Incl
```

Not solved

24.62 problem 62

Internal problem ID [10799]
Internal file name [OUTPUT/9746_Monday_June_06_2022_09_31_34_PM_1142874/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 62.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - a((n+2k-3)\,x + 3 - 2k)\,x^{-k}y = a^2\left((k+n-1)\,x^2 - (n+2k-3)\,x + k - 2\right)x^{1-2k}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(2*k^2*x+k*n*x-2*k^2-5*k*x-n*x+3*k+3
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(2*k^2*x^2+2*k*n*x^2-4*k^2*x-2*k*n*x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-a*((n+2*k-3)*x+3-2*k)*x^(-k)*y(x)=a^2*((n+k-1)*x^2-(n+2*k-3)*x+k-2)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-a*((n+2*k-3)*x+3-2*k)*x^(-k)*y[x]==a^2*((n+k-1)*x^2-(n+2*k-3)*x+k-2)*x^(1-
```

Timed out

24.63 problem 63

Internal problem ID [10800]
Internal file name [`OUTPUT/9747_Wednesday_June_08_2022_05_53_12_PM_9550685/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 63.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - \frac{a((2+n)\,x-2)\,x^{-\frac{2n+1}{n}}y}{n} = \frac{a^2((1+n)\,x^2-2x-n+1)\,x^{-\frac{2+3n}{n}}}{n}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(n^2*x+3*n*x-4*n+2*x-2)/(n*x*(n*x+2*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(n^2*x^2+3*n*x^2-3*n^2-4*n*x+2*x^2+n
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-a/n*((n+2)*x-2)*x^(-(2*n+1)/n)*y(x)=a^2/n*((n+1)*x^2-2*x-n+1)*x^(-(
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-a/n*((n+2)*x-2)*x^(-(2*n+1)/n)*y[x]==a^2/n*((n+1)*x^2-2*x-n+1)*x^(-(3*n+2)
```

Not solved

24.64 problem 64

Internal problem ID [10801]
Internal file name [`OUTPUT/9748_Wednesday_June_08_2022_05_53_20_PM_36010500/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 64.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - \frac{a\left(\frac{(n+4)x}{2+n} - 2\right) x^{-\frac{2n+1}{n}} y}{n} = \frac{a^2(2x^2 + (n^2 + n - 4)\,x - (n-1)\,(2+n))\, x^{-\frac{2+3n}{n}}}{n\,(2+n)}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(n^2*x-4*n^2+5*n*x-10*n+4*x-4)/(n*x*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(2*n^3*x-3*n^3+4*n^2*x+2*n*x^2-5*n^2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)-a/n*((n+4)/(n+2)*x-2)*x^(-(2*n+1)/n)*y(x)=a^2/(n*(n+2))*(2*x^2+(n^2
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-a/n*((n+4)/(n+2)*x-2)*x^(-(2*n+1)/n)*y[x]==a^2/(n*(n+2))*(2*x^2+(n^2+n-4)*
```

Not solved

24.65 problem 65

Internal problem ID [10802]
Internal file name [OUTPUT/9749_Wednesday_June_08_2022_05_53_31_PM_98114476/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 65.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' + \frac{a\left(\frac{(3n+5)x}{2} + \frac{n-1}{1+n}\right) x^{-\frac{n+4}{n+3}} y}{n+3} = -\frac{a^2\left((1+n)\, x^2 - \frac{(n^2+2n+5)x}{1+n} + \frac{4}{1+n}\right) x^{-\frac{n+5}{n+3}}}{2n+6}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(3*n^2*x+2*n^2+8*n*x+6*n+5*x-8)/(x*(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(n^3*x^2+3*n^2*x^2+2*n^2*x+3*n*x^2+4
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+a/(n+3)*((3*n+5)/(2)*x+(n-1)/(n+1))*x^(-(n+4)/(n+3))*y(x)=-a^2/(2*(
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+a/(n+3)*((3*n+5)/(2)*x+(n-1)/(n+1))*x^(-(n+4)/(n+3))*y[x]==-a^2/(2*(n+3))*
```

Timed out

24.66 problem 66

Internal problem ID [10803]
Internal file name [`OUTPUT/9750_Wednesday_June_08_2022_05_54_02_PM_41902957/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 66.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - a\left(\frac{2+n}{n} + b\,x^n\right)y = -\frac{a^2x\left(\frac{1+n}{n} + b\,x^n\right)}{n}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 192

```
dsolve(y(x)*diff(y(x),x)-a*((n+2)/n+b*x^n)*y(x)=-a^2/n*x*((n+1)/n+b*x^n),y(x), singsol=all)
```

$$-n\sqrt{-\frac{(n+1)^2}{n^2}}\left(\int^{\frac{2\arctan\left(\frac{2x^{n+1}abn+(n+1)(ax-y(x)n)}{\sqrt{-\frac{(n+1)^2}{n^2}}\,n(ax-y(x)n)}\right)}{\sqrt{-\frac{(n+1)^2}{n^2}}}}\tan\left(\frac{\_a\sqrt{-\frac{(n+1)^2}{n^2}}}{2}\right)e^{-\_a}d\_a\right) + (-2bn\,x^n-n-1)\,e^{-\frac{2\arctan\left(\frac{2x^{n+1}abn+(n+1)(ax-y(x)n)}{\sqrt{-\frac{(n+1)^2}{n^2}}\,n(ax-y(x)n)}\right)}{\sqrt{-\frac{(n+1)^2}{n^2}}}}+c_1=0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-a*((n+2)/n+b*x^n)*y[x]==-a^2/n*x*((n+1)/n+b*x^n),y[x],x,IncludeSingularSol
```

Not solved

24.67 problem 67

Internal problem ID [10804]
Internal file name [`OUTPUT/9751_Wednesday_June_08_2022_05_54_05_PM_43706231/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 67.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - (a\,e^{x} + b)\,y = c\,e^{2x} - ab\,e^{x} - b^{2}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 153

```
dsolve(y(x)*diff(y(x),x)=(a*exp(x)+b)*y(x)+c*exp(2*x)-a*b*exp(x)-b^2,y(x), singsol=all)
```

$$\sqrt{\frac{c\,e^{2x}-(b-y(x))\,(a\,e^{x}+b-y(x))}{(b-y(x))^{2}}}\,y(x)\,e^{-\frac{a\,\operatorname{arctanh}\left(\frac{(b-y(x))a-2\,e^{x}c}{\sqrt{a^{2}+4c}\,(b-y(x))}\right)}{\sqrt{a^{2}+4c}}} - b\left(\int^{\frac{e^{x}}{-b+y(x)}}\frac{\sqrt{\_a^{2}c+a\_a-1}\,e^{-\frac{a\,\operatorname{arctanh}\left(\frac{2c\_a+a}{\sqrt{a^{2}+4c}}\right)}{\sqrt{a^{2}+4c}}}}{\_a}d\_a\right)+c_{1}=0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==(a*Exp[x]+b)*y[x]+c*Exp[2*x]-a*b*Exp[x]-b^2,y[x],x,IncludeSingularSolutio
```

Not solved

24.68 problem 68

Internal problem ID [10805]
Internal file name [`OUTPUT/9752_Wednesday_June_08_2022_05_54_07_PM_90009543/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 68.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - \left(a(2\mu + \lambda)\,\mathrm{e}^{\lambda x} + b\right)\mathrm{e}^{\mu x} y = \left(-a^2\mu\,\mathrm{e}^{2\lambda x} - ab\,\mathrm{e}^{\lambda x} + c\right)\mathrm{e}^{2\mu x}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(a*lambda^2*exp(lambda*x+mu*x)+3*a*m
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(2*a^2*mu*exp(2*lambda*x+2*mu*x)*lam
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(a*lambda^2*exp(x*(lambda+mu))+3*a*m
      Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=(a*(2*mu+lambda)*exp(lambda*x)+b)*exp(mu*x)*y(x)+(-a^2*mu*exp(2*lam
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==(a*(2*\[Mu]+\[Lambda])*Exp[\[Lambda]*x]+b)*Exp[\[Mu]*x]*y[x]+(-a^2*\[Mu]*
```

Not solved

24.69 problem 69

Internal problem ID [10806]
Internal file name [`OUTPUT/9753_Wednesday_June_08_2022_05_54_10_PM_38000994/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - \left(a\,\mathrm{e}^{\lambda x} + b\right) y = c\left(a^2 \mathrm{e}^{2\lambda x} + ab(\lambda x + 1)\,\mathrm{e}^{\lambda x} + b^2 \lambda x\right)$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 257

```
dsolve(y(x)*diff(y(x),x)=(a*exp(lambda*x)+b)*y(x)+c*(a^2*exp(2*lambda*x)+a*b*(lambda*x+1)*ex
```

$$\frac{\sqrt{(4c\lambda+1)(3c\lambda+1)^2}\left(\frac{c\lambda}{2}+\frac{1}{6}\right)\ln\left(\frac{(3c\lambda+1)^2\left(b^2c\lambda^2x^2+2\,e^{x\lambda}abc\lambda x+e^{2x\lambda}a^2c+b\lambda xy(x)+a\,e^{x\lambda}y(x)-\lambda y(x)^2\right)c}{(9c\lambda+2)y(x)^2}\right)-3\left(c\lambda+\frac{1}{3}\right)}{}$$

$= 0$

✓ Solution by Mathematica

Time used: 0.494 (sec). Leaf size: 134

```
DSolve[y[x]*y'[x]==(a*Exp[\[Lambda]*x]+b)*y[x]+c*(a^2*Exp[2*\[Lambda]*x]+a*b*(\[Lambda]*x+1)
```

$$\text{Solve}\left[-\frac{\frac{2\arctan\left(\frac{\frac{2c\lambda y(x)}{ace^{\lambda x}+bc\lambda x}-1}{\sqrt{-4c\lambda-1}}\right)}{\sqrt{-4c\lambda-1}}+\log\left(-\frac{c\lambda y(x)^2}{(ace^{\lambda x}+bc\lambda x)^2}+\frac{y(x)}{ace^{\lambda x}+bc\lambda x}+1\right)}{2c\lambda}=\frac{\log\left(ace^{\lambda x}+bc\lambda x\right)}{c\lambda}+c_1,y(x)\right]$$

24.70 problem 70

Internal problem ID [10807]
Internal file name [`OUTPUT/9754_Wednesday_June_08_2022_05_54_12_PM_60456762/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 70.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - e^{\lambda x}(2a\lambda x + a + b)\, y = -e^{2\lambda x}\left(a^2\lambda\, x^2 + abx + c\right)}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 118

```
dsolve(y(x)*diff(y(x),x)=exp(lambda*x)*(2*a*lambda*x+a+b)*y(x)-exp(2*lambda*x)*(a^2*lambda*x
```

$$y(x) = \frac{\left(\tan\left(\frac{\operatorname{RootOf}\left(2ax\lambda\,\mathrm{e}^{\_Z+\_a}-\sqrt{-\frac{b^2-4c\lambda}{a^2}}\,\tan\left(\frac{\_a\sqrt{-\frac{b^2-4c\lambda}{a^2}}}{2}\right)\_Za+b\,\mathrm{e}^{\_Z+\_a}+2c_1a\,\mathrm{e}^{\_a}\right)\sqrt{\frac{-b^2+4c\lambda}{a^2}}}{2}\right)a\sqrt{\frac{-b^2+4c\lambda}{a^2}}+2xa}{2\lambda}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==Exp[\[Lambda]*x]*(2*a*\[Lambda]*x+a+b)*y[x]-Exp[2*\[Lambda]*x]*(a^2*\[Lam
```

Not solved

24.71 problem 71

Internal problem ID [10808]
Internal file name [`OUTPUT/9755_Wednesday_June_08_2022_05_54_27_PM_62639740/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 71.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - e^{ax}\left(2a\,x^2 + b + 2x\right)y = e^{2ax}\left(-a\,x^4 - b\,x^2 + c\right)}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(2*a^2*x^2+a*b+6*a*x+2)/(2*a*x^2+b+2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+2*y(x)*(a^2*x^4+a*b*x^2+2*a*x^3-a*c+b*x)/
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = -y(x)*(2*a^2*x^2+a*b+6*a*x+2)/(2*a*x^2+
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=exp(a*x)*(2*a*x^2+2*x+b)*y(x)+exp(2*a*x)*(-a*x^4-b*x^2+c),y(x), sin
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==Exp[a*x]*(2*a*x^2+2*x+b)*y[x]+Exp[2*a*x]*(-a*x^4-b*x^2+c),y[x],x,IncludeS
```

Not solved

24.72 problem 72

Internal problem ID [10809]
Internal file name [`OUTPUT/9756_Thursday_June_09_2022_12_57_29_AM_9550685/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 72.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' + a(2bx + 1)\,\mathrm{e}^{bx}y = -a^2b\,x^2\mathrm{e}^{2bx}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve(y(x)*diff(y(x),x)+a*(1+2*b*x)*exp(b*x)*y(x)=-a^2*b*x^2*exp(2*b*x),y(x), singsol=all)
```

$$y(x) = -\frac{e^{bx} a\left(bx \operatorname{RootOf}\left(-e^{\_Z} bx - \operatorname{expIntegral}_1\left(-\_Z\right) + c_1\right) - 1\right)}{\operatorname{RootOf}\left(-e^{\_Z} bx - \operatorname{expIntegral}_1\left(-\_Z\right) + c_1\right) b}$$

✓ Solution by Mathematica

Time used: 0.736 (sec). Leaf size: 59

```
DSolve[y[x]*y'[x]+a*(1+2*b*x)*Exp[b*x]*y[x]==-a^2*b*x^2*Exp[2*b*x],y[x],x,IncludeSingularSol
```

$$\text{Solve}\left[bxe^{\frac{ae^{bx}}{abxe^{bx}+by(x)}} = \text{ExpIntegralEi}\left(\frac{ae^{bx}}{abe^{bx}x + by(x)}\right) + c_1, y(x)\right]$$

24.73 problem 73

Internal problem ID [10810]
Internal file name [OUTPUT/9757_Thursday_June_09_2022_12_57_31_AM_11538751/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 73.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - a(1 + 2n + 2n(1 + n)\,x)\,e^{(1+n)x}y = -a^2n(1 + n)\,(xn + 1)\,x\,e^{2(1+n)x}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 130

```
dsolve(y(x)*diff(y(x),x)-a*(1+2*n+2*n*(n+1)*x)*exp((n+1)*x)*y(x)=-a^2*n*(n+1)*(1+n*x)*x*exp(
```

$$y(x) = \frac{e^{(n+1)x} a \left(1 + 2x n^2 + \left(\tan\left(\frac{\operatorname{RootOf}\left(2x n^2 e^{\_Z + \_a} - \tan\left(\frac{\_a\sqrt{-\frac{(n+1)^2}{n^2}}}{2}\right)\_Z\sqrt{-\frac{(n+1)^2}{n^2}}\, n + 2nx\, e^{\_Z + \_a} + n\, e^{\_Z + \_a} + 2c_1 n}{2}\right.\right.\right.}{2n+2}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]-a*(1+2*n+2*n*(n+1)*x)*Exp[(n+1)*x]*y[x]==-a^2*n*(n+1)*(1+n*x)*x*Exp[2*(n+1
```

Not solved

24.74 problem 74

Internal problem ID [10811]
Internal file name [`OUTPUT/9758_Thursday_June_09_2022_12_57_33_AM_22585268/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 74.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' + a\left(1 + 2b\sqrt{x}\right) e^{2b\sqrt{x}} y = -a^2 b\, x^{\frac{3}{2}} e^{4b\sqrt{x}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 307

```
dsolve(y(x)*diff(y(x),x)+a*(1+2*b*x^(1/2))*exp(2*b*x^(1/2))*y(x)=-a^2*b*x^(3/2)*exp(4*b*x^(1
```

$$\frac{\sqrt{\frac{a\,\mathrm{e}^{2b\sqrt{x}}}{b^2\left(\mathrm{e}^{2b\sqrt{x}}ax+y(x)\right)}}\sqrt{x}\,\mathrm{BesselI}\left(1,\sqrt{\frac{a\,\mathrm{e}^{2b\sqrt{x}}}{b^2\left(\mathrm{e}^{2b\sqrt{x}}ax+y(x)\right)}}\right)c_1 b-\mathrm{BesselK}\left(1,-\sqrt{\frac{a\,\mathrm{e}^{2b\sqrt{x}}}{b^2\left(\mathrm{e}^{2b\sqrt{x}}ax+y(x)\right)}}\right)\sqrt{\frac{a\,\mathrm{e}^{2b\sqrt{x}}}{b^2\left(\mathrm{e}^{2b\sqrt{x}}ax+y(x)\right)}}}{\mathrm{BesselI}\left(1,\sqrt{\frac{a\,\mathrm{e}^{2b\sqrt{x}}}{b^2\left(\mathrm{e}^{2b\sqrt{x}}ax+y(x)\right)}}\right)\sqrt{\frac{a\,\mathrm{e}^{2b\sqrt{x}}}{b^2\left(\mathrm{e}^{2b\sqrt{x}}ax+y(x)\right)}}\,b\sqrt{x}-\mathrm{Be}}$$

$= 0$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+a*(1+2*b*x^(1/2))*Exp[2*b*x^(1/2)]*y[x]==-a^2*b*x^(3/2)*exp(4*b*x^(1/2)),y
```

Not solved

24.75 problem 75

Internal problem ID [10812]
Internal file name [`OUTPUT/9759_Thursday_June_09_2022_12_57_35_AM_77570412/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 75.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - (a \cosh(x) + b) y = -ab \sinh(x) + c$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
   -> Calling odsolve with the ODE`, diff(y(x), x) = (sinh(y(x))*exp(-x*b)*a*b-exp(-x*b)*c-b
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
      trying separable
      trying inverse linear
      trying homogeneous types:
      trying Chini
      differential order: 1; looking for linear symmetries
      trying exact
      Looking for potential symmetries
      trying inverse_Riccati
      trying an equivalence to an Abel ODE
      differential order: 1; trying a linearization to 2nd order
      --- trying a change of variables {x -> y(x), y(x) -> x}
      differential order: 1; trying a linearization to 2nd order
      trying 1st order ODE linearizable_by_differentiation
      --- Trying Lie symmetry methods, 1st order ---
      `, `-> Computing symmetries using: way = 3
      `, `-> Computing symmetries using: way = 4
      `, `-> Computing symmetries using: way = 5
      trying symmetry patterns for 1st order ODEs
      -> trying a symmetry pattern of the form [F(x)*G(y), 0]
      -> trying a symmetry pattern of the form [0, F(x)*G(y)]
      -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
      `, `-> Computing symmetries using: way = HINT
         -> Calling odsolve with the ODE`, diff(f__1(y), y)-cosh(y)*f__1(y)/sinh(y), f__1(y)
            Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=(a*cosh(x)+b)*y(x)-a*b*sinh(x)+c,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==(a*Cosh[x]+b)*y[x]-a*b*Sinh[x]+c,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

24.76 problem 76

Internal problem ID [10813]
Internal file name [`OUTPUT/9760_Thursday_June_09_2022_12_57_43_AM_66301418/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 76.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - (a\sinh(x) + b)\,y = -ab\cosh(x) + c$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
   -> Calling odsolve with the ODE`, diff(y(x), x) = (cosh(y(x))*exp(-x*b)*a*b-exp(-x*b)*c-b
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
      trying separable
      trying inverse linear
      trying homogeneous types:
      trying Chini
      differential order: 1; looking for linear symmetries
      trying exact
      Looking for potential symmetries
      trying inverse_Riccati
      trying an equivalence to an Abel ODE
      differential order: 1; trying a linearization to 2nd order
      --- trying a change of variables {x -> y(x), y(x) -> x}
      differential order: 1; trying a linearization to 2nd order
      trying 1st order ODE linearizable_by_differentiation
      --- Trying Lie symmetry methods, 1st order ---
      `, `-> Computing symmetries using: way = 3
      `, `-> Computing symmetries using: way = 4
      `, `-> Computing symmetries using: way = 5
      trying symmetry patterns for 1st order ODEs
      -> trying a symmetry pattern of the form [F(x)*G(y), 0]
      -> trying a symmetry pattern of the form [0, F(x)*G(y)]
      -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
      `, `-> Computing symmetries using: way = HINT
         -> Calling odsolve with the ODE`, diff(f__1(y), y)-sinh(y)*f__1(y)/cosh(y), f__1(y)
            Methods for first order ODEs:
            --- Trying classification methods ---
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=(a*sinh(x)+b)*y(x)-a*b*cosh(x)+c,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==(a*Sinh[x]+b)*y[x]-a*b*Cosh[x]+c,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

24.77 problem 77

Internal problem ID [10814]
Internal file name [`OUTPUT/9761_Thursday_June_09_2022_12_57_50_AM_87747449/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 77.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - (2\ln(x) + a + 1)\, y = x\left(-\ln(x)^2 - a\ln(x) + b\right)}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 163

```
dsolve(y(x)*diff(y(x),x)=(2*ln(x)+a+1)*y(x)+x*( -(ln(x))^2-a*ln(x)+b),y(x), singsol=all)
```

$$y(x) = \frac{x\left(-\tanh\left(\frac{\mathrm{RootOf}\left(-\sqrt{a^2+4b}\,\tanh\left(\frac{\_Z\sqrt{a^2+4b}}{2}\right)\mathrm{e}^{\_Z}+\mathrm{e}^{-\frac{2\,\mathrm{arctanh}\left(\frac{2\_a-a}{\sqrt{a^2+4b}}\right)}{\sqrt{a^2+4b}}}\tanh\left(\frac{\_Z\sqrt{a^2+4b}}{2}\right)\sqrt{a^2+4b}+2\ln(x)\mathrm{e}^{\_Z}+\mathrm{e}^{\_Z}a-\mathrm{e}\cdots}{2}\right.\right.}{2}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==(2*Log[x]+a+1)*y[x]+x*( -(Log[x])^2-a*Log[x]+b),y[x],x,IncludeSingularSol
```

Not solved

24.78 problem 78

Internal problem ID [10815]
Internal file name [`OUTPUT/9762_Thursday_June_09_2022_12_57_51_AM_85664660/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 78.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$yy' - \left(2\ln(x)^2 + 2\ln(x) + a\right) y = x\left(-\ln(x)^4 - \ln(x)^2 a + b\right)$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+2*y(x)*(2*ln(x)+1)/(x*(2*ln(x)^2+2*ln(x)+
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(ln(x)^4+4*ln(x)^3+a*ln(x)^2+2*ln(x)
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = -2*y(x)*(2*ln(x)+1)/(x*(2*ln(x)^2+2*ln(
      Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=(2*(ln(x))^2+2*ln(x)+a)*y(x)+x*(- (ln(x))^4-a*(ln(x))^2+b),y(x), si
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==(2*(Log[x])^2+2*Log[x]+a)*y[x]+x*(- (Log[x])^4-a*(Log[x])^2+b),y[x],x,Inc
```

Not solved

24.79 problem 79

Internal problem ID [10816]
Internal file name [`OUTPUT/9763_Thursday_June_09_2022_12_57_53_AM_44345512/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - ax\cos\left(\lambda\, x^2\right) y = x}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(-2*x^2*lambda*sin(x^2*lambda)+cos(x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=a*x*cos(lambda*x^2)*y(x)+x,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==a*x*Cos[\[Lambda]*x^2]*y[x]+x,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

24.80 problem 80

Internal problem ID [10817]
Internal file name [OUTPUT/9764_Thursday_June_09_2022_12_57_54_AM_98898086/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.3-2. Equations of the form $yy' = f_1(x)y + f_0(x)$
Problem number: 80.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{yy' - ax\sin\left(\lambda\, x^2\right) y = x}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(2*cos(x^2*lambda)*lambda*x^2+sin(x^
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)=a*x*sin(lambda*x^2)*y(x)+x,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]==a*x*Sin[\[Lambda]*x^2]*y[x]+x,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

25 Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.4-2. Equations of the form $(g_1(x) + g_0(x))y' = f_2(x)y^2 + f_1(x)y + f_0(x)$

25.1 problem 1

Internal problem ID [10818]
Internal file name [`OUTPUT/9765_Thursday_June_09_2022_12_57_56_AM_13344031/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.4-2. Equations of the form $(g_1(x) + g_0(x))y' = f_2(x)y^2 + f_1(x)y + f_0(x)$
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{(Ay + Bx + a)\, y' + By = -kx - b}$$

25.1.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{By + kx + b}{Ay + Bx + a}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(By+kx+b)(b_3-a_2)}{Ay+Bx+a} - \frac{(By+kx+b)^2 a_3}{(Ay+Bx+a)^2} - \left(-\frac{k}{Ay+Bx+a} + \frac{(By+kx+b)B}{(Ay+Bx+a)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{B}{Ay+Bx+a} + \frac{(By+kx+b)A}{(Ay+Bx+a)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{A^2y^2b_2 + 2ABxyb_2 + AB\,y^2a_2 - AB\,y^2b_3 - Ak\,x^2b_2 + 2Akxya_2 - 2Akxyb_3 + Ak\,y^2a_3 + 2B^2x^2b_2 - 2B^2y}{}$$

$= 0$

Setting the numerator to zero gives

$$\begin{aligned} &A^2y^2b_2+2ABxyb_2+AB\,y^2a_2-AB\,y^2b_3-Ak\,x^2b_2+2Akxya_2-2Akxyb_3 \\ &+Ak\,y^2a_3+2B^2x^2b_2-2B^2y^2a_3+Bk\,x^2a_2-Bk\,x^2b_3-2Bkxya_3-k^2x^2a_3 \\ &+2Aayb_2-Abxb_2+Abya_2-2Abyb_3-Akxb_1+Akya_1+B^2xb_1-B^2ya_1 \\ &+3Baxb_2+Baya_2-Bbxb_3-3Bbya_3+2akxa_2-akxb_3+akya_3 \\ &-2bkxa_3-Abb_1+Bab_1-Bba_1+a^2b_2+aba_2-abb_3+aka_1-b^2a_3=0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&A^2b_2v_2^2 + ABa_2v_2^2 + 2ABb_2v_1v_2 - ABb_3v_2^2 + 2Aka_2v_1v_2 + Aka_3v_2^2 \\
&- Akb_2v_1^2 - 2Akb_3v_1v_2 - 2B^2a_3v_2^2 + 2B^2b_2v_1^2 + Bka_2v_1^2 - 2Bka_3v_1v_2 \\
&- Bkb_3v_1^2 - k^2a_3v_1^2 + 2Aab_2v_2 + Aba_2v_2 - Abb_2v_1 - 2Abb_3v_2 \\
&+ Aka_1v_2 - Akb_1v_1 - B^2a_1v_2 + B^2b_1v_1 + Baa_2v_2 + 3Bab_2v_1 \\
&- 3Bba_3v_2 - Bbb_3v_1 + 2aka_2v_1 + aka_3v_2 - akb_3v_1 - 2bka_3v_1 \\
&- Abb_1 + Bab_1 - Bba_1 + a^2b_2 + aba_2 - abb_3 + aka_1 - b^2a_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-Akb_2 + 2B^2b_2 + Bka_2 - Bkb_3 - k^2a_3)\, v_1^2 \\
&+ (2ABb_2 + 2Aka_2 - 2Akb_3 - 2Bka_3)\, v_1v_2 \\
&+ (-Abb_2 - Akb_1 + B^2b_1 + 3Bab_2 - Bbb_3 + 2aka_2 - akb_3 - 2bka_3)\, v_1 \\
&+ (A^2b_2 + ABa_2 - ABb_3 + Aka_3 - 2B^2a_3)\, v_2^2 \\
&+ (2Aab_2 + Aba_2 - 2Abb_3 + Aka_1 - B^2a_1 + Baa_2 - 3Bba_3 + aka_3)\, v_2 \\
&- Abb_1 + Bab_1 - Bba_1 + a^2b_2 + aba_2 - abb_3 + aka_1 - b^2a_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
2ABb_2 + 2Aka_2 - 2Akb_3 - 2Bka_3 &= 0 \\
-Akb_2 + 2B^2b_2 + Bka_2 - Bkb_3 - k^2a_3 &= 0 \\
A^2b_2 + ABa_2 - ABb_3 + Aka_3 - 2B^2a_3 &= 0 \\
2Aab_2 + Aba_2 - 2Abb_3 + Aka_1 - B^2a_1 + Baa_2 - 3Bba_3 + aka_3 &= 0 \\
-Abb_2 - Akb_1 + B^2b_1 + 3Bab_2 - Bbb_3 + 2aka_2 - akb_3 - 2bka_3 &= 0 \\
-Abb_1 + Bab_1 - Bba_1 + a^2b_2 + aba_2 - abb_3 + aka_1 - b^2a_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -\frac{ABbb_2 + Aakb_2 - Abkb_3 - 2B^2ab_2 + Bakb_3}{k\,(Ak - B^2)} \\
a_2 &= -\frac{2Bb_2 - kb_3}{k} \\
a_3 &= -\frac{Ab_2}{k} \\
b_1 &= \frac{Abb_2 - Bab_2 - Bbb_3 + akb_3}{Ak - B^2} \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{Akx - B^2x + Ab - Ba}{Ak - B^2} \\
\eta &= \frac{Aky - B^2y - Bb + ak}{Ak - B^2}
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= \frac{Aky - B^2y - Bb + ak}{Ak - B^2} - \left(-\frac{By + kx + b}{Ay + Bx + a}\right)\left(\frac{Akx - B^2x + Ab - Ba}{Ak - B^2}\right) \\
&= \frac{A^2k\,y^2 - A\,B^2y^2 + 2ABkxy + A\,k^2x^2 - 2B^3xy - B^2k\,x^2 + 2Aaky + 2Abkx - 2B^2ay - 2B^2bx + A\,b}{A^2ky - A\,B^2y + ABkx - x\,B^3 + Aak - B^2a} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{A^2k\,y^2-A\,B^2y^2+2ABkxy+A\,k^2x^2-2B^3xy-B^2k\,x^2+2Aaky+2Abkx-2B^2ay-2B^2bx+A\,b^2-2Bab+a^2k}{A^2ky-A\,B^2y+ABkx-x\,B^3+Aak-B^2a}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln\left(A^2k\,y^2 - A\,B^2y^2 + 2ABkxy + A\,k^2x^2 - 2B^3xy - B^2k\,x^2 + 2Aaky + 2Abkx - 2B^2ay - 2B^2bx + \right.}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{By + kx + b}{Ay + Bx + a}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{(By + kx + b)\,(Ak - B^2)}{A^2k\,y^2 + (-B^2y^2 + 2Bkxy + k^2x^2 + (2ay + 2bx)\,k + b^2)\,A - 2\left(B^2y + \left(\frac{kx}{2} + b\right)B - \frac{ak}{2}\right)(Bx + a)} \\
S_y &= \frac{(Ak - B^2)\,(Ay + Bx + a)}{A^2k\,y^2 + (-B^2y^2 + 2Bkxy + k^2x^2 + (2ay + 2bx)\,k + b^2)\,A - 2\left(B^2y + \left(\frac{kx}{2} + b\right)B - \frac{ak}{2}\right)(Bx + a)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln \left(A\,k^2 x^2+\left(-B^2 x^2+2ABxy+2Abx+\left(Ay+a\right)^2\right) k-\left(By+b\right)\left(2B^2 x+\left(Ay+2a\right) B-Ab\right)\right)}{2} = c_1$$

Which simplifies to

$$\frac{\ln \left(A\,k^2 x^2+\left(-B^2 x^2+2ABxy+2Abx+\left(Ay+a\right)^2\right) k-\left(By+b\right)\left(2B^2 x+\left(Ay+2a\right) B-Ab\right)\right)}{2} = c_1$$

Summary
The solution(s) found are the following

$$\frac{\ln \left(A\,k^2 x^2+\left(-B^2 x^2+2ABxy+2Abx+\left(Ay+a\right)^2\right) k-\left(By+b\right)\left(2B^2 x+\left(Ay+2a\right) B-Ab\right)\right)}{2} = c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln \left(A\,k^2 x^2+\left(-B^2 x^2+2ABxy+2Abx+\left(Ay+a\right)^2\right) k-\left(By+b\right)\left(2B^2 x+\left(Ay+2a\right) B-Ab\right)\right)}{2} = c_1$$

Verified OK.

25.1.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(Ay + Bx + a) \, \mathrm{d}y &= (-By - kx - b) \, \mathrm{d}x \\
(By + kx + b) \, \mathrm{d}x + (Ay + Bx + a) \, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= By + kx + b \\
N(x, y) &= Ay + Bx + a
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(By + kx + b) \\
&= B
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(Ay + Bx + a) \\
&= B
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int By+kx+b\,\mathrm{d}x$$
$$\phi=\frac{k\,x^2}{2}+(By+b)\,x+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=Bx+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=Ay+Bx+a$. Therefore equation (4) becomes

$$Ay+Bx+a=Bx+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=Ay+a$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int (Ay+a)\,\mathrm{d}y$$
$$f(y)=\frac{1}{2}A\,y^2+ay+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{k\,x^2}{2} + (By + b)\,x + \frac{A\,y^2}{2} + ay + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{k\,x^2}{2} + (By + b)\,x + \frac{A\,y^2}{2} + ay$$

Summary

The solution(s) found are the following

$$\frac{k\,x^2}{2} + (By + b)\,x + \frac{Ay^2}{2} + ay = c_1 \tag{1}$$

Verification of solutions

$$\frac{k\,x^2}{2} + (By + b)\,x + \frac{Ay^2}{2} + ay = c_1$$

Verified OK.

25.1.3 Maple step by step solution

Let's solve

$(Ay + Bx + a)\,y' + By = -kx - b$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

  ◦ Evaluate derivatives

    $B = B$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int (By+kx+b)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=Bxy+\frac{k\,x^2}{2}+bx+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $Ay+Bx+a=Bx+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=Ay+a$

- Solve for $f_1(y)$

  $f_1(y)=\frac{1}{2}A\,y^2+ay$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=Bxy+\frac{1}{2}k\,x^2+bx+\frac{1}{2}A\,y^2+ay$

- Substitute $F(x,y)$ into the solution of the ODE

  $Bxy+\frac{1}{2}k\,x^2+bx+\frac{1}{2}A\,y^2+ay=c_1$

- Solve for y

  $\left\{y=-\frac{Bx-\sqrt{-Ak\,x^2+B^2x^2-2Abx+2Bax+2Ac_1+a^2}+a}{A}, y=-\frac{Bx+\sqrt{-Ak\,x^2+B^2x^2-2Abx+2Bax+2Ac_1+a^2}+a}{A}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 87

```
dsolve((A*y(x)+B*x+a)*diff(y(x),x)+B*y(x)+k*x+b=0,y(x), singsol=all)
```

$$y(x) = \frac{-\sqrt{-\left(Ak - B^2\right)\left(\left(kx+b\right)A - B^2x - Ba\right)^2 c_1^2 + A} + \left(k(-Bx-a)A + x B^3 + a B^2\right)c_1}{Ac_1\left(Ak - B^2\right)}$$

✓ Solution by Mathematica

Time used: 18.19 (sec). Leaf size: 106

```
DSolve[(A*y[x]+B*x+a)*y'[x]+B*y[x]+k*x+b==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\frac{\sqrt{\frac{(a+Bx)^2}{A}+Ac_1-x(2b+kx)}}{\sqrt{\frac{1}{A}}} + a + Bx}{A}$$

$$y(x) \to -\frac{a+Bx}{A} + \sqrt{\frac{1}{A}}\sqrt{\frac{(a+Bx)^2}{A} + Ac_1 - x(2b+kx)}$$

25.2 problem 2

Internal problem ID [10819]
Internal file name [`OUTPUT/9766_Thursday_June_09_2022_12_57_57_AM_92691934/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.4-2. Equations of the form $(g_1(x) + g_0(x))y' = f_2(x)y^2 + f_1(x)y + f_0(x)$
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(y + ax + b)\, y' - \alpha y = \beta x + \gamma}$$

25.2.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{\alpha y + \beta x + \gamma}{ax + b + y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &+ \frac{(\alpha y + \beta x + \gamma)(b_3 - a_2)}{ax + b + y} - \frac{(\alpha y + \beta x + \gamma)^2 a_3}{(ax + b + y)^2} \\ &- \left(\frac{\beta}{ax + b + y} - \frac{(\alpha y + \beta x + \gamma) a}{(ax + b + y)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(\frac{\alpha}{ax + b + y} - \frac{\alpha y + \beta x + \gamma}{(ax + b + y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{a^2x^2b_2 - a\alpha\, x^2b_2 + a\alpha\, y^2a_3 - a\beta\, x^2a_2 + a\beta\, x^2b_3 - \alpha^2y^2a_3 - 2\alpha\beta xya_3 - \beta^2x^2a_3 - a\alpha xb_1 + a\alpha ya_1 + 2abx}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &a^2x^2b_2 - a\alpha\, x^2b_2 + a\alpha\, y^2a_3 - a\beta\, x^2a_2 + a\beta\, x^2b_3 - \alpha^2y^2a_3 - 2\alpha\beta xya_3 \\ &- \beta^2x^2a_3 - a\alpha xb_1 + a\alpha ya_1 + 2abxb_2 + a\gamma xb_3 + a\gamma ya_3 + 2axyb_2 - \alpha bxb_2 \\ &- \alpha bya_2 - 2\alpha\gamma ya_3 - \alpha\, y^2a_2 + \alpha\, y^2b_3 - 2b\beta xa_2 + b\beta xb_3 - b\beta ya_3 - 2\beta\gamma xa_3 \\ &+ \beta\, x^2b_2 - 2\beta xya_2 + 2\beta xyb_3 - \beta\, y^2a_3 + a\gamma a_1 - \alpha bb_1 + b^2b_2 - b\beta a_1 - b\gamma a_2 \\ &+ b\gamma b_3 + 2byb_2 + \beta xb_1 - \beta ya_1 - \gamma^2a_3 + \gamma xb_2 - \gamma ya_2 + 2\gamma yb_3 + y^2b_2 + \gamma b_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&a^{2} b_{2} v_{1}^{2}+a \alpha a_{3} v_{2}^{2}-a \alpha b_{2} v_{1}^{2}-a \beta a_{2} v_{1}^{2}+a \beta b_{3} v_{1}^{2}-\alpha^{2} a_{3} v_{2}^{2}-2 \alpha \beta a_{3} v_{1} v_{2} \\
&-\beta^{2} a_{3} v_{1}^{2}+a \alpha a_{1} v_{2}-a \alpha b_{1} v_{1}+2 a b b_{2} v_{1}+a \gamma a_{3} v_{2}+a \gamma b_{3} v_{1}+2 a b_{2} v_{1} v_{2} \\
&-\alpha b a_{2} v_{2}-\alpha b b_{2} v_{1}-2 \alpha \gamma a_{3} v_{2}-\alpha a_{2} v_{2}^{2}+\alpha b_{3} v_{2}^{2}-2 b \beta a_{2} v_{1}-b \beta a_{3} v_{2} \\
&+b \beta b_{3} v_{1}-2 \beta \gamma a_{3} v_{1}-2 \beta a_{2} v_{1} v_{2}-\beta a_{3} v_{2}^{2}+\beta b_{2} v_{1}^{2}+2 \beta b_{3} v_{1} v_{2} \\
&+a \gamma a_{1}-\alpha b b_{1}+b^{2} b_{2}-b \beta a_{1}-b \gamma a_{2}+b \gamma b_{3}+2 b b_{2} v_{2}-\beta a_{1} v_{2} \\
&+\beta b_{1} v_{1}-\gamma^{2} a_{3}-\gamma a_{2} v_{2}+\gamma b_{2} v_{1}+2 \gamma b_{3} v_{2}+b_{2} v_{2}^{2}+\gamma b_{1}=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(a^{2} b_{2}-a \alpha b_{2}-a \beta a_{2}+a \beta b_{3}-\beta^{2} a_{3}+\beta b_{2}\right) v_{1}^{2} \\
&+\left(-2 \alpha \beta a_{3}+2 a b_{2}-2 \beta a_{2}+2 \beta b_{3}\right) v_{1} v_{2} \\
&+\left(-a \alpha b_{1}+2 a b b_{2}+a \gamma b_{3}-\alpha b b_{2}-2 b \beta a_{2}+b \beta b_{3}-2 \beta \gamma a_{3}+\beta b_{1}+\gamma b_{2}\right) v_{1} \\
&+\left(a \alpha a_{3}-\alpha^{2} a_{3}-\alpha a_{2}+\alpha b_{3}-\beta a_{3}+b_{2}\right) v_{2}^{2} \\
&+\left(a \alpha a_{1}+a \gamma a_{3}-\alpha b a_{2}-2 \alpha \gamma a_{3}-b \beta a_{3}+2 b b_{2}-\beta a_{1}-\gamma a_{2}+2 \gamma b_{3}\right) v_{2} \\
&+a \gamma a_{1}-\alpha b b_{1}+b^{2} b_{2}-b \beta a_{1}-b \gamma a_{2}+b \gamma b_{3}-\gamma^{2} a_{3}+\gamma b_{1}=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2 \alpha \beta a_{3}+2 a b_{2}-2 \beta a_{2}+2 \beta b_{3} &=0 \\
a \alpha a_{3}-\alpha^{2} a_{3}-\alpha a_{2}+\alpha b_{3}-\beta a_{3}+b_{2} &=0 \\
a^{2} b_{2}-a \alpha b_{2}-a \beta a_{2}+a \beta b_{3}-\beta^{2} a_{3}+\beta b_{2} &=0 \\
a \gamma a_{1}-\alpha b b_{1}+b^{2} b_{2}-b \beta a_{1}-b \gamma a_{2}+b \gamma b_{3}-\gamma^{2} a_{3}+\gamma b_{1} &=0 \\
a \alpha a_{1}+a \gamma a_{3}-\alpha b a_{2}-2 \alpha \gamma a_{3}-b \beta a_{3}+2 b b_{2}-\beta a_{1}-\gamma a_{2}+2 \gamma b_{3} &=0 \\
-a \alpha b_{1}+2 a b b_{2}+a \gamma b_{3}-\alpha b b_{2}-2 b \beta a_{2}+b \beta b_{3}-2 \beta \gamma a_{3}+\beta b_{1}+\gamma b_{2} &=0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= \frac{a\alpha b a_3 - \alpha^2 b a_3 + \alpha b b_3 + \alpha\gamma a_3 - b\beta a_3 - \gamma b_3}{a\alpha - \beta} \\
a_2 &= a a_3 - \alpha a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= \frac{\alpha b\beta a_3 + a\gamma b_3 - b\beta b_3 - \beta\gamma a_3}{a\alpha - \beta} \\
b_2 &= \beta a_3 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{a\alpha x + b\alpha - \beta x - \gamma}{a\alpha - \beta} \\
\eta &= \frac{a\alpha y + a\gamma - b\beta - \beta y}{a\alpha - \beta}
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= \frac{a\alpha y + a\gamma - b\beta - \beta y}{a\alpha - \beta} - \left(\frac{\alpha y + \beta x + \gamma}{a x + b + y}\right)\left(\frac{a\alpha x + b\alpha - \beta x - \gamma}{a\alpha - \beta}\right) \\
&= \frac{a^2\alpha x y - a\,\alpha^2 x y - a\alpha\beta\, x^2 + a^2\gamma x + a\alpha b y - a\alpha\gamma x + a\alpha\, y^2 - ab\beta x - a\beta x y - \alpha^2 b y - \alpha b\beta x + \alpha\beta x y +}{a^2\alpha x + a\alpha b + a\alpha y - a\beta x - b\beta - \beta}
\end{aligned}
$$

$$
\xi = 0
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{a^2\alpha xy - a\,\alpha^2 xy - a\alpha\beta\, x^2 + a^2\gamma x + a\alpha by - a\alpha\gamma x + a\alpha\, y^2 - ab\beta x - a\beta xy - \alpha^2 by - \alpha b\beta x + \alpha\beta xy + \beta^2 x^2 + b\gamma a + a\gamma y - b\gamma\alpha + \alpha\gamma y - b^2\beta - 2b\beta y + 2\beta\gamma}{a^2\alpha x + a\alpha b + a\alpha y - a\beta x - b\beta - \beta y}}\end{aligned}$$

Which results in

$$S = (a\alpha - \beta)\left(\frac{\ln\left(a^2\alpha xy - a\,\alpha^2 xy - a\alpha\beta\, x^2 + a^2\gamma x + a\alpha by - a\alpha\gamma x + a\alpha\, y^2 - ab\beta x - a\beta xy - \alpha^2 by - \alpha\right.}{2a\alpha - 2}\right.$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\alpha y + \beta x + \gamma}{ax + b + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{(\alpha y + \beta x + \gamma)(a\alpha - \beta)}{-y(ax+b)\alpha^2 + (a^2xy + (-x^2\beta - \gamma x + y(b+y))a - (\beta x + \gamma)(b - y))\alpha + a^2\gamma x - (\beta x - \gamma)(b -} \\ S_y &= \frac{(a\alpha - \beta)(ax + b + y)}{-y(ax+b)\alpha^2 + (a^2xy + (-x^2\beta - \gamma x + y(b+y))a - (\beta x + \gamma)(b - y))\alpha + a^2\gamma x - (\beta x - \gamma)(b +}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\frac{\ln\left((-a\alpha\beta+\beta^2)x^2+(((-a+\alpha)y+2\gamma+(-\alpha-a)b)\beta+a(\alpha y+\gamma)(a-\alpha))x-(b+y)^2\beta+(ay+\gamma+b(a-\alpha))(\alpha y+\gamma)\right)\sqrt{-a^2+2a\alpha-\alpha^2-4\beta}}{2}+\arctan\left(\right.}{\sqrt{-a^2+2a\alpha-\alpha^2-4\beta}}$$

Which simplifies to

$$\frac{\frac{\ln\left((-a\alpha\beta+\beta^2)x^2+(((-a+\alpha)y+2\gamma+(-\alpha-a)b)\beta+a(\alpha y+\gamma)(a-\alpha))x-(b+y)^2\beta+(ay+\gamma+b(a-\alpha))(\alpha y+\gamma)\right)\sqrt{-a^2+2a\alpha-\alpha^2-4\beta}}{2}+\arctan\left(\right.}{\sqrt{-a^2+2a\alpha-\alpha^2-4\beta}}$$

Summary

The solution(s) found are the following

$$\frac{\frac{\ln\left((-a\alpha\beta+\beta^2)x^2+(((-a+\alpha)y+2\gamma+(-\alpha-a)b)\beta+a(\alpha y+\gamma)(a-\alpha))x-(b+y)^2\beta+(ay+\gamma+b(a-\alpha))(\alpha y+\gamma)\right)\sqrt{-a^2+2a\alpha-\alpha^2-4\beta}}{2}+\arctan\left(\right.}{\sqrt{-a^2+2a\alpha-\alpha^2-4\beta}} \tag{1}$$
$$= c_1$$

Verification of solutions

$$\frac{\frac{\ln\left((-a\alpha\beta+\beta^2)x^2+(((-a+\alpha)y+2\gamma+(-\alpha-a)b)\beta+a(\alpha y+\gamma)(a-\alpha))x-(b+y)^2\beta+(ay+\gamma+b(a-\alpha))(\alpha y+\gamma)\right)\sqrt{-a^2+2a\alpha-\alpha^2-4\beta}}{2}+\arctan\left(\right.}{\sqrt{-a^2+2a\alpha-\alpha^2-4\beta}}$$
$$= c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.219 (sec). Leaf size: 211

```
dsolve((y(x)+a*x+b)*diff(y(x),x)=alpha*y(x)+beta*x+gamma,y(x), singsol=all)
```

$$y(x) = \frac{((ax+b)\alpha - x\beta - \gamma)\sqrt{-a^2+2a\alpha-\alpha^2-4\beta}\tan\left(\mathrm{RootOf}\left(-2\sqrt{-a^2+2a\alpha-\alpha^2-4\beta}\ln(2)+\sqrt{-a^2\right.\right.}{}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(y[x]*a*x+b)*y'[x]==\[Alpha]*y[x]+\[Beta]*x+\[Gamma],y[x],x,IncludeSingularSolutions
```

Not solved

25.3 problem 3

Internal problem ID [10820]
Internal file name [`OUTPUT/9801_Sunday_June_19_2022_08_04_07_PM_9550685/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.4-2. Equations of the form $(g_1(x) + g_0(x))y' = f_2(x)y^2 + f_1(x)y + f_0(x)$
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\left(y + ak\,x^2 + bx + c\right)y' + y^2a - 2yakx - ym = k(k + b - m)\,x + s$$

Unable to determine ODE type.

✗ Solution by Maple

```
dsolve((y(x)+a*k*x^2+b*x+c)*diff(y(x),x)=-a*y(x)^2+2*a*k*x*y(x)+m*y(x)+k*(k+b-m)*x+s,y(x), s
```

No solution found

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(y[x]+a*k*x^2+b*x+c)*y'[x]==-a*y[x]^2+2*a*k*x*y[x]+m*y[x]+k*(k+b-m)*x+s,y[x],x,Includ
```

Timed out

25.4 problem 4

Internal problem ID [10821]
Internal file name [`OUTPUT/9802_Sunday_June_19_2022_09_25_19_PM_24093222/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.4-2. Equations of the form $(g_1(x) + g_0(x))y' = f_2(x)y^2 + f_1(x)y + f_0(x)$
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational, [_1st_order, `_with_symmetry_[F(x),G(x)]`],
    [_Abel, `2nd type`, `class A`]]
```

$$\boxed{(y + A\,x^n + a)\,y' + nA\,x^{n-1}y = -k\,x^m - b}$$

25.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(y + A\,x^{n} + a)\,\mathrm{d}y = \left(-nA\,x^{n-1}y - k\,x^{m} - b\right)\mathrm{d}x$$
$$\left(nA\,x^{n-1}y + k\,x^{m} + b\right)\mathrm{d}x + (y + A\,x^{n} + a)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = nA\,x^{n-1}y + k\,x^{m} + b$$
$$N(x, y) = y + A\,x^{n} + a$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(nA\,x^{n-1}y + k\,x^{m} + b\right)\\ &= A\,x^{n-1}n\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y + A\,x^{n} + a)\\ &= A\,x^{n-1}n\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int nA\, x^{n-1} y + k\, x^{m} + b \, \mathrm{d}x$$
$$\phi = bx + \frac{k\, x^{m+1}}{m+1} + Ay\, x^{n} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = A\, x^{n} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y + A\, x^{n} + a$. Therefore equation (4) becomes

$$y + A\, x^{n} + a = A\, x^{n} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = a + y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (a + y) \, \mathrm{d}y$$
$$f(y) = ay + \frac{1}{2} y^{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = bx + \frac{k\,x^{m+1}}{m+1} + Ay\,x^n + ay + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = bx + \frac{k\,x^{m+1}}{m+1} + Ay\,x^n + ay + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$bx + \frac{k\,x^{m+1}}{m+1} + Ayx^n + ay + \frac{y^2}{2} = c_1 \tag{1}$$

Verification of solutions

$$bx + \frac{k\,x^{m+1}}{m+1} + Ayx^n + ay + \frac{y^2}{2} = c_1$$

Verified OK.

25.4.2 Maple step by step solution

Let's solve

$(y + A\,x^n + a)\,y' + nA\,x^{n-1}y = -k\,x^m - b$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ∘ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - ∘ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

  - ∘ Evaluate derivatives

    $A\,x^{n-1}n = \frac{A\,x^n n}{x}$

- Simplify

  $A\,x^{n-1}n = A\,x^{n-1}n$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y)\,, N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (nA\,x^{n-1}y + k\,x^m + b)\,dx + f_1(y)$

- Evaluate integral

  $F(x,y) = bx + \frac{k\,x^{m+1}}{m+1} + Ay\,x^n + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $y + A\,x^n + a = A\,x^n + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = a + y$

- Solve for $f_1(y)$

  $f_1(y) = ay + \frac{1}{2}y^2$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = bx + \frac{k\,x^{m+1}}{m+1} + Ay\,x^n + ay + \frac{y^2}{2}$

- Substitute $F(x,y)$ into the solution of the ODE

  $bx + \frac{k\,x^{m+1}}{m+1} + Ay\,x^n + ay + \frac{y^2}{2} = c_1$

- Solve for y

  $\left\{y = -\frac{Am\,x^n + A\,x^n + am - \sqrt{A^2(x^n)^2m^2 + 2A^2(x^n)^2m + 2A\,x^n a\,m^2 + A^2(x^n)^2 + 4A\,x^n am + m^2a^2 - 2b\,m^2x + 2A\,x^n a - 2x^{m+1}km +}}{m+1}\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 149

```
dsolve((y(x)+A*x^n+a)*diff(y(x),x)+n*A*x^(n-1)*y(x)+k*x^m+b=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{(-2x^{1+m}k+(1+m)(x^{2n}A^2+2A\,x^n a+a^2-2bx-2c_1))(1+m)}+A(-m-1)\,x^n-am-a}{1+m}$$

$$y(x) = \frac{-\sqrt{(-2x^{1+m}k+(1+m)(x^{2n}A^2+2A\,x^n a+a^2-2bx-2c_1))(1+m)}+A(-m-1)\,x^n-am-a}{1+m}$$

✓ Solution by Mathematica

Time used: 21.171 (sec). Leaf size: 118

```
DSolve[(y[x]+A*x^n+a)*y'[x]+n*A*x^(n-1)*y[x]+k*x^m+b==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\sqrt{\frac{1}{x}}\sqrt{x\left((a+Ax^n)^2-\frac{2x(bm+b+kx^m)}{m+1}+c_1\right)}-a-Ax^n$$

$$y(x) \to \sqrt{\frac{1}{x}}\sqrt{x\left((a+Ax^n)^2-\frac{2x(bm+b+kx^m)}{m+1}+c_1\right)}-a-Ax^n$$

25.5 problem 5

Internal problem ID [10822]
Internal file name [`OUTPUT/9803_Sunday_June_19_2022_09_25_21_PM_56552507/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.4-2. Equations of the form $(g_1(x) + g_0(x))y' = f_2(x)y^2 + f_1(x)y + f_0(x)$
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{\left(y + a\,x^{1+n} + b\,x^{n}\right) y' - \left(x^{n} n a + c\,x^{n-1}\right) y = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(x^n*y(x)*a*n^2+x^(n-1)*y(x)*c*n-x^n*a*n*
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(x^(n+1)*x^n*a^2*n+2*x^(n+1)*x^(n-1)
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x), y(x)`      *** Sublevel 2 ***
```

✗ Solution by Maple

```
dsolve((y(x)+a*x^(n+1)+b*x^n)*diff(y(x),x)=(a*n*x^n+c*x^(n-1))*y(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(y[x]+a*x^(n+1)+b*x^n)*y'[x]==(a*n*x^n+c*x^(n-1))*y[x],y[x],x,IncludeSingularSolution
```

Not solved

25.6 problem 6

Internal problem ID [10823]
Internal file name [`OUTPUT/9804_Sunday_June_19_2022_09_25_42_PM_60888452/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.4-2. Equations of the form $(g_1(x) + g_0(x))y' = f_2(x)y^2 + f_1(x)y + f_0(x)$
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{xyy' - y^2a - yb = x^n c + s}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(x^n*y(x)*c*n-x^n*y(x)*c-y(x)*s+2*a*x)/(x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(-x*n+y(x))/x, y(x)`      *** Sublevel 2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)*(a*x^2-s)/(x*(a*x^2+b*x+s)), y(x)`
      Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(x*y(x)*diff(y(x),x)=a*y(x)^2+b*y(x)+c*x^n+s,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y[x]*y'[x]==a*y[x]^2+b*y[x]+c*x^n+s,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

25.7 problem 7

Internal problem ID [10824]
Internal file name [`OUTPUT/9805_Sunday_June_19_2022_09_25_56_PM_53983030/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 1, section 1.3. Abel Equations of the Second Kind. subsection 1.3.4-2. Equations of the form $(g_1(x) + g_0(x))y' = f_2(x)y^2 + f_1(x)y + f_0(x)$
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{xyy' + y^2 n - a(2n+1)\,xy - yb = -a^2 n\,x^2 - abx + c}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 224

```
dsolve(x*y(x)*diff(y(x),x)=-n*y(x)^2+a*(2*n+1)*x*y(x)+b*y(x)-a^2*n*x^2-a*b*x+c,y(x), singsol
```

$$\frac{\left(\frac{-ny(x)^2+(2axn+b)y(x)-a^2n\,x^2-abx+c}{(ax-y(x))^2}\right)^{-\frac{1}{2n}}\left(\frac{1}{ax-y(x)}\right)^{\frac{1}{n}}y(x)\,\mathrm{e}^{\frac{b\,\mathrm{arctanh}\left(\frac{-abx+by(x)+2c}{\sqrt{b^2+4cn}\,(-ax+y(x))}\right)}{\sqrt{b^2+4cn}\,n}}-\left(\left(\left(\int^{\frac{1}{ax-y(x)}}\left(\_a^2c-\_a\right.\right.\right.\right.}{x\left(ax-y\left(x\right)\right)}$$

$= 0$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y[x]*y'[x]==-n*y[x]^2+a*(2*n+1)*x*y[x]+b*y[x]-a^2*n*x^2-a*b*x+c,y[x],x,IncludeSingu
```

Not solved

26 Chapter 2, Second-Order Differential Equations. section 2.1.2 Equations Containing Power Functions. page 213

26.1 problem 1

Internal problem ID [10825]
Internal file name [`OUTPUT/9806_Sunday_June_19_2022_09_25_58_PM_6187875/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2 Equations Containing Power Functions. page 213
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + ay = 0}$$

26.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = a$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + a\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + a = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = a$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(a)} \\ &= \pm\sqrt{-a}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-a} \\ \lambda_2 &= -\sqrt{-a}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-a} \\ \lambda_2 &= -\sqrt{-a}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(\sqrt{-a})x} + c_2 e^{(-\sqrt{-a})x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\sqrt{-a}\,x} + c_2 \mathrm{e}^{-\sqrt{-a}\,x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\sqrt{-a}\,x} + c_2 \mathrm{e}^{-\sqrt{-a}\,x} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\sqrt{-a}\,x} + c_2 \mathrm{e}^{-\sqrt{-a}\,x}$$

Verified OK.

26.1.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + ay'y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + ay'y)\,dx = 0$$
$$\frac{y'^2}{2} + \frac{y^2 a}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-y^2 a + 2c_1} \tag{1}$$
$$y' = -\sqrt{-y^2 a + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-a\,y^2 + 2c_1}} dy = \int dx$$
$$\frac{\arctan\left(\frac{\sqrt{a}\,y}{\sqrt{-y^2 a + 2c_1}}\right)}{\sqrt{a}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-a\,y^2 + 2c_1}} dy = \int dx$$
$$-\frac{\arctan\left(\frac{\sqrt{a}\,y}{\sqrt{-y^2 a + 2c_1}}\right)}{\sqrt{a}} = x + c_3$$

Summary

The solution(s) found are the following

$$\frac{\arctan\left(\frac{\sqrt{a}\, y}{\sqrt{-y^{2} a+2 c_{1}}}\right)}{\sqrt{a}} = x + c_2 \tag{1}$$

$$-\frac{\arctan\left(\frac{\sqrt{a}\, y}{\sqrt{-y^{2} a+2 c_{1}}}\right)}{\sqrt{a}} = x + c_3 \tag{2}$$

Verification of solutions

$$\frac{\arctan\left(\frac{\sqrt{a}\, y}{\sqrt{-y^{2} a+2 c_{1}}}\right)}{\sqrt{a}} = x + c_2$$

Verified OK.

$$-\frac{\arctan\left(\frac{\sqrt{a}\, y}{\sqrt{-y^{2} a+2 c_{1}}}\right)}{\sqrt{a}} = x + c_3$$

Verified OK.

26.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + ay = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= a \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -a$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = (-a)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 17: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -a$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{-a}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{\sqrt{-a}\, x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\sqrt{-a}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{\sqrt{-a}\, x} \int \frac{1}{\mathrm{e}^{2\sqrt{-a}\, x}}\, dx \\ &= \mathrm{e}^{\sqrt{-a}\, x} \left(-\frac{\mathrm{e}^{-2\sqrt{-a}\, x}}{2\sqrt{-a}}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\sqrt{-a}\,x}\right) + c_2\left(\mathrm{e}^{\sqrt{-a}\,x}\left(-\frac{\mathrm{e}^{-2\sqrt{-a}\,x}}{2\sqrt{-a}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\sqrt{-a}\,x} - \frac{c_2 \mathrm{e}^{-\sqrt{-a}\,x}}{2\sqrt{-a}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{\sqrt{-a}\,x} - \frac{c_2 \mathrm{e}^{-\sqrt{-a}\,x}}{2\sqrt{-a}}
$$

Verified OK.

26.1.4 Maple step by step solution

Let's solve

$y'' + ay = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + a = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4a})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{-a}, -\sqrt{-a}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{\sqrt{-a}\,x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\sqrt{-a}\,x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\sqrt{-a}\, x} + c_2 e^{-\sqrt{-a}\, x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+a*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin\left(\sqrt{a}\, x\right) + c_2 \cos\left(\sqrt{a}\, x\right)$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 28

```
DSolve[y''[x]+a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 \cos\left(\sqrt{a}x\right) + c_2 \sin\left(\sqrt{a}x\right)$$

26.2 problem 2

Internal problem ID [10826]
Internal file name [`OUTPUT/9807_Sunday_June_19_2022_09_25_59_PM_19397890/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2 Equations Containing Power Functions. page 213
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - (ax + b)\, y = 0}$$

26.2.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + \left(-a\, x^3 - b\, x^2\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y' x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = -1$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = -c_1\sqrt{x}\ \text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\ \text{BesselY}\left(1, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = -c_1\sqrt{x}\ \text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\ \text{BesselY}\left(1, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = -c_1\sqrt{x}\ \text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\ \text{BesselY}\left(1, 2\sqrt{x}\right)$$

Verified OK.

26.2.2 Maple step by step solution

Let's solve

$y'' + (-ax - b)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$-a_0 b + 2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k b - a_{k-1} a\right) x^k\right) = 0$$

- Each term must be 0

$$-a_0 b + 2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - a_{k-1} a - a_k b = 0$$

- Shift index using $k->k+1$

$$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_k a - a_{k+1} b = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k a + a_{k+1} b}{k^2 + 5k + 6}, -a_0 b + 2a_2 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)-(a*x+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{AiryAi}\left(\frac{ax+b}{(-a)^{\frac{2}{3}}}\right) + c_2 \operatorname{AiryBi}\left(\frac{ax+b}{(-a)^{\frac{2}{3}}}\right)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 36

```
DSolve[y''[x]-(a*x+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{AiryAi}\left(\frac{b+ax}{a^{2/3}}\right) + c_2 \operatorname{AiryBi}\left(\frac{b+ax}{a^{2/3}}\right)$$

26.3 problem 3

Internal problem ID [10827]
Internal file name [`OUTPUT/9808_Sunday_June_19_2022_09_25_59_PM_52272737/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2 Equations Containing Power Functions. page 213
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - \left(a^2x^2 + a\right) y = 0}$$

26.3.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(-a^2x^4 - a\,x^2\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1\,\text{BesselJ}\,(n, \beta\,x^\gamma) + c_2\,\text{BesselY}\,(n, \beta\,x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = -1$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = -c_1\sqrt{x}\ \text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\ \text{BesselY}\left(1, 2\sqrt{x}\right)$$

Summary
The solution(s) found are the following

$$y = -c_1\sqrt{x}\ \text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\ \text{BesselY}\left(1, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = -c_1\sqrt{x}\ \text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\ \text{BesselY}\left(1, 2\sqrt{x}\right)$$

Verified OK.

26.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \left(-a^2x^2 - a\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -a^2x^2 - a \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a(a\,x^2 + 1)}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = a(a\,x^2 + 1)$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = \left(a(a\,x^2 + 1)\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 20: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx ax + \frac{1}{2x} - \frac{1}{8a\,x^3} + \frac{1}{16a^2x^5} - \frac{5}{128a^3x^7} + \frac{7}{256a^4x^9} - \frac{21}{1024a^5x^{11}} + \frac{33}{2048a^6x^{13}} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = a
$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= ax
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = a^2x^2
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a(a\,x^2+1)}{1} \\ &= Q + \frac{R}{1} \\ &= (a^2x^2 + a) + (0) \\ &= a^2x^2 + a \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is a. Now b can be found.

$$\begin{aligned} b &= (a) - (0) \\ &= a \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= ax \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{a}{a} - 1\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{a}{a} - 1\right) &= -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = a\left(a\,x^2+1\right)$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	ax	0	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + (ax) \\ &= ax \\ &= ax \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2(ax)(0) + \left((a) + (ax)^2 - \left(a\left(a\,x^2 + 1\right)\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int ax dx} \\ &= \mathrm{e}^{\frac{a\,x^2}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{\frac{a x^2}{2}} \end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{a x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= e^{\frac{a x^2}{2}} \int \frac{1}{e^{a x^2}} \, dx \\ &= e^{\frac{a x^2}{2}} \left(\frac{\sqrt{\pi} \, \operatorname{erf}\left(\sqrt{a}\, x\right)}{2\sqrt{a}} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{\frac{a x^2}{2}} \right) + c_2 \left(e^{\frac{a x^2}{2}} \left(\frac{\sqrt{\pi} \, \operatorname{erf}\left(\sqrt{a}\, x\right)}{2\sqrt{a}} \right) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{a x^2}{2}} + \frac{c_2 e^{\frac{a x^2}{2}} \sqrt{\pi} \, \operatorname{erf}\left(\sqrt{a}\, x\right)}{2\sqrt{a}} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\frac{a x^2}{2}} + \frac{c_2 e^{\frac{a x^2}{2}} \sqrt{\pi} \, \operatorname{erf}\left(\sqrt{a}\, x\right)}{2\sqrt{a}}$$

Verified OK.

26.3.3 Maple step by step solution

Let's solve

$y'' + (-a^2x^2 - a)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k + 2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$-a_0 a + 2a_2 + (6a_3 - a_1 a)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a a_k - a_{k-2}a^2\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - a_0 a = 0, 6a_3 - a_1 a = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = \frac{a_0 a}{2}, a_3 = \frac{a_1 a}{6}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_{k-2}a^2 - a a_k = 0$

- Shift index using $k- >k + 2$

$$\left((k+2)^2+3k+8\right) a_{k+4} - a_k a^2 - a a_{k+2} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{a(a a_k + a_{k+2})}{k^2+7k+12}, a_2 = \frac{a_0 a}{2}, a_3 = \frac{a_1 a}{6}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-(a^2*x^2+a)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{\frac{a x^2}{2}} \left(c_1 + \operatorname{erf}\left(\sqrt{a}\, x\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 43

```
DSolve[y''[x]-(a^2*x^2+a)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \text{ParabolicCylinderD}\left(-1, \sqrt{2}\sqrt{a}x\right) + c_2 \text{ParabolicCylinderD}\left(0, i\sqrt{2}\sqrt{a}x\right)$$

26.4 problem 4

Internal problem ID [10828]
Internal file name [`OUTPUT/9809_Sunday_June_19_2022_09_26_00_PM_50643366/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2 Equations Containing Power Functions. page 213
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - \left(a\,x^2 + b\right) y = 0}$$

26.4.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + \left(-a\,x^4 - b\,x^2\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y' x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = -1$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = -c_1\sqrt{x}\,\mathrm{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\,\mathrm{BesselY}\left(1, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = -c_1\sqrt{x}\,\mathrm{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\,\mathrm{BesselY}\left(1, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = -c_1\sqrt{x}\,\mathrm{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\,\mathrm{BesselY}\left(1, 2\sqrt{x}\right)$$

Verified OK.

26.4.2 Maple step by step solution

Let's solve

$y'' + (-a\,x^2 - b)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - ○ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$-a_0 b + 2a_2 + (6a_3 - a_1 b)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - a_k b - a_{k-2} a)\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 - a_0 b = 0, 6a_3 - a_1 b = 0]$$

- Solve for the dependent coefficient(s)

  $$\left\{a_2 = \frac{a_0 b}{2}, a_3 = \frac{a_1 b}{6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - a_{k-2} a - a_k b = 0$$

- Shift index using $k- >k+2$

  $$\left((k+2)^2 + 3k + 8\right) a_{k+4} - a_k a - a_{k+2} b = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{a_k a + a_{k+2} b}{k^2 + 7k + 12}, a_2 = \frac{a_0 b}{2}, a_3 = \frac{a_1 b}{6}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$2)-(a*x^2+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \operatorname{WhittakerM}\left(-\frac{b}{4\sqrt{a}}, \frac{1}{4}, \sqrt{a}\, x^2\right) + c_2 \operatorname{WhittakerW}\left(-\frac{b}{4\sqrt{a}}, \frac{1}{4}, \sqrt{a}\, x^2\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 68

```
DSolve[y''[x]-(a*x^2+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{ParabolicCylinderD}\left(-\frac{b}{2\sqrt{a}} - \frac{1}{2}, \sqrt{2}\sqrt[4]{a}x\right) + c_2 \operatorname{ParabolicCylinderD}\left(\frac{1}{2}\left(\frac{b}{\sqrt{a}} - 1\right), i\sqrt{2}\sqrt[4]{a}x\right)$$

26.5 problem 5

Internal problem ID [10829]
Internal file name [`OUTPUT/9810_Sunday_June_19_2022_09_26_01_PM_87384003/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2 Equations Containing Power Functions. page 213
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + a^3 x(-ax+2)\, y = 0}$$

26.5.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + \left(-a^4 x^4 + 2a^3 x^3\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y' x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1-2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= \frac{1}{2}\\ \beta &= 2\\ n &= -1\\ \gamma &= \frac{1}{2}\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = -c_1\sqrt{x}\,\mathrm{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\,\mathrm{BesselY}\left(1, 2\sqrt{x}\right)$$

Summary
The solution(s) found are the following

$$y = -c_1\sqrt{x}\,\mathrm{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\,\mathrm{BesselY}\left(1, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = -c_1\sqrt{x}\,\mathrm{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\,\mathrm{BesselY}\left(1, 2\sqrt{x}\right)$$

Verified OK.

26.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \left(-a^4x^2 + 2a^3x\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1\\ B &= 0\\ C &= -a^4x^2 + 2a^3x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^3x(ax-2)}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = a^3x(ax-2)$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = \left(a^3x(ax-2)\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 23: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a^2 x - a - \frac{1}{2x} - \frac{1}{2a\,x^2} - \frac{5}{8a^2x^3} - \frac{7}{8a^3x^4} - \frac{21}{16a^4x^5} - \frac{33}{16a^5x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a^2$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= a^2 x - a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^4x^2 - 2a^3x + a^2$$

This shows that the coefficient of 1 in the above is a^2. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^3x(ax-2)}{1} \\ &= Q + \frac{R}{1} \\ &= \left(a^4x^2 - 2a^3x\right) + (0) \\ &= a^4x^2 - 2a^3x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - \left(a^2\right) \\ &= -a^2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= a^2x - a \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-a^2}{a^2} - 1\right) &= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-a^2}{a^2} - 1\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = a^3x(ax-2)$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$a^2x - a$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(a^2 x - a\right) \\ &= -a^2 x + a \\ &= -a^2 x + a \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-a^2 x + a\right)(0) + \left(\left(-a^2\right) + \left(-a^2 x + a\right)^2 - \left(a^3 x (a x - 2)\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int (-a^2 x + a) dx} \\ &= \mathrm{e}^{-\frac{a x (a x - 2)}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-\frac{ax(ax-2)}{2}}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{ax(ax-2)}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \mathrm{e}^{-\frac{ax(ax-2)}{2}} \int \frac{1}{\mathrm{e}^{-ax(ax-2)}}\, dx \\
&= \mathrm{e}^{-\frac{ax(ax-2)}{2}} \left(-\frac{i\sqrt{\pi}\, \mathrm{e}^{-1} \operatorname{erf}(iax - i)}{2a}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{ax(ax-2)}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{ax(ax-2)}{2}} \left(-\frac{i\sqrt{\pi}\, \mathrm{e}^{-1} \operatorname{erf}(iax - i)}{2a}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{ax(ax-2)}{2}} - \frac{ic_2 \mathrm{e}^{-1-\frac{1}{2}a^2x^2+ax} \sqrt{\pi}\, \operatorname{erf}(iax - i)}{2a} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{ax(ax-2)}{2}} - \frac{ic_2 \mathrm{e}^{-1-\frac{1}{2}a^2x^2+ax} \sqrt{\pi}\, \operatorname{erf}(iax - i)}{2a}
$$

Verified OK.

26.5.3 Maple step by step solution

Let's solve

$y'' + (-a^4x^2 + 2a^3x)\, y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 1..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- > k + 2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + (2a_0a^3 + 6a_3)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + 2a^3a_{k-1} - a_{k-2}a^4)\, x^k\right) = 0$

- The coefficients of each power of x must be 0

$[2a_2 = 0, 2a_0a^3 + 6a_3 = 0]$

- Solve for the dependent coefficient(s)

$\left\{a_2 = 0, a_3 = -\frac{a_0a^3}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

$(k^2 + 3k + 2)\, a_{k+2} - a_{k-2}a^4 + 2a^3a_{k-1} = 0$

- Shift index using $k- > k + 2$

$$\left((k+2)^2+3k+8\right)a_{k+4}-a_k a^4+2a^3 a_{k+1}=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=\frac{a^3(a a_k-2a_{k+1})}{k^2+7k+12}, a_2=0, a_3=-\frac{a_0 a^3}{3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)+a^3*x*(2-a*x)*y(x)=0,y(x), singsol=all)
```

$$y(x)=\mathrm{e}^{-\frac{ax(ax-2)}{2}}(c_1+\mathrm{erf}(iax-i)\,c_2)$$

✓ Solution by Mathematica
Time used: 0.28 (sec). Leaf size: 50

```
DSolve[y''[x]+a^3*x*(2-a*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{e^{-\frac{1}{2}ax(ax-2)-1}\left(2eac_1-\sqrt{\pi}c_2\mathrm{erfi}(1-ax)\right)}{2a}$$

26.6 problem 6

Internal problem ID [10830]
Internal file name [`OUTPUT/9811_Sunday_June_19_2022_09_26_02_PM_61895003/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2 Equations Containing Power Functions. page 213
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - \left(a\,x^2 + bcx\right) y = 0}$$

26.6.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(-a\,x^4 - bc\,x^3\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}\left(n, \beta\,x^\gamma\right) + c_2 \operatorname{BesselY}\left(n, \beta\,x^\gamma\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = -1$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = -c_1\sqrt{x}\,\mathrm{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\,\mathrm{BesselY}\left(1, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = -c_1\sqrt{x}\,\mathrm{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\,\mathrm{BesselY}\left(1, 2\sqrt{x}\right) \quad (1)$$

Verification of solutions

$$y = -c_1\sqrt{x}\,\mathrm{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\,\mathrm{BesselY}\left(1, 2\sqrt{x}\right)$$

Verified OK.

26.6.2 Maple step by step solution

Let's solve

$y'' + (-a\,x^2 - bcx)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 1..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- > k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + (6a_3 - a_0 bc)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - a_{k-1}bc - a_{k-2}a)\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 - a_0 bc = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = 0, a_3 = \frac{a_0 bc}{6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - a_{k-1}bc - a_{k-2}a = 0$$

- Shift index using $k- > k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - a_{k+1}bc - a_k a = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{a_{k+1}bc + a_k a}{k^2 + 7k + 12}, a_2 = 0, a_3 = \frac{a_0 bc}{6}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: indirect Equivalence to 0F1 under \`\`^ @ Moebius\`\` is resolve
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 142

```
dsolve(diff(y(x),x$2)-(a*x^2+b*x*c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{x(ax+bc)}{2\sqrt{a}}} \left(2xac_2 \operatorname{hypergeom}\left(\left[-\frac{b^2c^2-12a^{\frac{3}{2}}}{16a^{\frac{3}{2}}}\right], \left[\frac{3}{2}\right], \frac{(2ax+bc)^2}{4a^{\frac{3}{2}}}\right) + cbc_2 \operatorname{hypergeom}\left(\left[-\frac{b^2c^2-12a^{\frac{3}{2}}}{16a^{\frac{3}{2}}}\right], \left[\frac{3}{2}\right], \frac{(2ax+bc)^2}{4a^{\frac{3}{2}}}\right) + \operatorname{hypergeom}\left(\left[-\frac{b^2c^2-4a^{\frac{3}{2}}}{16a^{\frac{3}{2}}}\right], \left[\frac{1}{2}\right], \frac{(2ax+bc)^2}{4a^{\frac{3}{2}}}\right) c_1\right)$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 92

```
DSolve[y''[x]-(a*x^2+b*x*c)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \operatorname{ParabolicCylinderD}\left(-\frac{b^2c^2}{8a^{3/2}} - \frac{1}{2}, \frac{i(bc+2ax)}{\sqrt{2}a^{3/4}}\right) + c_1 \operatorname{ParabolicCylinderD}\left(\frac{1}{8}\left(\frac{b^2c^2}{a^{3/2}} - 4\right), \frac{bc+2ax}{\sqrt{2}a^{3/4}}\right)$$

26.7 problem 7

Internal problem ID [10831]
Internal file name [OUTPUT/9812_Sunday_June_19_2022_09_26_03_PM_88912945/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2 Equations Containing Power Functions. page 213
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' - a\,x^n y = 0}$$

26.7.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - a\,x^2 x^n y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y' x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = \frac{2\sqrt{-a}}{2+n}$$
$$n = -\frac{1}{2+n}$$
$$\gamma = 1 + \frac{n}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(-\frac{1}{2+n}, \frac{2\sqrt{-a}\,x^{1+\frac{n}{2}}}{2+n}\right) + c_2\sqrt{x}\,\text{BesselY}\left(-\frac{1}{2+n}, \frac{2\sqrt{-a}\,x^{1+\frac{n}{2}}}{2+n}\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(-\frac{1}{2+n}, \frac{2\sqrt{-a}\,x^{1+\frac{n}{2}}}{2+n}\right) + c_2\sqrt{x}\,\text{BesselY}\left(-\frac{1}{2+n}, \frac{2\sqrt{-a}\,x^{1+\frac{n}{2}}}{2+n}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(-\frac{1}{2+n}, \frac{2\sqrt{-a}\,x^{1+\frac{n}{2}}}{2+n}\right) + c_2\sqrt{x}\,\text{BesselY}\left(-\frac{1}{2+n}, \frac{2\sqrt{-a}\,x^{1+\frac{n}{2}}}{2+n}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.281 (sec). Leaf size: 63

```
dsolve(diff(y(x),x$2)-a*x^n*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x}\left(\mathrm{BesselY}\left(\frac{1}{n+2}, \frac{2\sqrt{-a}\,x^{\frac{n}{2}+1}}{n+2}\right)c_2 + \mathrm{BesselJ}\left(\frac{1}{n+2}, \frac{2\sqrt{-a}\,x^{\frac{n}{2}+1}}{n+2}\right)c_1\right)$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 119

```
DSolve[y''[x]-a*x^n*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (n+2)^{-\frac{1}{n+2}}\sqrt{x}a^{\frac{1}{2n+4}}\left(c_1\,\mathrm{Gamma}\left(\frac{n+1}{n+2}\right)\mathrm{BesselI}\left(-\frac{1}{n+2}, \frac{2\sqrt{a}x^{\frac{n}{2}+1}}{n+2}\right) + c_2(-1)^{\frac{1}{n+2}}\,\mathrm{Gamma}\left(1+\frac{1}{n+2}\right)\mathrm{BesselI}\left(\frac{1}{n+2}, \frac{2\sqrt{a}x^{\frac{n}{2}+1}}{n+2}\right)\right)$$

26.8 problem 8

Internal problem ID [10832]
Internal file name [`OUTPUT/9813_Sunday_June_19_2022_09_26_04_PM_77415254/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2 Equations Containing Power Functions. page 213
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - a\left(a\,x^{2n} + n\,x^{n-1}\right) y = 0}$$

26.8.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + \left(-x^2 x^{2n} a^2 - x^n a n x\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y' x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = -1$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = -c_1 \sqrt{x}\, \text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2 \sqrt{x}\, \text{BesselY}\left(1, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = -c_1 \sqrt{x}\, \text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2 \sqrt{x}\, \text{BesselY}\left(1, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = -c_1 \sqrt{x}\, \text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2 \sqrt{x}\, \text{BesselY}\left(1, 2\sqrt{x}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- Equivalence, under non-integer power transformations successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 136

```
dsolve(diff(y(x),x$2)-a*(a*x^(2*n)+n*x^(n-1))*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^{-\frac{3n}{2}-1}(n+2)^2 \operatorname{WhittakerM}\left(\frac{n+2}{2n+2}, \frac{2n+3}{2n+2}, \frac{2a\,x^{n+1}}{n+1}\right)}{2} + \left(\left(\frac{n}{2}+1\right) x^{-\frac{3n}{2}-1} + a\,x^{-\frac{n}{2}}\right)(n+1)\,c_2 \operatorname{WhittakerM}\left(-\frac{n}{2n+2}, \frac{2n+3}{2n+2}, \frac{2a\,x^{n+1}}{n+1}\right) + c_1 e^{\frac{a\,x^{n+1}}{n+1}}$$

✓ Solution by Mathematica

Time used: 0.596 (sec). Leaf size: 81

```
DSolve[y''[x]-a*(a*x^(2*n)+n*x^(n-1))*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{ax^{n+1}}{n+1}}\left(c_2 - \frac{c_1 2^{-\frac{1}{n+1}} x \left(\frac{ax^{n+1}}{n+1}\right)^{-\frac{1}{n+1}} \Gamma\left(\frac{1}{n+1}, \frac{2ax^{n+1}}{n+1}\right)}{n+1}\right)$$

26.9 problem 9

Internal problem ID [10833]
Internal file name [`OUTPUT/9814_Sunday_June_19_2022_09_26_05_PM_30632592/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2 Equations Containing Power Functions. page 213
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - a\,x^{n-2}(a\,x^n + n + 1)\,y = 0}$$

26.9.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(-x^{2n}a^2 - x^n na - a\,x^n\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1\,\mathrm{BesselJ}\left(n, \beta\,x^{\gamma}\right) + c_2\,\mathrm{BesselY}\left(n, \beta\,x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = -1$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = -c_1 \sqrt{x}\, \mathrm{BesselJ}\left(1, 2\sqrt{x}\right) - c_2 \sqrt{x}\, \mathrm{BesselY}\left(1, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = -c_1 \sqrt{x}\, \mathrm{BesselJ}\left(1, 2\sqrt{x}\right) - c_2 \sqrt{x}\, \mathrm{BesselY}\left(1, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = -c_1 \sqrt{x}\, \mathrm{BesselJ}\left(1, 2\sqrt{x}\right) - c_2 \sqrt{x}\, \mathrm{BesselY}\left(1, 2\sqrt{x}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- Equivalence, under non-integer power transformations successful`
```

✓ Solution by Maple

Time used: 0.187 (sec). Leaf size: 113

```
dsolve(diff(y(x),x$2)-a*x^(n-2)*(a*x^n+n+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^{-\frac{3n}{2}+\frac{1}{2}}(n-1)^2 \operatorname{WhittakerM}\left(\frac{n-1}{2n}, \frac{2n-1}{2n}, \frac{2a\,x^n}{n}\right)}{2} + \left(\frac{(n-1)\,x^{-\frac{3n}{2}+\frac{1}{2}}}{2} + x^{-\frac{n}{2}+\frac{1}{2}} a\right) n c_2 \operatorname{WhittakerM}\left(-\frac{n+1}{2n}, \frac{2n-1}{2n}, \frac{2a\,x^n}{n}\right) + c_1 x\,\mathrm{e}^{\frac{a\,x^n}{n}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]-a*x^(n-2)*(a*x^n+n+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

26.10 problem 10

Internal problem ID [10834]
Internal file name [`OUTPUT/9815_Sunday_June_19_2022_09_26_07_PM_32066164/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2 Equations Containing Power Functions. page 213
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \left(a\,x^{2n} + b\,x^{n-1}\right) y = 0}$$

26.10.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + \left(x^2 a\,x^{2n} + x^n b x\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y' x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = -1$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = -c_1\sqrt{x}\,\text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\,\text{BesselY}\left(1, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = -c_1\sqrt{x}\,\text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\,\text{BesselY}\left(1, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = -c_1\sqrt{x}\,\text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\,\text{BesselY}\left(1, 2\sqrt{x}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 89

```
dsolve(diff(y(x),x$2)+(a*x^(2*n)+b*x^(n-1))*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^{-\frac{n}{2}}\left(c_1 \operatorname{WhittakerM}\left(-\frac{ib}{\sqrt{a}\,(2n+2)}, \frac{1}{2n+2}, \frac{2i\sqrt{a}\,x\,x^n}{n+1}\right) + c_2 \operatorname{WhittakerW}\left(-\frac{ib}{\sqrt{a}\,(2n+2)}, \frac{1}{2n+2}, \frac{2i\sqrt{a}\,x\,x^n}{n+1}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.399 (sec). Leaf size: 225

```
DSolve[y''[x]+(a*x^(2*n)+b*x^(n-1))*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2^{\frac{n}{2n+2}} x^{-n/2} \left(x^{n+1}\right)^{\frac{n}{2n+2}} e^{-\frac{\sqrt{a}x^{n+1}}{\sqrt{-(n+1)^2}}} \left(c_1 \operatorname{HypergeometricU}\left(-\frac{(n+1)\left(nb+b+\sqrt{a}n\sqrt{-(n+1)^2}\right)}{2\sqrt{a}\left(-(n+1)^2\right)^{3/2}}, \frac{n}{n+1}\right.\right.$$

27 Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$

27.1 problem 11

Internal problem ID [10835]
Internal file name [`OUTPUT/9816_Sunday_June_19_2022_09_26_08_PM_26525654/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + ay' + yb = 0}$$

27.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = a, C = b$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + a\lambda\, \mathrm{e}^{\lambda x} + b\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$a\lambda + \lambda^2 + b = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = a, C = b$ into the above gives

$$\lambda_{1,2} = \frac{-a}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{a^2 - (4)(1)(b)}$$
$$= -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4b}}{2}$$

Hence

$$\lambda_1 = -\frac{a}{2} + \frac{\sqrt{a^2 - 4b}}{2}$$
$$\lambda_2 = -\frac{a}{2} - \frac{\sqrt{a^2 - 4b}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{a}{2} + \frac{\sqrt{a^2 - 4b}}{2}$$
$$\lambda_2 = -\frac{a}{2} - \frac{\sqrt{a^2 - 4b}}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(-\frac{a}{2} + \frac{\sqrt{a^2-4b}}{2}\right)x} + c_2 e^{\left(-\frac{a}{2} - \frac{\sqrt{a^2-4b}}{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\left(-\frac{a}{2} + \frac{\sqrt{a^2-4b}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{a}{2} - \frac{\sqrt{a^2-4b}}{2}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\left(-\frac{a}{2} + \frac{\sqrt{a^2-4b}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{a}{2} - \frac{\sqrt{a^2-4b}}{2}\right)x} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\left(-\frac{a}{2} + \frac{\sqrt{a^2-4b}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{a}{2} - \frac{\sqrt{a^2-4b}}{2}\right)x}$$

Verified OK.

27.1.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' + ay' + yb &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= a \tag{3} \\
C &= b
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{a^2 - 4b}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= a^2 - 4b \\
t &= 4
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{a^2}{4} - b\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 26: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{a^2}{4} - b$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = e^{\frac{x\sqrt{a^2-4b}}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

$$
\begin{aligned}
&= z_1 e^{-\int \frac{1}{2}\frac{a}{1}\,dx} \\
&= z_1 e^{-\frac{ax}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{ax}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{\left(-a+\sqrt{a^2-4b}\right)x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{a}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-ax}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-x\sqrt{a^2-4b}}}{\sqrt{a^2-4b}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{\left(-a+\sqrt{a^2-4b}\right)x}{2}}\right) + c_2 \left(\mathrm{e}^{\frac{\left(-a+\sqrt{a^2-4b}\right)x}{2}} \left(-\frac{\mathrm{e}^{-x\sqrt{a^2-4b}}}{\sqrt{a^2-4b}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\frac{\left(-a+\sqrt{a^2-4b}\right)x}{2}} - \frac{c_2 \mathrm{e}^{-\frac{\left(a+\sqrt{a^2-4b}\right)x}{2}}}{\sqrt{a^2-4b}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{\frac{\left(-a+\sqrt{a^2-4b}\right)x}{2}} - \frac{c_2 \mathrm{e}^{-\frac{\left(a+\sqrt{a^2-4b}\right)x}{2}}}{\sqrt{a^2-4b}}
$$

Verified OK.

27.1.3 Maple step by step solution

Let's solve

$y'' + ay' + yb = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $ar + r^2 + b = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-a)\pm\left(\sqrt{a^2-4b}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{a}{2} - \frac{\sqrt{a^2-4b}}{2}, -\frac{a}{2} + \frac{\sqrt{a^2-4b}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{\left(-\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\left(-\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{\left(-\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}\right)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$2)+a*diff(y(x),x)+b*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{\frac{\left(-a+\sqrt{a^2-4b}\right)x}{2}} + c_2 \mathrm{e}^{-\frac{\left(a+\sqrt{a^2-4b}\right)x}{2}}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 47

```
DSolve[y''[x]+a*y'[x]+b*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{1}{2}x\left(\sqrt{a^2-4b}+a\right)}\left(c_2 e^{x\sqrt{a^2-4b}} + c_1\right)$$

27.2 problem 12

Internal problem ID [10836]
Internal file name [OUTPUT/9817_Sunday_June_19_2022_09_26_09_PM_36062604/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$y'' + ay' + (bx + c)\, y = 0$$

27.2.1 Maple step by step solution

Let's solve

$y'' + ay' + (bx + c)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- Shift index using $k->k-m$

  $$x^{m}\cdot y=\sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^{k}$$

- Convert y' to series expansion

  $$y'=\sum_{k=1}^{\infty} a_{k} k\, x^{k-1}$$

- Shift index using $k->k+1$

  $$y'=\sum_{k=0}^{\infty} a_{k+1}(k+1)\, x^{k}$$

- Convert y'' to series expansion

  $$y''=\sum_{k=2}^{\infty} a_{k} k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y''=\sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$a a_{1}+a_{0} c+2 a_{2}+\left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1)+a a_{k+1}(k+1)+a_{k} c+b a_{k-1}\right) x^{k}\right)=0$$

- Each term must be 0

  $$a a_{1}+a_{0} c+2 a_{2}=0$$

- Each term in the series must be 0, giving the recursion relation

  $$\left(k^{2}+3k+2\right) a_{k+2}+a a_{k+1} k+a a_{k+1}+b a_{k-1}+a_{k} c=0$$

- Shift index using $k->k+1$

  $$\left((k+1)^{2}+3k+5\right) a_{k+3}+a a_{k+2}(k+1)+a a_{k+2}+b a_{k}+a_{k+1} c=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y=\sum_{k=0}^{\infty} a_{k} x^{k}, a_{k+3}=-\frac{a k a_{k+2}+2 a a_{k+2}+b a_{k}+a_{k+1} c}{k^{2}+5k+6}, a a_{1}+a_{0} c+2 a_{2}=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
dsolve(diff(y(x),x$2)+a*diff(y(x),x)+(b*x+c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{ax}{2}} \left(\mathrm{AiryAi}\left(\frac{a^2 - 4bx - 4c}{4b^{\frac{2}{3}}} \right) c_1 + \mathrm{AiryBi}\left(\frac{a^2 - 4bx - 4c}{4b^{\frac{2}{3}}} \right) c_2 \right)$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 67

```
DSolve[y''[x]+a*y'[x]+(b*x+c)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{ax}{2}} \left(c_1 \,\mathrm{AiryAi}\left(\frac{a^2 - 4(c+bx)}{4(-b)^{2/3}} \right) + c_2 \,\mathrm{AiryBi}\left(\frac{a^2 - 4(c+bx)}{4(-b)^{2/3}} \right) \right)$$

27.3 problem 13

Internal problem ID [10837]
Internal file name [OUTPUT/9818_Sunday_June_19_2022_09_26_10_PM_23841255/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + ay' - (b\,x^2 + c)\,y = 0}$$

27.3.1 Maple step by step solution

Let's solve

$y'' + ay' + (-b\,x^2 - c)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert y' to series expansion

  $$y' = \sum_{k=1}^{\infty} a_k k \, x^{k-1}$$

- Shift index using $k- >k+1$

  $$y' = \sum_{k=0}^{\infty} a_{k+1}(k+1) \, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$$

  Rewrite ODE with series expansions

  $$a_1 a - a_0 c + 2a_2 + (2aa_2 - a_1 c + 6a_3) x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) + aa_{k+1}(k+1) - a_k c - ba_{k-}\right.$$

- The coefficients of each power of x must be 0

  $$[2a_2 + a_1 a - a_0 c = 0, 2aa_2 - a_1 c + 6a_3 = 0]$$

- Solve for the dependent coefficient(s)

  $$\left\{a_2 = -\frac{a_1 a}{2} + \frac{a_0 c}{2}, a_3 = \frac{1}{6}a_1 a^2 - \frac{1}{6}a_0 ac + \frac{1}{6}a_1 c\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2) a_{k+2} + aa_{k+1}k + aa_{k+1} - ba_{k-2} - a_k c = 0$$

- Shift index using $k- >k+2$

  $$\left((k+2)^2 + 3k + 8\right) a_{k+4} + aa_{k+3}(k+2) + aa_{k+3} - ba_k - a_{k+2}c = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{aka_{k+3} + 3aa_{k+3} - ba_k - a_{k+2}c}{k^2 + 7k + 12}, a_2 = -\frac{a_1 a}{2} + \frac{a_0 c}{2}, a_3 = \frac{1}{6}a_1 a^2 - \frac{1}{6}a_0 ac + \frac{1}{6}a_1 c\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.079 (sec). Leaf size: 74

```
dsolve(diff(y(x),x$2)+a*diff(y(x),x)-(b*x^2+c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{x\left(\sqrt{b}\,x+a\right)}{2}} x\left(\mathrm{KummerM}\left(\frac{a^2+12\sqrt{b}+4c}{16\sqrt{b}}, \frac{3}{2}, \sqrt{b}\,x^2\right) c_1 + \mathrm{KummerU}\left(\frac{a^2+12\sqrt{b}+4c}{16\sqrt{b}}, \frac{3}{2}, \sqrt{b}\,x^2\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.088 (sec). Leaf size: 96

```
DSolve[y''[x]+a*y'[x]-(b*x^2+c)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{1}{2}x\left(a+\sqrt{b}x\right)}\left(c_1 \operatorname{HermiteH}\left(\frac{-a^2-4\left(c+\sqrt{b}\right)}{8\sqrt{b}}, \sqrt[4]{b}x\right) + c_2 \operatorname{Hypergeometric1F1}\left(\frac{a^2+4\left(c+\sqrt{b}\right)}{16\sqrt{b}}, \frac{1}{2}, \sqrt{b}x^2\right)\right)$$

27.4 problem 14

Internal problem ID [10838]
Internal file name [`OUTPUT/9819_Sunday_June_19_2022_09_26_13_PM_40437218/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{y'' + ay' + b\left(-b\,x^2 + ax + 1\right) y = 0}$$

27.4.1 Solving using Kovacic algorithm

Writing the ode as

$$y'' + ay' + \left(-b^2x^2 + abx + b\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= a \\ C &= -b^2x^2 + abx + b \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4b^2x^2 - 4abx + a^2 - 4b}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4b^2x^2 - 4abx + a^2 - 4b \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(b^2x^2 - abx + \frac{1}{4}a^2 - b\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 30: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx bx - \frac{a}{2} - \frac{1}{2x} - \frac{a}{4b\,x^2} - \frac{a^2}{8b^2x^3} - \frac{a^3}{16b^3x^4} - \frac{1}{8b\,x^3} - \frac{a^4}{32b^4x^5} - \frac{3a}{16b^2x^4} - \frac{a^5}{64b^5x^6} - \frac{3a^2}{16b^3x^5} - \frac{5a^3}{32b^4x^6} - \frac{1}{16b^2x^5} - \cdots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = b$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= -\frac{a}{2} + bx \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}a^2 - abx + b^2x^2$$

This shows that the coefficient of 1 in the above is $\frac{a^2}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4b^2x^2 - 4abx + a^2 - 4b}{4} \\
&= Q + \frac{R}{4} \\
&= \left(b^2x^2 - abx + \frac{1}{4}a^2 - b\right) + (0) \\
&= b^2x^2 - abx + \frac{1}{4}a^2 - b
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{a^2}{4} - b$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{a^2}{4} - b\right) - \left(\frac{a^2}{4}\right) \\
&= -b
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= -\frac{a}{2} + bx \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-b}{b} - 1\right) &= -1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-b}{b} - 1\right) &= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = b^2x^2 - abx + \frac{1}{4}a^2 - b$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-\frac{a}{2}+bx$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(-\frac{a}{2} + bx\right) \\ &= \frac{a}{2} - bx \\ &= \frac{a}{2} - bx \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{a}{2} - bx\right)(0) + \left((-b) + \left(\frac{a}{2} - bx\right)^2 - \left(b^2x^2 - abx + \frac{1}{4}a^2 - b\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{a}{2} - bx\right) dx} \\
&= \mathrm{e}^{\frac{x(-bx+a)}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{a}{1} \, dx} \\
&= z_1 e^{-\frac{ax}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{ax}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{b x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{a}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-ax}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\sqrt{\pi}\, \mathrm{e}^{-\frac{a^2}{4b}} \operatorname{erf}\left(\frac{-2bx+a}{2\sqrt{-b}}\right)}{2\sqrt{-b}} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{b x^2}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{b x^2}{2}} \left(\frac{\sqrt{\pi}\, \mathrm{e}^{-\frac{a^2}{4b}} \operatorname{erf}\left(\frac{-2bx+a}{2\sqrt{-b}}\right)}{2\sqrt{-b}} \right) \right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{b\,x^2}{2}} + \frac{c_2 \sqrt{\pi}\, \mathrm{e}^{-\frac{2b^2x^2+a^2}{4b}} \operatorname{erf}\left(\frac{-2bx+a}{2\sqrt{-b}}\right)}{2\sqrt{-b}} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{b\,x^2}{2}} + \frac{c_2 \sqrt{\pi}\, \mathrm{e}^{-\frac{2b^2x^2+a^2}{4b}} \operatorname{erf}\left(\frac{-2bx+a}{2\sqrt{-b}}\right)}{2\sqrt{-b}}$$

Verified OK.

27.4.2 Maple step by step solution

Let's solve

$y'' + ay' + (-b^2x^2 + abx + b)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert y' to series expansion

    $y' = \sum_{k=1}^{\infty} a_k k\, x^{k-1}$

  - Shift index using $k- >k + 1$

    $y' = \sum_{k=0}^{\infty} a_{k+1}(k+1)\, x^k$

  - Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$a_1 a + a_0 b + 2a_2 + (a_0 ab + 2aa_2 + a_1 b + 6a_3)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + aa_{k+1}(k+1) + a_k b$$

- The coefficients of each power of x must be 0

$$[2a_2 + a_1 a + a_0 b = 0, a_0 ab + 2aa_2 + a_1 b + 6a_3 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{a_1 a}{2} - \frac{a_0 b}{2}, a_3 = \frac{1}{6}a_1 a^2 - \frac{1}{6}a_1 b\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + (ba_{k-1} + a_{k+1}(k+1))\, a - a_{k-2}b^2 + a_k b = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + (ba_{k+1} + a_{k+3}(k+3))\, a - a_k b^2 + a_{k+2} b = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{aba_{k+1} + aka_{k+3} - a_k b^2 + 3aa_{k+3} + a_{k+2} b}{k^2 + 7k + 12}, a_2 = -\frac{a_1 a}{2} - \frac{a_0 b}{2}, a_3 = \frac{1}{6}a_1 a^2 - \frac{1}{6}a_1 b\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+a*diff(y(x),x)+b*(-b*x^2+a*x+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x^2 b}{2}} \left(c_1 \operatorname{erf}\left(\frac{-2bx + a}{2\sqrt{-b}}\right) + c_2\right)$$

✓ Solution by Mathematica

Time used: 0.28 (sec). Leaf size: 67

```
DSolve[y''[x]+a*y'[x]+b*(-b*x^2+a*x+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-\frac{bx^2}{2}} \left(\frac{\sqrt{\pi} c_2 e^{-\frac{a^2}{4b}} \operatorname{erfi}\left(\frac{2bx-a}{2\sqrt{b}}\right)}{\sqrt{b}} + 2c_1\right)$$

27.5 problem 15

Internal problem ID [10839]
Internal file name [OUTPUT/9820_Sunday_June_19_2022_09_26_14_PM_9090230/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{y'' + ay' + bx\left(-x^3b + ax + 2\right) y = 0}$$

27.5.1 Solving using Kovacic algorithm

Writing the ode as

$$y'' + ay' + bx\left(-x^3b + ax + 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= a \\ C &= bx\left(-x^3b + ax + 2\right) \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4b^2x^4 - 4ab\,x^2 + a^2 - 8bx}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4b^2x^4 - 4ab\,x^2 + a^2 - 8bx \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}a^2 + b^2x^4 - ab\,x^2 - 2bx\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 32: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx b\,x^2 - \frac{a}{2} - \frac{1}{x} - \frac{a}{2b\,x^3} - \frac{a^2}{4b^2x^5} - \frac{1}{2b\,x^4} - \frac{a^3}{8b^3x^7} - \frac{3a}{4b^2x^6} - \frac{1}{2b^2x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = b$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= -\frac{a}{2} + b\,x^2 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}a^2 - ab\,x^2 + b^2x^4$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4b^2x^4 - 4ab\,x^2 + a^2 - 8bx}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}a^2 + b^2x^4 - ab\,x^2 - 2bx\right) + (0) \\ &= \frac{1}{4}a^2 + b^2x^4 - ab\,x^2 - 2bx \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-2b$. Now b can be found.

$$\begin{aligned} b &= (-2b) - (0) \\ &= -2b \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= -\frac{a}{2} + b\,x^2 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2b}{b} - 2\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2b}{b} - 2\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}a^2 + b^2x^4 - ab\,x^2 - 2bx$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-4	$-\frac{a}{2}+b\,x^2$	-2	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(-\frac{a}{2} + b\,x^2\right) \\ &= \frac{a}{2} - b\,x^2 \\ &= \frac{a}{2} - b\,x^2 \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{a}{2} - b\,x^2\right)(0) + \left((-2bx) + \left(\frac{a}{2} - b\,x^2\right)^2 - \left(\frac{1}{4}a^2 + b^2x^4 - ab\,x^2 - 2bx\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{a}{2} - b\,x^2\right) dx} \\
&= \mathrm{e}^{\frac{1}{2} a x - \frac{1}{3} x^3 b}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{a}{1} \, dx} \\
&= z_1 e^{-\frac{ax}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{ax}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x^3 b}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{a}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-ax}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \mathrm{e}^{-ax + \frac{2}{3} x^3 b} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^3 b}{3}}\right) + c_2 \left(\mathrm{e}^{-\frac{x^3 b}{3}} \left(\int \mathrm{e}^{-ax + \frac{2}{3} x^3 b} dx \right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{x^3 b}{3}} + c_2 \mathrm{e}^{-\frac{x^3 b}{3}} \left(\int \mathrm{e}^{-ax+\frac{2}{3}x^3 b} dx \right) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{x^3 b}{3}} + c_2 \mathrm{e}^{-\frac{x^3 b}{3}} \left(\int \mathrm{e}^{-ax+\frac{2}{3}x^3 b} dx \right)$$

Verified OK.

27.5.2 Maple step by step solution

Let's solve

$y'' + ay' + bx(-x^3 b + ax + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 1..4$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert y' to series expansion

  $y' = \sum_{k=1}^{\infty} a_k k\, x^{k-1}$

- ○ Shift index using $k- > k + 1$

  $y' = \sum_{k=0}^{\infty} a_{k+1}(k+1)\, x^k$

- ○ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$a_1a + 2a_2 + (2aa_2 + 2a_0b + 6a_3)\, x + (a_0ab + 3aa_3 + 2a_1b + 12a_4)\, x^2 + (a_1ab + 4aa_4 + 2ba_2 + 20$$

- The coefficients of each power of x must be 0

$$[2a_2 + a_1a = 0, 2aa_2 + 2a_0b + 6a_3 = 0, a_0ab + 3aa_3 + 2a_1b + 12a_4 = 0, a_1ab + 4aa_4 + 2ba_2 + 20a$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{a_1a}{2}, a_3 = \frac{a_1a^2}{6} - \frac{a_0b}{3}, a_4 = -\frac{1}{24}a_1a^3 - \frac{1}{6}a_1b, a_5 = \frac{1}{120}a_1a^4 + \frac{1}{30}a_1ab\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + (ba_{k-2} + a_{k+1}(k+1))\, a - a_{k-4}b^2 + 2ba_{k-1} = 0$$

- Shift index using $k- >k+4$

$$\left((k+4)^2 + 3k + 14\right) a_{k+6} + (ba_{k+2} + a_{k+5}(k+5))\, a - a_k b^2 + 2ba_{k+3} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{aba_{k+2} + aka_{k+5} - a_k b^2 + 5aa_{k+5} + 2ba_{k+3}}{k^2 + 11k + 30}, a_2 = -\frac{a_1a}{2}, a_3 = \frac{a_1a^2}{6} - \frac{a_0b}{3}, a_4 = -\frac{1}{24}a_1a\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+a*diff(y(x),x)+b*x*(-b*x^3+a*x+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\left(\int \mathrm{e}^{-ax+\frac{2}{3}x^3b}dx\right)c_1 + c_2\right)\mathrm{e}^{-\frac{x^3b}{3}}$$

✓ Solution by Mathematica

Time used: 0.913 (sec). Leaf size: 46

```
DSolve[y''[x]+a*y'[x]+b*x*(-b*x^3+a*x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{bx^3}{3}}\left(c_2\int_1^x e^{\frac{2}{3}bK[1]^3-aK[1]}dK[1] + c_1\right)$$

27.6 problem 16

Internal problem ID [10840]
Internal file name [`OUTPUT/9821_Sunday_June_19_2022_09_26_15_PM_78099300/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
    _with_symmetry_[0,F(x)]`]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + ay' + b\left(-b\,x^{2n} + a\,x^{n} + n\,x^{n-1}\right)y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 48

```
dsolve(diff(y(x),x$2)+a*diff(y(x),x)+b*(-b*x^(2*n)+a*x^n+n*x^(n-1))*y(x)=0,y(x), singsol=all
```

$$y(x) = \left(\left(\int \mathrm{e}^{\frac{2b\,x^{n+1}-xa(n+1)}{n+1}}dx\right)c_1 + c_2\right)\mathrm{e}^{-\frac{b\,x^{n+1}}{n+1}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+a*y'[x]+b*(-b*x^(2*n)+a*x^n+n*x^(n-1))*y[x]==0,y[x],x,IncludeSingularSolutions
```

Not solved

27.7 problem 17

Internal problem ID [10841]
Internal file name [`OUTPUT/9822_Sunday_June_19_2022_09_26_17_PM_68406290/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + ay' + b\left(-b\,x^{2n} - a\,x^{n} + n\,x^{n-1}\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
   <- 2nd order, integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.187 (sec). Leaf size: 51

```
dsolve(diff(y(x),x$2)+a*diff(y(x),x)+b*(-b*x^(2*n)-a*x^n+n*x^(n-1))*y(x)=0,y(x), singsol=all
```

$$y(x) = e^{-\frac{x(b\,x^n+a(n+1))}{n+1}} \left(c_1 + \left(\int e^{\frac{x(2b\,x^n+a(n+1))}{n+1}}dx\right) c_2\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+a*y'[x]+b*(-b*x^(2*n)-a*x^n+n*x^(n-1))*y[x]==0,y[x],x,IncludeSingularSolutions
```

Not solved

27.8 problem 18

Internal problem ID [10842]
Internal file name [`OUTPUT/9823_Sunday_June_19_2022_09_26_19_PM_9788249/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + y'x + (n-1)\,y = 0}$$

27.8.1 Maple step by step solution

Let's solve

$y'' + y'x + (n-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- > k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite DE with series expansions

  $$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+n-1)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} + a_k(k+n-1) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+n-1)}{k^2+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 104

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+(n-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\left(-2\left(-\frac{x^2}{2}+n+\frac{1}{2}\right)c_1 n \operatorname{KummerM}\left(-\frac{n}{2}+\frac{1}{2},\frac{3}{2},\frac{x^2}{2}\right)+2(-x^2+2n+1)\,c_2 \operatorname{KummerU}\left(-\frac{n}{2}+\frac{1}{2},\frac{3}{2},\frac{x^2}{2}\right)\right.}{n\,(n-1)}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 51

```
DSolve[y''[x]+x*y'[x]+(n-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{x^2}{2}}\left(c_1 \operatorname{HermiteH}\left(n-2,\frac{x}{\sqrt{2}}\right)+c_2 \operatorname{Hypergeometric1F1}\left(1-\frac{n}{2},\frac{1}{2},\frac{x^2}{2}\right)\right)$$

27.9 problem 19

Internal problem ID [10843]
Internal file name [`OUTPUT/9824_Sunday_June_19_2022_09_26_21_PM_41558290/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' - 2y'x + 2yn = 0}$$

27.9.1 Maple step by step solution

Let's solve

$y'' - 2y'x + 2yn = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite DE with series expansions

  $$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(k-n)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2+3k+2)\, a_{k+2} - 2a_k(k-n) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2a_k(k-n)}{k^2+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+2*n*y(x)=0,y(x), singsol=all)
```

$$y(x) = x\left(\mathrm{KummerU}\left(-\frac{n}{2}+\frac{1}{2},\frac{3}{2},x^2\right)c_2+\mathrm{KummerM}\left(-\frac{n}{2}+\frac{1}{2},\frac{3}{2},x^2\right)c_1\right)$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 27

```
DSolve[y''[x]-2*x*y'[x]+2*n*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1\,\mathrm{HermiteH}(n,x)+c_2\,\mathrm{Hypergeometric1F1}\left(-\frac{n}{2},\frac{1}{2},x^2\right)$$

27.10 problem 20

Internal problem ID [10844]
Internal file name [`OUTPUT/9825_Sunday_June_19_2022_09_26_22_PM_2454462/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + axy' + yb = 0}$$

27.10.1 Maple step by step solution

Let's solve

$y'' + axy' + yb = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}$$

- Shift index using $k- > k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k$$

  Rewrite DE with series expansions

  $$\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(ak+b)) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2+3k+2) a_{k+2} + a_k(ak+b) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(ak+b)}{k^2+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 58

```
dsolve(diff(y(x),x$2)+a*x*diff(y(x),x)+b*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{a x^2}{2}} x \left(\text{KummerM}\left(\frac{-b+2a}{2a}, \frac{3}{2}, \frac{a x^2}{2}\right) c_1 + \text{KummerU}\left(\frac{-b+2a}{2a}, \frac{3}{2}, \frac{a x^2}{2}\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 67

```
DSolve[y''[x]+a*x*y'[x]+b*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{ax^2}{2}} \left(c_1 \text{HermiteH}\left(\frac{b}{a} - 1, \frac{\sqrt{a}x}{\sqrt{2}}\right) + c_2 \text{Hypergeometric1F1}\left(\frac{a-b}{2a}, \frac{1}{2}, \frac{ax^2}{2}\right)\right)$$

27.11 problem 21

Internal problem ID [10845]
Internal file name [`OUTPUT/9826_Sunday_June_19_2022_09_26_24_PM_7626222/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + axy' + bxy = 0}$$

27.11.1 Maple step by step solution

Let's solve

$y'' + axy' + bxy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions

  - ○ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

- Shift index using $k->k-1$

  $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k}$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a a_k k + b a_{k-1}\right) x^{k}\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$\left(k^2 + 3k + 2\right) a_{k+2} + a a_k k + b a_{k-1} = 0$$

- Shift index using $k->k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} + a a_{k+1}(k+1) + b a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+3} = -\frac{a k a_{k+1} + a a_{k+1} + b a_k}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 70

```
dsolve(diff(y(x),x$2)+a*x*diff(y(x),x)+b*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{bx}{a}} \left(\text{KummerM}\left(\frac{b^2}{2a^3}, \frac{1}{2}, -\frac{(a^2x-2b)^2}{2a^3}\right) c_1 + \text{KummerU}\left(\frac{b^2}{2a^3}, \frac{1}{2}, -\frac{(a^2x-2b)^2}{2a^3}\right) c_2 \right)$$

✓ Solution by Mathematica

Time used: 0.106 (sec). Leaf size: 96

```
DSolve[y''[x]+a*x*y'[x]+b*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{bx}{a}-\frac{ax^2}{2}}\left(c_2\,\text{Hypergeometric1F1}\left(\frac{1}{2}-\frac{b^2}{2a^3},\frac{1}{2},\frac{(a^2x-2b)^2}{2a^3}\right)\right.$$
$$\left.+\,c_1\,\text{HermiteH}\left(\frac{b^2}{a^3}-1,\frac{a^2x-2b}{\sqrt{2}a^{3/2}}\right)\right)$$

27.12 problem 22

Internal problem ID [10846]
Internal file name [`OUTPUT/9827_Sunday_June_19_2022_09_26_25_PM_30342709/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$y'' + axy' + (bx + c)y = 0$$

27.12.1 Maple step by step solution

Let's solve

$y'' + axy' + (bx + c)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ∘ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$a_0 c + 2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(ak+c) + b a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

$$a_0 c + 2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + a a_k k + b a_{k-1} + a_k c = 0$$

- Shift index using $k->k+1$

$$\left((k+1)^2 + 3k + 5\right) a_{k+3} + a a_{k+1}(k+1) + b a_k + a_{k+1} c = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{ak a_{k+1} + a a_{k+1} + b a_k + a_{k+1} c}{k^2 + 5k + 6}, a_0 c + 2a_2 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 82

```
dsolve(diff(y(x),x$2)+a*x*diff(y(x),x)+(b*x+c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{bx}{a}} \left(\operatorname{KummerM}\left(\frac{a^2c+b^2}{2a^3}, \frac{1}{2}, -\frac{(a^2x-2b)^2}{2a^3}\right) c_1 + \operatorname{KummerU}\left(\frac{a^2c+b^2}{2a^3}, \frac{1}{2}, -\frac{(a^2x-2b)^2}{2a^3}\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 108

```
DSolve[y''[x]+a*x*y'[x]+(b*x+c)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{bx}{a}-\frac{ax^2}{2}}\left(c_2 \text{Hypergeometric1F1}\left(-\frac{-a^3+ca^2+b^2}{2a^3},\frac{1}{2},\frac{(a^2x-2b)^2}{2a^3}\right)+c_1 \text{HermiteH}\left(\frac{b^2}{a^3}+\frac{c}{a}-1,\frac{a^2x-2b}{\sqrt{2}a^{3/2}}\right)\right)$$

27.13 problem 23

Internal problem ID [10847]
Internal file name [`OUTPUT/9828_Sunday_June_19_2022_09_26_27_PM_54499846/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$y'' + 2axy' + \left(b\,x^4 + a^2x^2 + cx + a\right) y = 0$$

27.13.1 Maple step by step solution

Let's solve

$y'' + 2axy' + (b\,x^4 + a^2x^2 + cx + a)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..4$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- Shift index using $k->k-m$

  $$x^{m}\cdot y=\sum_{k=\max(0,-m)+m}^{\infty}a_{k-m}x^{k}$$

- Convert $x\cdot y'$ to series expansion

  $$x\cdot y'=\sum_{k=0}^{\infty}a_{k}k\,x^{k}$$

- Convert y'' to series expansion

  $$y''=\sum_{k=2}^{\infty}a_{k}k(k-1)\,x^{k-2}$$

- Shift index using $k->k+2$

  $$y''=\sum_{k=0}^{\infty}a_{k+2}(k+2)\,(k+1)\,x^{k}$$

  Rewrite ODE with series expansions

  $$a_{0}a+2a_{2}+(6a_{3}+3a_{1}a+a_{0}c)\,x+(a_{0}a^{2}+5aa_{2}+a_{1}c+12a_{4})\,x^{2}+(a_{1}a^{2}+7aa_{3}+a_{2}c+20a_{5})$$

- The coefficients of each power of x must be 0

  $$[2a_{2}+a_{0}a=0,6a_{3}+3a_{1}a+a_{0}c=0,a_{0}a^{2}+5aa_{2}+a_{1}c+12a_{4}=0,a_{1}a^{2}+7aa_{3}+a_{2}c+20a_{5}=$$

- Solve for the dependent coefficient(s)

  $$\left\{a_{2}=-\frac{a_{0}a}{2},a_{3}=-\frac{a_{1}a}{2}-\frac{a_{0}c}{6},a_{4}=\frac{a_{0}a^{2}}{8}-\frac{a_{1}c}{12},a_{5}=\frac{1}{8}a_{1}a^{2}+\frac{1}{12}a_{0}ac\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^{2}+3k+2)\,a_{k+2}+a_{k-2}a^{2}+aa_{k}(2k+1)+a_{k-4}b+a_{k-1}c=0$$

- Shift index using $k->k+4$

  $$\left((k+4)^{2}+3k+14\right)a_{k+6}+a_{k+2}a^{2}+aa_{k+4}(2k+9)+a_{k}b+a_{k+3}c=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y=\sum_{k=0}^{\infty}a_{k}x^{k},a_{k+6}=-\frac{a_{k+2}a^{2}+2aka_{k+4}+9aa_{k+4}+a_{k}b+a_{k+3}c}{k^{2}+11k+30},a_{2}=-\frac{a_{0}a}{2},a_{3}=-\frac{a_{1}a}{2}-\frac{a_{0}c}{6},a_{4}=\frac{a_{0}a^{2}}{8}-\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 80

```
dsolve(diff(y(x),x$2)+2*a*x*diff(y(x),x)+(b*x^4+a^2*x^2+c*x+a)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{\left(i\sqrt{b}\,x+\frac{3a}{2}\right)x^2}{3}} x \left(\operatorname{KummerM}\left(\frac{ic+4\sqrt{b}}{6\sqrt{b}}, \frac{4}{3}, \frac{2i\sqrt{b}\,x^3}{3}\right) c_1 + \operatorname{KummerU}\left(\frac{ic+4\sqrt{b}}{6\sqrt{b}}, \frac{4}{3}, \frac{2i\sqrt{b}\,x^3}{3}\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.322 (sec). Leaf size: 121

```
DSolve[y''[x]+2*a*x*y'[x]+(b*x^4+a^2*x^2+c*x+a)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{\sqrt[3]{2}\sqrt[3]{x^3}e^{\frac{1}{6}ix^2\left(2\sqrt{b}x+3ia\right)}\left(c_1 \operatorname{HypergeometricU}\left(\frac{1}{3}-\frac{ic}{6\sqrt{b}},\frac{2}{3},-\frac{2}{3}i\sqrt{b}x^3\right)+c_2 L_{\frac{ic}{6\sqrt{b}}-\frac{1}{3}}^{-\frac{1}{3}}\left(-\frac{2}{3}i\sqrt{b}x^3\right)\right)}{x}$$

27.14 problem 24

Internal problem ID [10848]
Internal file name [`OUTPUT/9829_Sunday_June_19_2022_09_26_29_PM_78344309/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + (ax + b)\, y' - ay = 0}$$

27.14.1 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B+2v'B'+B''v)+B(B'v+v'B)+CBv&=0\\
ABv''+\left(2AB'+B^2\right)v'+\left(AB''+BB'+CB\right)v&=0 \qquad (1)
\end{aligned}
$$

If the term $AB''+BB'+CB$ is zero, then this method works and can be used to solve

$$
ABv''+\left(2AB'+B^2\right)v'=0
$$

By Using $u=v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu'+\left(2AB'+B^2\right)u=0
$$

The above ode is first order ode which is solved for u. Now a new ode $v'=u$ is solved for v as first order ode. Then the final solution is obtain from $y=Bv$.

This method works only if the term $AB''+BB'+CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A&=1\\
B&=ax+b\\
C&=-a\\
F&=0
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB''+BB'+CB&=(1)\,(0)+(ax+b)\,(a)+(-a)\,(ax+b)\\
&=0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
ax+bv''+\left(2a+(ax+b)^2\right)v'=0
$$

Now by applying $v'=u$ the above becomes

$$
(ax+b)\,u'(x)+\left(a^2x^2+2abx+b^2+2a\right)u(x)=0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u'&=F(x,u)\\
&=f(x)g(u)\\
&=-\frac{\left(a^2x^2+2abx+b^2+2a\right)u}{ax+b}
\end{aligned}
$$

Where $f(x) = -\frac{a^2x^2+2abx+b^2+2a}{ax+b}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{a^2x^2+2abx+b^2+2a}{ax+b}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{a^2x^2+2abx+b^2+2a}{ax+b}\, dx \\
\ln(u) &= -\frac{a\,x^2}{2} - bx - 2\ln(ax+b) + c_1 \\
u &= e^{-\frac{a\,x^2}{2} - bx - 2\ln(ax+b) + c_1} \\
&= c_1 e^{-\frac{a\,x^2}{2} - bx - 2\ln(ax+b)}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= c_1 e^{-\frac{a\,x^2}{2} - bx - 2\ln(ax+b)}
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int c_1 e^{-\frac{a\,x^2}{2} - bx - 2\ln(ax+b)}\, \mathrm{d}x \\
&= c_1 \left(-\frac{e^{-\frac{(ax+b)^2}{2a} + \frac{b^2}{2a}}}{a\,(ax+b)} - \frac{\sqrt{\pi}\, e^{\frac{b^2}{2a}} \sqrt{2}\, \operatorname{erf}\left(\frac{\sqrt{2}\,(ax+b)}{2\sqrt{a}}\right)}{2a^{\frac{3}{2}}} \right) + c_2
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y(x) &= Bv \\
&= (ax+b) \left(c_1 \left(-\frac{e^{-\frac{(ax+b)^2}{2a} + \frac{b^2}{2a}}}{a\,(ax+b)} - \frac{\sqrt{\pi}\, e^{\frac{b^2}{2a}} \sqrt{2}\, \operatorname{erf}\left(\frac{\sqrt{2}\,(ax+b)}{2\sqrt{a}}\right)}{2a^{\frac{3}{2}}} \right) + c_2 \right) \\
&= -\frac{\sqrt{\pi}\, e^{\frac{b^2}{2a}} c_1 \sqrt{2}\,(ax+b)\, \operatorname{erf}\left(\frac{\sqrt{2}\,(ax+b)}{2\sqrt{a}}\right) + 2\, e^{-\frac{x(ax+2b)}{2}} c_1 \sqrt{a} - 2c_2 \left(a^{\frac{3}{2}} b + a^{\frac{5}{2}} x\right)}{2a^{\frac{3}{2}}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{\pi}\, e^{\frac{b^2}{2a}} c_1 \sqrt{2}\,(ax+b)\, \operatorname{erf}\left(\frac{\sqrt{2}\,(ax+b)}{2\sqrt{a}}\right) + 2\, e^{-\frac{x(ax+2b)}{2}} c_1 \sqrt{a} - 2c_2 \left(a^{\frac{3}{2}} b + a^{\frac{5}{2}} x\right)}{2a^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = -\frac{\sqrt{\pi}\, e^{\frac{b^2}{2a}} c_1 \sqrt{2}\,(ax+b)\, \operatorname{erf}\left(\frac{\sqrt{2}\,(ax+b)}{2\sqrt{a}}\right) + 2\, e^{-\frac{x(ax+2b)}{2}} c_1 \sqrt{a} - 2c_2 \left(a^{\frac{3}{2}} b + a^{\frac{5}{2}} x\right)}{2a^{\frac{3}{2}}}$$

Verified OK.

27.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + (ax + b)\, y' - ay = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= ax + b \\ C &= -a \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2abx + b^2 + 6a}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^2 + 2abx + b^2 + 6a \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{2}a + \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 40: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞.

Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{ax}{2}+\frac{b}{2}+\frac{3}{2x}-\frac{3b}{2a\,x^2}+\frac{3b^2}{2a^2x^3}-\frac{9}{4a\,x^3}-\frac{3b^3}{2a^3x^4}+\frac{27b}{4a^2x^4}+\frac{3b^4}{2a^4x^5}-\frac{27b^2}{2a^3x^5}-\frac{3b^5}{2a^5x^6}+\frac{27}{4a^2x^5}+\frac{45b^3}{2a^4x^6}-\frac{135b}{4a^3x^6}+ \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{ax}{2} + \frac{b}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2$$

This shows that the coefficient of 1 in the above is $\frac{b^2}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{a^2x^2 + 2abx + b^2 + 6a}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{3}{2}a + \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2\right) + (0) \\
&= \frac{3}{2}a + \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{3a}{2} + \frac{b^2}{4}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{3a}{2} + \frac{b^2}{4}\right) - \left(\frac{b^2}{4}\right) \\
&= \frac{3a}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{ax}{2} + \frac{b}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{3a}{2}}{\frac{a}{2}} - 1\right) &= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{3a}{2}}{\frac{a}{2}} - 1\right) &= -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{2}a + \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{ax}{2} + \frac{b}{2}$	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$
\begin{aligned}
d &= \alpha_\infty^+ \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= (+)[\sqrt{r}]_\infty \\
&= 0 + \left(\frac{ax}{2} + \frac{b}{2} \right) \\
&= \frac{ax}{2} + \frac{b}{2} \\
&= \frac{ax}{2} + \frac{b}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{ax}{2} + \frac{b}{2}\right)(1) + \left(\left(\frac{a}{2}\right) + \left(\frac{ax}{2} + \frac{b}{2}\right)^2 - \left(\frac{3}{2}a + \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2\right) \right) &= 0 \\
-aa_0 + b &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{ a_0 = \frac{b}{a} \right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{b}{a}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x+\frac{b}{a}\right) \mathrm{e}^{\int \left(\frac{ax}{2}+\frac{b}{2}\right)dx} \\
&= \left(x+\frac{b}{a}\right) \mathrm{e}^{\frac{1}{4}a\,x^2+\frac{1}{2}bx} \\
&= \frac{(ax+b)\,\mathrm{e}^{\frac{x(ax+2b)}{4}}}{a}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{ax+b}{1}\,dx} \\
&= z_1 e^{-\frac{1}{4}a\,x^2-\frac{1}{2}bx} \\
&= z_1 \left(\mathrm{e}^{-\frac{x(ax+2b)}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{ax+b}{a}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{ax+b}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{1}{2}a\,x^2-bx}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\left(\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{b^2}{2a}}(ax+b)\operatorname{erf}\left(\frac{\sqrt{2}\,(ax+b)}{2\sqrt{a}}\right)+2\,\mathrm{e}^{-\frac{x(ax+2b)}{2}}\sqrt{a}\right)\sqrt{a}}{2ax+2b}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{a x + b}{a}\right) \\
&\quad + c_2 \left(\frac{a x + b}{a} \left(-\frac{\left(\sqrt{2}\, \sqrt{\pi}\, \mathrm{e}^{\frac{b^2}{2a}} (a x + b) \operatorname{erf}\left(\frac{\sqrt{2}\, (a x + b)}{2 \sqrt{a}}\right) + 2\, \mathrm{e}^{-\frac{x(a x + 2b)}{2}} \sqrt{a}\right) \sqrt{a}}{2 a x + 2 b}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (a x + b)}{a} - \frac{c_2 \left(\sqrt{2}\, \sqrt{\pi}\, \mathrm{e}^{\frac{b^2}{2a}} (a x + b) \operatorname{erf}\left(\frac{\sqrt{2}\, (a x + b)}{2 \sqrt{a}}\right) + 2\, \mathrm{e}^{-\frac{x(a x + 2b)}{2}} \sqrt{a}\right)}{2 \sqrt{a}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 (a x + b)}{a} - \frac{c_2 \left(\sqrt{2}\, \sqrt{\pi}\, \mathrm{e}^{\frac{b^2}{2a}} (a x + b) \operatorname{erf}\left(\frac{\sqrt{2}\, (a x + b)}{2 \sqrt{a}}\right) + 2\, \mathrm{e}^{-\frac{x(a x + 2b)}{2}} \sqrt{a}\right)}{2 \sqrt{a}}
$$

Verified OK.

27.14.3 Maple step by step solution

Let's solve

$y'' + (a x + b) y' - a y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=\max(0, 1-m)}^{\infty} a_k k\, x^{k-1+m}$

  ◦ Shift index using $k- > k + 1 - m$

    $x^m \cdot y' = \sum_{k=\max(0, 1-m)+m-1}^{\infty} a_{k+1-m} (k + 1 - m)\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite DE with series expansions

  $$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k+1}(k+1)\, b + a a_k(k-1)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$k^2 a_{k+2} + (a a_k + a_{k+1} b + 3a_{k+2})\, k - a a_k + a_{k+1} b + 2a_{k+2} = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a a_k k + a_{k+1} b k - a a_k + a_{k+1} b}{k^2 + 3k + 2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 65

```
dsolve(diff(y(x),x$2)+(a*x+b)*diff(y(x),x)-a*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\frac{b^2}{2a}}\pi c_2(ax+b)\,\mathrm{erf}\left(\frac{\sqrt{2}\,(ax+b)}{2\sqrt{a}}\right) + \sqrt{\pi}\,\sqrt{2}\,\sqrt{a}\,\mathrm{e}^{-\frac{x(ax+2b)}{2}} c_2 + c_1(ax+b)$$

✓ Solution by Mathematica

Time used: 0.959 (sec). Leaf size: 82

```
DSolve[y''[x]+(a*x+b)*y'[x]-a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(ax+b)\left(-\frac{\sqrt{\frac{\pi}{2}}c_2\mathrm{erf}\left(\frac{ax+b}{\sqrt{2}\sqrt{a}}\right)}{a^{3/2}} - \frac{c_2 e^{-\frac{(ax+b)^2}{2a}}}{a(ax+b)} + c_1\right)}{b}$$

27.15 problem 25

Internal problem ID [10849]
Internal file name [`OUTPUT/9830_Sunday_June_19_2022_09_26_30_PM_79768022/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + (ax + b)\,y' + ay = 0}$$

27.15.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + (ax + b)\,y' + ay)\,dx = 0$$
$$(ax + b)\,y + y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= ax + b \\
q(x) &= c_1
\end{aligned}
$$

Hence the ode is

$$
(ax + b)\, y + y' = c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (ax+b)dx} \\
&= \mathrm{e}^{\frac{1}{2}a\,x^2+bx}
\end{aligned}
$$

Which simplifies to

$$
\mu = \mathrm{e}^{\frac{x(ax+2b)}{2}}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(c_1) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{x(ax+2b)}{2}} y\right) &= \left(\mathrm{e}^{\frac{x(ax+2b)}{2}}\right)(c_1) \\
\mathrm{d}\left(\mathrm{e}^{\frac{x(ax+2b)}{2}} y\right) &= \left(c_1 \mathrm{e}^{\frac{x(ax+2b)}{2}}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\frac{x(ax+2b)}{2}} y &= \int c_1 \mathrm{e}^{\frac{x(ax+2b)}{2}}\,\mathrm{d}x \\
\mathrm{e}^{\frac{x(ax+2b)}{2}} y &= -\frac{c_1\sqrt{\pi}\,\mathrm{e}^{-\frac{b^2}{2a}}\,\mathrm{erf}\left(-\frac{\sqrt{-2a}\,x}{2} + \frac{b}{\sqrt{-2a}}\right)}{\sqrt{-2a}} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{x(ax+2b)}{2}}$ results in

$$
y = -\frac{\mathrm{e}^{-\frac{x(ax+2b)}{2}} c_1\sqrt{\pi}\,\mathrm{e}^{-\frac{b^2}{2a}}\,\mathrm{erf}\left(-\frac{\sqrt{-2a}\,x}{2} + \frac{b}{\sqrt{-2a}}\right)}{\sqrt{-2a}} + \mathrm{e}^{-\frac{x(ax+2b)}{2}} c_2
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\mathrm{e}^{-\frac{x(ax+2b)}{2}} c_1\sqrt{\pi}\,\mathrm{e}^{-\frac{b^2}{2a}}\,\mathrm{erf}\left(-\frac{\sqrt{-2a}\,x}{2} + \frac{b}{\sqrt{-2a}}\right)}{\sqrt{-2a}} + \mathrm{e}^{-\frac{x(ax+2b)}{2}} c_2 \tag{1}
$$

Verification of solutions

$$y = -\frac{e^{-\frac{x(ax+2b)}{2}} c_1 \sqrt{\pi}\, e^{-\frac{b^2}{2a}} \operatorname{erf}\left(-\frac{\sqrt{-2a}\, x}{2} + \frac{b}{\sqrt{-2a}}\right)}{\sqrt{-2a}} + e^{-\frac{x(ax+2b)}{2}} c_2$$

Verified OK.

27.15.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + (ax + b)\, y' + ay = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + (ax + b)\, y' + ay)\, dx = 0$$
$$(ax + b)\, y + y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = ax + b$$
$$q(x) = c_1$$

Hence the ode is

$$(ax + b)\, y + y' = c_1$$

The integrating factor μ is

$$\mu = e^{\int (ax+b)dx}$$
$$= e^{\frac{1}{2} a\, x^2 + bx}$$

Which simplifies to

$$\mu = e^{\frac{x(ax+2b)}{2}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(c_1)$$
$$\frac{d}{dx}\left(e^{\frac{x(ax+2b)}{2}} y\right) = \left(e^{\frac{x(ax+2b)}{2}}\right)(c_1)$$
$$d\left(e^{\frac{x(ax+2b)}{2}} y\right) = \left(c_1 e^{\frac{x(ax+2b)}{2}}\right) dx$$

Integrating gives

$$e^{\frac{x(ax+2b)}{2}} y = \int c_1 e^{\frac{x(ax+2b)}{2}} dx$$
$$e^{\frac{x(ax+2b)}{2}} y = -\frac{c_1 \sqrt{\pi}\, e^{-\frac{b^2}{2a}} \operatorname{erf}\left(-\frac{\sqrt{-2a}\, x}{2} + \frac{b}{\sqrt{-2a}}\right)}{\sqrt{-2a}} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x(ax+2b)}{2}}$ results in

$$y = -\frac{e^{-\frac{x(ax+2b)}{2}} c_1 \sqrt{\pi}\, e^{-\frac{b^2}{2a}} \operatorname{erf}\left(-\frac{\sqrt{-2a}\, x}{2} + \frac{b}{\sqrt{-2a}}\right)}{\sqrt{-2a}} + e^{-\frac{x(ax+2b)}{2}} c_2$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{-\frac{x(ax+2b)}{2}} c_1 \sqrt{\pi}\, e^{-\frac{b^2}{2a}} \operatorname{erf}\left(-\frac{\sqrt{-2a}\, x}{2} + \frac{b}{\sqrt{-2a}}\right)}{\sqrt{-2a}} + e^{-\frac{x(ax+2b)}{2}} c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{e^{-\frac{x(ax+2b)}{2}} c_1 \sqrt{\pi}\, e^{-\frac{b^2}{2a}} \operatorname{erf}\left(-\frac{\sqrt{-2a}\, x}{2} + \frac{b}{\sqrt{-2a}}\right)}{\sqrt{-2a}} + e^{-\frac{x(ax+2b)}{2}} c_2$$

Verified OK.

27.15.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + (ax+b)\, y' + ay = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= ax + b \\ C &= a \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2abx + b^2 - 2a}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^2 + 2abx + b^2 - 2a \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{2}a + \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 42: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{ax}{2}+\frac{b}{2}-\frac{1}{2x}+\frac{b}{2a\,x^2}-\frac{b^2}{2a^2x^3}-\frac{1}{4a\,x^3}+\frac{b^3}{2a^3x^4}+\frac{3b}{4a^2x^4}-\frac{b^4}{2a^4x^5}-\frac{3b^2}{2a^3x^5}+\frac{b^5}{2a^5x^6}-\frac{1}{4a^2x^5}+\frac{5b^3}{2a^4x^6}+\frac{5b}{4a^3x^6}+ \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{ax}{2} + \frac{b}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2$$

This shows that the coefficient of 1 in the above is $\frac{b^2}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 2abx + b^2 - 2a}{4} \\ &= Q + \frac{R}{4} \\ &= \left(-\frac{1}{2}a + \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2\right) + (0) \\ &= -\frac{1}{2}a + \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{a}{2} + \frac{b^2}{4}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{a}{2} + \frac{b^2}{4}\right) - \left(\frac{b^2}{4}\right) \\ &= -\frac{a}{2}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{ax}{2} + \frac{b}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{a}{2}}{\frac{a}{2}} - 1\right) &&= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{a}{2}}{\frac{a}{2}} - 1\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{2}a + \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{ax}{2} + \frac{b}{2}$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^{-} \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{ax}{2} + \frac{b}{2}\right) \\
&= -\frac{ax}{2} - \frac{b}{2} \\
&= -\frac{ax}{2} - \frac{b}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{ax}{2} - \frac{b}{2}\right)(0) + \left(\left(-\frac{a}{2}\right) + \left(-\frac{ax}{2} - \frac{b}{2}\right)^2 - \left(-\frac{1}{2}a + \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{ax}{2} - \frac{b}{2}\right) dx} \\ &= e^{-\frac{x(ax+2b)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{ax+b}{1} \, dx} \\ &= z_1 e^{-\frac{1}{4}a\,x^2 - \frac{1}{2}bx} \\ &= z_1 \left(e^{-\frac{x(ax+2b)}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{x(ax+2b)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{ax+b}{1} dx}}{(y_1)^2} dx$$

$$= y_1 \int \frac{e^{-\frac{1}{2}a x^2 - bx}}{(y_1)^2} dx$$

$$= y_1 \left(-\frac{\sqrt{\pi}\, \mathrm{e}^{-\frac{b^2}{2a}} \sqrt{2}\, \mathrm{erf}\left(\frac{(ax+b)\sqrt{2}}{2\sqrt{-a}}\right)}{2\sqrt{-a}} \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\mathrm{e}^{-\frac{x(ax+2b)}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{x(ax+2b)}{2}} \left(-\frac{\sqrt{\pi}\, \mathrm{e}^{-\frac{b^2}{2a}} \sqrt{2}\, \mathrm{erf}\left(\frac{(ax+b)\sqrt{2}}{2\sqrt{-a}}\right)}{2\sqrt{-a}} \right)\right)$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{x(ax+2b)}{2}} - \frac{c_2 \sqrt{2}\, \sqrt{\pi}\, \mathrm{e}^{-\frac{(ax+b)^2}{2a}}\, \mathrm{erf}\left(\frac{(ax+b)\sqrt{2}}{2\sqrt{-a}}\right)}{2\sqrt{-a}} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{x(ax+2b)}{2}} - \frac{c_2 \sqrt{2}\, \sqrt{\pi}\, \mathrm{e}^{-\frac{(ax+b)^2}{2a}}\, \mathrm{erf}\left(\frac{(ax+b)\sqrt{2}}{2\sqrt{-a}}\right)}{2\sqrt{-a}}$$

Verified OK.

27.15.4 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 1 \\
q(x) &= ax + b \\
r(x) &= a \\
s(x) &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 0 \\
q'(x) &= a
\end{aligned}$$

Therefore (1) becomes

$$0 - (a) + (a) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx$$

Substituting the above values for p, q, r, s gives

$$(ax + b)\, y + y' = c_1$$

We now have a first order ode to solve which is

$$(ax + b)\, y + y' = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= ax + b \\
q(x) &= c_1
\end{aligned}$$

Hence the ode is

$$(ax + b)\, y + y' = c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int (ax+b)dx}$$
$$= \mathrm{e}^{\frac{1}{2}a\,x^2+bx}$$

Which simplifies to

$$\mu = \mathrm{e}^{\frac{x(ax+2b)}{2}}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\,(c_1)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{x(ax+2b)}{2}} y\right) = \left(\mathrm{e}^{\frac{x(ax+2b)}{2}}\right)(c_1)$$
$$\mathrm{d}\left(\mathrm{e}^{\frac{x(ax+2b)}{2}} y\right) = \left(c_1\mathrm{e}^{\frac{x(ax+2b)}{2}}\right)\,\mathrm{d}x$$

Integrating gives

$$\mathrm{e}^{\frac{x(ax+2b)}{2}} y = \int c_1\mathrm{e}^{\frac{x(ax+2b)}{2}}\,\mathrm{d}x$$
$$\mathrm{e}^{\frac{x(ax+2b)}{2}} y = -\frac{c_1\sqrt{\pi}\,\mathrm{e}^{-\frac{b^2}{2a}}\,\mathrm{erf}\left(-\frac{\sqrt{-2a}\,x}{2}+\frac{b}{\sqrt{-2a}}\right)}{\sqrt{-2a}} + c_2$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{x(ax+2b)}{2}}$ results in

$$y = -\frac{\mathrm{e}^{-\frac{x(ax+2b)}{2}} c_1\sqrt{\pi}\,\mathrm{e}^{-\frac{b^2}{2a}}\,\mathrm{erf}\left(-\frac{\sqrt{-2a}\,x}{2}+\frac{b}{\sqrt{-2a}}\right)}{\sqrt{-2a}} + \mathrm{e}^{-\frac{x(ax+2b)}{2}} c_2$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-\frac{x(ax+2b)}{2}} c_1\sqrt{\pi}\,\mathrm{e}^{-\frac{b^2}{2a}}\,\mathrm{erf}\left(-\frac{\sqrt{-2a}\,x}{2}+\frac{b}{\sqrt{-2a}}\right)}{\sqrt{-2a}} + \mathrm{e}^{-\frac{x(ax+2b)}{2}} c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{\mathrm{e}^{-\frac{x(ax+2b)}{2}} c_1\sqrt{\pi}\,\mathrm{e}^{-\frac{b^2}{2a}}\,\mathrm{erf}\left(-\frac{\sqrt{-2a}\,x}{2}+\frac{b}{\sqrt{-2a}}\right)}{\sqrt{-2a}} + \mathrm{e}^{-\frac{x(ax+2b)}{2}} c_2$$

Verified OK.

27.15.5 Maple step by step solution

Let's solve

$y'' + (ax + b)\, y' + ay = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  ◦ Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k + 1 - m)\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k - 1)\, x^{k-2}$

  ◦ Shift index using $k- >k + 2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k + 2)\,(k + 1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k + 2)\,(k + 1) + a_{k+1}(k + 1)\, b + a a_k(k + 1))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k + 1)\,(a_{k+2}(k + 2) + a_{k+1}b + a a_k) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a a_k + a_{k+1} b}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+(a*x+b)*diff(y(x),x)+a*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\operatorname{erf}\left(\frac{(ax+b)\sqrt{2}}{2\sqrt{-a}}\right)c_1 + c_2\right)\mathrm{e}^{-\frac{x(ax+2b)}{2}}$$

✓ Solution by Mathematica
Time used: 0.067 (sec). Leaf size: 79

```
DSolve[y''[x]+(a*x+b)*y'[x]+a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-\frac{(ax+b)^2}{2a}}\left(2\sqrt{a}c_2e^{\frac{b^2}{2a}} + \sqrt{2\pi}c_1\operatorname{erfi}\left(\frac{ax+b}{\sqrt{2}\sqrt{a}}\right)\right)}{2\sqrt{a}}$$

27.16 problem 26

Internal problem ID [10850]
Internal file name [`OUTPUT/9831_Sunday_June_19_2022_09_26_31_PM_50554008/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + (ax + b)\, y' + c(ax + b - c)\, y = 0}$$

27.16.1 Solving using Kovacic algorithm

Writing the ode as

$$y'' + (ax + b)\, y' + c(ax + b - c)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= ax + b \\ C &= c(ax + b - c) \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2abx - 4acx + b^2 - 4bc + 4c^2 + 2a}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^2 + 2abx - 4acx + b^2 - 4bc + 4c^2 + 2a \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{2}a + \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2 - acx - bc + c^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 44: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{b}{2} - c + \frac{ax}{2} + \frac{1}{2x} - \frac{2bc}{a^2x^3} + \frac{3b^2c}{a^3x^4} - \frac{6b\,c^2}{a^3x^4} - \frac{4b^3c}{a^4x^5} + \frac{12b^2c^2}{a^4x^5} - \frac{16b\,c^3}{a^4x^5} + \frac{6bc}{a^3x^5} + \frac{5b^4c}{a^5x^6} - \frac{20b^3c^2}{a^5x^6} + \frac{40b^2c^3}{a^5x^6} - \frac{40b\,c^4}{a^5x^6} - \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{b}{2} - c + \frac{ax}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}b^2 - bc + \frac{1}{2}abx + c^2 - acx + \frac{1}{4}a^2x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{4}b^2 - bc + c^2$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{a^2x^2 + 2abx - 4acx + b^2 - 4bc + 4c^2 + 2a}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{a^2x^2}{4} + \left(\frac{1}{2}ab - ac\right)x + \frac{a}{2} + \frac{b^2}{4} - bc + c^2\right) + (0) \\
&= \frac{a^2x^2}{4} + \left(\frac{1}{2}ab - ac\right)x + \frac{a}{2} + \frac{b^2}{4} - bc + c^2
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{1}{2}a + \frac{1}{4}b^2 - bc + c^2$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{1}{2}a + \frac{1}{4}b^2 - bc + c^2\right) - \left(\frac{1}{4}b^2 - bc + c^2\right) \\
&= \frac{a}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{b}{2} - c + \frac{ax}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{a}{2}}{\frac{a}{2}} - 1\right) &= 0 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{a}{2}}{\frac{a}{2}} - 1\right) &= -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{2}a + \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2 - acx - bc + c^2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{b}{2}-c+\frac{ax}{2}$	0	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=0$, and since there are no poles, then

$$\begin{aligned}d&=\alpha_\infty^{+}\\&=0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega&=(+)[\sqrt{r}]_\infty\\&=0+\left(\frac{b}{2}-c+\frac{ax}{2}\right)\\&=\frac{b}{2}-c+\frac{ax}{2}\\&=\frac{b}{2}-c+\frac{ax}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{b}{2}-c+\frac{ax}{2}\right)(0)+\left(\left(\frac{a}{2}\right)+\left(\frac{b}{2}-c+\frac{ax}{2}\right)^2-\left(\frac{1}{2}a+\frac{1}{4}a^2x^2+\frac{1}{2}abx+\frac{1}{4}b^2-acx-bc+c^2\right)\right)$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{b}{2} - c + \frac{ax}{2}\right) dx} \\
&= \mathrm{e}^{\frac{x(ax+2b-4c)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{ax+b}{1} \, dx} \\
&= z_1 e^{-\frac{1}{4} a x^2 - \frac{1}{2} bx} \\
&= z_1 \left(\mathrm{e}^{-\frac{x(ax+2b)}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-cx}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{ax+b}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{1}{2} a x^2 - bx}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\sqrt{\pi}\, \mathrm{e}^{\frac{(b-2c)^2}{2a}} \sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\,(ax+b-2c)}{2\sqrt{a}}\right)}{2\sqrt{a}} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-cx}\right) + c_2\left(\mathrm{e}^{-cx}\left(\frac{\sqrt{\pi}\,\mathrm{e}^{\frac{(b-2c)^2}{2a}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(ax+b-2c)}{2\sqrt{a}}\right)}{2\sqrt{a}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-cx} + \frac{c_2\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{-2acx+(b-2c)^2}{2a}}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(ax+b-2c)}{2\sqrt{a}}\right)}{2\sqrt{a}} \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^{-cx} + \frac{c_2\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{-2acx+(b-2c)^2}{2a}}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(ax+b-2c)}{2\sqrt{a}}\right)}{2\sqrt{a}}
$$

Verified OK.

27.16.2 Maple step by step solution

Let's solve

$y'' + (ax+b)\,y' + c(ax+b-c)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + a_1 b + a_0(bc - c^2) + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k+1}(k+1)\, b + a_k(ak + bc - c^2) + a_{k-1}ac\right.\right.$$

- Each term must be 0

$$2a_2 + a_1 b + a_0(bc - c^2) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+2} + (a a_k + a_{k+1} b + 3a_{k+2})\, k + 2a_{k+2} + (bc - c^2)\, a_k + a_{k-1}ac + a_{k+1}b = 0$$

- Shift index using $k- >k+1$

$$(k+1)^2 a_{k+3} + (a a_{k+1} + a_{k+2} b + 3a_{k+3})\,(k+1) + 2a_{k+3} + (bc - c^2)\, a_{k+1} + a_k ac + a_{k+2}b = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a_k ac + ak a_{k+1} + bc a_{k+1} + bk a_{k+2} - c^2 a_{k+1} + a a_{k+1} + 2a_{k+2} b}{k^2 + 5k + 6}, 2a_2 + a_1 b + a_0(bc - c^2) = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)+(a*x+b)*diff(y(x),x)+c*(a*x+b-c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-cx}\left(c_1 + \mathrm{erf}\left(\frac{\sqrt{2}\,(ax+b-2c)}{2\sqrt{a}}\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 70

```
DSolve[y''[x]+(a*x+b)*y'[x]+c*(a*x+b-c)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{1}{2}x(ax+2b-2c)}\left(c_1 \operatorname{HermiteH}\left(-1, \frac{b-2c+ax}{\sqrt{2}\sqrt{a}}\right) + c_2 e^{\frac{(ax+b-2c)^2}{2a}}\right)$$

27.17 problem 27

Internal problem ID [10851]
Internal file name [`OUTPUT/10107_Sunday_December_24_2023_05_12_22_PM_16279652/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + (ax + 2b)\, y' + \left(abx + b^2 - a\right) y = 0}$$

27.17.1 Solving using Kovacic algorithm

Writing the ode as

$$y'' + (ax + 2b)\, y' + \left(abx + b^2 - a\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= ax + 2b \\ C &= abx + b^2 - a \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a(a\,x^2 + 6)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a\left(a\,x^2 + 6\right) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a(a\,x^2 + 6)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 46: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{ax}{2} + \frac{3}{2x} - \frac{9}{4a\,x^3} + \frac{27}{4a^2x^5} - \frac{405}{16a^3x^7} + \frac{1701}{16a^4x^9} - \frac{15309}{32a^5x^{11}} + \frac{72171}{32a^6x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{ax}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{a^2x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{a(a\,x^2+6)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{3}{2}a + \frac{1}{4}a^2x^2\right) + (0) \\
&= \frac{3}{2}a + \frac{1}{4}a^2x^2
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{3a}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{3a}{2}\right) - (0) \\
&= \frac{3a}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{ax}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{3a}{2}}{\frac{a}{2}} - 1\right) &= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{3a}{2}}{\frac{a}{2}} - 1\right) &= -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a(a\,x^2+6)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{ax}{2}$	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{ax}{2}\right) \\ &= \frac{ax}{2} \\ &= \frac{ax}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{ax}{2}\right)(1) + \left(\left(\frac{a}{2}\right) + \left(\frac{ax}{2}\right)^2 - \left(\frac{a(a\,x^2+6)}{4}\right)\right) = 0$$
$$-aa_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, e^{\int \frac{ax}{2} dx} \\ &= (x) \, e^{\frac{a x^2}{4}} \\ &= x \, e^{\frac{a x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{ax+2b}{1} \, dx} \\ &= z_1 e^{-\frac{1}{4} a x^2 - bx} \\ &= z_1 \left(e^{-\frac{x(ax+4b)}{4}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = x \, e^{-bx}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{ax+2b}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{1}{2} a x^2 - 2bx}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{-\sqrt{a}\, \sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{\sqrt{2}\, \sqrt{a}\, x}{2}\right) x - 2 \, e^{-\frac{a x^2}{2}}}{2x} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-bx}\right) + c_2 \left(x\,\mathrm{e}^{-bx} \left(\frac{-\sqrt{a}\,\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{a}\,x}{2}\right) x - 2\,\mathrm{e}^{-\frac{a\,x^2}{2}}}{2x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-bx} - \frac{c_2 \mathrm{e}^{-bx} \left(\sqrt{a}\,\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{a}\,x}{2}\right) x + 2\,\mathrm{e}^{-\frac{a\,x^2}{2}}\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-bx} - \frac{c_2 \mathrm{e}^{-bx} \left(\sqrt{a}\,\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{a}\,x}{2}\right) x + 2\,\mathrm{e}^{-\frac{a\,x^2}{2}}\right)}{2}
$$

Verified OK.

27.17.2 Maple step by step solution

Let's solve

$y'' + (ax + 2b)\,y' + (abx + b^2 - a)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + 2a_1 b + a_0(b^2-a) + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_{k+1}(k+1)\, b + a_k(ak+b^2-a) + a_{k-1}ab\right.\right.$$

- Each term must be 0

$$2a_2 + 2a_1 b + a_0(b^2-a) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k b^2 + (a a_{k-1} + 2k a_{k+1} + 2a_{k+1})\, b + k^2 a_{k+2} + (a a_k + 3a_{k+2})\, k - a a_k + 2a_{k+2} = 0$$

- Shift index using $k->k+1$

$$a_{k+1} b^2 + (a a_k + 2(k+1)\, a_{k+2} + 2a_{k+2})\, b + (k+1)^2 a_{k+3} + (a a_{k+1} + 3a_{k+3})\,(k+1) - a a_{k+1} + 2a$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a_k ab + ak a_{k+1} + a_{k+1} b^2 + 2bk a_{k+2} + 4b a_{k+2}}{k^2+5k+6}, 2a_2 + 2a_1 b + a_0(b^2-a) = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 49

```
dsolve(diff(y(x),x$2)+(a*x+2*b)*diff(y(x),x)+(a*b*x-a+b^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = 2\,\mathrm{e}^{-\frac{x(ax+2b)}{2}} c_2 + \mathrm{e}^{-bx} x \left(c_2 \sqrt{a}\,\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{a}\,x}{2}\right) + c_1 \right)$$

✓ Solution by Mathematica
Time used: 0.405 (sec). Leaf size: 64

```
DSolve[y''[x]+(a*x+2*b)*y'[x]+(a*b*x-a+b^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to xe^{-bx} \left(-\sqrt{\frac{\pi}{2}} \sqrt{a} c_2 \operatorname{erf}\left(\frac{\sqrt{a}x}{\sqrt{2}}\right) - \frac{c_2 e^{-\frac{ax^2}{2}}}{x} + c_1 \right)$$

27.18 problem 28

Internal problem ID [10852]
Internal file name [OUTPUT/10108_Sunday_December_24_2023_05_12_23_PM_84918379/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$y'' + (ax + b)\, y' + (cx + d)\, y = 0$$

27.18.1 Maple step by step solution

Let's solve

$y'' + (ax + b)\, y' + (cx + d)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$a_1 b + a_0 d + 2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k+1}(k+1)\, b + a_k(ak+d) + a_{k-1} c\right) x^k\right) = 0$$

- Each term must be 0

$$a_1 b + a_0 d + 2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+2} + (a a_k + a_{k+1} b + 3a_{k+2})\, k + a_{k+1} b + a_{k-1} c + a_k d + 2a_{k+2} = 0$$

- Shift index using $k->k+1$

$$(k+1)^2 a_{k+3} + (a a_{k+1} + a_{k+2} b + 3a_{k+3})\,(k+1) + a_{k+2} b + a_k c + a_{k+1} d + 2a_{k+3} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{ak a_{k+1} + bk a_{k+2} + a a_{k+1} + 2a_{k+2} b + a_k c + a_{k+1} d}{k^2+5k+6}, a_1 b + a_0 d + 2a_2 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 98

```
dsolve(diff(y(x),x$2)+(a*x+b)*diff(y(x),x)+(c*x+d)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{cx}{a}} \left(\operatorname{KummerM}\left(\frac{d\,a^2 - abc + c^2}{2a^3}, \frac{1}{2}, -\frac{(a^2x + ab - 2c)^2}{2a^3} \right) c_1 + \operatorname{KummerU}\left(\frac{d\,a^2 - abc + c^2}{2a^3}, \frac{1}{2}, -\frac{(a^2x + ab - 2c)^2}{2a^3} \right) c_2 \right)$$

✓ Solution by Mathematica

Time used: 0.116 (sec). Leaf size: 132

```
DSolve[y''[x]+(a*x+b)*y'[x]+(c*x+d)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{cx}{a}-\frac{ax^2}{2}-bx}\left(c_2 \operatorname{Hypergeometric1F1}\left(\frac{a^3-da^2+bca-c^2}{2a^3},\frac{1}{2},\frac{(xa^2+ba-2c)^2}{2a^3}\right) + c_1 \operatorname{HermiteH}\left(\frac{-a^3+da^2-bca+c^2}{a^3},\frac{xa^2+ba-2c}{\sqrt{2}a^{3/2}}\right)\right)$$

27.19 problem 29

Internal problem ID [10853]
Internal file name [`OUTPUT/10109_Sunday_December_24_2023_05_12_23_PM_15034228/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y''+f(x)y'+g(x)y=0$
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$y'' + (ax + b)\, y' + c\big((a - c)\, x^2 + bx + 1\big)\, y = 0$$

27.19.1 Maple step by step solution

Let's solve

$y'' + (ax + b)\, y' + c\big((a - c)\, x^2 + bx + 1\big)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -c(a\, x^2 - c\, x^2 + bx + 1)\, y - (ax + b)\, y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + (ax + b)\, y' + c(a\, x^2 - c\, x^2 + bx + 1)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$a_1 b + a_0 c + 2a_2 + (6a_3 + 2a_2 b + a_1(a+c) + a_0 bc)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_{k+1}(k+1)\, b\right.$$

- The coefficients of each power of x must be 0

$$[2a_2 + a_1 b + a_0 c = 0, 6a_3 + 2a_2 b + a_1(a+c) + a_0 bc = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{a_1 b}{2} - \frac{a_0 c}{2}, a_3 = \frac{1}{6}a_1 b^2 - \frac{1}{6}a_1 a - \frac{1}{6}a_1 c\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-a_{k-2}c^2 + (a a_{k-2} + b a_{k-1} + a_k)\, c + k^2 a_{k+2} + (a a_k + a_{k+1} b + 3a_{k+2})\, k + a_{k+1} b + 2a_{k+2} = 0$$

- Shift index using $k->k+2$

$$-a_k c^2 + (a a_k + a_{k+1} b + a_{k+2})\, c + (k+2)^2 a_{k+4} + (a a_{k+2} + a_{k+3} b + 3a_{k+4})\,(k+2) + a_{k+3} b + 2a_k$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k ac + aka_{k+2} + bca_{k+1} + bka_{k+3} - a_k c^2 + 2aa_{k+2} + 3a_{k+3}b + ca_{k+2}}{k^2+7k+12}, a_2 = -\frac{a_1 b}{2} - \frac{a_0 c}{2}, a_3 = \frac{1}{6}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+(a*x+b)*diff(y(x),x)+c*((a-c)*x^2+b*x+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{c x^2}{2}} \left(c_1 + \operatorname{erf}\left(\frac{(-2c+a)\,x+b}{\sqrt{2a-4c}}\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.135 (sec). Leaf size: 81

```
DSolve[y''[x]+(a*x+b)*y'[x]+c*((a-c)*x^2+b*x+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{-\frac{1}{2}x(x(a-c)+2b)} \left(c_1 \operatorname{HermiteH}\left(-1, \frac{b+(a-2c)x}{\sqrt{2}\sqrt{a-2c}}\right) + c_2 e^{\frac{(x(a-2c)+b)^2}{2(a-2c)}}\right)$$

27.20 problem 30

Internal problem ID [10854]
Internal file name [`OUTPUT/10110_Sunday_December_24_2023_05_12_24_PM_53542549/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y''+f(x)y'+g(x)y=0$
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y''+2(ax+b)\,y'+\left(a^2x^2+2abx+c\right)y=0}$$

27.20.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=2ax+2b\\q(x)&=a^2x^2+2abx+c\end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= a^2x^2 + 2abx + c - \frac{(2ax+2b)'}{2} - \frac{(2ax+2b)^2}{4} \\
&= a^2x^2 + 2abx + c - \frac{(2a)}{2} - \frac{\left((2ax+2b)^2\right)}{4} \\
&= a^2x^2 + 2abx + c - (a) - \frac{(2ax+2b)^2}{4} \\
&= -b^2 - a + c
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{2ax+2b}{2}} \\
&= \mathrm{e}^{-\frac{x(ax+2b)}{2}}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, \mathrm{e}^{-\frac{x(ax+2b)}{2}} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
-\mathrm{e}^{-\frac{x(ax+2b)}{2}} \left(v(x)\, b^2 + av(x) - v(x)\, c - v''(x)\right) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = -1, B = 0, C = b^2 + a - c$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
-\lambda^2 \mathrm{e}^{\lambda x} + \left(b^2 + a - c\right) \mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$b^2 - \lambda^2 + a - c = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = -1, B = 0, C = b^2 + a - c$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(-1)} \pm \frac{1}{(2)(-1)}\sqrt{0^2 - (4)(-1)(b^2 + a - c)} \\ &= \pm - \sqrt{b^2 + a - c}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= + - \sqrt{b^2 + a - c} \\ \lambda_2 &= - - \sqrt{b^2 + a - c}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\sqrt{b^2 + a - c} \\ \lambda_2 &= \sqrt{b^2 + a - c}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ v(x) &= c_1 e^{\left(-\sqrt{b^2+a-c}\right)x} + c_2 e^{\left(\sqrt{b^2+a-c}\right)x}\end{aligned}$$

Or

$$v(x) = c_1 \mathrm{e}^{-\sqrt{b^2+a-c}\,x} + c_2 \mathrm{e}^{\sqrt{b^2+a-c}\,x}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x) \\ &= \left(c_1 \mathrm{e}^{-\sqrt{b^2+a-c}\,x} + c_2 \mathrm{e}^{\sqrt{b^2+a-c}\,x}\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \mathrm{e}^{-\frac{x(ax+2b)}{2}}$$

Hence (7) becomes

$$y = \left(c_1 \mathrm{e}^{-\sqrt{b^2+a-c}\,x} + c_2 \mathrm{e}^{\sqrt{b^2+a-c}\,x}\right) \mathrm{e}^{-\frac{x(ax+2b)}{2}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(c_1 \mathrm{e}^{-\sqrt{b^2+a-c}\,x} + c_2 \mathrm{e}^{\sqrt{b^2+a-c}\,x}\right) \mathrm{e}^{-\frac{x(ax+2b)}{2}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \left(c_1 \mathrm{e}^{-\sqrt{b^2+a-c}\,x} + c_2 \mathrm{e}^{\sqrt{b^2+a-c}\,x}\right) \mathrm{e}^{-\frac{x(ax+2b)}{2}}$$

Verified OK.

27.20.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + (2ax + 2b)\, y' + \left(a^2x^2 + 2abx + c\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2ax + 2b \\ C &= a^2x^2 + 2abx + c \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{b^2 + a - c}{1} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= b^2 + a - c \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(b^2 + a - c\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 50: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = b^2 + a - c$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{b^2+a-c}\,x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2ax+2b}{1}\,dx} \\
&= z_1 e^{-\frac{1}{2}a\,x^2 - bx} \\
&= z_1 \left(\mathrm{e}^{-\frac{x(ax+2b)}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x\left(ax-2\sqrt{b^2+a-c}+2b\right)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2ax+2b}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-a\,x^2-2bx}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2\sqrt{b^2+a-c}\,x}}{2\sqrt{b^2+a-c}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x\left(ax-2\sqrt{b^2+a-c}+2b\right)}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{x\left(ax-2\sqrt{b^2+a-c}+2b\right)}{2}} \left(-\frac{\mathrm{e}^{-2\sqrt{b^2+a-c}\,x}}{2\sqrt{b^2+a-c}}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{x\left(ax-2\sqrt{b^2+a-c}+2b\right)}{2}} - \frac{c_2 \mathrm{e}^{-\frac{x\left(ax+2\sqrt{b^2+a-c}+2b\right)}{2}}}{2\sqrt{b^2+a-c}} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{x\left(ax-2\sqrt{b^2+a-c}+2b\right)}{2}} - \frac{c_2 \mathrm{e}^{-\frac{x\left(ax+2\sqrt{b^2+a-c}+2b\right)}{2}}}{2\sqrt{b^2+a-c}}$$

Verified OK.

27.20.3 Maple step by step solution

Let's solve

$y'' + (2ax + 2b)\, y' + (a^2x^2 + 2abx + c)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ∘ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ∘ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ∘ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  ∘ Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k + 1 - m)\, x^k$

  ∘ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_1b + a_0c + 2a_2 + (6a_3 + 4a_2b + a_1(2a + c) + 2a_0ab)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + 2a_{k+1}(k +$$

- The coefficients of each power of x must be 0

$$[2a_2 + 2a_1b + a_0c = 0, 6a_3 + 4a_2b + a_1(2a + c) + 2a_0ab = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -a_1b - \frac{a_0c}{2}, a_3 = -\frac{1}{3}a_0ab + \frac{2}{3}a_1b^2 + \frac{1}{3}a_0bc - \frac{1}{3}a_1a - \frac{1}{6}a_1c\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$k^2a_{k+2} + (2aa_k + 2a_{k+1}b + 3a_{k+2})\, k + a_{k-2}a^2 + 2a_{k-1}ab + 2a_{k+1}b + a_kc + 2a_{k+2} = 0$$

- Shift index using $k- >k+2$

$$(k+2)^2\, a_{k+4} + (2aa_{k+2} + 2a_{k+3}b + 3a_{k+4})\,(k+2) + a_ka^2 + 2a_{k+1}ab + 2a_{k+3}b + a_{k+2}c + 2a_{k+4} =$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_kx^k, a_{k+4} = -\frac{a_ka^2 + 2a_{k+1}ab + 2aka_{k+2} + 2bka_{k+3} + 4aa_{k+2} + 6a_{k+3}b + a_{k+2}c}{k^2+7k+12}, a_2 = -a_1b - \frac{a_0c}{2}, a_3 = -\frac{1}{3}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve(diff(y(x),x$2)+2*(a*x+b)*diff(y(x),x)+(a^2*x^2+2*a*b*x+c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{x\left(ax-2\sqrt{b^2+a-c}+2b\right)}{2}} + c_2 \mathrm{e}^{-\frac{x\left(ax+2\sqrt{b^2+a-c}+2b\right)}{2}}$$

✓ Solution by Mathematica

Time used: 0.228 (sec). Leaf size: 86

```
DSolve[y''[x]+2*(a*x+b)*y'[x]+(a^2*x^2+2*a*b*x+c)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{e^{-\frac{1}{2}x\left(2\sqrt{a+b^2-c}+ax+2b\right)}\left(c_2 e^{2x\sqrt{a+b^2-c}} + 2c_1\sqrt{a+b^2-c}\right)}{2\sqrt{a+b^2-c}}$$

27.21 problem 31

Internal problem ID [10855]
Internal file name [`OUTPUT/10111_Sunday_December_24_2023_05_12_32_PM_77209391/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$y'' + (ax + b)\, y' + \left(\alpha\, x^2 + \beta x + \gamma\right) y = 0$$

27.21.1 Maple step by step solution

Let's solve

$y'' + (ax + b)\, y' + (\alpha\, x^2 + \beta x + \gamma)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$a_1 b + a_0\gamma + 2a_2 + (6a_3 + 2a_2 b + a_1(a+\gamma) + a_0\beta)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_{k+1}(k+1)\, b$$

- The coefficients of each power of x must be 0

  $$[2a_2 + a_1 b + a_0\gamma = 0, 6a_3 + 2a_2 b + a_1(a+\gamma) + a_0\beta = 0]$$

- Solve for the dependent coefficient(s)

  $$\left\{a_2 = -\frac{a_1 b}{2} - \frac{a_0\gamma}{2}, a_3 = \frac{1}{6}a_1 b^2 + \frac{1}{6}a_0 b\gamma - \frac{1}{6}a_1 a - \frac{1}{6}a_0\beta - \frac{1}{6}a_1\gamma\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$k^2 a_{k+2} + (a a_k + a_{k+1} b + 3a_{k+2})\, k + a_{k+1} b + a_{k-1}\beta + a_{k-2}\alpha + a_k\gamma + 2a_{k+2} = 0$$

- Shift index using $k- >k+2$

  $$(k+2)^2\, a_{k+4} + (a a_{k+2} + a_{k+3} b + 3a_{k+4})\,(k+2) + a_{k+3} b + a_{k+1}\beta + a_k\alpha + a_{k+2}\gamma + 2a_{k+4} = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{aka_{k+2} + bka_{k+3} + 2aa_{k+2} + a_k\alpha + 3a_{k+3}b + a_{k+1}\beta + a_{k+2}\gamma}{k^2+7k+12}, a_2 = -\frac{a_1 b}{2} - \frac{a_0\gamma}{2}, a_3 = \frac{1}{6}a_1 b^2\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: indirect Equivalence to 0F1 under \`\`^ @ Moebius\`\` is resolve
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 254

```
dsolve(diff(y(x),x$2)+(a*x+b)*diff(y(x),x)+(alpha*x^2+beta*x+gamma)*y(x)=0,y(x), singsol=all
```

$$y(x) = \mathrm{e}^{-\frac{x\left((ax+2b)\sqrt{a^2-4\alpha}+x\left(a^2-4\alpha\right)+2ab-4\beta\right)}{4\sqrt{a^2-4\alpha}}}\left(c_2\left(a^2x+ab-4\alpha x\right.\right.$$
$$\left.-2\beta\right)\mathrm{hypergeom}\left(\left[\frac{3(a^2-4\alpha)^{\frac{3}{2}}+a^3-2a^2\gamma+2(b\beta-2\alpha)a+2(-b^2+4\gamma)\alpha-2\beta^2}{4\left(a^2-4\alpha\right)^{\frac{3}{2}}}\right],\left[\frac{3}{2}\right],\frac{(a^2x+a}{2(}\right.$$
$$+\mathrm{hypergeom}\left(\left[\frac{(a^2-4\alpha)^{\frac{3}{2}}+a^3-2a^2\gamma+(2b\beta-4\alpha)a+(-2b^2+8\gamma)\alpha-2\beta^2}{4\left(a^2-4\alpha\right)^{\frac{3}{2}}}\right],\left[\frac{1}{2}\right],\frac{(a^2x+ab-4}{2(a^2-}\right.$$

✓ Solution by Mathematica

Time used: 0.467 (sec). Leaf size: 307

```
DSolve[y''[x]+(a*x+b)*y'[x]+(\[Alpha]*x^2+\[Beta]*x+\[Gamma])*y[x]==0,y[x],x,IncludeSingular
```

$y(x)$

$$\rightarrow \exp\left(-\frac{x\left(2b\sqrt{a^2-4\alpha}+a\left(x\sqrt{a^2-4\alpha}+2b\right)+a^2x-4(\beta+\alpha x)\right)}{4\sqrt{a^2-4\alpha}}\right)\left(c_1\,\text{HermiteH}\left(\frac{-a^3-\left(\sqrt{a^2-4\alpha\right.}}{}\right.\right.$$

27.22 problem 32

Internal problem ID [10856]
Internal file name [`OUTPUT/10112_Sunday_December_24_2023_05_12_34_PM_89403799/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + (ax + b)\, y' + c\left(-c\, x^{2n} + a\, x^{1+n} + b\, x^{n} + n\, x^{n-1}\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+(a*x+b)*diff(y(x),x)+c*(-c*x^(2*n)+a*x^(n+1)+b*x^n+n*x^(n-1) )*y(x)=0,
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+(a*x+b)*y'[x]+c*(-c*x^(2*n)+a*x^(n+1)+b*x^n+n*x^(n-1) )*y[x]==0,y[x],x,Include
```

Not solved

27.23 problem 33

Internal problem ID [10857]
Internal file name [`OUTPUT/10113_Sunday_December_24_2023_05_12_34_PM_15390749/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + a\left(-b^2 + x^2\right) y' - a(x + b)\, y = 0}$$

27.23.1 Solving using Kovacic algorithm

Writing the ode as

$$y'' + a\left(-b^2 + x^2\right) y' - a(x + b)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \left(-b^2 + x^2\right) a \\ C &= -a(x + b) \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a(a\,b^4 - 2a\,b^2x^2 + a\,x^4 + 4b + 8x)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a\left(a\,b^4 - 2a\,b^2x^2 + a\,x^4 + 4b + 8x\right) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a(a\,b^4 - 2a\,b^2x^2 + a\,x^4 + 4b + 8x)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 53: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx -\frac{a\,b^2}{2} + \frac{a\,x^2}{2} + \frac{b^5}{x^6} + \frac{2b^4}{x^5} + \frac{b^3}{x^4} + \frac{2b^2}{x^3} + \frac{b}{x^2} + \frac{2}{x} - \frac{13b^2}{a\,x^6} - \frac{4b}{a\,x^5} - \frac{4}{a\,x^4} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= -\frac{1}{2} a\,b^2 + \frac{1}{2} a\,x^2 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}a^2b^4 - \frac{1}{2}a^2b^2x^2 + \frac{1}{4}a^2x^4$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{a(a\,b^4 - 2a\,b^2x^2 + a\,x^4 + 4b + 8x)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{a^2x^4}{4} - \frac{a^2b^2x^2}{2} + 2ax + \frac{ab(b^3a+4)}{4}\right) + (0) \\
&= \frac{a^2x^4}{4} - \frac{a^2b^2x^2}{2} + 2ax + \frac{ab(b^3a+4)}{4}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $2a$. Now b can be found.

$$\begin{aligned}
b &= (2a) - (0) \\
&= 2a
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= -\frac{1}{2}a\,b^2 + \frac{1}{2}a\,x^2 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2a}{\frac{a}{2}} - 2\right) &&= 1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2a}{\frac{a}{2}} - 2\right) &&= -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a(a\,b^4 - 2a\,b^2x^2 + a\,x^4 + 4b + 8x)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$-\frac{1}{2}a\,b^2+\frac{1}{2}a\,x^2$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=1$, and since there are no poles, then

$$\begin{aligned}d&=\alpha_\infty^+\\&=1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega&=(+)[\sqrt{r}]_\infty\\&=0+\left(-\frac{1}{2}a\,b^2+\frac{1}{2}a\,x^2\right)\\&=-\frac{1}{2}a\,b^2+\frac{1}{2}a\,x^2\\&=-\frac{a(b^2-x^2)}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2}a\,b^2+\frac{1}{2}a\,x^2\right)(1)+\left((ax)+\left(-\frac{1}{2}a\,b^2+\frac{1}{2}a\,x^2\right)^2-\left(\frac{a(a\,b^4-2a\,b^2x^2+a\,x^4+4b+8x)}{4}\right)\right)=$$

$$-a(a_0+b)\,(x+b)=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -b\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - b$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x-b)\, e^{\int \left(-\frac{1}{2} a\, b^2 + \frac{1}{2} a\, x^2\right) dx} \\ &= (x-b)\, e^{-\frac{a\left(b^2 x - \frac{1}{3} x^3\right)}{2}} \\ &= -(-x+b)\, e^{-\frac{a x \left(b^2 - \frac{x^2}{3}\right)}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{\left(-b^2 + x^2\right) a}{1} \, dx} \\ &= z_1 e^{-\frac{a\left(-b^2 x + \frac{1}{3} x^3\right)}{2}} \\ &= z_1 \left(e^{\frac{a x \left(3 b^2 - x^2\right)}{6}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = x - b$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{\left(-b^2 + x^2\right) a}{1} \, dx}}{\left(y_1\right)^2} \, dx \\ &= y_1 \int \frac{e^{a\, b^2 x - \frac{1}{3} a\, x^3}}{\left(y_1\right)^2} \, dx \\ &= y_1 \left(\int \frac{e^{a\, b^2 x - \frac{1}{3} a\, x^3}}{\left(-x+b\right)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x-b) + c_2\left(x - b\left(\int \frac{e^{a\,b^2 x - \frac{1}{3} a\,x^3}}{(-x+b)^2} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x-b) - c_2(-x+b)\left(\int \frac{e^{a\,b^2 x - \frac{1}{3} a\,x^3}}{(-x+b)^2} dx\right) \tag{1}
$$

Verification of solutions

$$
y = c_1(x-b) - c_2(-x+b)\left(\int \frac{e^{a\,b^2 x - \frac{1}{3} a\,x^3}}{(-x+b)^2} dx\right)
$$

Verified OK.

27.23.2 Maple step by step solution

Let's solve

$y'' + a(-b^2 + x^2)\, y' - a(x+b)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = a(x+b)\, y + a(b^2 - x^2)\, y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - a(b^2 - x^2)\, y' - a(x+b)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- Shift index using $k->k-m$

$$x^{m}\cdot y=\sum_{k=\max(0,-m)+m}^{\infty}a_{k-m}x^{k}$$

- Convert $x^{m}\cdot y'$ to series expansion for $m=0..2$

$$x^{m}\cdot y'=\sum_{k=\max(0,1-m)}^{\infty}a_{k}k\,x^{k-1+m}$$

- Shift index using $k->k+1-m$

$$x^{m}\cdot y'=\sum_{k=\max(0,1-m)+m-1}^{\infty}a_{k+1-m}(k+1-m)\,x^{k}$$

- Convert y'' to series expansion

$$y''=\sum_{k=2}^{\infty}a_{k}k(k-1)\,x^{k-2}$$

- Shift index using $k->k+2$

$$y''=\sum_{k=0}^{\infty}a_{k+2}(k+2)\,(k+1)\,x^{k}$$

Rewrite ODE with series expansions

$$-a_{1}a\,b^{2}-aa_{0}b+2a_{2}+\left(\sum_{k=1}^{\infty}\left(a_{k+2}(k+2)\,(k+1)-a_{k+1}(k+1)\,a\,b^{2}-aa_{k}b+a_{k-1}a(k-2)\right)x^{k}\right)$$

- Each term must be 0

$$-a_{1}a\,b^{2}-aa_{0}b+2a_{2}=0$$

- Each term in the series must be 0, giving the recursion relation

$$\left(\left(-b^{2}a_{k+1}+a_{k-1}\right)k-b^{2}a_{k+1}-ba_{k}-2a_{k-1}\right)a+a_{k+2}(k+2)\,(k+1)=0$$

- Shift index using $k->k+1$

$$\left(\left(-b^{2}a_{k+2}+a_{k}\right)(k+1)-b^{2}a_{k+2}-ba_{k+1}-2a_{k}\right)a+a_{k+3}(k+3)\,(k+2)=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty}a_{k}x^{k},a_{k+3}=\frac{\left(b^{2}ka_{k+2}+2b^{2}a_{k+2}+ba_{k+1}-a_{k}k+a_{k}\right)a}{(k+3)(k+2)},-a_{1}a\,b^{2}-aa_{0}b+2a_{2}=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.297 (sec). Leaf size: 141

```
dsolve(diff(y(x),x$2)+a*(x^2-b^2)*diff(y(x),x)-a*(x+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{HeunT}\left(-\frac{a 3^{\frac{2}{3}} b}{(a^2)^{\frac{1}{3}}}, -6 \operatorname{csgn}(a), -\frac{a^2 b^2 3^{\frac{1}{3}}}{(a^2)^{\frac{2}{3}}}, \frac{3^{\frac{2}{3}} (a^2)^{\frac{1}{6}} x}{3}\right) e^{\frac{x(3b^2-x^2)\operatorname{csgn}(a) a(\operatorname{csgn}(a)+1)}{6}} + c_2 \operatorname{HeunT}\left(-\frac{a 3^{\frac{2}{3}} b}{(a^2)^{\frac{1}{3}}}, 6 \operatorname{csgn}(a), -\frac{a^2 b^2 3^{\frac{1}{3}}}{(a^2)^{\frac{2}{3}}}, -\frac{3^{\frac{2}{3}} (a^2)^{\frac{1}{6}} x}{3}\right) e^{\frac{x(3b^2-x^2)\operatorname{csgn}(a) a(\operatorname{csgn}(a)-1)}{6}}$$

✓ Solution by Mathematica

Time used: 3.893 (sec). Leaf size: 55

```
DSolve[y''[x]+a*(x^2-b^2)*y'[x]-a*(x+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(b-x)\left(c_2 \int_1^x \frac{e^{ab^2 K[1]-\frac{1}{3} a K[1]^3}}{(b-K[1])^2} dK[1] + c_1\right)}{b}$$

27.24 problem 34

Internal problem ID [10858]
Internal file name [`OUTPUT/10114_Sunday_December_24_2023_05_12_35_PM_24279716/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \left(a\,x^2 + b\right) y' + c\left(a\,x^2 + b - c\right) y = 0}$$

27.24.1 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \left(a\,x^2 + b\right) y' + c\left(a\,x^2 + b - c\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= a\,x^2 + b \\ C &= c\left(a\,x^2 + b - c\right) \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^4 + 2ab\,x^2 - 4ac\,x^2 + 4ax + b^2 - 4bc + 4c^2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^4 + 2ab\,x^2 - 4ac\,x^2 + 4ax + b^2 - 4bc + 4c^2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(ax + \frac{1}{4}a^2x^4 + \frac{1}{2}ab\,x^2 + \frac{1}{4}b^2 - ac\,x^2 - bc + c^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 55: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{a\,x^2}{2} + \frac{b}{2} - c + \frac{1}{x} - \frac{b}{a\,x^3} + \frac{2c}{a\,x^3} - \frac{1}{a\,x^4} + \frac{b^2}{a^2x^5} - \frac{4bc}{a^2x^5} + \frac{4c^2}{a^2x^5} + \frac{3b}{a^2x^6} - \frac{6c}{a^2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{b}{2} - c + \frac{a\,x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}b^2 - bc + \frac{1}{2}ab\,x^2 + c^2 - ac\,x^2 + \frac{1}{4}a^2x^4$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{a^2x^4 + 2ab\,x^2 - 4ac\,x^2 + 4ax + b^2 - 4bc + 4c^2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{a^2x^4}{4} + \left(\frac{1}{2}ab - ac\right)x^2 + ax + \frac{b^2}{4} - bc + c^2\right) + (0) \\
&= \frac{a^2x^4}{4} + \left(\frac{1}{2}ab - ac\right)x^2 + ax + \frac{b^2}{4} - bc + c^2
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is a. Now b can be found.

$$\begin{aligned}
b &= (a) - (0) \\
&= a
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{b}{2} - c + \frac{a\,x^2}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{a}{\frac{a}{2}} - 2\right) = 0 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{a}{\frac{a}{2}} - 2\right) = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = ax + \frac{1}{4}a^2x^4 + \frac{1}{2}ab\,x^2 + \frac{1}{4}b^2 - ac\,x^2 - bc + c^2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{b}{2}-c+\frac{a\,x^2}{2}$	0	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=0$, and since there are no poles, then

$$\begin{aligned}d&=\alpha_\infty^+\\&=0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega&=(+)[\sqrt{r}]_\infty\\&=0+\left(\frac{b}{2}-c+\frac{a\,x^2}{2}\right)\\&=\frac{b}{2}-c+\frac{a\,x^2}{2}\\&=\frac{b}{2}-c+\frac{a\,x^2}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{b}{2}-c+\frac{a\,x^2}{2}\right)(0)+\left((ax)+\left(\frac{b}{2}-c+\frac{a\,x^2}{2}\right)^2-\left(ax+\frac{1}{4}a^2x^4+\frac{1}{2}ab\,x^2+\frac{1}{4}b^2-ac\,x^2-bc+c\right.\right.$$

0

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{b}{2} - c + \frac{a\,x^2}{2}\right) dx} \\
&= \mathrm{e}^{\frac{x\left(a\,x^2 + 3b - 6c\right)}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{a\,x^2 + b}{1} \, dx} \\
&= z_1 e^{-\frac{1}{6} a\,x^3 - \frac{1}{2} b x} \\
&= z_1 \left(\mathrm{e}^{-\frac{x\left(a\,x^2 + 3b\right)}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-cx}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{a\,x^2 + b}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{1}{3} a\,x^3 - bx}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \mathrm{e}^{-\frac{x\left(a\,x^2 + 3b - 6c\right)}{3}} \, dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-cx}\right) + c_2 \left(\mathrm{e}^{-cx} \left(\int \mathrm{e}^{-\frac{x\left(a\,x^2 + 3b - 6c\right)}{3}} \, dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-cx} + c_2 e^{-cx} \left(\int e^{-\frac{x(a\,x^2+3b-6c)}{3}} dx \right) \tag{1}$$

Verification of solutions

$$y = c_1 e^{-cx} + c_2 e^{-cx} \left(\int e^{-\frac{x(a\,x^2+3b-6c)}{3}} dx \right)$$

Verified OK.

27.24.2 Maple step by step solution

Let's solve

$y'' + (a\,x^2 + b)\,y' + c(a\,x^2 + b - c)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  ◦ Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k + 1 - m)\, x^k$

  ◦ Convert y'' to series expansion

$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$

  Rewrite ODE with series expansions

  $2a_2 + a_1 b + c a_0(b-c) + (6a_3 + 2a_2 b + c a_1(b-c))\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_{k+1}(k+1)\, b\right.$

- The coefficients of each power of x must be 0

  $[2a_2 + a_1 b + c a_0(b-c) = 0, 6a_3 + 2a_2 b + c a_1(b-c) = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = -\frac{1}{2}a_0 bc + \frac{1}{2}a_0 c^2 - \frac{1}{2}a_1 b, a_3 = \frac{1}{6}a_0 b^2 c - \frac{1}{6}a_0 b\, c^2 + \frac{1}{6}a_1 b^2 - \frac{1}{6}a_1 bc + \frac{1}{6}a_1 c^2\right\}$

- Each term in the series must be 0, giving the recursion relation

  $k^2 a_{k+2} + (a a_{k-1} + a_{k+1} b + 3a_{k+2})\, k + 2a_{k+2} + (c a_{k-2} - a_{k-1})\, a + (c a_k + a_{k+1})\, b - a_k c^2 = 0$

- Shift index using $k- >k+2$

  $(k+2)^2\, a_{k+4} + (a a_{k+1} + b a_{k+3} + 3a_{k+4})\,(k+2) + 2a_{k+4} + (c a_k - a_{k+1})\, a + (c a_{k+2} + a_{k+3})\, b - a$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k a c + a k a_{k+1} + b c a_{k+2} + b k a_{k+3} - a_{k+2} c^2 + a a_{k+1} + 3b a_{k+3}}{k^2+7k+12}, a_2 = -\frac{1}{2}a_0 bc + \frac{1}{2}a_0 c^2 - \frac{1}{2}a_1 b\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 134

```
dsolve(diff(y(x),x$2)+(a*x^2+b)*diff(y(x),x)+c*(a*x^2+b-c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{HeunT}\left(0, -3\operatorname{csgn}(a), \frac{a(b-2c)\,3^{\frac{1}{3}}}{(a^2)^{\frac{2}{3}}}, \frac{3^{\frac{2}{3}}(a^2)^{\frac{1}{6}}x}{3}\right) e^{-\frac{x\left(\left(a\,x^2+3b\right)\operatorname{csgn}(a)+a\,x^2+3b-6c\right)\operatorname{csgn}(a)}{6}} + c_2 \operatorname{HeunT}\left(0, 3\operatorname{csgn}(a), \frac{a(b-2c)\,3^{\frac{1}{3}}}{(a^2)^{\frac{2}{3}}}, -\frac{3^{\frac{2}{3}}(a^2)^{\frac{1}{6}}x}{3}\right) e^{-\frac{x\left(\left(a\,x^2+3b\right)\operatorname{csgn}(a)-a\,x^2-3b+6c\right)\operatorname{csgn}(a)}{6}}$$

✓ Solution by Mathematica

Time used: 0.915 (sec). Leaf size: 46

```
DSolve[y''[x]+(a*x^2+b)*y'[x]+c*(a*x^2+b-c)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-cx}\left(c_2 \int_1^x e^{-\frac{1}{3}K[1]\left(aK[1]^2+3b-6c\right)}dK[1] + c_1\right)$$

27.25 problem 35

Internal problem ID [10859]
Internal file name [`OUTPUT/10115_Sunday_December_24_2023_05_12_36_PM_4955882/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \left(a\,x^2 + 2b\right) y' + \left(ab\,x^2 - ax + b^2\right) y = 0}$$

27.25.1 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \left(a\,x^2 + 2b\right) y' + \left(\left(b\,x^2 - x\right) a + b^2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= a\,x^2 + 2b \\ C &= \left(b\,x^2 - x\right) a + b^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{ax(a\,x^3 + 8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= ax\left(a\,x^3 + 8\right) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{ax(a\,x^3 + 8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 57: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 4 \\
&= -4
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{2} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{a\,x^2}{2} + \frac{2}{x} - \frac{4}{a\,x^4} + \frac{16}{a^2 x^7} - \frac{80}{a^3 x^{10}} + \frac{448}{a^4 x^{13}} - \frac{2688}{a^5 x^{16}} + \frac{16896}{a^6 x^{19}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\
&= \frac{a\,x^2}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{a^2x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{ax(a\,x^3 + 8)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(2ax + \frac{1}{4}a^2x^4\right) + (0) \\
&= 2ax + \frac{1}{4}a^2x^4
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $2a$. Now b can be found.

$$\begin{aligned}
b &= (2a) - (0) \\
&= 2a
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{a\,x^2}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2a}{\frac{a}{2}} - 2\right) &= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2a}{\frac{a}{2}} - 2\right) &= -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{ax(a\,x^3 + 8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{a\,x^2}{2}$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{a\,x^2}{2}\right) \\ &= \frac{a\,x^2}{2} \\ &= \frac{a\,x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{a\,x^2}{2}\right)(1) + \left((ax) + \left(\frac{a\,x^2}{2}\right)^2 - \left(\frac{ax(a\,x^3+8)}{4}\right)\right) &= 0 \\ -axa_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x)\, \mathrm{e}^{\int \frac{a\,x^2}{2} dx} \\ &= (x)\, \mathrm{e}^{\frac{a\,x^3}{6}} \\ &= x\, \mathrm{e}^{\frac{a\,x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{a\,x^2+2b}{1}\, dx} \\ &= z_1 e^{-\frac{1}{6}a\,x^3 - bx} \\ &= z_1 \left(\mathrm{e}^{-\frac{x\left(a\,x^2+6b\right)}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-bx}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{a\,x^2+2b}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{1}{3}a\,x^3 - 2bx}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{a\,x^3}{3}}}{x^2} dx\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x\,\mathrm{e}^{-bx}\right) + c_2\left(x\,\mathrm{e}^{-bx}\left(\int \frac{\mathrm{e}^{-\frac{a\,x^3}{3}}}{x^2}dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1x\,\mathrm{e}^{-bx} + c_2x\,\mathrm{e}^{-bx}\left(\int \frac{\mathrm{e}^{-\frac{a\,x^3}{3}}}{x^2}dx\right) \tag{1}
$$

Verification of solutions

$$
y = c_1x\,\mathrm{e}^{-bx} + c_2x\,\mathrm{e}^{-bx}\left(\int \frac{\mathrm{e}^{-\frac{a\,x^3}{3}}}{x^2}dx\right)
$$

Verified OK.

27.25.2 Maple step by step solution

Let's solve

$y'' + (a\,x^2 + 2b)\,y' + ((b\,x^2 - x)\,a + b^2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-ab\,x^2 + ax - b^2)\,y - (a\,x^2 + 2b)\,y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + (a\,x^2 + 2b)\,y' + (ab\,x^2 - ax + b^2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_kx^k$

□ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_kx^{k+m}$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$$

Rewrite ODE with series expansions

$$a_0 b^2 + 2a_1 b + 2a_2 + (a_1 b^2 - a_0 a + 4a_2 b + 6a_3)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) + 2a_{k+1}(k+1)\, b +\right.$$

- The coefficients of each power of x must be 0

$$[2a_2 + a_0 b^2 + 2a_1 b = 0, a_1 b^2 - a_0 a + 4a_2 b + 6a_3 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{1}{2}a_0 b^2 - a_1 b, a_3 = \frac{1}{3}a_0 b^3 + \frac{1}{2}a_1 b^2 + \frac{1}{6}a_0 a\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k b^2 + (a a_{k-2} + 2k a_{k+1} + 2a_{k+1})\, b + k^2 a_{k+2} + (a_{k-1} a + 3a_{k+2})\, k - 2a_{k-1} a + 2a_{k+2} = 0$$

- Shift index using $k->k+2$

$$a_{k+2} b^2 + (a_k a + 2(k+2)\, a_{k+3} + 2a_{k+3})\, b + (k+2)^2\, a_{k+4} + (a_{k+1} a + 3a_{k+4})\,(k+2) - 2a_{k+1} a + 2$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k a b + a k a_{k+1} + a_{k+2} b^2 + 2bk a_{k+3} + 6b a_{k+3}}{k^2 + 7k + 12}, a_2 = -\frac{1}{2}a_0 b^2 - a_1 b, a_3 = \frac{1}{3}a_0 b^3 + \frac{1}{2}a_1 b^2\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 91

```
dsolve(diff(y(x),x$2)+(a*x^2+2*b)*diff(y(x),x)+(a*b*x^2-a*x+b^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{5\left(3^{\frac{2}{3}}c_2a(a\,x^3)^{\frac{1}{3}}\,(a\,x^3+2)\,\mathrm{e}^{-\frac{x\left(a\,x^2+6b\right)}{6}} + \frac{9x^2\left(c_2a^2x\,\mathrm{e}^{-bx}\,\mathrm{WhittakerM}\left(\frac{1}{3},\frac{5}{6},\frac{a\,x^3}{3}\right)+c_1\mathrm{e}^{\frac{x\left(a\,x^2-6b\right)}{6}}\right)}{5}\right)\mathrm{e}^{-\frac{a\,x^3}{6}}}{9x}$$

✓ Solution by Mathematica

Time used: 0.407 (sec). Leaf size: 51

```
DSolve[y''[x]+(a*x^2+2*b)*y'[x]+(a*b*x^2-a*x+b^2)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{9}e^{-bx}\left(9c_1x - 3^{2/3}c_2\sqrt[3]{ax^3}\Gamma\left(-\frac{1}{3},\frac{ax^3}{3}\right)\right)$$

27.26 problem 36

Internal problem ID [10860]
Internal file name [`OUTPUT/10116_Sunday_December_24_2023_05_12_36_PM_17868219/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \left(2x^2 + a\right) y' + \left(x^4 + a\,x^2 + b + 2x\right) y = 0}$$

27.26.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= 2x^2 + a\\ q(x) &= x^4 + a\,x^2 + b + 2x\end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= x^4 + a\,x^2 + b + 2x - \frac{(2x^2+a)'}{2} - \frac{(2x^2+a)^2}{4} \\
&= x^4 + a\,x^2 + b + 2x - \frac{(4x)}{2} - \frac{\left((2x^2+a)^2\right)}{4} \\
&= x^4 + a\,x^2 + b + 2x - (2x) - \frac{(2x^2+a)^2}{4} \\
&= b - \frac{a^2}{4}
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\,z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2}dx\right)} \\
&= e^{-\int \frac{2x^2+a}{2}} \\
&= \mathrm{e}^{-\frac{1}{3}x^3-\frac{1}{2}ax}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\,\mathrm{e}^{-\frac{1}{3}x^3-\frac{1}{2}ax} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
-\mathrm{e}^{-\frac{x(2x^2+3a)}{6}}\left(v(x)\,a^2 - 4v(x)\,b - 4v''(x)\right) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = -4, B = 0, C = a^2 - 4b$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
-4\lambda^2\mathrm{e}^{\lambda x} + \left(a^2 - 4b\right)\mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$a^2 - 4\lambda^2 - 4b = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = -4, B = 0, C = a^2 - 4b$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(-4)} \pm \frac{1}{(2)(-4)}\sqrt{0^2 - (4)(-4)(a^2 - 4b)}\\ &= \pm - \frac{\sqrt{a^2 - 4b}}{2}\end{aligned}$$

Hence

$$\lambda_1 = + - \frac{\sqrt{a^2 - 4b}}{2}$$

$$\lambda_2 = - - \frac{\sqrt{a^2 - 4b}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{\sqrt{a^2 - 4b}}{2}$$

$$\lambda_2 = \frac{\sqrt{a^2 - 4b}}{2}$$

Since roots are real and distinct, then the solution is

$$v(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$v(x) = c_1 e^{\left(-\frac{\sqrt{a^2-4b}}{2}\right)x} + c_2 e^{\left(\frac{\sqrt{a^2-4b}}{2}\right)x}$$

Or

$$v(x) = c_1 \mathrm{e}^{-\frac{x\sqrt{a^2-4b}}{2}} + c_2 \mathrm{e}^{\frac{x\sqrt{a^2-4b}}{2}}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x)\\ &= \left(c_1 \mathrm{e}^{-\frac{x\sqrt{a^2-4b}}{2}} + c_2 \mathrm{e}^{\frac{x\sqrt{a^2-4b}}{2}}\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \mathrm{e}^{-\frac{1}{3}x^3-\frac{1}{2}ax}$$

Hence (7) becomes

$$y = \left(c_1 \mathrm{e}^{-\frac{x\sqrt{a^2-4b}}{2}} + c_2 \mathrm{e}^{\frac{x\sqrt{a^2-4b}}{2}}\right) \mathrm{e}^{-\frac{1}{3}x^3-\frac{1}{2}ax}$$

Summary

The solution(s) found are the following

$$y = \left(c_1 \mathrm{e}^{-\frac{x\sqrt{a^2-4b}}{2}} + c_2 \mathrm{e}^{\frac{x\sqrt{a^2-4b}}{2}}\right) \mathrm{e}^{-\frac{1}{3}x^3-\frac{1}{2}ax} \tag{1}$$

Verification of solutions

$$y = \left(c_1 \mathrm{e}^{-\frac{x\sqrt{a^2-4b}}{2}} + c_2 \mathrm{e}^{\frac{x\sqrt{a^2-4b}}{2}}\right) \mathrm{e}^{-\frac{1}{3}x^3-\frac{1}{2}ax}$$

Verified OK.

27.26.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + (2x^2 + a) y' + (x^4 + a x^2 + b + 2x) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x^2 + a \\ C &= x^4 + a x^2 + b + 2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2 - 4b}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = a^2 - 4b$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2}{4} - b\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 59: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{a^2}{4} - b$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\frac{x\sqrt{a^2-4b}}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2+a}{1}\,dx} \\
&= z_1 e^{-\frac{1}{3}x^3-\frac{1}{2}ax} \\
&= z_1 \left(\mathrm{e}^{-\frac{1}{3}x^3-\frac{1}{2}ax}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^3}{3}-\frac{ax}{2}+\frac{x\sqrt{a^2-4b}}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+a}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{2}{3}x^3-ax}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-x\sqrt{a^2-4b}}}{\sqrt{a^2-4b}}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^3}{3}-\frac{ax}{2}+\frac{x\sqrt{a^2-4b}}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{x^3}{3}-\frac{ax}{2}+\frac{x\sqrt{a^2-4b}}{2}} \left(-\frac{\mathrm{e}^{-x\sqrt{a^2-4b}}}{\sqrt{a^2-4b}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x^3}{3}-\frac{ax}{2}+\frac{x\sqrt{a^2-4b}}{2}} - \frac{c_2 \mathrm{e}^{-\frac{x\left(2x^2+3\sqrt{a^2-4b}+3a\right)}{6}}}{\sqrt{a^2-4b}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x^3}{3}-\frac{ax}{2}+\frac{x\sqrt{a^2-4b}}{2}} - \frac{c_2 \mathrm{e}^{-\frac{x\left(2x^2+3\sqrt{a^2-4b}+3a\right)}{6}}}{\sqrt{a^2-4b}}
$$

Verified OK.

27.26.3 Maple step by step solution

Let's solve

$y'' + (2x^2 + a)\, y' + (x^4 + a\, x^2 + b + 2x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..4$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$a_1 a + a_0 b + 2a_2 + (2aa_2 + a_1 b + 2a_0 + 6a_3)\, x + (a_0 a + 3aa_3 + a_2 b + 4a_1 + 12a_4)\, x^2 + (a_1 a + 4aa$$

- The coefficients of each power of x must be 0

  $$[2a_2 + a_1 a + a_0 b = 0, 2aa_2 + a_1 b + 2a_0 + 6a_3 = 0, a_0 a + 3aa_3 + a_2 b + 4a_1 + 12a_4 = 0, a_1 a + 4aa$$

- Solve for the dependent coefficient(s)

  $$\left\{a_2 = -\frac{a_1 a}{2} - \frac{a_0 b}{2}, a_3 = \frac{1}{6}a_1 a^2 + \frac{1}{6}a_0 ab - \frac{1}{6}a_1 b - \frac{1}{3}a_0, a_4 = -\frac{1}{24}a_1 a^3 - \frac{1}{24}a_0 a^2 b + \frac{1}{12}a_1 ab + \frac{1}{24}a_0 b^2\right.$$

- Each term in the series must be 0, giving the recursion relation

  $$k^2 a_{k+2} + (aa_{k+1} + 2a_{k-1} + 3a_{k+2})\, k + 2a_{k+2} + (a_{k-2} + a_{k+1})\, a + a_k b + a_{k-4} = 0$$

- Shift index using $k->k+4$

  $$(k+4)^2 a_{k+6} + (aa_{k+5} + 2a_{k+3} + 3a_{k+6})\,(k+4) + 2a_{k+6} + (a_{k+2} + a_{k+5})\, a + a_{k+4} b + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{aka_{k+5} + aa_{k+2} + 5aa_{k+5} + a_{k+4}b + 2ka_{k+3} + a_k + 8a_{k+3}}{k^2 + 11k + 30}, a_2 = -\frac{a_1 a}{2} - \frac{a_0 b}{2}, a_3 = \frac{1}{6}a_1 a^2 +\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 57

```
dsolve(diff(y(x),x$2)+(2*x^2+a)*diff(y(x),x)+(x^4+a*x^2+2*x+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{\frac{x\left(-2x^2+3\sqrt{a^2-4b}-3a\right)}{6}} + c_2 \mathrm{e}^{-\frac{x\left(2x^2+3\sqrt{a^2-4b}+3a\right)}{6}}$$

✓ Solution by Mathematica

Time used: 0.217 (sec). Leaf size: 79

```
DSolve[y''[x]+(2*x^2+a)*y'[x]+(x^4+a*x^2+2*x+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{e^{-\frac{1}{6}x\left(3\sqrt{a^2-4b}+3a+2x^2\right)}\left(c_2 e^{x\sqrt{a^2-4b}} + c_1\sqrt{a^2-4b}\right)}{\sqrt{a^2-4b}}$$

27.27 problem 37

Internal problem ID [10861]
Internal file name [`OUTPUT/10117_Sunday_December_24_2023_05_12_38_PM_85001203/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$y'' + (a\,x^2 + bx)\,y' + (\alpha\,x^2 + \beta x + \gamma)\,y = 0$$

27.27.1 Maple step by step solution

Let's solve

$y'' + x(ax + b)\,y' + (\alpha\,x^2 + \beta x + \gamma)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$$

Rewrite ODE with series expansions

$$a_0\gamma + 2a_2 + (6a_3 + a_1(b+\gamma) + a_0\beta)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(bk+\gamma) + a_{k-1}(a(k-1)\right.$$

- The coefficients of each power of x must be 0

$$[2a_2 + a_0\gamma = 0, 6a_3 + a_1(b+\gamma) + a_0\beta = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{a_0\gamma}{2}, a_3 = -\frac{1}{6}a_1 b - \frac{1}{6}a_0\beta - \frac{1}{6}a_1\gamma\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+2} + (a a_{k-1} + b a_k + 3a_{k+2})\, k + (-a+\beta)\, a_{k-1} + a_{k-2}\alpha + a_k\gamma + 2a_{k+2} = 0$$

- Shift index using $k->k+2$

$$(k+2)^2 a_{k+4} + (a a_{k+1} + b a_{k+2} + 3a_{k+4})(k+2) + (-a+\beta)\, a_{k+1} + a_k\alpha + a_{k+2}\gamma + 2a_{k+4} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{ak a_{k+1} + bk a_{k+2} + a a_{k+1} + a_k\alpha + 2b a_{k+2} + \beta a_{k+1} + a_{k+2}\gamma}{k^2+7k+12}, a_2 = -\frac{a_0\gamma}{2}, a_3 = -\frac{1}{6}a_1 b - \frac{1}{6}a_0\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.265 (sec). Leaf size: 271

```
dsolve(diff(y(x),x$2)+(a*x^2+b*x)*diff(y(x),x)+(alpha*x^2+beta*x+gamma)*y(x)=0,y(x), singsol
```

$$
\begin{aligned}
y(x) &= c_1 \mathrm{e}^{-\frac{\operatorname{csgn}(a)x\left(2a^2x^2\operatorname{csgn}(a)+3abx\ \operatorname{csgn}(a)+2a^2x^2+3abx-12\alpha\right)}{12a}} \operatorname{HeunT}\left(\frac{3^{\frac{2}{3}}(2a^2\gamma-ab\beta+\alpha b^2+2\alpha^2)}{2a^2\left(a^2\right)^{\frac{1}{3}}},\right. \\
&\left. -\frac{3(a^2-\beta a+b\alpha)\operatorname{csgn}(a)}{a^2}, -\frac{3^{\frac{1}{3}}(b^2+8\alpha)}{4\left(a^2\right)^{\frac{2}{3}}}, \frac{3^{\frac{2}{3}}a(2ax+b)}{6\left(a^2\right)^{\frac{5}{6}}}\right) \\
&+ c_2 \mathrm{e}^{-\frac{\operatorname{csgn}(a)x\left(2a^2x^2\operatorname{csgn}(a)+3abx\ \operatorname{csgn}(a)-2a^2x^2-3abx+12\alpha\right)}{12a}} \operatorname{HeunT}\left(\frac{3^{\frac{2}{3}}(2a^2\gamma-ab\beta+\alpha b^2+2\alpha^2)}{2a^2\left(a^2\right)^{\frac{1}{3}}}, \frac{3(a^2-\beta a+b\alpha)\ \mathrm{c}}{a^2}\right. \\
&\left. -\frac{3^{\frac{1}{3}}(b^2+8\alpha)}{4\left(a^2\right)^{\frac{2}{3}}}, -\frac{3^{\frac{2}{3}}a(2ax+b)}{6\left(a^2\right)^{\frac{5}{6}}}\right)
\end{aligned}
$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+(a*x^2+b*x)*y'[x]+(\[Alpha]*x^2+\[Beta]*x+\[Gamma])*y[x]==0,y[x],x,IncludeSing
```

Not solved

27.28 problem 38

Internal problem ID [10862]
Internal file name [`OUTPUT/10118_Sunday_December_24_2023_05_12_38_PM_96398998/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \left(ab\,x^2 + bx + 2a\right)y' + a^2\left(b\,x^2 + 1\right)y = 0}$$

27.28.1 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \left(\left(a\,x^2 + x\right)b + 2a\right)y' + a^2\left(b\,x^2 + 1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \left(a\,x^2 + x\right)b + 2a \\ C &= a^2\left(b\,x^2 + 1\right) \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{b(a^2 b\, x^4 + 2ab\, x^3 + b\, x^2 + 8ax + 2)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= b\left(a^2 b\, x^4 + 2ab\, x^3 + b\, x^2 + 8ax + 2\right) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{b(a^2 b\, x^4 + 2ab\, x^3 + b\, x^2 + 8ax + 2)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 62: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{ab\,x^2}{2} + \frac{bx}{2} + \frac{2}{x} - \frac{3}{2a\,x^2} + \frac{3}{2a^2x^3} - \frac{4}{ab\,x^4} - \frac{3}{2a^3x^4} + \frac{10}{a^2b\,x^5} + \frac{3}{2a^4x^5} - \frac{73}{4a^3b\,x^6} - \frac{3}{2a^5x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{ab}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{1}{2}bx + \frac{1}{2}ab\,x^2 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}b^2x^2 + \frac{1}{2}a\,b^2x^3 + \frac{1}{4}a^2b^2x^4$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{b(a^2b\,x^4 + 2ab\,x^3 + b\,x^2 + 8ax + 2)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(2abx + \frac{1}{2}b + \frac{1}{4}a^2b^2x^4 + \frac{1}{2}a\,b^2x^3 + \frac{1}{4}b^2x^2\right) + (0) \\
&= 2abx + \frac{1}{2}b + \frac{1}{4}a^2b^2x^4 + \frac{1}{2}a\,b^2x^3 + \frac{1}{4}b^2x^2
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $2ab$. Now b can be found.

$$\begin{aligned}
b &= (2ab) - (0) \\
&= 2ab
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}bx + \frac{1}{2}ab\,x^2 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2ab}{\frac{ab}{2}} - 2\right) &&= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2ab}{\frac{ab}{2}} - 2\right) &&= -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{b(a^2b\,x^4 + 2ab\,x^3 + b\,x^2 + 8ax + 2)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-4	$\frac{1}{2}bx+\frac{1}{2}ab\,x^2$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=1$, and since there are no poles, then

$$\begin{aligned} d&=\alpha_\infty^{+}\\ &=1 \end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega&=(+)[\sqrt{r}]_\infty\\ &=0+\left(\frac{1}{2}bx+\frac{1}{2}ab\,x^2\right)\\ &=\frac{1}{2}bx+\frac{1}{2}ab\,x^2\\ &=\frac{bx(ax+1)}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2}bx+\frac{1}{2}ab\,x^2\right)(1)+\left(\left(abx+\frac{1}{2}b\right)+\left(\frac{1}{2}bx+\frac{1}{2}ab\,x^2\right)^2-\left(\frac{b(a^2b\,x^4+2ab\,x^3+b\,x^2+8ax+2)}{4}\right.\right.$$

$$-bx(aa_0-1)$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{a}\right) e^{\int \left(\frac{1}{2}bx + \frac{1}{2}ab\,x^2\right) dx} \\
&= \left(x + \frac{1}{a}\right) e^{\frac{b\left(\frac{1}{3}a\,x^3 + \frac{1}{2}x^2\right)}{2}} \\
&= \frac{(ax+1)\, e^{\frac{1}{6}ab\,x^3 + \frac{1}{4}b\,x^2}}{a}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\left(a\,x^2 + x\right)b + 2a}{1} \, dx} \\
&= z_1 e^{-\frac{1}{6}ab\,x^3 - \frac{1}{4}b\,x^2 - ax} \\
&= z_1 \left(e^{-\frac{x\left(ab\,x^2 + \frac{3}{2}bx + 6a\right)}{6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(ax+1)\, e^{-ax}}{a}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{\left(a\,x^2+x\right)b+2a}{1}\,dx}}{\left(y_1\right)^2}\,dx$$
$$= y_1 \int \frac{e^{-\frac{x\left(ab\,x^2+\frac{3}{2}bx+6a\right)}{3}}}{\left(y_1\right)^2}\,dx$$
$$= y_1 \left(\int \frac{a^2 \mathrm{e}^{-\frac{x^2\left(ax+\frac{3}{2}\right)b}{3}}}{\left(ax+1\right)^2}dx\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a}\right) + c_2 \left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a} \left(\int \frac{a^2 \mathrm{e}^{-\frac{x^2\left(ax+\frac{3}{2}\right)b}{3}}}{\left(ax+1\right)^2}dx\right)\right)$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1(ax+1)\,\mathrm{e}^{-ax}}{a} + c_2(ax+1)\,\mathrm{e}^{-ax} a \left(\int \frac{\mathrm{e}^{-\frac{x^2\left(ax+\frac{3}{2}\right)b}{3}}}{\left(ax+1\right)^2}dx\right) \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1(ax+1)\,\mathrm{e}^{-ax}}{a} + c_2(ax+1)\,\mathrm{e}^{-ax} a \left(\int \frac{\mathrm{e}^{-\frac{x^2\left(ax+\frac{3}{2}\right)b}{3}}}{\left(ax+1\right)^2}dx\right)$$

Verified OK.

27.28.2 Maple step by step solution

Let's solve

$y'' + ((a\,x^2+x)\,b+2a)\,y' + a^2(b\,x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -(ab\,x^2+bx+2a)\,y' - a^2(b\,x^2+1)\,y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + (ab\,x^2+bx+2a)\,y' + a^2(b\,x^2+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- > k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

- ○ Shift index using $k- > k+1-m$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

- ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- > k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$a_0a^2 + 2a_1a + 2a_2 + (6a_3 + 4aa_2 + a_1(a^2 + b))\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + 2aa_{k+1}(k+1) + \right.$$

- The coefficients of each power of x must be 0

$$[2a_2 + a_0a^2 + 2a_1a = 0, 6a_3 + 4aa_2 + a_1(a^2 + b) = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{1}{2}a_0a^2 - a_1a, a_3 = \frac{1}{3}a_0a^3 + \frac{1}{2}a_1a^2 - \frac{1}{6}a_1b\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(ba_{k-2} + a_k)\, a^2 + ((ba_{k-1} + 2a_{k+1})\, k - ba_{k-1} + 2a_{k+1})\, a + k^2a_{k+2} + (ba_k + 3a_{k+2})\, k + 2a_{k+2} = 0$$

- Shift index using $k- >k+2$

$$(ba_k + a_{k+2})\, a^2 + ((ba_{k+1} + 2a_{k+3})\,(k+2) - ba_{k+1} + 2a_{k+3})\, a + (k+2)^2\, a_{k+4} + (ba_{k+2} + 3a_{k+4})$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_kx^k, a_{k+4} = -\frac{a_ka^2b + abka_{k+1} + a^2a_{k+2} + aba_{k+1} + 2aka_{k+3} + bka_{k+2} + 6aa_{k+3} + 2ba_{k+2}}{k^2 + 7k + 12}, a_2 = -\frac{1}{2}a_0a^2 - a_1\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.36 (sec). Leaf size: 294

```
dsolve(diff(y(x),x$2)+(a*b*x^2+b*x+2*a)*diff(y(x),x)+a^2*(b*x^2+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{x\left(2a^2b^2x^2+2ab\,x^2\sqrt{a^2b^2}+3a\,b^2x+3bx\sqrt{a^2b^2}+12a\sqrt{a^2b^2}\right)}{12\sqrt{a^2b^2}}}\,\mathrm{HeunT}\left(\frac{b3^{\frac{2}{3}}}{2\left(a^2b^2\right)^{\frac{1}{3}}}, -\frac{6ab}{\sqrt{a^2b^2}}, -\frac{b^2 3^{\frac{1}{3}}}{4\left(a^2b^2\right)^{\frac{2}{3}}}, \frac{3^{\frac{2}{3}}a\,b^2(2ax+1)}{6\left(a^2b^2\right)^{\frac{5}{6}}}\right) + c_2 \mathrm{e}^{-\frac{\left(-2a^2b^2x^2+2ab\,x^2\sqrt{a^2b^2}-3a\,b^2x+3bx\sqrt{a^2b^2}+12a\sqrt{a^2b^2}\right)x}{12\sqrt{a^2b^2}}}\,\mathrm{HeunT}\left(\frac{b3^{\frac{2}{3}}}{2\left(a^2b^2\right)^{\frac{1}{3}}}, \frac{6ab}{\sqrt{a^2b^2}}, -\frac{b^2 3^{\frac{1}{3}}}{4\left(a^2b^2\right)^{\frac{2}{3}}}, -\frac{\left(ax+\frac{1}{2}\right)3^{\frac{2}{3}}b^2a}{3\left(a^2b^2\right)^{\frac{5}{6}}}\right)$$

✓ Solution by Mathematica

Time used: 2.136 (sec). Leaf size: 57

```
DSolve[y''[x]+(a*b*x^2+b*x+2*a)*y'[x]+a^2*(b*x^2+1)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to e^{-ax}(ax+1)\left(c_2\int_1^x \frac{e^{-\frac{1}{6}bK[1]^2(2aK[1]+3)}}{(aK[1]+1)^2}dK[1]+c_1\right)$$

27.29 problem 39

Internal problem ID [10863]
Internal file name [`OUTPUT/10119_Sunday_December_24_2023_05_12_39_PM_10954655/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + \left(a\,x^2 + bx + c\right) y' + x\left(ab\,x^2 + bc + 2a\right) y = 0}$$

27.29.1 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \left(a\,x^2 + bx + c\right) y' + x\left(\left(a\,x^2 + c\right) b + 2a\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= a\,x^2 + bx + c \\ C &= x\left(\left(a\,x^2 + c\right) b + 2a\right) \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^4 - 2ab\,x^3 + 2ac\,x^2 + b^2x^2 - 2bcx - 4ax + c^2 + 2b}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^4 - 2ab\,x^3 + 2ac\,x^2 + b^2x^2 - 2bcx - 4ax + c^2 + 2b \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-ax + \frac{1}{2}b + \frac{1}{4}a^2x^4 - \frac{1}{2}ab\,x^3 + \frac{1}{2}ac\,x^2 + \frac{1}{4}b^2x^2 - \frac{1}{2}bcx + \frac{1}{4}c^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 64: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 4 \\
&= -4
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{2} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{a\,x^2}{2} - \frac{bx}{2} + \frac{c}{2} - \frac{1}{x} - \frac{b}{2a\,x^2} + \frac{c}{a\,x^3} - \frac{b^2}{2a^2x^3} + \frac{3bc}{2a^2x^4} - \frac{b^3}{2a^3x^4} - \frac{1}{a\,x^4} - \frac{c^2}{a^2x^5} + \frac{2b^2c}{a^3x^5} - \frac{b^4}{2a^4x^5} - \frac{2b}{a^2x^5} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\
&= \frac{1}{2}c - \frac{1}{2}bx + \frac{1}{2}a\,x^2 \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}c^2 - \frac{1}{2}bcx + \frac{1}{2}ac\,x^2 + \frac{1}{4}b^2x^2 - \frac{1}{2}ab\,x^3 + \frac{1}{4}a^2x^4$$

This shows that the coefficient of x in the above is $-\frac{bc}{2}$. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{a^2x^4 - 2ab\,x^3 + 2ac\,x^2 + b^2x^2 - 2bcx - 4ax + c^2 + 2b}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{a^2x^4}{4} - \frac{ab\,x^3}{2} + \left(\frac{ac}{2} + \frac{b^2}{4}\right)x^2 + \left(-a - \frac{bc}{2}\right)x + \frac{b}{2} + \frac{c^2}{4}\right) + (0) \\
&= \frac{a^2x^4}{4} - \frac{ab\,x^3}{2} + \left(\frac{ac}{2} + \frac{b^2}{4}\right)x^2 + \left(-a - \frac{bc}{2}\right)x + \frac{b}{2} + \frac{c^2}{4}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-a - \frac{bc}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-a - \frac{bc}{2}\right) - \left(-\frac{bc}{2}\right) \\
&= -a
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}c - \frac{1}{2}bx + \frac{1}{2}a\,x^2 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-a}{\frac{a}{2}} - 2\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-a}{\frac{a}{2}} - 2\right) &&= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -ax + \frac{1}{2}b + \frac{1}{4}a^2x^4 - \frac{1}{2}ab\,x^3 + \frac{1}{2}ac\,x^2 + \frac{1}{4}b^2x^2 - \frac{1}{2}bcx + \frac{1}{4}c^2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{1}{2}c - \frac{1}{2}bx + \frac{1}{2}a\,x^2$	-2	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{1}{2}c - \frac{1}{2}bx + \frac{1}{2}a\,x^2\right) \\ &= -\frac{1}{2}c + \frac{1}{2}bx - \frac{1}{2}a\,x^2 \\ &= -\frac{1}{2}c + \frac{1}{2}bx - \frac{1}{2}a\,x^2 \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2}c + \frac{1}{2}bx - \frac{1}{2}a\,x^2\right)(0) + \left(\left(\frac{b}{2} - ax\right) + \left(-\frac{1}{2}c + \frac{1}{2}bx - \frac{1}{2}a\,x^2\right)^2 - \left(-ax + \frac{1}{2}b + \frac{1}{4}a^2x^4 - \frac{1}{2}c\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2}c+\frac{1}{2}bx-\frac{1}{2}a\,x^2\right) dx} \\
&= \mathrm{e}^{-\frac{\left(a\,x^2-\frac{3}{2}bx+3c\right)x}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{a\,x^2+bx+c}{1} \, dx} \\
&= z_1 e^{-\frac{1}{6}a\,x^3-\frac{1}{4}b\,x^2-\frac{1}{2}cx} \\
&= z_1 \left(\mathrm{e}^{-\frac{x\left(a\,x^2+\frac{3}{2}bx+3c\right)}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{1}{3}a\,x^3-cx}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{a\,x^2+bx+c}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x\left(a\,x^2+\frac{3}{2}bx+3c\right)}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \mathrm{e}^{\frac{1}{3}a\,x^3-\frac{1}{2}b\,x^2+cx} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{1}{3}a\,x^3-cx}\right) + c_2 \left(\mathrm{e}^{-\frac{1}{3}a\,x^3-cx} \left(\int \mathrm{e}^{\frac{1}{3}a\,x^3-\frac{1}{2}b\,x^2+cx} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-\frac{1}{3} a x^3 - c x} + c_2 e^{-\frac{x(a x^2 + 3c)}{3}} \left(\int e^{\frac{1}{3} a x^3 - \frac{1}{2} b x^2 + c x} dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 e^{-\frac{1}{3} a x^3 - c x} + c_2 e^{-\frac{x(a x^2 + 3c)}{3}} \left(\int e^{\frac{1}{3} a x^3 - \frac{1}{2} b x^2 + c x} dx\right)$$

Verified OK.

27.29.2 Maple step by step solution

Let's solve

$y'' + (a x^2 + b x + c) y' + x((a x^2 + c) b + 2a) y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = -(a x^2 + b x + c) y' - x(a b x^2 + b c + 2a) y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + (a x^2 + b x + c) y' + x(a b x^2 + b c + 2a) y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 1..3$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$a_1c + 2a_2 + (6a_3 + 2a_2c + a_1b + a_0(bc+2a))\, x + (12a_4 + 3a_3c + 2a_2b + a_1(bc+3a))\, x^2 + \left(\sum_{k=3}^{\infty} (\right.$$

- The coefficients of each power of x must be 0

$$[2a_2 + a_1c = 0, 6a_3 + 2a_2c + a_1b + a_0(bc+2a) = 0, 12a_4 + 3a_3c + 2a_2b + a_1(bc+3a) = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{a_1c}{2}, a_3 = -\frac{1}{6}a_0bc + \frac{1}{6}a_1c^2 - \frac{1}{3}a_0a - \frac{1}{6}a_1b, a_4 = \frac{1}{24}a_0b\,c^2 - \frac{1}{24}a_1c^3 + \frac{1}{12}a_0ac + \frac{1}{24}a_1bc - \frac{1}{4}a\right.$$

- Each term in the series must be 0, giving the recursion relation

$$k^2a_{k+2} + (aa_{k-1} + ba_k + a_{k+1}c + 3a_{k+2})\, k + (bc+a)\, a_{k-1} + a_{k-3}ab + a_{k+1}c + 2a_{k+2} = 0$$

- Shift index using $k->k+3$

$$(k+3)^2\, a_{k+5} + (aa_{k+2} + ba_{k+3} + a_{k+4}c + 3a_{k+5})\,(k+3) + (bc+a)\, a_{k+2} + a_kab + a_{k+4}c + 2a_{k+5}$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_kx^k, a_{k+5} = -\frac{a_kab + aka_{k+2} + bca_{k+2} + bka_{k+3} + cka_{k+4} + 4aa_{k+2} + 3ba_{k+3} + 4a_{k+4}c}{k^2+9k+20}, a_2 = -\frac{a_1c}{2}, a_3 = -\frac{1}{6}a\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 169

```
dsolve(diff(y(x),x$2)+(a*x^2+b*x+c)*diff(y(x),x)+x*(a*b*x^2+b*c+2*a)*y(x)=0,y(x), singsol=al
```

$$y(x) = c_1 \mathrm{e}^{-\frac{x\,\mathrm{csgn}(a)\left(\left(a\,x^2+\frac{3}{2}bx+3c\right)\mathrm{csgn}(a)+a\,x^2-\frac{3bx}{2}+3c\right)}{6}} \mathrm{HeunT}\left(0, 3\,\mathrm{csgn}(a), \frac{3^{\frac{1}{3}}(4ac-b^2)}{4\,(a^2)^{\frac{2}{3}}}, \frac{3^{\frac{2}{3}}a(2ax-b)}{6\,(a^2)^{\frac{5}{6}}}\right) + c_2 \mathrm{e}^{-\frac{x\left(\left(a\,x^2+\frac{3}{2}bx+3c\right)\mathrm{csgn}(a)-a\,x^2+\frac{3bx}{2}-3c\right)\mathrm{csgn}(a)}{6}} \mathrm{HeunT}\left(0, -3\,\mathrm{csgn}(a), \frac{3^{\frac{1}{3}}(4ac-b^2)}{4\,(a^2)^{\frac{2}{3}}}, -\frac{3^{\frac{2}{3}}\left(ax-\frac{b}{2}\right)a}{3\,(a^2)^{\frac{5}{6}}}\right)$$

✓ Solution by Mathematica

Time used: 1.085 (sec). Leaf size: 59

```
DSolve[y''[x]+(a*x^2+b*x+c)*y'[x]+x*(a*b*x^2+b*c+2*a)*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to e^{-\frac{1}{3}x(ax^2+3c)}\left(c_2 \int_1^x \exp\left(\frac{1}{6}K[1](6c+K[1](2aK[1]-3b))\right) dK[1] + c_1\right)$$

27.30 problem 40

Internal problem ID [10864]
Internal file name [`OUTPUT/10120_Sunday_December_24_2023_05_12_40_PM_89868459/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \left(a\,x^2 + bx + c\right) y' + \left(ab\,x^3 + ac\,x^2 + b\right) y = 0}$$

27.30.1 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \left(a\,x^2 + bx + c\right) y' + \left(x^2(bx + c)\,a + b\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= a\,x^2 + bx + c \\ C &= x^2(bx + c)\,a + b \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^4 - 2ab\,x^3 - 2ac\,x^2 + b^2x^2 + 2bcx + 4ax + c^2 - 2b}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^4 - 2ab\,x^3 - 2ac\,x^2 + b^2x^2 + 2bcx + 4ax + c^2 - 2b \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(ax - \frac{1}{2}b + \frac{1}{4}a^2x^4 - \frac{1}{2}ab\,x^3 - \frac{1}{2}ac\,x^2 + \frac{1}{4}b^2x^2 + \frac{1}{2}bcx + \frac{1}{4}c^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 66: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{a\,x^2}{2} - \frac{bx}{2} - \frac{c}{2} + \frac{1}{x} + \frac{b}{2a\,x^2} + \frac{c}{a\,x^3} + \frac{b^2}{2a^2x^3} + \frac{3bc}{2a^2x^4} + \frac{b^3}{2a^3x^4} - \frac{1}{a\,x^4} + \frac{c^2}{a^2x^5} + \frac{2b^2c}{a^3x^5} + \frac{b^4}{2a^4x^5} - \frac{2b}{a^2x^5} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= -\frac{1}{2}c - \frac{1}{2}bx + \frac{1}{2}a\,x^2 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}c^2 + \frac{1}{2}bcx - \frac{1}{2}ac\,x^2 + \frac{1}{4}b^2x^2 - \frac{1}{2}ab\,x^3 + \frac{1}{4}a^2x^4$$

This shows that the coefficient of x in the above is $\frac{bc}{2}$. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{a^2x^4 - 2ab\,x^3 - 2ac\,x^2 + b^2x^2 + 2bcx + 4ax + c^2 - 2b}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{a^2x^4}{4} - \frac{ab\,x^3}{2} + \left(-\frac{ac}{2} + \frac{b^2}{4}\right)x^2 + \left(\frac{bc}{2} + a\right)x - \frac{b}{2} + \frac{c^2}{4}\right) + (0) \\
&= \frac{a^2x^4}{4} - \frac{ab\,x^3}{2} + \left(-\frac{ac}{2} + \frac{b^2}{4}\right)x^2 + \left(\frac{bc}{2} + a\right)x - \frac{b}{2} + \frac{c^2}{4}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{bc}{2} + a$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{bc}{2} + a\right) - \left(\frac{bc}{2}\right) \\
&= a
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= -\frac{1}{2}c - \frac{1}{2}bx + \frac{1}{2}a\,x^2 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{a}{\frac{a}{2}} - 2\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{a}{\frac{a}{2}} - 2\right) &&= -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = ax - \frac{1}{2}b + \frac{1}{4}a^2x^4 - \frac{1}{2}ab\,x^3 - \frac{1}{2}ac\,x^2 + \frac{1}{4}b^2x^2 + \frac{1}{2}bcx + \frac{1}{4}c^2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$-\frac{1}{2}c-\frac{1}{2}bx+\frac{1}{2}a\,x^2$	0	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(-\frac{1}{2}c - \frac{1}{2}bx + \frac{1}{2}a\,x^2\right) \\ &= -\frac{1}{2}c - \frac{1}{2}bx + \frac{1}{2}a\,x^2 \\ &= -\frac{1}{2}c - \frac{1}{2}bx + \frac{1}{2}a\,x^2 \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2}c - \frac{1}{2}bx + \frac{1}{2}a\,x^2\right)(0) + \left(\left(ax - \frac{b}{2}\right) + \left(-\frac{1}{2}c - \frac{1}{2}bx + \frac{1}{2}a\,x^2\right)^2 - \left(ax - \frac{1}{2}b + \frac{1}{4}a^2x^4 - \frac{1}{2}ab\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2}c-\frac{1}{2}bx+\frac{1}{2}a\,x^2\right) dx} \\
&= \mathrm{e}^{-\frac{1}{2}cx-\frac{1}{4}b\,x^2+\frac{1}{6}a\,x^3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{a\,x^2+bx+c}{1} \, dx} \\
&= z_1 e^{-\frac{1}{6}a\,x^3-\frac{1}{4}b\,x^2-\frac{1}{2}cx} \\
&= z_1 \left(\mathrm{e}^{-\frac{x\left(a\,x^2+\frac{3}{2}bx+3c\right)}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x(bx+2c)}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{a\,x^2+bx+c}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x\left(a\,x^2+\frac{3}{2}bx+3c\right)}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \mathrm{e}^{-\frac{\left(a\,x^2-\frac{3}{2}bx-3c\right)x}{3}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x(bx+2c)}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{x(bx+2c)}{2}} \left(\int \mathrm{e}^{-\frac{\left(a\,x^2-\frac{3}{2}bx-3c\right)x}{3}} dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{x(bx+2c)}{2}} + c_2 \mathrm{e}^{-\frac{x(bx+2c)}{2}} \left(\int \mathrm{e}^{-\frac{\left(a\,x^2-\frac{3}{2}bx-3c\right)x}{3}} dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{x(bx+2c)}{2}} + c_2 \mathrm{e}^{-\frac{x(bx+2c)}{2}} \left(\int \mathrm{e}^{-\frac{\left(a\,x^2-\frac{3}{2}bx-3c\right)x}{3}} dx\right)$$

Verified OK.

27.30.2 Maple step by step solution

Let's solve

$y'' + (a\,x^2 + bx + c)\,y' + (x^2(bx + c)\,a + b)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-ab\,x^3 - ac\,x^2 - b)\,y - (a\,x^2 + bx + c)\,y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + (a\,x^2 + bx + c)\,y' + (ab\,x^3 + ac\,x^2 + b)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..3$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$
- Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
- Shift index using $k->k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $a_0 b + a_1 c + 2a_2 + (2a_1 b + 2a_2 c + 6a_3)\, x + (a_0 ac + a_1 a + 3a_2 b + 3a_3 c + 12a_4)\, x^2 + \left(\sum_{k=3}^{\infty} (a_{k+2}(k\right.$

- The coefficients of each power of x must be 0

  $[2a_2 + a_1 c + a_0 b = 0, 2a_1 b + 2a_2 c + 6a_3 = 0, a_0 ac + a_1 a + 3a_2 b + 3a_3 c + 12a_4 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = -\frac{a_0 b}{2} - \frac{a_1 c}{2}, a_3 = \frac{1}{6}a_0 bc + \frac{1}{6}a_1 c^2 - \frac{1}{3}a_1 b, a_4 = -\frac{1}{24}a_0 b\, c^2 - \frac{1}{24}a_1 c^3 - \frac{1}{12}a_0 ac + \frac{1}{8}a_0 b^2 + \frac{5}{24}a_1 b\right.$
- Each term in the series must be 0, giving the recursion relation

  $k^2 a_{k+2} + (a a_{k-1} + a_k b + a_{k+1} c + 3a_{k+2})\, k + (b a_{k-3} + c a_{k-2} - a_{k-1})\, a + a_k b + a_{k+1} c + 2a_{k+2} = 0$
- Shift index using $k->k+3$

  $(k+3)^2\, a_{k+5} + (a a_{k+2} + a_{k+3} b + a_{k+4} c + 3a_{k+5})\,(k+3) + (a_k b + a_{k+1} c - a_{k+2})\, a + a_{k+3} b + a_{k+4}$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+5} = -\frac{a_k ab + ac a_{k+1} + ak a_{k+2} + bk a_{k+3} + ck a_{k+4} + 2a a_{k+2} + 4a_{k+3} b + 4a_{k+4} c}{k^2 + 9k + 20}, a_2 = -\frac{a_0 b}{2} - \frac{a_1 c}{2}, a_3 =\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 165

```
dsolve(diff(y(x),x$2)+(a*x^2+b*x+c)*diff(y(x),x)+(a*b*x^3+a*c*x^2+b)*y(x)=0,y(x), singsol=al
```

$$y(x) = c_1 \mathrm{e}^{-\frac{x\,\mathrm{csgn}(a)\left(\left(a\,x^2+\frac{3}{2}bx+3c\right)\mathrm{csgn}(a)+a\,x^2-\frac{3bx}{2}-3c\right)}{6}}\,\mathrm{HeunT}\left(0,-3\,\mathrm{csgn}(a),-\frac{3^{\frac{1}{3}}(4ac+b^2)}{4\,(a^2)^{\frac{2}{3}}},\frac{3^{\frac{2}{3}}a(2ax-b)}{6\,(a^2)^{\frac{5}{6}}}\right) + c_2 \mathrm{e}^{-\frac{x\,\mathrm{csgn}(a)\left(\left(a\,x^2+\frac{3}{2}bx+3c\right)\mathrm{csgn}(a)-a\,x^2+\frac{3bx}{2}+3c\right)}{6}}\,\mathrm{HeunT}\left(0,3\,\mathrm{csgn}(a),-\frac{3^{\frac{1}{3}}(4ac+b^2)}{4\,(a^2)^{\frac{2}{3}}},-\frac{3^{\frac{2}{3}}\left(ax-\frac{b}{2}\right)a}{3\,(a^2)^{\frac{5}{6}}}\right)$$

✓ Solution by Mathematica

Time used: 1.096 (sec). Leaf size: 57

```
DSolve[y''[x]+(a*x^2+b*x+c)*y'[x]+(a*b*x^3+a*c*x^2+b)*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to e^{-\frac{1}{2}x(bx+2c)}\left(c_2\int_1^x \exp\left(\frac{1}{6}K[1](6c+K[1](3b-2aK[1]))\right)dK[1]+c_1\right)$$

27.31 problem 41

Internal problem ID [10865]
Internal file name [`OUTPUT/10121_Sunday_December_24_2023_05_12_41_PM_73326468/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \left(a\,x^3 + 2b\right) y' + \left(ab\,x^3 - a\,x^2 + b^2\right) y = 0}$$

27.31.1 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \left(a\,x^3 + 2b\right) y' + \left(ab\,x^3 - a\,x^2 + b^2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= a\,x^3 + 2b \\ C &= ab\,x^3 - a\,x^2 + b^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a\,x^2(a\,x^4 + 10)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a\,x^2\left(a\,x^4 + 10\right) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a\,x^2(a\,x^4 + 10)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 68: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 6 \\ &= -6 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -6 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -6$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{6}{2} = 3$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{3} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^3$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{a\,x^3}{2} + \frac{5}{2x} - \frac{25}{4a\,x^5} + \frac{125}{4a^2x^9} - \frac{3125}{16a^3x^{13}} + \frac{21875}{16a^4x^{17}} - \frac{328125}{32a^5x^{21}} + \frac{2578125}{32a^6x^{25}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 3$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{3} a_i x^i \\ &= \frac{a\,x^3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^2 = x^2$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{a^2 x^6}{4}$$

This shows that the coefficient of x^2 in the above is 0. Now we need to find the coefficient of x^2 in r. How this is done depends on if $v = 0$ or not. Since $v = 3$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^2 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a\,x^2(a\,x^4 + 10)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{5}{2}a\,x^2 + \frac{1}{4}a^2x^6\right) + (0) \\ &= \frac{5}{2}a\,x^2 + \frac{1}{4}a^2x^6 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{5a}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5a}{2}\right) - (0) \\ &= \frac{5a}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{a\,x^3}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{5a}{2}}{\frac{a}{2}} - 3\right) &= 1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{5a}{2}}{\frac{a}{2}} - 3\right) &= -4 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a\,x^2(a\,x^4 + 10)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-6	$\frac{a\,x^3}{2}$	1	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{a\,x^3}{2}\right) \\ &= \frac{a\,x^3}{2} \\ &= \frac{a\,x^3}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{a\,x^3}{2}\right)(1) + \left(\left(\frac{3a\,x^2}{2}\right) + \left(\frac{a\,x^3}{2}\right)^2 - \left(\frac{a\,x^2(a\,x^4+10)}{4}\right)\right) = 0$$

$$-a\,x^2 a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= (x) \, \mathrm{e}^{\int \frac{a\,x^3}{2} dx} \\
&= (x) \, \mathrm{e}^{\frac{a\,x^4}{8}} \\
&= x \, \mathrm{e}^{\frac{a\,x^4}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{a\,x^3+2b}{1}\, dx} \\
&= z_1 e^{-\frac{1}{8}a\,x^4 - bx} \\
&= z_1 \left(\mathrm{e}^{-\frac{x\left(a\,x^3+8b\right)}{8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x \, \mathrm{e}^{-bx}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{a\,x^3+2b}{1}\, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{1}{4}a\,x^4 - 2bx}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{a\,x^4}{4}}}{x^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x \,\mathrm{e}^{-bx}\right) + c_2 \left(x \,\mathrm{e}^{-bx} \left(\int \frac{\mathrm{e}^{-\frac{a\,x^4}{4}}}{x^2} dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x \,\mathrm{e}^{-bx} + c_2 x \,\mathrm{e}^{-bx} \left(\int \frac{\mathrm{e}^{-\frac{a\,x^4}{4}}}{x^2} dx\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x \,\mathrm{e}^{-bx} + c_2 x \,\mathrm{e}^{-bx} \left(\int \frac{\mathrm{e}^{-\frac{a\,x^4}{4}}}{x^2} dx\right)
$$

Verified OK.

27.31.2 Maple step by step solution

Let's solve

$y'' + (a\,x^3 + 2b)\,y' + (ab\,x^3 - a\,x^2 + b^2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ∘ Convert $x^m \cdot y$ to series expansion for $m = 0..3$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ∘ Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ∘ Convert $x^m \cdot y'$ to series expansion for $m = 0..3$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$a_0 b^2 + 2a_1 b + 2a_2 + (a_1 b^2 + 4a_2 b + 6a_3)\, x + (a_2 b^2 - a_0 a + 6a_3 b + 12a_4)\, x^2 + \left(\sum_{k=3}^{\infty} (a_{k+2}(k+2)\,(\right.$$

- The coefficients of each power of x must be 0

$$[2a_2 + a_0 b^2 + 2a_1 b = 0, a_1 b^2 + 4a_2 b + 6a_3 = 0, a_2 b^2 - a_0 a + 6a_3 b + 12a_4 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{1}{2} a_0 b^2 - a_1 b, a_3 = \frac{1}{3} a_0 b^3 + \frac{1}{2} a_1 b^2, a_4 = -\frac{1}{8} a_0 b^4 - \frac{1}{6} a_1 b^3 + \frac{1}{12} a_0 a\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k b^2 + (a a_{k-3} + 2k a_{k+1} + 2a_{k+1})\, b + k^2 a_{k+2} + (a_{k-2} a + 3a_{k+2})\, k - 3a_{k-2} a + 2a_{k+2} = 0$$

- Shift index using $k- >k+3$

$$a_{k+3} b^2 + (a_k a + 2(k+3)\, a_{k+4} + 2a_{k+4})\, b + (k+3)^2\, a_{k+5} + (a_{k+1} a + 3a_{k+5})\,(k+3) - 3a_{k+1} a + 2$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+5} = -\frac{a_k ab + ak a_{k+1} + a_{k+3} b^2 + 2bk a_{k+4} + 8b a_{k+4}}{k^2 + 9k + 20}, a_2 = -\frac{1}{2} a_0 b^2 - a_1 b, a_3 = \frac{1}{3} a_0 b^3 + \frac{1}{2} a_1 b^2,\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 85

```
dsolve(diff(y(x),x$2)+(a*x^3+2*b)*diff(y(x),x)+(a*b*x^3-a*x^2+b^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\frac{7\,2^{\frac{1}{4}}c_2 a (x^4 a)^{\frac{3}{8}}(x^4 a+3)\mathrm{e}^{-\frac{x(a\,x^3+4b)}{4}}}{8} + \mathrm{e}^{-\frac{x(a\,x^3+8b)}{8}}\,\mathrm{WhittakerM}\left(\frac{3}{8},\frac{7}{8},\frac{x^4 a}{4}\right)c_2 a^2 x^4 + \mathrm{e}^{-bx}c_1 x^{\frac{5}{2}}}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.431 (sec). Leaf size: 51

```
DSolve[y''[x]+(a*x^3+2*b)*y'[x]+(a*b*x^3-a*x^2+b^2)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{8}e^{-bx}\left(8c_1 x - \sqrt{2}c_2\sqrt[4]{ax^4}\,\Gamma\left(-\frac{1}{4},\frac{ax^4}{4}\right)\right)$$

27.32 problem 42

Internal problem ID [10866]
Internal file name [`OUTPUT/10122_Sunday_December_24_2023_05_12_41_PM_63694197/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y''+f(x)y'+g(x)y=0$
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y''+\left(a\,x^{3}+bx\right)y'+2\left(2a\,x^{2}+b\right)y=0}$$

27.32.1 Solving using Kovacic algorithm

Writing the ode as

$$y''+x\left(a\,x^{2}+b\right)y'+\left(4a\,x^{2}+2b\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1\\ B&=x\left(a\,x^{2}+b\right)\\ C&=4a\,x^{2}+2b \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^6 + 2ab\,x^4 + b^2x^2 - 10a\,x^2 - 6b}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^6 + 2ab\,x^4 + b^2x^2 - 10a\,x^2 - 6b \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{5}{2}a\,x^2 - \frac{3}{2}b + \frac{1}{4}a^2x^6 + \frac{1}{2}ab\,x^4 + \frac{1}{4}b^2x^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 70: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 6 \\ &= -6 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -6 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -6$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{6}{2} = 3$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{3} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^3$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{a\,x^3}{2} + \frac{bx}{2} - \frac{5}{2x} + \frac{b}{a\,x^3} - \frac{b^2}{a^2x^5} - \frac{25}{4a\,x^5} + \frac{b^3}{a^3x^7} + \frac{45b}{4a^2x^7} - \frac{b^4}{a^4x^9} - \frac{69b^2}{4a^3x^9} - \frac{125}{4a^2x^9} + \frac{b^5}{a^5x^{11}} + \frac{97b^3}{4a^4x^{11}} + \frac{100b}{a^3x^{11}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 3$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{3} a_i x^i \\ &= \frac{1}{2}bx + \frac{1}{2}a\,x^3 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^2 = x^2$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}b^2x^2 + \frac{1}{2}ab\,x^4 + \frac{1}{4}a^2x^6$$

This shows that the coefficient of x^2 in the above is $\frac{b^2}{4}$. Now we need to find the coefficient of x^2 in r. How this is done depends on if $v = 0$ or not. Since $v = 3$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^2 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{a^2x^6 + 2ab\,x^4 + b^2x^2 - 10a\,x^2 - 6b}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{a^2x^6}{4} + \frac{ab\,x^4}{2} + \left(-\frac{5a}{2} + \frac{b^2}{4}\right)x^2 - \frac{3b}{2}\right) + (0) \\
&= \frac{a^2x^6}{4} + \frac{ab\,x^4}{2} + \left(-\frac{5a}{2} + \frac{b^2}{4}\right)x^2 - \frac{3b}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5a}{2} + \frac{b^2}{4}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{5a}{2} + \frac{b^2}{4}\right) - \left(\frac{b^2}{4}\right) \\
&= -\frac{5a}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}bx + \frac{1}{2}a\,x^3 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5a}{2}}{\frac{a}{2}} - 3\right) &&= -4 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5a}{2}}{\frac{a}{2}} - 3\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{2}a\,x^2 - \frac{3}{2}b + \frac{1}{4}a^2x^6 + \frac{1}{2}ab\,x^4 + \frac{1}{4}b^2x^2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-6	$\frac{1}{2}bx+\frac{1}{2}a\,x^3$	-4	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=1$, and since there are no poles then

$$\begin{aligned}d&=\alpha_\infty^-\\&=1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega&=(-)[\sqrt{r}]_\infty\\&=0+(-)\left(\frac{1}{2}bx+\frac{1}{2}a\,x^3\right)\\&=-\frac{1}{2}bx-\frac{1}{2}a\,x^3\\&=-\frac{x(a\,x^2+b)}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2}bx-\frac{1}{2}a\,x^3\right)(1)+\left(\left(-\frac{b}{2}-\frac{3a\,x^2}{2}\right)+\left(-\frac{1}{2}bx-\frac{1}{2}a\,x^3\right)^2-\left(-\frac{5}{2}a\,x^2-\frac{3}{2}b+\frac{1}{4}a^2x^6+\frac{1}{2}ab\,x\right.\right.$$

$a_0($

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= (x)\,\mathrm{e}^{\int \left(-\frac{1}{2}bx - \frac{1}{2}a\,x^3\right)dx} \\ &= (x)\,\mathrm{e}^{-\frac{\left(a\,x^2+b\right)^2}{8a}} \\ &= x\,\mathrm{e}^{-\frac{\left(a\,x^2+b\right)^2}{8a}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x\left(a\,x^2+b\right)}{1}\,dx} \\ &= z_1 e^{-\frac{\left(a\,x^2+b\right)^2}{8a}} \\ &= z_1\left(\mathrm{e}^{-\frac{\left(a\,x^2+b\right)^2}{8a}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-\frac{\left(a\,x^2+b\right)^2}{4a}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{x\left(a\,x^2+b\right)}{1}dx}}{\left(y_1\right)^2}\, dx$$
$$= y_1 \int \frac{e^{-\frac{\left(a\,x^2+b\right)^2}{4a}}}{\left(y_1\right)^2}\, dx$$
$$= y_1 \left(\int \frac{\mathrm{e}^{\frac{\left(a\,x^2+b\right)^2}{4a}}}{x^2} dx\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(x\,\mathrm{e}^{-\frac{\left(a\,x^2+b\right)^2}{4a}}\right) + c_2 \left(x\,\mathrm{e}^{-\frac{\left(a\,x^2+b\right)^2}{4a}} \left(\int \frac{\mathrm{e}^{\frac{\left(a\,x^2+b\right)^2}{4a}}}{x^2} dx\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{-\frac{\left(a\,x^2+b\right)^2}{4a}} + c_2 x\,\mathrm{e}^{-\frac{\left(a\,x^2+b\right)^2}{4a}} \left(\int \frac{\mathrm{e}^{\frac{\left(a\,x^2+b\right)^2}{4a}}}{x^2} dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{-\frac{\left(a\,x^2+b\right)^2}{4a}} + c_2 x\,\mathrm{e}^{-\frac{\left(a\,x^2+b\right)^2}{4a}} \left(\int \frac{\mathrm{e}^{\frac{\left(a\,x^2+b\right)^2}{4a}}}{x^2} dx\right)$$

Verified OK.

27.32.2 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$y'' + x\left(a\,x^2 + b\right) y' + \left(4a\,x^2 + 2b\right) y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= x\left(a\,x^2 + b\right) \\ q(x) &= 4a\,x^2 + 2b \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q = 0 \\ \xi'' - \left(x\left(a\,x^2 + b\right)\xi(x)\right)' + \left(\left(4a\,x^2 + 2b\right)\xi(x)\right) = 0 \\ \xi''(x) - x\left(a\,x^2 + b\right)\xi'(x) + \left(a\,x^2 + b\right)\xi(x) = 0 \end{aligned}$$

Which is solved for $\xi(x)$. In normal form the ode

$$\xi''(x) + \left(-a\,x^3 - bx\right)\xi'(x) + \left(a\,x^2 + b\right)\xi(x) = 0 \tag{1}$$

Becomes

$$\xi''(x) + p(x)\,\xi'(x) + q(x)\,\xi(x) = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -a\,x^3 - bx \\ q(x) &= a\,x^2 + b \end{aligned}$$

Applying change of variables on the depndent variable $\xi(x) = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not $\xi(x)$.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{n(-a\,x^3-bx)}{x}+a\,x^2+b=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(x)+\left(\frac{2}{x}-a\,x^3-bx\right)v'(x)&=0\\ v''(x)+\left(\frac{2}{x}-a\,x^3-bx\right)v'(x)&=0\end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x)+\left(\frac{2}{x}-a\,x^3-bx\right)u(x)=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\ &=f(x)g(u)\\ &=\frac{u(a\,x^4+b\,x^2-2)}{x}\end{aligned}$$

Where $f(x)=\frac{a\,x^4+b\,x^2-2}{x}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du&=\frac{a\,x^4+b\,x^2-2}{x}\,dx\\ \int\frac{1}{u}\,du&=\int\frac{a\,x^4+b\,x^2-2}{x}\,dx\\ \ln(u)&=\frac{a\,x^4}{4}+\frac{b\,x^2}{2}-2\ln(x)+c_1\\ u&=\mathrm{e}^{\frac{a\,x^4}{4}+\frac{b\,x^2}{2}-2\ln(x)+c_1}\\ &=c_1\mathrm{e}^{\frac{a\,x^4}{4}+\frac{b\,x^2}{2}-2\ln(x)}\end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= \int c_1 \mathrm{e}^{\frac{a\,x^4}{4}+\frac{b\,x^2}{2}-2\ln(x)} dx + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
\xi(x) &= v(x)\, x^n \\
&= \left(\int c_1 \mathrm{e}^{\frac{a\,x^4}{4}+\frac{b\,x^2}{2}-2\ln(x)} dx + c_2\right) x \\
&= \left(c_1 \left(\int \frac{\mathrm{e}^{\frac{x^2\left(a\,x^2+2b\right)}{4}}}{x^2} dx\right) + c_2\right) x
\end{aligned}
$$

The original ode (2) now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)} \\
y' + y\left(x\left(a\,x^2+b\right) - \frac{c_3 \mathrm{e}^{\frac{a\,x^4}{4}+\frac{b\,x^2}{2}-2\ln(x)} x + \int c_3 \mathrm{e}^{\frac{a\,x^4}{4}+\frac{b\,x^2}{2}-2\ln(x)} dx + c_2}{\left(\int c_3 \mathrm{e}^{\frac{a\,x^4}{4}+\frac{b\,x^2}{2}-2\ln(x)} dx + c_2\right) x}\right) &= 0
\end{aligned}
$$

Which is now a first order ode. This is now solved for y. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= -\frac{y\left(\left(\int c_3 \mathrm{e}^{\frac{a\,x^4}{4}+\frac{b\,x^2}{2}-2\ln(x)} dx\right) a\,x^4 + c_2 a\,x^4 + \left(\int c_3 \mathrm{e}^{\frac{a\,x^4}{4}+\frac{b\,x^2}{2}-2\ln(x)} dx\right) b\,x^2 + c_2 b\,x^2 - c_3 \mathrm{e}^{\frac{a\,x^4}{4}+\frac{b\,x^2}{2}-2\ln(x}}{\left(\int c_3 \mathrm{e}^{\frac{a\,x^4}{4}+\frac{b\,x^2}{2}-2\ln(x)} dx + c_2\right) x}
\end{aligned}
$$

Where $f(x) = -\frac{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) a x^4 + c_2 a x^4 + \left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) b x^2 + c_2 b x^2 - c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} x - \left(\int c_3 e^{\frac{a x^4}{4}+\cdots}\right.}{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx + c_2\right) x}$

and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\, dy = -\frac{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) a x^4 + c_2 a x^4 + \left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) b x^2 + c_2 b x^2 - c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\cdots}}{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx + c_2\right) x}$$

$$\int \frac{1}{y}\, dy = \int -\frac{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) a x^4 + c_2 a x^4 + \left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) b x^2 + c_2 b x^2 - c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}\cdots}}{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx + c_2\right) x}$$

$$\ln(y) = \int -\frac{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) a x^4 + c_2 a x^4 + \left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) b x^2 + c_2 b x^2 - c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}\cdots}}{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx + c_2\right) x}$$

$$y = e^{\int -\frac{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) a x^4 + c_2 a x^4 + \left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) b x^2 + c_2 b x^2 - c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} x - \left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln\cdots}\right.}{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx + c_2\right) x}}$$

$$= c_3 e^{\int -\frac{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) a x^4 + c_2 a x^4 + \left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) b x^2 + c_2 b x^2 - c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} x - \left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-\cdots}\right.}{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx + c_2\right) x}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_3 e^{\int -\frac{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) a x^4 + c_2 a x^4 + \left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) b x^2 + c_2 b x^2 - c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} x - \left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) - \cdots}{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx + c_2\right) x}}$$

Summary

The solution(s) found are the following

$$y = c_3 e^{\int -\frac{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) a x^4 + c_2 a x^4 + \left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) b x^2 + c_2 b x^2 - c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} x - \left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) - \cdots}{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx + c_2\right) x}} \tag{1}$$

Verification of solutions

$$y = c_3 e^{\int -\frac{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) a x^4 + c_2 a x^4 + \left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) b x^2 + c_2 b x^2 - c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} x - \left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx\right) - \cdots}{\left(\int c_3 e^{\frac{a x^4}{4}+\frac{b x^2}{2}-2\ln(x)} dx + c_2\right) x}}$$

Verified OK.

27.32.3 Maple step by step solution

Let's solve

$y'' + x(a\,x^2 + b)\,y' + (4a\,x^2 + 2b)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

- ◦ Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

- ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ◦ Shift index using $k- >k + 2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_0 b + 2a_2 + (3a_1 b + 6a_3)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k b(k+2) + a_{k-2} a(k+2)\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_0 b + 2a_2 = 0, 3a_1 b + 6a_3 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = -a_0 b, a_3 = -\frac{a_1 b}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\left(a_{k-2}a + a_k b + k a_{k+2} + a_{k+2}\right) = 0$

- Shift index using $k- >k+2$

  $(k+4)\left(a_k a + a_{k+2} b + (k+2) a_{k+4} + a_{k+4}\right) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k a + a_{k+2} b}{k+3}, a_2 = -a_0 b, a_3 = -\frac{a_1 b}{2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.406 (sec). Leaf size: 70

```
dsolve(diff(y(x),x$2)+(a*x^3+b*x)*diff(y(x),x)+2*(2*a*x^2+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{(a\,x^2+2b)x^2}{4}} \left(\mathrm{HeunB}\left(\frac{1}{2}, \frac{b}{\sqrt{a}}, \frac{5}{2}, -\frac{3b}{2\sqrt{a}}, \frac{\sqrt{a}\,x^2}{2}\right) c_1 x + \mathrm{HeunB}\left(-\frac{1}{2}, \frac{b}{\sqrt{a}}, \frac{5}{2}, -\frac{3b}{2\sqrt{a}}, \frac{\sqrt{a}\,x^2}{2}\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 2.589 (sec). Leaf size: 63

```
DSolve[y''[x]+(a*x^3+b*x)*y'[x]+2*(2*a*x^2+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to xe^{-\frac{1}{4}x^2(ax^2+2b)}\left(c_2\int_1^x \frac{e^{\frac{1}{4}(aK[1]^4+2bK[1]^2)}}{K[1]^2}dK[1]+c_1\right)$$

27.33 problem 43

Internal problem ID [10867]
Internal file name [`OUTPUT/10123_Sunday_December_24_2023_05_12_43_PM_85035290/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 43.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \left(ab\,x^3 + b\,x^2 + 2a\right) y' + a^2\left(x^3 b + 1\right) y = 0}$$

27.33.1 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \left(\left(a\,x^3 + x^2\right) b + 2a\right) y' + a^2\left(x^3 b + 1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \left(a\,x^3 + x^2\right) b + 2a \\ C &= a^2\left(x^3 b + 1\right) \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{bx(a^2b\,x^5 + 2ab\,x^4 + x^3b + 10ax + 4)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= bx\left(a^2b\,x^5 + 2ab\,x^4 + x^3b + 10ax + 4\right) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{bx(a^2b\,x^5 + 2ab\,x^4 + x^3b + 10ax + 4)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 72: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 6 \\ &= -6 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -6 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -6$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{6}{2} = 3$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{3} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^3$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{ab\,x^3}{2} + \frac{b\,x^2}{2} + \frac{5}{2x} - \frac{3}{2a\,x^2} + \frac{3}{2a^2x^3} - \frac{25}{4ab\,x^5} - \frac{3}{2a^3x^4} + \frac{55}{4a^2b\,x^6} + \frac{3}{2a^4x^5} - \frac{3}{2a^5x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{ab}{2}$$

From Eq. (9) the sum up to $v = 3$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{3} a_i x^i \\ &= \frac{1}{2}b\,x^2 + \frac{1}{2}ab\,x^3 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^2 = x^2$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}b^2x^4 + \frac{1}{2}a\,b^2x^5 + \frac{1}{4}a^2b^2x^6$$

This shows that the coefficient of x^2 in the above is 0. Now we need to find the coefficient of x^2 in r. How this is done depends on if $v = 0$ or not. Since $v = 3$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^2 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{bx(a^2b\,x^5 + 2ab\,x^4 + x^3b + 10ax + 4)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{5}{2}ab\,x^2 + bx + \frac{1}{4}a^2b^2x^6 + \frac{1}{2}a\,b^2x^5 + \frac{1}{4}b^2x^4\right) + (0) \\
&= \frac{5}{2}ab\,x^2 + bx + \frac{1}{4}a^2b^2x^6 + \frac{1}{2}a\,b^2x^5 + \frac{1}{4}b^2x^4
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{5ab}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{5ab}{2}\right) - (0) \\
&= \frac{5ab}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}b\,x^2 + \frac{1}{2}ab\,x^3 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5ab}{2}}{\frac{ab}{2}} - 3\right) &&= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5ab}{2}}{\frac{ab}{2}} - 3\right) &&= -4
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{bx(a^2b\,x^5 + 2ab\,x^4 + x^3b + 10ax + 4)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-6	$\frac{1}{2}b\,x^2 + \frac{1}{2}ab\,x^3$	1	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{1}{2}b\,x^2 + \frac{1}{2}ab\,x^3\right) \\ &= \frac{1}{2}b\,x^2 + \frac{1}{2}ab\,x^3 \\ &= \frac{b\,x^2(ax+1)}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2}b\,x^2 + \frac{1}{2}ab\,x^3\right)(1) + \left(\left(\frac{3}{2}ab\,x^2 + bx\right) + \left(\frac{1}{2}b\,x^2 + \frac{1}{2}ab\,x^3\right)^2 - \left(\frac{bx(a^2b\,x^5 + 2ab\,x^4 + x^3b + 10}{4}\right.\right.$$

$$-b\,x^2(a$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{a}\right) e^{\int \left(\frac{1}{2} b x^2 + \frac{1}{2} ab x^3\right) dx} \\
&= \left(x + \frac{1}{a}\right) e^{\frac{b\left(\frac{1}{4} a x^4 + \frac{1}{3} x^3\right)}{2}} \\
&= \frac{(ax+1) e^{\frac{1}{8} ab x^4 + \frac{1}{6} x^3 b}}{a}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\left(a x^3 + x^2\right) b + 2a}{1} \, dx} \\
&= z_1 e^{-\frac{1}{8} ab x^4 - \frac{1}{6} x^3 b - ax} \\
&= z_1 \left(e^{-\frac{x\left(ab x^3 + \frac{4}{3} b x^2 + 8a\right)}{8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(ax+1) e^{-ax}}{a}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{(a\,x^3+x^2)b+2a}{1}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{-\frac{x\left(ab\,x^3+\frac{4}{3}b\,x^2+8a\right)}{4}}}{(y_1)^2}\,dx$$

$$= y_1 \left(\int \frac{a^2 \mathrm{e}^{-\frac{x^3\left(ax+\frac{4}{3}\right)b}{4}}}{(ax+1)^2}dx\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a}\right) + c_2 \left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a}\left(\int \frac{a^2 \mathrm{e}^{-\frac{x^3\left(ax+\frac{4}{3}\right)b}{4}}}{(ax+1)^2}dx\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(ax+1)\,\mathrm{e}^{-ax}}{a} + c_2(ax+1)\,\mathrm{e}^{-ax} a \left(\int \frac{\mathrm{e}^{-\frac{x^3\left(ax+\frac{4}{3}\right)b}{4}}}{(ax+1)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1(ax+1)\,\mathrm{e}^{-ax}}{a} + c_2(ax+1)\,\mathrm{e}^{-ax} a \left(\int \frac{\mathrm{e}^{-\frac{x^3\left(ax+\frac{4}{3}\right)b}{4}}}{(ax+1)^2}dx\right)$$

Verified OK.

27.33.2 Maple step by step solution

Let's solve

$y'' + ((a\,x^3 + x^2)\,b + 2a)\,y' + a^2(x^3b + 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -(ab\,x^3 + b\,x^2 + 2a)\,y' - a^2(x^3b + 1)\,y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + (ab\,x^3 + b\,x^2 + 2a)\,y' + a^2(x^3b + 1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ∘ Convert $x^m \cdot y$ to series expansion for $m = 0..3$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ∘ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ∘ Convert $x^m \cdot y'$ to series expansion for $m = 0..3$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

- ∘ Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k + 1 - m)\, x^k$

- ∘ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k - 1)\, x^{k-2}$

- ∘ Shift index using $k- >k + 2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k + 2)\,(k + 1)\, x^k$

Rewrite ODE with series expansions

$$a_0a^2+2a_1a+2a_2+\left(a_1a^2+4aa_2+6a_3\right)x+\left(a_2a^2+6aa_3+a_1b+12a_4\right)x^2+\left(\sum_{k=3}^{\infty}\left(a_{k+2}(k+2)\right.\right.$$

- The coefficients of each power of x must be 0

$$[2a_2+a_0a^2+2a_1a=0, a_1a^2+4aa_2+6a_3=0, a_2a^2+6aa_3+a_1b+12a_4=0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2=-\frac{1}{2}a_0a^2-a_1a, a_3=\frac{1}{3}a_0a^3+\frac{1}{2}a_1a^2, a_4=-\frac{1}{8}a_0a^4-\frac{1}{6}a_1a^3-\frac{1}{12}a_1b\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(ba_{k-3}+a_k)a^2+(a_{k-2}(k-2)b+2a_{k+1}(k+1))a+a_{k-1}(k-1)b+a_{k+2}(k+2)(k+1)=0$$

- Shift index using $k->k+3$

$$(ba_k+a_{k+3})a^2+(a_{k+1}(k+1)b+2a_{k+4}(k+4))a+a_{k+2}(k+2)b+a_{k+5}(k+5)(k+4)=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty}a_kx^k, a_{k+5}=-\frac{a_ka^2b+abka_{k+1}+a^2a_{k+3}+aba_{k+1}+2aka_{k+4}+bka_{k+2}+8aa_{k+4}+2ba_{k+2}}{(k+5)(k+4)}, a_2=-\frac{1}{2}a_0a^2-a_1\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
   No special function solution was found.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.437 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$2)+(a*b*x^3+b*x^2+2*a)*diff(y(x),x)+a^2*(b*x^3+1)*y(x)=0,y(x), singsol=al
```

$$y(x) = e^{-ax}\left(c_2\left(\int \frac{e^{-\frac{b\,x^3\left(ax+\frac{4}{3}\right)}{4}}}{(ax+1)^2}dx\right)+c_1\right)(ax+1)$$

✓ Solution by Mathematica

Time used: 3.356 (sec). Leaf size: 57

```
DSolve[y''[x]+(a*b*x^3+b*x^2+2*a)*y'[x]+a^2*(b*x^3+1)*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to e^{-ax}(ax+1)\left(c_2 \int_1^x \frac{e^{-\frac{1}{12}bK[1]^3(3aK[1]+4)}}{(aK[1]+1)^2}dK[1] + c_1\right)$$

27.34 problem 44

Internal problem ID [10868]
Internal file name [`OUTPUT/10124_Sunday_December_24_2023_05_12_44_PM_40460358/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 44.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + a\,x^n y' = 0}$$

27.34.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + a\,x^n p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= -a\,x^n p \end{aligned}$$

Where $f(x) = -a\,x^n$ and $g(p) = p$. Integrating both sides gives

$$\frac{1}{p}\,dp = -a\,x^n\,dx$$
$$\int \frac{1}{p}\,dp = \int -a\,x^n\,dx$$
$$\ln(p) = -\frac{a\,x^{1+n}}{1+n} + c_1$$
$$p = e^{-\frac{a\,x^{1+n}}{1+n}+c_1}$$
$$= c_1 e^{-\frac{a\,x^{1+n}}{1+n}}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_1 e^{-\frac{a\,x^{1+n}}{1+n}}$$

Integrating both sides gives

$$y = \int c_1 e^{-\frac{a\,x^{1+n}}{1+n}}\,dx$$
$$= \frac{c_1\left(\frac{a}{1+n}\right)^{-\frac{1}{1+n}}\left(\frac{(1+n)^2 x^{\frac{1}{1+n}+\frac{n}{1+n}-1-n}\left(\frac{a}{1+n}\right)^{\frac{1}{1+n}}\left(\frac{x^{1+n}a\,n^2}{1+n}+\frac{2x^{1+n}an}{1+n}+n^2+\frac{a\,x^{1+n}}{1+n}+3n+2\right)\left(\frac{a\,x^{1+n}}{1+n}\right)^{-\frac{2+n}{2(1+n)}}e^{-\frac{a\,x^{1+n}}{2(1+n)}}\,\text{Whitta}}{(2+n)(3+2n)a}\right.}{}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(\frac{a}{1+n}\right)^{-\frac{1}{1+n}}\left(\frac{(1+n)^2 x^{\frac{1}{1+n}+\frac{n}{1+n}-1-n}\left(\frac{a}{1+n}\right)^{\frac{1}{1+n}}\left(\frac{x^{1+n}a\,n^2}{1+n}+\frac{2x^{1+n}an}{1+n}+n^2+\frac{a\,x^{1+n}}{1+n}+3n+2\right)\left(\frac{a\,x^{1+n}}{1+n}\right)^{-\frac{2+n}{2(1+n)}}e^{-\frac{a\,x^{1+n}}{2(1+n)}}\,\text{Whittaker}}{(2+n)(3+2n)a}\right.}{} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(\frac{a}{1+n}\right)^{-\frac{1}{1+n}}\left(\frac{(1+n)^2 x^{\frac{1}{1+n}+\frac{n}{1+n}-1-n}\left(\frac{a}{1+n}\right)^{\frac{1}{1+n}}\left(\frac{x^{1+n}a\,n^2}{1+n}+\frac{2x^{1+n}an}{1+n}+n^2+\frac{a\,x^{1+n}}{1+n}+3n+2\right)\left(\frac{a\,x^{1+n}}{1+n}\right)^{-\frac{2+n}{2(1+n)}}e^{-\frac{a\,x^{1+n}}{2(1+n)}}\,\text{Whittaker}}{(2+n)(3+2n)a}\right.}{} + c_2$$

Verified OK.

27.34.2 Maple step by step solution

Let's solve

$y'' + a\,x^n y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + a\,x^n u(x) = 0$

- Separate variables

  $\frac{u'(x)}{u(x)} = -a\,x^n$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)}dx = \int -a\,x^n dx + c_1$

- Evaluate integral

  $\ln\left(u(x)\right) = -\frac{a\,x^{1+n}}{1+n} + c_1$

- Solve for $u(x)$

  $u(x) = \mathrm{e}^{-\frac{-c_1 n + a\,x^{1+n} - c_1}{1+n}}$

- Solve 1st ODE for $u(x)$

  $u(x) = \mathrm{e}^{-\frac{-c_1 n + a\,x^{1+n} - c_1}{1+n}}$

- Make substitution $u = y'$

  $y' = \mathrm{e}^{-\frac{-c_1 n + a\,x^{1+n} - c_1}{1+n}}$

- Integrate both sides to solve for y

  $\int y' dx = \int \mathrm{e}^{-\frac{-c_1 n + a\,x^{1+n} - c_1}{1+n}} dx + c_2$

- Compute integrals

  $$y = \frac{\mathrm{e}^{-\frac{-c_1 n - c_1}{1+n}} \left(\frac{a}{1+n}\right)^{-\frac{1}{1+n}} \left(\frac{(1+n)^2 x^{\frac{1}{1+n}+\frac{n}{1+n}-1-n} \left(\frac{a}{1+n}\right)^{\frac{1}{1+n}} \left(\frac{x^{1+n} a\,n^2}{1+n} + \frac{2x^{1+n} a n}{1+n} + n^2 + \frac{a\,x^{1+n}}{1+n} + 3n + 2\right) \left(\frac{a\,x^{1+n}}{1+n}\right)^{-\frac{2+n}{2(1+n)}} \mathrm{e}^{-\frac{a\,x^{1+n}}{2(1+n)}} \cdots}{(2+n)(3+2n)a} \right.}{\cdots}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
<- LODE missing y successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 244

```
dsolve(diff(y(x),x$2)+a*x^n*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^{-n}\left(\left(\frac{a x x^{n}}{n+1}\right)^{\frac{-n-2}{2n+2}} c_2 \left(\frac{a}{n+1}\right)^{\frac{1}{n+1}} \mathrm{e}^{-\frac{x^{n} a x}{2n+2}} (n+2)^2 (n+1)^2 \operatorname{WhittakerM}\left(\frac{n+2}{2n+2}, \frac{2n+3}{2n+2}, \frac{a x x^{n}}{n+1}\right) + \left(\frac{a x x^{n}}{n+1}\right)^{\frac{-n-2}{2n+2}} c_2 \left(\frac{a}{n+ \cdots}\right.}{(n+2)(2n \cdots}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 56

```
DSolve[y''[x]+a*x^n*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{c_1 x \left(\frac{a x^{n+1}}{n+1}\right)^{-\frac{1}{n+1}} \Gamma\left(\frac{1}{n+1}, \frac{a x^{n+1}}{n+1}\right)}{n+1}$$

27.35 problem 45

Internal problem ID [10869]
Internal file name [`OUTPUT/10125_Sunday_December_24_2023_05_13_38_PM_31232604/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 45.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + a\,x^n y' + y x^{n-1} b = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.407 (sec). Leaf size: 96

```
dsolve(diff(y(x),x$2)+a*x^n*diff(y(x),x)+b*x^(n-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x\left(\text{KummerU}\left(\frac{1+n-\frac{b}{a}}{n+1}, \frac{n+2}{n+1}, \frac{a x\, x^n}{n+1}\right) c_2 + \text{KummerM}\left(\frac{1+n-\frac{b}{a}}{n+1}, \frac{n+2}{n+1}, \frac{a x\, x^n}{n+1}\right) c_1\right) \mathrm{e}^{-\frac{a x\, x^n}{n+1}}$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 120

```
DSolve[y''[x]+a*x^n*y'[x]+b*x^(n-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \left(\frac{1}{n}+1\right)^{-\frac{1}{n+1}} n^{-\frac{1}{n+1}} a^{\frac{1}{n+1}} (x^n)^{\frac{1}{n}} \text{Hypergeometric1F1}\left(\frac{a+b}{na+a}, \frac{n+2}{n+1}, -\frac{a(x^n)^{1+\frac{1}{n}}}{n+1}\right) + c_1 \text{Hypergeometric1F1}\left(\frac{b}{na+a}, \frac{n}{n+1}, -\frac{a(x^n)^{1+\frac{1}{n}}}{n+1}\right)$$

27.36 problem 46

Internal problem ID [10870]
Internal file name [`OUTPUT/10126_Sunday_December_24_2023_05_13_38_PM_911658/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 46.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2a\,x^n y' + a\left(a\,x^{2n} + n\,x^{n-1}\right)y = 0}$$

27.36.1 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 2a\,x^n$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2a\,x^n\,dx} \\ &= \mathrm{e}^{\frac{a\,x^{1+n}}{1+n}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\mathrm{e}^{\frac{a\,x^{1+n}}{1+n}} y\right)'' &= 0
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{\frac{a\,x^{1+n}}{1+n}} y\right)' = c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{\frac{a\,x^{1+n}}{1+n}} y\right) = c_1 x + c_2
$$

Hence the solution is

$$
y = \frac{c_1 x + c_2}{\mathrm{e}^{\frac{a\,x^{1+n}}{1+n}}}
$$

Or

$$
y = c_1 x\,\mathrm{e}^{-\frac{a\,x^{n+1}}{n+1}} + c_2 \mathrm{e}^{-\frac{a\,x^{n+1}}{n+1}}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{a\,x^{n+1}}{n+1}} + c_2 \mathrm{e}^{-\frac{a\,x^{n+1}}{n+1}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{a\,x^{n+1}}{n+1}} + c_2 \mathrm{e}^{-\frac{a\,x^{n+1}}{n+1}}
$$

Verified OK.

27.36.2 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= 2a\,x^n \\
q(x) &= x^{2n}a^2 + \frac{an\,x^n}{x}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= x^{2n}a^2 + \frac{an\,x^n}{x} - \frac{(2a\,x^n)'}{2} - \frac{(2a\,x^n)^2}{4} \\
&= x^{2n}a^2 + \frac{an\,x^n}{x} - \frac{\left(\frac{2an\,x^n}{x}\right)}{2} - \frac{(4x^{2n}a^2)}{4} \\
&= x^{2n}a^2 + \frac{an\,x^n}{x} - \left(\frac{an\,x^n}{x}\right) - x^{2n}a^2 \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\,z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{2a\,x^n}{2}} \\
&= \mathrm{e}^{-\frac{a\,x^{n+1}}{n+1}}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\,\mathrm{e}^{-\frac{a\,x^{n+1}}{n+1}} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(x)\,\mathrm{e}^{-\frac{a\,x^{n+1}}{n+1}} = 0
$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$
v(x) = c_1 x + c_2
$$

Now that $v(x)$ is known, then

$$
\begin{aligned}
y &= v(x)\,z(x) \\
&= (c_1 x + c_2)\,(z(x))
\end{aligned} \tag{7}
$$

But from (5)

$$z(x) = e^{-\frac{a\,x^{n+1}}{n+1}}$$

Hence (7) becomes

$$y = e^{-\frac{a\,x^{n+1}}{n+1}}(c_1x + c_2)$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{a\,x^{n+1}}{n+1}}(c_1x + c_2) \tag{1}$$

Verification of solutions

$$y = e^{-\frac{a\,x^{n+1}}{n+1}}(c_1x + c_2)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+2*a*x^n*diff(y(x),x)+a*(a*x^(2*n)+n*x^(n-1))*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{a\,x^{n+1}}{n+1}}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.112 (sec). Leaf size: 28

```
DSolve[y''[x]+2*a*x^n*y'[x]+a*(a*x^(2*n)+n*x^(n-1))*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to (c_2x + c_1)e^{-\frac{ax^{n+1}}{n+1}}$$

27.37 problem 47

Internal problem ID [10871]
Internal file name [`OUTPUT/10127_Sunday_December_24_2023_05_13_44_PM_32460967/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 47.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + a\,x^{n} y' + \left(b\,x^{2n} + c\,x^{n-1}\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 171

```
dsolve(diff(y(x),x$2)+a*x^n*diff(y(x),x)+(b*x^(2*n)+c*x^(n-1))*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{x^{n+1}\left(a+\sqrt{a^2-4b}\right)}{2n+2}} x \left(\mathrm{KummerM}\left(\frac{(n+2)\sqrt{a^2-4b}+an-2c}{\sqrt{a^2-4b}\,(2n+2)}, \frac{n+2}{n+1}, \frac{\sqrt{a^2-4b}\,x^{n+1}}{n+1}\right) c_1 + \mathrm{KummerU}\left(\frac{(n+2)\sqrt{a^2-4b}+an-2c}{\sqrt{a^2-4b}\,(2n+2)}, \frac{n+2}{n+1}, \frac{\sqrt{a^2-4b}\,x^{n+1}}{n+1}\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.492 (sec). Leaf size: 333

```
DSolve[y''[x]+a*x^n*y'[x]+(b*x^(2*n)+c*x^(n-1))*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$y(x)$

$$\rightarrow 2^{\frac{n}{2n+2}} x^{-n/2} \left(x^{n+1}\right)^{\frac{n}{2n+2}} \exp\left(-\frac{1}{2} x^{n+1} \left(\frac{\sqrt{a^2-4b}}{\sqrt{(n+1)^2}} + \frac{a}{n+1}\right)\right) \left(c_1 \text{HypergeometricU}\left(\frac{n\left(\sqrt{(n+1)^2} a^2\right.}{}\right.\right.$$

$$\left. + c_2 L^{-\frac{1}{n+1}}_{\frac{2\sqrt{a^2-4b}c(n+1)-n\left(\sqrt{(n+1)^2}a^2+\sqrt{a^2-4b}(n+1)a-4b\sqrt{(n+1)^2}\right)}{2\left(a^2-4b\right)(n+1)\sqrt{(n+1)^2}}} \left(\frac{\sqrt{a^2-4b}x^{n+1}}{\sqrt{(n+1)^2}}\right)\right)$$

27.38 problem 48

Internal problem ID [10872]
Internal file name [`OUTPUT/10128_Sunday_December_24_2023_05_13_45_PM_9092565/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 48.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + a\,x^{n}y' - b\left(a\,x^{m+n} + b\,x^{2m} + m\,x^{m-1}\right)y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+a*x^n*diff(y(x),x)-b*(a*x^(n+m)+b*x^(2*m)+m*x^(m-1))*y(x)=0,y(x), sing
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+a*x^n*y'[x]-b*(a*x^(n+m)+b*x^(2*m)+m*x^(m-1))*y[x]==0,y[x],x,IncludeSingularSo
```

Not solved

27.39 problem 49

Internal problem ID [10873]
Internal file name [`OUTPUT/10129_Sunday_December_24_2023_05_13_46_PM_18930705/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 49.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + 2a\,x^n y' + \left(x^{2n}a^2 + b\,x^{2m} + an\,x^{n-1} + c\,x^{m-1}\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.406 (sec). Leaf size: 147

```
dsolve(diff(y(x),x$2)+2*a*x^n*diff(y(x),x)+(a^2*x^(2*n)+b*x^(2*m)+a*n*x^(n-1)+c*x^(m-1))*y(x
```

$$y(x) = x\left(\text{KummerM}\left(\frac{(m+2)\sqrt{b}+ic}{\sqrt{b}\,(2m+2)}, \frac{m+2}{1+m}, \frac{2i\sqrt{b}\,x^{1+m}}{1+m}\right)c_1 + \text{KummerU}\left(\frac{(m+2)\sqrt{b}+ic}{\sqrt{b}\,(2m+2)}, \frac{m+2}{1+m}, \frac{2i\sqrt{b}\,x^{1+m}}{1+m}\right)c_2\right)\mathrm{e}^{\frac{-i(n+1)\sqrt{b}\,x^{1+m}-x^{n+1}a(1+m)}{(n+1)(1+m)}}$$

✓ Solution by Mathematica

Time used: 0.376 (sec). Leaf size: 236

```
DSolve[y''[x]+2*a*x^n*y'[x]+(a^2*x^(2*n)+b*x^(2*m)+a*n*x^(n-1)+c*x^(m-1))*y[x]==0,y[x],x,Inc
```

$y(x)$

$$\rightarrow 2^{\frac{m}{2m+2}} x^{-m/2} \left(x^{m+1}\right)^{\frac{m}{2m+2}} \exp\left(-x\left(\frac{ax^n}{n+1} + \frac{\sqrt{b}x^m}{\sqrt{-(m+1)^2}}\right)\right) \left(c_1 \operatorname{HypergeometricU}\left(-\frac{(m+1)\left(mc+\right.}{2\sqrt{b}\left(\right.}\right.\right.$$

27.40 problem 50

Internal problem ID [10874]
Internal file name [`OUTPUT/10130_Sunday_December_24_2023_05_13_59_PM_72771316/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 50.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + (a\,x^n + b)\,y' + c(a\,x^n + b - c)\,y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+(a*x^n+b)*diff(y(x),x)+c*(a*x^n+b-c)*y(x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+(a*x^n+b)*y'[x]+c*(a*x^n+b-c)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

27.41 problem 51

Internal problem ID [10875]
Internal file name [`OUTPUT/10131_Sunday_December_24_2023_05_14_00_PM_4386285/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 51.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$y'' + (a\,x^n + 2b)\,y' + \left(ab\,x^n - a\,x^{n-1} + b^2\right) y = 0$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- Equivalence, under non-integer power transformations successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 167

```
dsolve(diff(y(x),x$2)+(a*x^n+2*b)*diff(y(x),x)+(a*b*x^n-a*x^(n-1)+b^2)*y(x)=0,y(x), singsol=
```

$$y(x) = \mathrm{e}^{-\frac{(a\,x^n+2(n+1)b)x}{2n+2}} c_2(n+1)\left(a\,x^{-\frac{n}{2}} + x^{-\frac{3n}{2}-1}n\right) \mathrm{WhittakerM}\left(\frac{-n-2}{2n+2}, \frac{2n+1}{2n+2}, \frac{a\,x^{n+1}}{n+1}\right) + c_2 n^2 x^{-\frac{3n}{2}-1} \mathrm{e}^{-\frac{(a\,x^n+2(n+1)b)x}{2n+2}} \mathrm{WhittakerM}\left(\frac{n}{2n+2}, \frac{2n+1}{2n+2}, \frac{a\,x^{n+1}}{n+1}\right) + c_1 \mathrm{e}^{-bx} x$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+(a*x^n+2*b)*y'[x]+(a*b*x^n-a*x^(n-1)+b^2)*y[x]==0,y[x],x,IncludeSingularSoluti
```

Not solved

27.42 problem 52

Internal problem ID [10876]
Internal file name [OUTPUT/10132_Sunday_December_24_2023_05_14_01_PM_22677750/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 52.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + \left(ab\,x^{n} + b\,x^{n-1} + 2a\right) y' + a^{2}(b\,x^{n} + 1)\,y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+(a*b*x^n+b*x^(n-1)+2*a)*diff(y(x),x)+a^2*(b*x^n+1)*y(x)=0,y(x), singso
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+(a*b*x^n+b*x^(n-1)+2*a)*y'[x]+a^2*(b*x^n+1)*y[x]==0,y[x],x,IncludeSingularSolu
```

Not solved

27.43 problem 53

Internal problem ID [10877]
Internal file name [OUTPUT/10133_Sunday_December_24_2023_05_14_02_PM_16480413/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 53.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$y'' + \left(ab\,x^{n} + 2b\,x^{n-1} - a^{2}x\right) y' + a\left(ab\,x^{n} + b\,x^{n-1} - a^{2}x\right) y = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+(a*b*x^n+2*b*x^(n-1)-a^2*x)*diff(y(x),x)+a*(a*b*x^n+b*x^(n-1)-a^2*x)*y
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+(a*b*x^n+2*b*x^(n-1)-a^2*x)*y'[x]+a*(a*b*x^n+b*x^(n-1)-a^2*x)*y[x]==0,y[x],x,I
```

Not solved

27.44 problem 54

Internal problem ID [10878]
Internal file name [`OUTPUT/10134_Sunday_December_24_2023_05_14_03_PM_1011614/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 54.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + x^n \left(a\,x^2 + (ac+b)\,x + bc\right) y' - x^n (ax+b)\,y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   <- linear symmetries successful`
```

✓ Solution by Maple

Time used: 0.484 (sec). Leaf size: 78

```
dsolve(diff(y(x),x$2)+x^n*(a*x^2+(a*c+b)*x+b*c)*diff(y(x),x)-x^n*(a*x+b)*y(x)=0,y(x), singsol
```

$$y(x) = -(x+c)\left(\left(\int \frac{e^{-\frac{\left(a\,x^{2}(n+2)(n+1)+(ac+b)x(3+n)(n+1)+bc(3+n)(n+2)\right)x^{n+1}}{(3+n)(n+1)(n+2)}}}{(x+c)^{2}}dx\right)c_1+c_2\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+x^n*(a*x^2+(a*c+b)*x+b*c)*y'[x]-x^n*(a*x+b)*y[x]==0,y[x],x,IncludeSingularSolu
```

Not solved

27.45 problem 55

Internal problem ID [10879]
Internal file name [`OUTPUT/10135_Sunday_December_24_2023_05_14_04_PM_3647982/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y''+f(x)y'+g(x)y=0$
Problem number: 55.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y''+(a\,x^n+b\,x^m)\,y'-\left(a\,x^{n-1}+b\,x^{m-1}\right)y=0}$$

27.45.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$y''+(a\,x^n+b\,x^m)\,y'+\frac{(-a\,x^n-b\,x^m)\,y}{x}=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=a\,x^n+b\,x^m$$
$$q(x)=\frac{-a\,x^n-b\,x^m}{x}$$

Applying change of variables on the depndent variable $y=v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right)v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right)v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{n(a\,x^n+b\,x^m)}{x}+\frac{-a\,x^n-b\,x^m}{x}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x)+\left(\frac{2}{x}+a\,x^n+b\,x^m\right)v'(x)=0$$

$$v''(x)+\left(\frac{2}{x}+a\,x^n+b\,x^m\right)v'(x)=0 \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x)+\left(\frac{2}{x}+a\,x^n+b\,x^m\right)u(x)=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=-\frac{u(ax\,x^n+x^m bx+2)}{x}\end{aligned}$$

Where $f(x)=-\frac{ax\,x^n+x^m bx+2}{x}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du&=-\frac{ax\,x^n+x^m bx+2}{x}\,dx\\ \int\frac{1}{u}\,du&=\int-\frac{ax\,x^n+x^m bx+2}{x}\,dx\\ \ln(u)&=-2\ln(x)-\frac{ax\,\mathrm{e}^{n\ln(x)}}{n+1}-\frac{bx\,\mathrm{e}^{m\ln(x)}}{m+1}+c_1\\ u&=\mathrm{e}^{-2\ln(x)-\frac{ax\,\mathrm{e}^{n\ln(x)}}{n+1}-\frac{bx\,\mathrm{e}^{m\ln(x)}}{m+1}+c_1}\\ &=c_1\mathrm{e}^{-2\ln(x)-\frac{ax\,\mathrm{e}^{n\ln(x)}}{n+1}-\frac{bx\,\mathrm{e}^{m\ln(x)}}{m+1}}\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 \mathrm{e}^{-\frac{a\,x^{n+1}}{n+1}} \mathrm{e}^{-\frac{b\,x^{m+1}}{m+1}}}{x^2}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= \int \frac{c_1 \mathrm{e}^{-\frac{a\,x^{n+1}}{n+1}} \mathrm{e}^{-\frac{b\,x^{m+1}}{m+1}}}{x^2} dx + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(\int \frac{c_1 \mathrm{e}^{-\frac{a\,x^{n+1}}{n+1}} \mathrm{e}^{-\frac{b\,x^{m+1}}{m+1}}}{x^2} dx + c_2\right) x \\ &= \left(c_1 \left(\int \frac{\mathrm{e}^{-\frac{(a\,x^n (m+1)+b\,x^m (n+1))x}{(n+1)(m+1)}}}{x^2} dx\right) + c_2\right) x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(\int \frac{c_1 \mathrm{e}^{-\frac{a\,x^{n+1}}{n+1}} \mathrm{e}^{-\frac{b\,x^{m+1}}{m+1}}}{x^2} dx + c_2\right) x \tag{1}$$

Verification of solutions

$$y = \left(\int \frac{c_1 \mathrm{e}^{-\frac{a\,x^{n+1}}{n+1}} \mathrm{e}^{-\frac{b\,x^{m+1}}{m+1}}}{x^2} dx + c_2\right) x$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
      One independent solution has integrals. Trying a hypergeometric solution free of integ
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   No hypergeometric solution was found.
   <- linear_1 successful
   <- 2nd order, integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 47

```
dsolve(diff(y(x),x$2)+(a*x^n+b*x^m)*diff(y(x),x)-(a*x^(n-1)+b*x^(m-1))*y(x)=0,y(x), singsol=
```

$$y(x) = x\left(c_1 + c_2\left(\int \frac{\mathrm{e}^{-\frac{(b(n+1)x^m + a(1+m)x^n)x}{(1+m)(n+1)}}}{x^2}dx\right)\right)$$

✓ Solution by Mathematica

Time used: 1.216 (sec). Leaf size: 55

```
DSolve[y''[x]+(a*x^n+b*x^m)*y'[x]-(a*x^(n-1)+b*x^(m-1))*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to x\left(c_2 \int_1^x \frac{\exp\left(K[1]\left(-\frac{bK[1]^m}{m+1} - \frac{aK[1]^n}{n+1}\right)\right)}{K[1]^2}dK[1] + c_1\right)$$

27.46 problem 56

Internal problem ID [10880]
Internal file name [`OUTPUT/10136_Sunday_December_24_2023_05_14_06_PM_36119654/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y''+f(x)y'+g(x)y=0$
Problem number: 56.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{y''+(a\,x^n+b\,x^m)\,y'+\left(an\,x^{n-1}+x^{m-1}bm\right)y=0}$$

27.46.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int\left(y''+(a\,x^n+b\,x^m)\,y'+\frac{(x^n na+b\,x^m m)\,y}{x}\right)dx=0$$
$$\frac{(ax\,x^n+x^m bx)\,y}{x}+y'=c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x) = a\,x^n + b\,x^m$$
$$q(x) = c_1$$

Hence the ode is

$$y' + (a\,x^n + b\,x^m)\,y = c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int (a\,x^n + b\,x^m)dx}$$
$$= \mathrm{e}^{\frac{a\,x^{n+1}}{n+1} + \frac{b\,x^{m+1}}{m+1}}$$

Which simplifies to

$$\mu = \mathrm{e}^{\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\,(c_1)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}} y\right) = \left(\mathrm{e}^{\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}}\right)(c_1)$$
$$\mathrm{d}\left(\mathrm{e}^{\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}} y\right) = \left(c_1 \mathrm{e}^{\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}}\right)\,\mathrm{d}x$$

Integrating gives

$$\mathrm{e}^{\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}} y = \int c_1 \mathrm{e}^{\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}}\,\mathrm{d}x$$
$$\mathrm{e}^{\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}} y = \int c_1 \mathrm{e}^{\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}}\,dx + c_2$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}}$ results in

$$y = \mathrm{e}^{-\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}} \left(\int c_1 \mathrm{e}^{\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}}\,dx\right) + c_2 \mathrm{e}^{-\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}}$$

which simplifies to

$$y = \mathrm{e}^{-\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}} \left(c_1 \left(\int \mathrm{e}^{\frac{(a\,x^n (m+1) + b\,x^m (n+1))x}{(n+1)(m+1)}}\,dx\right) + c_2\right)$$

Summary
The solution(s) found are the following

$$y = e^{-\frac{(a\,x^n(m+1)+b\,x^m(n+1))x}{(n+1)(m+1)}}\left(c_1\left(\int e^{\frac{(a\,x^n(m+1)+b\,x^m(n+1))x}{(n+1)(m+1)}}dx\right)+c_2\right) \tag{1}$$

Verification of solutions

$$y = e^{-\frac{(a\,x^n(m+1)+b\,x^m(n+1))x}{(n+1)(m+1)}}\left(c_1\left(\int e^{\frac{(a\,x^n(m+1)+b\,x^m(n+1))x}{(n+1)(m+1)}}dx\right)+c_2\right)$$

Verified OK.

27.46.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + (a\,x^n + b\,x^m)\,y' + \frac{(x^n na + b\,x^m m)\,y}{x} = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(y'' + (a\,x^n + b\,x^m)\,y' + \frac{(x^n na + b\,x^m m)\,y}{x}\right)dx = 0$$
$$\frac{(ax\,x^n + x^m bx)\,y}{x} + y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = a\,x^n + b\,x^m$$
$$q(x) = c_1$$

Hence the ode is

$$y' + (a\,x^n + b\,x^m)\,y = c_1$$

The integrating factor μ is

$$\mu = e^{\int (a\,x^n + b\,x^m)dx}$$
$$= e^{\frac{a\,x^{n+1}}{n+1}+\frac{b\,x^{m+1}}{m+1}}$$

Which simplifies to

$$\mu = \mathrm{e}^{\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu) (c_1) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}} y\right) &= \left(\mathrm{e}^{\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}}\right) (c_1) \\
\mathrm{d}\left(\mathrm{e}^{\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}} y\right) &= \left(c_1 \mathrm{e}^{\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}}\right) \mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\mathrm{e}^{\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}} y &= \int c_1 \mathrm{e}^{\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}} \,\mathrm{d}x \\
\mathrm{e}^{\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}} y &= \int c_1 \mathrm{e}^{\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}} dx + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}}$ results in

$$y = \mathrm{e}^{-\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}} \left(\int c_1 \mathrm{e}^{\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}} dx\right) + c_2 \mathrm{e}^{-\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}}$$

which simplifies to

$$y = \mathrm{e}^{-\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}} \left(c_1 \left(\int \mathrm{e}^{\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}} dx\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}} \left(c_1 \left(\int \mathrm{e}^{\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}} dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}} \left(c_1 \left(\int \mathrm{e}^{\frac{\left(a\,x^{n} (m+1)+b\,x^{m} (n+1)\right) x}{(n+1) (m+1)}} dx\right) + c_2\right)$$

Verified OK.

27.46.3 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)$$

is exact if

$$p''(x)-q'(x)+r(x)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 1\\
q(x) &= a\,x^{n}+b\,x^{m}\\
r(x) &= \frac{x^{n} n a+b\,x^{m} m}{x}\\
s(x) &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 0\\
q'(x) &= \frac{a n\,x^{n}}{x}+\frac{x^{m} b m}{x}
\end{aligned}$$

Therefore (1) becomes

$$0-\left(\frac{a n\,x^{n}}{x}+\frac{x^{m} b m}{x}\right)+\left(\frac{x^{n} n a+b\,x^{m} m}{x}\right)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+\left(q(x)-p'(x)\right)y=\int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$y'+\left(a\,x^{n}+b\,x^{m}\right)y=c_1$$

We now have a first order ode to solve which is

$$y'+\left(a\,x^{n}+b\,x^{m}\right)y=c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= a\,x^{n}+b\,x^{m}\\ q(x) &= c_1\end{aligned}$$

Hence the ode is

$$y' + (a\,x^{n}+b\,x^{m})\,y = c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int (a\,x^{n}+b\,x^{m})dx}\\ &= \mathrm{e}^{\frac{a\,x^{n+1}}{n+1}+\frac{b\,x^{m+1}}{m+1}}\end{aligned}$$

Which simplifies to

$$\mu = \mathrm{e}^{\frac{(a\,x^{n}(m+1)+b\,x^{m}(n+1))x}{(n+1)(m+1)}}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(c_1)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{(a\,x^{n}(m+1)+b\,x^{m}(n+1))x}{(n+1)(m+1)}}y\right) &= \left(\mathrm{e}^{\frac{(a\,x^{n}(m+1)+b\,x^{m}(n+1))x}{(n+1)(m+1)}}\right)(c_1)\\ \mathrm{d}\left(\mathrm{e}^{\frac{(a\,x^{n}(m+1)+b\,x^{m}(n+1))x}{(n+1)(m+1)}}y\right) &= \left(c_1\mathrm{e}^{\frac{(a\,x^{n}(m+1)+b\,x^{m}(n+1))x}{(n+1)(m+1)}}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{\frac{(a\,x^{n}(m+1)+b\,x^{m}(n+1))x}{(n+1)(m+1)}}y &= \int c_1\mathrm{e}^{\frac{(a\,x^{n}(m+1)+b\,x^{m}(n+1))x}{(n+1)(m+1)}}\,\mathrm{d}x\\ \mathrm{e}^{\frac{(a\,x^{n}(m+1)+b\,x^{m}(n+1))x}{(n+1)(m+1)}}y &= \int c_1\mathrm{e}^{\frac{(a\,x^{n}(m+1)+b\,x^{m}(n+1))x}{(n+1)(m+1)}}dx + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{(a\,x^{n}(m+1)+b\,x^{m}(n+1))x}{(n+1)(m+1)}}$ results in

$$y = \mathrm{e}^{-\frac{(a\,x^{n}(m+1)+b\,x^{m}(n+1))x}{(n+1)(m+1)}}\left(\int c_1\mathrm{e}^{\frac{(a\,x^{n}(m+1)+b\,x^{m}(n+1))x}{(n+1)(m+1)}}dx\right) + c_2\mathrm{e}^{-\frac{(a\,x^{n}(m+1)+b\,x^{m}(n+1))x}{(n+1)(m+1)}}$$

which simplifies to

$$y = e^{-\frac{(a\,x^n(m+1)+b\,x^m(n+1))x}{(n+1)(m+1)}}\left(c_1\left(\int e^{\frac{(a\,x^n(m+1)+b\,x^m(n+1))x}{(n+1)(m+1)}}dx\right)+c_2\right)$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{(a\,x^n(m+1)+b\,x^m(n+1))x}{(n+1)(m+1)}}\left(c_1\left(\int e^{\frac{(a\,x^n(m+1)+b\,x^m(n+1))x}{(n+1)(m+1)}}dx\right)+c_2\right) \tag{1}$$

Verification of solutions

$$y = e^{-\frac{(a\,x^n(m+1)+b\,x^m(n+1))x}{(n+1)(m+1)}}\left(c_1\left(\int e^{\frac{(a\,x^n(m+1)+b\,x^m(n+1))x}{(n+1)(m+1)}}dx\right)+c_2\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
No hypergeometric solution was found.
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 72

```
dsolve(diff(y(x),x$2)+(a*x^n+b*x^m)*diff(y(x),x)+(a*n*x^(n-1)+b*m*x^(m-1))*y(x)=0,y(x), sing
```

$$y(x) = \left(c_1\left(\int e^{\frac{(b(n+1)x^m+a(1+m)x^n)x}{(1+m)(n+1)}}dx\right)+c_2\right)e^{-\frac{(b(n+1)x^m+a(1+m)x^n)x}{(1+m)(n+1)}}$$

✓ Solution by Mathematica

Time used: 0.139 (sec). Leaf size: 74

```
DSolve[y''[x]+(a*x^n+b*x^m)*y'[x]+(a*n*x^(n-1)+b*m*x^(m-1))*y[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to e^{x\left(-\frac{ax^n}{n+1}-\frac{bx^m}{m+1}\right)}\left(\int_1^x \exp\left(K[1]\left(\frac{bK[1]^m}{m+1}+\frac{aK[1]^n}{n+1}\right)\right)c_1dK[1]+c_2\right)$$

27.47 problem 57

Internal problem ID [10881]
Internal file name [OUTPUT/10137_Sunday_December_24_2023_05_14_34_PM_28021992/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 57.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + (a\,x^n + b\,x^m)\,y' + \left(a(n+1)\,x^{n-1} + b(m+1)\,x^{m-1}\right)y = 0}$$

27.47.1 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$y'' + (a\,x^n + b\,x^m)\,y' + \frac{(b(m+1)\,x^m + a(n+1)\,x^n)\,y}{x} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= a\,x^n + b\,x^m \\ q(x) &= \frac{b(m+1)\,x^m + a(n+1)\,x^n}{x} \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\xi'' - (\xi\, p)' + \xi q = 0$$
$$\xi'' - ((a\,x^n + b\,x^m)\,\xi(x))' + \left(\frac{(b(m+1)\,x^m + a(n+1)\,x^n)\,\xi(x)}{x}\right) = 0$$
$$\xi''(x) + (-a\,x^n - b\,x^m)\,\xi'(x) + \left(\frac{b(m+1)\,x^m + a(n+1)\,x^n}{x} - \frac{an\,x^n}{x} - \frac{x^m bm}{x}\right)\xi(x) = 0$$

Which is solved for $\xi(x)$. In normal form the ode

$$-\xi''(x)\,x + (a\,x^n + b\,x^m)\,\xi'(x)\,x + (-a\,x^n - b\,x^m)\,\xi(x) = 0 \tag{1}$$

Becomes

$$\xi''(x) + p(x)\,\xi'(x) + q(x)\,\xi(x) = 0 \tag{2}$$

Where

$$p(x) = -a\,x^n - b\,x^m$$
$$q(x) = \frac{a\,x^n + b\,x^m}{x}$$

Applying change of variables on the depndent variable $\xi(x) = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not $\xi(x)$.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(-a\,x^n - b\,x^m)}{x} + \frac{a\,x^n + b\,x^m}{x} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \left(\frac{2}{x} - a\,x^n - b\,x^m\right)v'(x) = 0$$
$$v''(x) + \left(\frac{2}{x} - a\,x^n - b\,x^m\right)v'(x) = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \left(\frac{2}{x} - a\,x^{n} - b\,x^{m}\right) u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(ax\,x^{n} + x^{m}bx - 2)}{x} \end{aligned}$$

Where $f(x) = \frac{ax\,x^{n}+x^{m}bx-2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= \frac{ax\,x^{n} + x^{m}bx - 2}{x}\,dx \\ \int \frac{1}{u}\,du &= \int \frac{ax\,x^{n} + x^{m}bx - 2}{x}\,dx \\ \ln(u) &= -2\ln(x) + \frac{ax\,\mathrm{e}^{n\ln(x)}}{n+1} + \frac{bx\,\mathrm{e}^{m\ln(x)}}{m+1} + c_1 \\ u &= \mathrm{e}^{-2\ln(x)+\frac{ax\,\mathrm{e}^{n\ln(x)}}{n+1}+\frac{bx\,\mathrm{e}^{m\ln(x)}}{m+1}+c_1} \\ &= c_1\mathrm{e}^{-2\ln(x)+\frac{ax\,\mathrm{e}^{n\ln(x)}}{n+1}+\frac{bx\,\mathrm{e}^{m\ln(x)}}{m+1}} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1\mathrm{e}^{\frac{ax\,x^{n}}{n+1}}\,\mathrm{e}^{\frac{b\,x^{m}x}{m+1}}}{x^{2}}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= \int \frac{c_1\mathrm{e}^{\frac{ax\,x^{n}}{n+1}}\,\mathrm{e}^{\frac{b\,x^{m}x}{m+1}}}{x^{2}}\,dx + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
\xi(x) &= v(x)\, x^{n} \\
&= \left(\int \frac{c_1 \mathrm{e}^{\frac{a x\, x^{n}}{n+1}}\, \mathrm{e}^{\frac{b\, x^{m} x}{m+1}}}{x^{2}} d x + c_2\right) x \\
&= \left(c_1 \left(\int \frac{\mathrm{e}^{\frac{\left(a\, x^{n} (m+1)+b\, x^{m} (n+1)\right) x}{(n+1)(m+1)}}}{x^{2}} d x\right) + c_2\right) x
\end{aligned}
$$

The original ode (2) now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y \xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, d x \\
y' + y \left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, d x}{\xi\,(x)}
\end{aligned}
$$

$$
y' + y \left(a\, x^{n} + b\, x^{m} - \frac{\frac{c_3 \mathrm{e}^{\frac{a x\, x^{n}}{n+1}}\, \mathrm{e}^{\frac{b\, x^{m} x}{m+1}}}{x} + \int \frac{c_3 \mathrm{e}^{\frac{a x\, x^{n}}{n+1}}\, \mathrm{e}^{\frac{b\, x^{m} x}{m+1}}}{x^{2}} d x + c_2}{\left(\int \frac{c_3 \mathrm{e}^{\frac{a x\, x^{n}}{n+1}}\, \mathrm{e}^{\frac{b\, x^{m} x}{m+1}}}{x^{2}} d x + c_2\right) x} \right) = 0
$$

Which is now a first order ode. This is now solved for y. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x) g(y) \\
&= -\frac{y \left(a\, x^{n} x^{2} \left(\int \frac{c_3 \mathrm{e}^{\frac{a x\, x^{n}}{n+1}}\, \mathrm{e}^{\frac{b\, x^{m} x}{m+1}}}{x^{2}} d x\right) + a\, x^{n} x^{2} c_2 + b\, x^{m} x^{2} \left(\int \frac{c_3 \mathrm{e}^{\frac{a x\, x^{n}}{n+1}}\, \mathrm{e}^{\frac{b\, x^{m} x}{m+1}}}{x^{2}} d x\right) + b\, x^{m} x^{2} c_2 - c_3 \mathrm{e}^{\frac{a x\, x^{n}}{n+1}}\, \mathrm{e}^{\frac{b\, x^{m} x}{m+1}}}{x^{2} \left(\int \frac{c_3 \mathrm{e}^{\frac{a x\, x^{n}}{n+1}}\, \mathrm{e}^{\frac{b\, x^{m} x}{m+1}}}{x^{2}} d x + c_2\right)}
\end{aligned}
$$

Where $f(x) = -\frac{a\,x^n x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + a\,x^n x^2 c_2 + b\,x^m x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + b\,x^m x^2 c_2 - c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}} - \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \cdots}{x^2}\right.}{x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx + c_2\right)}$

and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\,dy = -\frac{a\,x^n x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + a\,x^n x^2 c_2 + b\,x^m x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + b\,x^m x^2 c_2 - c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}} \cdots}{x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx + c_2\right)}$$

$$\int \frac{1}{y}\,dy = \int -\frac{a\,x^n x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + a\,x^n x^2 c_2 + b\,x^m x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + b\,x^m x^2 c_2 - c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \cdots}{x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx + c_2\right)}$$

$$\ln(y) = \int -\frac{a\,x^n x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + a\,x^n x^2 c_2 + b\,x^m x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + b\,x^m x^2 c_2 - c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \cdots}{x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx + c_2\right)}$$

$$y = \mathrm{e}^{\int -\frac{a\,x^n x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + a\,x^n x^2 c_2 + b\,x^m x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + b\,x^m x^2 c_2 - c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}} - \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) \cdots}{x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx + c_2\right)}}$$

$$= c_3 \mathrm{e}^{\int -\frac{a\,x^n x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + a\,x^n x^2 c_2 + b\,x^m x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + b\,x^m x^2 c_2 - c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}} - \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2} \cdots\right.}{x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx + c_2\right)}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_3 \mathrm{e}^{\int -\frac{a\,x^n x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + a\,x^n x^2 c_2 + b\,x^m x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + b\,x^m x^2 c_2 - c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}} - \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) x - c_2 x}{x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx + c_2\right)} dx}$$

Summary

The solution(s) found are the following

$$y = c_3 \mathrm{e}^{\int -\frac{a\,x^n x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + a\,x^n x^2 c_2 + b\,x^m x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) + b\,x^m x^2 c_2 - c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}} - \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx\right) x - c_2 x}{x^2 \left(\int \frac{c_3 \mathrm{e}^{\frac{ax\,x^n}{n+1}} \mathrm{e}^{\frac{b\,x^m x}{m+1}}}{x^2}dx + c_2\right)} dx} \tag{1}$$

Verification of solutions

$$
y = \int -\frac{a\,x^{n}x^{2}\left(\int \frac{c_3\,\mathrm{e}^{\frac{ax\,x^{n}}{n+1}}\,\mathrm{e}^{\frac{b\,x^{m}x}{m+1}}}{x^{2}}dx\right)+a\,x^{n}x^{2}c_2+b\,x^{m}x^{2}\left(\int \frac{c_3\,\mathrm{e}^{\frac{ax\,x^{n}}{n+1}}\,\mathrm{e}^{\frac{b\,x^{m}x}{m+1}}}{x^{2}}dx\right)+b\,x^{m}x^{2}c_2-c_3\,\mathrm{e}^{\frac{ax\,x^{n}}{n+1}}\,\mathrm{e}^{\frac{b\,x^{m}x}{m+1}}-\left(\int \frac{c_3\,\mathrm{e}^{\frac{ax\,x^{n}}{n+1}}\,\mathrm{e}^{\frac{b\,x^{m}x}{m+1}}}{x^{2}}dx\right)x-c_2x}{x^{2}\left(\int \frac{c_3\,\mathrm{e}^{\frac{ax\,x^{n}}{n+1}}\,\mathrm{e}^{\frac{b\,x^{m}x}{m+1}}}{x^{2}}dx+c_2\right)}\,dx
$$

$$
= c_3\mathrm{e}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
   <- 2nd order, integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.313 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)+(a*x^n+b*x^m)*diff(y(x),x)+(a*(n+1)*x^(n-1)+b*(m+1)*x^(m-1))*y(x)=0,y(
```

$$y(x) = x\left(c_1 + \left(\int \frac{e^{\frac{(b(n+1)x^m + a(1+m)x^n)x}{(1+m)(n+1)}}}{x^2}dx\right)c_2\right)e^{-\frac{(b(n+1)x^m + a(1+m)x^n)x}{(1+m)(n+1)}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+(a*x^n+b*x^m)*y'[x]+(a*(n+1)*x^(n-1)+b*(m+1)*x^(m-1))*y[x]==0,y[x],x,IncludeSi
```

Not solved

27.48 problem 58

Internal problem ID [10882]
Internal file name [`OUTPUT/10138_Sunday_December_24_2023_05_14_43_PM_8065640/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 58.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + (a\,x^n + b\,x^m)\,y' + c(a\,x^n + b\,x^m - c)\,y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+(a*x^n+b*x^m)*diff(y(x),x)+c*(a*x^n+b*x^m-c)*y(x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+(a*x^n+b*x^m)*y'[x]+c*(a*x^n+b*x^m-c)*y[x]==0,y[x],x,IncludeSingularSolutions
```

Not solved

27.49 problem 59

Internal problem ID [10883]
Internal file name [`OUTPUT/10139_Sunday_December_24_2023_05_14_43_PM_65340826/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 59.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + (a\,x^n + b\,x^m)\,y' + \left(x^{m+n}ab + b(m+1)\,x^{m-1} - a\,x^{n-1}\right)y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+(a*x^n+b*x^m)*diff(y(x),x)+(a*b*x^(n+m)+b*(m+1)*x^(m-1)-a*x^(n-1))*y(x
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+(a*x^n+b*x^m)*y'[x]+(a*b*x^(n+m)+b*(m+1)*x^(m-1)-a*x^(n-1))*y[x]==0,y[x],x,Inc
```

Not solved

27.50 problem 60

Internal problem ID [10884]
Internal file name [`OUTPUT/10140_Sunday_December_24_2023_05_14_44_PM_40192406/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-2 Equation of form $y'' + f(x)y' + g(x)y = 0$
Problem number: 60.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + (a\,x^n + b\,x^m + c)\,y' + \left(x^{m+n}ab + x^m bc + an\,x^{n-1}\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+(a*x^n+b*x^m+c)*diff(y(x),x)+(a*b*x^(n+m)+b*c*x^m+a*n*x^(n-1))*y(x)=0,
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+(a*x^n+b*x^m+c)*y'[x]+(a*b*x^(n+m)+b*c*x^m+a*n*x^(n-1))*y[x]==0,y[x],x,Include
```

Not solved

28 Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y'' + f(x)y' + g(x)y = 0$

28.1 problem 61

Internal problem ID [10885]
Internal file name [`OUTPUT/10141_Sunday_December_24_2023_05_14_45_PM_71262164/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 61.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode", "second__order__change__of__variable__on__x__method__1", "second__order__change__of__variable__on__x__method__2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{xy''+\frac{y'}{2}+ay=0}$$

28.1.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$xy''+\frac{y'}{2}+ay=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{a}{x}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$
$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{2x}dx\right)}dx \\
&= \int e^{-\frac{\ln(x)}{2}}\,dx \\
&= \int \frac{1}{\sqrt{x}}dx \\
&= 2\sqrt{x}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{a}{x}}{\frac{1}{x}} \\
&= a
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + a y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = a$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + a\, e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + a = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = a$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(a)} \\ &= \pm\sqrt{-a}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-a} \\ \lambda_2 &= -\sqrt{-a}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-a} \\ \lambda_2 &= -\sqrt{-a}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\ y(\tau) &= c_1 e^{(\sqrt{-a})\tau} + c_2 e^{(-\sqrt{-a})\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{\sqrt{-a}\,\tau} + c_2 e^{-\sqrt{-a}\,\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \mathrm{e}^{2\sqrt{-a}\,\sqrt{x}} + c_2 \mathrm{e}^{-2\sqrt{-a}\,\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2\sqrt{-a}\,\sqrt{x}} + c_2 \mathrm{e}^{-2\sqrt{-a}\,\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{2\sqrt{-a}\,\sqrt{x}} + c_2 \mathrm{e}^{-2\sqrt{-a}\,\sqrt{x}}$$

Verified OK.

28.1.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$xy'' + \frac{y'}{2} + ay = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{2x} \\ q(x) &= \frac{a}{x} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{a}{x}}}{c} \\ \tau'' &= -\frac{a}{2c\sqrt{\frac{a}{x}}\, x^2} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{a}{2c\sqrt{\frac{a}{x}}\,x^2} + \frac{1}{2x}\frac{\sqrt{\frac{a}{x}}}{c}}{\left(\frac{\sqrt{\frac{a}{x}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{a}{x}}dx}{c} \\
&= \frac{2x\sqrt{\frac{a}{x}}}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos\left(2\sqrt{a}\,\sqrt{x}\right) + c_2 \sin\left(2\sqrt{a}\,\sqrt{x}\right)
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos\left(2\sqrt{a}\,\sqrt{x}\right) + c_2 \sin\left(2\sqrt{a}\,\sqrt{x}\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 \cos\left(2\sqrt{a}\,\sqrt{x}\right) + c_2 \sin\left(2\sqrt{a}\,\sqrt{x}\right)
$$

Verified OK.

28.1.3 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \frac{y'x}{2} + yax = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= \frac{1}{4} \\ \beta &= 2\sqrt{a} \\ n &= \frac{1}{2} \\ \gamma &= \frac{1}{2} \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x^{\frac{1}{4}} \sin(2\sqrt{a}\,\sqrt{x})}{\sqrt{\pi}\,\sqrt{\sqrt{a}\,\sqrt{x}}} - \frac{c_2 x^{\frac{1}{4}} \cos(2\sqrt{a}\,\sqrt{x})}{\sqrt{\pi}\,\sqrt{\sqrt{a}\,\sqrt{x}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{4}} \sin(2\sqrt{a}\,\sqrt{x})}{\sqrt{\pi}\,\sqrt{\sqrt{a}\,\sqrt{x}}} - \frac{c_2 x^{\frac{1}{4}} \cos(2\sqrt{a}\,\sqrt{x})}{\sqrt{\pi}\,\sqrt{\sqrt{a}\,\sqrt{x}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{4}} \sin(2\sqrt{a}\,\sqrt{x})}{\sqrt{\pi}\,\sqrt{\sqrt{a}\,\sqrt{x}}} - \frac{c_2 x^{\frac{1}{4}} \cos(2\sqrt{a}\,\sqrt{x})}{\sqrt{\pi}\,\sqrt{\sqrt{a}\,\sqrt{x}}}$$

Verified OK.

28.1.4 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + \frac{y'}{2} + ay = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= \frac{1}{2} \\ C &= a \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16ax - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16ax - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16ax - 3}{16x^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 75: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} - \frac{a}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\
&= \{1, 2, 3\}
\end{aligned}
$$

Since the order of r at ∞ is $1 < 2$ then

$$
E_\infty = \{1\}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$
e_1 = 1, \; e_\infty = 1
$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{1}{2}(1 - (1)) \\
&= 0
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x - c} \\
&= \frac{1}{2}\left(\frac{1}{(x - (0))}\right) \\
&= \frac{1}{2x}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}
$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{16ax + 1}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 4\sqrt{-ax}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+4\sqrt{-ax}}{4x} dx} \\ &= x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{-ax}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{1}{2}}{x} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{4}} \\ &= z_1 \left(\frac{1}{x^{\frac{1}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2\sqrt{-ax}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{1}{2}}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{-ax}\left(-1 + \mathrm{e}^{-4\sqrt{-ax}}\right)}{2\sqrt{x}\, a} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2\sqrt{-ax}}\right) + c_2 \left(\mathrm{e}^{2\sqrt{-ax}} \left(\frac{\sqrt{-ax}\left(-1 + \mathrm{e}^{-4\sqrt{-ax}}\right)}{2\sqrt{x}\, a} \right) \right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2\sqrt{-ax}} + \frac{c_2 \sqrt{-ax}\left(-\mathrm{e}^{2\sqrt{-ax}} + \mathrm{e}^{-2\sqrt{-ax}}\right)}{2\sqrt{x}\, a} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{2\sqrt{-ax}} + \frac{c_2 \sqrt{-ax}\left(-\mathrm{e}^{2\sqrt{-ax}} + \mathrm{e}^{-2\sqrt{-ax}}\right)}{2\sqrt{x}\, a}$$

Verified OK.

28.1.5 Maple step by step solution

Let's solve

$y''x + \frac{y'}{2} + ay = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{2x} - \frac{ay}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2x} + \frac{ay}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{2x}, P_3(x) = \frac{a}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x + 2ay + y' = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  ◦ Shift index using $k- >k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+1+2r) + 2a a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+1+r)\left(k+\frac{1}{2}+r\right) a_{k+1} + 2a a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{2a a_k}{(k+1+r)(2k+1+2r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{2a a_k}{(k+1)(2k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a a_k}{(k+1)(2k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{2a a_k}{\left(k+\frac{3}{2}\right)(2k+2)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{2a a_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} b_k x^k\right)+\left(\sum_{k=0}^{\infty} c_k x^{k+\frac{1}{2}}\right), b_{1+k}=-\frac{2ab_k}{(1+k)(2k+1)}, c_{1+k}=-\frac{2ac_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x*diff(y(x),x$2)+1/2*diff(y(x),x)+a*y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1\sin\left(2\sqrt{x}\,\sqrt{a}\right)+c_2\cos\left(2\sqrt{x}\,\sqrt{a}\right)$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 38

```
DSolve[x*y''[x]+1/2*y'[x]+a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_1\cos\left(2\sqrt{a}\sqrt{x}\right)+c_2\sin\left(2\sqrt{a}\sqrt{x}\right)$$

28.2 problem 62

Internal problem ID [10886]
Internal file name [`OUTPUT/10142_Sunday_December_24_2023_05_14_46_PM_39518988/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 62.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy''+ay'+yb=0}$$

28.2.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y''+axy'+bxy=0 \tag{1}$$

Bessel ode has the form

$$x^2y''+y'x+\left(-n^2+x^2\right)y=0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y''+(1-2\alpha)\,xy'+\left(\beta^2\gamma^2x^{2\gamma}-n^2\gamma^2+\alpha^2\right)y=0 \tag{3}$$

With the standard solution

$$y=x^{\alpha}(c_1\,\mathrm{BesselJ}\left(n,\beta\,x^{\gamma}\right)+c_2\,\mathrm{BesselY}\left(n,\beta\,x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2} - \frac{a}{2}$$
$$\beta = 2\sqrt{b}$$
$$n = -a + 1$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-a+1, 2\sqrt{b}\,\sqrt{x}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-a+1, 2\sqrt{b}\,\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-a+1, 2\sqrt{b}\,\sqrt{x}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-a+1, 2\sqrt{b}\,\sqrt{x}\right) \quad (1)$$

Verification of solutions

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-a+1, 2\sqrt{b}\,\sqrt{x}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-a+1, 2\sqrt{b}\,\sqrt{x}\right)$$

Verified OK.

28.2.2 Maple step by step solution

Let's solve

$y''x + ay' + yb = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{ay'}{x} - \frac{by}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{ay'}{x} + \frac{by}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{a}{x}, P_3(x) = \frac{b}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\Big|_{x=0} = a$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\Big|_{x=0} = 0$$

- $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$y''x + ay' + yb = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\,(k + r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1 + r + a)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k + 1 + r)\,(k + r + a) + a_k b\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-1 + r + a) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, -a+1\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r+a)+a_k b=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k b}{(k+1+r)(k+r+a)}$

- Recursion relation for $r=0$

  $a_{k+1}=-\frac{a_k b}{(k+1)(k+a)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=-\frac{a_k b}{(k+1)(k+a)}\right]$

- Recursion relation for $r=-a+1$

  $a_{k+1}=-\frac{a_k b}{(k+2-a)(k+1)}$

- Solution for $r=-a+1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-a+1}, a_{k+1}=-\frac{a_k b}{(k+2-a)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} c_k x^k\right)+\left(\sum_{k=0}^{\infty} d_k x^{k-a+1}\right), c_{1+k}=-\frac{c_k b}{(1+k)(k+a)}, d_{1+k}=-\frac{d_k b}{(k+2-a)(1+k)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 83

```
dsolve(x*diff(y(x),x$2)+a*diff(y(x),x)+b*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(-\sqrt{x}\,\mathrm{BesselJ}\left(a+1, 2\sqrt{b}\,\sqrt{x}\right)\sqrt{b}\,c_1 - \sqrt{x}\,\mathrm{BesselY}\left(a+1, 2\sqrt{b}\,\sqrt{x}\right)\sqrt{b}\,c_2 + a\left(\mathrm{BesselJ}\left(a, 2\sqrt{b}\,\sqrt{x}\right)\right.\right.}{\sqrt{b}}$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 77

```
DSolve[x*y''[x]+a*y'[x]+b*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to b^{\frac{1}{2}-\frac{a}{2}} x^{\frac{1}{2}-\frac{a}{2}} \left(c_2 \,\mathrm{Gamma}(2-a)\,\mathrm{BesselJ}\left(1-a, 2\sqrt{b}\sqrt{x}\right) + c_1 \,\mathrm{Gamma}(a)\,\mathrm{BesselJ}\left(a-1, 2\sqrt{b}\sqrt{x}\right)\right)$$

28.3 problem 63

Internal problem ID [10887]
Internal file name [`OUTPUT/10143_Sunday_December_24_2023_05_14_47_PM_22794471/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 63.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''+ay'+bxy=0}$$

28.3.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y''+axy'+b\,x^2y=0 \tag{1}$$

Bessel ode has the form

$$x^2y''+y'x+\left(-n^2+x^2\right)y=0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y''+(1-2\alpha)\,xy'+\left(\beta^2\gamma^2x^{2\gamma}-n^2\gamma^2+\alpha^2\right)y=0 \tag{3}$$

With the standard solution

$$y=x^{\alpha}(c_1\,\mathrm{BesselJ}\left(n,\beta\,x^{\gamma}\right)+c_2\,\mathrm{BesselY}\left(n,\beta\,x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2} - \frac{a}{2}$$
$$\beta = \sqrt{b}$$
$$n = \frac{1}{2} - \frac{a}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(\frac{1}{2} - \frac{a}{2}, x\sqrt{b}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(\frac{1}{2} - \frac{a}{2}, x\sqrt{b}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(\frac{1}{2} - \frac{a}{2}, x\sqrt{b}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(\frac{1}{2} - \frac{a}{2}, x\sqrt{b}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(\frac{1}{2} - \frac{a}{2}, x\sqrt{b}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(\frac{1}{2} - \frac{a}{2}, x\sqrt{b}\right)$$

Verified OK.

28.3.2 Maple step by step solution

Let's solve

$y''x + ay' + bxy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{ay'}{x} - yb$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{ay'}{x} + yb = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{a}{x}, P_3(x) = b\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\Big|_{x=0} = a$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\Big|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x + ay' + bxy = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k- >k - 1$

  $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k + r + 1)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-1+r+a)\, x^{-1+r} + a_1(1+r)(r+a)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(k+r+a) + b a_{k-1}\right) x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r+a) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, -a+1\}$
- Each term must be 0

  $a_1(1+r)(r+a) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+r+a) + b a_{k-1} = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(k+1+r+a) + b a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{b a_k}{(k+2+r)(k+1+r+a)}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{b a_k}{(k+2)(k+1+a)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{b a_k}{(k+2)(k+1+a)}, a_1 a = 0\right]$
- Recursion relation for $r = -a+1$

  $a_{k+2} = -\frac{b a_k}{(k+3-a)(k+2)}$
- Solution for $r = -a+1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-a+1}, a_{k+2} = -\frac{b a_k}{(k+3-a)(k+2)}, a_1(-a+2) = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} c_k x^k\right) + \left(\sum_{k=0}^{\infty} d_k x^{k-a+1}\right), c_{k+2} = -\frac{b c_k}{(k+2)(k+1+a)}, c_1 a = 0, d_{k+2} = -\frac{b d_k}{(k+3-a)(k+2)}, d_1(-a\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(x*diff(y(x),x$2)+a*diff(y(x),x)+b*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\operatorname{BesselJ}\left(\frac{a}{2} - \frac{1}{2}, \sqrt{b}\, x\right) c_1 + \operatorname{BesselY}\left(\frac{a}{2} - \frac{1}{2}, \sqrt{b}\, x\right) c_2\right) x^{-\frac{a}{2}+\frac{1}{2}}$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 54

```
DSolve[x*y''[x]+a*y'[x]+b*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^{\frac{1}{2}-\frac{a}{2}} \left(c_1 \operatorname{BesselJ}\left(\frac{a-1}{2}, \sqrt{b}x\right) + c_2 \operatorname{BesselY}\left(\frac{a-1}{2}, \sqrt{b}x\right)\right)$$

28.4 problem 64

Internal problem ID [10888]
Internal file name [`OUTPUT/10144_Sunday_December_24_2023_05_14_49_PM_65619040/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 64.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''+ay'+(bx+c)\,y=0}$$

28.4.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y''+axy'+\left(b\,x^2+cx\right)y=0 \tag{1}$$

Bessel ode has the form

$$x^2y''+y'x+\left(-n^2+x^2\right)y=0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y''+(1-2\alpha)\,xy'+\left(\beta^2\gamma^2x^{2\gamma}-n^2\gamma^2+\alpha^2\right)y=0 \tag{3}$$

With the standard solution

$$y=x^\alpha(c_1\,\mathrm{BesselJ}\left(n,\beta\,x^\gamma\right)+c_2\,\mathrm{BesselY}\left(n,\beta\,x^\gamma\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2} - \frac{a}{2}$$
$$\beta = 2$$
$$n = -a + 1$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-a+1, 2\sqrt{x}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-a+1, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-a+1, 2\sqrt{x}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-a+1, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-a+1, 2\sqrt{x}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-a+1, 2\sqrt{x}\right)$$

Verified OK.

28.4.2 Maple step by step solution

Let's solve

$y''x + ay' + (bx + c)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(bx+c)y}{x} - \frac{ay'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{ay'}{x} + \frac{(bx+c)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{a}{x}, P_3(x) = \frac{bx+c}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\Big|_{x=0} = a$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x + ay' + (bx + c)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$$

- Shift index using $k- > k + 1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r-1}$$

- Shift index using $k- > k + 1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\, (k + r)\, x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0 r(-1+r+a)\, x^{-1+r} + (a_1(1+r)\,(r+a) + a_0 c)\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k+r+a) + a_k c -\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r+a) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, -a+1\}$

- Each term must be 0

  $a_1(1+r)\,(r+a) + a_0 c = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r+a) + a_k c + b a_{k-1} = 0$

- Shift index using $k- > k+1$

  $a_{k+2}(k+2+r)\,(k+1+r+a) + a_{k+1} c + b a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{b a_k + a_{k+1} c}{(k+2+r)(k+1+r+a)}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{b a_k + a_{k+1} c}{(k+2)(k+1+a)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{b a_k + a_{k+1} c}{(k+2)(k+1+a)}, a_1 a + a_0 c = 0\right]$

- Recursion relation for $r = -a+1$

  $a_{k+2} = -\frac{b a_k + a_{k+1} c}{(k+3-a)(k+2)}$

- Solution for $r = -a+1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-a+1}, a_{k+2} = -\frac{b a_k + a_{k+1} c}{(k+3-a)(k+2)}, a_1(-a+2) + a_0 c = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} d_k x^k\right) + \left(\sum_{k=0}^{\infty} e_k x^{k-a+1}\right), d_{k+2} = -\frac{b d_k + c d_{1+k}}{(k+2)(k+1+a)}, a d_1 + c d_0 = 0, e_{k+2} = -\frac{b e_k + c e_{1+k}}{(k+3-a)(k+2)},\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 66

```
dsolve(x*diff(y(x),x$2)+a*diff(y(x),x)+(b*x+c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-i\sqrt{b}\,x}\left(\mathrm{KummerU}\left(\frac{ic+a\sqrt{b}}{2\sqrt{b}}, a, 2i\sqrt{b}\,x\right)c_2 + \mathrm{KummerM}\left(\frac{ic+a\sqrt{b}}{2\sqrt{b}}, a, 2i\sqrt{b}\,x\right)c_1\right)$$

✓ Solution by Mathematica

Time used: 0.099 (sec). Leaf size: 85

```
DSolve[x*y''[x]+a*y'[x]+(b*x+c)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-i\sqrt{b}x}\left(c_1 \,\mathrm{HypergeometricU}\left(\frac{1}{2}\left(a+\frac{ic}{\sqrt{b}}\right), a, 2i\sqrt{b}x\right) + c_2 L^{a-1}_{-\frac{a}{2}-\frac{ic}{2\sqrt{b}}}\left(2i\sqrt{b}x\right)\right)$$

28.5 problem 65

Internal problem ID [10889]
Internal file name [`OUTPUT/10145_Sunday_December_24_2023_05_15_08_PM_11268123/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 65.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{xy''+ny'+b\,x^{1-2n}y=0}$$

28.5.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$xy''+ny'+b\,x^{1-2n}y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=\frac{n}{x}$$
$$q(x)=b\,x^{-2n}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{n}{x}dx\right)}dx \\ &= \int e^{-n\ln(x)}\,dx \\ &= \int x^{-n}dx \\ &= -\frac{x^{1-n}}{n-1}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{b\,x^{-2n}}{x^{-2n}} \\ &= b\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + b y(\tau) &= 0\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = b$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + b\, e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + b = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = b$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(b)}\\ &= \pm\sqrt{-b}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-b}\\ \lambda_2 &= -\sqrt{-b}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-b}\\ \lambda_2 &= -\sqrt{-b}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1\tau} + c_2 e^{\lambda_2\tau}\\ y(\tau) &= c_1 e^{(\sqrt{-b})\tau} + c_2 e^{(-\sqrt{-b})\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{\sqrt{-b}\,\tau} + c_2 e^{-\sqrt{-b}\,\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \mathrm{e}^{-\frac{\sqrt{-b}\, x^{1-n}}{n-1}} + c_2 \mathrm{e}^{\frac{\sqrt{-b}\, x^{1-n}}{n-1}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{\sqrt{-b}\, x^{1-n}}{n-1}} + c_2 \mathrm{e}^{\frac{\sqrt{-b}\, x^{1-n}}{n-1}} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{\sqrt{-b}\, x^{1-n}}{n-1}} + c_2 \mathrm{e}^{\frac{\sqrt{-b}\, x^{1-n}}{n-1}}$$

Verified OK.

28.5.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$xy'' + ny' + b\, x^{1-2n} y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{n}{x} \\ q(x) &= b\, x^{-2n} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{b\, x^{-2n}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{b\, x^{-2n} n}{c\sqrt{b\, x^{-2n}}\, x}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{b\,x^{-2n}n}{c\sqrt{b\,x^{-2n}}\,x} + \frac{n}{x}\frac{\sqrt{b\,x^{-2n}}}{c}}{\left(\frac{\sqrt{b\,x^{-2n}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \sqrt{b\,x^{-2n}}dx}{c} \\ &= -\frac{x\sqrt{b\,x^{-2n}}}{c\,(n-1)} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\frac{x^{1-n}\sqrt{b}}{n-1}\right) - c_2 \sin\left(\frac{x^{1-n}\sqrt{b}}{n-1}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\frac{x^{1-n}\sqrt{b}}{n-1}\right) - c_2 \sin\left(\frac{x^{1-n}\sqrt{b}}{n-1}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(\frac{x^{1-n}\sqrt{b}}{n-1}\right) - c_2 \sin\left(\frac{x^{1-n}\sqrt{b}}{n-1}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve(x*diff(y(x),x$2)+n*diff(y(x),x)+b*x^(1-2*n)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin\left(\frac{x^{-n+1}\sqrt{b}}{n-1}\right) + c_2 \cos\left(\frac{x^{-n+1}\sqrt{b}}{n-1}\right)$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 52

```
DSolve[x*y''[x]+n*y'[x]+b*x^(1-2*n)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(\frac{\sqrt{b}x^{1-n}}{n-1}\right) + c_2 \sin\left(\frac{\sqrt{b}x^{1-n}}{1-n}\right)$$

28.6 problem 66

Internal problem ID [10890]
Internal file name [`OUTPUT/10146_Sunday_December_24_2023_05_15_09_PM_18821174/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 66.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

Unable to solve or complete the solution.

$$\boxed{xy''+(1-3n)\,y'-a^2n^2x^{-1+2n}y=0}$$

28.6.1 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$xy''+(1-3n)\,y'-a^2n^2x^{-1+2n}y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1-3n}{x} \\ q(x) &= -x^{2n-2}a^2n^2 \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(\frac{(1-3n)\,\xi(x)}{x}\right)' + \left(-x^{2n-2}a^2n^2\xi(x)\right) &= 0 \\
\xi''(x) - \frac{(1-3n)\,\xi'(x)}{x} + \left(\frac{1-3n}{x^2} - x^{2n-2}a^2n^2\right)\xi(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- Equivalence, under non-integer power transformations successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 62

```
dsolve(x*diff(y(x),x$2)+(1-3*n)*diff(y(x),x)-a^2*n^2*x^(2*n-1)*y(x)=0,y(x), singsol=all)
```

$$
y(x) = c_2 \mathrm{e}^{-a\,x^n}\left(a\,x^n + x^{-n}\sqrt{x^{2n}}\right) - \mathrm{e}^{a\,x^n} c_1 \left(a\,x^n - x^{-n}\sqrt{x^{2n}}\right)
$$

✓ Solution by Mathematica

Time used: 0.196 (sec). Leaf size: 77

```
DSolve[x*y''[x]+(1-3*n)*y'[x]-a^2*n^2*x^(2*n-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$
y(x) \to \left(c_1 - \frac{3}{8} i a c_2 \sqrt{x^{2n}}\right)\cosh\left(a\sqrt{x^{2n}}\right) + \frac{1}{8}\left(3 i c_2 - 8 a c_1 \sqrt{x^{2n}}\right)\sinh\left(a\sqrt{x^{2n}}\right)
$$

28.7 problem 67

Internal problem ID [10891]
Internal file name [`OUTPUT/10147_Sunday_December_24_2023_05_15_10_PM_64440882/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 67.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy''+ay'+b\,x^n y=0}$$

28.7.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y''+axy'+x^n bxy=0 \tag{1}$$

Bessel ode has the form

$$x^2y''+y'x+\left(-n^2+x^2\right)y=0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y''+(1-2\alpha)\,xy'+\left(\beta^2\gamma^2x^{2\gamma}-n^2\gamma^2+\alpha^2\right)y=0 \tag{3}$$

With the standard solution

$$y=x^\alpha(c_1\,\text{BesselJ}\,(n,\beta\,x^\gamma)+c_2\,\text{BesselY}\,(n,\beta\,x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2} - \frac{a}{2}$$
$$\beta = \frac{2\sqrt{b}}{n+1}$$
$$n = -\frac{a-1}{n+1}$$
$$\gamma = \frac{n}{2} + \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-\frac{a-1}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-\frac{a-1}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-\frac{a-1}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-\frac{a-1}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-\frac{a-1}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-\frac{a-1}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.218 (sec). Leaf size: 71

```
dsolve(x*diff(y(x),x$2)+a*diff(y(x),x)+b*x^n*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\text{BesselY}\left(\frac{a-1}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_2 + \text{BesselJ}\left(\frac{a-1}{n+1}, \frac{2\sqrt{b}\, x^{\frac{n}{2}+\frac{1}{2}}}{n+1}\right) c_1\right) x^{-\frac{a}{2}+\frac{1}{2}}$$

✓ Solution by Mathematica

Time used: 0.245 (sec). Leaf size: 165

```
DSolve[x*y''[x]+a*y'[x]+b*x^n*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\frac{1}{n} + 1\right)^{\frac{a-1}{n+1}} n^{\frac{a-1}{n+1}} b^{\frac{1-a}{2n+2}} (x^n)^{-\frac{a-1}{2n}} \left(c_2 \,\text{Gamma}\left(\frac{-a+n+2}{n+1}\right) \text{BesselJ}\left(\frac{1-a}{n+1}, \frac{2\sqrt{b}(x^n)^{\frac{n+1}{2n}}}{n+1}\right) + c_1 \,\text{Gamma}\left(\frac{a+n}{n+1}\right) \text{BesselJ}\left(\frac{a-1}{n+1}, \frac{2\sqrt{b}(x^n)^{\frac{n+1}{2n}}}{n+1}\right)\right)$$

28.8 problem 68

Internal problem ID [10892]
Internal file name [`OUTPUT/10148_Sunday_December_24_2023_05_15_11_PM_52030711/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 68.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''+ay'+b\,x^n\left(-b\,x^{n+1}+a+n\right)y=0}$$

28.8.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y''+axy'+\left(-x^{2n}b^2x^2+x^nabx+x^nbnx\right)y=0 \tag{1}$$

Bessel ode has the form

$$x^2y''+y'x+\left(-n^2+x^2\right)y=0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y''+(1-2\alpha)\,xy'+\left(\beta^2\gamma^2x^{2\gamma}-n^2\gamma^2+\alpha^2\right)y=0 \tag{3}$$

With the standard solution

$$y=x^{\alpha}(c_1\,\mathrm{BesselJ}\left(n,\beta\,x^{\gamma}\right)+c_2\,\mathrm{BesselY}\left(n,\beta\,x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2} - \frac{a}{2}$$
$$\beta = 2$$
$$n = -a + 1$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-a+1, 2\sqrt{x}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-a+1, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-a+1, 2\sqrt{x}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-a+1, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-a+1, 2\sqrt{x}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-a+1, 2\sqrt{x}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- Equivalence, under non-integer power transformations successful`
```

✓ Solution by Maple

Time used: 0.594 (sec). Leaf size: 166

```
dsolve(x*diff(y(x),x$2)+a*diff(y(x),x)+b*x^n*(-b*x^(n+1)+a+n)*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\left(\left((a-n-2)\,x^{-\frac{3n}{2}-\frac{a}{2}-1}+2b\,x^{-\frac{n}{2}-\frac{a}{2}}\right)c_2(n+1)\,\mathrm{WhittakerM}\left(\frac{-a-n}{2n+2},\frac{-a+2n+3}{2n+2},-\frac{2b\,x^{n+1}}{n+1}\right) + x^{-\frac{3n}{2}-\frac{a}{2}-1}c_2(a-n-2)^2\,\mathrm{WhittakerM}\left(\frac{n+2-a}{2n+2},\frac{-a+2n+3}{2n+2},-\frac{2b\,x^{n+1}}{n+1}\right)\right) + c_1\mathrm{e}^{-\frac{b\,x^{n+1}}{n+1}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y''[x]+a*y'[x]+b*x^n*(-b*x^(n+1)+a+n)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

Not solved

28.9 problem 69

Internal problem ID [10893]
Internal file name [`OUTPUT/10149_Sunday_December_24_2023_05_15_33_PM_49026519/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 69.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{xy''+axy'+ay=0}$$

28.9.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (xy''+axy'+ay)\,dx=0$$
$$(ax-1)\,y+y'x=c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{-ax+1}{x}\\q(x) &= \frac{c_1}{x}\end{aligned}$$

Hence the ode is

$$y' - \frac{(-ax+1)\,y}{x} = \frac{c_1}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{-ax+1}{x}dx}\\&= \mathrm{e}^{ax-\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{\mathrm{e}^{ax}}{x}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x}\right)\\\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{ax}y}{x}\right) &= \left(\frac{\mathrm{e}^{ax}}{x}\right)\left(\frac{c_1}{x}\right)\\\mathrm{d}\left(\frac{\mathrm{e}^{ax}y}{x}\right) &= \left(\frac{c_1\mathrm{e}^{ax}}{x^2}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{\mathrm{e}^{ax}y}{x} &= \int \frac{c_1\mathrm{e}^{ax}}{x^2}\,\mathrm{d}x\\\frac{\mathrm{e}^{ax}y}{x} &= c_1a\left(-\frac{\mathrm{e}^{ax}}{ax} - \mathrm{expIntegral}_1\left(-ax\right)\right) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{ax}}{x}$ results in

$$y = x\,\mathrm{e}^{-ax}c_1a\left(-\frac{\mathrm{e}^{ax}}{ax} - \mathrm{expIntegral}_1\left(-ax\right)\right) + c_2x\,\mathrm{e}^{-ax}$$

which simplifies to

$$y = -\,\mathrm{expIntegral}_1\left(-ax\right)c_1ax\,\mathrm{e}^{-ax} - c_1 + c_2x\,\mathrm{e}^{-ax}$$

Summary
The solution(s) found are the following

$$y = -\operatorname{expIntegral}_1(-ax)\, c_1 a x\, e^{-ax} - c_1 + c_2 x\, e^{-ax} \tag{1}$$

Verification of solutions

$$y = -\operatorname{expIntegral}_1(-ax)\, c_1 a x\, e^{-ax} - c_1 + c_2 x\, e^{-ax}$$

Verified OK.

28.9.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' + axy' + ay = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' + axy' + ay)\, dx = 0$$
$$(ax - 1)\, y + y'x = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-ax + 1}{x}$$
$$q(x) = \frac{c_1}{x}$$

Hence the ode is

$$y' - \frac{(-ax + 1)\, y}{x} = \frac{c_1}{x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-ax+1}{x} dx}$$
$$= e^{ax - \ln(x)}$$

Which simplifies to

$$\mu = \frac{e^{ax}}{x}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{ax} y}{x}\right) &= \left(\frac{\mathrm{e}^{ax}}{x}\right)\left(\frac{c_1}{x}\right)\\ \mathrm{d}\left(\frac{\mathrm{e}^{ax} y}{x}\right) &= \left(\frac{c_1 \mathrm{e}^{ax}}{x^2}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{\mathrm{e}^{ax} y}{x} &= \int \frac{c_1 \mathrm{e}^{ax}}{x^2}\,\mathrm{d}x\\ \frac{\mathrm{e}^{ax} y}{x} &= c_1 a\left(-\frac{\mathrm{e}^{ax}}{ax} - \mathrm{expIntegral}_1(-ax)\right) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{ax}}{x}$ results in

$$y = x\,\mathrm{e}^{-ax} c_1 a\left(-\frac{\mathrm{e}^{ax}}{ax} - \mathrm{expIntegral}_1(-ax)\right) + c_2 x\,\mathrm{e}^{-ax}$$

which simplifies to

$$y = -\mathrm{expIntegral}_1(-ax)\, c_1 a x\,\mathrm{e}^{-ax} - c_1 + c_2 x\,\mathrm{e}^{-ax}$$

Summary

The solution(s) found are the following

$$y = -\mathrm{expIntegral}_1(-ax)\, c_1 a x\,\mathrm{e}^{-ax} - c_1 + c_2 x\,\mathrm{e}^{-ax} \tag{1}$$

Verification of solutions

$$y = -\mathrm{expIntegral}_1(-ax)\, c_1 a x\,\mathrm{e}^{-ax} - c_1 + c_2 x\,\mathrm{e}^{-ax}$$

Verified OK.

28.9.3 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + axy' + ay = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= x\\ B &= ax \\ C &= a\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a(ax - 4)}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a(ax - 4) \\ t &= 4x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a(ax - 4)}{4x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 80: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{a}{2} - \frac{1}{x} - \frac{1}{a\,x^2} - \frac{2}{a^2x^3} - \frac{5}{a^3x^4} - \frac{14}{a^4x^5} - \frac{42}{a^5x^6} - \frac{132}{a^6x^7} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{a}{2}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{a}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{a^2}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a(ax-4)}{4x} \\ &= Q + \frac{R}{4x} \\ &= \left(\frac{a^2}{4}\right) + \left(-\frac{a}{x}\right) \\ &= \frac{a^2}{4} - \frac{a}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is $-4a$. Dividing this by leading coefficient in t which is 4 gives $-a$. Now b can be found.

$$\begin{aligned} b &= (-a) - (0) \\ &= -a \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{a}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-a}{\frac{a}{2}} - 0\right) &= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-a}{\frac{a}{2}} - 0\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a(ax-4)}{4x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{a}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{x} + (-)\left(\frac{a}{2}\right) \\
&= \frac{1}{x} - \frac{a}{2} \\
&= \frac{1}{x} - \frac{a}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{x} - \frac{a}{2}\right)(0) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} - \frac{a}{2}\right)^2 - \left(\frac{a(ax-4)}{4x}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{x} - \frac{a}{2}\right) dx} \\
&= x \, e^{-\frac{ax}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{ax}{x}\,dx} \\
&= z_1 e^{-\frac{ax}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{ax}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\,\mathrm{e}^{-ax}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{ax}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-ax}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1(-ax)\,ax - \mathrm{e}^{ax}}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-ax}\right) + c_2 \left(x\,\mathrm{e}^{-ax}\left(\frac{-\operatorname{expIntegral}_1(-ax)\,ax - \mathrm{e}^{ax}}{x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-ax} + c_2\left(-\operatorname{expIntegral}_1(-ax)\,ax\,\mathrm{e}^{-ax} - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-ax} + c_2\left(-\operatorname{expIntegral}_1(-ax)\,ax\,\mathrm{e}^{-ax} - 1\right)
$$

Verified OK.

28.9.4 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x) &= x\\ q(x) &= ax\\ r(x) &= a\\ s(x) &= 0\end{aligned}$$

Hence

$$\begin{aligned}p''(x) &= 0\\ q'(x) &= a\end{aligned}$$

Therefore (1) becomes

$$0 - (a) + (a) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$(ax - 1)\,y + y'x = c_1$$

We now have a first order ode to solve which is

$$(ax - 1)\,y + y'x = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{-ax+1}{x}\\
q(x) &= \frac{c_1}{x}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{(-ax+1)\, y}{x} = \frac{c_1}{x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{-ax+1}{x}dx}\\
&= \mathrm{e}^{ax-\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{\mathrm{e}^{ax}}{x}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{ax} y}{x}\right) &= \left(\frac{\mathrm{e}^{ax}}{x}\right)\left(\frac{c_1}{x}\right)\\
\mathrm{d}\left(\frac{\mathrm{e}^{ax} y}{x}\right) &= \left(\frac{c_1 \mathrm{e}^{ax}}{x^2}\right)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\mathrm{e}^{ax} y}{x} &= \int \frac{c_1 \mathrm{e}^{ax}}{x^2}\, \mathrm{d}x\\
\frac{\mathrm{e}^{ax} y}{x} &= c_1 a\left(-\frac{\mathrm{e}^{ax}}{ax} - \mathrm{expIntegral}_1\left(-ax\right)\right) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{ax}}{x}$ results in

$$
y = x\,\mathrm{e}^{-ax} c_1 a\left(-\frac{\mathrm{e}^{ax}}{ax} - \mathrm{expIntegral}_1\left(-ax\right)\right) + c_2 x\,\mathrm{e}^{-ax}
$$

which simplifies to

$$
y = -\,\mathrm{expIntegral}_1\left(-ax\right) c_1 a x\,\mathrm{e}^{-ax} - c_1 + c_2 x\,\mathrm{e}^{-ax}
$$

Summary
The solution(s) found are the following

$$y = -\operatorname{expIntegral}_1(-ax)\, c_1 a x\, e^{-ax} - c_1 + c_2 x\, e^{-ax} \tag{1}$$

Verification of solutions

$$y = -\operatorname{expIntegral}_1(-ax)\, c_1 a x\, e^{-ax} - c_1 + c_2 x\, e^{-ax}$$

Verified OK.

28.9.5 Maple step by step solution

Let's solve

$y''x + axy' + ay = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -ay' - \frac{ay}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + ay' + \frac{ay}{x} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = a, P_3(x) = \frac{a}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + axy' + ay = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r) + a a_k(k+1+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+1+r)\,(a_{k+1}(k+r) + a a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a a_k}{k+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a a_k}{k}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a a_k}{k}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a a_k}{k+1}$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a a_k}{k+1}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} b_k x^k\right) + \left(\sum_{k=0}^{\infty} c_k x^{1+k}\right), b_{1+k} = -\frac{a b_k}{k}, c_{1+k} = -\frac{a c_k}{1+k}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(x*diff(y(x),x$2)+a*x*diff(y(x),x)+a*y(x)=0,y(x), singsol=all)
```

$$y(x) = \text{expIntegral}_1(-ax) c_1 a x \, e^{-ax} + c_1 + c_2 x \, e^{-ax}$$

✓ Solution by Mathematica

Time used: 0.177 (sec). Leaf size: 35

```
DSolve[x*y''[x]+a*x*y'[x]+a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-ax}(a c_2 x \, \text{ExpIntegralEi}(ax) - c_2 e^{ax} + c_1 x)$$

28.10 problem 70

Internal problem ID [10894]
Internal file name [`OUTPUT/10150_Sunday_December_24_2023_05_15_35_PM_2570596/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 70.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_Laguerre]`

Unable to solve or complete the solution.

$$xy''+(-x+b)\,y'-ay=0$$

28.10.1 Maple step by step solution

Let's solve

$y''x+(-x+b)\,y'-ay=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{ay}{x}-\frac{(-x+b)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(-x+b)y'}{x}-\frac{ay}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x)=\frac{-x+b}{x}, P_3(x)=-\frac{a}{x}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=b$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

- ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x+(-x+b)\,y'-ay=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

    $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ◦ Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ◦ Convert $x\cdot y''$ to series expansion

    $x\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  - ◦ Shift index using $k->k+1$

    $x\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-1+r+b)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r+b)-a_k(k+r+a)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-1+r+b) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, -b+1\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r+b) - a_k(k+r+a) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+a)}{(k+1+r)(k+r+b)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+a)}{(k+1)(k+b)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(k+a)}{(k+1)(k+b)}\right]$

- Recursion relation for $r = -b+1$

  $a_{k+1} = \frac{a_k(k-b+1+a)}{(k+2-b)(k+1)}$

- Solution for $r = -b+1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-b+1}, a_{k+1} = \frac{a_k(k-b+1+a)}{(k+2-b)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} c_k x^k\right) + \left(\sum_{k=0}^{\infty} d_k x^{k-b+1}\right), c_{1+k} = \frac{c_k(k+a)}{(1+k)(k+b)}, d_{1+k} = \frac{d_k(k-b+1+a)}{(k+2-b)(1+k)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x$2)+(b-x)*diff(y(x),x)-a*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{KummerM}(a, b, x) + c_2 \operatorname{KummerU}(a, b, x)$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 24

```
DSolve[x*y''[x]+(b-x)*y'[x]-a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 \operatorname{HypergeometricU}(a, b, x) + c_2 L_{-a}^{b-1}(x)$$

28.11 problem 71

Internal problem ID [10895]
Internal file name [`OUTPUT/10151_Sunday_December_24_2023_05_15_36_PM_79692819/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 71.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{xy''+(ax+b)\,y'+c((a-c)\,x+b)\,y=0}$$

28.11.1 Maple step by step solution

Let's solve

$y''x+(ax+b)\,y'+c((a-c)\,x+b)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{c(ax-cx+b)y}{x}-\frac{(ax+b)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(ax+b)y'}{x}+\frac{c(ax-cx+b)y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ○ Define functions

$$\left[P_2(x) = \frac{ax+b}{x}, P_3(x) = \frac{c(ax-cx+b)}{x}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = b$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x + (ax+b)\,y' + c(ax-cx+b)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+r+b)\, x^{-1+r} + (a_1(1+r)(r+b) + a_0(ar+bc))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r+b)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+r+b) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, -b+1\}$$

- Each term must be 0

$$a_1(1+r)(r+b) + a_0(ar+bc) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r+b) + aka_k + ara_k + (ba_k + a_{k-1}(a-c))\, c = 0$$

- Shift index using $k- > k+1$

$$a_{k+2}(k+2+r)(k+1+r+b) + a(k+1)\, a_{k+1} + ara_{k+1} + (ba_{k+1} + a_k(a-c))\, c = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k ac + aka_{k+1} + ara_{k+1} + bca_{k+1} - a_k c^2 + aa_{k+1}}{(k+2+r)(k+1+r+b)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k ac + aka_{k+1} + bca_{k+1} - a_k c^2 + aa_{k+1}}{(k+2)(k+1+b)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k ac + aka_{k+1} + bca_{k+1} - a_k c^2 + aa_{k+1}}{(k+2)(k+1+b)}, a_0 bc + a_1 b = 0\right]$$

- Recursion relation for $r = -b+1$

$$a_{k+2} = -\frac{a_k ac + aka_{k+1} + a(-b+1)a_{k+1} + bca_{k+1} - a_k c^2 + aa_{k+1}}{(k+3-b)(k+2)}$$

- Solution for $r = -b+1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-b+1}, a_{k+2} = -\frac{a_k ac + aka_{k+1} + a(-b+1)a_{k+1} + bca_{k+1} - a_k c^2 + aa_{k+1}}{(k+3-b)(k+2)}, a_1(2-b) + a_0(a(-b+1) +\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} d_k x^k\right) + \left(\sum_{k=0}^{\infty} e_k x^{k-b+1}\right), d_{k+2} = -\frac{acd_k + akd_{1+k} + bcd_{1+k} - c^2 d_k + ad_{1+k}}{(k+2)(k+1+b)}, bcd_0 + bd_1 = 0, e_{k+2} =\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
dsolve(x*diff(y(x),x$2)+(a*x+b)*diff(y(x),x)+c*((a-c)*x+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-cx} + c_2 x^{-\frac{b}{2}} \operatorname{WhittakerM}\left(-\frac{b}{2}, \frac{1}{2} - \frac{b}{2}, (-2c + a)\, x\right) \mathrm{e}^{-\frac{ax}{2}}$$

✓ Solution by Mathematica

Time used: 0.333 (sec). Leaf size: 50

```
DSolve[x*y''[x]+(a*x+b)*y'[x]+c*((a-c)*x+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-cx} \left(c_1 - c_2 x^{1-b} (x(a-2c))^{b-1} \Gamma(1-b, (a-2c)x)\right)$$

28.12 problem 72

Internal problem ID [10896]
Internal file name [`OUTPUT/10152_Sunday_December_24_2023_05_15_37_PM_13557728/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 72.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''+(2ax+b)\,y'+a(ax+b)\,y=0}$$

28.12.1 Solving using Kovacic algorithm

Writing the ode as

$$xy''+(2ax+b)\,y'+a(ax+b)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2ax+b \\ C &= (ax+b)\,a \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{b(-2+b)}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= b(-2+b) \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{b(-2+b)}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 84: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{b(-2+b)}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{b(-2+b)}{4}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \left\{2, 2 - 2\sqrt{(b-1)^2}, 2 + 2\sqrt{(b-1)^2}\right\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{b(-2+b)}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{1}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\left\{2, 2-2\sqrt{(b-1)^2}, 2+2\sqrt{(b-1)^2}\right\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 2, \; e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2-(2)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{2}{(x-(0))}\right) \\ &= \frac{1}{x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{x} - \frac{b(-2+b)}{4x^2} = 0$$

Solving for ω gives

$$\omega = -\frac{-2+b}{2x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx}\\ &= \mathrm{e}^{\int -\frac{-2+b}{2x} dx}\\ &= x^{1-\frac{b}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2ax+b}{x} \, dx}\\ &= z_1 e^{-ax-\frac{b\ln(x)}{2}}\\ &= z_1 \left(x^{-\frac{b}{2}}\mathrm{e}^{-ax}\right)\end{aligned}$$

Which simplifies to

$$y_1 = x^{-b+1}\mathrm{e}^{-ax}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2ax+b}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2ax-b\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^{b-1}}{b-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-b+1} \mathrm{e}^{-ax}\right) + c_2 \left(x^{-b+1} \mathrm{e}^{-ax} \left(\frac{x^{b-1}}{b-1}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x^{-b+1} \mathrm{e}^{-ax} + \frac{c_2 \mathrm{e}^{-ax}}{b-1} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{-b+1} \mathrm{e}^{-ax} + \frac{c_2 \mathrm{e}^{-ax}}{b-1}
$$

Verified OK.

28.12.2 Maple step by step solution

Let's solve

$y''x + (2ax+b)\,y' + a(ax+b)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(ax+b)ay}{x} - \frac{(2ax+b)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2ax+b)y'}{x} + \frac{(ax+b)ay}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2ax+b}{x}, P_3(x) = \frac{(ax+b)a}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = b$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $y''x + (2ax+b)\,y' + a(ax+b)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+r+b) x^{-1+r} + (a_1(1+r)(r+b) + a a_0(2r+b)) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r+b)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+r+b) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, -b+1\}$$

- Each term must be 0

$$a_1(1+r)(r+b) + a a_0(2r+b) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r+b) + a a_k(2k+2r+b) + a_{k-1} a^2 = 0$$

- Shift index using $k- >k+1$

$$a_{k+2}(k+2+r)(k+1+r+b) + a a_{k+1}(2k+2+2r+b) + a_k a^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a(a a_k + b a_{k+1} + 2k a_{k+1} + 2r a_{k+1} + 2a_{k+1})}{(k+2+r)(k+1+r+b)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a(a a_k + b a_{k+1} + 2k a_{k+1} + 2a_{k+1})}{(k+2)(k+1+b)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a(a a_k + b a_{k+1} + 2k a_{k+1} + 2a_{k+1})}{(k+2)(k+1+b)}, a a_0 b + a_1 b = 0\right]$$

- Recursion relation for $r = -b+1$

$$a_{k+2} = -\frac{a(a a_k + b a_{k+1} + 2k a_{k+1} + 2(-b+1) a_{k+1} + 2a_{k+1})}{(k+3-b)(k+2)}$$

- Solution for $r = -b+1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-b+1}, a_{k+2} = -\frac{a(a a_k + b a_{k+1} + 2k a_{k+1} + 2(-b+1) a_{k+1} + 2a_{k+1})}{(k+3-b)(k+2)}, a_1(2-b) + a a_0(2-b) = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} c_k x^k\right) + \left(\sum_{k=0}^{\infty} d_k x^{k-b+1}\right), c_{k+2} = -\frac{a(ac_k + bc_{1+k} + 2kc_{1+k} + 2c_{1+k})}{(k+2)(k+1+b)}, abc_0 + bc_1 = 0, d_{k+2} = -\frac{a}{}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(x*diff(y(x),x$2)+(2*a*x+b)*diff(y(x),x)+a*(a*x+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-ax}\left(c_1 + x^{-b+1}c_2\right)$$

✓ Solution by Mathematica

Time used: 0.233 (sec). Leaf size: 70

```
DSolve[x*y''[x]+(2*a*x+b)*y'[x]+a*(a*x+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ax}x^{\frac{1}{2}\left(-b-\sqrt{(b-1)^2}+1\right)}\left(c_2 x^{\sqrt{(b-1)^2}} + \sqrt{(b-1)^2}c_1\right)}{\sqrt{(b-1)^2}}$$

28.13 problem 73

Internal problem ID [10897]
Internal file name [`OUTPUT/10153_Sunday_December_24_2023_05_15_38_PM_12477976/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 73.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$xy''+((a+b)x+n+m)y'+(abx+an+bm)y=0$$

28.13.1 Maple step by step solution

Let's solve

$y''x+((a+b)x+n+m)y'+((bx+n)a+bm)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(abx+an+bm)y}{x}-\frac{(ax+bx+m+n)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(ax+bx+m+n)y'}{x}+\frac{(abx+an+bm)y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{ax+bx+m+n}{x}, P_3(x) = \frac{abx+an+bm}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\bigg|_{x=0} = m + n$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x + (ax + bx + m + n)\, y' + (abx + an + bm)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+r+m+n)\, x^{-1+r} + (a_1(1+r)(r+m+n) + a_0(an+ar+bm+br))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}($$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+r+m+n) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, -m-n+1\}$$

- Each term must be 0

$$a_1(1+r)(r+m+n) + a_0(an+ar+bm+br) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r+m+n) + a_k(a+b)\, k + a_k(a+b)\, r + (an+bm)\, a_k + a_{k-1}ab = 0$$

- Shift index using $k- >k+1$

$$a_{k+2}(k+2+r)(k+1+r+m+n) + a_{k+1}(a+b)(k+1) + a_{k+1}(a+b)\, r + (an+bm)\, a_{k+1} + a$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k ab + aka_{k+1} + ana_{k+1} + ara_{k+1} + bka_{k+1} + bma_{k+1} + bra_{k+1} + aa_{k+1} + ba_{k+1}}{(k+2+r)(k+1+r+m+n)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k ab + aka_{k+1} + ana_{k+1} + bka_{k+1} + bma_{k+1} + aa_{k+1} + ba_{k+1}}{(k+2)(k+1+m+n)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k ab + aka_{k+1} + ana_{k+1} + bka_{k+1} + bma_{k+1} + aa_{k+1} + ba_{k+1}}{(k+2)(k+1+m+n)}, a_1(m+n) + a_0(an+bm) =\right.$$

- Recursion relation for $r = -m-n+1$

$$a_{k+2} = -\frac{a_k ab + aka_{k+1} + ana_{k+1} + a(-m-n+1)a_{k+1} + bka_{k+1} + bma_{k+1} + b(-m-n+1)a_{k+1} + aa_{k+1} + ba_{k+1}}{(k+3-m-n)(k+2)}$$

- Solution for $r = -m-n+1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-m-n+1}, a_{k+2} = -\frac{a_k ab + aka_{k+1} + ana_{k+1} + a(-m-n+1)a_{k+1} + bka_{k+1} + bma_{k+1} + b(-m-n+1)a_{k+1} + aa_{k+}}{(k+3-m-n)(k+2)}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} c_k x^k\right) + \left(\sum_{k=0}^{\infty} d_k x^{k-m-n+1}\right), c_{k+2} = -\frac{abc_k + akc_{1+k} + anc_{1+k} + bkc_{1+k} + bmc_{1+k} + ac_{1+k} + bc_{1+k}}{(k+2)(k+1+m+n)}, c_1(r\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 39

```
dsolve(x*diff(y(x),x$2)+((a+b)*x+n+m)*diff(y(x),x)+(a*b*x+a*n+b*m)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-ax}(\mathrm{KummerU}(m, n+m, (a-b)x)c_2 + \mathrm{KummerM}(m, n+m, (a-b)x)c_1)$$

✓ Solution by Mathematica

Time used: 0.131 (sec). Leaf size: 46

```
DSolve[x*y''[x]+((a+b)*x+n+m)*y'[x]+(a*b*x+a*n+b*m)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to e^{-ax}(c_1 \,\mathrm{HypergeometricU}(m, m+n, (a-b)x) + c_2 L_{-m}^{m+n-1}((a-b)x))$$

28.14 problem 74

Internal problem ID [10898]
Internal file name [`OUTPUT/10154_Sunday_December_24_2023_05_15_39_PM_52312670/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 74.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{xy''+(ax+b)\,y'+(cx+d)\,y=0}$$

28.14.1 Maple step by step solution

Let's solve

$y''x+(ax+b)\,y'+(cx+d)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(cx+d)y}{x}-\frac{(ax+b)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(ax+b)y'}{x}+\frac{(cx+d)y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x)=\frac{ax+b}{x}, P_3(x)=\frac{cx+d}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x=0$

  $$\left(x\cdot P_2(x)\right)\Big|_{x=0}=b$$

- $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $$\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $$y''x+(ax+b)\,y'+(cx+d)\,y=0$$

- Assume series solution for y

  $$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m=0..1$

  $$x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

  $$x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m=0..1$

  $$x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+r+b)\, x^{-1+r} + (a_1(1+r)(r+b) + a_0(ar+d))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r+b) +$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+r+b) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, -b+1\}$$

- Each term must be 0

$$a_1(1+r)(r+b) + a_0(ar+d) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r+b) + aka_k + ara_k + a_{k-1}c + a_k d = 0$$

- Shift index using $k- >k+1$

$$a_{k+2}(k+2+r)(k+1+r+b) + a(k+1)\, a_{k+1} + ara_{k+1} + a_k c + a_{k+1}d = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{aka_{k+1} + ara_{k+1} + aa_{k+1} + a_k c + a_{k+1}d}{(k+2+r)(k+1+r+b)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{aka_{k+1} + aa_{k+1} + a_k c + a_{k+1}d}{(k+2)(k+1+b)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{aka_{k+1} + aa_{k+1} + a_k c + a_{k+1}d}{(k+2)(k+1+b)}, a_1 b + a_0 d = 0\right]$$

- Recursion relation for $r = -b+1$

$$a_{k+2} = -\frac{aka_{k+1} + a(-b+1)a_{k+1} + aa_{k+1} + a_k c + a_{k+1}d}{(k+3-b)(k+2)}$$

- Solution for $r = -b+1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-b+1}, a_{k+2} = -\frac{aka_{k+1} + a(-b+1)a_{k+1} + aa_{k+1} + a_k c + a_{k+1}d}{(k+3-b)(k+2)}, a_1(2-b) + a_0(a(-b+1)+d) = 0\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} e_k x^k\right) + \left(\sum_{k=0}^{\infty} f_k x^{k-b+1}\right), e_{k+2} = -\frac{ake_{1+k} + ae_{1+k} + ce_k + de_{1+k}}{(k+2)(k+1+b)}, be_1 + de_0 = 0, f_{k+2} = -\frac{akf_1}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 109

```
dsolve(x*diff(y(x),x$2)+(a*x+b)*diff(y(x),x)+(c*x+d)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x\left(\sqrt{a^2-4c}+a\right)}{2}} \left(\operatorname{KummerM}\left(\frac{b\sqrt{a^2-4c}+ab-2d}{2\sqrt{a^2-4c}}, b, \sqrt{a^2-4c}\,x\right) c_1 + \operatorname{KummerU}\left(\frac{b\sqrt{a^2-4c}+ab-2d}{2\sqrt{a^2-4c}}, b, \sqrt{a^2-4c}\,x\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.135 (sec). Leaf size: 135

```
DSolve[x*y''[x]+(a*x+b)*y'[x]+(c*x+d)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{1}{2}x\left(\sqrt{a^2-4c}+a\right)}\left(c_1 \text{HypergeometricU}\left(\frac{ab+\sqrt{a^2-4c}b-2d}{2\sqrt{a^2-4c}}, b, \sqrt{a^2-4c}x\right) + c_2 L^{b-1}_{-\frac{ab+\sqrt{a^2-4c}b-2d}{2\sqrt{a^2-4c}}}\left(\sqrt{a^2-4c}x\right)\right)$$

28.15 problem 75

Internal problem ID [10899]
Internal file name [`OUTPUT/10155_Sunday_December_24_2023_05_15_40_PM_71475027/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 75.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''-(ax+1)\,y'-b\,x^2(bx+a)\,y=0}$$

28.15.1 Solving using Kovacic algorithm

Writing the ode as

$$xy''+(-ax-1)\,y'+\left(-b^2x^3-ab\,x^2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -ax-1 \\ C &= -b^2x^3-ab\,x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4b^2x^4 + 4ab\,x^3 + a^2x^2 + 2ax + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4b^2x^4 + 4ab\,x^3 + a^2x^2 + 2ax + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4b^2x^4 + 4ab\,x^3 + a^2x^2 + 2ax + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 88: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = b^2x^2 + abx + \frac{a^2}{4} + \frac{3}{4x^2} + \frac{a}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx bx+\frac{a}{2}+\frac{a}{4b\,x^2}-\frac{a^2}{8b^2x^3}+\frac{a^3}{16b^3x^4}+\frac{3}{8b\,x^3}-\frac{a^4}{32b^4x^5}-\frac{3a}{16b^2x^4}+\frac{a^5}{64b^5x^6}+\frac{a^2}{16b^3x^5}-\frac{a^6}{128b^6x^7}-\frac{3a^4}{128b^5x^7}-\frac{3a}{32b^3} \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = b$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{a}{2} + bx \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = b^2x^2 + abx + \frac{1}{4}a^2$$

This shows that the coefficient of 1 in the above is $\frac{a^2}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4b^2x^4 + 4ab\,x^3 + a^2x^2 + 2ax + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(b^2x^2 + abx + \frac{1}{4}a^2\right) + \left(\frac{2ax+3}{4x^2}\right) \\ &= b^2x^2 + abx + \frac{a^2}{4} + \frac{2ax+3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $\frac{a^2}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{a^2}{4}\right) - \left(\frac{a^2}{4}\right) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{a}{2} + bx \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{b} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{b} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4b^2x^4 + 4ab\,x^3 + a^2x^2 + 2ax + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{a}{2} + bx$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)\left(\frac{a}{2} + bx\right) \\
&= -\frac{1}{2x} - \frac{a}{2} - bx \\
&= -\frac{1}{2x} - \frac{a}{2} - bx
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{2x} - \frac{a}{2} - bx\right)(0) + \left(\left(\frac{1}{2x^2} - b\right) + \left(-\frac{1}{2x} - \frac{a}{2} - bx\right)^2 - \left(\frac{4b^2x^4 + 4ab\, x^3 + a^2x^2 + 2ax + 3}{4x^2}\right.\right.
$$

$$
0 =
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{a}{2} - bx\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x(bx+a)}{2}}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-ax-1}{x}\, dx} \\
&= z_1 e^{\frac{ax}{2} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{\frac{ax}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{b\,x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-ax-1}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{ax+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{a\sqrt{\pi}\,\mathrm{e}^{-\frac{a^2}{4b}}\,\mathrm{erf}\left(\frac{2bx+a}{2\sqrt{-b}}\right)+2\,\mathrm{e}^{x(bx+a)}\sqrt{-b}}{4\,(-b)^{\frac{3}{2}}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{b\,x^2}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{b\,x^2}{2}}\left(-\frac{a\sqrt{\pi}\,\mathrm{e}^{-\frac{a^2}{4b}}\,\mathrm{erf}\left(\frac{2bx+a}{2\sqrt{-b}}\right)+2\,\mathrm{e}^{x(bx+a)}\sqrt{-b}}{4\,(-b)^{\frac{3}{2}}}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-\frac{b\,x^2}{2}} - \frac{c_2\mathrm{e}^{-\frac{b\,x^2}{2}}\left(a\sqrt{\pi}\,\mathrm{e}^{-\frac{a^2}{4b}}\,\mathrm{erf}\left(\frac{2bx+a}{2\sqrt{-b}}\right)+2\,\mathrm{e}^{x(bx+a)}\sqrt{-b}\right)}{4\,(-b)^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{-\frac{b\,x^2}{2}} - \frac{c_2\mathrm{e}^{-\frac{b\,x^2}{2}}\left(a\sqrt{\pi}\,\mathrm{e}^{-\frac{a^2}{4b}}\,\mathrm{erf}\left(\frac{2bx+a}{2\sqrt{-b}}\right)+2\,\mathrm{e}^{x(bx+a)}\sqrt{-b}\right)}{4\,(-b)^{\frac{3}{2}}}$$

Verified OK.

28.15.2 Maple step by step solution

Let's solve

$y''x+(-ax-1)\,y'+(-b^2x^3-ab\,x^2)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=bx(bx+a)\,y+\frac{(ax+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(ax+1)y'}{x}-bx(bx+a)\,y=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{ax+1}{x},P_3(x)=-bx(bx+a)\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x+(-ax-1)\,y'-b\,x^2(bx+a)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=2..3$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r)\, x^{-1+r} + (a_1(1+r)\,(-1+r) - a_0 a r)\, x^r + (a_2(2+r)\, r - a a_1(1+r))\, x^{1+r} + (a_3(3+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 2\}$$

- The coefficients of each power of x must be 0

  $$[a_1(1+r)\,(-1+r) - a_0 a r = 0, a_2(2+r)\, r - a a_1(1+r) = 0, a_3(3+r)\,(1+r) - a a_2(2+r) - a_0$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1 = \frac{a_0 a r}{r^2-1}, a_2 = \frac{a^2 a_0}{r^2+r-2}, a_3 = \frac{a a_0 (a^2+br-b)}{r^3+3r^2-r-3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)\,(k+r-1) + (-b a_{k-2} - k a_k - r a_k)\, a - a_{k-3} b^2 = 0$$

- Shift index using $k->k+3$

  $$a_{k+4}(k+4+r)\,(k+2+r) + (-b a_{k+1} - (k+3)\, a_{k+3} - r a_{k+3})\, a - a_k b^2 = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+4} = \frac{a b a_{k+1} + a k a_{k+3} + a r a_{k+3} + a_k b^2 + 3 a a_{k+3}}{(k+4+r)(k+2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+4} = \frac{ab a_{k+1} + ak a_{k+3} + a_k b^2 + 3a a_{k+3}}{(k+4)(k+2)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{ab a_{k+1} + ak a_{k+3} + a_k b^2 + 3a a_{k+3}}{(k+4)(k+2)}, a_1 = 0, a_2 = -\frac{a^2 a_0}{2}, a_3 = -\frac{a a_0 (a^2 - b)}{3}\right]$$

- Recursion relation for $r = 2$

$$a_{k+4} = \frac{ab a_{k+1} + ak a_{k+3} + a_k b^2 + 5a a_{k+3}}{(k+6)(k+4)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+4} = \frac{ab a_{k+1} + ak a_{k+3} + a_k b^2 + 5a a_{k+3}}{(k+6)(k+4)}, a_1 = \frac{2a a_0}{3}, a_2 = \frac{a^2 a_0}{4}, a_3 = \frac{a a_0 (a^2 + b)}{15}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} c_k x^k\right) + \left(\sum_{k=0}^{\infty} d_k x^{k+2}\right), c_{k+4} = \frac{abc_{1+k} + akc_{k+3} + b^2 c_k + 3ac_{k+3}}{(k+4)(k+2)}, c_1 = 0, c_2 = -\frac{a^2 c_0}{2}, c_3 = -\frac{ac_0(}{}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 73

```
dsolve(x*diff(y(x),x$2)-(a*x+1)*diff(y(x),x)-b*x^2*(b*x+a)*y(x)=0,y(x), singsol=all)
```

$$y(x) = -e^{-\frac{2b^2x^2+a^2}{4b}} \sqrt{\pi}\, \sqrt{-b}\, \operatorname{erf}\left(\frac{2bx+a}{2\sqrt{-b}}\right) c_2 a + 2\, e^{\frac{1}{2}x^2 b + ax} c_2 b + c_1 e^{-\frac{x^2 b}{2}}$$

✓ Solution by Mathematica

Time used: 0.444 (sec). Leaf size: 88

```
DSolve[x*y''[x]-(a*x+1)*y'[x]-b*x^2*(b*x+a)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-\frac{bx^2}{2}}\left(2\sqrt{b}\left(c_2e^{x(a+bx)}+2bc_1\right)-\sqrt{\pi}ac_2e^{-\frac{a^2}{4b}}\operatorname{erfi}\left(\frac{a+2bx}{2\sqrt{b}}\right)\right)}{4b^{3/2}}$$

28.16 problem 76

Internal problem ID [10900]
Internal file name [`OUTPUT/10156_Sunday_December_31_2023_11_03_00_AM_16279652/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 76.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''-(2ax+1)\,y'+\left(x^3b+a^2x+a\right)y=0}$$

28.16.1 Solving using Kovacic algorithm

Writing the ode as

$$xy''+(-2ax-1)\,y'+\left(x^3b+a^2x+a\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2ax-1 \\ C &= x^3b+a^2x+a \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4b\,x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4b\,x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4b\,x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 90: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 4\\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -b\,x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i\sqrt{b}\,x - \frac{3i}{8\sqrt{b}\,x^3} - \frac{9i}{128b^{\frac{3}{2}}x^7} - \frac{27i}{1024b^{\frac{5}{2}}x^{11}} - \frac{405i}{32768b^{\frac{7}{2}}x^{15}} - \frac{1701i}{262144b^{\frac{9}{2}}x^{19}} - \frac{15309i}{4194304b^{\frac{11}{2}}x^{23}} - \frac{72171i}{335554432b^{\frac{13}{2}}x^{27}} \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i\sqrt{b}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= i\sqrt{b}\,x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -b\,x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-4b\,x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(-b\,x^2\right) + \left(\frac{3}{4x^2}\right) \\ &= -b\,x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i\sqrt{b}\, x \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i\sqrt{b}} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i\sqrt{b}} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-4b\, x^4 + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$i\sqrt{b}\, x$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)\left(i\sqrt{b}\, x\right) \\
&= -\frac{1}{2x} - i\sqrt{b}\, x \\
&= \frac{-2i\sqrt{b}\, x^2 - 1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - i\sqrt{b}\, x\right)(0) + \left(\left(\frac{1}{2x^2} - i\sqrt{b}\right) + \left(-\frac{1}{2x} - i\sqrt{b}\, x\right)^2 - \left(\frac{-4b\, x^4 + 3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - i\sqrt{b}\, x\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{i\sqrt{b}\, x^2}{2}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2ax-1}{x}\, dx} \\
&= z_1 e^{ax + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{ax}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{ax-\frac{i\sqrt{b}\,x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2ax-1}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2ax+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{i\mathrm{e}^{i\sqrt{b}\,x^2}}{2\sqrt{b}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{ax-\frac{i\sqrt{b}\,x^2}{2}}\right) + c_2 \left(\mathrm{e}^{ax-\frac{i\sqrt{b}\,x^2}{2}} \left(-\frac{i\mathrm{e}^{i\sqrt{b}\,x^2}}{2\sqrt{b}}\right)\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{ax-\frac{i\sqrt{b}\,x^2}{2}} - \frac{ic_2 \mathrm{e}^{\frac{x\left(i\sqrt{b}\,x+2a\right)}{2}}}{2\sqrt{b}} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 \mathrm{e}^{ax-\frac{i\sqrt{b}\,x^2}{2}} - \frac{ic_2 \mathrm{e}^{\frac{x\left(i\sqrt{b}\,x+2a\right)}{2}}}{2\sqrt{b}}$$

Verified OK.

28.16.2 Maple step by step solution

Let's solve

$y''x+(-2ax-1)\,y'+(x^3b+a^2x+a)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x^3b+a^2x+a)y}{x}+\frac{(2ax+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(2ax+1)y'}{x}+\frac{(x^3b+a^2x+a)y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{2ax+1}{x}, P_3(x)=\frac{x^3b+a^2x+a}{x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $y''x+(-2ax-1)\,y'+(x^3b+a^2x+a)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..3$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r)\, x^{-1+r} + (a_1(1+r)\,(-1+r) - a a_0(-1+2r))\, x^r + (a_2(2+r)\, r - a a_1(1+2r) + a_0 a^2$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 2\}$$

- The coefficients of each power of x must be 0

  $$[a_1(1+r)\,(-1+r) - a a_0(-1+2r) = 0, a_2(2+r)\, r - a a_1(1+2r) + a_0 a^2 = 0, a_3(3+r)\,(1+r)$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1 = \frac{a a_0(-1+2r)}{r^2-1}, a_2 = \frac{3a^2 r a_0}{r^3+2r^2-r-2}, a_3 = \frac{2a^3 a_0(1+2r)}{r^4+5r^3+5r^2-5r-6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)\,(k+r-1) + a_{k-1}a^2 - 2a_k\left(k+r-\frac{1}{2}\right) a + a_{k-3}b = 0$$

- Shift index using $k->k+3$

  $$a_{k+4}(k+4+r)\,(k+2+r) + a_{k+2}a^2 - 2a_{k+3}\left(k+\frac{5}{2}+r\right) a + a_k b = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+4} = -\frac{a_{k+2}a^2 - 2aka_{k+3} - 2ara_{k+3} - 5aa_{k+3} + a_k b}{(k+4+r)(k+2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+4} = -\frac{a_{k+2}a^2 - 2aka_{k+3} - 5aa_{k+3} + a_k b}{(k+4)(k+2)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_{k+2}a^2 - 2aka_{k+3} - 5aa_{k+3} + a_k b}{(k+4)(k+2)}, a_1 = a_0 a, a_2 = 0, a_3 = -\frac{a^3 a_0}{3}\right]$$

- Recursion relation for $r = 2$

$$a_{k+4} = -\frac{a_{k+2}a^2 - 2aka_{k+3} - 9aa_{k+3} + a_k b}{(k+6)(k+4)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+4} = -\frac{a_{k+2}a^2 - 2aka_{k+3} - 9aa_{k+3} + a_k b}{(k+6)(k+4)}, a_1 = a_0 a, a_2 = \frac{a_0 a^2}{2}, a_3 = \frac{a^3 a_0}{6}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} c_k x^k\right) + \left(\sum_{k=0}^{\infty} d_k x^{k+2}\right), c_{k+4} = -\frac{a^2 c_{k+2} - 2akc_{k+3} - 5ac_{k+3} + bc_k}{(k+4)(k+2)}, c_1 = c_0 a, c_2 = 0, c_3 = -\frac{a^3 c_0}{3}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
dsolve(x*diff(y(x),x$2)-(2*a*x+1)*diff(y(x),x)+(b*x^3+a^2*x+a)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{ax + \frac{x^2\sqrt{-b}}{2}} + c_2 e^{ax - \frac{x^2\sqrt{-b}}{2}}$$

✓ Solution by Mathematica

Time used: 0.276 (sec). Leaf size: 59

```
DSolve[x*y''[x]-(2*a*x+1)*y'[x]+(b*x^3+a^2*x+a)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{2} e^{ax - \frac{1}{2} i \sqrt{b} x^2} \left(2c_1 - \frac{i c_2 e^{i\sqrt{b} x^2}}{\sqrt{b}}\right)$$

28.17 problem 77

Internal problem ID [10901]
Internal file name [`OUTPUT/10157_Sunday_December_31_2023_11_03_01_AM_84918379/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 77.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_y]]`

$$\boxed{xy''+(ax+b)\,y'=-cx\left(-c\,x^2+ax+b+1\right)}$$

28.17.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x)=y'$$

Then

$$p'(x)=y''$$

Hence the ode becomes

$$p'(x)\,x+(ax+b)\,p(x)+cx\left(-c\,x^2+ax+b+1\right)=0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x)+p(x)p(x)=q(x)$$

Where here

$$p(x) = -\frac{-ax-b}{x}$$
$$q(x) = -c(-c\,x^2+ax+b+1)$$

Hence the ode is

$$p'(x) - \frac{(-ax-b)\,p(x)}{x} = -c(-c\,x^2+ax+b+1)$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{-ax-b}{x}dx}$$
$$= \mathrm{e}^{ax+b\ln(x)}$$

Which simplifies to

$$\mu = x^b \mathrm{e}^{ax}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) = (\mu)\left(-c(-c\,x^2+ax+b+1)\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^b\mathrm{e}^{ax}p\right) = \left(x^b\mathrm{e}^{ax}\right)\left(-c(-c\,x^2+ax+b+1)\right)$$
$$\mathrm{d}\left(x^b\mathrm{e}^{ax}p\right) = \left(-c(-c\,x^2+ax+b+1)\,x^b\mathrm{e}^{ax}\right)\,\mathrm{d}x$$

Integrating gives

$$x^b\mathrm{e}^{ax}p = \int -c(-c\,x^2+ax+b+1)\,x^b\mathrm{e}^{ax}\,\mathrm{d}x$$
$$x^b\mathrm{e}^{ax}p = -\frac{(-a)^{-b}\,c^2\left(x^b(-a)^b\,b(b^2+3b+2)\,\Gamma(b)\,(-ax)^{-b} - x^b(-a)^b\,(a^2x^2-abx-2ax+b^2+3b+2)\,\mathrm{e}^{a}}{a^3}$$

Dividing both sides by the integrating factor $\mu = x^b\mathrm{e}^{ax}$ results in

$$p(x) = x^{-b}\mathrm{e}^{-ax}\left(-\frac{(-a)^{-b}\,c^2\left(x^b(-a)^b\,b(b^2+3b+2)\,\Gamma(b)\,(-ax)^{-b} - x^b(-a)^b\,(a^2x^2-abx-2ax+b^2+3b\right.}{a^3}\right.$$

which simplifies to

$$p(x) = \frac{c^2\mathrm{e}^{-ax}((b^3+3b^2+2b)\,\Gamma(b,-ax) - \Gamma(b+3))\,(-ax)^{-b} + x^{-b}c_1a^3\mathrm{e}^{-ax} - c((-a^2x^2+x(2+b)\,a-b^2}{a^3}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c^2 e^{-ax}((b^3 + 3b^2 + 2b)\,\Gamma(b, -ax) - \Gamma(b+3))\,(-ax)^{-b} + x^{-b}c_1 a^3 e^{-ax} - c((-a^2x^2 + x(2+b)\,a - b^2 -}{a^3}$$

Integrating both sides gives

$$y = \int \frac{(-ax)^{-b}\, b^3 c^2 \Gamma(b, -ax)\, e^{-ax} + a^2c^2x^2 + 3(-ax)^{-b}\, b^2c^2\Gamma(b, -ax)\, e^{-ax} - a^3cx - a\,c^2bx + x^{-b}c_1a^3e^{-a}}{a^3}$$

$$= \int \frac{(-ax)^{-b}\, b^3 c^2 \Gamma(b, -ax)\, e^{-ax} + a^2c^2x^2 + 3(-ax)^{-b}\, b^2c^2\Gamma(b, -ax)\, e^{-ax} - a^3cx - a\,c^2bx + x^{-b}c_1a^3e^{-a}}{a^3}$$

Summary

The solution(s) found are the following

$$y = \int \frac{(-ax)^{-b}\, b^3 c^2 \Gamma(b, -ax)\, e^{-ax} + a^2c^2x^2 + 3(-ax)^{-b}\, b^2c^2\Gamma(b, -ax)\, e^{-ax} - a^3cx - a\,c^2bx + x^{-b}c_1a^3e^{-ax} -}{a^3} + c_2 \tag{1}$$

Verification of solutions

$$y = \int \frac{(-ax)^{-b}\, b^3 c^2 \Gamma(b, -ax)\, e^{-ax} + a^2c^2x^2 + 3(-ax)^{-b}\, b^2c^2\Gamma(b, -ax)\, e^{-ax} - a^3cx - a\,c^2bx + x^{-b}c_1a^3e^{-ax} -}{a^3} + c_2$$

Verified OK.

28.17.2 Maple step by step solution

Let's solve

$y''x + (ax + b)\, y' = -cx(-c\,x^2 + ax + b + 1)$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x)\, x + (ax + b)\, u(x) = -cx(-c\,x^2 + ax + b + 1)$

- Isolate the derivative

  $u'(x) = -\frac{(ax+b)u(x)}{x} - c(-c\,x^2 + ax + b + 1)$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

$$u'(x) + \frac{(ax+b)u(x)}{x} = -c(-c\,x^2 + ax + b + 1)$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(u'(x) + \frac{(ax+b)u(x)}{x}\right) = -\mu(x)\,c(-c\,x^2 + ax + b + 1)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

$$\mu(x)\left(u'(x) + \frac{(ax+b)u(x)}{x}\right) = \mu'(x)\,u(x) + \mu(x)\,u'(x)$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{\mu(x)(ax+b)}{x}$$

- Solve to find the integrating factor

$$\mu(x) = x^b \mathrm{e}^{ax}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,u(x))\right) dx = \int -\mu(x)\,c(-c\,x^2 + ax + b + 1)\,dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,u(x) = \int -\mu(x)\,c(-c\,x^2 + ax + b + 1)\,dx + c_1$$

- Solve for $u(x)$

$$u(x) = \frac{\int -\mu(x)c(-c\,x^2+ax+b+1)dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = x^b \mathrm{e}^{ax}$

$$u(x) = \frac{\int -c(-c\,x^2+ax+b+1)x^b\mathrm{e}^{ax}dx + c_1}{x^b\mathrm{e}^{ax}}$$

- Evaluate the integrals on the rhs

$$u(x) = \frac{-\frac{(-a)^{-b}c^2\left(x^b(-a)^b b\left(b^2+3b+2\right)\Gamma(b)(-ax)^{-b} - x^b(-a)^b\left(a^2x^2 - abx - 2ax + b^2 + 3b + 2\right)\mathrm{e}^{ax} - x^b(-a)^b b\left(b^2+3b+2\right)(-ax)^{-b}\Gamma(b,-ax)\right)}{a^3} - \frac{(-\cdots}{\cdots}}{\cdots}$$

- Simplify

$$u(x) = \frac{c^2\mathrm{e}^{-ax}\left(\left(b^3+3b^2+2b\right)\Gamma(b,-ax) - \Gamma(b+3)\right)(-ax)^{-b} + x^{-b}c_1a^3\mathrm{e}^{-ax} - c\left(\left(-a^2x^2+x(2+b)a-b^2-3b-2\right)c+a^3x\right)}{a^3}$$

- Solve 1st ODE for $u(x)$

$$u(x) = \frac{c^2\mathrm{e}^{-ax}\left(\left(b^3+3b^2+2b\right)\Gamma(b,-ax) - \Gamma(b+3)\right)(-ax)^{-b} + x^{-b}c_1a^3\mathrm{e}^{-ax} - c\left(\left(-a^2x^2+x(2+b)a-b^2-3b-2\right)c+a^3x\right)}{a^3}$$

- Make substitution $u = y'$

$$y' = \frac{c^2\mathrm{e}^{-ax}\left(\left(b^3+3b^2+2b\right)\Gamma(b,-ax) - \Gamma(b+3)\right)(-ax)^{-b} + x^{-b}c_1a^3\mathrm{e}^{-ax} - c\left(\left(-a^2x^2+x(2+b)a-b^2-3b-2\right)c+a^3x\right)}{a^3}$$

- Integrate both sides to solve for y

$$\int y'dx = \int \frac{c^2\mathrm{e}^{-ax}\left(\left(b^3+3b^2+2b\right)\Gamma(b,-ax) - \Gamma(b+3)\right)(-ax)^{-b} + x^{-b}c_1a^3\mathrm{e}^{-ax} - c\left(\left(-a^2x^2+x(2+b)a-b^2-3b-2\right)c+a^3x\right)}{a^3}dx + c_2$$

- Compute integrals

$$y = \int \frac{c^2 \mathrm{e}^{-ax}((b^3+3b^2+2b)\Gamma(b,-ax)-\Gamma(b+3))(-ax)^{-b}+x^{-b}c_1a^3\mathrm{e}^{-ax}-c((-a^2x^2+x(2+b)a-b^2-3b-2)c+a^3x)}{a^3}dx + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(-c^2*_a^3+c*_a^2*a+_a*_b(_a)*a+_a*b*c
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 115

```
dsolve(x*diff(y(x),x$2)+(a*x+b)*diff(y(x),x)+c*x*(-c*x^2+a*x+b+1)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2a^3 - \left(\int \left(-c^2((b^3+3b^2+2b)\,\Gamma(b,-ax) - \Gamma(b+3))\,\mathrm{e}^{-ax}(-ax)^{-b} - \mathrm{e}^{-ax}x^{-b}c_1a^3 + ((-b^2+(ax-3)\,b\right.\right.}{a^3}$$

✓ Solution by Mathematica

Time used: 61.322 (sec). Leaf size: 92

```
DSolve[x*y''[x]+(a*x+b)*y'[x]+c*x*(-c*x^2+a*x+b+1)==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \int_1^x e^{-aK[1]}K[1]^{-b}\left(\frac{c(-((b+1)\Gamma(b+1,-aK[1])a^2)+\Gamma(b+2,-aK[1])a^2+c\Gamma(b+3,-aK[1]))\,K[1]^b(}{a^3} + c_1\right)dK[1] + c_2$$

28.18 problem 78

Internal problem ID [10902]
Internal file name [`OUTPUT/10158_Sunday_December_31_2023_11_03_03_AM_20057251/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 78.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$xy''-(2ax+1)\,y'+yb\,x^3=0$$

28.18.1 Maple step by step solution

Let's solve

$y''x+(-2ax-1)\,y'+yb\,x^3=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-x^2by+\frac{(2ax+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(2ax+1)y'}{x}+x^2by=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{2ax+1}{x}, P_3(x) = b\,x^2\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (-2ax - 1)\,y' + yb\,x^3 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

  $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

- Shift index using $k- >k-3$

  $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\,x^{-1+r} + (a_1(1+r)\,(-1+r) - 2a_0 a r)\,x^r + (a_2(2+r)\,r - 2a a_1(1+r))\,x^{1+r} + (a_3(3$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 2\}$$

- The coefficients of each power of x must be 0

  $$[a_1(1+r)\,(-1+r) - 2a_0 a r = 0, a_2(2+r)\,r - 2a a_1(1+r) = 0, a_3(3+r)\,(1+r) - 2a a_2(2+r) =$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1 = \frac{2a_0 a r}{r^2-1}, a_2 = \frac{4a^2 a_0}{r^2+r-2}, a_3 = \frac{8a^3 a_0}{r^3+3r^2-r-3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)\,(k+r-1) - 2a a_k(k+r) + a_{k-3} b = 0$$

- Shift index using $k- >k+3$

  $$a_{k+4}(k+4+r)\,(k+2+r) - 2a a_{k+3}(k+r+3) + a_k b = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+4} = \frac{2aka_{k+3} + 2ara_{k+3} + 6aa_{k+3} - a_k b}{(k+4+r)(k+2+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+4} = \frac{2aka_{k+3} + 6aa_{k+3} - a_k b}{(k+4)(k+2)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2aka_{k+3} + 6aa_{k+3} - a_k b}{(k+4)(k+2)}, a_1 = 0, a_2 = -2a^2 a_0, a_3 = -\frac{8a^3 a_0}{3}\right]$$

- Recursion relation for $r = 2$

  $$a_{k+4} = \frac{2aka_{k+3} + 10aa_{k+3} - a_k b}{(k+6)(k+4)}$$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+4} = \frac{2aka_{k+3} + 10aa_{k+3} - a_k b}{(k+6)(k+4)}, a_1 = \frac{4aa_0}{3}, a_2 = a^2 a_0, a_3 = \frac{8a^3 a_0}{15}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}c_kx^k\right)+\left(\sum_{k=0}^{\infty}d_kx^{k+2}\right),c_{k+4}=\frac{2akc_{k+3}+6ac_{k+3}-bc_k}{(k+4)(k+2)},c_1=0,c_2=-2a^2c_0,c_3=-\frac{8a^3c_0}{3},d_{k\cdot}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.188 (sec). Leaf size: 106

```
dsolve(x*diff(y(x),x$2)-(2*a*x+1)*diff(y(x),x)+b*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^2 \operatorname{HeunB}\left(2, 0, \frac{a^2}{\sqrt{-b}}, -\frac{2ia}{(-b)^{\frac{1}{4}}}, i(-b)^{\frac{1}{4}} x\right) e^{ax+\frac{x^2\sqrt{-b}}{2}} \left(c_1 + c_2 \left(\int \frac{e^{-x^2\sqrt{-b}}}{\operatorname{HeunB}\left(2, 0, \frac{a^2}{\sqrt{-b}}, -\frac{2ia}{(-b)^{\frac{1}{4}}}, i\,(-b)^{\frac{1}{4}} x\right)^2 x^3} dx\right)\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y''[x]-(2*a*x+1)*y'[x]+b*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

28.19 problem 79

Internal problem ID [10903]
Internal file name [`OUTPUT/10159_Sunday_December_31_2023_11_03_03_AM_54718002/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 79.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{xy''+\left(ab\,x^2+b-5\right)y'+2a^2(-2+b)\,x^3y=0}$$

28.19.1 Maple step by step solution

Let's solve

$y''x+\left(ab\,x^2+b-5\right)y'+2a^2(-2+b)\,x^3y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{\left(ab\,x^2+b-5\right)y'}{x}-2x^2a^2(-2+b)\,y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{\left(ab\,x^2+b-5\right)y'}{x}+2x^2a^2(-2+b)\,y=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{ab\,x^2+b-5}{x}, P_3(x) = 2a^2x^2(-2+b)\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = b - 5$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (ab\,x^2 + b - 5)\,y' + 2a^2(-2+b)\,x^3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

  $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

- Shift index using $k- >k-3$

  $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-6+r+b)\, x^{-1+r} + a_1(1+r)(-5+r+b)\, x^r + (a_2(2+r)(-4+r+b) + a_0 abr)\, x^{1+r} + (a_3$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-6+r+b) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, -b+6\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)(-5+r+b) = 0, a_2(2+r)(-4+r+b) + a_0 abr = 0, a_3(3+r)(-3+r+b) + a_1(1+$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = -\frac{a_0 abr}{rb+r^2+2b-2r-8}, a_3 = 0\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-5+r+b) + a_{k-1}(k+r-1)\, ab + 2a_{k-3}a^2(-2+b) = 0$

- Shift index using $k->k+3$

  $a_{k+4}(k+4+r)(k-2+r+b) + a_{k+2}(k+2+r)\, ab + 2a_k a^2(-2+b) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{a(2aba_k+bka_{k+2}+bra_{k+2}-4aa_k+2ba_{k+2})}{(k+4+r)(k-2+r+b)}$

- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{a(2aba_k+bka_{k+2}-4aa_k+2ba_{k+2})}{(k+4)(k-2+b)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a(2aba_k+bka_{k+2}-4aa_k+2ba_{k+2})}{(k+4)(k-2+b)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

- Recursion relation for $r = -b+6$

  $a_{k+4} = -\frac{a(2aba_k+bka_{k+2}+b(-b+6)a_{k+2}-4aa_k+2ba_{k+2})}{(k+10-b)(k+4)}$

- Solution for $r = -b+6$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-b+6}, a_{k+4} = -\frac{a(2aba_k+bka_{k+2}+b(-b+6)a_{k+2}-4aa_k+2ba_{k+2})}{(k+10-b)(k+4)}, a_1 = 0, a_2 = -\frac{a_0 ab(-b+6)}{(-b+6)b+(-b+6)^2+}\right.$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} c_k x^k\right) + \left(\sum_{k=0}^{\infty} d_k x^{k-b+6}\right), c_{k+4} = -\frac{a(2abc_k+bkc_{k+2}-4ac_k+2bc_{k+2})}{(k+4)(k-2+b)}, c_1 = 0, c_2 = 0, c_3 = 0, d_k\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.187 (sec). Leaf size: 92

```
dsolve(x*diff(y(x),x$2)+(a*b*x^2+b-5)*diff(y(x),x)+2*a^2*(b-2)*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(-3\,\mathrm{KummerU}\left(\frac{b}{2}+1, -2+\frac{b}{2}, \frac{a(b-4)x^2}{2}\right) c_2 b + \left(a(b-4)\,x^2+b+4\right) c_2\,\mathrm{KummerU}\left(\frac{b}{2}, -2+\frac{b}{2}, \frac{a(b-4)x^2}{2}\right)\right.}{2}$$

✓ Solution by Mathematica

Time used: 3.578 (sec). Leaf size: 67

```
DSolve[x*y''[x]+(a*b*x^2+b-5)*y'[x]+2*a^2*(b-2)*x^3*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to e^{-ax^2}\left(ax^2+1\right)\left(c_2\int_1^x \frac{e^{-\frac{1}{2}a(b-4)K[1]^2}K[1]^{5-b}}{\left(aK[1]^2+1\right)^2}dK[1]+c_1\right)$$

28.20 problem 80

Internal problem ID [10904]
Internal file name [`OUTPUT/10160_Sunday_December_31_2023_11_03_04_AM_75682043/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 80.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy''+\left(a\,x^2+bx\right)y'-\left(ac\,x^2+\left(bc+c^2+a\right)x+b+2c\right)y=0}$$

28.20.1 Solving using Kovacic algorithm

Writing the ode as

$$xy''+x(ax+b)\,y'+\left(-c^2x+\left(-a\,x^2-bx-2\right)c-ax-b\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= (ax+b)\,x \\ C &= -c^2x+\left(-a\,x^2-bx-2\right)c-ax-b \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^3 a^2 + 2ab\,x^2 + 4ac\,x^2 + b^2 x + 4bcx + 4c^2 x + 6ax + 4b + 8c}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^3 a^2 + 2ab\,x^2 + 4ac\,x^2 + b^2 x + 4bcx + 4c^2 x + 6ax + 4b + 8c$$
$$t = 4x$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^3 a^2 + 2ab\,x^2 + 4ac\,x^2 + b^2 x + 4bcx + 4c^2 x + 6ax + 4b + 8c}{4x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 95: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{b}{2} + c + \frac{ax}{2} + \frac{3}{2x} + \frac{2bc}{a^2x^3} - \frac{3b^2c}{a^3x^4} - \frac{6b\,c^2}{a^3x^4} + \frac{4b^3c}{a^4x^5} + \frac{12b^2c^2}{a^4x^5} + \frac{16b\,c^3}{a^4x^5} - \frac{22bc}{a^3x^5} - \frac{5b^4c}{a^5x^6} - \frac{20b^3c^2}{a^5x^6} - \frac{40b^2c^3}{a^5x^6} - \frac{40b\,c^4}{a^5x^6} + \cdots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{b}{2} + c + \frac{ax}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}b^2 + bc + \frac{1}{2}abx + c^2 + acx + \frac{1}{4}a^2x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{4}b^2 + bc + c^2$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^3a^2 + 2ab\,x^2 + 4ac\,x^2 + b^2x + 4bcx + 4c^2x + 6ax + 4b + 8c}{4x} \\
&= Q + \frac{R}{4x} \\
&= \left(\frac{a^2x^2}{4} + \left(\frac{1}{2}ab + ac\right)x + \frac{b^2}{4} + bc + c^2 + \frac{3a}{2}\right) + \left(\frac{4b+8c}{4x}\right) \\
&= \frac{a^2x^2}{4} + \left(\frac{1}{2}ab + ac\right)x + \frac{b^2}{4} + bc + c^2 + \frac{3a}{2} + \frac{4b+8c}{4x}
\end{aligned}$$

We see that the coefficient of the term 1 in the quotient is $\frac{1}{4}b^2 + bc + c^2 + \frac{3}{2}a$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{1}{4}b^2 + bc + c^2 + \frac{3}{2}a\right) - \left(\frac{1}{4}b^2 + bc + c^2\right) \\
&= \frac{3a}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{b}{2} + c + \frac{ax}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) \quad = \frac{1}{2}\left(\frac{\frac{3a}{2}}{\frac{a}{2}} - 1\right) \quad = 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{\frac{3a}{2}}{\frac{a}{2}} - 1\right) = -2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^3a^2 + 2ab\,x^2 + 4ac\,x^2 + b^2x + 4bcx + 4c^2x + 6ax + 4b + 8c}{4x}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{b}{2} + c + \frac{ax}{2}$	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} + \left(\frac{b}{2} + c + \frac{ax}{2}\right) \\
&= \frac{1}{x} + \frac{b}{2} + c + \frac{ax}{2} \\
&= \frac{1}{x} + \frac{b}{2} + c + \frac{ax}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{x} + \frac{b}{2} + c + \frac{ax}{2}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{a}{2}\right) + \left(\frac{1}{x} + \frac{b}{2} + c + \frac{ax}{2}\right)^2 - \left(\frac{x^3 a^2 + 2ab\,x^2 + 4ac\,x^2 + b^2 x
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x} + \frac{b}{2} + c + \frac{ax}{2}\right) dx} \\
&= x\,\mathrm{e}^{\frac{x(ax+2b+4c)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{(ax+b)x}{x} \, dx} \\
&= z_1 e^{-\frac{1}{4}a\,x^2 - \frac{1}{2}bx} \\
&= z_1 \left(\mathrm{e}^{-\frac{x(ax+2b)}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{cx}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{(ax+b)x}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{1}{2}a\,x^2-bx}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x(ax+2b+4c)}{2}}}{x^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\,\mathrm{e}^{cx}) + c_2\left(x\,\mathrm{e}^{cx}\left(\int \frac{\mathrm{e}^{-\frac{x(ax+2b+4c)}{2}}}{x^2}dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{cx} + c_2 x\,\mathrm{e}^{cx}\left(\int \frac{\mathrm{e}^{-\frac{x(ax+2b+4c)}{2}}}{x^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{cx} + c_2 x\,\mathrm{e}^{cx}\left(\int \frac{\mathrm{e}^{-\frac{x(ax+2b+4c)}{2}}}{x^2}dx\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 8.546 (sec). Leaf size: 34

```
dsolve(x*diff(y(x),x$2)+(a*x^2+b*x)*diff(y(x),x)-(a*c*x^2+(a+b*c+c^2)*x+b+2*c)*y(x)=0,y(x),
```

$$y(x) = e^{cx} x \left(c_1 + c_2 \left(\int \frac{e^{-\frac{x(ax+2b+4c)}{2}}}{x^2} dx\right)\right)$$

✓ Solution by Mathematica

Time used: 3.129 (sec). Leaf size: 49

```
DSolve[x*y''[x]+(a*x^2+b*x)*y'[x]-(a*c*x^2+(a+b*c+c^2)*x+b+2*c)*y[x]==0,y[x],x,IncludeSingul
```

$$y(x) \to xe^{cx} \left(c_2 \int_1^x \frac{e^{-\frac{1}{2}K[1](2b+4c+aK[1])}}{K[1]^2} dK[1] + c_1 \right)$$

28.21 problem 81

Internal problem ID [10905]
Internal file name [`OUTPUT/10161_Sunday_December_31_2023_11_03_05_AM_43663572/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 81.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''+\left(a\,x^{2}+bx+2\right)y'+yb=0}$$

28.21.1 Solving using Kovacic algorithm

Writing the ode as

$$xy''+\left(a\,x^{2}+bx+2\right)y'+yb=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= a\,x^{2}+bx+2 \\ C &= b \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2abx + b^2 + 6a}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^2 + 2abx + b^2 + 6a \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2 + \frac{3}{2}a\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 96: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{ax}{2} + \frac{b}{2} + \frac{3}{2x} - \frac{3b}{2a\,x^2} + \frac{3b^2}{2a^2x^3} - \frac{9}{4a\,x^3} - \frac{3b^3}{2a^3x^4} + \frac{27b}{4a^2x^4} + \frac{3b^4}{2a^4x^5} - \frac{27b^2}{2a^3x^5} - \frac{3b^5}{2a^5x^6} + \frac{27}{4a^2x^5} + \frac{45b^3}{2a^4x^6} - \frac{135b}{4a^3x^6} + \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{ax}{2} + \frac{b}{2} \qquad (10)\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2$$

This shows that the coefficient of 1 in the above is $\frac{b^2}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{a^2x^2 + 2abx + b^2 + 6a}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2 + \frac{3}{2}a\right) + (0) \\
&= \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2 + \frac{3}{2}a
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{3a}{2} + \frac{b^2}{4}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{3a}{2} + \frac{b^2}{4}\right) - \left(\frac{b^2}{4}\right) \\
&= \frac{3a}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{ax}{2} + \frac{b}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{3a}{2}}{\frac{a}{2}} - 1\right) &= 1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{\frac{3a}{2}}{\frac{a}{2}} - 1\right) = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2 + \frac{3}{2}a$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{ax}{2}+\frac{b}{2}$	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{ax}{2} + \frac{b}{2}\right) \\ &= \frac{ax}{2} + \frac{b}{2} \\ &= \frac{ax}{2} + \frac{b}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{ax}{2} + \frac{b}{2}\right)(1) + \left(\left(\frac{a}{2}\right) + \left(\frac{ax}{2} + \frac{b}{2}\right)^2 - \left(\frac{1}{4}a^2x^2 + \frac{1}{2}abx + \frac{1}{4}b^2 + \frac{3}{2}a\right)\right) = 0$$

$$-aa_0 + b = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{b}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{b}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{b}{a}\right) \mathrm{e}^{\int \left(\frac{ax}{2} + \frac{b}{2}\right) dx} \\
&= \left(x + \frac{b}{a}\right) \mathrm{e}^{\frac{1}{4} a\, x^2 + \frac{1}{2} b x} \\
&= \frac{(ax + b)\, \mathrm{e}^{\frac{x(ax+2b)}{4}}}{a}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{a\, x^2 + bx + 2}{x} \, dx} \\
&= z_1 e^{-\frac{a\, x^2}{4} - \frac{bx}{2} - \ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x(ax+2b)}{4}}}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{ax + b}{xa}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{a\,x^2+bx+2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{a\,x^2}{2}-bx-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\left(\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{b^2}{2a}}(ax+b)\,\mathrm{erf}\left(\frac{\sqrt{2}\,(ax+b)}{2\sqrt{a}}\right)+2\,\mathrm{e}^{-\frac{x(ax+2b)}{2}}\sqrt{a}\right)\sqrt{a}}{2ax+2b}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{ax+b}{xa}\right) \\
&\quad + c_2\left(\frac{ax+b}{xa}\left(-\frac{\left(\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{b^2}{2a}}(ax+b)\,\mathrm{erf}\left(\frac{\sqrt{2}\,(ax+b)}{2\sqrt{a}}\right)+2\,\mathrm{e}^{-\frac{x(ax+2b)}{2}}\sqrt{a}\right)\sqrt{a}}{2ax+2b}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(ax+b)}{xa} - \frac{c_2\left(\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{b^2}{2a}}(ax+b)\,\mathrm{erf}\left(\frac{\sqrt{2}\,(ax+b)}{2\sqrt{a}}\right)+2\,\mathrm{e}^{-\frac{x(ax+2b)}{2}}\sqrt{a}\right)}{2\sqrt{a}\,x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(ax+b)}{xa} - \frac{c_2\left(\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{b^2}{2a}}(ax+b)\,\mathrm{erf}\left(\frac{\sqrt{2}\,(ax+b)}{2\sqrt{a}}\right)+2\,\mathrm{e}^{-\frac{x(ax+2b)}{2}}\sqrt{a}\right)}{2\sqrt{a}\,x}
$$

Verified OK.

28.21.2 Maple step by step solution

Let's solve

$y''x + (a\,x^2 + bx + 2)\,y' + yb = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{by}{x} - \frac{(a\,x^2+bx+2)\,y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(a\,x^2+bx+2)\,y'}{x} + \frac{by}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{a\,x^2+bx+2}{x}, P_3(x) = \frac{b}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x + (a\,x^2 + bx + 2)\,y' + yb = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x\cdot y''$ to series expansion

  $x\cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x\cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(1+r)\, x^{-1+r} + (a_1(1+r)\,(2+r) + a_0b(1+r))\, x^{r} + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k+2+r) + a_kb($

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) + a_0b(1+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+2+r) + a_kb(k+1+r) + a_{k-1}(k+r-1)\, a = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a_{k+1}b(k+2+r) + a_k(k+r)\, a = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{aka_k + ara_k + bka_{k+1} + bra_{k+1} + 2ba_{k+1}}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

  $a_{k+2} = -\frac{aka_k + bka_{k+1} - aa_k + ba_{k+1}}{(k+1)(k+2)}$

- Solution for $r=-1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{aka_k + bka_{k+1} - aa_k + ba_{k+1}}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r=0$

  $a_{k+2} = -\frac{aka_k + bka_{k+1} + 2ba_{k+1}}{(k+2)(k+3)}$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{ak a_k+bk a_{k+1}+2b a_{k+1}}{(k+2)(k+3)}, a_0 b+2a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} c_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} d_k x^k\right), c_{k+2}=-\frac{ak c_k+bk c_{1+k}-a c_k+b c_{1+k}}{(1+k)(k+2)}, 0=0, d_{k+2}=-\frac{ak d_k+bk d_{1+k}+2b d_1}{(k+2)(k+3)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
dsolve(x*diff(y(x),x$2)+(a*x^2+b*x+2)*diff(y(x),x)+b*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{e^{\frac{b^2}{2a}}\pi c_2(ax+b)\operatorname{erf}\left(\frac{\sqrt{2}\,(ax+b)}{2\sqrt{a}}\right)+\sqrt{\pi}\,\sqrt{2}\,\sqrt{a}\,e^{-\frac{x(ax+2b)}{2}}c_2+c_1(ax+b)}{x}$$

✓ Solution by Mathematica

Time used: 0.535 (sec). Leaf size: 85

```
DSolve[x*y''[x]+(a*x^2+b*x+2)*y'[x]+b*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{(ax+b)\left(-\frac{\sqrt{\frac{\pi}{2}}c_2\operatorname{erf}\left(\frac{ax+b}{\sqrt{2}\sqrt{a}}\right)}{a^{3/2}}-\frac{c_2e^{-\frac{(ax+b)^2}{2a}}}{a(ax+b)}+c_1\right)}{bx}$$

28.22 problem 82

Internal problem ID [10906]
Internal file name [`OUTPUT/10162_Sunday_December_31_2023_11_03_06_AM_31797229/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 82.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{xy''+\left(a\,x^{2}+bx+c\right)y'+\left(2ax+b\right)y=0}$$

28.22.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(xy''+\left(a\,x^{2}+bx+c\right)y'+\left(2ax+b\right)y\right)dx=0$$
$$\left(a\,x^{2}+bx+c-1\right)y+y'x=c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{-a\,x^{2}-bx-c+1}{x}\\
q(x) &= \frac{c_1}{x}
\end{aligned}
$$

Hence the ode is

$$
y'-\frac{\left(-a\,x^{2}-bx-c+1\right)y}{x} = \frac{c_1}{x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{-a\,x^{2}-bx-c+1}{x}dx}\\
&= \mathrm{e}^{\frac{a\,x^{2}}{2}+bx+(c-1)\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = x^{c-1}\mathrm{e}^{\frac{x(ax+2b)}{2}}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{c-1}\mathrm{e}^{\frac{x(ax+2b)}{2}}y\right) &= \left(x^{c-1}\mathrm{e}^{\frac{x(ax+2b)}{2}}\right)\left(\frac{c_1}{x}\right)\\
\mathrm{d}\left(x^{c-1}\mathrm{e}^{\frac{x(ax+2b)}{2}}y\right) &= \left(c_1x^{c-2}\mathrm{e}^{\frac{x(ax+2b)}{2}}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^{c-1}\mathrm{e}^{\frac{x(ax+2b)}{2}}y &= \int c_1x^{c-2}\mathrm{e}^{\frac{x(ax+2b)}{2}}\,\mathrm{d}x\\
x^{c-1}\mathrm{e}^{\frac{x(ax+2b)}{2}}y &= \int c_1x^{c-2}\mathrm{e}^{\frac{x(ax+2b)}{2}}dx + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^{c-1}\mathrm{e}^{\frac{x(ax+2b)}{2}}$ results in

$$
y = x^{-c+1}\mathrm{e}^{-\frac{x(ax+2b)}{2}}\left(\int c_1x^{c-2}\mathrm{e}^{\frac{x(ax+2b)}{2}}dx\right) + c_2x^{-c+1}\mathrm{e}^{-\frac{x(ax+2b)}{2}}
$$

which simplifies to

$$
y = x^{-c+1}\mathrm{e}^{-\frac{x(ax+2b)}{2}}\left(c_1\left(\int x^{c-2}\mathrm{e}^{\frac{1}{2}a\,x^{2}+bx}dx\right)+c_2\right)
$$

Summary
The solution(s) found are the following

$$y = x^{-c+1} \mathrm{e}^{-\frac{x(ax+2b)}{2}} \left(c_1 \left(\int x^{c-2} \mathrm{e}^{\frac{1}{2} a\, x^2 + bx} dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = x^{-c+1} \mathrm{e}^{-\frac{x(ax+2b)}{2}} \left(c_1 \left(\int x^{c-2} \mathrm{e}^{\frac{1}{2} a\, x^2 + bx} dx\right) + c_2\right)$$

Verified OK.

28.22.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' + \left(a\, x^2 + bx + c\right) y' + (2ax + b)\, y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(xy'' + \left(a\, x^2 + bx + c\right) y' + (2ax + b)\, y\right) dx = 0$$
$$\left(a\, x^2 + bx + c - 1\right) y + y'x = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-a\, x^2 - bx - c + 1}{x}$$
$$q(x) = \frac{c_1}{x}$$

Hence the ode is

$$y' - \frac{\left(-a\, x^2 - bx - c + 1\right) y}{x} = \frac{c_1}{x}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{-a\, x^2 - bx - c + 1}{x} dx}$$
$$= \mathrm{e}^{\frac{a\, x^2}{2} + bx + (c-1)\ln(x)}$$

Which simplifies to

$$\mu = x^{c-1} e^{\frac{x(ax+2b)}{2}}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x}\right) \\
\frac{d}{dx}\left(x^{c-1} e^{\frac{x(ax+2b)}{2}} y\right) &= \left(x^{c-1} e^{\frac{x(ax+2b)}{2}}\right)\left(\frac{c_1}{x}\right) \\
d\left(x^{c-1} e^{\frac{x(ax+2b)}{2}} y\right) &= \left(c_1 x^{c-2} e^{\frac{x(ax+2b)}{2}}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
x^{c-1} e^{\frac{x(ax+2b)}{2}} y &= \int c_1 x^{c-2} e^{\frac{x(ax+2b)}{2}} dx \\
x^{c-1} e^{\frac{x(ax+2b)}{2}} y &= \int c_1 x^{c-2} e^{\frac{x(ax+2b)}{2}} dx + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^{c-1} e^{\frac{x(ax+2b)}{2}}$ results in

$$y = x^{-c+1} e^{-\frac{x(ax+2b)}{2}} \left(\int c_1 x^{c-2} e^{\frac{x(ax+2b)}{2}} dx\right) + c_2 x^{-c+1} e^{-\frac{x(ax+2b)}{2}}$$

which simplifies to

$$y = x^{-c+1} e^{-\frac{x(ax+2b)}{2}} \left(c_1 \left(\int x^{c-2} e^{\frac{1}{2} a x^2 + bx} dx\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = x^{-c+1} e^{-\frac{x(ax+2b)}{2}} \left(c_1 \left(\int x^{c-2} e^{\frac{1}{2} a x^2 + bx} dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = x^{-c+1} e^{-\frac{x(ax+2b)}{2}} \left(c_1 \left(\int x^{c-2} e^{\frac{1}{2} a x^2 + bx} dx\right) + c_2\right)$$

Verified OK.

28.22.3 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x \\ q(x) &= a\,x^2 + bx + c \\ r(x) &= 2ax + b \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 2ax + b \end{aligned}$$

Therefore (1) becomes

$$0 - (2ax + b) + (2ax + b) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$\left(a\,x^2 + bx + c - 1\right) y + y'x = c_1$$

We now have a first order ode to solve which is

$$\left(a\,x^2 + bx + c - 1\right) y + y'x = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{-a\,x^2-bx-c+1}{x}\\
q(x) &= \frac{c_1}{x}
\end{aligned}
$$

Hence the ode is

$$
y'-\frac{(-a\,x^2-bx-c+1)\,y}{x}=\frac{c_1}{x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{-a\,x^2-bx-c+1}{x}dx}\\
&= \mathrm{e}^{\frac{a\,x^2}{2}+bx+(c-1)\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = x^{c-1}\mathrm{e}^{\frac{x(ax+2b)}{2}}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{c-1}\mathrm{e}^{\frac{x(ax+2b)}{2}}y\right) &= \left(x^{c-1}\mathrm{e}^{\frac{x(ax+2b)}{2}}\right)\left(\frac{c_1}{x}\right)\\
\mathrm{d}\left(x^{c-1}\mathrm{e}^{\frac{x(ax+2b)}{2}}y\right) &= \left(c_1x^{c-2}\mathrm{e}^{\frac{x(ax+2b)}{2}}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^{c-1}\mathrm{e}^{\frac{x(ax+2b)}{2}}y &= \int c_1x^{c-2}\mathrm{e}^{\frac{x(ax+2b)}{2}}\,\mathrm{d}x\\
x^{c-1}\mathrm{e}^{\frac{x(ax+2b)}{2}}y &= \int c_1x^{c-2}\mathrm{e}^{\frac{x(ax+2b)}{2}}dx + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^{c-1}\mathrm{e}^{\frac{x(ax+2b)}{2}}$ results in

$$
y = x^{-c+1}\mathrm{e}^{-\frac{x(ax+2b)}{2}}\left(\int c_1x^{c-2}\mathrm{e}^{\frac{x(ax+2b)}{2}}dx\right)+c_2x^{-c+1}\mathrm{e}^{-\frac{x(ax+2b)}{2}}
$$

which simplifies to

$$
y = x^{-c+1}\mathrm{e}^{-\frac{x(ax+2b)}{2}}\left(c_1\left(\int x^{c-2}\mathrm{e}^{\frac{1}{2}a\,x^2+bx}dx\right)+c_2\right)
$$

Summary
The solution(s) found are the following

$$y = x^{-c+1} e^{-\frac{x(ax+2b)}{2}} \left(c_1 \left(\int x^{c-2} e^{\frac{1}{2} a x^2 + bx} dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = x^{-c+1} e^{-\frac{x(ax+2b)}{2}} \left(c_1 \left(\int x^{c-2} e^{\frac{1}{2} a x^2 + bx} dx\right) + c_2\right)$$

Verified OK.

28.22.4 Maple step by step solution

Let's solve

$y'' x + (a x^2 + bx + c) y' + (2ax + b) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2ax+b)y}{x} - \frac{(a x^2+bx+c)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(a x^2+bx+c)y'}{x} + \frac{(2ax+b)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{a x^2+bx+c}{x}, P_3(x) = \frac{2ax+b}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = c$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x+(a\,x^{2}+bx+c)\,y'+(2ax+b)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_{k} x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^{m}\cdot y$ to series expansion for $m=0..1$

    $x^{m}\cdot y=\sum_{k=0}^{\infty} a_{k} x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^{m}\cdot y'$ to series expansion for $m=0..2$

    $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x\cdot y''$ to series expansion

    $x\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,x^{k+r-1}$

  - Shift index using $k->k+1$

    $x\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}r(-1+r+c)\,x^{-1+r}+(a_{1}(1+r)\,(r+c)+a_{0}b(1+r))\,x^{r}+\left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k+r+c)\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $r(-1+r+c)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,-c+1\}$

- Each term must be 0

  $a_{1}(1+r)\,(r+c)+a_{0}b(1+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1+r)\left(a_{k+1}(k+r+c)+a_k b+a_{k-1}a\right)=0$
- Shift index using $k->k+1$

  $(k+r+2)\left(a_{k+2}(k+1+r+c)+a_{k+1}b+a_k a\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k a+a_{k+1}b}{k+1+r+c}$
- Recursion relation for $r=0$

  $a_{k+2}=-\frac{a_k a+a_{k+1}b}{k+1+c}$
- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k a+a_{k+1}b}{k+1+c}, a_0 b+a_1 c=0\right]$
- Recursion relation for $r=-c+1$

  $a_{k+2}=-\frac{a_k a+a_{k+1}b}{k+2}$
- Solution for $r=-c+1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-c+1}, a_{k+2}=-\frac{a_k a+a_{k+1}b}{k+2}, a_1(-c+2)+a_0 b(-c+2)=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} d_k x^k\right)+\left(\sum_{k=0}^{\infty} e_k x^{k-c+1}\right), d_{k+2}=-\frac{a d_k+b d_{1+k}}{k+1+c}, b d_0+c d_1=0, e_{k+2}=-\frac{a e_k+b e_{1+k}}{k+2}, e_1(-c\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
No hypergeometric solution was found.
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 46

```
dsolve(x*diff(y(x),x$2)+(a*x^2+b*x+c)*diff(y(x),x)+(2*a*x+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^{-c+1} \mathrm{e}^{-\frac{x(ax+2b)}{2}} \left(c_1 \left(\int x^{c-2} \mathrm{e}^{\frac{1}{2} a\, x^2+bx} dx\right) + c_2\right)$$

✓ Solution by Mathematica

Time used: 1.772 (sec). Leaf size: 63

```
DSolve[x*y''[x]+(a*x^2+b*x+c)*y'[x]+(2*a*x+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to x^{1-c} e^{-\frac{1}{2}x(ax+2b)} \left(c_2 \int_1^x e^{\frac{1}{2}aK[1]^2+bK[1]} K[1]^{c-2} dK[1] + c_1\right)$$

28.23 problem 83

Internal problem ID [10907]
Internal file name [`OUTPUT/10163_Sunday_December_31_2023_11_03_11_AM_95428243/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 83.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$
\boxed{xy''+\left(a\,x^{2}+bx+c\right)y'+\left(c-1\right)\left(ax+b\right)y=0}
$$

28.23.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$
xy''+\left(a\,x^{2}+bx+c\right)y'+\left(c-1\right)\left(ax+b\right)y=0 \tag{1}
$$

Becomes

$$
y''+p(x)\,y'+q(x)\,y=0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x)&=\frac{a\,x^{2}+bx+c}{x}\\
q(x)&=\frac{\left(c-1\right)\left(ax+b\right)}{x}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(a\,x^2 + bx + c)}{x^2} + \frac{(c-1)\,(ax+b)}{x} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -c + 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{-2c+2}{x} + \frac{a\,x^2 + bx + c}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(a\,x^2 + bx - c + 2)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(a\,x^2 + bx - c + 2)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(a\,x^2 + bx - c + 2)\, u}{x} \end{aligned}$$

Where $f(x) = -\frac{a\,x^2+bx-c+2}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{a\,x^2+bx-c+2}{x}\,dx\\
\int \frac{1}{u}\,du &= \int -\frac{a\,x^2+bx-c+2}{x}\,dx\\
\ln\left(u\right) &= -\frac{a\,x^2}{2}-bx-\left(-c+2\right)\ln\left(x\right)+c_1\\
u &= \mathrm{e}^{-\frac{a\,x^2}{2}-bx-\left(-c+2\right)\ln\left(x\right)+c_1}\\
&= c_1\mathrm{e}^{-\frac{a\,x^2}{2}-bx-\left(-c+2\right)\ln\left(x\right)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_1\mathrm{e}^{-\frac{a\,x^2}{2}}\mathrm{e}^{-bx}x^c}{x^2}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x)\\
v(x) &= \int u(x)\,dx + c_2\\
&= \int \frac{c_1\mathrm{e}^{-\frac{a\,x^2}{2}}\mathrm{e}^{-bx}x^c}{x^2}dx + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n\\
&= \left(\int \frac{c_1\mathrm{e}^{-\frac{a\,x^2}{2}}\mathrm{e}^{-bx}x^c}{x^2}dx + c_2\right)x^{-c+1}\\
&= x^{-c+1}\left(c_1\left(\int x^{c-2}\mathrm{e}^{-\frac{1}{2}a\,x^2-bx}dx\right)+c_2\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(\int \frac{c_1\mathrm{e}^{-\frac{a\,x^2}{2}}\mathrm{e}^{-bx}x^c}{x^2}dx + c_2\right)x^{-c+1} \tag{1}
$$

Verification of solutions

$$y = \left(\int \frac{c_1 \mathrm{e}^{-\frac{a\,x^2}{2}} \mathrm{e}^{-bx} x^c}{x^2} dx + c_2\right) x^{-c+1}$$

Verified OK.

28.23.2 Maple step by step solution

Let's solve

$y''x + (a\,x^2 + bx + c)\,y' + (c-1)\,(ax+b)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(c-1)(ax+b)y}{x} - \frac{(a\,x^2+bx+c)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(a\,x^2+bx+c)y'}{x} + \frac{(c-1)(ax+b)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{a\,x^2+bx+c}{x}, P_3(x) = \frac{(c-1)(ax+b)}{x}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = c$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (a\,x^2 + bx + c)\,y' + (c-1)\,(ax+b)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+r+c)\, x^{-1+r} + (a_1(1+r)\,(r+c) + a_0 b(-1+r+c))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k+r\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+r+c) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, -c+1\}$$

- Each term must be 0

$$a_1(1+r)\,(r+c) + a_0 b(-1+r+c) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(k+r+c) + a_k b(k+r+c-1) + a_{k-1} a(k-2+r+c) = 0$$

- Shift index using $k- >k + 1$

$$a_{k+2}(k+2+r)(k+1+r+c)+a_{k+1}b(k+r+c)+a_k a(k+r+c-1)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a_k ac+aka_k+ara_k+bca_{k+1}+bka_{k+1}+bra_{k+1}-a_k a}{(k+2+r)(k+1+r+c)}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{a_k ac+aka_k+bca_{k+1}+bka_{k+1}-a_k a}{(k+2)(k+1+c)}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+2}=-\frac{a_k ac+aka_k+bca_{k+1}+bka_{k+1}-a_k a}{(k+2)(k+1+c)}, a_1 c+a_0 b(c-1)=0\right]$$

- Recursion relation for $r=-c+1$

$$a_{k+2}=-\frac{a_k ac+aka_k+a(-c+1)a_k+bca_{k+1}+bka_{k+1}+b(-c+1)a_{k+1}-a_k a}{(k+3-c)(k+2)}$$

- Solution for $r=-c+1$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-c+1}, a_{k+2}=-\frac{a_k ac+aka_k+a(-c+1)a_k+bca_{k+1}+bka_{k+1}+b(-c+1)a_{k+1}-a_k a}{(k+3-c)(k+2)}, a_1(-c+2)=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}d_k x^k\right)+\left(\sum_{k=0}^{\infty}e_k x^{k-c+1}\right), d_{k+2}=-\frac{acd_k+akd_k+bcd_{1+k}+bkd_{1+k}-ad_k}{(k+2)(k+1+c)}, d_1 c+d_0 b(c-1)=0, e\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.297 (sec). Leaf size: 102

```
dsolve(x*diff(y(x),x$2)+(a*x^2+b*x+c)*diff(y(x),x)+(c-1)*(a*x+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{x(ax+2b)}{2}} \left(\mathrm{HeunB}\left(c-1, \frac{b\sqrt{2}}{\sqrt{a}}, c-3, -\frac{\sqrt{2}\, b(c-2)}{\sqrt{a}}, \frac{\sqrt{2}\,\sqrt{a}\, x}{2}\right) c_1 + \mathrm{HeunB}\left(-c+1, \frac{b\sqrt{2}}{\sqrt{a}}, c-3, -\frac{\sqrt{2}\, b(c-2)}{\sqrt{a}}, \frac{\sqrt{2}\,\sqrt{a}\, x}{2}\right) x^{-c+1} c_2\right)$$

✓ Solution by Mathematica

Time used: 1.579 (sec). Leaf size: 49

```
DSolve[x*y''[x]+(a*x^2+b*x+c)*y'[x]+(c-1)*(a*x+b)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to x^{1-c}\left(c_2\int_1^x e^{-\frac{1}{2}K[1](2b+aK[1])}K[1]^{c-2}dK[1]+c_1\right)$$

28.24 problem 84

Internal problem ID [10908]
Internal file name [`OUTPUT/10164_Sunday_December_31_2023_11_03_13_AM_15472296/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 84.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$
\boxed{xy''+\left(a\,x^{2}+bx+c\right)y'+\left(A\,x^{2}+Bx+\mathrm{C0}\right)y=0}
$$

28.24.1 Maple step by step solution

Let's solve

$y''x+\left(a\,x^{2}+bx+c\right)y'+\left(A\,x^{2}+Bx+C0\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{\left(A\,x^{2}+Bx+C0\right)y}{x}-\frac{\left(a\,x^{2}+bx+c\right)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{\left(a\,x^{2}+bx+c\right)y'}{x}+\frac{\left(A\,x^{2}+Bx+C0\right)y}{x}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{a x^2+bx+c}{x}, P_3(x) = \frac{A x^2+Bx+C0}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = c$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (a x^2 + bx + c) y' + (A x^2 + Bx + C0) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r) x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1) x^{k+r-1}$

- Shift index using $k- >k + 1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+r+c)\,x^{-1+r} + (a_1(1+r)\,(r+c) + a_0(br+C0))\,x^r + (a_2(2+r)\,(1+r+c) + a_1(br+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r+c) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, -c+1\}$
- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(r+c) + a_0(br+C0) = 0, a_2(2+r)\,(1+r+c) + a_1(br+C0+b) + a_0(ar+B) = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_1 = -\frac{a_0(br+C0)}{rc+r^2+c+r}, a_2 = -\frac{a_0(a\,r^2c+a\,r^3-b^2r^2+Brc+B\,r^2-2br\,C0+arc+a\,r^2-b^2r+Bc+Br-C0^2-C0b)}{r^2c^2+2r^3c+r^4+3r\,c^2+7r^2c+4r^3+2c^2+7rc+5r^2+2c+2r}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r+c) + a_k((k+r)\,b+C0) + a_{k-1}((k+r-1)\,a+B) + Aa_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+3}(k+3+r)\,(k+2+r+c) + a_{k+2}((k+2+r)\,b+C0) + a_{k+1}((k+1+r)\,a+B) + Aa_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+3} = -\frac{aka_{k+1}+ara_{k+1}+bka_{k+2}+bra_{k+2}+Aa_k+Ba_{k+1}+C0a_{k+2}+aa_{k+1}+2ba_{k+2}}{(k+3+r)(k+2+r+c)}$
- Recursion relation for $r = 0$

  $a_{k+3} = -\frac{aka_{k+1}+bka_{k+2}+Aa_k+Ba_{k+1}+C0a_{k+2}+aa_{k+1}+2ba_{k+2}}{(k+3)(k+2+c)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{aka_{k+1}+bka_{k+2}+Aa_k+Ba_{k+1}+C0a_{k+2}+aa_{k+1}+2ba_{k+2}}{(k+3)(k+2+c)}, a_1 = -\frac{a_0\,C0}{c}, a_2 = -\frac{a_0\,(Bc-C}{2c^2+}\right.$
- Recursion relation for $r = -c+1$

  $a_{k+3} = -\frac{aka_{k+1}+a(-c+1)a_{k+1}+bka_{k+2}+b(-c+1)a_{k+2}+Aa_k+Ba_{k+1}+C0a_{k+2}+aa_{k+1}+2ba_{k+2}}{(k+4-c)(k+3)}$
- Solution for $r = -c+1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-c+1}, a_{k+3} = -\frac{aka_{k+1}+a(-c+1)a_{k+1}+bka_{k+2}+b(-c+1)a_{k+2}+Aa_k+Ba_{k+1}+C0a_{k+2}+aa_{k+1}+2ba_{k+2}}{(k+4-c)(k+3)}, a\right.$
- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} d_k x^k\right) + \left(\sum_{k=0}^{\infty} e_k x^{k-c+1}\right), d_{k+3} = -\frac{akd_{1+k}+bkd_{k+2}+Ad_k+Bd_{1+k}+C0d_{k+2}+ad_{1+k}+2bd_{k+2}}{(k+3)(k+2+c)}, d_1 = \right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.281 (sec). Leaf size: 186

```
dsolve(x*diff(y(x),x$2)+(a*x^2+b*x+c)*diff(y(x),x)+(A*x^2+B*x+C0)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\frac{x\left(-a^2x-2ab+2A\right)}{2a}}\left(x^{-c+1}\,\mathrm{HeunB}\left(-c+1, -\frac{\sqrt{2}\,(-ab+2A)}{a^{\frac{3}{2}}}, \frac{(-c-1)\,a^3+2B\,a^2-2Aab+2A^2}{a^3}, \frac{(bc-2\,\mathrm{C0})\sqrt{2}}{\sqrt{a}}, \frac{\sqrt{2}\,\sqrt{a}\,x}{2}\right)c_2 + \mathrm{HeunB}\left(c-1, -\frac{\sqrt{2}\,(-ab+2A)}{a^{\frac{3}{2}}}, \frac{(-c-1)\,a^3+2B\,a^2-2Aab+2A^2}{a^3}, \frac{(bc-2\,\mathrm{C0})\sqrt{2}}{\sqrt{a}}, \frac{\sqrt{2}\,\sqrt{a}\,x}{2}\right)c_1\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y''[x]+(a*x^2+b*x+c)*y'[x]+(A*x^2+B*x+C0)*y[x]==0,y[x],x,IncludeSingularSolutions -
```

Not solved

28.25 problem 85

Internal problem ID [10909]
Internal file name [`OUTPUT/10165_Sunday_December_31_2023_11_03_13_AM_53294476/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 85.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{xy''+(a\,x^2+bx+2)\,y'+(c\,x^2+dx+b)\,y=0}$$

28.25.1 Maple step by step solution

Let's solve

$y''x+(a\,x^2+bx+2)\,y'+(c\,x^2+dx+b)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(c\,x^2+dx+b)y}{x}-\frac{(a\,x^2+bx+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(a\,x^2+bx+2)y'}{x}+\frac{(c\,x^2+dx+b)y}{x}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = \frac{a\,x^2+bx+2}{x}, P_3(x) = \frac{c\,x^2+dx+b}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (a\,x^2 + bx + 2)\,y' + (c\,x^2 + dx + b)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r)\, x^{-1+r} + \left(a_1(1+r)(2+r) + a_0 b(1+r)\right) x^r + \left(a_2(2+r)(3+r) + a_1 b(2+r) + a_0(ar$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)(2+r) + a_0 b(1+r) = 0, a_2(2+r)(3+r) + a_1 b(2+r) + a_0(ar+d) = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = -\frac{a_0 b}{2+r}, a_2 = -\frac{a_0(ar-b^2+d)}{r^2+5r+6}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+2+r) + a_k b(k+1+r) + a_{k-1}((k+r-1)a+d) + a_{k-2}c = 0$

- Shift index using $k- >k+2$

  $a_{k+3}(k+3+r)(k+4+r) + a_{k+2} b(k+3+r) + a_{k+1}((k+1+r)a+d) + a_k c = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+3} = -\frac{aka_{k+1}+ara_{k+1}+bka_{k+2}+bra_{k+2}+aa_{k+1}+3ba_{k+2}+a_k c+da_{k+1}}{(k+3+r)(k+4+r)}$

- Recursion relation for $r = -1$

  $a_{k+3} = -\frac{aka_{k+1}+bka_{k+2}+2ba_{k+2}+a_k c+da_{k+1}}{(k+2)(k+3)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+3} = -\frac{aka_{k+1}+bka_{k+2}+2ba_{k+2}+a_k c+da_{k+1}}{(k+2)(k+3)}, a_1 = -a_0 b, a_2 = -\frac{a_0(-b^2-a+d)}{2}\right]$

- Recursion relation for $r = 0$

  $a_{k+3} = -\frac{aka_{k+1}+bka_{k+2}+aa_{k+1}+3ba_{k+2}+a_k c+da_{k+1}}{(k+3)(k+4)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{aka_{k+1}+bka_{k+2}+aa_{k+1}+3ba_{k+2}+a_k c+da_{k+1}}{(k+3)(k+4)}, a_1 = -\frac{a_0 b}{2}, a_2 = -\frac{a_0(-b^2+d)}{6}\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} e_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} f_k x^k\right), e_{k+3} = -\frac{ake_{1+k}+bke_{k+2}+2be_{k+2}+ce_k+de_{1+k}}{(k+2)(k+3)}, e_1 = -e_0 b, e_2 = -\frac{e_0(-\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: indirect Equivalence to 0F1 under \`\`^ @ Moebius\`\` is resolve
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 143

```
dsolve(x*diff(y(x),x$2)+(a*x^2+b*x+2)*diff(y(x),x)+(c*x^2+d*x+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{-\frac{x(a^2x+2ab-2c)}{2a}}\left(\text{hypergeom}\left(\left[\frac{3a^3-d\,a^2+abc-c^2}{2a^3}\right],\left[\frac{3}{2}\right],\frac{(a^2x+ab-2c)^2}{2a^3}\right)(a^2x+ab-2c)\,c_2 + c_1\,\text{hypergeom}\left(\left[\frac{2a}{\right.\right.\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.155 (sec). Leaf size: 134

```
DSolve[x*y''[x]+(a*x^2+b*x+2)*y'[x]+(c*x^2+d*x+b)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{e^{-\frac{1}{2}x\left(-\frac{2c}{a}+ax+2b\right)}\left(c_2\,\text{Hypergeometric1F1}\left(-\frac{-2a^3+da^2-bca+c^2}{2a^3},\frac{1}{2},\frac{(xa^2+ba-2c)^2}{2a^3}\right)+c_1\,\text{HermiteH}\left(\frac{-2a^3+da^2-bc}{a^3}\right.\right.}{x}$$

28.26 problem 86

Internal problem ID [10910]
Internal file name [`OUTPUT/10166_Sunday_December_31_2023_11_03_14_AM_61480679/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 86.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''+\left(a\,x^{3}+b\right)y'+a(b-1)\,x^{2}y=0}$$

28.26.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$xy''+\left(a\,x^{3}+b\right)y'+a(b-1)\,x^{2}y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{a\,x^{3}+b}{x} \\ q(x) &= ax(b-1) \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(a\, x^3 + b)}{x^2} + ax(b-1) = 0 \tag{5}$$

Solving (5) for n gives

$$n = -b + 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{-2b+2}{x} + \frac{a\, x^3 + b}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(a\, x^3 - b + 2)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(a\, x^3 - b + 2)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(a\, x^3 - b + 2)\, u}{x} \end{aligned}$$

Where $f(x) = -\frac{a\,x^3-b+2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\frac{1}{u}\,du = -\frac{a\,x^3-b+2}{x}\,dx$$
$$\int \frac{1}{u}\,du = \int -\frac{a\,x^3-b+2}{x}\,dx$$
$$\ln(u) = -\frac{a\,x^3}{3} - (2-b)\ln(x) + c_1$$
$$u = \mathrm{e}^{-\frac{a\,x^3}{3}-(2-b)\ln(x)+c_1}$$
$$= c_1 \mathrm{e}^{-\frac{a\,x^3}{3}-(2-b)\ln(x)}$$

Which simplifies to

$$u(x) = \frac{c_1 \mathrm{e}^{-\frac{a\,x^3}{3}} x^b}{x^2}$$

Now that $u(x)$ is known, then

$$v'(x) = u(x)$$
$$v(x) = \int u(x)\,dx + c_2$$
$$= 3^{-\frac{4}{3}+\frac{b}{3}} a^{-\frac{b}{3}+\frac{1}{3}} c_1 \left(\frac{3^{-\frac{b}{6}+\frac{8}{3}} x^{2+b} a^{\frac{2}{3}+\frac{b}{3}} (a\,x^3)^{-\frac{1}{3}-\frac{b}{6}} \mathrm{e}^{-\frac{a\,x^3}{6}} \operatorname{WhittakerM}\left(\frac{1}{3}+\frac{b}{6}, \frac{b}{6}+\frac{5}{6}, \frac{a\,x^3}{3}\right)}{(b-1)(2+b)(5+b)} + \frac{3^{-\frac{b}{6}+\frac{8}{3}} x^{-4+}}{}\right.$$

Hence

$$y = v(x)\,x^n$$
$$= \left(3^{-\frac{4}{3}+\frac{b}{3}} a^{-\frac{b}{3}+\frac{1}{3}} c_1 \left(\frac{3^{-\frac{b}{6}+\frac{8}{3}} x^{2+b} a^{\frac{2}{3}+\frac{b}{3}} (a\,x^3)^{-\frac{1}{3}-\frac{b}{6}} \mathrm{e}^{-\frac{a\,x^3}{6}} \operatorname{WhittakerM}\left(\frac{1}{3}+\frac{b}{6}, \frac{b}{6}+\frac{5}{6}, \frac{a\,x^3}{3}\right)}{(b-1)(2+b)(5+b)} + \frac{3^{-\frac{b}{6}+\frac{8}{3}} x^{-4+b} a}{}\right.\right.$$
$$= \frac{3^{\frac{4}{3}+\frac{b}{6}} a c_1 x^3 (a\,x^3)^{-\frac{1}{3}-\frac{b}{6}} \mathrm{e}^{-\frac{a\,x^3}{6}} \operatorname{WhittakerM}\left(\frac{1}{3}+\frac{b}{6}, \frac{b}{6}+\frac{5}{6}, \frac{a\,x^3}{3}\right) + (5+b)\left(c_1(a\,x^3+b+2)\,\mathrm{e}^{-\frac{a\,x^3}{3}} + c_2 x^{-}\right.}{(2+b)(5+b)(b-1)}$$

Summary
The solution(s) found are the following

$$y = \left(3^{-\frac{4}{3}+\frac{b}{3}} a^{-\frac{b}{3}+\frac{1}{3}} c_1 \left(\frac{3^{-\frac{b}{6}+\frac{8}{3}} x^{2+b} a^{\frac{2}{3}+\frac{b}{3}} (a\,x^3)^{-\frac{1}{3}-\frac{b}{6}} \mathrm{e}^{-\frac{a\,x^3}{6}} \operatorname{WhittakerM}\left(\frac{1}{3}+\frac{b}{6}, \frac{b}{6}+\frac{5}{6}, \frac{a\,x^3}{3}\right)}{(b-1)\,(2+b)\,(5+b)} + \frac{3^{-\frac{b}{6}+\frac{8}{3}} x^{-4+b} a^{-\frac{4}{3}+\frac{b}{3}} (a\,x^3+b+2)\,(a\,x^3)^{-\frac{1}{3}-\frac{b}{6}} \mathrm{e}^{-\frac{a\,x^3}{6}} \operatorname{WhittakerM}\left(\frac{4}{3}+\frac{b}{6}, \frac{b}{6}+\frac{5}{6}, \frac{a\,x^3}{3}\right)}{(b-1)\,(2+b)}\right) + c_2\right) x^{-b+1} \tag{1}$$

Verification of solutions

$$y = \left(3^{-\frac{4}{3}+\frac{b}{3}} a^{-\frac{b}{3}+\frac{1}{3}} c_1 \left(\frac{3^{-\frac{b}{6}+\frac{8}{3}} x^{2+b} a^{\frac{2}{3}+\frac{b}{3}} (a\,x^3)^{-\frac{1}{3}-\frac{b}{6}} \mathrm{e}^{-\frac{a\,x^3}{6}} \operatorname{WhittakerM}\left(\frac{1}{3}+\frac{b}{6}, \frac{b}{6}+\frac{5}{6}, \frac{a\,x^3}{3}\right)}{(b-1)\,(2+b)\,(5+b)} + \frac{3^{-\frac{b}{6}+\frac{8}{3}} x^{-4+b} a^{-\frac{4}{3}+\frac{b}{3}} (a\,x^3+b+2)\,(a\,x^3)^{-\frac{1}{3}-\frac{b}{6}} \mathrm{e}^{-\frac{a\,x^3}{6}} \operatorname{WhittakerM}\left(\frac{4}{3}+\frac{b}{6}, \frac{b}{6}+\frac{5}{6}, \frac{a\,x^3}{3}\right)}{(b-1)\,(2+b)}\right) + c_2\right) x^{-b+1}$$

Verified OK.

28.26.2 Maple step by step solution

Let's solve

$y''x + (a\,x^3+b)\,y' + a(b-1)\,x^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(a\,x^3+b)\,y'}{x} - ax(b-1)\,y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(a\,x^3+b)\,y'}{x} + ax(b-1)\,y = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{a\,x^3+b}{x}, P_3(x) = ax(b-1)\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = b$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $y''x + (a\,x^3 + b)\,y' + a(b-1)\,x^2y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^2 \cdot y$ to series expansion

    $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+2}$
  - Shift index using $k- >k-2$

    $x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+r+b) x^{-1+r} + a_1(1+r)(r+b) x^r + a_2(2+r)(1+r+b) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_{k+1}(k+1+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+r+b) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, -b+1\}$$

- The coefficients of each power of x must be 0

$$[a_1(1+r)(r+b) = 0, a_2(2+r)(1+r+b) = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_1 = 0, a_2 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r+b) + a_{k-2} a(k-3+r+b) = 0$$

- Shift index using $k- >k+2$

$$a_{k+3}(k+3+r)(k+2+r+b) + a_k a(k+r+b-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = -\frac{a_k a(k+r+b-1)}{(k+3+r)(k+2+r+b)}$$

- Recursion relation for $r = 0$

$$a_{k+3} = -\frac{a_k a(k-1+b)}{(k+3)(k+2+b)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a_k a(k-1+b)}{(k+3)(k+2+b)}, a_1 = 0, a_2 = 0\right]$$

- Recursion relation for $r = -b+1$

$$a_{k+3} = -\frac{a_k a k}{(k+4-b)(k+3)}$$

- Solution for $r = -b+1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-b+1}, a_{k+3}=-\frac{a_k a k}{(k+4-b)(k+3)}, a_1=0, a_2=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} c_k x^k\right)+\left(\sum_{k=0}^{\infty} d_k x^{k-b+1}\right), c_{k+3}=-\frac{c_k a(k-1+b)}{(k+3)(k+2+b)}, c_1=0, c_2=0, d_{k+3}=-\frac{d_k a k}{(k+4-b)(k+3)}, d\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 107

```
dsolve(x*diff(y(x),x$2)+(a*x^3+b)*diff(y(x),x)+a*(b-1)*x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{9c_2a^2x^{-\frac{b}{2}+3}e^{-\frac{a\,x^3}{6}}\operatorname{WhittakerM}\left(\frac{1}{3}+\frac{b}{6},\frac{b}{6}+\frac{5}{6},\frac{a\,x^3}{3}\right)+\left(a\,x^{-\frac{b}{2}+3}+x^{-\frac{b}{2}}(b+2)\right)c_2e^{-\frac{a\,x^3}{3}}a3^{-\frac{b}{6}+\frac{2}{3}}(b+5)\,(a}{9x}$$

✓ Solution by Mathematica

Time used: 0.424 (sec). Leaf size: 60

```
DSolve[x*y''[x]+(a*x^3+b)*y'[x]+a*(b-1)*x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_1x^{1-b}-3^{\frac{b-4}{3}}c_2\left(ax^3\right)^{\frac{1}{3}-\frac{b}{3}}\Gamma\left(\frac{b-1}{3},\frac{ax^3}{3}\right)$$

28.27 problem 87

Internal problem ID [10911]
Internal file name [`OUTPUT/10167_Sunday_December_31_2023_11_03_15_AM_41317649/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 87.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{xy''+x\left(a\,x^2+b\right)y'+\left(3a\,x^2+b\right)y=0}$$

28.27.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(xy''+x\left(a\,x^2+b\right)y'+\left(3a\,x^2+b\right)y\right)dx=0$$
$$\left(a\,x^3+bx-1\right)y+y'x=c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{-a\,x^{3}-bx+1}{x}\\
q(x) &= \frac{c_1}{x}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{\left(-a\,x^{3}-bx+1\right)y}{x} = \frac{c_1}{x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{-a\,x^{3}-bx+1}{x}dx}\\
&= \mathrm{e}^{\frac{a\,x^{3}}{3}+bx-\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}}{x}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}y}{x}\right) &= \left(\frac{\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}}{x}\right)\left(\frac{c_1}{x}\right)\\
\mathrm{d}\left(\frac{\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}y}{x}\right) &= \left(\frac{c_1\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}}{x^{2}}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}y}{x} &= \int \frac{c_1\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}}{x^{2}}\,\mathrm{d}x\\
\frac{\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}y}{x} &= \int \frac{c_1\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}}{x^{2}}dx + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}}{x}$ results in

$$
y = x\,\mathrm{e}^{-\frac{x\left(a\,x^{2}+3b\right)}{3}}\left(\int \frac{c_1\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}}{x^{2}}dx\right) + c_2 x\,\mathrm{e}^{-\frac{x\left(a\,x^{2}+3b\right)}{3}}
$$

which simplifies to

$$y = x\,{\mathrm e}^{-\frac{x \left(a\,x^{2}+3 b \right)}{3}} \left(c_{1} \left(\int \frac{{\mathrm e}^{\frac{x \left(a\,x^{2}+3 b \right)}{3}}}{x^{2}}d x \right)+c_{2}\right)$$

Summary
The solution(s) found are the following

$$y = x\,{\mathrm e}^{-\frac{x \left(a\,x^{2}+3 b \right)}{3}} \left(c_{1} \left(\int \frac{{\mathrm e}^{\frac{x \left(a\,x^{2}+3 b \right)}{3}}}{x^{2}}d x \right)+c_{2}\right) \tag{1}$$

Verification of solutions

$$y = x\,{\mathrm e}^{-\frac{x \left(a\,x^{2}+3 b \right)}{3}} \left(c_{1} \left(\int \frac{{\mathrm e}^{\frac{x \left(a\,x^{2}+3 b \right)}{3}}}{x^{2}}d x \right)+c_{2}\right)$$

Verified OK.

28.27.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x y''+x \left(a\,x^{2}+b \right) y'+\left(3 a\,x^{2}+b \right) y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x y''+x \left(a\,x^{2}+b \right) y'+\left(3 a\,x^{2}+b \right) y\right)d x = 0$$
$$\left(a\,x^{3}+b x -1\right) y+y' x = c_{1}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-a\,x^{3}-b x +1}{x}$$
$$q(x) = \frac{c_{1}}{x}$$

Hence the ode is

$$y' - \frac{(-a\,x^3 - bx + 1)\,y}{x} = \frac{c_1}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{-a\,x^3 - bx + 1}{x}dx}\\ &= e^{\frac{a\,x^3}{3} + bx - \ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{e^{\frac{x\left(a\,x^2+3b\right)}{3}}}{x}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x}\right)\\ \frac{d}{dx}\left(\frac{e^{\frac{x\left(a\,x^2+3b\right)}{3}}y}{x}\right) &= \left(\frac{e^{\frac{x\left(a\,x^2+3b\right)}{3}}}{x}\right)\left(\frac{c_1}{x}\right)\\ d\left(\frac{e^{\frac{x\left(a\,x^2+3b\right)}{3}}y}{x}\right) &= \left(\frac{c_1 e^{\frac{x\left(a\,x^2+3b\right)}{3}}}{x^2}\right)dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{e^{\frac{x\left(a\,x^2+3b\right)}{3}}y}{x} &= \int \frac{c_1 e^{\frac{x\left(a\,x^2+3b\right)}{3}}}{x^2}\,dx\\ \frac{e^{\frac{x\left(a\,x^2+3b\right)}{3}}y}{x} &= \int \frac{c_1 e^{\frac{x\left(a\,x^2+3b\right)}{3}}}{x^2}dx + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{e^{\frac{x\left(a\,x^2+3b\right)}{3}}}{x}$ results in

$$y = x\,e^{-\frac{x\left(a\,x^2+3b\right)}{3}}\left(\int \frac{c_1 e^{\frac{x\left(a\,x^2+3b\right)}{3}}}{x^2}dx\right) + c_2 x\,e^{-\frac{x\left(a\,x^2+3b\right)}{3}}$$

which simplifies to

$$y = x\,e^{-\frac{x\left(a\,x^2+3b\right)}{3}}\left(c_1\left(\int \frac{e^{\frac{x\left(a\,x^2+3b\right)}{3}}}{x^2}dx\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = x\,\mathrm{e}^{-\frac{x\left(a\,x^{2}+3b\right)}{3}} \left(c_{1} \left(\int \frac{\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}}{x^{2}}dx\right)+c_{2}\right) \tag{1}$$

Verification of solutions

$$y = x\,\mathrm{e}^{-\frac{x\left(a\,x^{2}+3b\right)}{3}} \left(c_{1} \left(\int \frac{\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}}{x^{2}}dx\right)+c_{2}\right)$$

Verified OK.

28.27.3 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + x\left(a\,x^{2}+b\right)y' + \left(3a\,x^{2}+b\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x\left(a\,x^{2}+b\right) \\ C &= 3a\,x^{2}+b \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^{2} - 4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^{5}a^{2} + 2ab\,x^{3} - 8a\,x^{2} + b^{2}x - 4b}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^5 a^2 + 2ab\, x^3 - 8a\, x^2 + b^2 x - 4b \\
t &= 4x
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^5 a^2 + 2ab\, x^3 - 8a\, x^2 + b^2 x - 4b}{4x}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 103: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 1 - 5 \\
&= -4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= 1 \\
\alpha_c^{-} &= 1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{2} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{a\,x^2}{2} + \frac{b}{2} - \frac{2}{x} + \frac{b}{a\,x^3} - \frac{4}{a\,x^4} - \frac{b^2}{a^2 x^5} + \frac{8b}{a^2 x^6} + \frac{b^3}{a^3 x^7} - \frac{16}{a^2 x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\
&= \frac{a\,x^2}{2} + \frac{b}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}a^2 x^4 + \frac{1}{2}ab\,x^2 + \frac{1}{4}b^2$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^5 a^2 + 2ab\,x^3 - 8a\,x^2 + b^2 x - 4b}{4x} \\
&= Q + \frac{R}{4x} \\
&= \left(\frac{1}{4}a^2 x^4 + \frac{1}{2}ab\,x^2 - 2ax + \frac{1}{4}b^2\right) + \left(-\frac{b}{x}\right) \\
&= \frac{a^2 x^4}{4} + \frac{ab\,x^2}{2} - 2ax + \frac{b^2}{4} - \frac{b}{x}
\end{aligned}$$

We see that the coefficient of the term 1 in the quotient is $-2a$. Now b can be found.

$$\begin{aligned}
b &= (-2a) - (0) \\
&= -2a
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{a\,x^2}{2} + \frac{b}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2a}{\frac{a}{2}} - 2\right) &&= -3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2a}{\frac{a}{2}} - 2\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^5 a^2 + 2ab\,x^3 - 8a\,x^2 + b^2 x - 4b}{4x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-4	$\frac{a\,x^2}{2} + \frac{b}{2}$	-3	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - (\alpha_{c_1}^{-}) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{x} + (-)\left(\frac{a\,x^2}{2} + \frac{b}{2} \right) \\ &= \frac{1}{x} - \frac{a\,x^2}{2} - \frac{b}{2} \\ &= \frac{1}{x} - \frac{a\,x^2}{2} - \frac{b}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{a\,x^2}{2} - \frac{b}{2}\right)(0) + \left(\left(-\frac{1}{x^2} - ax\right) + \left(\frac{1}{x} - \frac{a\,x^2}{2} - \frac{b}{2}\right)^2 - \left(\frac{x^5a^2 + 2ab\,x^3 - 8a\,x^2 + b^2x - 4b}{4x}\right.\right.$$

$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{a\,x^2}{2} - \frac{b}{2}\right) dx} \\
&= x\,\mathrm{e}^{-\frac{x\left(a\,x^2+3b\right)}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x\left(a\,x^2+b\right)}{x} \, dx} \\
&= z_1 e^{-\frac{1}{2}bx - \frac{1}{6}a\,x^3} \\
&= z_1 \left(\mathrm{e}^{-\frac{x\left(a\,x^2+3b\right)}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\,\mathrm{e}^{-\frac{x\left(a\,x^2+3b\right)}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x\left(a\,x^2+b\right)}{x} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{1}{3}a\,x^3 - bx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x\left(a\,x^2+3b\right)}{3}}}{x^2} \, dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x\left(a\,x^{2}+3b\right)}{3}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x\left(a\,x^{2}+3b\right)}{3}}\left(\int \frac{\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}}{x^{2}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x\left(a\,x^{2}+3b\right)}{3}} + c_2 x\,\mathrm{e}^{-\frac{x\left(a\,x^{2}+3b\right)}{3}}\left(\int \frac{\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}}{x^{2}}dx\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x\left(a\,x^{2}+3b\right)}{3}} + c_2 x\,\mathrm{e}^{-\frac{x\left(a\,x^{2}+3b\right)}{3}}\left(\int \frac{\mathrm{e}^{\frac{x\left(a\,x^{2}+3b\right)}{3}}}{x^{2}}dx\right)
$$

Verified OK.

28.27.4 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x \\
q(x) &= x\left(a\,x^{2} + b\right) \\
r(x) &= 3a\,x^{2} + b \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 3a\,x^{2} + b
\end{aligned}
$$

Therefore (1) becomes

$$
0 - \left(3a\,x^{2} + b\right) + \left(3a\,x^{2} + b\right) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y'x + \left(x\left(a\, x^2 + b\right) - 1\right) y = c_1$$

We now have a first order ode to solve which is

$$y'x + \left(x\left(a\, x^2 + b\right) - 1\right) y = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{-a\, x^3 - bx + 1}{x} \\ q(x) &= \frac{c_1}{x} \end{aligned}$$

Hence the ode is

$$y' - \frac{(-a\, x^3 - bx + 1)\, y}{x} = \frac{c_1}{x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{-a\, x^3 - bx + 1}{x} dx} \\ &= \mathrm{e}^{\frac{a\, x^3}{3} + bx - \ln(x)} \end{aligned}$$

Which simplifies to

$$\mu = \frac{\mathrm{e}^{\frac{x\left(a\, x^2 + 3b\right)}{3}}}{x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{c_1}{x}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{\frac{x(a\,x^2+3b)}{3}} y}{x}\right) = \left(\frac{\mathrm{e}^{\frac{x(a\,x^2+3b)}{3}}}{x}\right)\left(\frac{c_1}{x}\right)$$
$$\mathrm{d}\left(\frac{\mathrm{e}^{\frac{x(a\,x^2+3b)}{3}} y}{x}\right) = \left(\frac{c_1 \mathrm{e}^{\frac{x(a\,x^2+3b)}{3}}}{x^2}\right)\mathrm{d}x$$

Integrating gives

$$\frac{\mathrm{e}^{\frac{x(a\,x^2+3b)}{3}} y}{x} = \int \frac{c_1 \mathrm{e}^{\frac{x(a\,x^2+3b)}{3}}}{x^2}\,\mathrm{d}x$$
$$\frac{\mathrm{e}^{\frac{x(a\,x^2+3b)}{3}} y}{x} = \int \frac{c_1 \mathrm{e}^{\frac{x(a\,x^2+3b)}{3}}}{x^2} dx + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{\frac{x(a\,x^2+3b)}{3}}}{x}$ results in

$$y = x\,\mathrm{e}^{-\frac{x(a\,x^2+3b)}{3}}\left(\int \frac{c_1 \mathrm{e}^{\frac{x(a\,x^2+3b)}{3}}}{x^2} dx\right) + c_2 x\,\mathrm{e}^{-\frac{x(a\,x^2+3b)}{3}}$$

which simplifies to

$$y = x\,\mathrm{e}^{-\frac{x(a\,x^2+3b)}{3}}\left(c_1\left(\int \frac{\mathrm{e}^{\frac{x(a\,x^2+3b)}{3}}}{x^2} dx\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = x\,\mathrm{e}^{-\frac{x(a\,x^2+3b)}{3}}\left(c_1\left(\int \frac{\mathrm{e}^{\frac{x(a\,x^2+3b)}{3}}}{x^2} dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = x\,\mathrm{e}^{-\frac{x(a\,x^2+3b)}{3}}\left(c_1\left(\int \frac{\mathrm{e}^{\frac{x(a\,x^2+3b)}{3}}}{x^2} dx\right) + c_2\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
No hypergeometric solution was found.
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 42

```
dsolve(x*diff(y(x),x$2)+x*(a*x^2+b)*diff(y(x),x)+(3*a*x^2+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x\,\mathrm{e}^{-\frac{x\left(a\,x^2+3b\right)}{3}}\left(c_1\left(\int \frac{\mathrm{e}^{\frac{x\left(a\,x^2+3b\right)}{3}}}{x^2}dx\right)+c_2\right)$$

✓ Solution by Mathematica

Time used: 2.43 (sec). Leaf size: 56

```
DSolve[x*y''[x]+x*(a*x^2+b)*y'[x]+(3*a*x^2+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to xe^{-\frac{ax^3}{3}-bx}\left(c_2\int_1^x \frac{e^{\frac{1}{3}aK[1]^3+bK[1]}}{K[1]^2}dK[1]+c_1\right)$$

28.28 problem 88

Internal problem ID [10912]
Internal file name [OUTPUT/10168_Sunday_December_31_2023_11_03_18_AM_67796438/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 88.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''+\left(a\,x^3+b\,x^2+2\right)y'+bxy=0}$$

28.28.1 Solving using Kovacic algorithm

Writing the ode as

$$xy''+\left(a\,x^3+b\,x^2+2\right)y'+bxy=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= a\,x^3+b\,x^2+2 \\ C &= bx \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^4 + 2ab\,x^3 + b^2x^2 + 8ax + 2b}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^4 + 2ab\,x^3 + b^2x^2 + 8ax + 2b \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}a^2x^4 + \frac{1}{2}ab\,x^3 + \frac{1}{4}b^2x^2 + 2ax + \frac{1}{2}b\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 104: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{a\,x^2}{2} + \frac{bx}{2} + \frac{2}{x} - \frac{3b}{2a\,x^2} + \frac{3b^2}{2a^2x^3} - \frac{3b^3}{2a^3x^4} - \frac{4}{a\,x^4} + \frac{3b^4}{2a^4x^5} + \frac{10b}{a^2x^5} - \frac{3b^5}{2a^5x^6} - \frac{73b^2}{4a^3x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{1}{2}bx + \frac{1}{2}a\,x^2 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}b^2x^2 + \frac{1}{2}ab\,x^3 + \frac{1}{4}a^2x^4$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{a^2x^4 + 2ab\,x^3 + b^2x^2 + 8ax + 2b}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}a^2x^4 + \frac{1}{2}ab\,x^3 + \frac{1}{4}b^2x^2 + 2ax + \frac{1}{2}b\right) + (0) \\
&= \frac{1}{4}a^2x^4 + \frac{1}{2}ab\,x^3 + \frac{1}{4}b^2x^2 + 2ax + \frac{1}{2}b
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $2a$. Now b can be found.

$$\begin{aligned}
b &= (2a) - (0) \\
&= 2a
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}bx + \frac{1}{2}a\,x^2 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2a}{\frac{a}{2}} - 2\right) &= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2a}{\frac{a}{2}} - 2\right) &= -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}a^2x^4 + \frac{1}{2}ab\,x^3 + \frac{1}{4}b^2x^2 + 2ax + \frac{1}{2}b$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{1}{2}bx+\frac{1}{2}a\,x^2$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=1$, and since there are no poles, then

$$\begin{aligned}d&=\alpha_\infty^+\\&=1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega&=(+)[\sqrt{r}]_\infty\\&=0+\left(\frac{1}{2}bx+\frac{1}{2}a\,x^2\right)\\&=\frac{1}{2}bx+\frac{1}{2}a\,x^2\\&=\frac{(ax+b)\,x}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2}bx+\frac{1}{2}a\,x^2\right)(1)+\left(\left(ax+\frac{b}{2}\right)+\left(\frac{1}{2}bx+\frac{1}{2}a\,x^2\right)^2-\left(\frac{1}{4}a^2x^4+\frac{1}{2}ab\,x^3+\frac{1}{4}b^2x^2+2ax+\frac{1}{2}b\right)\right.$$

$$-x(aa_0-b)=$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{b}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{b}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x + \frac{b}{a}\right) e^{\int \left(\frac{1}{2} b x + \frac{1}{2} a\, x^2\right) dx} \\ &= \left(x + \frac{b}{a}\right) e^{\frac{1}{6} a\, x^3 + \frac{1}{4} b\, x^2} \\ &= \frac{(ax + b)\, e^{\frac{1}{6} a\, x^3 + \frac{1}{4} b\, x^2}}{a} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{a\, x^3 + b\, x^2 + 2}{x} \, dx} \\ &= z_1 e^{-\frac{a\, x^3}{6} - \frac{b\, x^2}{4} - \ln(x)} \\ &= z_1 \left(\frac{e^{-\frac{1}{6} a\, x^3 - \frac{1}{4} b\, x^2}}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{ax + b}{xa}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{a\,x^3+b\,x^2+2}{x}dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-\frac{a\,x^3}{3}-\frac{b\,x^2}{2}-2\ln(x)}}{(y_1)^2}\,dx$$
$$= y_1 \left(\int \frac{\mathrm{e}^{-\frac{1}{3}a\,x^3-\frac{1}{2}b\,x^2}a^2}{(ax+b)^2}dx\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{ax+b}{xa}\right) + c_2\left(\frac{ax+b}{xa}\left(\int \frac{\mathrm{e}^{-\frac{1}{3}a\,x^3-\frac{1}{2}b\,x^2}a^2}{(ax+b)^2}dx\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(ax+b)}{xa} + \frac{c_2(ax+b)\,a\left(\int \frac{\mathrm{e}^{-\frac{1}{3}a\,x^3-\frac{1}{2}b\,x^2}}{(ax+b)^2}dx\right)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(ax+b)}{xa} + \frac{c_2(ax+b)\,a\left(\int \frac{\mathrm{e}^{-\frac{1}{3}a\,x^3-\frac{1}{2}b\,x^2}}{(ax+b)^2}dx\right)}{x}$$

Verified OK.

28.28.2 Maple step by step solution

Let's solve

$y''x + (a\,x^3 + b\,x^2 + 2)\,y' + bxy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(a\,x^3+b\,x^2+2)y'}{x} - yb$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(a\,x^{3}+b\,x^{2}+2)y'}{x}+yb=0$

□ Check to see if $x_{0}=0$ is a regular singular point

  - Define functions

    $\left[P_{2}(x)=\frac{a\,x^{3}+b\,x^{2}+2}{x}, P_{3}(x)=b\right]$

  - $x\cdot P_{2}(x)$ is analytic at $x=0$

    $\left(x\cdot P_{2}(x)\right)\bigg|_{x=0}=2$

  - $x^{2}\cdot P_{3}(x)$ is analytic at $x=0$

    $\left(x^{2}\cdot P_{3}(x)\right)\bigg|_{x=0}=0$

  - $x=0$is a regular singular point

    Check to see if $x_{0}=0$ is a regular singular point

    $x_{0}=0$

- Multiply by denominators

  $y''x+\left(a\,x^{3}+b\,x^{2}+2\right)y'+bxy=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_{k}x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x\cdot y$ to series expansion

    $x\cdot y=\sum_{k=0}^{\infty}a_{k}x^{k+r+1}$

  - Shift index using $k->k-1$

    $x\cdot y=\sum_{k=1}^{\infty}a_{k-1}x^{k+r}$

  - Convert $x^{m}\cdot y'$ to series expansion for $m=0..3$

    $x^{m}\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\,x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r)\,x^{-1+r} + a_1(1+r)\,(2+r)\,x^{r} + (a_2(2+r)\,(3+r) + b a_0(1+r))\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_{k+1}(k+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- The coefficients of each power of x must be 0

$$[a_1(1+r)\,(2+r) = 0, a_2(2+r)\,(3+r) + b a_0(1+r) = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_1 = 0, a_2 = -\frac{b a_0(1+r)}{r^2+5r+6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+r+1)\,(k+2+r) + b a_{k-1}(k+r) + a_{k-2}(k-2+r)\,a = 0$$

- Shift index using $k->k+2$

$$a_{k+3}(k+3+r)\,(k+4+r) + b a_{k+1}(k+2+r) + a_k(k+r)\,a = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = -\frac{aka_k + ara_k + bka_{k+1} + bra_{k+1} + 2ba_{k+1}}{(k+3+r)(k+4+r)}$$

- Recursion relation for $r = -1$

$$a_{k+3} = -\frac{aka_k + bka_{k+1} - aa_k + ba_{k+1}}{(k+2)(k+3)}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+3} = -\frac{aka_k + bka_{k+1} - aa_k + ba_{k+1}}{(k+2)(k+3)}, a_1 = 0, a_2 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+3} = -\frac{aka_k + bka_{k+1} + 2ba_{k+1}}{(k+3)(k+4)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a k a_k + b k a_{k+1} + 2 b a_{k+1}}{(k+3)(k+4)}, a_1 = 0, a_2 = -\frac{a_0 b}{6}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} c_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} d_k x^k\right), c_{k+3} = -\frac{a k c_k + b k c_{1+k} - a c_k + b c_{1+k}}{(k+2)(k+3)}, c_1 = 0, c_2 = 0, d_{k+3} = -\frac{a k d_k + b k}{(k+}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.281 (sec). Leaf size: 143

```
dsolve(x*diff(y(x),x$2)+(a*x^3+b*x^2+2)*diff(y(x),x)+b*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \mathrm{e}^{-\frac{\operatorname{csgn}(a)x^2(\operatorname{csgn}(a)+1)\left(ax+\frac{3b}{2}\right)}{6}} \operatorname{HeunT}\left(\frac{3^{\frac{2}{3}}b}{2(a^2)^{\frac{1}{3}}}, -6\operatorname{csgn}(a), -\frac{b^2 3^{\frac{1}{3}}}{4(a^2)^{\frac{2}{3}}}, \frac{3^{\frac{2}{3}}a(2ax+b)}{6(a^2)^{\frac{5}{6}}}\right) + c_2 \mathrm{e}^{-\frac{\operatorname{csgn}(a)x^2(\operatorname{csgn}(a)-1)\left(ax+\frac{3b}{2}\right)}{6}}}{x}$$

✓ Solution by Mathematica

Time used: 1.962 (sec). Leaf size: 58

```
DSolve[x*y''[x]+(a*x^3+b*x^2+2)*y'[x]+b*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(ax+b)\left(c_2 \int_1^x \frac{e^{-\frac{1}{6}K[1]^2(3b+2aK[1])}}{(b+aK[1])^2} dK[1] + c_1\right)}{bx}$$

28.29 problem 89

Internal problem ID [10913]
Internal file name [`OUTPUT/10169_Sunday_December_31_2023_11_03_19_AM_96435749/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 89.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''+\left(ab\,x^{3}+b\,x^{2}+ax-1\right)y'+a^{2}b\,x^{3}y=0}$$

28.29.1 Solving using Kovacic algorithm

Writing the ode as

$$xy''+\left(ab\,x^{3}+b\,x^{2}+ax-1\right)y'+a^{2}b\,x^{3}y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= ab\,x^{3}+b\,x^{2}+ax-1 \\ C &= a^{2}b\,x^{3} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2b^2x^6 + 2a\,b^2x^5 - 2a^2b\,x^4 + b^2x^4 + 4ab\,x^3 + a^2x^2 - 2ax + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2b^2x^6 + 2a\,b^2x^5 - 2a^2b\,x^4 + b^2x^4 + 4ab\,x^3 + a^2x^2 - 2ax + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2b^2x^6 + 2a\,b^2x^5 - 2a^2b\,x^4 + b^2x^4 + 4ab\,x^3 + a^2x^2 - 2ax + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 106: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 6 \\
&= -4
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{a^2 b^2 x^4}{4} + \frac{a\, b^2 x^3}{2} - \frac{a^2 b\, x^2}{2} + \frac{b^2 x^2}{4} + abx + \frac{a^2}{4} + \frac{3}{4x^2} - \frac{a}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{2} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{ab\,x^2}{2} + \frac{bx}{2} - \frac{a}{2} + \frac{3}{2x} - \frac{3}{2a\,x^2} + \frac{1}{b\,x^3} + \frac{3}{2a^2x^3} - \frac{4}{ab\,x^4} + \frac{1}{b^2x^5} - \frac{3}{2a^3x^4} + \frac{10}{a^2b\,x^5} + \frac{3}{2a^4x^5} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{ab}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= -\frac{1}{2}a + \frac{1}{2}bx + \frac{1}{2}ab\,x^2 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}a^2 - \frac{1}{2}abx - \frac{1}{2}a^2b\,x^2 + \frac{1}{4}b^2x^2 + \frac{1}{2}a\,b^2x^3 + \frac{1}{4}a^2b^2x^4$$

This shows that the coefficient of x in the above is $-\frac{ab}{2}$. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2b^2x^6 + 2a\,b^2x^5 - 2a^2b\,x^4 + b^2x^4 + 4ab\,x^3 + a^2x^2 - 2ax + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{a^2b^2x^4}{4} + \frac{a\,b^2x^3}{2} + \left(-\frac{1}{2}a^2b + \frac{1}{4}b^2\right)x^2 + abx + \frac{a^2}{4}\right) + \left(\frac{-2ax+3}{4x^2}\right) \\ &= \frac{a^2b^2x^4}{4} + \frac{a\,b^2x^3}{2} + \left(-\frac{1}{2}a^2b + \frac{1}{4}b^2\right)x^2 + abx + \frac{a^2}{4} + \frac{-2ax+3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is ab. Now b can be found.

$$\begin{aligned} b &= (ab) - \left(-\frac{ab}{2}\right) \\ &= \frac{3ab}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= -\frac{1}{2}a + \frac{1}{2}bx + \frac{1}{2}ab\,x^2 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{3ab}{2}}{\frac{ab}{2}} - 2\right) &&= \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{3ab}{2}}{\frac{ab}{2}} - 2\right) &&= -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{a^2b^2x^6 + 2a\,b^2x^5 - 2a^2b\,x^4 + b^2x^4 + 4ab\,x^3 + a^2x^2 - 2ax + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$-\frac{1}{2}a + \frac{1}{2}bx + \frac{1}{2}ab\,x^2$	$\frac{1}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + \left(-\frac{1}{2}a + \frac{1}{2}bx + \frac{1}{2}ab\,x^2\right) \\
&= -\frac{1}{2x} - \frac{a}{2} + \frac{bx}{2} + \frac{ab\,x^2}{2} \\
&= \frac{(b\,x^2 - 1)\,(ax + 1)}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{2x} - \frac{a}{2} + \frac{bx}{2} + \frac{ab\,x^2}{2}\right)(1) + \left(\left(\frac{1}{2x^2} + \frac{b}{2} + abx\right) + \left(-\frac{1}{2x} - \frac{a}{2} + \frac{bx}{2} + \frac{ab\,x^2}{2}\right)^2 - \left(\frac{a^2b^2x^6 +
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{1}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{1}{a}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= \left(x + \frac{1}{a}\right) e^{\int \left(-\frac{1}{2x} - \frac{a}{2} + \frac{bx}{2} + \frac{ab\,x^2}{2}\right) dx} \\
&= \left(x + \frac{1}{a}\right) e^{\frac{b\,x^2}{4} + \frac{ab\,x^3}{6} - \frac{ax}{2} - \frac{\ln(x)}{2}} \\
&= \frac{(ax + 1)\,e^{\frac{1}{4}b\,x^2 + \frac{1}{6}ab\,x^3 - \frac{1}{2}ax}}{a\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{ab\,x^3+b\,x^2+ax-1}{x}\,dx} \\
&= z_1 e^{-\frac{ab\,x^3}{6}-\frac{b\,x^2}{4}-\frac{ax}{2}+\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x\left(ab\,x^2+\frac{3}{2}bx+3a\right)}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(ax+1)\,\mathrm{e}^{-ax}}{a}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{ab\,x^3+b\,x^2+ax-1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{ab\,x^3}{3}-\frac{b\,x^2}{2}-ax+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{x\,a^2\mathrm{e}^{-\frac{\left(ab\,x^2+\frac{3}{2}bx-3a\right)x}{3}}}{(ax+1)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a}\right) + c_2\left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a}\left(\int \frac{x\,a^2\mathrm{e}^{-\frac{\left(ab\,x^2+\frac{3}{2}bx-3a\right)x}{3}}}{(ax+1)^2}dx\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{c_1(ax+1)\,\mathrm{e}^{-ax}}{a} + c_2(ax+1)\,\mathrm{e}^{-ax}a\left(\int \frac{x\,\mathrm{e}^{-\frac{\left(ab\,x^2+\frac{3}{2}bx-3a\right)x}{3}}}{(ax+1)^2}dx\right) \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(ax+1)\,\mathrm{e}^{-ax}}{a} + c_2(ax+1)\,\mathrm{e}^{-ax} a \left(\int \frac{x\,\mathrm{e}^{-\frac{\left(ab\,x^2+\frac{3}{2}bx-3a\right)x}{3}}}{(ax+1)^2} dx\right)$$

Verified OK.

28.29.2 Maple step by step solution

Let's solve

$y''x + (ab\,x^3 + b\,x^2 + ax - 1)\,y' + a^2 b\,x^3 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -a^2 b\,x^2 y - \frac{(ab\,x^3+b\,x^2+ax-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(ab\,x^3+b\,x^2+ax-1)y'}{x} + a^2 b\,x^2 y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{ab\,x^3+b\,x^2+ax-1}{x}, P_3(x) = a^2 b\,x^2\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x + (ab\,x^3 + b\,x^2 + ax - 1)\,y' + a^2 b\,x^3 y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^3 \cdot y$ to series expansion

    $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$
  - Shift index using $k->k-3$

    $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, x^{-1+r} + (a_1(1+r)\,(-1+r) + a_0 a r)\, x^r + (a_2(2+r)\, r + a a_1(1+r) + a_0 b r)\, x^{1+r} + ($

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(-1+r) + a_0 a r = 0, a_2(2+r)\, r + a a_1(1+r) + a_0 b r = 0, a_3(3+r)\,(1+r) + a a_2(2+$
- Solve for the dependent coefficient(s)

  $\left\{a_1 = -\frac{a_0 a r}{r^2-1}, a_2 = \frac{a_0(a^2-br+b)}{r^2+r-2}, a_3 = -\frac{a a_0(b r^2+a^2-3br+b)}{r^3+3r^2-r-3}\right\}$

- Each term in the series must be 0, giving the recursion relation

$$\left(a^2 a_{k-3}+a_{k-2}(k+r-2)\,a+a_{k-1}(k+r-1)\right)b+a a_k(k+r)+a_{k+1}(k+1+r)\,(k+r-1)=0$$

- Shift index using $k->k+3$

$$\left(a^2 a_k+a_{k+1}(k+1+r)\,a+a_{k+2}(k+2+r)\right)b+a a_{k+3}(k+r+3)+a_{k+4}(k+4+r)\,(k+2+r)$$

- Recursion relation that defines series solution to ODE

$$a_{k+4}=-\frac{a^2 b a_k+abk a_{k+1}+abr a_{k+1}+ab a_{k+1}+ak a_{k+3}+ar a_{k+3}+bk a_{k+2}+br a_{k+2}+3a a_{k+3}+2b a_{k+2}}{(k+4+r)(k+2+r)}$$

- Recursion relation for $r=0$

$$a_{k+4}=-\frac{a^2 b a_k+abk a_{k+1}+ab a_{k+1}+ak a_{k+3}+bk a_{k+2}+3a a_{k+3}+2b a_{k+2}}{(k+4)(k+2)}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=-\frac{a^2 b a_k+abk a_{k+1}+ab a_{k+1}+ak a_{k+3}+bk a_{k+2}+3a a_{k+3}+2b a_{k+2}}{(k+4)(k+2)}, a_1=0, a_2=-\frac{a_0(a^2+b)}{2}, a_3\right.$$

- Recursion relation for $r=2$

$$a_{k+4}=-\frac{a^2 b a_k+abk a_{k+1}+3ab a_{k+1}+ak a_{k+3}+bk a_{k+2}+5a a_{k+3}+4b a_{k+2}}{(k+6)(k+4)}$$

- Solution for $r=2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+4}=-\frac{a^2 b a_k+abk a_{k+1}+3ab a_{k+1}+ak a_{k+3}+bk a_{k+2}+5a a_{k+3}+4b a_{k+2}}{(k+6)(k+4)}, a_1=-\frac{2a a_0}{3}, a_2=\frac{a_0(a^2}{4}\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} c_k x^k\right)+\left(\sum_{k=0}^{\infty} d_k x^{k+2}\right), c_{k+4}=-\frac{a^2 b c_k+abk c_{1+k}+ab c_{1+k}+ak c_{k+3}+bk c_{k+2}+3a c_{k+3}+2b c_{k+2}}{(k+4)(k+2)}, c_1=0\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
   No special function solution was found.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.313 (sec). Leaf size: 48

```
dsolve(x*diff(y(x),x$2)+(a*b*x^3+b*x^2+a*x-1)*diff(y(x),x)+a^2*b*x^3*y(x)=0,y(x), singsol=al
```

$$y(x) = e^{-ax}\left(c_2\left(\int \frac{x\,e^{-\frac{\left(a b\,x^2+\frac{3}{2} b x-3a\right)x}{3}}}{\left(ax+1\right)^2}dx\right)+c_1\right)\left(ax+1\right)$$

✓ Solution by Mathematica

Time used: 4.606 (sec). Leaf size: 72

```
DSolve[x*y''[x]+(a*b*x^3+b*x^2+a*x-1)*y'[x]+a^2*b*x^3*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{e^{-ax}(ax+1)\left(c_2\int_1^x \frac{a^2\exp\left(-\frac{1}{6}K[1](3bK[1]+2a(bK[1]^2-3))\right)K[1]}{(aK[1]+1)^2}dK[1]+c_1\right)}{a}$$

28.30 problem 90

Internal problem ID [10914]
Internal file name [`OUTPUT/10170_Sunday_December_31_2023_11_03_20_AM_5802521/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 90.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''+\left(a\,x^{3}+b\,x^{2}+cx+d\right)y'+(d-1)\left(a\,x^{2}+bx+c\right)y=0}$$

28.30.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$xy''+\left(a\,x^{3}+b\,x^{2}+cx+d\right)y'+(d-1)\left(a\,x^{2}+bx+c\right)y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}
p(x)&=\frac{a\,x^{3}+b\,x^{2}+cx+d}{x}\\
q(x)&=\frac{(d-1)\left(a\,x^{2}+bx+c\right)}{x}
\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(a\,x^3 + b\,x^2 + cx + d)}{x^2} + \frac{(d-1)\,(a\,x^2 + bx + c)}{x} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -d + 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{-2d+2}{x} + \frac{a\,x^3 + b\,x^2 + cx + d}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(a\,x^3 + b\,x^2 + cx - d + 2)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(a\,x^3 + b\,x^2 + cx - d + 2)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(a\,x^3 + b\,x^2 + cx - d + 2)\, u}{x} \end{aligned}$$

Where $f(x) = -\frac{a\,x^3+b\,x^2+cx-d+2}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{a\,x^3+b\,x^2+cx-d+2}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{a\,x^3+b\,x^2+cx-d+2}{x}\,dx \\
\ln(u) &= -\frac{a\,x^3}{3}-\frac{b\,x^2}{2}-cx-(2-d)\ln(x)+c_1 \\
u &= \mathrm{e}^{-\frac{a\,x^3}{3}-\frac{b\,x^2}{2}-cx-(2-d)\ln(x)+c_1} \\
&= c_1\mathrm{e}^{-\frac{a\,x^3}{3}-\frac{b\,x^2}{2}-cx-(2-d)\ln(x)}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= \int c_1\mathrm{e}^{-\frac{a\,x^3}{3}-\frac{b\,x^2}{2}-cx-(2-d)\ln(x)}dx + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(\int c_1\mathrm{e}^{-\frac{a\,x^3}{3}-\frac{b\,x^2}{2}-cx-(2-d)\ln(x)}dx + c_2\right)x^{-d+1} \\
&= x^{-d+1}\left(c_1\left(\int x^{d-2}\mathrm{e}^{-\frac{1}{3}a\,x^3-\frac{1}{2}b\,x^2-cx}dx\right)+c_2\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(\int c_1\mathrm{e}^{-\frac{a\,x^3}{3}-\frac{b\,x^2}{2}-cx-(2-d)\ln(x)}dx + c_2\right)x^{-d+1} \tag{1}
$$

Verification of solutions

$$
y = \left(\int c_1\mathrm{e}^{-\frac{a\,x^3}{3}-\frac{b\,x^2}{2}-cx-(2-d)\ln(x)}dx + c_2\right)x^{-d+1}
$$

Verified OK.

28.30.2 Maple step by step solution

Let's solve

$y''x + (a\,x^3 + b\,x^2 + cx + d)\,y' + (d-1)\,(a\,x^2 + bx + c)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(d-1)(a\,x^2+bx+c)y}{x} - \frac{(a\,x^3+b\,x^2+cx+d)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(a\,x^3+b\,x^2+cx+d)y'}{x} + \frac{(d-1)(a\,x^2+bx+c)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{a\,x^3+b\,x^2+cx+d}{x}, P_3(x) = \frac{(d-1)(a\,x^2+bx+c)}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = d$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (a\,x^3 + b\,x^2 + cx + d)\,y' + (d-1)\,(a\,x^2 + bx + c)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-1+r+d)\, x^{-1+r} + (a_1(1+r)\,(r+d) + a_0 c(-1+r+d))\, x^r + (a_2(2+r)\,(1+r+d) + a_1 c$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-1+r+d) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, -d+1\}$$

- The coefficients of each power of x must be 0

  $$[a_1(1+r)\,(r+d) + a_0 c(-1+r+d) = 0, a_2(2+r)\,(1+r+d) + a_1 c(r+d) + b a_0(-1+r+d) =$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1 = -\frac{a_0 c(-1+r+d)}{rd+r^2+d+r}, a_2 = -\frac{a_0 \left(bdr+b\,r^2-c^2 d-c^2 r+bd+c^2-b\right)}{r^2 d+r^3+3rd+4r^2+2d+5r+2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)\,(k+r+d) + a_k c(k+r+d-1) + b a_{k-1}(k-2+r+d) + a_{k-2} a(k-3+r+d)$$

- Shift index using $k->k+2$

  $$a_{k+3}(k+3+r)\,(k+2+r+d) + a_{k+2} c(k+1+r+d) + b a_{k+1}(k+r+d) + a_k a(k+r+d-1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = -\frac{a_k a d + a k a_k + a r a_k + b d a_{k+1} + b k a_{k+1} + b r a_{k+1} + c d a_{k+2} + c k a_{k+2} + c r a_{k+2} - a_k a + c a_{k+2}}{(k+3+r)(k+2+r+d)}$$

- Recursion relation for $r = 0$

$$a_{k+3} = -\frac{a_k a d + a k a_k + b d a_{k+1} + b k a_{k+1} + c d a_{k+2} + c k a_{k+2} - a_k a + c a_{k+2}}{(k+3)(k+2+d)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a_k a d + a k a_k + b d a_{k+1} + b k a_{k+1} + c d a_{k+2} + c k a_{k+2} - a_k a + c a_{k+2}}{(k+3)(k+2+d)}, a_1 = -\frac{a_0 c(d-1)}{d}, a_2 = -\frac{a_0 (\right.$$

- Recursion relation for $r = -d + 1$

$$a_{k+3} = -\frac{a_k a d + a k a_k + a(-d+1) a_k + b d a_{k+1} + b k a_{k+1} + b(-d+1) a_{k+1} + c d a_{k+2} + c k a_{k+2} + c(-d+1) a_{k+2} - a_k a + c a_{k+2}}{(k+4-d)(k+3)}$$

- Solution for $r = -d + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-d+1}, a_{k+3} = -\frac{a_k a d + a k a_k + a(-d+1) a_k + b d a_{k+1} + b k a_{k+1} + b(-d+1) a_{k+1} + c d a_{k+2} + c k a_{k+2} + c(-d+1) a_{k+2}}{(k+4-d)(k+3)}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} e_k x^k\right) + \left(\sum_{k=0}^{\infty} f_k x^{k-d+1}\right), e_{k+3} = -\frac{a d e_k + a k e_k + b d e_{1+k} + b k e_{1+k} + c d e_{k+2} + c k e_{k+2} - a e_k + c e_{k+2}}{(k+3)(k+2+d)}, e_1 =\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
   No special function solution was found.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.359 (sec). Leaf size: 42

```
dsolve(x*diff(y(x),x$2)+(a*x^3+b*x^2+c*x+d)*diff(y(x),x)+(d-1)*(a*x^2+b*x+c)*y(x)=0,y(x), si
```

$$y(x) = x^{-d+1}\left(\left(\int x^{d-2}\mathrm{e}^{-\frac{1}{3}a\,x^3-\frac{1}{2}x^2b-cx}dx\right)c_2+c_1\right)$$

✓ Solution by Mathematica

Time used: 1.839 (sec). Leaf size: 57

```
DSolve[x*y''[x]+(a*x^3+b*x^2+c*x+d)*y'[x]+(d-1)*(a*x^2+b*x+c)*y[x]==0,y[x],x,IncludeSingular
```

$$y(x) \to x^{1-d}\left(c_2\int_1^x \exp\left(-\frac{1}{6}K[1](6c+K[1](3b+2aK[1]))\right)K[1]^{d-2}dK[1]+c_1\right)$$

28.31 problem 91

Internal problem ID [10915]
Internal file name [OUTPUT/10171_Sunday_December_31_2023_11_03_21_AM_13522463/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 91.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{xy''+a\,x^{n}y'+\left(ab\,x^{n}-a\,x^{n-1}-b^{2}x+2b\right)y=0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x$2)+a*x^n*diff(y(x),x)+(a*b*x^n-a*x^(n-1)-b^2*x+2*b)*y(x)=0,y(x), singso
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y''[x]+a*x^n*y'[x]+(a*b*x^n-a*x^(n-1)-b^2*x+2*b)*y[x]==0,y[x],x,IncludeSingularSolu
```

Not solved

28.32 problem 92

Internal problem ID [10916]
Internal file name [`OUTPUT/10172_Sunday_December_31_2023_11_03_22_AM_74756653/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 92.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''+(a\,x^n+2)\,y'+x^{n-1}ay=0}$$

28.32.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$xy''+(a\,x^n+2)\,y'+x^{n-1}ay=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=\frac{a\,x^n+2}{x}$$
$$q(x)=a\,x^{n-2}$$

Applying change of variables on the depndent variable $y=v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right)v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right)v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(a\,x^n + 2)}{x^2} + a\,x^{n-2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \left(-\frac{2}{x} + \frac{a\,x^n + 2}{x}\right) v'(x) = 0$$
$$v''(x) + a\,x^{n-1} v'(x) = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + a\,x^{n-1} u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -a\,x^{n-1} u \end{aligned}$$

Where $f(x) = -a\,x^{n-1}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -a\,x^{n-1}\,dx \\ \int \frac{1}{u}\,du &= \int -a\,x^{n-1}\,dx \\ \ln(u) &= -\frac{a\,x^n}{n} + c_1 \\ u &= e^{-\frac{a\,x^n}{n} + c_1} \\ &= c_1 e^{-\frac{a\,x^n}{n}} \end{aligned}$$

Now that $u(x)$ is known, then

$$v'(x) = u(x)$$

$$v(x) = \int u(x)\, dx + c_2$$

$$= \frac{c_1 \left(\frac{a}{n}\right)^{-\frac{1}{n}} \left(\frac{n^3 x^{1-n} \left(\frac{a}{n}\right)^{\frac{1}{n}} (a\, x^n + n + 1) \left(\frac{a\, x^n}{n}\right)^{-\frac{n+1}{2n}} \mathrm{e}^{-\frac{a\, x^n}{2n}} \operatorname{WhittakerM}\left(\frac{1}{n} - \frac{n+1}{2n}, \frac{n+1}{2n} + \frac{1}{2}, \frac{a\, x^n}{n}\right)}{(n+1)(2n+1)a} + \frac{n^2 x^{1-n} \left(\frac{a}{n}\right)^{\frac{1}{n}} (n+1) \left(\frac{a\, x^n}{n}\right)}{}\right.}{n}$$

Hence

$$y = v(x)\, x^n$$

$$= \frac{\frac{c_1 \left(\frac{a}{n}\right)^{-\frac{1}{n}} \left(\frac{n^3 x^{1-n} \left(\frac{a}{n}\right)^{\frac{1}{n}} (a\, x^n + n + 1) \left(\frac{a\, x^n}{n}\right)^{-\frac{n+1}{2n}} \mathrm{e}^{-\frac{a\, x^n}{2n}} \operatorname{WhittakerM}\left(\frac{1}{n} - \frac{n+1}{2n}, \frac{n+1}{2n} + \frac{1}{2}, \frac{a\, x^n}{n}\right)}{(n+1)(2n+1)a} + \frac{n^2 x^{1-n} \left(\frac{a}{n}\right)^{\frac{1}{n}} (n+1) \left(\frac{a\, x^n}{n}\right)^{-\frac{n+1}{2n}} \mathrm{e}^{-\frac{a\, x^n}{2n}} \operatorname{Wh}}{a(2n+1)}\right.}{n}}{x}$$

$$= \frac{n c_1 x^{1-n} \mathrm{e}^{-\frac{a x^n}{2n}} \left(\frac{a x^n}{n}\right)^{-\frac{n+1}{2n}} (n+1)^2 \operatorname{WhittakerM}\left(\frac{n+1}{2n}, \frac{2n+1}{2n}, \frac{a x^n}{n}\right) + \left((n+1)\, x^{1-n} + ax\right) \left(\frac{a x^n}{n}\right)^{-\frac{n+1}{2n}} \mathrm{e}^{-\frac{a x^n}{2n}}}{(n+1)\,(2n+1)\, ax}$$

Summary

The solution(s) found are the following

$$y = \frac{\frac{c_1 \left(\frac{a}{n}\right)^{-\frac{1}{n}} \left(\frac{n^3 x^{1-n} \left(\frac{a}{n}\right)^{\frac{1}{n}} (a\, x^n + n + 1) \left(\frac{a\, x^n}{n}\right)^{-\frac{n+1}{2n}} \mathrm{e}^{-\frac{a\, x^n}{2n}} \operatorname{WhittakerM}\left(\frac{1}{n} - \frac{n+1}{2n}, \frac{n+1}{2n} + \frac{1}{2}, \frac{a\, x^n}{n}\right)}{(n+1)(2n+1)a} + \frac{n^2 x^{1-n} \left(\frac{a}{n}\right)^{\frac{1}{n}} (n+1) \left(\frac{a\, x^n}{n}\right)^{-\frac{n+1}{2n}} \mathrm{e}^{-\frac{a\, x^n}{2n}} \operatorname{Whitta}}{a(2n+1)}\right.}{n}}{x} \tag{1}$$

Verification of solutions

$$y = \frac{\frac{c_1 \left(\frac{a}{n}\right)^{-\frac{1}{n}} \left(\frac{n^3 x^{1-n} \left(\frac{a}{n}\right)^{\frac{1}{n}} (a\, x^n + n + 1) \left(\frac{a\, x^n}{n}\right)^{-\frac{n+1}{2n}} \mathrm{e}^{-\frac{a\, x^n}{2n}} \operatorname{WhittakerM}\left(\frac{1}{n} - \frac{n+1}{2n}, \frac{n+1}{2n} + \frac{1}{2}, \frac{a\, x^n}{n}\right)}{(n+1)(2n+1)a} + \frac{n^2 x^{1-n} \left(\frac{a}{n}\right)^{\frac{1}{n}} (n+1) \left(\frac{a\, x^n}{n}\right)^{-\frac{n+1}{2n}} \mathrm{e}^{-\frac{a\, x^n}{2n}} \operatorname{Whitta}}{a(2n+1)}\right.}{n}}{x}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- Equivalence, under non-integer power transformations successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 122

```
dsolve(x*diff(y(x),x$2)+(a*x^n+2)*diff(y(x),x)+a*x^(n-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{n c_2 \mathrm{e}^{-\frac{a x^n}{2n}} \left((n+1)\, x^{-\frac{3n}{2}+\frac{1}{2}} + x^{-\frac{n}{2}+\frac{1}{2}} a \right) \mathrm{WhittakerM}\left(-\frac{n-1}{2n}, \frac{2n+1}{2n}, \frac{a x^n}{n}\right) + c_2 x^{-\frac{3n}{2}+\frac{1}{2}} \mathrm{e}^{-\frac{a x^n}{2n}} (n+1)^2 \,\mathrm{Whitta}}{x}$$

✓ Solution by Mathematica

Time used: 0.099 (sec). Leaf size: 62

```
DSolve[x*y''[x]+(a*x^n+2)*y'[x]+a*x^(n-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-1)^{-1/n} n^{\frac{1}{n}-1} a^{-1/n} (x^n)^{-1/n} \left(c_1 (-1)^{\frac{1}{n}} \Gamma\left(\frac{1}{n}, 0, \frac{a x^n}{n}\right) + c_2 n \right)$$

28.33 problem 93

Internal problem ID [10917]
Internal file name [OUTPUT/10173_Sunday_December_31_2023_11_03_23_AM_45823441/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 93.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$
\boxed{xy''+(x^n+1-n)\,y'+b\,x^{-1+2n}y=0}
$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- Equivalence, under non-integer power transformations successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 53

```
dsolve(x*diff(y(x),x$2)+(x^n+1-n)*diff(y(x),x)+b*x^(2*n-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{x^n}{2n}} \left(c_1 \sinh\left(\frac{x^n \sqrt{\frac{-4b+1}{n^2}}}{2} \right) + c_2 \cosh\left(\frac{x^n \sqrt{\frac{-4b+1}{n^2}}}{2} \right) \right)$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 53

```
DSolve[x*y''[x]+(x^n+1-n)*y'[x]+b*x^(2*n-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{-\frac{(\sqrt{1-4b}+1)x^n}{2n}} \left(c_2 e^{\frac{\sqrt{1-4b}x^n}{n}} + c_1 \right)$$

28.34 problem 94

Internal problem ID [10918]
Internal file name [`OUTPUT/10174_Sunday_December_31_2023_11_03_24_AM_32701816/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 94.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{xy''+(a\,x^n+b)\,y'+yx^{n-1}an=0}$$

28.34.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(xy''+(a\,x^n+b)\,y'+yx^{n-1}an\right)dx=0$$
$$(a\,x^n+b-1)\,y+y'x=c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{-a\,x^{n}-b+1}{x}\\
q(x) &= \frac{c_1}{x}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{(-a\,x^{n}-b+1)\,y}{x} = \frac{c_1}{x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{-a\,x^{n}-b+1}{x}dx}\\
&= \mathrm{e}^{\frac{a\,x^{n}+(b-1)\ln(x^{n})}{n}}
\end{aligned}
$$

Which simplifies to

$$
\mu = (x^{n})^{\frac{b-1}{n}}\,\mathrm{e}^{\frac{a\,x^{n}}{n}}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left((x^{n})^{\frac{b-1}{n}}\,\mathrm{e}^{\frac{a\,x^{n}}{n}}y\right) &= \left((x^{n})^{\frac{b-1}{n}}\,\mathrm{e}^{\frac{a\,x^{n}}{n}}\right)\left(\frac{c_1}{x}\right)\\
\mathrm{d}\left((x^{n})^{\frac{b-1}{n}}\,\mathrm{e}^{\frac{a\,x^{n}}{n}}y\right) &= \left(\frac{c_1(x^{n})^{\frac{b-1}{n}}\,\mathrm{e}^{\frac{a\,x^{n}}{n}}}{x}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
(x^{n})^{\frac{b-1}{n}}\,\mathrm{e}^{\frac{a\,x^{n}}{n}}y &= \int \frac{c_1(x^{n})^{\frac{b-1}{n}}\,\mathrm{e}^{\frac{a\,x^{n}}{n}}}{x}\,\mathrm{d}x\\
(x^{n})^{\frac{b-1}{n}}\,\mathrm{e}^{\frac{a\,x^{n}}{n}}y &= \int \frac{c_1(x^{n})^{\frac{b-1}{n}}\,\mathrm{e}^{\frac{a\,x^{n}}{n}}}{x}dx + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = (x^{n})^{\frac{b-1}{n}}\,\mathrm{e}^{\frac{a\,x^{n}}{n}}$ results in

$$
y = \mathrm{e}^{-\frac{a\,x^{n}}{n}}(x^{n})^{\frac{-b+1}{n}}\left(\int \frac{c_1(x^{n})^{\frac{b-1}{n}}\,\mathrm{e}^{\frac{a\,x^{n}}{n}}}{x}dx\right) + c_2\mathrm{e}^{-\frac{a\,x^{n}}{n}}(x^{n})^{\frac{-b+1}{n}}
$$

which simplifies to

$$
y = (x^{n})^{\frac{-b+1}{n}}\,\mathrm{e}^{-\frac{a\,x^{n}}{n}}\left(c_1\left(\int \frac{(x^{n})^{\frac{b-1}{n}}\,\mathrm{e}^{\frac{a\,x^{n}}{n}}}{x}dx\right) + c_2\right)
$$

Summary
The solution(s) found are the following

$$y = (x^n)^{\frac{-b+1}{n}} \mathrm{e}^{-\frac{a\,x^n}{n}} \left(c_1 \left(\int \frac{(x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a\,x^n}{n}}}{x} dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = (x^n)^{\frac{-b+1}{n}} \mathrm{e}^{-\frac{a\,x^n}{n}} \left(c_1 \left(\int \frac{(x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a\,x^n}{n}}}{x} dx\right) + c_2\right)$$

Verified OK.

28.34.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' + (a\,x^n + b)\,y' + yx^{n-1}an = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(xy'' + (a\,x^n + b)\,y' + yx^{n-1}an\right) dx = 0$$

$$(a\,x^n + b - 1)\,y + y'x = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-a\,x^n - b + 1}{x}$$

$$q(x) = \frac{c_1}{x}$$

Hence the ode is

$$y' - \frac{(-a\,x^n - b + 1)\,y}{x} = \frac{c_1}{x}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{-a\,x^n - b + 1}{x} dx}$$

$$= \mathrm{e}^{\frac{a\,x^n + (b-1)\ln(x^n)}{n}}$$

Which simplifies to

$$\mu = (x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left((x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}} y\right) &= \left((x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}}\right)\left(\frac{c_1}{x}\right) \\
\mathrm{d}\left((x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}} y\right) &= \left(\frac{c_1 (x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}}}{x}\right) \mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
(x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}} y &= \int \frac{c_1 (x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}}}{x} \,\mathrm{d}x \\
(x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}} y &= \int \frac{c_1 (x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}}}{x} dx + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = (x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}}$ results in

$$y = \mathrm{e}^{-\frac{a x^n}{n}} (x^n)^{\frac{-b+1}{n}} \left(\int \frac{c_1 (x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}}}{x} dx\right) + c_2 \mathrm{e}^{-\frac{a x^n}{n}} (x^n)^{\frac{-b+1}{n}}$$

which simplifies to

$$y = (x^n)^{\frac{-b+1}{n}} \mathrm{e}^{-\frac{a x^n}{n}} \left(c_1 \left(\int \frac{(x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}}}{x} dx\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = (x^n)^{\frac{-b+1}{n}} \mathrm{e}^{-\frac{a x^n}{n}} \left(c_1 \left(\int \frac{(x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}}}{x} dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = (x^n)^{\frac{-b+1}{n}} \mathrm{e}^{-\frac{a x^n}{n}} \left(c_1 \left(\int \frac{(x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}}}{x} dx\right) + c_2\right)$$

Verified OK.

28.34.3 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x \\ q(x) &= a\,x^n + b \\ r(x) &= an\,x^{n-1} \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= \frac{an\,x^n}{x} \end{aligned}$$

Therefore (1) becomes

$$0 - \left(\frac{an\,x^n}{x}\right) + \left(an\,x^{n-1}\right) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + \left(q(x) - p'(x)\right)y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + \left(q(x) - p'(x)\right)y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$\left(a\,x^n + b - 1\right)y + y'x = c_1$$

We now have a first order ode to solve which is

$$\left(a\,x^n + b - 1\right)y + y'x = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{-a\,x^n - b + 1}{x} \\ q(x) &= \frac{c_1}{x} \end{aligned}$$

Hence the ode is

$$y' - \frac{(-a\,x^n - b + 1)\,y}{x} = \frac{c_1}{x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{-a\,x^n - b + 1}{x}dx} \\ &= e^{\frac{a\,x^n + (b-1)\ln(x^n)}{n}} \end{aligned}$$

Which simplifies to

$$\mu = (x^n)^{\frac{b-1}{n}}\, e^{\frac{a\,x^n}{n}}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x}\right) \\ \frac{d}{dx}\left((x^n)^{\frac{b-1}{n}}\, e^{\frac{a\,x^n}{n}}\, y\right) &= \left((x^n)^{\frac{b-1}{n}}\, e^{\frac{a\,x^n}{n}}\right)\left(\frac{c_1}{x}\right) \\ d\left((x^n)^{\frac{b-1}{n}}\, e^{\frac{a\,x^n}{n}}\, y\right) &= \left(\frac{c_1 (x^n)^{\frac{b-1}{n}}\, e^{\frac{a\,x^n}{n}}}{x}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} (x^n)^{\frac{b-1}{n}}\, e^{\frac{a\,x^n}{n}}\, y &= \int \frac{c_1 (x^n)^{\frac{b-1}{n}}\, e^{\frac{a\,x^n}{n}}}{x}\, dx \\ (x^n)^{\frac{b-1}{n}}\, e^{\frac{a\,x^n}{n}}\, y &= \int \frac{c_1 (x^n)^{\frac{b-1}{n}}\, e^{\frac{a\,x^n}{n}}}{x} dx + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = (x^n)^{\frac{b-1}{n}}\, e^{\frac{a\,x^n}{n}}$ results in

$$y = e^{-\frac{a\,x^n}{n}} (x^n)^{\frac{-b+1}{n}} \left(\int \frac{c_1 (x^n)^{\frac{b-1}{n}}\, e^{\frac{a\,x^n}{n}}}{x} dx\right) + c_2 e^{-\frac{a\,x^n}{n}} (x^n)^{\frac{-b+1}{n}}$$

which simplifies to

$$y = (x^n)^{\frac{-b+1}{n}} \mathrm{e}^{-\frac{a x^n}{n}} \left(c_1 \left(\int \frac{(x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}}}{x} dx\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = (x^n)^{\frac{-b+1}{n}} \mathrm{e}^{-\frac{a x^n}{n}} \left(c_1 \left(\int \frac{(x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}}}{x} dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = (x^n)^{\frac{-b+1}{n}} \mathrm{e}^{-\frac{a x^n}{n}} \left(c_1 \left(\int \frac{(x^n)^{\frac{b-1}{n}} \mathrm{e}^{\frac{a x^n}{n}}}{x} dx\right) + c_2\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form is not straightforward to achieve - returning hypergeometric solution
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 53

```
dsolve(x*diff(y(x),x$2)+(a*x^n+b)*diff(y(x),x)+a*n*x^(n-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{a x^n}{n}} \left(\mathrm{hypergeom}\left(\left[\frac{b-1}{n}\right], \left[\frac{b+n-1}{n}\right], \frac{a x^n}{n}\right) c_1 + x^{-b+1} c_2\right)$$

✓ Solution by Mathematica

Time used: 0.244 (sec). Leaf size: 121

```
DSolve[x*y''[x]+(a*x^n+b)*y'[x]+a*n*x^(n-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-1)^{-\frac{b}{n}} n^{\frac{b-n-1}{n}} a^{\frac{1-b}{n}} e^{-\frac{ax^n}{n}} (x^n)^{\frac{1-b}{n}} \left(-(b-1)c_1(-1)^{\frac{1}{n}}\Gamma\left(\frac{b-1}{n}, -\frac{ax^n}{n}\right) + c_2 n(-1)^{b/n} + (b-1)c_1(-1)^{\frac{1}{n}} \operatorname{Gamma}\left(\frac{b-1}{n}\right)\right)$$

28.35 problem 95

Internal problem ID [10919]
Internal file name [`OUTPUT/10175_Sunday_December_31_2023_11_03_31_AM_34888450/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 95.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''+\left(a\,x^{n}+b\right)y'+a(b-1)\,x^{n-1}y=0}$$

28.35.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$xy''+\left(a\,x^{n}+b\right)y'+a(b-1)\,x^{n-1}y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=\frac{a\,x^{n}+b}{x}\\ q(x)&=a\,x^{n-2}(b-1)\end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\,x^{n}$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right)v'(x)+\left(\frac{n(n-1)}{x^{2}}+\frac{np}{x}+q\right)v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(a\,x^n + b)}{x^2} + a\,x^{n-2}(b-1) = 0 \tag{5}$$

Solving (5) for n gives

$$n = -b + 1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \left(\frac{-2b+2}{x} + \frac{a\,x^n + b}{x}\right) v'(x) = 0$$

$$v''(x) + \frac{(-b + 2 + a\,x^n)\, v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(-b + 2 + a\,x^n)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(-b+2+a\,x^n)\,u}{x} \end{aligned}$$

Where $f(x) = -\frac{-b+2+a\,x^n}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{-b+2+a\,x^n}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{-b+2+a\,x^n}{x}\,dx \\ \ln(u) &= -\frac{a\,x^n + (2-b)\ln(x^n)}{n} + c_1 \\ u &= \mathrm{e}^{-\frac{a\,x^n + (2-b)\ln(x^n)}{n} + c_1} \\ &= c_1 \mathrm{e}^{-\frac{a\,x^n + (2-b)\ln(x^n)}{n}} \end{aligned}$$

Which simplifies to

$$u(x) = c_1 \mathrm{e}^{-\frac{a\,x^n}{n}} \left(x^n\right)^{-\frac{2}{n}} \left(x^n\right)^{\frac{b}{n}}$$

Now that $u(x)$ is known, then

$$v'(x) = u(x)$$

$$v(x) = \int u(x)\, dx + c_2$$

$$= \frac{\left(\frac{a}{n}\right)^{-\frac{b}{n}+\frac{1}{n}} c_1 \left(x^n\right)^{\frac{-2+b}{n}} x^{2-b} \left(\frac{n^3 x^{-n+b-1} \left(\frac{a}{n}\right)^{\frac{b}{n}-\frac{1}{n}} (a\,x^n+b+n-1) \left(\frac{a\,x^n}{n}\right)^{-\frac{b+n-1}{2n}} \mathrm{e}^{-\frac{a\,x^n}{2n}} \operatorname{WhittakerM}\left(\frac{b-1}{n}-\frac{b+n-1}{2n}, \frac{b+n-1}{2n}+\right.}{(b-1)(b+n-1)(b+2n-1)a}}{n}$$

Hence

$$y = v(x)\, x^n$$

$$= \left(\frac{\left(\frac{a}{n}\right)^{-\frac{b}{n}+\frac{1}{n}} c_1 \left(x^n\right)^{\frac{-2+b}{n}} x^{2-b} \left(\frac{n^3 x^{-n+b-1} \left(\frac{a}{n}\right)^{\frac{b}{n}-\frac{1}{n}} (a\,x^n+b+n-1) \left(\frac{a\,x^n}{n}\right)^{-\frac{b+n-1}{2n}} \mathrm{e}^{-\frac{a\,x^n}{2n}} \operatorname{WhittakerM}\left(\frac{b-1}{n}-\frac{b+n-1}{2n}, \frac{b+n-1}{2n}+\frac{1}{2}\right.}{(b-1)(b+n-1)(b+2n-1)a}}{n}\right.$$

$$= \frac{x^{-b+1} \left(\left(\frac{a\,x^n}{n}\right)^{-\frac{b+n-1}{2n}} n^2 c_1 \mathrm{e}^{-\frac{a\,x^n}{2n}} \left(x^n\right)^{\frac{-2+b}{n}} \left((b+n-1)\,x^{1-n}+ax\right) \operatorname{WhittakerM}\left(\frac{-n+b-1}{2n}, \frac{b+2n-1}{2n}, \frac{a\,x^n}{n}\right) + \right.}{a\,(b-}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \left(\frac{\left(\frac{a}{n}\right)^{-\frac{b}{n}+\frac{1}{n}} c_1 \left(x^n\right)^{\frac{-2+b}{n}} x^{2-b} \left(\frac{n^3 x^{-n+b-1} \left(\frac{a}{n}\right)^{\frac{b}{n}-\frac{1}{n}} (a\,x^n+b+n-1) \left(\frac{a\,x^n}{n}\right)^{-\frac{b+n-1}{2n}} \mathrm{e}^{-\frac{a\,x^n}{2n}} \operatorname{WhittakerM}\left(\frac{b-1}{n}-\frac{b+n-1}{2n}, \frac{b+n-1}{2n}+\frac{1}{2}, \right.}{(b-1)(b+n-1)(b+2n-1)a}}{n} + c_2\right) x^{-b+1} \tag{1}$$

Verification of solutions

$$y = \left(\frac{\left(\frac{a}{n}\right)^{-\frac{b}{n}+\frac{1}{n}} c_1 \left(x^n\right)^{\frac{-2+b}{n}} x^{2-b} \left(\frac{n^3 x^{-n+b-1} \left(\frac{a}{n}\right)^{\frac{b}{n}-\frac{1}{n}} (a\,x^n+b+n-1) \left(\frac{a\,x^n}{n}\right)^{-\frac{b+n-1}{2n}} \mathrm{e}^{-\frac{a\,x^n}{2n}} \operatorname{WhittakerM}\left(\frac{b-1}{n}-\frac{b+n-1}{2n}, \frac{b+n-1}{2n}+\frac{1}{2}, \frac{a\,x^n}{n}\right.}{(b-1)(b+n-1)(b+2n-1)a}}{n} + c_2\right) x^{-b+1}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- Equivalence, under non-integer power transformations successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 143

```
dsolve(x*diff(y(x),x$2)+(a*x^n+b)*diff(y(x),x)+a*(b-1)*x^(n-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{a\,x^n}{2n}} n c_2 \left((b+n-1)\, x^{-\frac{3n}{2}+\frac{1}{2}-\frac{b}{2}} + a\, x^{\frac{1}{2}-\frac{b}{2}-\frac{n}{2}} \right) \mathrm{WhittakerM}\left(\frac{b-n-1}{2n}, \frac{b+2n-1}{2n}, \frac{a\,x^n}{n}\right) + x^{-\frac{3n}{2}+\frac{1}{2}-\frac{b}{2}} \mathrm{e}^{-\frac{a\,x^n}{2n}} c_2 (b+n-1)^2\, \mathrm{WhittakerM}\left(\frac{b+n-1}{2n}, \frac{b+2n-1}{2n}, \frac{a\,x^n}{n}\right) + c_1 x^{-b+1}$$

✓ Solution by Mathematica

Time used: 0.182 (sec). Leaf size: 90

```
DSolve[x*y''[x]+(a*x^n+b)*y'[x]+a*(b-1)*x^(n-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to (-1)^{-\frac{b}{n}} n^{\frac{b-n-1}{n}} a^{\frac{1-b}{n}} (x^n)^{\frac{1-b}{n}} \left((b-1)c_1(-1)^{b/n}\Gamma\left(\frac{b-1}{n}, 0, \frac{ax^n}{n}\right) + c_2(-1)^{\frac{1}{n}} n \right)$$

28.36 problem 96

Internal problem ID [10920]
Internal file name [OUTPUT/10176_Sunday_December_31_2023_11_03_32_AM_23395983/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 96.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''+(a\,x^n+b)\,y'+a(b+n-1)\,x^{n-1}y=0}$$

28.36.1 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$xy''+(a\,x^n+b)\,y'+a(b+n-1)\,x^{n-1}y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=r(x) \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{a\,x^n+b}{x} \\
q(x) &= a\,x^{n-2}(b+n-1) \\
r(x) &= 0
\end{aligned}$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(\frac{(a\,x^n + b)\,\xi(x)}{x}\right)' + \left(a\,x^{n-2}(b+n-1)\,\xi(x)\right) &= 0 \\
\xi''(x) - \frac{(a\,x^n + b)\,\xi'(x)}{x} + \left(-\frac{an\,x^n}{x^2} + \frac{a\,x^n + b}{x^2} + a\,x^{n-2}(b+n-1)\right)\xi(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. In normal form the ode

$$
\xi''(x)\,x^2 + (-a\,x^n - b)\,\xi'(x)\,x + b(a\,x^n + 1)\,\xi(x) = 0 \tag{1}
$$

Becomes

$$
\xi''(x) + p(x)\,\xi'(x) + q(x)\,\xi(x) = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{-a\,x^n - b}{x} \\
q(x) &= \frac{b(a\,x^n + 1)}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $\xi(x) = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not $\xi(x)$.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$
\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$
\frac{n(n-1)}{x^2} + \frac{n(-a\,x^n - b)}{x^2} + \frac{b(a\,x^n + 1)}{x^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = b \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \left(\frac{2b}{x} + \frac{-a\,x^n - b}{x}\right) v'(x) &= 0 \\
v''(x) + \frac{(b - a\,x^n)\,v'(x)}{x} &= 0
\end{aligned} \tag{7}
$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(b - a\,x^n)\,u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(-b + a\,x^n)}{x} \end{aligned}$$

Where $f(x) = \frac{-b+a\,x^n}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= \frac{-b + a\,x^n}{x}\,dx \\ \int \frac{1}{u}\,du &= \int \frac{-b + a\,x^n}{x}\,dx \\ \ln(u) &= \frac{a\,x^n}{n} - \frac{b\ln(x^n)}{n} + c_1 \\ u &= \mathrm{e}^{\frac{a\,x^n}{n} - \frac{b\ln(x^n)}{n} + c_1} \\ &= c_1 \mathrm{e}^{\frac{a\,x^n}{n} - \frac{b\ln(x^n)}{n}} \end{aligned}$$

Which simplifies to

$$u(x) = c_1 \mathrm{e}^{\frac{a\,x^n}{n}} (x^n)^{-\frac{b}{n}}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= \int c_1 \mathrm{e}^{\frac{a\,x^n}{n}} (x^n)^{-\frac{b}{n}}\,dx + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
\xi(x) &= v(x)\, x^{n} \\
&= \left(\int c_1 \mathrm{e}^{\frac{a\,x^{n}}{n}} \left(x^{n}\right)^{-\frac{b}{n}} dx + c_2\right) x^{b} \\
&= \left(c_1 \left(\int \mathrm{e}^{\frac{a\,x^{n}}{n}} \left(x^{n}\right)^{-\frac{b}{n}} dx\right) + c_2\right) x^{b}
\end{aligned}
$$

The original ode (2) now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}
\end{aligned}
$$

$$
y' + y\left(\frac{a\,x^{n} + b}{x} - \frac{\left(c_3 \mathrm{e}^{\frac{a\,x^{n}}{n}} \left(x^{n}\right)^{-\frac{b}{n}} x^{b} + \frac{\left(\int c_3 \mathrm{e}^{\frac{a\,x^{n}}{n}} \left(x^{n}\right)^{-\frac{b}{n}} dx + c_2\right) x^{b} b}{x}\right) x^{-b}}{\int c_3 \mathrm{e}^{\frac{a\,x^{n}}{n}} \left(x^{n}\right)^{-\frac{b}{n}} dx + c_2}\right) = 0
$$

Which is now a first order ode. This is now solved for y. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= -\frac{y\left(x^{n}\left(x^{n}\right)^{\frac{b}{n}} \left(\int c_3 \mathrm{e}^{\frac{a\,x^{n}}{n}} \left(x^{n}\right)^{-\frac{b}{n}} dx\right) a + a\,x^{n}\left(x^{n}\right)^{\frac{b}{n}} c_2 - c_3 \mathrm{e}^{\frac{a\,x^{n}}{n}} x\right) \left(x^{n}\right)^{-\frac{b}{n}}}{x\left(\int c_3 \mathrm{e}^{\frac{a\,x^{n}}{n}} \left(x^{n}\right)^{-\frac{b}{n}} dx + c_2\right)}
\end{aligned}
$$

Where $f(x) = -\frac{\left(x^{n} (x^{n})^{\frac{b}{n}} \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx\right) a + a x^{n} (x^{n})^{\frac{b}{n}} c_2 - c_3 \mathrm{e}^{\frac{a x^{n}}{n}} x\right) (x^{n})^{-\frac{b}{n}}}{x \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx + c_2\right)}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= -\frac{\left(x^{n} (x^{n})^{\frac{b}{n}} \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx\right) a + a x^{n} (x^{n})^{\frac{b}{n}} c_2 - c_3 \mathrm{e}^{\frac{a x^{n}}{n}} x\right) (x^{n})^{-\frac{b}{n}}}{x \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx + c_2\right)} dx \\
\int \frac{1}{y}\, dy &= \int -\frac{\left(x^{n} (x^{n})^{\frac{b}{n}} \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx\right) a + a x^{n} (x^{n})^{\frac{b}{n}} c_2 - c_3 \mathrm{e}^{\frac{a x^{n}}{n}} x\right) (x^{n})^{-\frac{b}{n}}}{x \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx + c_2\right)} dx \\
\ln(y) &= \int -\frac{\left(x^{n} (x^{n})^{\frac{b}{n}} \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx\right) a + a x^{n} (x^{n})^{\frac{b}{n}} c_2 - c_3 \mathrm{e}^{\frac{a x^{n}}{n}} x\right) (x^{n})^{-\frac{b}{n}}}{x \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx + c_2\right)} dx + c_3 \\
y &= \mathrm{e}^{\int -\frac{\left(x^{n} (x^{n})^{\frac{b}{n}} \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx\right) a + a x^{n} (x^{n})^{\frac{b}{n}} c_2 - c_3 \mathrm{e}^{\frac{a x^{n}}{n}} x\right) (x^{n})^{-\frac{b}{n}}}{x \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx + c_2\right)} dx + c_3} \\
&= c_3 \mathrm{e}^{\int -\frac{\left(x^{n} (x^{n})^{\frac{b}{n}} \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx\right) a + a x^{n} (x^{n})^{\frac{b}{n}} c_2 - c_3 \mathrm{e}^{\frac{a x^{n}}{n}} x\right) (x^{n})^{-\frac{b}{n}}}{x \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx + c_2\right)} dx}
\end{aligned}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = c_3 \mathrm{e}^{\int -\frac{\left(x^{n} (x^{n})^{\frac{b}{n}} \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx\right) a + a x^{n} (x^{n})^{\frac{b}{n}} c_2 - c_3 \mathrm{e}^{\frac{a x^{n}}{n}} x\right) (x^{n})^{-\frac{b}{n}}}{x \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx + c_2\right)} dx}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_3 \mathrm{e}^{\int -\frac{\left(x^{n} (x^{n})^{\frac{b}{n}} \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx\right) a + a x^{n} (x^{n})^{\frac{b}{n}} c_2 - c_3 \mathrm{e}^{\frac{a x^{n}}{n}} x\right) (x^{n})^{-\frac{b}{n}}}{x \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx + c_2\right)} dx} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_3 \mathrm{e}^{\int -\frac{\left(x^{n} (x^{n})^{\frac{b}{n}} \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx\right) a + a x^{n} (x^{n})^{\frac{b}{n}} c_2 - c_3 \mathrm{e}^{\frac{a x^{n}}{n}} x\right) (x^{n})^{-\frac{b}{n}}}{x \left(\int c_3 \mathrm{e}^{\frac{a x^{n}}{n}} (x^{n})^{-\frac{b}{n}} dx + c_2\right)} dx}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
      Solution has integrals. Trying a special function solution free of integrals...
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         <- Kummer successful
      <- special function solution successful
         Solution using Kummer functions still has integrals. Trying a hypergeometric soluti
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
         -> Trying to convert hypergeometric functions to elementary form...
         <- elementary form could result into a too large expression - returning special fun
      <- Kovacics algorithm successful
<- Equivalence, under non-integer power transformations successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 56

```
dsolve(x*diff(y(x),x$2)+(a*x^n+b)*diff(y(x),x)+a*(b+n-1)*x^(n-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{a\,x^n}{n}} \left(c_1 + x^{-b+1} c_2 \,\mathrm{hypergeom}\left(\left[\frac{-b+1}{n}\right], \left[\frac{-b+n+1}{n}\right], \frac{a\,x^n}{n}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.178 (sec). Leaf size: 93

```
DSolve[x*y''[x]+(a*x^n+b)*y'[x]+a*(b+n-1)*x^(n-1)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{(-1)^{-1/n}e^{-\frac{ax^n}{n}}\left((b-1)c_2(-1)^{b/n}\Gamma\left(\frac{1-b}{n},-\frac{ax^n}{n}\right)-(b-1)c_2(-1)^{b/n}\operatorname{Gamma}\left(\frac{1-b}{n}\right)+c_1(-1)^{\frac{1}{n}}n\right)}{n}$$

28.37 problem 97

Internal problem ID [10921]
Internal file name [`OUTPUT/10177_Sunday_December_31_2023_11_03_35_AM_43007570/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y'' + f(x)y' + g(x)y = 0$
Problem number: 97.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{xy'' + (a\,x^n + b)\,y' + c(a\,x^n - cx + b)\,y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x$2)+(a*x^n+b)*diff(y(x),x)+c*(a*x^n-c*x+b)*y(x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y''[x]+(a*x^n+b)*y'[x]+c*(a*x^n-c*x+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

Not solved

28.38 problem 98

Internal problem ID [10922]
Internal file name [`OUTPUT/10178_Sunday_December_31_2023_11_03_36_AM_49956565/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 98.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{xy'' + (ab\,x^n + b - 3n + 1)\,y' + a^2n(-n+b)\,x^{-1+2n}y = 0}$$

✗ Solution by Maple

```
dsolve(x*diff(y(x),x$2)+(a*b*x^n+b-3*n+1)*diff(y(x),x)+a^2*n*(b-n)*x^(2*n-1)*y(x)=0,y(x), si
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y''[x]+(a*b*x^n+b-3*n+1)*y'[x]+a^2*n*(b-n)*x^(2*n-1)*y[x]==0,y[x],x,IncludeSingular
```

Not solved

28.39 problem 99

Internal problem ID [10923]
Internal file name [OUTPUT/10179_Sunday_December_31_2023_11_03_36_AM_5864548/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 99.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{xy''+(a\,x^n+b)\,y'+\left(x^{-1+2n}c+d\,x^{n-1}\right)y=0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 156

```
dsolve(x*diff(y(x),x$2)+(a*x^n+b)*diff(y(x),x)+(c*x^(2*n-1)+d*x^(n-1))*y(x)=0,y(x), singsol=
```

$$y(x) = e^{-\frac{x^n\left(\sqrt{a^2-4c}+a\right)}{2n}} \left(\text{KummerU}\left(\frac{(b+n-1)\sqrt{a^2-4c}+a(b+n-1)-2d}{2\sqrt{a^2-4c}\,n}, \frac{b+n-1}{n}, \frac{\sqrt{a^2-4c}\,x^n}{n}\right)c_2 + \text{KummerM}\left(\frac{(b+n-1)\sqrt{a^2-4c}+a(b+n-1)-2d}{2\sqrt{a^2-4c}\,n}, \frac{b+n-1}{n}, \frac{\sqrt{a^2-4c}\,x^n}{n}\right)c_1\right)$$

✓ Solution by Mathematica

Time used: 0.38 (sec). Leaf size: 255

```
DSolve[x*y''[x]+(a*x^n+b)*y'[x]+(c*x^(2*n-1)+d*x^(n-1))*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to 2^{\frac{b+n-1}{2n}} x^{\frac{1}{2}-\frac{n}{2}} (x^n)^{\frac{n-1}{2n}} e^{-\frac{\left(\sqrt{a^2-4c}+a\right)x^n}{2n}} \left(c_1 \, \text{HypergeometricU}\left(\frac{(b+n-1)a^2 + \sqrt{a^2-4c}(b+n-1)a - 2\sqrt{}}{2\left(a^2-4c\right)n} \right. \right.$$
$$\left. + c_2 L^{\frac{b-1}{n}}_{-\frac{(b+n-1)a^2+\sqrt{a^2-4c}(b+n-1)a-2\sqrt{a^2-4c}d-4c(b+n-1)}{2\left(a^2-4c\right)n}} \left(\frac{\sqrt{a^2-4c}x^n}{n} \right) \right)$$

28.40 problem 100

Internal problem ID [10924]
Internal file name [`OUTPUT/10180_Sunday_December_31_2023_11_03_37_AM_34545972/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 100.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{xy''+\left(a\,x^{n}+b\,x^{n-1}+2\right)y'+yx^{n-2}b=0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 53

```
dsolve(x*diff(y(x),x$2)+(a*x^n+b*x^(n-1)+2)*diff(y(x),x)+(b*x^(n-2))*y(x)=0,y(x), singsol=al
```

$$y(x) = \frac{(ax+b)\left(c_2\left(\int \frac{e^{-\frac{(ax(n-1)+bn)x^{n-1}}{n(n-1)}}}{(ax+b)^2}dx\right)+c_1\right)}{x}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y''[x]+(a*x^n+b*x^(n-1)+2)*y'[x]+(b*x^(n-2))*y[x]==0,y[x],x,IncludeSingularSolution
```

Not solved

28.41 problem 101

Internal problem ID [10925]
Internal file name [OUTPUT/10181_Sunday_December_31_2023_11_03_38_AM_61021467/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 101.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{xy''+(a\,x^n+bx)\,y'+\left(ab\,x^n+an\,x^{n-1}-b\right)y=0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x$2)+(a*x^n+b*x)*diff(y(x),x)+(a*b*x^n+a*n*x^(n-1)-b)*y(x)=0,y(x), singso
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y''[x]+(a*x^n+b*x)*y'[x]+(a*b*x^n+a*n*x^(n-1)-b)*y[x]==0,y[x],x,IncludeSingularSolu
```

Not solved

28.42 problem 102

Internal problem ID [10926]
Internal file name [`OUTPUT/10182_Sunday_December_31_2023_11_03_39_AM_77566012/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 102.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{xy''+\left(ab\,x^{n}+b\,x^{n-1}+ax-1\right)y'+a^{2}b\,x^{n}y=0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x$2)+(a*b*x^n+b*x^(n-1)+a*x-1)*diff(y(x),x)+(a^2*b*x^n)*y(x)=0,y(x), sing
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y''[x]+(a*b*x^n+b*x^(n-1)+a*x-1)*y'[x]+(a^2*b*x^n)*y[x]==0,y[x],x,IncludeSingularSo
```

Not solved

28.43 problem 103

Internal problem ID [10927]
Internal file name [`OUTPUT/10183_Sunday_December_31_2023_11_03_40_AM_86763740/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 103.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy''+\left(a\,x^{n}+b\,x^{m}+c\right)y'+\left(c-1\right)\left(a\,x^{n-1}+b\,x^{m-1}\right)y=0}$$

28.43.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$xy''+\left(a\,x^{n}+b\,x^{m}+c\right)y'+\frac{\left(c-1\right)\left(a\,x^{n}+b\,x^{m}\right)y}{x}=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=\frac{a\,x^{n}+b\,x^{m}+c}{x}$$
$$q(x)=\frac{\left(c-1\right)\left(a\,x^{n}+b\,x^{m}\right)}{x^{2}}$$

Applying change of variables on the depndent variable $y=v(x)\,x^{n}$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right)v'(x)+\left(\frac{n(n-1)}{x^{2}}+\frac{np}{x}+q\right)v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(a\,x^n + b\,x^m + c)}{x^2} + \frac{(c-1)\,(a\,x^n + b\,x^m)}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -c + 1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \left(\frac{-2c+2}{x} + \frac{a\,x^n + b\,x^m + c}{x}\right) v'(x) = 0$$

$$v''(x) + \frac{(-c + 2 + a\,x^n + b\,x^m)\,v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(-c + 2 + a\,x^n + b\,x^m)\,u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(-c + 2 + a\,x^n + b\,x^m)\,u}{x} \end{aligned}$$

Where $f(x) = -\frac{-c+2+a\,x^n+b\,x^m}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{-c + 2 + a\,x^n + b\,x^m}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{-c + 2 + a\,x^n + b\,x^m}{x}\,dx \\ \ln(u) &= (c-2)\ln(x) - \frac{b\,e^{m\ln(x)}}{m} - \frac{a\,e^{n\ln(x)}}{n} + c_1 \\ u &= e^{(c-2)\ln(x) - \frac{b\,e^{m\ln(x)}}{m} - \frac{a\,e^{n\ln(x)}}{n} + c_1} \\ &= c_1 e^{(c-2)\ln(x) - \frac{b\,e^{m\ln(x)}}{m} - \frac{a\,e^{n\ln(x)}}{n}} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^c \mathrm{e}^{-\frac{b\,x^m}{m}} \mathrm{e}^{-\frac{a\,x^n}{n}}}{x^2}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= \int \frac{c_1 x^c \mathrm{e}^{-\frac{b\,x^m}{m}} \mathrm{e}^{-\frac{a\,x^n}{n}}}{x^2} dx + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(\int \frac{c_1 x^c \mathrm{e}^{-\frac{b\,x^m}{m}} \mathrm{e}^{-\frac{a\,x^n}{n}}}{x^2} dx + c_2\right) x^{-c+1} \\ &= x^{-c+1} \left(c_1 \left(\int x^{c-2} \mathrm{e}^{-\frac{b\,x^m}{m} - \frac{a\,x^n}{n}} dx\right) + c_2\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \left(\int \frac{c_1 x^c \mathrm{e}^{-\frac{b\,x^m}{m}} \mathrm{e}^{-\frac{a\,x^n}{n}}}{x^2} dx + c_2\right) x^{-c+1} \tag{1}$$

<u>Verification of solutions</u>

$$y = \left(\int \frac{c_1 x^c \mathrm{e}^{-\frac{b\,x^m}{m}} \mathrm{e}^{-\frac{a\,x^n}{n}}}{x^2} dx + c_2\right) x^{-c+1}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x$2)+(a*x^n+b*x^m+c)*diff(y(x),x)+(c-1)*(a*x^(n-1)+b*x^(m-1))*y(x)=0,y(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y''[x]+(a*x^n+b*x^m+c)*y'[x]+(c-1)*(a*x^(n-1)+b*x^(m-1))*y[x]==0,y[x],x,IncludeSing
```

Not solved

28.44 problem 104

Internal problem ID [10928]
Internal file name [`OUTPUT/10184_Sunday_December_31_2023_11_03_41_AM_16119579/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 104.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{xy''+\left(x^{m+n}ab+x^{n}na+b\,x^{m}+1-2n\right)y'+a^{2}bn\,x^{2n+m-1}y=0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x$2)+(a*b*x^(n+m)+a*n*x^n+b*x^m+1-2*n)*diff(y(x),x)+a^2*b*n*x^(2*n+m-1)*y
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y''[x]+(a*b*x^(n+m)+a*n*x^n+b*x^m+1-2*n)*y'[x]+a^2*b*n*x^(2*n+m-1)*y[x]==0,y[x],x,I
```

Not solved

28.45 problem 105

Internal problem ID [10929]
Internal file name [`OUTPUT/10185_Sunday_December_31_2023_11_03_43_AM_1225267/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 105.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x+a)\,y''+(bx+c)\,y'+yb=0}$$

28.45.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int ((x+a)\,y''+(bx+c)\,y'+yb)\,dx=0$$
$$(bx+c-1)\,y+(x+a)\,y'=c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x) = -\frac{-bx - c + 1}{x + a}$$
$$q(x) = \frac{c_1}{x + a}$$

Hence the ode is

$$y' - \frac{(-bx - c + 1)\, y}{x + a} = \frac{c_1}{x + a}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-bx-c+1}{x+a}dx}$$
$$= e^{bx + (-ab+c-1)\ln(x+a)}$$

Which simplifies to

$$\mu = (x + a)^{-ab+c-1}\, e^{bx}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x + a}\right)$$
$$\frac{d}{dx}\left((x + a)^{-ab+c-1}\, e^{bx} y\right) = \left((x + a)^{-ab+c-1}\, e^{bx}\right)\left(\frac{c_1}{x + a}\right)$$
$$d\left((x + a)^{-ab+c-1}\, e^{bx} y\right) = \left(c_1 (x + a)^{-ab+c-2}\, e^{bx}\right)\, dx$$

Integrating gives

$$(x + a)^{-ab+c-1}\, e^{bx} y = \int c_1 (x + a)^{-ab+c-2}\, e^{bx}\, dx$$
$$(x + a)^{-ab+c-1}\, e^{bx} y = \int c_1 (x + a)^{-ab+c-2}\, e^{bx} dx + c_2$$

Dividing both sides by the integrating factor $\mu = (x + a)^{-ab+c-1}\, e^{bx}$ results in

$$y = (x + a)^{ab-c+1}\, e^{-bx}\left(\int c_1 (x + a)^{-ab+c-2}\, e^{bx} dx\right) + c_2 (x + a)^{ab-c+1}\, e^{-bx}$$

which simplifies to

$$y = (x + a)^{ab-c+1}\, e^{-bx}\left(c_1\left(\int (x + a)^{-ab+c-2}\, e^{bx} dx\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = (x+a)^{ab-c+1} \mathrm{e}^{-bx} \left(c_1 \left(\int (x+a)^{-ab+c-2} \mathrm{e}^{bx} dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = (x+a)^{ab-c+1} \mathrm{e}^{-bx} \left(c_1 \left(\int (x+a)^{-ab+c-2} \mathrm{e}^{bx} dx\right) + c_2\right)$$

Verified OK.

28.45.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(x+a)\, y'' + (bx+c)\, y' + yb = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int ((x+a)\, y'' + (bx+c)\, y' + yb)\, dx = 0$$
$$(bx+c-1)\, y + (x+a)\, y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-bx-c+1}{x+a}$$
$$q(x) = \frac{c_1}{x+a}$$

Hence the ode is

$$y' - \frac{(-bx-c+1)\, y}{x+a} = \frac{c_1}{x+a}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{-bx-c+1}{x+a} dx}$$
$$= \mathrm{e}^{bx+(-ab+c-1)\ln(x+a)}$$

Which simplifies to

$$\mu = (x+a)^{-ab+c-1}\, e^{bx}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x+a}\right) \\
\frac{d}{dx}\left((x+a)^{-ab+c-1}\, e^{bx} y\right) &= \left((x+a)^{-ab+c-1}\, e^{bx}\right)\left(\frac{c_1}{x+a}\right) \\
d\left((x+a)^{-ab+c-1}\, e^{bx} y\right) &= \left(c_1(x+a)^{-ab+c-2}\, e^{bx}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
(x+a)^{-ab+c-1}\, e^{bx} y &= \int c_1(x+a)^{-ab+c-2}\, e^{bx}\, dx \\
(x+a)^{-ab+c-1}\, e^{bx} y &= \int c_1(x+a)^{-ab+c-2}\, e^{bx} dx + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = (x+a)^{-ab+c-1}\, e^{bx}$ results in

$$y = (x+a)^{ab-c+1}\, e^{-bx}\left(\int c_1(x+a)^{-ab+c-2}\, e^{bx} dx\right) + c_2(x+a)^{ab-c+1}\, e^{-bx}$$

which simplifies to

$$y = (x+a)^{ab-c+1}\, e^{-bx}\left(c_1\left(\int (x+a)^{-ab+c-2}\, e^{bx} dx\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = (x+a)^{ab-c+1}\, e^{-bx}\left(c_1\left(\int (x+a)^{-ab+c-2}\, e^{bx} dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = (x+a)^{ab-c+1}\, e^{-bx}\left(c_1\left(\int (x+a)^{-ab+c-2}\, e^{bx} dx\right) + c_2\right)$$

Verified OK.

28.45.3 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x + a \\ q(x) &= bx + c \\ r(x) &= b \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= b \end{aligned}$$

Therefore (1) becomes

$$0 - (b) + (b) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$(bx + c - 1)\,y + (x + a)\,y' = c_1$$

We now have a first order ode to solve which is

$$(bx + c - 1)\,y + (x + a)\,y' = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-bx - c + 1}{x + a}$$
$$q(x) = \frac{c_1}{x + a}$$

Hence the ode is

$$y' - \frac{(-bx - c + 1)\, y}{x + a} = \frac{c_1}{x + a}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{-bx-c+1}{x+a}dx} \\ &= \mathrm{e}^{bx+(-ab+c-1)\ln(x+a)}\end{aligned}$$

Which simplifies to

$$\mu = (x + a)^{-ab+c-1}\, \mathrm{e}^{bx}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x + a}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left((x + a)^{-ab+c-1}\, \mathrm{e}^{bx} y\right) &= \left((x + a)^{-ab+c-1}\, \mathrm{e}^{bx}\right)\left(\frac{c_1}{x + a}\right) \\ \mathrm{d}\left((x + a)^{-ab+c-1}\, \mathrm{e}^{bx} y\right) &= \left(c_1 (x + a)^{-ab+c-2}\, \mathrm{e}^{bx}\right)\, \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}(x + a)^{-ab+c-1}\, \mathrm{e}^{bx} y &= \int c_1 (x + a)^{-ab+c-2}\, \mathrm{e}^{bx}\, \mathrm{d}x \\ (x + a)^{-ab+c-1}\, \mathrm{e}^{bx} y &= \int c_1 (x + a)^{-ab+c-2}\, \mathrm{e}^{bx} dx + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = (x + a)^{-ab+c-1}\, \mathrm{e}^{bx}$ results in

$$y = (x + a)^{ab-c+1}\, \mathrm{e}^{-bx}\left(\int c_1 (x + a)^{-ab+c-2}\, \mathrm{e}^{bx} dx\right) + c_2 (x + a)^{ab-c+1}\, \mathrm{e}^{-bx}$$

which simplifies to

$$y = (x + a)^{ab-c+1}\, \mathrm{e}^{-bx}\left(c_1\left(\int (x + a)^{-ab+c-2}\, \mathrm{e}^{bx} dx\right) + c_2\right)$$

Summary
The solution(s) found are the following

$$y = (x+a)^{ab-c+1} \mathrm{e}^{-bx} \left(c_1 \left(\int (x+a)^{-ab+c-2} \mathrm{e}^{bx} dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = (x+a)^{ab-c+1} \mathrm{e}^{-bx} \left(c_1 \left(\int (x+a)^{-ab+c-2} \mathrm{e}^{bx} dx\right) + c_2\right)$$

Verified OK.

28.45.4 Maple step by step solution

Let's solve

$(x+a)\, y'' + (bx+c)\, y' + yb = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{by}{x+a} - \frac{(bx+c)y'}{x+a}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(bx+c)y'}{x+a} + \frac{by}{x+a} = 0$

□ Check to see if $x_0 = -a$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{bx+c}{x+a}, P_3(x) = \frac{b}{x+a}\right]$

  - $(x+a) \cdot P_2(x)$ is analytic at $x = -a$

    $\left((x+a) \cdot P_2(x)\right)\Big|_{x=-a} = -ab + c$

  - $(x+a)^2 \cdot P_3(x)$ is analytic at $x = -a$

    $\left((x+a)^2 \cdot P_3(x)\right)\Big|_{x=-a} = 0$

  - $x = -a$is a regular singular point

  Check to see if $x_0 = -a$ is a regular singular point

  $x_0 = -a$

- Multiply by denominators

$(x+a)\,y''+(bx+c)\,y'+yb=0$

- Change variables using $x=u-a$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(-ab+bu+c)\left(\frac{d}{du}y(u)\right)+by(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

  - Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-a_0r(ab-c-r+1)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(ab-c-k-r)+ba_k(k+1+r)\right)u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(ab-c-r+1)=0$

- Values of r that satisfy the indicial equation

$r\in\{0,ab-c+1\}$

- Each term in the series must be 0, giving the recursion relation

$-(k+1+r)\,(a_{k+1}(ab-c-k-r)-ba_k)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{ba_k}{ab-c-k-r}$

- Recursion relation for $r=0$

$a_{k+1} = \frac{b a_k}{ab-c-k}$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{b a_k}{ab-c-k}\right]$$

- Revert the change of variables $u = x + a$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+a)^k, a_{k+1} = \frac{b a_k}{ab-c-k}\right]$$

- Recursion relation for $r = ab - c + 1$

$a_{k+1} = \frac{b a_k}{-k-1}$

- Solution for $r = ab - c + 1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{ab-c+k+1}, a_{k+1} = \frac{b a_k}{-k-1}\right]$$

- Revert the change of variables $u = x + a$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+a)^{ab-c+k+1}, a_{k+1} = \frac{b a_k}{-k-1}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} d_k (x+a)^k\right) + \left(\sum_{k=0}^{\infty} e_k (x+a)^{ab-c+k+1}\right), d_{1+k} = \frac{b d_k}{ab-c-k}, e_{1+k} = \frac{b e_k}{-k-1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form is not straightforward to achieve - returning hypergeometric solution
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 78

```
dsolve((x+a)*diff(y(x),x$2)+(b*x+c)*diff(y(x),x)+b*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\left(-(a+x)^{ab-c+1} c_1 + (\Gamma(-ab+c) + \Gamma(-ab+c-1, -b(a+x))(ab-c+1)) b(a+x) c_2(-b(a+x))^{ab-c}\right) e^{-bx}$$

✓ Solution by Mathematica

Time used: 0.559 (sec). Leaf size: 90

```
DSolve[(x+a)*y''[x]+(b*x+c)*y'[x]+b*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-b(a+x)}(a+x)^{1-c}(-b(a+x))^{-c} \left(c_1 e^{ab}(a+x)^{ab}(-b(a+x))^c + bc_2(-b(a+x))^{ab}(a+x)^c\Gamma(-ab+c-1, -b(a+x))\right)$$

28.46 problem 106

Internal problem ID [10930]
Internal file name [`OUTPUT/10186_Sunday_December_31_2023_11_03_44_AM_97751028/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 106.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$(a_1x+a_0)\,y''+(b_1x+b_0)\,y'-mb_1y=0$$

28.46.1 Maple step by step solution

Let's solve

$(a_1x+a_0)\,y''+(b_1x+b_0)\,y'-mb_1y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{b_1my}{a_1x+a_0}-\frac{(b_1x+b_0)y'}{a_1x+a_0}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(b_1x+b_0)y'}{a_1x+a_0}-\frac{b_1my}{a_1x+a_0}=0$

☐ Check to see if $x_0=-\frac{a_0}{a_1}$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{b_1 x + b_0}{a_1 x + a_0}, P_3(x) = -\frac{b_1 m}{a_1 x + a_0}\right]$

- ◦ $\left(x + \frac{a_0}{a_1}\right) \cdot P_2(x)$ is analytic at $x = -\frac{a_0}{a_1}$

  $\left(\left(x + \frac{a_0}{a_1}\right) \cdot P_2(x)\right)\bigg|_{x = -\frac{a_0}{a_1}} = \frac{-\frac{b_1 a_0}{a_1} + b_0}{a_1}$

- ◦ $\left(x + \frac{a_0}{a_1}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{a_0}{a_1}$

  $\left(\left(x + \frac{a_0}{a_1}\right)^2 \cdot P_3(x)\right)\bigg|_{x = -\frac{a_0}{a_1}} = 0$

- ◦ $x = -\frac{a_0}{a_1}$ is a regular singular point

  Check to see if $x_0 = -\frac{a_0}{a_1}$ is a regular singular point

  $x_0 = -\frac{a_0}{a_1}$

- • Multiply by denominators

  $(a_1 x + a_0) y'' + (b_1 x + b_0) y' - m b_1 y = 0$

- • Change variables using $x = u - \frac{a_0}{a_1}$ so that the regular singular point is at $u = 0$

  $a_1 u \left(\frac{d^2}{du^2} y(u)\right) + \left(b_1 u - \frac{b_1 a_0}{a_1} + b_0\right) \left(\frac{d}{du} y(u)\right) - b_1 m y(u) = 0$

- • Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k + r) u^{k+r-1+m}$

  - ◦ Shift index using $k- > k + 1 - m$

    $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r) u^{k+r}$

  - ◦ Convert $u \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k + r)(k + r - 1) u^{k+r-1}$

  - ◦ Shift index using $k- > k + 1$

    $u \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)(k + r) u^{k+r}$

Rewrite ODE with series expansions

$$-\frac{a_0 r\left(-a_1^2 r+a_0 b_1+a_1^2-a_1 b_0\right) u^{-1+r}}{a_1}+\left(\sum_{k=0}^{\infty}\left(-\frac{a_{k+1}(k+1+r)\left(-a_1^2(k+1)-a_1^2 r+a_0 b_1+a_1^2-a_1 b_0\right)}{a_1}+a_k b_1(k+r-m)\right) u$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-\frac{r\left(-a_1^2 r+a_0 b_1+a_1^2-a_1 b_0\right)}{a_1}=0$$

- Values of r that satisfy the indicial equation

$$r \in\left\{0, \frac{a_0 b_1+a_1^2-a_1 b_0}{a_1^2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$\frac{a_{k+1}(k+1+r)(k+r) a_1^2+\left(b_0(k+1+r) a_{k+1}+a_k b_1(k+r-m)\right) a_1-a_0 b_1 a_{k+1}(k+1+r)}{a_1}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k b_1(k+r-m) a_1}{(k+1+r)\left(-a_1^2 k-a_1^2 r+a_0 b_1-a_1 b_0\right)}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{a_k b_1(k-m) a_1}{(k+1)\left(-a_1^2 k+a_0 b_1-a_1 b_0\right)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k b_1(k-m) a_1}{(k+1)\left(-a_1^2 k+a_0 b_1-a_1 b_0\right)}\right]$$

- Revert the change of variables $u=x+\frac{a_0}{a_1}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(x+\frac{a_0}{a_1}\right)^k, a_{k+1}=\frac{a_k b_1(k-m) a_1}{(k+1)\left(-a_1^2 k+a_0 b_1-a_1 b_0\right)}\right]$$

- Recursion relation for $r=\frac{a_0 b_1+a_1^2-a_1 b_0}{a_1^2}$

$$a_{k+1}=\frac{a_k b_1\left(k+\frac{a_0 b_1+a_1^2-a_1 b_0}{a_1^2}-m\right) a_1}{\left(k+1+\frac{a_0 b_1+a_1^2-a_1 b_0}{a_1^2}\right)\left(-a_1^2 k-a_1^2\right)}$$

- Solution for $r=\frac{a_0 b_1+a_1^2-a_1 b_0}{a_1^2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{a_0 b_1+a_1^2-a_1 b_0}{a_1^2}}, a_{k+1}=\frac{a_k b_1\left(k+\frac{a_0 b_1+a_1^2-a_1 b_0}{a_1^2}-m\right) a_1}{\left(k+1+\frac{a_0 b_1+a_1^2-a_1 b_0}{a_1^2}\right)\left(-a_1^2 k-a_1^2\right)}\right]$$

- Revert the change of variables $u=x+\frac{a_0}{a_1}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(x+\frac{a_0}{a_1}\right)^{k+\frac{a_0 b_1+a_1^2-a_1 b_0}{a_1^2}}, a_{k+1}=\frac{a_k b_1\left(k+\frac{a_0 b_1+a_1^2-a_1 b_0}{a_1^2}-m\right) a_1}{\left(k+1+\frac{a_0 b_1+a_1^2-a_1 b_0}{a_1^2}\right)\left(-a_1^2 k-a_1^2\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x+\frac{a_0}{a_1}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x+\frac{a_0}{a_1}\right)^{k+\frac{a_0 b_1 + a_1^2 - a_1 b_0}{a_1^2}}\right), a_{1+k} = \frac{a_k b_1 (-m+k) a_1}{(1+k)(-a_1^2 k + a_0 b_1 - a_1 b_0)}, b_{1+k} =\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 101

```
dsolve((a__1*x+a__0)*diff(y(x),x$2)+(b__1*x+b__0)*diff(y(x),x)-m*b__1*y(x)=0,y(x), singsol=a
```

$$y(x) = (a_1 x + a_0)^{\frac{a_0 b_1 + a_1^2 - a_1 b_0}{a_1^2}} e^{-\frac{b_1 x}{a_1}} \left(\mathrm{KummerM}\left(1+m, \frac{a_0 b_1 + 2a_1^2 - a_1 b_0}{a_1^2}, \frac{b_1 (a_1 x + a_0)}{a_1^2}\right) c_1 + \mathrm{KummerU}\left(1+m, \frac{a_0 b_1 + 2a_1^2 - a_1 b_0}{a_1^2}, \frac{b_1 (a_1 x + a_0)}{a_1^2}\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.278 (sec). Leaf size: 102

```
DSolve[(a1*x+a0)*y''[x]+(b1*x+b0)*y'[x]-m*b1*y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{-\frac{\text{b1}x}{\text{a1}}}(\text{a0}+\text{a1}x)^{\frac{\text{a0b1}+\text{a1}^2-\text{a1b0}}{\text{a1}^2}}\left(c_1 \text{HypergeometricU}\left(m+1,-\frac{\text{b0}}{\text{a1}}+\frac{\text{a0b1}}{\text{a1}^2}+2,\frac{\text{b1}(\text{a0}+\text{a1}x)}{\text{a1}^2}\right)+c_2 L_{-m-1}^{\frac{\text{a1}^2-\text{b0a1}+\text{a0b1}}{\text{a1}^2}}\left(\frac{\text{b1}(\text{a0}+\text{a1}x)}{\text{a1}^2}\right)\right)$$

28.47 problem 107

Internal problem ID [10931]
Internal file name [OUTPUT/10187_Sunday_December_31_2023_11_03_45_AM_79548268/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 107.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$(ax+b)\,y''+s(cx+d)\,y'-s^2((a+c)\,x+b+d)\,y=0$$

28.47.1 Maple step by step solution

Let's solve

$(ax+b)\,y''+s(cx+d)\,y'-s^2((a+c)\,x+b+d)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{s^2(ax+cx+b+d)y}{ax+b}-\frac{s(cx+d)y'}{ax+b}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{s(cx+d)y'}{ax+b}-\frac{s^2(ax+cx+b+d)y}{ax+b}=0$

□ Check to see if $x_0=-\frac{b}{a}$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{s(cx+d)}{ax+b}, P_3(x) = -\frac{s^2(ax+cx+b+d)}{ax+b}\right]$

- $\left(x+\frac{b}{a}\right) \cdot P_2(x)$ is analytic at $x = -\frac{b}{a}$

  $\left(\left(x+\frac{b}{a}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{b}{a}} = \frac{s\left(-\frac{bc}{a}+d\right)}{a}$

- $\left(x+\frac{b}{a}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{b}{a}$

  $\left(\left(x+\frac{b}{a}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{b}{a}} = 0$

- $x = -\frac{b}{a}$ is a regular singular point

  Check to see if $x_0 = -\frac{b}{a}$ is a regular singular point

  $x_0 = -\frac{b}{a}$

- Multiply by denominators

  $(ax+b)\, y'' + s(cx+d)\, y' - s^2(ax+cx+b+d)\, y = 0$

- Change variables using $x = u - \frac{b}{a}$ so that the regular singular point is at $u = 0$

  $au\left(\frac{d^2}{du^2}y(u)\right) + \left(scu - \frac{csb}{a} + ds\right)\left(\frac{d}{du}y(u)\right) + \left(-s^2au - s^2cu + \frac{s^2bc}{a} - d\,s^2\right) y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- >k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)u^{k+r-1}$$

- Shift index using $k->k+1$

$$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)u^{k+r}$$

Rewrite ODE with series expansions

$$\frac{a_0 r(a^2r+dsa-bcs-a^2)u^{-1+r}}{a}+\left(\frac{a_1(1+r)(a^2r+dsa-bcs)}{a}+\frac{a_0 s(acr-dsa+bcs)}{a}\right)u^r+\left(\sum_{k=1}^{\infty}\left(\frac{a_{k+1}(k+1+r)(a^2(k+1)+a^2r}{a}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\frac{r(a^2r+dsa-bcs-a^2)}{a}=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{-dsa+bcs+a^2}{a^2}\right\}$$

- Each term must be 0

$$\frac{a_1(1+r)(a^2r+dsa-bcs)}{a}+\frac{a_0 s(acr-dsa+bcs)}{a}=0$$

- Each term in the series must be 0, giving the recursion relation

$$\frac{(-s^2a_{k-1}+a_{k+1}(k+1+r)(k+r))a^2+s((-ca_{k-1}-da_k)s+d(k+1+r)a_{k+1}+ca_k(k+r))a-c(-a_ks+a_{k+1}(k+1+r))bs}{a}=0$$

- Shift index using $k->k+1$

$$\frac{(-s^2a_k+a_{k+2}(k+2+r)(k+1+r))a^2+s((-ca_k-da_{k+1})s+d(k+2+r)a_{k+2}+ca_{k+1}(k+1+r))a-c(-a_{k+1}s+a_{k+2}(k+2+r))bs}{a}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=\frac{s(a^2sa_k-acka_{k+1}-acra_{k+1}+acsa_k+adsa_{k+1}-bcsa_{k+1}-aca_{k+1})}{(k+2+r)(a^2k+a^2r+dsa-bcs+a^2)}$$

- Recursion relation for $r=0$

$$a_{k+2}=\frac{s(a^2sa_k-acka_{k+1}+acsa_k+adsa_{k+1}-bcsa_{k+1}-aca_{k+1})}{(k+2)(a^2k+dsa-bcs+a^2)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=\frac{s(a^2sa_k-acka_{k+1}+acsa_k+adsa_{k+1}-bcsa_{k+1}-aca_{k+1})}{(k+2)(a^2k+dsa-bcs+a^2)},\frac{a_1(dsa-bcs)}{a}+\frac{a_0s(-dsa+bcs)}{a}=0\right.$$

- Revert the change of variables $u=x+\frac{b}{a}$

$$\left[y=\sum_{k=0}^{\infty}a_k\left(x+\frac{b}{a}\right)^k,a_{k+2}=\frac{s(a^2sa_k-acka_{k+1}+acsa_k+adsa_{k+1}-bcsa_{k+1}-aca_{k+1})}{(k+2)(a^2k+dsa-bcs+a^2)},\frac{a_1(dsa-bcs)}{a}+\frac{a_0s(-dsa+bcs)}{a}\right.$$

- Recursion relation for $r=\frac{-dsa+bcs+a^2}{a^2}$

$$a_{k+2} = \frac{s\left(a^2 s a_k - ack a_{k+1} - \frac{c\left(-dsa+bcs+a^2\right)a_{k+1}}{a} + acsa_k + adsa_{k+1} - bcsa_{k+1} - aca_{k+1}\right)}{\left(k+2+\frac{-dsa+bcs+a^2}{a^2}\right)(a^2k+2a^2)}$$

- Solution for $r = \frac{-dsa+bcs+a^2}{a^2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{-dsa+bcs+a^2}{a^2}}, a_{k+2} = \frac{s\left(a^2 s a_k - ack a_{k+1} - \frac{c\left(-dsa+bcs+a^2\right)a_{k+1}}{a} + acsa_k + adsa_{k+1} - bcsa_{k+1} - aca_{k+1}\right)}{\left(k+2+\frac{-dsa+bcs+a^2}{a^2}\right)(a^2k+2a^2)}\right.$$

- Revert the change of variables $u = x + \frac{b}{a}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{b}{a}\right)^{k+\frac{-dsa+bcs+a^2}{a^2}}, a_{k+2} = \frac{s\left(a^2 s a_k - ack a_{k+1} - \frac{c\left(-dsa+bcs+a^2\right)a_{k+1}}{a} + acsa_k + adsa_{k+1} - bcsa_{k+1} - aca\right.}{\left(k+2+\frac{-dsa+bcs+a^2}{a^2}\right)(a^2k+2a^2)}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} e_k \left(x+\frac{b}{a}\right)^{k}\right) + \left(\sum_{k=0}^{\infty} f_k \left(x+\frac{b}{a}\right)^{k+\frac{-dsa+bcs+a^2}{a^2}}\right), e_{k+2} = \frac{s(a^2 se_k - acke_{1+k} + acse_k + adse_{1+k} - bcse_1}{(k+2)(a^2k+dsa-bcs+a^2)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 166

```
dsolve((a*x+b)*diff(y(x),x$2)+s*(c*x+d)*diff(y(x),x)-s^2*((a+c)*x+b+d)*y(x)=0,y(x), singsol=
```

$$y(x) = \frac{\left(\left(\left(-c_1+c_2\right)a^2+adsc_1-bcsc_1\right)\Gamma\left(\frac{-dsa+bcs+a^2}{a^2},\frac{s(2a+c)(ax+b)}{a^2}\right)+\Gamma\left(\frac{-dsa+bcs+2a^2}{a^2}\right)c_1a^2\right)(ax+b)^{\frac{-dsa+bcs+\cdots}{a^2}}}{a^2}$$

✓ Solution by Mathematica

Time used: 1.269 (sec). Leaf size: 122

```
DSolve[(a*x+b)*y''[x]+s*(c*x+d)*y'[x]-s^2*((a+c)*x+b+d)*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to c_1 e^{sx} - \frac{c_2 e^{s\left(\frac{b(2a+c)}{a^2}+x\right)}(ax+b)^{\frac{s(bc-ad)}{a^2}+1}\left(\frac{s(2a+c)(ax+b)}{a^2}\right)^{\frac{s(ad-bc)}{a^2}-1}\Gamma\left(\frac{a^2-dsa+bcs}{a^2},\frac{(2a+c)s(b+ax)}{a^2}\right)}{a}$$

28.48 problem 108

Internal problem ID [10932]
Internal file name [OUTPUT/10188_Sunday_December_31_2023_11_03_46_AM_19163397/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 108.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$(a_2x+b_2)\,y''+(a_1x+b_1)\,y'+(a_0x+b_0)\,y=0$$

28.48.1 Maple step by step solution

Let's solve
$(a_2x+b_2)\,y''+(a_1x+b_1)\,y'+(a_0x+b_0)\,y=0$

- Highest derivative means the order of the ODE is 2
  y''
- Isolate 2nd derivative
  $y''=-\frac{(a_0x+b_0)y}{a_2x+b_2}-\frac{(a_1x+b_1)y'}{a_2x+b_2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $y''+\frac{(a_1x+b_1)y'}{a_2x+b_2}+\frac{(a_0x+b_0)y}{a_2x+b_2}=0$

□ Check to see if $x_0=-\frac{b_2}{a_2}$ is a regular singular point
  ◦ Define functions

$$\left[P_2(x) = \frac{a_1 x+b_1}{a_2 x+b_2}, P_3(x) = \frac{a_0 x+b_0}{a_2 x+b_2}\right]$$

- $\left(x+\frac{b_2}{a_2}\right) \cdot P_2(x)$ is analytic at $x = -\frac{b_2}{a_2}$

  $$\left(\left(x+\frac{b_2}{a_2}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{b_2}{a_2}} = \frac{-\frac{a_1 b_2}{a_2}+b_1}{a_2}$$

- $\left(x+\frac{b_2}{a_2}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{b_2}{a_2}$

  $$\left(\left(x+\frac{b_2}{a_2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{b_2}{a_2}} = 0$$

- $x = -\frac{b_2}{a_2}$ is a regular singular point

  Check to see if $x_0 = -\frac{b_2}{a_2}$ is a regular singular point

  $x_0 = -\frac{b_2}{a_2}$

- Multiply by denominators

  $$(a_2 x + b_2) y'' + (a_1 x + b_1) y' + (a_0 x + b_0) y = 0$$

- Change variables using $x = u - \frac{b_2}{a_2}$ so that the regular singular point is at $u = 0$

  $$a_2 u\left(\frac{d^2}{du^2} y(u)\right) + \left(a_1 u - \frac{a_1 b_2}{a_2} + b_1\right) \left(\frac{d}{du} y(u)\right) + \left(a_0 u - \frac{a_0 b_2}{a_2} + b_0\right) y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r) u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) u^{k+r-1}$$

- Shift index using $k- >k+1$

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) u^{k+r}$$

  Rewrite ODE with series expansions

  $$-\frac{a_0 r(-a_2^2 r+a_1 b_2+a_2^2-a_2 b_1) u^{-1+r}}{a_2} + \left(-\frac{a_1(1+r)(-a_2^2 r+a_1 b_2-a_2 b_1)}{a_2} - \frac{a_0(-a_1 a_2 r+a_0 b_2-a_2 b_0)}{a_2}\right) u^r + \left(\sum_{k=1}^{\infty} \left(-\frac{a_{k+1}(\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-\frac{r(-a_2^2 r+a_1 b_2+a_2^2-a_2 b_1)}{a_2} = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{a_1 b_2+a_2^2-a_2 b_1}{a_2^2}\right\}$$

- Each term must be 0

  $$-\frac{a_1(1+r)(-a_2^2 r+a_1 b_2-a_2 b_1)}{a_2} - \frac{a_0(-a_1 a_2 r+a_0 b_2-a_2 b_0)}{a_2} = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$\frac{a_{k+1}(k+1+r)(k+r)a_2^2+(b_1(k+1+r)a_{k+1}+k a_1 a_k+r a_1 a_k+a_{k-1}a_0+a_k b_0)a_2-b_2(a_1(k+1+r)a_{k+1}+a_k a_0)}{a_2} = 0$$

- Shift index using $k- >k+1$

  $$\frac{a_{k+2}(k+2+r)(k+1+r)a_2^2+(b_1(k+2+r)a_{k+2}+(k+1)a_1 a_{k+1}+r a_1 a_{k+1}+a_k a_0+a_{k+1}b_0)a_2-b_2(a_1(k+2+r)a_{k+2}+a_{k+1}a_0)}{a_2} = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{a_1 a_2 k a_{k+1}+a_1 a_2 r a_{k+1}+a_0 a_2 a_k-a_0 b_2 a_{k+1}+a_1 a_2 a_{k+1}+a_2 b_0 a_{k+1}}{(k+2+r)(-a_2^2 k-a_2^2 r+a_1 b_2-a_2^2-a_2 b_1)}$$

- Recursion relation for $r=0$

  $$a_{k+2} = \frac{a_1 a_2 k a_{k+1}+a_0 a_2 a_k-a_0 b_2 a_{k+1}+a_1 a_2 a_{k+1}+a_2 b_0 a_{k+1}}{(k+2)(-a_2^2 k+a_1 b_2-a_2^2-a_2 b_1)}$$

- Solution for $r=0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = \frac{a_1 a_2 k a_{k+1}+a_0 a_2 a_k-a_0 b_2 a_{k+1}+a_1 a_2 a_{k+1}+a_2 b_0 a_{k+1}}{(k+2)(-a_2^2 k+a_1 b_2-a_2^2-a_2 b_1)}, -\frac{a_1(a_1 b_2-a_2 b_1)}{a_2} - \frac{a_0(a_0 b_2-a_2 b_0)}{a_2} =\right.$$

- Revert the change of variables $u = x+\frac{b_2}{a_2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{b_2}{a_2}\right)^k, a_{k+2} = \frac{a_1 a_2 k a_{k+1}+a_0 a_2 a_k-a_0 b_2 a_{k+1}+a_1 a_2 a_{k+1}+a_2 b_0 a_{k+1}}{(k+2)(-a_2^2 k+a_1 b_2-a_2^2-a_2 b_1)}, -\frac{a_1(a_1 b_2-a_2 b_1)}{a_2} - \frac{a_0(a_0 b_2-a\right.}{a_2}$$

- Recursion relation for $r = \frac{a_1 b_2 + a_2^2 - a_2 b_1}{a_2^2}$

$$a_{k+2} = \frac{a_1 a_2 k a_{k+1} + \frac{a_1 \left(a_1 b_2 + a_2^2 - a_2 b_1\right) a_{k+1}}{a_2} + a_0 a_2 a_k - a_0 b_2 a_{k+1} + a_1 a_2 a_{k+1} + a_2 b_0 a_{k+1}}{\left(k+2+\frac{a_1 b_2 + a_2^2 - a_2 b_1}{a_2^2}\right)\left(-a_2^2 k - 2a_2^2\right)}$$

- Solution for $r = \frac{a_1 b_2 + a_2^2 - a_2 b_1}{a_2^2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{a_1 b_2 + a_2^2 - a_2 b_1}{a_2^2}}, a_{k+2} = \frac{a_1 a_2 k a_{k+1} + \frac{a_1 \left(a_1 b_2 + a_2^2 - a_2 b_1\right) a_{k+1}}{a_2} + a_0 a_2 a_k - a_0 b_2 a_{k+1} + a_1 a_2 a_{k+1} + a_2 b_0 a_{k+1}}{\left(k+2+\frac{a_1 b_2 + a_2^2 - a_2 b_1}{a_2^2}\right)\left(-a_2^2 k - 2a_2^2\right)}\right.$$

- Revert the change of variables $u = x + \frac{b_2}{a_2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{b_2}{a_2}\right)^{k+\frac{a_1 b_2 + a_2^2 - a_2 b_1}{a_2^2}}, a_{k+2} = \frac{a_1 a_2 k a_{k+1} + \frac{a_1 \left(a_1 b_2 + a_2^2 - a_2 b_1\right) a_{k+1}}{a_2} + a_0 a_2 a_k - a_0 b_2 a_{k+1} + a_1 a_2 a_{k+1} + a_2}{\left(k+2+\frac{a_1 b_2 + a_2^2 - a_2 b_1}{a_2^2}\right)\left(-a_2^2 k - 2a_2^2\right)}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x + \frac{b_2}{a_2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + \frac{b_2}{a_2}\right)^{k+\frac{a_1 b_2 + a_2^2 - a_2 b_1}{a_2^2}}\right), a_{k+2} = \frac{a_1 a_2 k a_{1+k} + a_0 a_2 a_k - a_0 b_2 a_{1+k} + a_1 a}{(k+2)\left(-a_2^2 k + a_1 b_2 - a_2^2 -\right.}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 248

```
dsolve((a__2*x+b__2)*diff(y(x),x$2)+(a__1*x+b__1)*diff(y(x),x)+(a__0*x+b__0)*y(x)=0,y(x), si
```

$$y(x) = (a_2x + b_2)^{\frac{a_1b_2+a_2^2-a_2b_1}{a_2^2}} e^{-\frac{\left(\sqrt{-4a_0a_2+a_1^2}+a_1\right)x}{2a_2}} \left(\text{KummerM}\left(\frac{(a_1b_2 + 2a_2^2 - a_2b_1)\sqrt{-4a_0a_2 + a_1^2} - 2a_2^2b_0 + (2a_0b_2 -}{2\sqrt{-4a_0a_2 + a_1^2}\,a_2^2}\right.\right.$$

$$+\text{KummerU}\left(\frac{(a_1b_2 + 2a_2^2 - a_2b_1)\sqrt{-4a_0a_2 + a_1^2} - 2a_2^2b_0 + (2a_0b_2 + b_1a_1)a_2 - a_1^2b_2}{2\sqrt{-4a_0a_2 + a_1^2}\,a_2^2}, \frac{a_1b_2 + 2a_2^2 - a_2b}{a_2^2}\right.$$

✓ Solution by Mathematica

Time used: 0.526 (sec). Leaf size: 301

```
DSolve[(a2*x+b2)*y''[x]+(a1*x+b1)*y'[x]+(a0*x+b0)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to e^{-\frac{x\left(\sqrt{\mathrm{a1}^2-4\mathrm{a0a2}}+\mathrm{a1}\right)}{2\mathrm{a2}}}(\mathrm{a2}x + \mathrm{b2})^{\frac{\mathrm{a1b2}+\mathrm{a2}^2-\mathrm{a2b1}}{\mathrm{a2}^2}}\left(c_1 \,\mathrm{HypergeometricU}\left(\frac{2\left(\sqrt{\mathrm{a1}^2-4\mathrm{a0a2}}-\mathrm{b0}\right)\mathrm{a2}^2+\left(\mathrm{a1b1}-\sqrt{\mathrm{a1}^2-4\mathrm{a0a2}}\mathrm{b1}+2\right.}{2\mathrm{a2}^2\sqrt{\mathrm{a1}^2-4\mathrm{a0a2}}}\right.\right.$$

$$\left.-\frac{\mathrm{b1}}{\mathrm{a2}}+\frac{\mathrm{a1b2}}{\mathrm{a2}^2}+2,\frac{\sqrt{\mathrm{a1}^2-4\mathrm{a0a2}}(\mathrm{b2}+\mathrm{a2}x)}{\mathrm{a2}^2}\right)$$

$$\left.+c_2 L^{\frac{\mathrm{a2}^2-\mathrm{b1a2}+\mathrm{a1b2}}{\mathrm{a2}^2}}_{\frac{-2\left(\sqrt{\mathrm{a1}^2-4\mathrm{a0a2}}-\mathrm{b0}\right)\mathrm{a2}^2+\left(-\mathrm{a1b1}+\sqrt{\mathrm{a1}^2-4\mathrm{a0a2}}\mathrm{b1}-2\mathrm{a0b2}\right)\mathrm{a2}+\mathrm{a1}\left(\mathrm{a1}-\sqrt{\mathrm{a1}^2-4\mathrm{a0a2}}\right)\mathrm{b2}}{2\mathrm{a2}^2\sqrt{\mathrm{a1}^2-4\mathrm{a0a2}}}}\left(\frac{\sqrt{\mathrm{a1}^2-4\mathrm{a0a2}}(\mathrm{b2}+\mathrm{a2}x)}{\mathrm{a2}^2}\right)\right)$$

28.49 problem 109

Internal problem ID [10933]
Internal file name [`OUTPUT/10189_Sunday_December_31_2023_11_03_49_AM_24274775/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-3 Equation of form $(ax+b)y''+f(x)y'+g(x)y=0$
Problem number: 109.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{(x+\gamma)\,y''+(a\,x^n+b\,x^m+c)\,y'+\left(an\,x^{n-1}+x^{m-1}bm\right)y=0}$$

28.49.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int\left((x+\gamma)\,y''+(a\,x^n+b\,x^m+c)\,y'+\frac{(x^n na+b\,x^m m)\,y}{x}\right)dx=0$$
$$\frac{(ax\,x^n+x^m bx+cx-x)\,y}{x}+(x+\gamma)\,y'=c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x) = -\frac{-a\,x^n - b\,x^m - c + 1}{x+\gamma}$$
$$q(x) = \frac{c_1}{x+\gamma}$$

Hence the ode is

$$y' - \frac{(-a\,x^n - b\,x^m - c + 1)\,y}{x+\gamma} = \frac{c_1}{x+\gamma}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{-a\,x^n - b\,x^m - c+1}{x+\gamma}dx}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{c_1}{x+\gamma}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\int -\frac{-a\,x^n - b\,x^m - c+1}{x+\gamma}dx} y\right) = \left(\mathrm{e}^{\int -\frac{-a\,x^n - b\,x^m - c+1}{x+\gamma}dx}\right)\left(\frac{c_1}{x+\gamma}\right)$$
$$\mathrm{d}\left(\mathrm{e}^{\int -\frac{-a\,x^n - b\,x^m - c+1}{x+\gamma}dx} y\right) = \left(\frac{c_1 \mathrm{e}^{\int \frac{a\,x^n + b\,x^m + c-1}{x+\gamma}dx}}{x+\gamma}\right)\mathrm{d}x$$

Integrating gives

$$\mathrm{e}^{\int -\frac{-a\,x^n - b\,x^m - c+1}{x+\gamma}dx} y = \int \frac{c_1 \mathrm{e}^{\int \frac{a\,x^n + b\,x^m + c-1}{x+\gamma}dx}}{x+\gamma}\,\mathrm{d}x$$
$$\mathrm{e}^{\int -\frac{-a\,x^n - b\,x^m - c+1}{x+\gamma}dx} y = \int \frac{c_1 \mathrm{e}^{\int \frac{a\,x^n + b\,x^m + c-1}{x+\gamma}dx}}{x+\gamma}dx + c_2$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{-a\,x^n - b\,x^m - c+1}{x+\gamma}dx}$ results in

$$y = \mathrm{e}^{-\left(\int \frac{a\,x^n + b\,x^m + c-1}{x+\gamma}dx\right)}\left(\int \frac{c_1 \mathrm{e}^{\int \frac{a\,x^n + b\,x^m + c-1}{x+\gamma}dx}}{x+\gamma}dx\right) + c_2 \mathrm{e}^{-\left(\int \frac{a\,x^n + b\,x^m + c-1}{x+\gamma}dx\right)}$$

which simplifies to

$$y = \mathrm{e}^{-\left(\int \frac{a\,x^n + b\,x^m + c-1}{x+\gamma}dx\right)}\left(c_1\left(\int \frac{\mathrm{e}^{\int \frac{a\,x^n + b\,x^m + c-1}{x+\gamma}dx}}{x+\gamma}dx\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\left(\int \frac{a\,x^n + b\,x^m + c-1}{x+\gamma}dx\right)}\left(c_1\left(\int \frac{\mathrm{e}^{\int \frac{a\,x^n + b\,x^m + c-1}{x+\gamma}dx}}{x+\gamma}dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\left(\int \frac{a\,x^n + b\,x^m + c - 1}{x+\gamma} dx\right)} \left(c_1 \left(\int \frac{\mathrm{e}^{\int \frac{a\,x^n + b\,x^m + c - 1}{x+\gamma} dx}}{x+\gamma} dx\right) + c_2\right)$$

Verified OK.

28.49.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(x+\gamma)\,y'' + (a\,x^n + b\,x^m + c)\,y' + \frac{(x^n n a + b\,x^m m)\,y}{x} = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left((x+\gamma)\,y'' + (a\,x^n + b\,x^m + c)\,y' + \frac{(x^n n a + b\,x^m m)\,y}{x}\right) dx = 0$$
$$\frac{(ax\,x^n + x^m bx + cx - x)\,y}{x} + (x+\gamma)\,y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-a\,x^n - b\,x^m - c + 1}{x+\gamma}$$
$$q(x) = \frac{c_1}{x+\gamma}$$

Hence the ode is

$$y' - \frac{(-a\,x^n - b\,x^m - c + 1)\,y}{x+\gamma} = \frac{c_1}{x+\gamma}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{-a\,x^n - b\,x^m - c + 1}{x+\gamma} dx}$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x+\gamma}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\int -\frac{-a\,x^n-b\,x^m-c+1}{x+\gamma}dx} y\right) &= \left(\mathrm{e}^{\int -\frac{-a\,x^n-b\,x^m-c+1}{x+\gamma}dx}\right)\left(\frac{c_1}{x+\gamma}\right) \\
\mathrm{d}\left(\mathrm{e}^{\int -\frac{-a\,x^n-b\,x^m-c+1}{x+\gamma}dx} y\right) &= \left(\frac{c_1 \mathrm{e}^{\int \frac{a\,x^n+b\,x^m+c-1}{x+\gamma}dx}}{x+\gamma}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\int -\frac{-a\,x^n-b\,x^m-c+1}{x+\gamma}dx} y &= \int \frac{c_1 \mathrm{e}^{\int \frac{a\,x^n+b\,x^m+c-1}{x+\gamma}dx}}{x+\gamma}\,\mathrm{d}x \\
\mathrm{e}^{\int -\frac{-a\,x^n-b\,x^m-c+1}{x+\gamma}dx} y &= \int \frac{c_1 \mathrm{e}^{\int \frac{a\,x^n+b\,x^m+c-1}{x+\gamma}dx}}{x+\gamma}dx + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{-a\,x^n-b\,x^m-c+1}{x+\gamma}dx}$ results in

$$
y = \mathrm{e}^{-\left(\int \frac{a\,x^n+b\,x^m+c-1}{x+\gamma}dx\right)}\left(\int \frac{c_1 \mathrm{e}^{\int \frac{a\,x^n+b\,x^m+c-1}{x+\gamma}dx}}{x+\gamma}dx\right) + c_2 \mathrm{e}^{-\left(\int \frac{a\,x^n+b\,x^m+c-1}{x+\gamma}dx\right)}
$$

which simplifies to

$$
y = \mathrm{e}^{-\left(\int \frac{a\,x^n+b\,x^m+c-1}{x+\gamma}dx\right)}\left(c_1\left(\int \frac{\mathrm{e}^{\int \frac{a\,x^n+b\,x^m+c-1}{x+\gamma}dx}}{x+\gamma}dx\right) + c_2\right)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-\left(\int \frac{a\,x^n+b\,x^m+c-1}{x+\gamma}dx\right)}\left(c_1\left(\int \frac{\mathrm{e}^{\int \frac{a\,x^n+b\,x^m+c-1}{x+\gamma}dx}}{x+\gamma}dx\right) + c_2\right) \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{-\left(\int \frac{a\,x^n+b\,x^m+c-1}{x+\gamma}dx\right)}\left(c_1\left(\int \frac{\mathrm{e}^{\int \frac{a\,x^n+b\,x^m+c-1}{x+\gamma}dx}}{x+\gamma}dx\right) + c_2\right)
$$

Verified OK.

28.49.3 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x + \gamma \\ q(x) &= a\, x^{n} + b\, x^{m} + c \\ r(x) &= \frac{x^{n} n a + b\, x^{m} m}{x} \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= \frac{a n\, x^{n}}{x} + \frac{b\, x^{m} m}{x} \end{aligned}$$

Therefore (1) becomes

$$0 - \left(\frac{a n\, x^{n}}{x} + \frac{b\, x^{m} m}{x}\right) + \left(\frac{x^{n} n a + b\, x^{m} m}{x}\right) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\, y' + \left(q(x) - p'(x)\right) y\right)' = s(x)$$

Integrating gives

$$p(x)\, y' + \left(q(x) - p'(x)\right) y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$(x + \gamma)\, y' + \left(a\, x^{n} + b\, x^{m} + c - 1\right) y = c_1$$

We now have a first order ode to solve which is

$$(x + \gamma)\, y' + \left(a\, x^{n} + b\, x^{m} + c - 1\right) y = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{-a\,x^{n} - b\,x^{m} - c + 1}{x + \gamma}\\ q(x) &= \frac{c_1}{x + \gamma}\end{aligned}$$

Hence the ode is

$$y' - \frac{\left(-a\,x^{n} - b\,x^{m} - c + 1\right) y}{x + \gamma} = \frac{c_1}{x + \gamma}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{-a\,x^{n} - b\,x^{m} - c + 1}{x + \gamma}dx}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x + \gamma}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\int -\frac{-a\,x^{n} - b\,x^{m} - c + 1}{x + \gamma}dx} y\right) &= \left(\mathrm{e}^{\int -\frac{-a\,x^{n} - b\,x^{m} - c + 1}{x + \gamma}dx}\right)\left(\frac{c_1}{x + \gamma}\right)\\ \mathrm{d}\left(\mathrm{e}^{\int -\frac{-a\,x^{n} - b\,x^{m} - c + 1}{x + \gamma}dx} y\right) &= \left(\frac{c_1 \mathrm{e}^{\int \frac{a\,x^{n} + b\,x^{m} + c - 1}{x + \gamma}dx}}{x + \gamma}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{\int -\frac{-a\,x^{n} - b\,x^{m} - c + 1}{x + \gamma}dx} y &= \int \frac{c_1 \mathrm{e}^{\int \frac{a\,x^{n} + b\,x^{m} + c - 1}{x + \gamma}dx}}{x + \gamma}\,\mathrm{d}x\\ \mathrm{e}^{\int -\frac{-a\,x^{n} - b\,x^{m} - c + 1}{x + \gamma}dx} y &= \int \frac{c_1 \mathrm{e}^{\int \frac{a\,x^{n} + b\,x^{m} + c - 1}{x + \gamma}dx}}{x + \gamma}dx + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{-a\,x^{n} - b\,x^{m} - c + 1}{x + \gamma}dx}$ results in

$$y = \mathrm{e}^{-\left(\int \frac{a\,x^{n} + b\,x^{m} + c - 1}{x + \gamma}dx\right)}\left(\int \frac{c_1 \mathrm{e}^{\int \frac{a\,x^{n} + b\,x^{m} + c - 1}{x + \gamma}dx}}{x + \gamma}dx\right) + c_2 \mathrm{e}^{-\left(\int \frac{a\,x^{n} + b\,x^{m} + c - 1}{x + \gamma}dx\right)}$$

which simplifies to

$$y = \mathrm{e}^{-\left(\int \frac{a\,x^{n} + b\,x^{m} + c - 1}{x + \gamma}dx\right)}\left(c_1\left(\int \frac{\mathrm{e}^{\int \frac{a\,x^{n} + b\,x^{m} + c - 1}{x + \gamma}dx}}{x + \gamma}dx\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\left(\int \frac{a\,x^{n}+b\,x^{m}+c-1}{x+\gamma}dx\right)} \left(c_1 \left(\int \frac{\mathrm{e}^{\int \frac{a\,x^{n}+b\,x^{m}+c-1}{x+\gamma}dx}}{x+\gamma}dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\left(\int \frac{a\,x^{n}+b\,x^{m}+c-1}{x+\gamma}dx\right)} \left(c_1 \left(\int \frac{\mathrm{e}^{\int \frac{a\,x^{n}+b\,x^{m}+c-1}{x+\gamma}dx}}{x+\gamma}dx\right) + c_2\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 63

```
dsolve((x+gamma)*diff(y(x),x$2)+(a*x^n+b*x^m+c)*diff(y(x),x)+(a*n*x^(n-1)+b*m*x^(m-1))*y(x)=
```

$$y(x) = \left(c_1 \left(\int \frac{\mathrm{e}^{\int \frac{a\,x^{n}+b\,x^{m}+c-1}{x+\gamma}dx}}{x+\gamma}dx\right) + c_2\right) \mathrm{e}^{-\left(\int \frac{a\,x^{n}+b\,x^{m}+c-1}{x+\gamma}dx\right)}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(x+\[Gamma])*y''[x]+(a*x^n+b*x^m+c)*y'[x]+(a*n*x^(n-1)+b*m*x^(m-1))*y[x]==0,y[x],x,In
```

Not solved

29 Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$

29.1 problem 110

Internal problem ID [10934]
Internal file name [`OUTPUT/10190_Sunday_December_31_2023_11_03_51_AM_72514253/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 110.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + ay = 0}$$

29.1.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 0rx^{r-1} + a\,x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 0\,x^r + a\,x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 0 + a = 0$$

Or

$$r^2 + a - r = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{2} - \frac{\sqrt{1-4a}}{2} \\
r_2 &= \frac{1}{2} + \frac{\sqrt{1-4a}}{2}
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$
y = c_1 y_1 + c_2 y_2
$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$
y = c_1 x^{\frac{1}{2} - \frac{\sqrt{1-4a}}{2}} + c_2 x^{\frac{1}{2} + \frac{\sqrt{1-4a}}{2}}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{2} - \frac{\sqrt{1-4a}}{2}} + c_2 x^{\frac{1}{2} + \frac{\sqrt{1-4a}}{2}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{1}{2} - \frac{\sqrt{1-4a}}{2}} + c_2 x^{\frac{1}{2} + \frac{\sqrt{1-4a}}{2}}
$$

Verified OK.

29.1.2 Solving using Kovacic algorithm

Writing the ode as

$$
x^2 y'' + a y = 0 \tag{1}
$$

$$
A y'' + B y' + C y = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= 0 \\
C &= a
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -a \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{a}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 113: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{a}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -a$. Hence

$$\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \left\{2, 2 - 2\sqrt{1 - 4a}, 2 + 2\sqrt{1 - 4a}\right\}
\end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{a}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -1$. Hence

$$\begin{aligned}
E_\infty &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{2\}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{2, 2-2\sqrt{1-4a}, 2+2\sqrt{1-4a}\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 2,\; e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2-(2)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{2}{(x-(0))}\right) \\ &= \frac{1}{x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{x} + \frac{a}{x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + \sqrt{1 - 4a}}{2x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+\sqrt{1-4a}}{2x} dx} \\ &= x^{\frac{1}{2}+\frac{\sqrt{1-4a}}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= x^{\frac{1}{2}+\frac{\sqrt{1-4a}}{2}}\end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{2}+\frac{\sqrt{1-4a}}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= x^{\frac{1}{2}+\frac{\sqrt{1-4a}}{2}} \int \frac{1}{x^{1+\sqrt{1-4a}}} \, dx \\
&= x^{\frac{1}{2}+\frac{\sqrt{1-4a}}{2}} \left(-\frac{x^{-\sqrt{1-4a}}}{\sqrt{1-4a}} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{\frac{1}{2}+\frac{\sqrt{1-4a}}{2}} \right) + c_2 \left(x^{\frac{1}{2}+\frac{\sqrt{1-4a}}{2}} \left(-\frac{x^{-\sqrt{1-4a}}}{\sqrt{1-4a}} \right) \right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{2}+\frac{\sqrt{1-4a}}{2}} - \frac{c_2 x^{\frac{1}{2}-\frac{\sqrt{1-4a}}{2}}}{\sqrt{1-4a}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{2}+\frac{\sqrt{1-4a}}{2}} - \frac{c_2 x^{\frac{1}{2}-\frac{\sqrt{1-4a}}{2}}}{\sqrt{1-4a}}$$

Verified OK.

29.1.3 Maple step by step solution

Let's solve

$y''x^2 + ay = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{ay}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{ay}{x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=0, P_3(x)=\frac{a}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=a$

  ◦ $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $y''x^2+ay=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite DE with series expansions

  ◦ Convert $x^2\cdot y''$ to series expansion

    $x^2\cdot y''=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

    Rewrite DE with series expansions

    $\sum_{k=0}^{\infty}a_k(k^2+2kr+r^2+a-k-r)\,x^{k+r}=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k^2+a-k)=0$

- Recursion relation that defines series solution to ODE

$a_k = 0$

- Recursion relation for $r = 0$

  $a_k = 0$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_k = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(x^2*diff(y(x),x$2)+a*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x}\left(c_1 x^{\frac{\sqrt{-4a+1}}{2}} + c_2 x^{-\frac{\sqrt{-4a+1}}{2}}\right)$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 42

```
DSolve[x^2*y''[x]+a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^{\frac{1}{2}-\frac{1}{2}\sqrt{1-4a}}\left(c_2 x^{\sqrt{1-4a}} + c_1\right)$$

29.2 problem 111

Internal problem ID [10935]
Internal file name [`OUTPUT/10191_Sunday_December_31_2023_11_03_52_AM_93987930/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 111.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + (ax + b)\,y = 0}$$

29.2.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + (ax + b)\,y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \,\text{BesselJ}\,(n, \beta\, x^{\gamma}) + c_2 \,\text{BesselY}\,(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2\sqrt{a}$$
$$n = \sqrt{-4b+1}$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\sqrt{-4b+1}, 2\sqrt{a}\,\sqrt{x}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\sqrt{-4b+1}, 2\sqrt{a}\,\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\sqrt{-4b+1}, 2\sqrt{a}\,\sqrt{x}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\sqrt{-4b+1}, 2\sqrt{a}\,\sqrt{x}\right) \quad (1)$$

Verification of solutions

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\sqrt{-4b+1}, 2\sqrt{a}\,\sqrt{x}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\sqrt{-4b+1}, 2\sqrt{a}\,\sqrt{x}\right)$$

Verified OK.

29.2.2 Maple step by step solution

Let's solve

$y''x^2 + (ax+b)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(ax+b)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(ax+b)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{ax+b}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = b$$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$y''x^2 + (ax + b)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (r^2 + b - r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k^2 + 2kr + r^2 + b - k - r) + a_{k-1} a\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 + b - r = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{2} - \frac{\sqrt{-4b+1}}{2}, \frac{\sqrt{-4b+1}}{2} + \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2 + (2r - 1)\, k + r^2 + b - r\right) a_k + a_{k-1} a = 0$$

- Shift index using $k- > k+1$

$$\left(\left(k+1\right)^{2}+\left(2r-1\right)\left(k+1\right)+r^{2}+b-r\right)a_{k+1}+a_{k}a=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{a_{k}a}{k^{2}+2kr+r^{2}+b+k+r}$$

- Recursion relation for $r=\frac{1}{2}-\frac{\sqrt{-4b+1}}{2}$

$$a_{k+1}=-\frac{a_{k}a}{k^{2}+2k\left(\frac{1}{2}-\frac{\sqrt{-4b+1}}{2}\right)+\left(\frac{1}{2}-\frac{\sqrt{-4b+1}}{2}\right)^{2}+b+k+\frac{1}{2}-\frac{\sqrt{-4b+1}}{2}}$$

- Solution for $r=\frac{1}{2}-\frac{\sqrt{-4b+1}}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_{k}x^{k+\frac{1}{2}-\frac{\sqrt{-4b+1}}{2}},a_{k+1}=-\frac{a_{k}a}{k^{2}+2k\left(\frac{1}{2}-\frac{\sqrt{-4b+1}}{2}\right)+\left(\frac{1}{2}-\frac{\sqrt{-4b+1}}{2}\right)^{2}+b+k+\frac{1}{2}-\frac{\sqrt{-4b+1}}{2}}\right]$$

- Recursion relation for $r=\frac{\sqrt{-4b+1}}{2}+\frac{1}{2}$

$$a_{k+1}=-\frac{a_{k}a}{k^{2}+2k\left(\frac{\sqrt{-4b+1}}{2}+\frac{1}{2}\right)+\left(\frac{\sqrt{-4b+1}}{2}+\frac{1}{2}\right)^{2}+b+k+\frac{\sqrt{-4b+1}}{2}+\frac{1}{2}}$$

- Solution for $r=\frac{\sqrt{-4b+1}}{2}+\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_{k}x^{k+\frac{\sqrt{-4b+1}}{2}+\frac{1}{2}},a_{k+1}=-\frac{a_{k}a}{k^{2}+2k\left(\frac{\sqrt{-4b+1}}{2}+\frac{1}{2}\right)+\left(\frac{\sqrt{-4b+1}}{2}+\frac{1}{2}\right)^{2}+b+k+\frac{\sqrt{-4b+1}}{2}+\frac{1}{2}}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}c_{k}x^{k+\frac{1}{2}-\frac{\sqrt{-4b+1}}{2}}\right)+\left(\sum_{k=0}^{\infty}d_{k}x^{k+\frac{\sqrt{-4b+1}}{2}+\frac{1}{2}}\right),c_{1+k}=-\frac{c_{k}a}{k^{2}+2k\left(\frac{1}{2}-\frac{\sqrt{-4b+1}}{2}\right)+\left(\frac{1}{2}-\frac{\sqrt{-4b+1}}{2}\right)^{2}+b+k+$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(x^2*diff(y(x),x$2)+(a*x+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\text{BesselJ}\left(\sqrt{-4b+1}, 2\sqrt{x}\,\sqrt{a}\right)c_1 + \text{BesselY}\left(\sqrt{-4b+1}, 2\sqrt{x}\,\sqrt{a}\right)c_2\right)\sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.147 (sec). Leaf size: 95

```
DSolve[x^2*y''[x]+(a*x+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{a}\sqrt{x}\Big(c_1 \text{Gamma}\left(1-\sqrt{1-4b}\right)\text{BesselJ}\left(-\sqrt{1-4b}, 2\sqrt{a}\sqrt{x}\right) + c_2 \text{Gamma}\left(\sqrt{1-4b}+1\right)\text{BesselJ}\left(\sqrt{1-4b}, 2\sqrt{a}\sqrt{x}\right)\Big)$$

29.3 problem 112

Internal problem ID [10936]
Internal file name [`OUTPUT/10192_Sunday_December_31_2023_11_03_53_AM_39922936/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 112.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(a^2x^2 - n(n+1)\right) y = 0}$$

29.3.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(a^2x^2 - n^2 - n\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= a \\
n &= -n - \frac{1}{2} \\
\gamma &= 1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\ \mathrm{BesselJ}\left(-n - \frac{1}{2}, ax\right) + c_2\sqrt{x}\ \mathrm{BesselY}\left(-n - \frac{1}{2}, ax\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\ \mathrm{BesselJ}\left(-n - \frac{1}{2}, ax\right) + c_2\sqrt{x}\ \mathrm{BesselY}\left(-n - \frac{1}{2}, ax\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\ \mathrm{BesselJ}\left(-n - \frac{1}{2}, ax\right) + c_2\sqrt{x}\ \mathrm{BesselY}\left(-n - \frac{1}{2}, ax\right)$$

Verified OK.

29.3.2 Maple step by step solution

Let's solve

$y''x^2 + (a^2x^2 - n^2 - n)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(a^2x^2-n^2-n)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(a^2x^2-n^2-n)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = 0, P_3(x) = \frac{a^2x^2-n^2-n}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -n^2 - n$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2 + (a^2x^2 - n^2 - n)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(r+n)\,(r-1-n)\,x^r + a_1(r+n+1)\,(r-n)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(r+n+k)\,(r-1-n+k) + a_k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(r+n)\,(r-1-n) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-n, n+1\}$

- Each term must be 0

  $a_1(r+n+1)(r-n)=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(r+n+k)(r-1-n+k)+a_{k-2}a^2=0$
- Shift index using $k->k+2$

  $a_{k+2}(r+n+k+2)(r+1-n+k)+a_k a^2=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k a^2}{(r+n+k+2)(r+1-n+k)}$
- Recursion relation for $r=-n$

  $a_{k+2}=-\frac{a_k a^2}{(k+2)(-2n+1+k)}$
- Solution for $r=-n$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-n}, a_{k+2}=-\frac{a_k a^2}{(k+2)(-2n+1+k)}, a_1=0\right]$
- Recursion relation for $r=n+1$

  $a_{k+2}=-\frac{a_k a^2}{(2n+3+k)(k+2)}$
- Solution for $r=n+1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+n+1}, a_{k+2}=-\frac{a_k a^2}{(2n+3+k)(k+2)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} b_k x^{k-n}\right)+\left(\sum_{k=0}^{\infty} c_k x^{k+n+1}\right), b_{k+2}=-\frac{b_k a^2}{(k+2)(-2n+1+k)}, b_1=0, c_{k+2}=-\frac{c_k a^2}{(2n+3+k)(k+2)}, c_1\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)+(a^2*x^2-n*(n+1))*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\text{BesselJ}\left(n+\frac{1}{2}, ax\right)c_1 + \text{BesselY}\left(n+\frac{1}{2}, ax\right)c_2\right)\sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 36

```
DSolve[x^2*y''[x]+(a^2*x^2-n*(n+1))*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x}\left(c_1 \text{BesselJ}\left(n+\frac{1}{2}, ax\right) + c_2 \text{BesselY}\left(n+\frac{1}{2}, ax\right)\right)$$

29.4 problem 113

Internal problem ID [10937]
Internal file name [`OUTPUT/10193_Sunday_December_31_2023_11_03_54_AM_32349464/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 113.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - \left(a^2x^2 + n(n+1)\right) y = 0}$$

29.4.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(-a^2x^2 - n^2 - n\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= \frac{1}{2} \\ \beta &= ia \\ n &= -n - \frac{1}{2} \\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(-n - \frac{1}{2}, iax\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(-n - \frac{1}{2}, iax\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(-n - \frac{1}{2}, iax\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(-n - \frac{1}{2}, iax\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(-n - \frac{1}{2}, iax\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(-n - \frac{1}{2}, iax\right)$$

Verified OK.

29.4.2 Maple step by step solution

Let's solve

$y''x^2 + (-a^2x^2 - n^2 - n)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(a^2x^2+n^2+n)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(a^2x^2+n^2+n)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = 0, P_3(x) = -\frac{a^2x^2+n^2+n}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -n^2 - n$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2 + (-a^2x^2 - n^2 - n)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(r+n)\,(r-1-n)\, x^r + a_1(r+n+1)\,(r-n)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(r+n+k)\,(r-1-n+k) - a_k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(r+n)\,(r-1-n) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-n, n+1\}$

- Each term must be 0

  $a_1(r+n+1)(r-n)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(r+n+k)(r-1-n+k)-a_{k-2}a^2=0$

- Shift index using $k->k+2$

  $a_{k+2}(r+n+k+2)(r+1-n+k)-a_k a^2=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_k a^2}{(r+n+k+2)(r+1-n+k)}$

- Recursion relation for $r=-n$

  $a_{k+2}=\frac{a_k a^2}{(k+2)(-2n+1+k)}$

- Solution for $r=-n$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-n}, a_{k+2}=\frac{a_k a^2}{(k+2)(-2n+1+k)}, a_1=0\right]$

- Recursion relation for $r=n+1$

  $a_{k+2}=\frac{a_k a^2}{(2n+3+k)(k+2)}$

- Solution for $r=n+1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+n+1}, a_{k+2}=\frac{a_k a^2}{(2n+3+k)(k+2)}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} b_k x^{k-n}\right)+\left(\sum_{k=0}^{\infty} c_k x^{k+n+1}\right), b_{k+2}=\frac{b_k a^2}{(k+2)(-2n+1+k)}, b_1=0, c_{k+2}=\frac{c_k a^2}{(2n+3+k)(k+2)}, c_1=0\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(x^2*diff(y(x),x$2)-(a^2*x^2+n*(n+1))*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x}\left(\text{BesselJ}\left(n+\frac{1}{2}, \sqrt{-a^2}\,x\right)c_1 + \text{BesselY}\left(n+\frac{1}{2}, \sqrt{-a^2}\,x\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 42

```
DSolve[x^2*y''[x]-(a^2*x^2+n*(n+1))*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x}\left(c_1\,\text{BesselJ}\left(n+\frac{1}{2}, -iax\right) + c_2\,\text{BesselY}\left(n+\frac{1}{2}, -iax\right)\right)$$

29.5 problem 114

Internal problem ID [10938]
Internal file name [`OUTPUT/10194_Sunday_December_31_2023_11_03_55_AM_29510659/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 114.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - \left(a^2x^2 + 2abx + b^2 - b\right) y = 0}$$

29.5.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(-a^2x^2 - 2abx - b^2 + b\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = 1 - 2b$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\ \text{BesselJ}\left(1-2b, 2\sqrt{x}\right) + c_2\sqrt{x}\ \text{BesselY}\left(1-2b, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\ \text{BesselJ}\left(1-2b, 2\sqrt{x}\right) + c_2\sqrt{x}\ \text{BesselY}\left(1-2b, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\ \text{BesselJ}\left(1-2b, 2\sqrt{x}\right) + c_2\sqrt{x}\ \text{BesselY}\left(1-2b, 2\sqrt{x}\right)$$

Verified OK.

29.5.2 Maple step by step solution

Let's solve

$y''x^2 + (-a^2x^2 - 2abx - b^2 + b)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(a^2x^2+2abx+b^2-b)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(a^2x^2+2abx+b^2-b)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{a^2x^2+2abx+b^2-b}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -b^2 + b$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x^2 + \left(-a^2x^2 - 2abx - b^2 + b\right) y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- > k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0(r - 1 + b)(-r + b)\, x^r + \left(-a_1(r + b)(-r - 1 + b) - 2a_0ab\right) x^{1+r} + \left(\sum_{k=2}^{\infty} (-a_k(r - 1 + k + b\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-(r - 1 + b)(-r + b) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{b, -b + 1\}$$

- Each term must be 0

  $$-a_1(r + b)(-r - 1 + b) - 2a_0ab = 0$$

- Solve for the dependent coefficient(s)

  $a_1 = -\frac{2a_0ab}{b^2-r^2-b-r}$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(r-1+k+b)\,(-r-k+b)-2a_{k-1}ab-a_{k-2}a^2 = 0$

- Shift index using $k- >k+2$

  $-a_{k+2}(r+1+k+b)\,(-r-k-2+b)-2a_{k+1}ab-a_k a^2 = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a(aa_k+2ba_{k+1})}{(r+1+k+b)(-r-k-2+b)}$

- Recursion relation for $r = b$

  $a_{k+2} = -\frac{a(aa_k+2ba_{k+1})}{(2b+1+k)(-k-2)}$

- Solution for $r = b$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+b}, a_{k+2} = -\frac{a(aa_k+2ba_{k+1})}{(2b+1+k)(-k-2)}, a_1 = aa_0\right]$

- Recursion relation for $r = -b+1$

  $a_{k+2} = -\frac{a(aa_k+2ba_{k+1})}{(k+2)(2b-3-k)}$

- Solution for $r = -b+1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-b+1}, a_{k+2} = -\frac{a(aa_k+2ba_{k+1})}{(k+2)(2b-3-k)}, a_1 = -\frac{2a_0ab}{b^2-(-b+1)^2-1}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} c_k x^{k+b}\right) + \left(\sum_{k=0}^{\infty} d_k x^{k-b+1}\right), c_{k+2} = -\frac{a(ac_k+2bc_{1+k})}{(2b+1+k)(-k-2)}, c_1 = ac_0, d_{k+2} = -\frac{a(ad_k+2bd_{1+k})}{(k+2)(2b-3-k)}, d\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 29

```
dsolve(x^2*diff(y(x),x$2)-(a^2*x^2+2*a*b*x+b^2-b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^b e^{ax} + c_2 \operatorname{WhittakerM}\left(-b, \frac{1}{2} - b, 2ax\right)$$

✓ Solution by Mathematica
Time used: 0.055 (sec). Leaf size: 38

```
DSolve[x^2*y''[x]-(a^2*x^2+2*a*b*x+b^2-b)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 M_{-b,b-\frac{1}{2}}(2ax) + c_2 W_{-b,b-\frac{1}{2}}(2ax)$$

29.6 problem 115

Internal problem ID [10939]
Internal file name [`OUTPUT/10195_Sunday_December_31_2023_11_04_23_AM_68992916/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 115.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(a\,x^2 + bx + c\right) y = 0}$$

29.6.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(a\,x^2 + bx + c\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = \sqrt{-4c+1}$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\ \mathrm{BesselJ}\left(\sqrt{-4c+1}, 2\sqrt{x}\right) + c_2\sqrt{x}\ \mathrm{BesselY}\left(\sqrt{-4c+1}, 2\sqrt{x}\right)$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x}\ \mathrm{BesselJ}\left(\sqrt{-4c+1}, 2\sqrt{x}\right) + c_2\sqrt{x}\ \mathrm{BesselY}\left(\sqrt{-4c+1}, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\ \mathrm{BesselJ}\left(\sqrt{-4c+1}, 2\sqrt{x}\right) + c_2\sqrt{x}\ \mathrm{BesselY}\left(\sqrt{-4c+1}, 2\sqrt{x}\right)$$

Verified OK.

29.6.2 Maple step by step solution

Let's solve

$y''x^2 + (a\,x^2 + bx + c)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(a\,x^2+bx+c)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(a\,x^2+bx+c)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{a\,x^2+bx+c}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\Bigg|_{x=0} = 0$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = c$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2 + (a\,x^2 + bx + c)\,y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(r^2 + c - r)\,x^r + \left((r^2 + c + r)\,a_1 + a_0 b\right)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k^2 + 2kr + r^2 + c - k - r) + b a_{k-1} +\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 + c - r = 0$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{2} - \frac{\sqrt{-4c+1}}{2}, \frac{\sqrt{-4c+1}}{2} + \frac{1}{2}\right\}$$

- Each term must be 0

  $(r^2 + c + r)\,a_1 + a_0 b = 0$

- Solve for the dependent coefficient(s)

$a_1 = -\frac{a_0 b}{r^2+c+r}$

- Each term in the series must be 0, giving the recursion relation

$(k^2 + (2r-1)k + r^2 + c - r)\, a_k + a_{k-2} a + b a_{k-1} = 0$

- Shift index using $k- > k+2$

$\left((k+2)^2 + (2r-1)(k+2) + r^2 + c - r\right) a_{k+2} + a_k a + b a_{k+1} = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k a + b a_{k+1}}{k^2+2kr+r^2+c+3k+3r+2}$

- Recursion relation for $r = \frac{1}{2} - \frac{\sqrt{-4c+1}}{2}$

$a_{k+2} = -\frac{a_k a + b a_{k+1}}{k^2+2k\left(\frac{1}{2}-\frac{\sqrt{-4c+1}}{2}\right)+\left(\frac{1}{2}-\frac{\sqrt{-4c+1}}{2}\right)^2+c+3k+\frac{7}{2}-\frac{3\sqrt{-4c+1}}{2}}$

- Solution for $r = \frac{1}{2} - \frac{\sqrt{-4c+1}}{2}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}-\frac{\sqrt{-4c+1}}{2}}, a_{k+2} = -\frac{a_k a + b a_{k+1}}{k^2+2k\left(\frac{1}{2}-\frac{\sqrt{-4c+1}}{2}\right)+\left(\frac{1}{2}-\frac{\sqrt{-4c+1}}{2}\right)^2+c+3k+\frac{7}{2}-\frac{3\sqrt{-4c+1}}{2}}, a_1 = -\frac{}{\left(\frac{1}{2}-\frac{\sqrt{-4c+}}{2}\right.}\right.$

- Recursion relation for $r = \frac{\sqrt{-4c+1}}{2} + \frac{1}{2}$

$a_{k+2} = -\frac{a_k a + b a_{k+1}}{k^2+2k\left(\frac{\sqrt{-4c+1}}{2}+\frac{1}{2}\right)+\left(\frac{\sqrt{-4c+1}}{2}+\frac{1}{2}\right)^2+c+3k+\frac{3\sqrt{-4c+1}}{2}+\frac{7}{2}}$

- Solution for $r = \frac{\sqrt{-4c+1}}{2} + \frac{1}{2}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{\sqrt{-4c+1}}{2}+\frac{1}{2}}, a_{k+2} = -\frac{a_k a + b a_{k+1}}{k^2+2k\left(\frac{\sqrt{-4c+1}}{2}+\frac{1}{2}\right)+\left(\frac{\sqrt{-4c+1}}{2}+\frac{1}{2}\right)^2+c+3k+\frac{3\sqrt{-4c+1}}{2}+\frac{7}{2}}, a_1 = -\frac{}{\left(\frac{\sqrt{-4c+1}}{2}+\right.}\right.$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} d_k x^{k+\frac{1}{2}-\frac{\sqrt{-4c+1}}{2}}\right) + \left(\sum_{k=0}^{\infty} e_k x^{k+\frac{\sqrt{-4c+1}}{2}+\frac{1}{2}}\right), d_{k+2} = -\frac{a d_k + b d_{1+k}}{k^2+2k\left(\frac{1}{2}-\frac{\sqrt{-4c+1}}{2}\right)+\left(\frac{1}{2}-\frac{\sqrt{-4c+1}}{2}\right)^2+c+3k}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 57

```
dsolve(x^2*diff(y(x),x$2)+(a*x^2+b*x+c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{WhittakerM}\left(-\frac{ib}{2\sqrt{a}}, \frac{\sqrt{-4c+1}}{2}, 2i\sqrt{a}\, x\right) + c_2 \operatorname{WhittakerW}\left(-\frac{ib}{2\sqrt{a}}, \frac{\sqrt{-4c+1}}{2}, 2i\sqrt{a}\, x\right)$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 88

```
DSolve[x^2*y''[x]+(a*x^2+b*x+c)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 M_{-\frac{ib}{2\sqrt{a}},-\frac{1}{2}i\sqrt{4c-1}}\left(2i\sqrt{a}x\right) + c_2 W_{-\frac{ib}{2\sqrt{a}},-\frac{1}{2}i\sqrt{4c-1}}\left(2i\sqrt{a}x\right)$$

29.7 problem 116

Internal problem ID [10940]
Internal file name [`OUTPUT/10196_Sunday_December_31_2023_11_05_12_AM_19421269/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 116.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - \left(a\,x^3 + \frac{5}{16}\right)y = 0}$$

29.7.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(-a\,x^3 - \frac{5}{16}\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\,x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\,x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = \frac{2\sqrt{-a}}{3}$$
$$n = -\frac{1}{2}$$
$$\gamma = \frac{3}{2}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{x}\,\sqrt{2}\,\sqrt{6}\,\cos\left(\frac{2\sqrt{-a}\,x^{\frac{3}{2}}}{3}\right)}{2\sqrt{\pi}\,\sqrt{\sqrt{-a}\,x^{\frac{3}{2}}}} + \frac{c_2\sqrt{x}\,\sqrt{2}\,\sqrt{6}\,\sin\left(\frac{2\sqrt{-a}\,x^{\frac{3}{2}}}{3}\right)}{2\sqrt{\pi}\,\sqrt{\sqrt{-a}\,x^{\frac{3}{2}}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{x}\,\sqrt{2}\,\sqrt{6}\,\cos\left(\frac{2\sqrt{-a}\,x^{\frac{3}{2}}}{3}\right)}{2\sqrt{\pi}\,\sqrt{\sqrt{-a}\,x^{\frac{3}{2}}}} + \frac{c_2\sqrt{x}\,\sqrt{2}\,\sqrt{6}\,\sin\left(\frac{2\sqrt{-a}\,x^{\frac{3}{2}}}{3}\right)}{2\sqrt{\pi}\,\sqrt{\sqrt{-a}\,x^{\frac{3}{2}}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{x}\,\sqrt{2}\,\sqrt{6}\,\cos\left(\frac{2\sqrt{-a}\,x^{\frac{3}{2}}}{3}\right)}{2\sqrt{\pi}\,\sqrt{\sqrt{-a}\,x^{\frac{3}{2}}}} + \frac{c_2\sqrt{x}\,\sqrt{2}\,\sqrt{6}\,\sin\left(\frac{2\sqrt{-a}\,x^{\frac{3}{2}}}{3}\right)}{2\sqrt{\pi}\,\sqrt{\sqrt{-a}\,x^{\frac{3}{2}}}}$$

Verified OK.

29.7.2 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + \left(-a\,x^3 - \frac{5}{16}\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -a\,x^3 - \frac{5}{16} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16a\, x^3 + 5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 16a\, x^3 + 5 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{16a\, x^3 + 5}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 120: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 3 \\
&= -1
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [2]
$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = ax + \frac{5}{16x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{-1, 2, 5\}
\end{aligned}
$$

Since the order of r at ∞ is $-1 < 2$ then

$$
E_\infty = \{-1\}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
-1	$\{-1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1, \; e_\infty = -1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(-1 - (-1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right) \\ &= -\frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= -\frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 + \frac{w}{2x} + \frac{-16a\,x^3 + 1}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{-1 + 4x\sqrt{ax}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{-1+4x\sqrt{ax}}{4x} dx} \\ &= \frac{\mathrm{e}^{\frac{2x\sqrt{ax}}{3}}}{x^{\frac{1}{4}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \frac{\mathrm{e}^{\frac{2x\sqrt{ax}}{3}}}{x^{\frac{1}{4}}}\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\frac{2x\sqrt{ax}}{3}}}{x^{\frac{1}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \frac{e^{\frac{2x\sqrt{ax}}{3}}}{x^{\frac{1}{4}}} \int \frac{1}{\frac{e^{\frac{4x\sqrt{ax}}{3}}}{\sqrt{x}}}\, dx \\
&= \frac{e^{\frac{2x\sqrt{ax}}{3}}}{x^{\frac{1}{4}}} \left(-\frac{\sqrt{x}\left(-1 + e^{-\frac{4x\sqrt{ax}}{3}}\right)}{2\sqrt{ax}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\frac{2x\sqrt{ax}}{3}}}{x^{\frac{1}{4}}}\right) + c_2 \left(\frac{e^{\frac{2x\sqrt{ax}}{3}}}{x^{\frac{1}{4}}} \left(-\frac{\sqrt{x}\left(-1 + e^{-\frac{4x\sqrt{ax}}{3}}\right)}{2\sqrt{ax}}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{\frac{2x\sqrt{ax}}{3}}}{x^{\frac{1}{4}}} + \frac{c_2 x^{\frac{1}{4}}\left(e^{\frac{2x\sqrt{ax}}{3}} - e^{-\frac{2x\sqrt{ax}}{3}}\right)}{2\sqrt{ax}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{\frac{2x\sqrt{ax}}{3}}}{x^{\frac{1}{4}}} + \frac{c_2 x^{\frac{1}{4}}\left(e^{\frac{2x\sqrt{ax}}{3}} - e^{-\frac{2x\sqrt{ax}}{3}}\right)}{2\sqrt{ax}}$$

Verified OK.

29.7.3 Maple step by step solution

Let's solve

$y''x^2 + \left(-a\,x^3 - \frac{5}{16}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(16a\,x^3+5)y}{16x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(16a\,x^3+5)y}{16x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = -\frac{16a\,x^3+5}{16x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{5}{16}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16y''x^2 + (-16a\,x^3 - 5)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..3$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\left(k+r-1\right) x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+4r)\left(-5+4r\right) x^{r}+a_{1}(5+4r)\left(-1+4r\right) x^{1+r}+a_{2}(9+4r)\left(3+4r\right) x^{2+r}+\left(\sum_{k=3}^{\infty} (a_{k}(4k+\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+4r)\left(-5+4r\right)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{1}{4}, \frac{5}{4}\right\}$

- The coefficients of each power of x must be 0

  $[a_{1}(5+4r)\left(-1+4r\right)=0, a_{2}(9+4r)\left(3+4r\right)=0]$

- Solve for the dependent coefficient(s)

  $\{a_{1}=0, a_{2}=0\}$

- Each term in the series must be 0, giving the recursion relation

  $16\left(k+r-\frac{5}{4}\right)\left(k+r+\frac{1}{4}\right) a_{k}-16a_{k-3}a=0$

- Shift index using $k->k+3$

  $16\left(k+\frac{7}{4}+r\right)\left(k+\frac{13}{4}+r\right) a_{k+3}-16a_{k}a=0$

- Recursion relation that defines series solution to ODE

  $a_{k+3}=\frac{16a_{k}a}{(4k+7+4r)(4k+13+4r)}$

- Recursion relation for $r=-\frac{1}{4}$

  $a_{k+3}=\frac{16a_{k}a}{(4k+6)(4k+12)}$

- Solution for $r=-\frac{1}{4}$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k-\frac{1}{4}}, a_{k+3}=\frac{16a_{k}a}{(4k+6)(4k+12)}, a_{1}=0, a_{2}=0\right]$

- Recursion relation for $r=\frac{5}{4}$

  $a_{k+3}=\frac{16a_{k}a}{(4k+12)(4k+18)}$

- Solution for $r = \frac{5}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{4}}, a_{k+3} = \frac{16 a_k a}{(4k+12)(4k+18)}, a_1 = 0, a_2 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} c_k x^{k+\frac{5}{4}}\right), b_{k+3} = \frac{16 b_k a}{(4k+6)(4k+12)}, b_1 = 0, b_2 = 0, c_{k+3} = \frac{16 c_k a}{(4k+12)(4k+18)}, c_1$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 31

```
dsolve(x^2*diff(y(x),x$2)-(a*x^3+5/16)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sinh\left(\frac{2x^{\frac{3}{2}}\sqrt{a}}{3}\right) + c_2 \cosh\left(\frac{2x^{\frac{3}{2}}\sqrt{a}}{3}\right)}{x^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 60

```
DSolve[x^2*y''[x]-(a*x^3+5/16)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-\frac{2}{3}\sqrt{a}x^{3/2}}\left(2c_1 e^{\frac{4}{3}\sqrt{a}x^{3/2}} - \frac{c_2}{\sqrt{a}}\right)}{2\sqrt[4]{x}}$$

29.8 problem 117

Internal problem ID [10941]
Internal file name [`OUTPUT/10197_Sunday_December_31_2023_11_05_15_AM_29391360/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 117.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - \left(a^2x^4 + a(-1+2b)\,x^2 + b(1+b)\right)y = 0}$$

29.8.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(-a^2x^4 - 2ab\,x^2 + a\,x^2 - b^2 - b\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1\,\text{BesselJ}\,(n, \beta\,x^{\gamma}) + c_2\,\text{BesselY}\,(n, \beta\,x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = -2b - 1$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\ \text{BesselJ}\left(-2b-1, 2\sqrt{x}\right) + c_2\sqrt{x}\ \text{BesselY}\left(-2b-1, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\ \text{BesselJ}\left(-2b-1, 2\sqrt{x}\right) + c_2\sqrt{x}\ \text{BesselY}\left(-2b-1, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\ \text{BesselJ}\left(-2b-1, 2\sqrt{x}\right) + c_2\sqrt{x}\ \text{BesselY}\left(-2b-1, 2\sqrt{x}\right)$$

Verified OK.

29.8.2 Maple step by step solution

Let's solve

$y''x^2 + \left(-a^2x^4 - 2\left(b - \frac{1}{2}\right)x^2a - b^2 - b\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(a^2x^4 + 2ab\,x^2 - a\,x^2 + b^2 + b)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(a^2x^4 + 2ab\,x^2 - a\,x^2 + b^2 + b)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{a^2x^4 + 2ab\,x^2 - a\,x^2 + b^2 + b}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\Big|_{x=0} = 0$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -b^2 - b$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2 + (-a^2x^4 - 2ab\,x^2 + a\,x^2 - b^2 - b)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..4$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(r+b)\,(-r+1+b)\,x^r - a_1(r+1+b)\,(-r+b)\,x^{1+r} + (-a_2(r+2+b)\,(-r-1+b) - a_0a($

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(r+b)\,(-r+1+b) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-b, 1+b\}$

- The coefficients of each power of x must be 0

  $[-a_1(r+1+b)\,(-r+b) = 0, -a_2(r+2+b)\,(-r-1+b) - a_0a(-1+2b) = 0, -a_3(r+3+b)\,($

- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = -\frac{a_0 a(-1+2b)}{b^2-r^2+b-3r-2}, a_3 = 0\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(r+k+b)(-r+1-k+b) - a(a a_{k-4} + 2b a_{k-2} - a_{k-2}) = 0$

- Shift index using $k- >k+4$

  $-a_{k+4}(r+k+4+b)(-r-3-k+b) - a(a_k a + 2b a_{k+2} - a_{k+2}) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{a(a_k a+2b a_{k+2}-a_{k+2})}{(r+k+4+b)(-r-3-k+b)}$

- Recursion relation for $r = -b$

  $a_{k+4} = -\frac{a(a_k a+2b a_{k+2}-a_{k+2})}{(k+4)(2b-3-k)}$

- Solution for $r = -b$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-b}, a_{k+4} = -\frac{a(a_k a+2b a_{k+2}-a_{k+2})}{(k+4)(2b-3-k)}, a_1 = 0, a_2 = -\frac{a_0 a(-1+2b)}{-2+4b}, a_3 = 0\right]$

- Recursion relation for $r = 1+b$

  $a_{k+4} = -\frac{a(a_k a+2b a_{k+2}-a_{k+2})}{(5+2b+k)(-k-4)}$

- Solution for $r = 1+b$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1+b}, a_{k+4} = -\frac{a(a_k a+2b a_{k+2}-a_{k+2})}{(5+2b+k)(-k-4)}, a_1 = 0, a_2 = -\frac{a_0 a(-1+2b)}{b^2-(1+b)^2-2b-5}, a_3 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} c_k x^{k-b}\right) + \left(\sum_{k=0}^{\infty} d_k x^{k+1+b}\right), c_{k+4} = -\frac{a(a c_k+2b c_{k+2}-c_{k+2})}{(k+4)(2b-3-k)}, c_1 = 0, c_2 = -\frac{c_0 a(-1+2b)}{-2+4b}, c_3 =\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 38

```
dsolve(x^2*diff(y(x),x$2)-(a^2*x^4+a*(2*b-1)*x^2+b*(b+1))*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^{-b} \mathrm{e}^{-\frac{a\,x^2}{2}} \left(c_2 \Gamma\left(b+\frac{1}{2}\right) - c_2 \Gamma\left(b+\frac{1}{2}, -a\,x^2\right) + c_1\right)$$

✓ Solution by Mathematica
Time used: 0.45 (sec). Leaf size: 66

```
DSolve[x^2*y''[x]-(a^2*x^4+a*(2*b-1)*x^2+b*(b+1))*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \rightarrow \frac{1}{2} e^{-\frac{ax^2}{2}} x^{-b} \left(a c_2 x^{2b+3} (-ax^2)^{-b-\frac{3}{2}} \Gamma\left(b+\frac{1}{2}, -ax^2\right) + 2c_1\right)$$

29.9 problem 118

Internal problem ID [10942]
Internal file name [`OUTPUT/10198_Sunday_December_31_2023_11_06_30_AM_50299776/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 118.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + (a\,x^n + b)\,y = 0}$$

29.9.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + (a\,x^n + b)\,y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\,x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\,x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = \frac{2\sqrt{a}}{n}$$
$$n = \frac{\sqrt{-4b+1}}{n}$$
$$\gamma = \frac{n}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{\sqrt{-4b+1}}{n}, \frac{2\sqrt{a}\,x^{\frac{n}{2}}}{n}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{\sqrt{-4b+1}}{n}, \frac{2\sqrt{a}\,x^{\frac{n}{2}}}{n}\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{\sqrt{-4b+1}}{n}, \frac{2\sqrt{a}\,x^{\frac{n}{2}}}{n}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{\sqrt{-4b+1}}{n}, \frac{2\sqrt{a}\,x^{\frac{n}{2}}}{n}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{\sqrt{-4b+1}}{n}, \frac{2\sqrt{a}\,x^{\frac{n}{2}}}{n}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{\sqrt{-4b+1}}{n}, \frac{2\sqrt{a}\,x^{\frac{n}{2}}}{n}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 63

```
dsolve(x^2*diff(y(x),x$2)+(a*x^n+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\operatorname{BesselJ}\left(\frac{\sqrt{-4b+1}}{n}, \frac{2\sqrt{a}\,x^{\frac{n}{2}}}{n}\right)c_1 + \operatorname{BesselY}\left(\frac{\sqrt{-4b+1}}{n}, \frac{2\sqrt{a}\,x^{\frac{n}{2}}}{n}\right)c_2\right)\sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.343 (sec). Leaf size: 351

```
DSolve[x^2*y''[x]+(a*x^n+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to n^{-\frac{\sqrt{(1-4b)n^2}+i\sqrt{4b-1}n+n}{n^2}} a^{\frac{-\sqrt{(1-4b)n^2}-i\sqrt{4b-1}n+n}{2n^2}} (x^n)^{\frac{-\sqrt{(1-4b)n^2}-i\sqrt{4b-1}n+n}{2n^2}} \left(c_2 n^{\frac{2\sqrt{(1-4b)n^2}}{n^2}} a^{\frac{i\sqrt{4b-1}}{n}} (x^n)^{\frac{i\sqrt{4b-1}}{n}} \operatorname{Gamma} + c_1 n^{\frac{2i\sqrt{4b-1}}{n}} a^{\frac{\sqrt{(1-4b)n^2}}{n^2}} (x^n)^{\frac{\sqrt{(1-4b)n^2}}{n^2}} \operatorname{Gamma}\left(1 - \frac{\sqrt{1-4b}}{n}\right) \operatorname{BesselJ}\left(-\frac{\sqrt{(1-4b)n^2}}{n^2}, \frac{2\sqrt{a}\sqrt{x^n}}{n}\right)\right)$$

29.10 problem 119

Internal problem ID [10943]
Internal file name [`OUTPUT/10199_Sunday_December_31_2023_11_06_31_AM_12918249/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 119.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - \left(x^{2n}a^2 + a(2b+n-1)\,x^n + b(b-1)\right)y = 0}$$

29.10.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(-x^{2n}a^2 - 2ab\,x^n - x^n na + a\,x^n - b^2 + b\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = 1 - 2b$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(1-2b, 2\sqrt{x}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(1-2b, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(1-2b, 2\sqrt{x}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(1-2b, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(1-2b, 2\sqrt{x}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(1-2b, 2\sqrt{x}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- Equivalence, under non-integer power transformations successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 134

```
dsolve(x^2*diff(y(x),x$2)-(a^2*x^(2*n)+a*(2*b+n-1)*x^n+b*(b-1))*y(x)=0,y(x), singsol=all)
```

$$y(x) = 2\left(b-\frac{1}{2}-\frac{n}{2}\right)^{2} c_2 x^{-\frac{3n}{2}+\frac{1}{2}} \operatorname{WhittakerM}\left(\frac{n-2b+1}{2n}, -\frac{2b-2n-1}{2n}, \frac{2a\,x^{n}}{n}\right) + n\left(\left(-b+\frac{1}{2}+\frac{n}{2}\right) x^{-\frac{3n}{2}+\frac{1}{2}} + x^{-\frac{n}{2}+\frac{1}{2}} a\right) c_2 \operatorname{WhittakerM}\left(-\frac{2b+n-1}{2n}, -\frac{2b-2n-1}{2n}, \frac{2a\,x^{n}}{n}\right) + c_1 x^{b} \mathrm{e}^{\frac{a\,x^{n}}{n}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*y''[x]-(a^2*x^(2*n)+a*(2*b+n-1)*x^n+b*(b-1))*y[x]==0,y[x],x,IncludeSingularSoluti
```

Not solved

29.11 problem 120

Internal problem ID [10944]
Internal file name [`OUTPUT/10200_Sunday_December_31_2023_11_06_58_AM_48794329/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 120.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(a\,x^{2n} + b\,x^n + c\right)y = 0}$$

29.11.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(a\,x^{2n} + b\,x^n + c\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= 2 \\
n &= \sqrt{-4c+1} \\
\gamma &= \frac{1}{2}
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\sqrt{-4c+1}, 2\sqrt{x}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\sqrt{-4c+1}, 2\sqrt{x}\right)
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\sqrt{-4c+1}, 2\sqrt{x}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\sqrt{-4c+1}, 2\sqrt{x}\right) \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\sqrt{-4c+1}, 2\sqrt{x}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\sqrt{-4c+1}, 2\sqrt{x}\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 90

```
dsolve(x^2*diff(y(x),x$2)+(a*x^(2*n)+b*x^n+c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^{-\frac{n}{2}}\sqrt{x}\left(\operatorname{WhittakerW}\left(-\frac{ib}{2\sqrt{a}\,n}, \frac{i\sqrt{4c-1}}{2n}, \frac{2i\sqrt{a}\,x^{n}}{n}\right)c_2 + \operatorname{WhittakerM}\left(-\frac{ib}{2\sqrt{a}\,n}, \frac{i\sqrt{4c-1}}{2n}, \frac{2i\sqrt{a}\,x^{n}}{n}\right)c_1\right)$$

✓ Solution by Mathematica

Time used: 0.313 (sec). Leaf size: 236

```
DSolve[x^2*y''[x]+(a*x^(2*n)+b*x^n+c)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2^{\frac{\sqrt{(1-4c)n^2}+n^2}{2n^2}} x^{\frac{1}{2}-\frac{n}{2}} e^{\frac{i\sqrt{a}x^n}{n}} (x^n)^{\frac{\sqrt{(1-4c)n^2}+n^2}{2n^2}} \left(c_1 \operatorname{HypergeometricU}\left(\frac{1}{2}\left(-\frac{ib}{\sqrt{a}n}+\frac{\sqrt{(1-4c)n^2}}{n^2}+1\right), \frac{\sqrt{(1-}}{n} \cdots + 1, -\frac{2i\sqrt{a}x^n}{n}\right) + c_2 L_{\frac{1}{2}\left(\frac{ib}{\sqrt{a}n}-\frac{\sqrt{(1-4c)n^2}}{n^2}-1\right)}^{\frac{\sqrt{(1-4c)n^2}}{n^2}}\left(-\frac{2i\sqrt{a}x^n}{n}\right)\right)$$

29.12 problem 121

Internal problem ID [10945]
Internal file name [`OUTPUT/10201_Sunday_December_31_2023_11_07_54_AM_35512270/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 121.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(a\,x^{3n} + b\,x^{2n} + \frac{1}{4} - \frac{n^2}{4}\right)y = 0}$$

29.12.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(a\,x^{3n} + b\,x^{2n} + \frac{1}{4} - \frac{n^2}{4}\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\,x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\,x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = n$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\ \text{BesselJ}\left(n, 2\sqrt{x}\right) + c_2\sqrt{x}\ \text{BesselY}\left(n, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\ \text{BesselJ}\left(n, 2\sqrt{x}\right) + c_2\sqrt{x}\ \text{BesselY}\left(n, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\ \text{BesselJ}\left(n, 2\sqrt{x}\right) + c_2\sqrt{x}\ \text{BesselY}\left(n, 2\sqrt{x}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: indirect Equivalence to 0F1 under \`\`^ @ Moebius\`\` is resolve
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.297 (sec). Leaf size: 177

```
dsolve(x^2*diff(y(x),x$2)+(a*x^(3*n)+b*x^(2*n)+1/4-1/4*n^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(\frac{2\,3^{\frac{5}{6}}\pi c_2 (a\,x^n+b) \operatorname{BesselI}\left(\frac{1}{3}, \frac{2\sqrt{\frac{-x^{3n}a^3-3x^{2n}a^2b-3x^n a\,b^2-b^3}{n^2a^2}}}{3}\right)}{3} + c_1 \operatorname{BesselI}\left(-\frac{1}{3}, \frac{2\sqrt{\frac{-x^{3n}a^3-3x^{2n}a^2b-3x^n a\,b^2-b^3}{n^2a^2}}}{3}\right)\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}}}{3\left(-\frac{(a\,x^n+b)^3}{a^2n^2}\right)^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*y''[x]+(a*x^(3*n)+b*x^(2*n)+1/4-1/4*n^2)*y[x]==0,y[x],x,IncludeSingularSolutions
```

Not solved

29.13 problem 122

Internal problem ID [10946]
Internal file name [`OUTPUT/10202_Sunday_December_31_2023_11_08_17_AM_31922755/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 122.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x^2y'' + \left(a\,x^{2n}(b\,x^n + c)^m + \frac{1}{4} - \frac{n^2}{4}\right)y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✗ Solution by Maple

```
dsolve(x^2*diff(y(x),x$2)+(a*x^(2*n)*(b*x^n+c)^m+1/4-1/4*n^2)*y(x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*y''[x]+(a*x^(2*n)*(b*x^n+c)^m+1/4-1/4*n^2)*y[x]==0,y[x],x,IncludeSingularSolution
```

Not solved

29.14 problem 123

Internal problem ID [10947]
Internal file name [`OUTPUT/10203_Sunday_December_31_2023_11_08_18_AM_20323320/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 123.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + axy' + yb = 0}$$

29.14.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + axrx^{r-1} + b\,x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + ar\,x^r + b\,x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + ar + b = 0$$

Or

$$r^{2}+(a-1)\,r+b=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned}
r_1 &= -\frac{a}{2}+\frac{1}{2}-\frac{\sqrt{a^{2}-2a-4b+1}}{2}\\
r_2 &= -\frac{a}{2}+\frac{1}{2}+\frac{\sqrt{a^{2}-2a-4b+1}}{2}
\end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y=c_1y_1+c_2y_2$$

Where $y_1=x^{r_1}$ and $y_2=x^{r_2}$. Hence

$$y=c_1x^{-\frac{a}{2}+\frac{1}{2}-\frac{\sqrt{a^{2}-2a-4b+1}}{2}}+c_2x^{-\frac{a}{2}+\frac{1}{2}+\frac{\sqrt{a^{2}-2a-4b+1}}{2}}$$

Summary
The solution(s) found are the following

$$y=c_1x^{-\frac{a}{2}+\frac{1}{2}-\frac{\sqrt{a^{2}-2a-4b+1}}{2}}+c_2x^{-\frac{a}{2}+\frac{1}{2}+\frac{\sqrt{a^{2}-2a-4b+1}}{2}} \tag{1}$$

Verification of solutions

$$y=c_1x^{-\frac{a}{2}+\frac{1}{2}-\frac{\sqrt{a^{2}-2a-4b+1}}{2}}+c_2x^{-\frac{a}{2}+\frac{1}{2}+\frac{\sqrt{a^{2}-2a-4b+1}}{2}}$$

Verified OK.

29.14.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^{2}y''+axy'+yb=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{a}{x}\\
q(x) &= \frac{b}{x^{2}}
\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{a}{x}dx\right)}dx \\
&= \int e^{-a\ln(x)}\,dx \\
&= \int x^{-a}dx \\
&= -\frac{x^{-a+1}}{a-1}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{b}{x^2}}{x^{-2a}} \\
&= b\,x^{-2+2a}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{gathered}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0 \\
\frac{d^2}{d\tau^2}y(\tau) + b\,x^{-2+2a}y(\tau) = 0
\end{gathered}$$

But in terms of τ

$$b\,x^{-2+2a} = \frac{b}{\left(a-1\right)^{2} \tau^{2}}$$

Hence the above ode becomes

$$\frac{d^{2}}{d\tau^{2}}y(\tau) + \frac{b y(\tau)}{\left(a-1\right)^{2} \tau^{2}} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^{2}}{d\tau^{2}}y(\tau)\right) \left(a-1\right)^{2} \tau^{2} + b y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^{r}$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\left(a-1\right)^{2} \tau^{2}(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + b\,\tau^{r} = 0$$

Simplifying gives

$$\left(a-1\right)^{2} r(r-1)\,\tau^{r} + 0\,\tau^{r} + b\,\tau^{r} = 0$$

Since $\tau^{r} \neq 0$ then dividing throughout by τ^{r} gives

$$\left(a-1\right)^{2} r(r-1) + 0 + b = 0$$

Or

$$\left(a^{2}-2a+1\right) r^{2} + \left(-a^{2}+2a-1\right) r + b = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{-a+1+\sqrt{a^{2}-2a-4b+1}}{2\left(a-1\right)}$$
$$r_2 = \frac{a-1+\sqrt{a^{2}-2a-4b+1}}{-2+2a}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{-\frac{-a+1+\sqrt{a^{2}-2a-4b+1}}{2(a-1)}} + c_2 \tau^{\frac{a-1+\sqrt{a^{2}-2a-4b+1}}{-2+2a}}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(-\frac{x^{-a+1}}{a-1}\right)^{\frac{a-1-\sqrt{a^2-2a-4b+1}}{-2+2a}} + c_2 \left(-\frac{x^{-a+1}}{a-1}\right)^{\frac{a-1+\sqrt{a^2-2a-4b+1}}{-2+2a}}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(-\frac{x^{-a+1}}{a-1}\right)^{\frac{a-1-\sqrt{a^2-2a-4b+1}}{-2+2a}} + c_2 \left(-\frac{x^{-a+1}}{a-1}\right)^{\frac{a-1+\sqrt{a^2-2a-4b+1}}{-2+2a}} \tag{1}$$

Verification of solutions

$$y = c_1 \left(-\frac{x^{-a+1}}{a-1}\right)^{\frac{a-1-\sqrt{a^2-2a-4b+1}}{-2+2a}} + c_2 \left(-\frac{x^{-a+1}}{a-1}\right)^{\frac{a-1+\sqrt{a^2-2a-4b+1}}{-2+2a}}$$

Verified OK.

29.14.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + axy' + yb = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{a}{x}$$
$$q(x) = \frac{b}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{na}{x^2}+\frac{b}{x^2}=0 \tag{5}$$

Solving (5) for n gives

$$n=-\frac{a}{2}+\frac{1}{2}+\frac{\sqrt{a^2-2a-4b+1}}{2} \tag{6}$$

Substituting this value in (3) gives

$$v''(x)+\left(\frac{-a+1+\sqrt{a^2-2a-4b+1}}{x}+\frac{a}{x}\right)v'(x)=0$$

$$v''(x)+\frac{\left(1+\sqrt{a^2-2a-4b+1}\right)v'(x)}{x}=0 \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x)+\frac{\left(1+\sqrt{a^2-2a-4b+1}\right)u(x)}{x}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=\frac{\left(-1-\sqrt{a^2-2a-4b+1}\right)u}{x}\end{aligned}$$

Where $f(x)=\frac{-1-\sqrt{a^2-2a-4b+1}}{x}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du&=\frac{-1-\sqrt{a^2-2a-4b+1}}{x}\,dx\\ \int\frac{1}{u}\,du&=\int\frac{-1-\sqrt{a^2-2a-4b+1}}{x}\,dx\\ \ln(u)&=\left(-1-\sqrt{a^2-2a-4b+1}\right)\ln(x)+c_1\\ u&=\mathrm{e}^{\left(-1-\sqrt{a^2-2a-4b+1}\right)\ln(x)+c_1}\\&=c_1\mathrm{e}^{\left(-1-\sqrt{a^2-2a-4b+1}\right)\ln(x)}\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-\sqrt{a^2-2a-4b+1}}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1 x^{-\sqrt{a^2-2a-4b+1}}}{\sqrt{a^2-2a-4b+1}} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(-\frac{c_1 x^{-\sqrt{a^2-2a-4b+1}}}{\sqrt{a^2-2a-4b+1}} + c_2\right) x^{-\frac{a}{2}+\frac{1}{2}+\frac{\sqrt{a^2-2a-4b+1}}{2}} \\ &= -\frac{x^{-\frac{a}{2}+\frac{1}{2}-\frac{\sqrt{a^2-2a-4b+1}}{2}} \left(-c_2\sqrt{a^2-2a-4b+1}\, x^{\sqrt{a^2-2a-4b+1}} + c_1\right)}{\sqrt{a^2-2a-4b+1}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{c_1 x^{-\sqrt{a^2-2a-4b+1}}}{\sqrt{a^2-2a-4b+1}} + c_2\right) x^{-\frac{a}{2}+\frac{1}{2}+\frac{\sqrt{a^2-2a-4b+1}}{2}} \tag{1}$$

Verification of solutions

$$y = \left(-\frac{c_1 x^{-\sqrt{a^2-2a-4b+1}}}{\sqrt{a^2-2a-4b+1}} + c_2\right) x^{-\frac{a}{2}+\frac{1}{2}+\frac{\sqrt{a^2-2a-4b+1}}{2}}$$

Verified OK.

29.14.4 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + axy' + yb = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= ax \\ C &= b \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2 - 2a - 4b}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2 - 2a - 4b \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2 - 2a - 4b}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 123: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{\frac{1}{4}a^2 - \frac{1}{2}a - b}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{1}{4}a^2 - \frac{1}{2}a - b$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \left\{2, 2-2\sqrt{a^2-2a-4b+1}, 2+2\sqrt{a^2-2a-4b+1}\right\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{a^2 - 2a - 4b}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{1}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\left\{2, 2-2\sqrt{a^2-2a-4b+1}, 2+2\sqrt{a^2-2a-4b+1}\right\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 2, \; e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (2)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\theta = \frac{1}{2} \sum_{c\in\Gamma} \frac{e_c}{x-c}$$
$$= \frac{1}{2}\left(\frac{2}{(x-(0))}\right)$$
$$= \frac{1}{x}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\phi = \theta + \frac{p'}{p}$$
$$= \frac{1}{x}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{x} + \frac{-a^2 + 2a + 4b}{4x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + \sqrt{a^2 - 2a - 4b + 1}}{2x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \,dx} \\
&= \mathrm{e}^{\int \frac{1+\sqrt{a^2-2a-4b+1}}{2x}dx} \\
&= x^{\frac{1}{2}+\frac{\sqrt{a^2-2a-4b+1}}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{ax}{x^2}\,dx} \\
&= z_1 e^{-\frac{a\ln(x)}{2}} \\
&= z_1\left(x^{-\frac{a}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{-\frac{a}{2}+\frac{1}{2}+\frac{\sqrt{a^2-2a-4b+1}}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{ax}{x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-a\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{x^{-\sqrt{a^2-2a-4b+1}}}{\sqrt{a^2-2a-4b+1}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{-\frac{a}{2}+\frac{1}{2}+\frac{\sqrt{a^2-2a-4b+1}}{2}}\right) + c_2\left(x^{-\frac{a}{2}+\frac{1}{2}+\frac{\sqrt{a^2-2a-4b+1}}{2}}\left(-\frac{x^{-\sqrt{a^2-2a-4b+1}}}{\sqrt{a^2-2a-4b+1}}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x^{-\frac{a}{2}+\frac{1}{2}+\frac{\sqrt{a^2-2a-4b+1}}{2}} - \frac{c_2 x^{-\frac{a}{2}+\frac{1}{2}-\frac{\sqrt{a^2-2a-4b+1}}{2}}}{\sqrt{a^2-2a-4b+1}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{-\frac{a}{2}+\frac{1}{2}+\frac{\sqrt{a^2-2a-4b+1}}{2}} - \frac{c_2 x^{-\frac{a}{2}+\frac{1}{2}-\frac{\sqrt{a^2-2a-4b+1}}{2}}}{\sqrt{a^2-2a-4b+1}}$$

Verified OK.

29.14.5 Maple step by step solution

Let's solve

$y''x^2 + axy' + yb = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{ay'}{x} - \frac{by}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{ay'}{x} + \frac{by}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{a}{x}, P_3(x) = \frac{b}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = a$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = b$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x^2 + axy' + yb = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} a_k (ak + ar + k^2 + 2kr + r^2 + b - k - r)\, x^{k+r} = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + (a-1)\,k + b)\, a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_k = 0$
- Recursion relation for $r = 0$

  $a_k = 0$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_k = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 52

```
dsolve(x^2*diff(y(x),x$2)+a*x*diff(y(x),x)+b*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^{-\frac{a}{2}}\sqrt{x}\left(x^{\frac{\sqrt{a^2-2a-4b+1}}{2}}c_1 + x^{-\frac{\sqrt{a^2-2a-4b+1}}{2}}c_2\right)$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 57

```
DSolve[x^2*y''[x]+a*x*y'[x]+b*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^{\frac{1}{2}\left(-\sqrt{a^2-2a-4b+1}-a+1\right)}\left(c_2 x^{\sqrt{a^2-2a-4b+1}} + c_1\right)$$

29.15 problem 124

Internal problem ID [10948]
Internal file name [`OUTPUT/10204_Sunday_December_31_2023_11_08_22_AM_32927791/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 124.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y'x + \left(x^2 - \left(n + \frac{1}{2}\right)^2\right) y = 0}$$

29.15.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + y'x + \left(x^2 - n^2 - n - \frac{1}{4}\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = 1$$
$$n = -n - \frac{1}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}\left(-n - \frac{1}{2}, x\right) + c_2 \operatorname{BesselY}\left(-n - \frac{1}{2}, x\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}\left(-n - \frac{1}{2}, x\right) + c_2 \operatorname{BesselY}\left(-n - \frac{1}{2}, x\right) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}\left(-n - \frac{1}{2}, x\right) + c_2 \operatorname{BesselY}\left(-n - \frac{1}{2}, x\right)$$

Verified OK.

29.15.2 Maple step by step solution

Let's solve

$y''x^2 + y'x + \left(x^2 - n^2 - n - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(4n^2 - 4x^2 + 4n + 1) y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(4n^2 - 4x^2 + 4n + 1) y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$$\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{4n^2-4x^2+4n+1}{4x^2}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\bigg|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = -n^2 - n - \frac{1}{4}$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $$x_0 = 0$$

- Multiply by denominators

  $$4y''x^2 + 4y'x + (-4n^2 + 4x^2 - 4n - 1)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2r+1+2n)\,(2r-1-2n)\, x^r + a_1(2r+3+2n)\,(2r+1-2n)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2r+1+2n+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(2r+1+2n)(2r-1-2n) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-n-\frac{1}{2}, n+\frac{1}{2}\right\}$

- Each term must be 0

  $a_1(2r+3+2n)(2r+1-2n) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2r+1+2n+2k)(2r-1-2n+2k) + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(2r+5+2n+2k)(2r+3-2n+2k) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(2r+5+2n+2k)(2r+3-2n+2k)}$

- Recursion relation for $r = -n-\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{(4+2k)(-4n+2+2k)}$

- Solution for $r = -n-\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-n-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{(4+2k)(-4n+2+2k)}, a_1 = 0\right]$

- Recursion relation for $r = n+\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{(4n+6+2k)(4+2k)}$

- Solution for $r = n+\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+n+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{(4n+6+2k)(4+2k)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-n-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+n+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{(4+2k)(-4n+2+2k)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(4n+6+2k)(4-\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-(n+1/2)^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}\left(n+\frac{1}{2}, x\right) + c_2 \operatorname{BesselY}\left(n+\frac{1}{2}, x\right)$$

✓ Solution by Mathematica

Time used: 0.387 (sec). Leaf size: 26

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-(n+1/2)^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}\left(n+\frac{1}{2}, x\right) + c_2 \operatorname{BesselY}\left(n+\frac{1}{2}, x\right)$$

29.16 problem 125

Internal problem ID [10949]
Internal file name [`OUTPUT/10205_Sunday_December_31_2023_11_08_24_AM_80160225/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 125.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y'x - \left(x^2 + \left(n + \frac{1}{2}\right)^2\right) y = 0}$$

29.16.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + y'x + \left(-x^2 - n^2 - n - \frac{1}{4}\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= 0 \\
\beta &= i \\
n &= -n - \frac{1}{2} \\
\gamma &= 1
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = c_1 \operatorname{BesselJ}\left(-n - \frac{1}{2}, ix\right) + c_2 \operatorname{BesselY}\left(-n - \frac{1}{2}, ix\right)
$$

Summary

The solution(s) found are the following

$$
y = c_1 \operatorname{BesselJ}\left(-n - \frac{1}{2}, ix\right) + c_2 \operatorname{BesselY}\left(-n - \frac{1}{2}, ix\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 \operatorname{BesselJ}\left(-n - \frac{1}{2}, ix\right) + c_2 \operatorname{BesselY}\left(-n - \frac{1}{2}, ix\right)
$$

Verified OK.

29.16.2 Maple step by step solution

Let's solve

$y''x^2 + y'x + \left(-x^2 - n^2 - n - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(4n^2+4x^2+4n+1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(4n^2+4x^2+4n+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$$\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{4n^2+4x^2+4n+1}{4x^2}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -n^2 - n - \frac{1}{4}$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $$x_0 = 0$$

- Multiply by denominators

  $$4y''x^2 + 4y'x + \left(-4n^2 - 4x^2 - 4n - 1\right) y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

  - Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2r + 1 + 2n)\,(2r - 1 - 2n)\, x^r + a_1(2r + 3 + 2n)\,(2r + 1 - 2n)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2r + 1 + 2n +$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2r+1+2n)(2r-1-2n)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-n-\frac{1}{2}, n+\frac{1}{2}\right\}$$

- Each term must be 0

$$a_1(2r+3+2n)(2r+1-2n)=0$$

- Solve for the dependent coefficient(s)

$$a_1=0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(2r+1+2n+2k)(2r-1-2n+2k)-4a_{k-2}=0$$

- Shift index using $k->k+2$

$$a_{k+2}(2r+5+2n+2k)(2r+3-2n+2k)-4a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=\frac{4a_k}{(2r+5+2n+2k)(2r+3-2n+2k)}$$

- Recursion relation for $r=-n-\frac{1}{2}$

$$a_{k+2}=\frac{4a_k}{(4+2k)(-4n+2+2k)}$$

- Solution for $r=-n-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-n-\frac{1}{2}}, a_{k+2}=\frac{4a_k}{(4+2k)(-4n+2+2k)}, a_1=0\right]$$

- Recursion relation for $r=n+\frac{1}{2}$

$$a_{k+2}=\frac{4a_k}{(4n+6+2k)(4+2k)}$$

- Solution for $r=n+\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+n+\frac{1}{2}}, a_{k+2}=\frac{4a_k}{(4n+6+2k)(4+2k)}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-n-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+n+\frac{1}{2}}\right), a_{k+2}=\frac{4a_k}{(4+2k)(-4n+2+2k)}, a_1=0, b_{k+2}=\frac{4b_k}{(4n+6+2k)(4+2k)}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-(x^2+(n+1/2)^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselI}\left(n + \frac{1}{2}, x\right) + c_2 \operatorname{BesselK}\left(n + \frac{1}{2}, x\right)$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 34

```
DSolve[x^2*y''[x]+x*y'[x]-(x^2+(n+1/2)^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}\left(n + \frac{1}{2}, -ix\right) + c_2 \operatorname{BesselY}\left(n + \frac{1}{2}, -ix\right)$$

29.17 problem 126

Internal problem ID [10950]
Internal file name [`OUTPUT/10206_Sunday_December_31_2023_11_08_26_AM_33239479/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 126.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[_Bessel]`

$$\boxed{x^2y'' + y'x + (-\nu^2 + x^2)\, y = 0}$$

29.17.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + y'x + (-\nu^2 + x^2)\, y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = 1$$
$$n = \nu$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}(\nu, x) + c_2 \operatorname{BesselY}(\nu, x)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}(\nu, x) + c_2 \operatorname{BesselY}(\nu, x) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}(\nu, x) + c_2 \operatorname{BesselY}(\nu, x)$$

Verified OK.

29.17.2 Maple step by step solution

Let's solve

$y''x^2 + y'x + (-\nu^2 + x^2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(\nu^2 - x^2) y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(\nu^2 - x^2) y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{\nu^2 - x^2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\Big|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\Big|_{x=0} = -\nu^2$$

- $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$y''x^2 + y'x + (-\nu^2 + x^2)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(\nu + r)\,(-\nu + r)\, x^r + a_1(1 + \nu + r)\,(1 - \nu + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k + \nu + r)\,(k - \nu + r) + a_{k-2}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(\nu + r)\,(-\nu + r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{\nu, -\nu\}$$

- Each term must be 0

  $a_1(1+\nu+r)(1-\nu+r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+\nu+r)(k-\nu+r)+a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+2+\nu+r)(k+2-\nu+r)+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+2+\nu+r)(k+2-\nu+r)}$

- Recursion relation for $r=\nu$

  $a_{k+2}=-\frac{a_k}{(k+2+2\nu)(k+2)}$

- Solution for $r=\nu$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\nu}, a_{k+2}=-\frac{a_k}{(k+2+2\nu)(k+2)}, a_1=0\right]$

- Recursion relation for $r=-\nu$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+2-2\nu)}$

- Solution for $r=-\nu$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\nu}, a_{k+2}=-\frac{a_k}{(k+2)(k+2-2\nu)}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\nu}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\nu}\right), a_{k+2}=-\frac{a_k}{(k+2+2\nu)(k+2)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+2)(k+2-2\nu)}, b_1=0\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-nu^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}(\nu, x) + c_2 \operatorname{BesselY}(\nu, x)$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 26

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-\[Nu])*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}\left(\sqrt{\nu}, x\right) + c_2 \operatorname{BesselY}\left(\sqrt{\nu}, x\right)$$

29.18 problem 127

Internal problem ID [10951]
Internal file name [`OUTPUT/10207_Sunday_December_31_2023_11_08_27_AM_25912510/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 127.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Bessel, _modified]]
```

$$\boxed{x^2y'' + y'x - (\nu^2 + x^2)\, y = 0}$$

29.18.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + y'x + (-\nu^2 - x^2)\, y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = i$$
$$n = \nu$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}(\nu, ix) + c_2 \operatorname{BesselY}(\nu, ix)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}(\nu, ix) + c_2 \operatorname{BesselY}(\nu, ix) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}(\nu, ix) + c_2 \operatorname{BesselY}(\nu, ix)$$

Verified OK.

29.18.2 Maple step by step solution

Let's solve

$y''x^2 + y'x + (-\nu^2 - x^2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(\nu^2+x^2)y}{x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(\nu^2+x^2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{\nu^2+x^2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\nu^2$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x^2 + y'x + \left(-\nu^2 - x^2\right) y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(\nu + r)\,(-\nu + r)\, x^r + a_1(1 + \nu + r)\,(1 - \nu + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k + \nu + r)\,(k - \nu + r) - a_{k-2}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(\nu + r)\,(-\nu + r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{\nu, -\nu\}$$

- Each term must be 0

  $a_1(1+\nu+r)(1-\nu+r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+\nu+r)(k-\nu+r)-a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+2+\nu+r)(k+2-\nu+r)-a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_k}{(k+2+\nu+r)(k+2-\nu+r)}$

- Recursion relation for $r=\nu$

  $a_{k+2}=\frac{a_k}{(k+2+2\nu)(k+2)}$

- Solution for $r=\nu$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\nu}, a_{k+2}=\frac{a_k}{(k+2+2\nu)(k+2)}, a_1=0\right]$

- Recursion relation for $r=-\nu$

  $a_{k+2}=\frac{a_k}{(k+2)(k+2-2\nu)}$

- Solution for $r=-\nu$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\nu}, a_{k+2}=\frac{a_k}{(k+2)(k+2-2\nu)}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\nu}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\nu}\right), a_{k+2}=\frac{a_k}{(k+2+2\nu)(k+2)}, a_1=0, b_{k+2}=\frac{b_k}{(k+2)(k+2-2\nu)}, b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-(x^2+nu^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselI}(\nu, x) + c_2 \operatorname{BesselK}(\nu, x)$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 34

```
DSolve[x^2*y''[x]+x*y'[x]-(x^2+\[Nu])*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}\left(\sqrt{\nu}, -ix\right) + c_2 \operatorname{BesselY}\left(\sqrt{\nu}, -ix\right)$$

29.19 problem 128

Internal problem ID [10952]
Internal file name [`OUTPUT/10208_Sunday_December_31_2023_11_08_29_AM_14546133/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 128.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + 2y'x - \left(a^2x^2 + 2\right)y = 0}$$

29.19.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + 2y'x + \left(-a^2x^2 - 2\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= -\frac{1}{2}\\ \beta &= ia\\ n &= -\frac{3}{2}\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = -\frac{ic_1\sqrt{2}\,(\sinh(ax)\,ax - \cosh(ax))}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{iax}\,a} - \frac{c_2\sqrt{2}\,(-\cosh(ax)\,ax + \sinh(ax))}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{iax}\,a}$$

Summary

The solution(s) found are the following

$$y = -\frac{ic_1\sqrt{2}\,(\sinh(ax)\,ax - \cosh(ax))}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{iax}\,a} - \frac{c_2\sqrt{2}\,(-\cosh(ax)\,ax + \sinh(ax))}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{iax}\,a} \tag{1}$$

Verification of solutions

$$y = -\frac{ic_1\sqrt{2}\,(\sinh(ax)\,ax - \cosh(ax))}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{iax}\,a} - \frac{c_2\sqrt{2}\,(-\cosh(ax)\,ax + \sinh(ax))}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{iax}\,a}$$

Verified OK.

29.19.2 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 2y'x + \left(-a^2x^2 - 2\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= x^2\\ B &= 2x\\ C &= -a^2x^2 - 2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 129: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = a^2 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{2}{x^2}\right) \\ &= a^2 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{a^2x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)\,(a) \\
&= -\frac{1}{x} - a \\
&= \frac{-ax-1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - a\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - a\right)^2 - \left(\frac{a^2x^2+2}{x^2}\right)\right) &= 0 \\
\frac{2aa_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{1}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{1}{a}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{a}\right) e^{\int \left(-\frac{1}{x} - a\right) dx} \\
&= \left(x + \frac{1}{a}\right) e^{-ax - \ln(x)} \\
&= \frac{(ax+1)\, e^{-ax}}{ax}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(ax+1)\,\mathrm{e}^{-ax}}{a\,x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(ax-1)\,\mathrm{e}^{2ax}}{2\,(ax+1)\,a}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a\,x^2}\right) + c_2 \left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a\,x^2}\left(\frac{(ax-1)\,\mathrm{e}^{2ax}}{2\,(ax+1)\,a}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(ax+1)\,\mathrm{e}^{-ax}}{a\,x^2} + \frac{c_2(ax-1)\,\mathrm{e}^{ax}}{2a^2x^2} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(ax+1)\,\mathrm{e}^{-ax}}{a\,x^2} + \frac{c_2(ax-1)\,\mathrm{e}^{ax}}{2a^2x^2}$$

Verified OK.

29.19.3 Maple step by step solution

Let's solve

$y''x^2 + 2y'x + (-a^2x^2 - 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(a^2x^2+2)y}{x^2} - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} - \frac{(a^2x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -\frac{a^2x^2+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -2$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2 + 2y'x + (-a^2x^2 - 2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\, (k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-1+r)\, x^r + a_1(3+r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)\,(k+r-1) - a_{k-2}a^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 1\}$

- Each term must be 0

  $a_1(3+r)\, r = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)\,(k+r-1) - a_{k-2}a^2 = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}(k+4+r)\,(k+1+r) - a_k a^2 = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k a^2}{(k+4+r)(k+1+r)}$

- Recursion relation for $r = -2$

$a_{k+2} = \frac{a_k a^2}{(k+2)(k-1)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = \frac{a_k a^2}{(k+2)(k-1)}, a_1 = 0\right]$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{a_k a^2}{(k+5)(k+2)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k a^2}{(k+5)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} c_k x^{1+k}\right), b_{k+2} = \frac{b_k a^2}{(k+2)(k-1)}, b_1 = 0, c_{k+2} = \frac{c_k a^2}{(k+5)(k+2)}, c_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve(x^2*diff(y(x),x$2)+2*x*diff(y(x),x)-(a^2*x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 e^{-ax}(ax+1) + c_1 e^{ax}(ax-1)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 29

```
DSolve[x^2*y''[x]+2*x*y'[x]-(a^2*x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 j_{-2}(iax) - c_2 y_{-2}(iax)$$

29.20 problem 129

Internal problem ID [10953]
Internal file name [`OUTPUT/10209_Sunday_December_31_2023_11_08_32_AM_83452150/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 129.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - 2axy' + \left(b^2x^2 + a(a+1)\right) y = 0}$$

29.20.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2a}{x} \\ q(x) &= \frac{b^2x^2 + a^2 + a}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{b^2x^2 + a^2 + a}{x^2} - \frac{\left(-\frac{2a}{x}\right)'}{2} - \frac{\left(-\frac{2a}{x}\right)^2}{4} \\
&= \frac{b^2x^2 + a^2 + a}{x^2} - \frac{\left(\frac{2a}{x^2}\right)}{2} - \frac{\left(\frac{4a^2}{x^2}\right)}{4} \\
&= \frac{b^2x^2 + a^2 + a}{x^2} - \left(\frac{a}{x^2}\right) - \frac{a^2}{x^2} \\
&= b^2
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{2a}{x}}{2}} \\
&= x^a
\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x^a \tag{4}$$

Applying this change of variable to the original ode results in

$$x^{a+2}\left(v(x)\, b^2 + v''(x)\right) = 0$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = b^2$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + b^2 \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$b^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = b^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(b^2)}\\ &= \pm\sqrt{-b^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-b^2}\\ \lambda_2 &= -\sqrt{-b^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-b^2}\\ \lambda_2 &= -\sqrt{-b^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ v(x) &= c_1 e^{\left(\sqrt{-b^2}\right)x} + c_2 e^{\left(-\sqrt{-b^2}\right)x}\end{aligned}$$

Or

$$v(x) = c_1 \mathrm{e}^{\sqrt{-b^2}\,x} + c_2 \mathrm{e}^{-\sqrt{-b^2}\,x}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\,z(x)\\ &= \left(c_1 \mathrm{e}^{\sqrt{-b^2}\,x} + c_2 \mathrm{e}^{-\sqrt{-b^2}\,x}\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^a$$

Hence (7) becomes

$$y = \left(c_1 \mathrm{e}^{\sqrt{-b^2}\,x} + c_2 \mathrm{e}^{-\sqrt{-b^2}\,x}\right) x^a$$

Summary
The solution(s) found are the following

$$y = \left(c_1 \mathrm{e}^{\sqrt{-b^2}\, x} + c_2 \mathrm{e}^{-\sqrt{-b^2}\, x}\right) x^a \tag{1}$$

Verification of solutions

$$y = \left(c_1 \mathrm{e}^{\sqrt{-b^2}\, x} + c_2 \mathrm{e}^{-\sqrt{-b^2}\, x}\right) x^a$$

Verified OK.

29.20.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - 2axy' + \left(b^2 x^2 + a^2 + a\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, xy' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = a + \frac{1}{2}$$
$$\beta = b$$
$$n = -\frac{1}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x^{a+\frac{1}{2}} \sqrt{2}\, \cos(bx)}{\sqrt{\pi}\, \sqrt{bx}} + \frac{c_2 x^{a+\frac{1}{2}} \sqrt{2}\, \sin(bx)}{\sqrt{\pi}\, \sqrt{bx}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^{a+\frac{1}{2}} \sqrt{2}\, \cos(bx)}{\sqrt{\pi}\, \sqrt{bx}} + \frac{c_2 x^{a+\frac{1}{2}} \sqrt{2}\, \sin(bx)}{\sqrt{\pi}\, \sqrt{bx}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{a+\frac{1}{2}} \sqrt{2}\, \cos(bx)}{\sqrt{\pi}\, \sqrt{bx}} + \frac{c_2 x^{a+\frac{1}{2}} \sqrt{2}\, \sin(bx)}{\sqrt{\pi}\, \sqrt{bx}}$$

Verified OK.

29.20.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 2axy' + \left(b^2 x^2 + a^2 + a\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2ax \\ C &= b^2 x^2 + a^2 + a \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-b^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -b^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (-b^2)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 131: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -b^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{-b^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2ax}{x^2}\,dx} \\ &= z_1 e^{a\ln(x)} \\ &= z_1(x^a) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\sqrt{-b^2}\,x} x^a$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2ax}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2a\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\sqrt{-b^2}\,\mathrm{e}^{-2\sqrt{-b^2}\,x}}{2b^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{\sqrt{-b^2}\,x} x^a\right) + c_2\left(\mathrm{e}^{\sqrt{-b^2}\,x} x^a \left(\frac{\sqrt{-b^2}\,\mathrm{e}^{-2\sqrt{-b^2}\,x}}{2b^2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\sqrt{-b^2}\,x} x^a + \frac{c_2 x^a \sqrt{-b^2}\,\mathrm{e}^{-\sqrt{-b^2}\,x}}{2b^2} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\sqrt{-b^2}\, x} x^a + \frac{c_2 x^a \sqrt{-b^2}\, e^{-\sqrt{-b^2}\, x}}{2b^2}$$

Verified OK.

29.20.4 Maple step by step solution

Let's solve

$y''x^2 - 2axy' + (b^2x^2 + a^2 + a)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(b^2x^2+a^2+a)y}{x^2} + \frac{2ay'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2ay'}{x} + \frac{(b^2x^2+a^2+a)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2a}{x}, P_3(x) = \frac{b^2x^2+a^2+a}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2a$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = a^2 + a$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x^2 - 2axy' + (b^2x^2 + a^2 + a)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-r+1+a)\,(-r+a)\, x^r + a_1(-r+a)\,(-r-1+a)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(-r+1-k+a)\,(-r-k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-r+1+a)\,(-r+a) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{a, a+1\}$

- Each term must be 0

  $a_1(-r+a)\,(-r-1+a) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(-r+1-k+a)\,(-r-k+a) + a_{k-2}b^2 = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(-r-1-k+a)\,(-r-k-2+a) + a_k b^2 = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k b^2}{(-r-1-k+a)(-r-k-2+a)}$

- Recursion relation for $r = a$

  $a_{k+2} = -\frac{a_k b^2}{(-1-k)(-k-2)}$

- Solution for $r = a$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+a}, a_{k+2} = -\frac{a_k b^2}{(-1-k)(-k-2)}, a_1 = 0\right]$

- Recursion relation for $r = a+1$

  $a_{k+2} = -\frac{a_k b^2}{(-k-2)(-3-k)}$

- Solution for $r = a+1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+a+1}, a_{k+2} = -\frac{a_k b^2}{(-k-2)(-3-k)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} c_k x^{k+a}\right) + \left(\sum_{k=0}^{\infty} d_k x^{k+a+1}\right), c_{k+2} = -\frac{c_k b^2}{(-1-k)(-k-2)}, c_1 = 0, d_{k+2} = -\frac{d_k b^2}{(-k-2)(-k-3)}, d_1 =\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(x^2*diff(y(x),x$2)-2*a*x*diff(y(x),x)+(b^2*x^2+a*(a+1))*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^a(c_1 \sin(bx) + c_2 \cos(bx))$$

✓ Solution by Mathematica

Time used: 0.066 (sec). Leaf size: 42

```
DSolve[x^2*y''[x]-2*a*x*y'[x]+(b^2*x^2+a*(a+1))*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to c_1 x^a e^{-ibx} - \frac{i c_2 x^a e^{ibx}}{2b}$$

29.21 problem 130

Internal problem ID [10954]
Internal file name [`OUTPUT/10210_Sunday_December_31_2023_11_08_35_AM_56902748/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 130.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - 2axy' + (-b^2x^2 + a(a+1))\, y = 0}$$

29.21.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2a}{x} \\ q(x) &= \frac{-b^2x^2 + a^2 + a}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{-b^2x^2 + a^2 + a}{x^2} - \frac{\left(-\frac{2a}{x}\right)'}{2} - \frac{\left(-\frac{2a}{x}\right)^2}{4} \\
&= \frac{-b^2x^2 + a^2 + a}{x^2} - \frac{\left(\frac{2a}{x^2}\right)}{2} - \frac{\left(\frac{4a^2}{x^2}\right)}{4} \\
&= \frac{-b^2x^2 + a^2 + a}{x^2} - \left(\frac{a}{x^2}\right) - \frac{a^2}{x^2} \\
&= -b^2
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{2a}{x}}{2}} \\
&= x^a
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$y = v(x)\, x^a \tag{4}$$

Applying this change of variable to the original ode results in

$$-x^{a+2}\left(v(x)\, b^2 - v''(x)\right) = 0$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = -1, B = 0, C = b^2$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$-\lambda^2 \mathrm{e}^{\lambda x} + b^2 \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$b^2 - \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = -1, B = 0, C = b^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(-1)} \pm \frac{1}{(2)(-1)}\sqrt{0^2 - (4)(-1)(b^2)} \\ &= \pm - \sqrt{b^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= + - \sqrt{b^2} \\ \lambda_2 &= - - \sqrt{b^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\sqrt{b^2} \\ \lambda_2 &= \sqrt{b^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ v(x) &= c_1 e^{\left(-\sqrt{b^2}\right)x} + c_2 e^{\left(\sqrt{b^2}\right)x}\end{aligned}$$

Or

$$v(x) = c_1 \mathrm{e}^{-\sqrt{b^2}\, x} + c_2 \mathrm{e}^{\sqrt{b^2}\, x}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x) \\ &= \left(c_1 \mathrm{e}^{-\sqrt{b^2}\, x} + c_2 \mathrm{e}^{\sqrt{b^2}\, x}\right)(z(x)) \qquad (7)\end{aligned}$$

But from (5)

$$z(x) = x^a$$

Hence (7) becomes

$$y = \left(c_1 \mathrm{e}^{-\sqrt{b^2}\, x} + c_2 \mathrm{e}^{\sqrt{b^2}\, x}\right) x^a$$

Summary
The solution(s) found are the following

$$y = \left(c_1 \mathrm{e}^{-\sqrt{b^2}\, x} + c_2 \mathrm{e}^{\sqrt{b^2}\, x}\right) x^a \tag{1}$$

Verification of solutions

$$y = \left(c_1 \mathrm{e}^{-\sqrt{b^2}\, x} + c_2 \mathrm{e}^{\sqrt{b^2}\, x}\right) x^a$$

Verified OK.

29.21.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - 2axy' + \left(-b^2 x^2 + a^2 + a\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y' x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, xy' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= a + \frac{1}{2} \\ \beta &= ib \\ n &= -\frac{1}{2} \\ \gamma &= 1 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x^{a+\frac{1}{2}} \sqrt{2}\, \cosh(bx)}{\sqrt{\pi}\, \sqrt{ibx}} + \frac{i c_2 x^{a+\frac{1}{2}} \sqrt{2}\, \sinh(bx)}{\sqrt{\pi}\, \sqrt{ibx}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^{a+\frac{1}{2}} \sqrt{2}\, \cosh(bx)}{\sqrt{\pi}\, \sqrt{ibx}} + \frac{i c_2 x^{a+\frac{1}{2}} \sqrt{2}\, \sinh(bx)}{\sqrt{\pi}\, \sqrt{ibx}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{a+\frac{1}{2}} \sqrt{2}\, \cosh(bx)}{\sqrt{\pi}\, \sqrt{ibx}} + \frac{i c_2 x^{a+\frac{1}{2}} \sqrt{2}\, \sinh(bx)}{\sqrt{\pi}\, \sqrt{ibx}}$$

Verified OK.

29.21.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 2axy' + (-b^2x^2 + a^2 + a)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2ax \\ C &= -b^2x^2 + a^2 + a \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{b^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= b^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (b^2)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 133: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = b^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{\sqrt{b^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2ax}{x^2}\,dx} \\
&= z_1 e^{a\ln(x)} \\
&= z_1 (x^a)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\mathrm{csgn}(b)bx} x^a
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2ax}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2a\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{csgn}(b)\,\mathrm{e}^{-2\,\mathrm{csgn}(b)bx}}{2b}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\mathrm{csgn}(b)bx} x^a\right) + c_2 \left(\mathrm{e}^{\mathrm{csgn}(b)bx} x^a \left(-\frac{\mathrm{csgn}(b)\,\mathrm{e}^{-2\,\mathrm{csgn}(b)bx}}{2b}\right)\right)
\end{aligned}
$$

Simplifying the solution $y = c_1 \mathrm{e}^{\mathrm{csgn}(b)bx} x^a - \frac{c_2 x^a\,\mathrm{csgn}(b)\mathrm{e}^{-\,\mathrm{csgn}(b)bx}}{2b}$ to $y = c_1 \mathrm{e}^{bx} x^a - \frac{c_2 x^a \mathrm{e}^{-bx}}{2b}$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{bx} x^a - \frac{c_2 x^a \mathrm{e}^{-bx}}{2b} \tag{1}
$$

Verification of solutions

$$y = c_1 e^{bx} x^a - \frac{c_2 x^a e^{-bx}}{2b}$$

Verified OK.

29.21.4 Maple step by step solution

Let's solve

$y'' x^2 - 2axy' + (-b^2x^2 + a^2 + a)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(-b^2x^2+a^2+a)y}{x^2} + \frac{2ay'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2ay'}{x} + \frac{(-b^2x^2+a^2+a)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2a}{x}, P_3(x) = \frac{-b^2x^2+a^2+a}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -2a$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = a^2 + a$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y'' x^2 - 2axy' + (-b^2x^2 + a^2 + a)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-r + 1 + a)\,(-r + a)\, x^r + a_1(-r + a)\,(-r - 1 + a)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(-r + 1 - k + a)\,(-r - k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-r + 1 + a)\,(-r + a) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{a, a + 1\}$

- Each term must be 0

  $a_1(-r + a)\,(-r - 1 + a) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(-r + 1 - k + a)\,(-r - k + a) - a_{k-2}b^2 = 0$

- Shift index using $k- > k + 2$

  $a_{k+2}(-r - 1 - k + a)\,(-r - k - 2 + a) - a_k b^2 = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k b^2}{(-r-1-k+a)(-r-k-2+a)}$

- Recursion relation for $r = a$

$a_{k+2} = \frac{a_k b^2}{(-1-k)(-k-2)}$

- Solution for $r = a$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+a}, a_{k+2} = \frac{a_k b^2}{(-1-k)(-k-2)}, a_1 = 0\right]$$

- Recursion relation for $r = a + 1$

$a_{k+2} = \frac{a_k b^2}{(-k-2)(-3-k)}$

- Solution for $r = a + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+a+1}, a_{k+2} = \frac{a_k b^2}{(-k-2)(-3-k)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} c_k x^{k+a}\right) + \left(\sum_{k=0}^{\infty} d_k x^{k+a+1}\right), c_{k+2} = \frac{c_k b^2}{(-1-k)(-k-2)}, c_1 = 0, d_{k+2} = \frac{d_k b^2}{(-k-2)(-k-3)}, d_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(x^2*diff(y(x),x$2)-2*a*x*diff(y(x),x)+(-b^2*x^2+a*(a+1))*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^a (c_1 \sinh(bx) + c_2 \cosh(bx))$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 35

```
DSolve[x^2*y''[x]-2*a*x*y'[x]+(-b^2*x^2+a*(a+1))*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to c_1 x^a e^{-bx} + \frac{c_2 x^a e^{bx}}{2b}$$

29.22 problem 131

Internal problem ID [10955]
Internal file name [`OUTPUT/10211_Sunday_December_31_2023_11_08_37_AM_37366069/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 131.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \lambda xy' + \left(a\,x^2 + bx + c\right) y = 0}$$

29.22.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \lambda xy' + \left(a\,x^2 + bx + c\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}\left(n, \beta\, x^\gamma\right) + c_2 \operatorname{BesselY}\left(n, \beta\, x^\gamma\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= \frac{1}{2}-\frac{\lambda}{2} \\ \beta &= 2 \\ n &= \sqrt{\lambda^{2}-4c-2\lambda+1} \\ \gamma &= \frac{1}{2}\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_{1} x^{\frac{1}{2}-\frac{\lambda}{2}} \operatorname{BesselJ}\left(\sqrt{\lambda^{2}-4c-2\lambda+1}, 2\sqrt{x}\right)+c_{2} x^{\frac{1}{2}-\frac{\lambda}{2}} \operatorname{BesselY}\left(\sqrt{\lambda^{2}-4c-2\lambda+1}, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & c_{1} x^{\frac{1}{2}-\frac{\lambda}{2}} \operatorname{BesselJ}\left(\sqrt{\lambda^{2}-4c-2\lambda+1}, 2\sqrt{x}\right) \\ & + c_{2} x^{\frac{1}{2}-\frac{\lambda}{2}} \operatorname{BesselY}\left(\sqrt{\lambda^{2}-4c-2\lambda+1}, 2\sqrt{x}\right)\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = {} & c_{1} x^{\frac{1}{2}-\frac{\lambda}{2}} \operatorname{BesselJ}\left(\sqrt{\lambda^{2}-4c-2\lambda+1}, 2\sqrt{x}\right) \\ & + c_{2} x^{\frac{1}{2}-\frac{\lambda}{2}} \operatorname{BesselY}\left(\sqrt{\lambda^{2}-4c-2\lambda+1}, 2\sqrt{x}\right)\end{aligned}$$

Verified OK.

29.22.2 Maple step by step solution

Let's solve

$y'' x^{2}+\lambda x y'+\left(a\,x^{2}+b x+c\right) y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{\left(a\,x^{2}+b x+c\right) y}{x^{2}}-\frac{\lambda y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{\lambda y'}{x}+\frac{\left(a\,x^{2}+b x+c\right) y}{x^{2}}=0$

□ Check to see if $x_{0}=0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{\lambda}{x}, P_3(x) = \frac{a\,x^2+bx+c}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \lambda$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = c$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2 + \lambda xy' + (a\,x^2 + bx + c)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(\lambda r + r^2 + c - r)\,x^r + ((\lambda r + r^2 + c + \lambda + r)\,a_1 + a_0 b)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k^2 + \lambda k + 2kr + \lambda r +\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\lambda r + r^2 + c - r = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2} - \frac{\lambda}{2} - \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}, \frac{1}{2} - \frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}\right\}$
- Each term must be 0

  $(\lambda r + r^2 + c + \lambda + r) a_1 + a_0 b = 0$
- Solve for the dependent coefficient(s)

  $a_1 = -\frac{a_0 b}{\lambda r + r^2 + c + \lambda + r}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + (2r + \lambda - 1) k + r^2 + (\lambda - 1) r + c) a_k + a_{k-2} a + b a_{k-1} = 0$
- Shift index using $k- >k + 2$

  $\left((k+2)^2 + (2r + \lambda - 1)(k+2) + r^2 + (\lambda - 1) r + c\right) a_{k+2} + a_k a + b a_{k+1} = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k a + b a_{k+1}}{k^2 + \lambda k + 2kr + \lambda r + r^2 + c + 3k + 2\lambda + 3r + 2}$
- Recursion relation for $r = \frac{1}{2} - \frac{\lambda}{2} - \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}$

  $a_{k+2} = -\frac{a_k a + b a_{k+1}}{k^2 + \lambda k + 2k\left(\frac{1}{2} - \frac{\lambda}{2} - \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}\right) + \lambda\left(\frac{1}{2} - \frac{\lambda}{2} - \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}\right) + \left(\frac{1}{2} - \frac{\lambda}{2} - \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}\right)^2 + c + 3k + \frac{\lambda}{2} + \frac{7}{2} - \frac{3\sqrt{\lambda^2-4c-}}{2}}$
- Solution for $r = \frac{1}{2} - \frac{\lambda}{2} - \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}-\frac{\lambda}{2}-\frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}}, a_{k+2} = -\frac{a_k a + b a_{k+1}}{k^2 + \lambda k + 2k\left(\frac{1}{2} - \frac{\lambda}{2} - \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}\right) + \lambda\left(\frac{1}{2} - \frac{\lambda}{2} - \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}\right) + \left(\frac{1}{2} - \frac{\lambda}{2}\right.}\right.$
- Recursion relation for $r = \frac{1}{2} - \frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}$

  $a_{k+2} = -\frac{a_k a + b a_{k+1}}{k^2 + \lambda k + 2k\left(\frac{1}{2} - \frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}\right) + \lambda\left(\frac{1}{2} - \frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}\right) + \left(\frac{1}{2} - \frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}\right)^2 + c + 3k + \frac{\lambda}{2} + \frac{7}{2} + \frac{3\sqrt{\lambda^2-4c-}}{2}}$
- Solution for $r = \frac{1}{2} - \frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}}, a_{k+2} = -\frac{a_k a + b a_{k+1}}{k^2 + \lambda k + 2k\left(\frac{1}{2} - \frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}\right) + \lambda\left(\frac{1}{2} - \frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}\right) + \left(\frac{1}{2} - \frac{\lambda}{2}\right.}\right.$
- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} d_k x^{k+\frac{1}{2}-\frac{\lambda}{2}-\frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}}\right)+\left(\sum_{k=0}^{\infty} e_k x^{k+\frac{1}{2}-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4c-2\lambda+1}}{2}}\right), d_{k+2}=-\frac{}{k^2+\lambda k+2k\left(\frac{1}{2}-\frac{\lambda}{2}-\frac{\sqrt{\lambda^2-}}{}\right.}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 75

```
dsolve(x^2*diff(y(x),x$2)+lambda*x*diff(y(x),x)+(a*x^2+b*x+c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^{-\frac{\lambda}{2}} \left(\text{WhittakerW}\left(-\frac{ib}{2\sqrt{a}}, \frac{\sqrt{\lambda^2 - 4c - 2\lambda + 1}}{2}, 2i\sqrt{a}\, x \right) c_2 + \text{WhittakerM}\left(-\frac{ib}{2\sqrt{a}}, \frac{\sqrt{\lambda^2 - 4c - 2\lambda + 1}}{2}, 2i\sqrt{a}\, x \right) c_1 \right)$$

✓ Solution by Mathematica

Time used: 0.224 (sec). Leaf size: 159

```
DSolve[x^2*y''[x]+\[Lambda]*x*y'[x]+(a*x^2+b*x+c)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to e^{-i\sqrt{a}x} x^{\frac{1}{2}\left(\sqrt{(\lambda-1)^2-4c}-\lambda+1\right)} \left(c_1 \text{HypergeometricU}\left(\frac{1}{2}\left(\frac{ib}{\sqrt{a}} + \sqrt{(\lambda-1)^2-4c} + 1 \right), \sqrt{(\lambda-1)^2-4c} + 1, 2i\sqrt{a}x \right) + c_2 L_{\frac{1}{2}\left(-\frac{ib}{\sqrt{a}} - \sqrt{(\lambda-1)^2-4c} - 1\right)}^{\sqrt{(\lambda-1)^2-4c}} \left(2i\sqrt{a}x\right) \right)$$

29.23 problem 132

Internal problem ID [10956]
Internal file name [`OUTPUT/10212_Sunday_December_31_2023_11_09_42_AM_75486021/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 132.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + axy' + (b\,x^n + c)\,y = 0}$$

29.23.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + axy' + (b\,x^n + c)\,y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\,x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\,x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= \frac{1}{2}-\frac{a}{2} \\
\beta &= \frac{2\sqrt{b}}{n} \\
n &= \frac{\sqrt{a^{2}-2a-4c+1}}{n} \\
\gamma &= \frac{n}{2}
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(\frac{\sqrt{a^{2}-2a-4c+1}}{n}, \frac{2\sqrt{b}\, x^{\frac{n}{2}}}{n}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(\frac{\sqrt{a^{2}-2a-4c+1}}{n}, \frac{2\sqrt{b}\, x^{\frac{n}{2}}}{n}\right)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(\frac{\sqrt{a^{2}-2a-4c+1}}{n}, \frac{2\sqrt{b}\, x^{\frac{n}{2}}}{n}\right) \\
& + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(\frac{\sqrt{a^{2}-2a-4c+1}}{n}, \frac{2\sqrt{b}\, x^{\frac{n}{2}}}{n}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(\frac{\sqrt{a^{2}-2a-4c+1}}{n}, \frac{2\sqrt{b}\, x^{\frac{n}{2}}}{n}\right) \\
& + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(\frac{\sqrt{a^{2}-2a-4c+1}}{n}, \frac{2\sqrt{b}\, x^{\frac{n}{2}}}{n}\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 80

```
dsolve(x^2*diff(y(x),x$2)+a*x*diff(y(x),x)+(b*x^n+c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^{-\frac{a}{2}}\sqrt{x}\left(\operatorname{BesselY}\left(\frac{\sqrt{a^2-2a-4c+1}}{n}, \frac{2\sqrt{b}\,x^{\frac{n}{2}}}{n}\right)c_2 + \operatorname{BesselJ}\left(\frac{\sqrt{a^2-2a-4c+1}}{n}, \frac{2\sqrt{b}\,x^{\frac{n}{2}}}{n}\right)c_1\right)$$

✓ Solution by Mathematica

Time used: 0.287 (sec). Leaf size: 168

```
DSolve[x^2*y''[x]+a*x*y'[x]+(b*x^n+c)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to n^{\frac{a-1}{n}} b^{-\frac{a-1}{2n}} (x^n)^{-\frac{a-1}{2n}} \left(c_1 \operatorname{Gamma}\left(1 - \frac{\sqrt{a^2-2a-4c+1}}{n}\right) \operatorname{BesselJ}\left(-\frac{\sqrt{a^2-2a-4c+1}}{n}, \frac{2\sqrt{b}\sqrt{x^n}}{n}\right) + c_2 \operatorname{Gamma}\left(\frac{n+\sqrt{a^2-2a-4c+1}}{n}\right) \operatorname{BesselJ}\left(\frac{\sqrt{a^2-2a-4c+1}}{n}, \frac{2\sqrt{b}\sqrt{x^n}}{n}\right)\right)$$

29.24 problem 133

Internal problem ID [10957]
Internal file name [OUTPUT/10213_Sunday_December_31_2023_11_09_46_AM_10467483/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 133.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + axy' + x^n(b\,x^n + c)\,y = 0}$$

29.24.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + axy' + \left(b\,x^{2n} + x^n c\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1\,\mathrm{BesselJ}\,(n, \beta\,x^{\gamma}) + c_2\,\mathrm{BesselY}\,(n, \beta\,x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= \frac{1}{2} - \frac{a}{2} \\
\beta &= 2 \\
n &= -a + 1 \\
\gamma &= \frac{1}{2}
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-a+1, 2\sqrt{x}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-a+1, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-a+1, 2\sqrt{x}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-a+1, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselJ}\left(-a+1, 2\sqrt{x}\right) + c_2 x^{\frac{1}{2}-\frac{a}{2}} \operatorname{BesselY}\left(-a+1, 2\sqrt{x}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.218 (sec). Leaf size: 82

```
dsolve(x^2*diff(y(x),x$2)+a*x*diff(y(x),x)+x^n*(b*x^n+c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\text{WhittakerM}\left(-\frac{ic}{2n\sqrt{b}}, \frac{a-1}{2n}, \frac{2i\sqrt{b}\,x^n}{n}\right)c_1 + \text{WhittakerW}\left(-\frac{ic}{2n\sqrt{b}}, \frac{a-1}{2n}, \frac{2i\sqrt{b}\,x^n}{n}\right)c_2\right)x^{-\frac{a}{2}-\frac{n}{2}+\frac{1}{2}}$$

✓ Solution by Mathematica

Time used: 0.269 (sec). Leaf size: 165

```
DSolve[x^2*y''[x]+a*x*y'[x]+x^n*(b*x^n+c)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2^{\frac{a+n-1}{2n}}x^{\frac{1}{2}(-a-n+1)}(x^n)^{\frac{a+n-1}{2n}}e^{\frac{i\sqrt{b}x^n}{n}}\left(c_1\,\text{HypergeometricU}\left(-\frac{-a+\frac{ic}{\sqrt{b}}-n+1}{2n}, \frac{a+n-1}{n}, -\frac{2i\sqrt{b}x^n}{n}\right) + c_2L^{\frac{a-1}{n}}_{-\frac{a-\frac{ic}{\sqrt{b}}+n-1}{2n}}\left(-\frac{2i\sqrt{b}x^n}{n}\right)\right)$$

29.25 problem 134

Internal problem ID [10958]
Internal file name [`OUTPUT/10214_Sunday_December_31_2023_11_10_05_AM_77954774/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 134.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^2y'' + (ax + b)\,y' + yc = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 114

```
dsolve(x^2*diff(y(x),x$2)+(a*x+b)*diff(y(x),x)+c*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^{-\frac{\sqrt{a^2-2a-4c+1}}{2}-\frac{a}{2}+\frac{1}{2}}\left(\operatorname{KummerU}\left(-\frac{1}{2}+\frac{\sqrt{a^2-2a-4c+1}}{2}+\frac{a}{2}, 1+\sqrt{a^2-2a-4c+1}, \frac{b}{x}\right)c_2 + \operatorname{KummerM}\left(-\frac{1}{2}+\frac{\sqrt{a^2-2a-4c+1}}{2}+\frac{a}{2}, 1+\sqrt{a^2-2a-4c+1}, \frac{b}{x}\right)c_1\right)$$

✓ Solution by Mathematica

Time used: 0.574 (sec). Leaf size: 243

```
DSolve[x^2*y''[x]+(a*x+b)*y'[x]+c*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow$$

$$-i^{-\sqrt{a^2-2a-4c+1}+a+1} b^{\frac{1}{2}\left(-\sqrt{a^2-2a-4c+1}+a-1\right)} \left(\frac{1}{x}\right)^{\frac{1}{2}\left(-\sqrt{a^2-2a-4c+1}+a-1\right)} \left(c_2 i^{2\sqrt{a^2-2a-4c+1}} b^{\sqrt{a^2-2a-4c+1}} \left(\frac{1}{x}\right)^{\sqrt{}}\right.$$

$$\left. +1, \frac{b}{x}\right) + c_1 \text{Hypergeometric1F1}\left(\frac{1}{2}\left(a - \sqrt{a^2-2a-4c+1} - 1\right), 1 \right.$$

$$\left. -\sqrt{a^2-2a-4c+1}, \frac{b}{x}\right)\Bigg)$$

29.26 problem 135

Internal problem ID [10959]
Internal file name [`OUTPUT/10215_Sunday_December_31_2023_11_10_06_AM_80813510/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 135.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x^2y'' + a\,x^2y' + \left(b\,x^2 + cx + d\right)y = 0}$$

29.26.1 Maple step by step solution

Let's solve

$y''x^2 + a\,x^2y' + (b\,x^2 + cx + d)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -ay' - \frac{(b\,x^2+cx+d)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + ay' + \frac{(b\,x^2+cx+d)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$$\left[P_2(x)=a, P_3(x)=\frac{b\,x^2+cx+d}{x^2}\right]$$

- $x \cdot P_2(x)$ is analytic at $x=0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0}=0$$

- $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0}=d$$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $$y''x^2+a\,x^2y'+\left(b\,x^2+cx+d\right)y=0$$

- Assume series solution for y

  $$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m=0..2$

  $$x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

  $$x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y'$ to series expansion

  $$x^2 \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$$

- Shift index using $k->k-1$

  $$x^2 \cdot y'=\sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(r^2+d-r)\,x^r+\left(\left(r^2+d+r\right)a_1+a_0(ar+c)\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k\left(k^2+2kr+r^2+d-k-r\right)+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2+d-r=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{\frac{1}{2}-\frac{\sqrt{1-4d}}{2},\frac{1}{2}+\frac{\sqrt{1-4d}}{2}\right\}$$

- Each term must be 0

$$\left(r^2+d+r\right)a_1+a_0(ar+c)=0$$

- Solve for the dependent coefficient(s)

$$a_1=-\frac{a_0(ar+c)}{r^2+d+r}$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2+(2r-1)\,k+r^2+d-r\right)a_k+(ak+ar-a+c)\,a_{k-1}+a_{k-2}b=0$$

- Shift index using $k->k+2$

$$\left((k+2)^2+(2r-1)\,(k+2)+r^2+d-r\right)a_{k+2}+(a(k+2)+ar-a+c)\,a_{k+1}+a_kb=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{aka_{k+1}+ara_{k+1}+aa_{k+1}+a_kb+ca_{k+1}}{k^2+2kr+r^2+d+3k+3r+2}$$

- Recursion relation for $r=\frac{1}{2}-\frac{\sqrt{1-4d}}{2}$

$$a_{k+2}=-\frac{aka_{k+1}+a\left(\frac{1}{2}-\frac{\sqrt{1-4d}}{2}\right)a_{k+1}+aa_{k+1}+a_kb+ca_{k+1}}{k^2+2k\left(\frac{1}{2}-\frac{\sqrt{1-4d}}{2}\right)+\left(\frac{1}{2}-\frac{\sqrt{1-4d}}{2}\right)^2+d+3k+\frac{7}{2}-\frac{3\sqrt{1-4d}}{2}}$$

- Solution for $r=\frac{1}{2}-\frac{\sqrt{1-4d}}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}-\frac{\sqrt{1-4d}}{2}},a_{k+2}=-\frac{aka_{k+1}+a\left(\frac{1}{2}-\frac{\sqrt{1-4d}}{2}\right)a_{k+1}+aa_{k+1}+a_kb+ca_{k+1}}{k^2+2k\left(\frac{1}{2}-\frac{\sqrt{1-4d}}{2}\right)+\left(\frac{1}{2}-\frac{\sqrt{1-4d}}{2}\right)^2+d+3k+\frac{7}{2}-\frac{3\sqrt{1-4d}}{2}},a_1=-\frac{a_0\left(a\left(\frac{1}{2}-\frac{\sqrt{1-}}{2}\right.\right.}{\left(\frac{1}{2}-\frac{\sqrt{1-4d}}{2}\right)^2+d}\right.$$

- Recursion relation for $r=\frac{1}{2}+\frac{\sqrt{1-4d}}{2}$

$$a_{k+2}=-\frac{aka_{k+1}+a\left(\frac{1}{2}+\frac{\sqrt{1-4d}}{2}\right)a_{k+1}+aa_{k+1}+a_kb+ca_{k+1}}{k^2+2k\left(\frac{1}{2}+\frac{\sqrt{1-4d}}{2}\right)+\left(\frac{1}{2}+\frac{\sqrt{1-4d}}{2}\right)^2+d+3k+\frac{7}{2}+\frac{3\sqrt{1-4d}}{2}}$$

- Solution for $r=\frac{1}{2}+\frac{\sqrt{1-4d}}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}+\frac{\sqrt{1-4d}}{2}},a_{k+2}=-\frac{aka_{k+1}+a\left(\frac{1}{2}+\frac{\sqrt{1-4d}}{2}\right)a_{k+1}+aa_{k+1}+a_kb+ca_{k+1}}{k^2+2k\left(\frac{1}{2}+\frac{\sqrt{1-4d}}{2}\right)+\left(\frac{1}{2}+\frac{\sqrt{1-4d}}{2}\right)^2+d+3k+\frac{7}{2}+\frac{3\sqrt{1-4d}}{2}},a_1=-\frac{a_0\left(a\left(\frac{1}{2}+\frac{\sqrt{1-}}{2}\right.\right.}{\left(\frac{1}{2}+\frac{\sqrt{1-4d}}{2}\right)^2+d}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} e_k x^{k+\frac{1}{2}-\frac{\sqrt{1-4d}}{2}}\right) + \left(\sum_{k=0}^{\infty} f_k x^{k+\frac{1}{2}+\frac{\sqrt{1-4d}}{2}}\right), e_{k+2} = -\frac{ake_{1+k}+a\left(\frac{1}{2}-\frac{\sqrt{1-4d}}{2}\right)e_{1+k}+ae_{1+k}+e_kb+ce}{k^2+2k\left(\frac{1}{2}-\frac{\sqrt{1-4d}}{2}\right)+\left(\frac{1}{2}-\frac{\sqrt{1-4d}}{2}\right)^2+d+3k+\frac{7}{2}-}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 79

```
dsolve(x^2*diff(y(x),x$2)+a*x^2*diff(y(x),x)+(b*x^2+c*x+d)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{ax}{2}}\left(\text{WhittakerM}\left(\frac{c}{\sqrt{a^2-4b}}, \frac{\sqrt{1-4d}}{2}, \sqrt{a^2-4b}\,x\right)c_1 + \text{WhittakerW}\left(\frac{c}{\sqrt{a^2-4b}}, \frac{\sqrt{1-4d}}{2}, \sqrt{a^2-4b}\,x\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.228 (sec). Leaf size: 157

```
DSolve[x^2*y''[x]+a*x^2*y'[x]+(b*x^2+c*x+d)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^{\frac{1}{2}(\sqrt{1-4d}+1)}e^{-\frac{1}{2}x\left(\sqrt{a^2-4b}+a\right)}\left(c_1 \operatorname{HypergeometricU}\left(\frac{1}{2}\left(-\frac{2c}{\sqrt{a^2-4b}}+\sqrt{1-4d}+1\right),\sqrt{1-4d}+1,\sqrt{a^2-4b}x\right)+c_2 L_{\frac{c}{\sqrt{a^2-4b}}-\frac{1}{2}\sqrt{1-4d}-\frac{1}{2}}^{\sqrt{1-4d}}\left(\sqrt{a^2-4b}x\right)\right)$$

29.27 problem 136

Internal problem ID [10960]
Internal file name [`OUTPUT/10216_Sunday_December_31_2023_11_10_06_AM_94347684/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 136.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^2y'' + \left(a\,x^2 + b\right) y' + c\left(\left(a - c\right) x^2 + b\right) y = 0}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunD  ODE, case  c = 0
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.485 (sec). Leaf size: 243

```
dsolve(x^2*diff(y(x),x$2)+(a*x^2+b)*diff(y(x),x)+c*((a-c)*x^2+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x}\left(\operatorname{HeunD}\left(-4\sqrt{-b\left(-2c+a\right)}, -4\sqrt{-b\left(-2c+a\right)}-1 + \left(-4a+8c\right)b, 8\sqrt{-b\left(-2c+a\right)}, -4\sqrt{-b\left(-2c+a\right)}+1 + \left(-8c+4a\right)b, \frac{\sqrt{-b\left(-2c+a\right)}\,x-b}{\sqrt{-b\left(-2c+a\right)}\,x+b}\right)\mathrm{e}^{-x(a-c)}c_2 + \operatorname{HeunD}\left(4\sqrt{-b\left(-2c+a\right)}, -4\sqrt{-b\left(-2c+a\right)}-1+\left(-4a+8c\right)b, 8\sqrt{-b\left(-2c+a\right)}, -4\sqrt{-b\left(-2c+a\right)}+1 + \left(-8c+4a\right)b, \frac{\sqrt{-b\left(-2c+a\right)}\,x-b}{\sqrt{-b\left(-2c+a\right)}\,x+b}\right)\mathrm{e}^{\frac{-c\,x^2+b}{x}}c_1\right)$$

✓ Solution by Mathematica

Time used: 1.026 (sec). Leaf size: 44

```
DSolve[x^2*y''[x]+(a*x^2+b)*y'[x]+c*((a-c)*x^2+b)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to e^{-cx}\left(c_2\int_1^x e^{\frac{b}{K[1]}-aK[1]+2cK[1]}dK[1]+c_1\right)$$

29.28 problem 137

Internal problem ID [10961]
Internal file name [OUTPUT/10217_Sunday_December_31_2023_11_10_08_AM_81546286/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 137.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(a\,x^2 + bx\right)y' - yb = 0}$$

29.28.1 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$x^2y'' + x(ax + b)\,y' - yb = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{ax + b}{x} \\
q(x) &= -\frac{b}{x^2} \\
r(x) &= 0
\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(\frac{(ax+b)\,\xi(x)}{x}\right)' + \left(-\frac{b\xi(x)}{x^2}\right) &= 0 \\
\xi''(x) - \frac{(ax+b)\,\xi'(x)}{x} + \left(-\frac{a}{x} + \frac{ax+b}{x^2} - \frac{b}{x^2}\right)\xi(x) &= 0
\end{aligned}$$

Which is solved for $\xi(x)$. This is second order ode with missing dependent variable $\xi(x)$. Let

$$p(x) = \xi'(x)$$

Then

$$p'(x) = \xi''(x)$$

Hence the ode becomes

$$p'(x)\,x + (-ax-b)\,p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned}
p' &= F(x,p) \\
&= f(x)g(p) \\
&= \frac{p(ax+b)}{x}
\end{aligned}$$

Where $f(x) = \frac{ax+b}{x}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= \frac{ax+b}{x}\,dx \\
\int \frac{1}{p}\,dp &= \int \frac{ax+b}{x}\,dx \\
\ln(p) &= ax + b\ln(x) + c_1 \\
p &= \mathrm{e}^{ax+b\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{ax+b\ln(x)}
\end{aligned}$$

Which simplifies to

$$p(x) = c_1 x^b \mathrm{e}^{ax}$$

Since $p = \xi'(x)$ then the new first order ode to solve is

$$\xi'(x) = c_1 x^b \mathrm{e}^{ax}$$

Integrating both sides gives

$$
\begin{aligned}
\xi(x) &= \int c_1 x^b \mathrm{e}^{ax}\, \mathrm{d}x \\
&= -\frac{c_1(-a)^{-b}\left(x^b(-a)^b b\Gamma(b)\,(-ax)^{-b} - x^b(-a)^b\,\mathrm{e}^{ax} - x^b(-a)^b\, b(-ax)^{-b}\,\Gamma(b,-ax)\right)}{a} + c_2
\end{aligned}
$$

The original ode (2) now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x) \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)}{\xi\,(}
\end{aligned}
$$

$$
y' + y\left(\frac{ax+b}{x} + \frac{c_3(-a)^{-b}\left(-\frac{x^b b(-a)^b \mathrm{e}^{ax}}{x} - x^b(-a)^b\, a\,\mathrm{e}^{ax} - x^b(-a)^b\, b(-ax)^{-b}\, a(-ax)^{b-1}\,\mathrm{e}^{ax}\right)}{a\left(-\frac{c_3(-a)^{-b}\left(x^b(-a)^b b\Gamma(b)(-ax)^{-b} - x^b(-a)^b \mathrm{e}^{ax} - x^b(-a)^b b(-ax)^{-b}\Gamma(b,-ax)\right)}{a} + c_2\right)}\right) = 0
$$

Which is now a first order ode. This is now solved for y. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{y\Big(c_3(-a)^{-b}\, a\, x^b(-a)^b\, b(-ax)^{-b}\,(-ax)^{b-1}\,\mathrm{e}^{ax}x + x^b(-a)^{-b}\,(-a)^b\,(-ax)^{-b}\,\Gamma(b)\, c_3abx - x^b(-a)^{-b}\,(-a}{x\,\Big(-c_3\,(-a)^{-b}\,x^b\,(-a)^b\, b\Gamma\,(b)\,(-ax)^{-b} + c_3\,(-}
\end{aligned}
$$

Where $f(x) = \frac{c_3(-a)^{-b} a\, x^b(-a)^b b(-ax)^{-b}(-ax)^{b-1}\mathrm{e}^{ax}x + x^b(-a)^{-b}(-a)^b(-ax)^{-b}\Gamma(b)c_3abx - x^b(-a)^{-b}(-a)^b(-ax)^{-b}\Gamma(b,-ax)c_3abx + x}{\Big(-c_3(-a)^{-b}x^b(-a)^b b\Gamma(b)(-ax)^{-b} + c_3(-a)^{-b}x^b(-a)^b b(-ax)^{-b}\Gamma(b,-a}$
and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= \frac{c_3(-a)^{-b}\, a\, x^b(-a)^b\, b(-ax)^{-b}\,(-ax)^{b-1}\,\mathrm{e}^{ax}x + x^b(-a)^{-b}\,(-a)^b\,(-ax)^{-b}\,\Gamma(b)\, c_3abx - x^b(-a)^{-b}\,(}{\Big(-c_3\,(-a)^{-b}\,x^b\,(-a)^b\, b\Gamma\,(b)\,(-ax)^{-b} + c_3\,(} \\
\int \frac{1}{y}\, dy &= \int \frac{c_3(-a)^{-b}\, a\, x^b(-a)^b\, b(-ax)^{-b}\,(-ax)^{b-1}\,\mathrm{e}^{ax}x + x^b(-a)^{-b}\,(-a)^b\,(-ax)^{-b}\,\Gamma(b)\, c_3abx - x^b(-a)^{-}}{\Big(-c_3\,(-a)^{-b}\,x^b\,(-a)^b\, b\Gamma\,(b)\,(-ax)^{-b} +} \\
\ln{(y)} &= \int \frac{c_3(-a)^{-b}\, a\, x^b(-a)^b\, b(-ax)^{-b}\,(-ax)^{b-1}\,\mathrm{e}^{ax}x + x^b(-a)^{-b}\,(-a)^b\,(-ax)^{-b}\,\Gamma(b)\, c_3abx - x^b(-a)^{-}}{\Big(-c_3\,(-a)^{-b}\,x^b\,(-a)^b\, b\Gamma\,(b)\,(-ax)^{-b} +} \\
y &= \mathrm{e}^{\int \frac{c_3(-a)^{-b} a\, x^b(-a)^b b(-ax)^{-b}(-ax)^{b-1}\mathrm{e}^{ax}x + x^b(-a)^{-b}(-a)^b(-ax)^{-b}\Gamma(b)c_3abx - x^b(-a)^{-b}(-a)^b(-ax)^{-b}\Gamma(b,-ax)c_3abx + x^b(-a)^{-b}(-a)^b}{\big(-c_3(-a)^{-b}x^b(-a)^b b\Gamma(b)(-ax)^{-b} + c_3(-a)^{-b}x^b(-a)^b b(-ax)^{-b}\Gamma(b,-ax) + c_3(-a)^{-b}x}} \\
&= c_3\mathrm{e}^{\int \frac{c_3(-a)^{-b} a\, x^b(-a)^b b(-ax)^{-b}(-ax)^{b-1}\mathrm{e}^{ax}x + x^b(-a)^{-b}(-a)^b(-ax)^{-b}\Gamma(b)c_3abx - x^b(-a)^{-b}(-a)^b(-ax)^{-b}\Gamma(b,-ax)c_3abx + x^b(-a)^{-b}(-}{\big(-c_3(-a)^{-b}x^b(-a)^b b\Gamma(b)(-ax)^{-b} + c_3(-a)^{-b}x^b(-a)^b b(-ax)^{-b}\Gamma(b,-ax) + c_3(-a)^{-}}}
\end{aligned}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_3 e^{\int \frac{c_3(-a)^{-b} a x^b (-a)^b b (-ax)^{-b} (-ax)^{b-1} e^{ax} x + x^b (-a)^{-b} (-a)^b (-ax)^{-b} \Gamma(b) c_3 a b x - x^b (-a)^{-b} (-a)^b (-ax)^{-b} \Gamma(b, -ax) c_3 a b x + x^b (-a)^{-b} (-a)^b (-ax)^{-b} \cdots}{\left(-c_3 (-a)^{-b} x^b (-a)^b b \Gamma(b) (-ax)^{-b} + c_3 (-a)^{-b} x^b (-a)^b b (-ax)^{-b} \Gamma(b, -ax) + c_3 (-a)^{-b} x^b (-a)^b e^{a \cdots}\right.}}$$

Summary

The solution(s) found are the following

$$y = c_3 e^{\int \frac{c_3(-a)^{-b} a x^b (-a)^b b (-ax)^{-b} (-ax)^{b-1} e^{ax} x + x^b (-a)^{-b} (-a)^b (-ax)^{-b} \Gamma(b) c_3 a b x - x^b (-a)^{-b} (-a)^b (-ax)^{-b} \Gamma(b, -ax) c_3 a b x + x^b (-a)^{-b} (-a)^b (-ax)^{-b} \cdots}{\left(-c_3 (-a)^{-b} x^b (-a)^b b \Gamma(b) (-ax)^{-b} + c_3 (-a)^{-b} x^b (-a)^b b (-ax)^{-b} \Gamma(b, -ax) + c_3 (-a)^{-b} x^b (-a)^b e^{a \cdots}\right.}} \tag{1}$$

Verification of solutions

$$y = c_3 e^{\int \frac{c_3(-a)^{-b} a x^b (-a)^b b (-ax)^{-b} (-ax)^{b-1} e^{ax} x + x^b (-a)^{-b} (-a)^b (-ax)^{-b} \Gamma(b) c_3 a b x - x^b (-a)^{-b} (-a)^b (-ax)^{-b} \Gamma(b, -ax) c_3 a b x + x^b (-a)^{-b} (-a)^b (-ax)^{-b} \cdots}{\left(-c_3 (-a)^{-b} x^b (-a)^b b \Gamma(b) (-ax)^{-b} + c_3 (-a)^{-b} x^b (-a)^b b (-ax)^{-b} \Gamma(b, -ax) + c_3 (-a)^{-b} x^b (-a)^b e^{a \cdots}\right.}}$$

Verified OK.

29.28.2 Maple step by step solution

Let's solve

$y'' x^2 + x(ax + b) y' - yb = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(ax+b)y'}{x} + \frac{by}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(ax+b)y'}{x} - \frac{by}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{ax+b}{x}, P_3(x) = -\frac{b}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = b$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -b$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y'' x^2 + x(ax+b)\, y' - yb = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $$a_0(r-1)\,(b+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+b+r) + a a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(r-1)\,(b+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, -b\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+b+r) + a a_{k-1}) = 0$

- Shift index using $k- > k+1$

  $(k+r)\,(a_{k+1}(k+1+b+r) + a a_k) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{a a_k}{k+1+b+r}$

- Recursion relation for $r = 1$

$a_{k+1} = -\frac{a a_k}{k+2+b}$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a a_k}{k+2+b}\right]$$

- Recursion relation for $r = -b$

$a_{k+1} = -\frac{a a_k}{k+1}$

- Solution for $r = -b$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-b}, a_{k+1} = -\frac{a a_k}{k+1}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} c_k x^{1+k}\right) + \left(\sum_{k=0}^{\infty} d_k x^{k-b}\right), c_{1+k} = -\frac{a c_k}{k+2+b}, d_{1+k} = -\frac{a d_k}{1+k}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
dsolve(x^2*diff(y(x),x$2)+(a*x^2+b*x)*diff(y(x),x)-b*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\mathrm{e}^{-ax} c_2(\Gamma(b, -ax)\, b - \Gamma(b+1))\,(-ax)^{-b} + c_1 x^{-b} \mathrm{e}^{-ax} - c_2$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 43

```
DSolve[x^2*y''[x]+(a*x^2+b*x)*y'[x]-b*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-ax}\left(\frac{c_1(-ax)^{-b}\Gamma(b+1,-ax)}{a}+c_2x^{-b}\right)$$

29.29 problem 138

Internal problem ID [10962]
Internal file name [`OUTPUT/10218_Sunday_December_31_2023_11_10_11_AM_65170388/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 138.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$x^2y'' + \left(a\,x^2 + bx\right) y' + \left(k(a-k)\,x^2 + (an + bk - 2kn)\,x + n(-n + b - 1)\right) y = 0$$

29.29.1 Maple step by step solution

Let's solve

$y''x^2 + x(ax+b)\,y' + \left(-n^2 + ((a-2k)\,x + b - 1)\,n + kx((a-k)\,x + b)\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(ak\,x^2 - k^2x^2 + anx + kxb - 2knx + nb - n^2 - n)y}{x^2} - \frac{(ax+b)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(ax+b)y'}{x} + \frac{(ak\,x^2 - k^2x^2 + anx + kxb - 2knx + nb - n^2 - n)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$$\left[P_2(x)=\frac{ax+b}{x}, P_3(x)=\frac{ak\,x^2-k^2x^2+anx+kxb-2knx+nb-n^2-n}{x^2}\right]$$

- $x \cdot P_2(x)$ is analytic at $x=0$

  $$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=b$$

- $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=nb-n^2-n$$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $$y''x^2+x(ax+b)\,y'+\left(ak\,x^2-k^2x^2+anx+kxb-2knx+nb-n^2-n\right)y=0$$

- Assume series solution for y

  $$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m=0..2$

  $$x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

  $$x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m=1..2$

  $$x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(n+r)(b+r-1-n)x^r + (a_1(1+n+r)(b+r-n) + a_0(an+ar+bk-2kn))x^{1+r} + \left(\sum_{k=2}^{\infty}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(n+r)(b+r-1-n) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-n, n-b+1\}$$

- Each term must be 0

$$a_1(1+n+r)(b+r-n) + a_0(an+ar+bk-2kn) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{a_0(an+ar+bk-2kn)}{nb+br-n^2+r^2+b-n+r}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+n+r)(k+b+r-1-n) + ((k+n+r-1)a + k(-2n+b))a_{k-1} + ka_{k-2}(a-k) = 0$$

- Shift index using $k- >k+2$

$$a_{k+2}(k+2+n+r)(k+1+b+r-n) + ((k+1+n+r)a + k(-2n+b))a_{k+1} + ka_k(a-k) =$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k ak + ak a_{k+1} + an a_{k+1} + ar a_{k+1} + bk a_{k+1} - a_k k^2 - 2kn a_{k+1} + a a_{k+1}}{(k+2+n+r)(k+1+b+r-n)}$$

- Recursion relation for $r = -n$

$$a_{k+2} = -\frac{a_k ak + ak a_{k+1} + bk a_{k+1} - a_k k^2 - 2kn a_{k+1} + a a_{k+1}}{(k+2)(k+1+b-2n)}$$

- Solution for $r = -n$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-n}, a_{k+2} = -\frac{a_k ak + ak a_{k+1} + bk a_{k+1} - a_k k^2 - 2kn a_{k+1} + a a_{k+1}}{(k+2)(k+1+b-2n)}, a_1 = -\frac{a_0(bk-2kn)}{-2n+b}\right]$$

- Recursion relation for $r = n-b+1$

$$a_{k+2} = -\frac{a_k ak + ak a_{k+1} + an a_{k+1} + a(n-b+1)a_{k+1} + bk a_{k+1} - a_k k^2 - 2kn a_{k+1} + a a_{k+1}}{(k+3+2n-b)(k+2)}$$

- Solution for $r = n-b+1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+n-b+1}, a_{k+2} = -\frac{a_k ak + ak a_{k+1} + an a_{k+1} + a(n-b+1)a_{k+1} + bk a_{k+1} - a_k k^2 - 2kn a_{k+1} + a a_{k+1}}{(k+3+2n-b)(k+2)}, a_1 = -\frac{}{nb}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{m=0}^{\infty} c_m x^{m-n}\right) + \left(\sum_{m=0}^{\infty} d_m x^{m+n-b+1}\right), c_{m+2} = -\frac{akc_m + amc_{m+1} + bkc_{m+1} - k^2 c_m - 2knc_{m+1} + ac_{m+1}}{(m+2)(m+1+b-2n)}, c\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 48

```
dsolve(x^2*diff(y(x),x$2)+(a*x^2+b*x)*diff(y(x),x)+(k*(a-k)*x^2+(a*n+b*k-2*k*n)*x+n*(b-n-1))
```

$$y(x) = c_1 x^{-n} e^{-kx} + c_2 x^{-\frac{b}{2}} \operatorname{WhittakerM}\left(-\frac{b}{2}+n, -\frac{b}{2}+n+\frac{1}{2}, (-2k+a)\,x\right) e^{-\frac{ax}{2}}$$

✓ Solution by Mathematica

Time used: 0.504 (sec). Leaf size: 64

```
DSolve[x^2*y''[x]+(a*x^2+b*x)*y'[x]+(k*(a-k)*x^2+(a*n+b*k-2*k*n)*x+n*(b-n-1))*y[x]==0,y[x],x
```

$$y(x) \to e^{-kx} x^{-n} \left(c_1 - c_2 x^{-b+2n+1} (x(a-2k))^{b-2n-1} \Gamma(-b+2n+1, (a-2k)x)\right)$$

29.30 problem 139

Internal problem ID [10963]
Internal file name [OUTPUT/10219_Sunday_December_31_2023_11_10_12_AM_23575013/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 139.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$a_2x^2y'' + (a_1x^2 + b_1x)\, y' + (x^2a_0 + b_0x + c_0)\, y = 0$$

29.30.1 Maple step by step solution

Let's solve

$a_2x^2y'' + x(a_1x + b_1)\, y' + (x^2a_0 + b_0x + c_0)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2a_0+b_0x+c_0)y}{a_2x^2} - \frac{(a_1x+b_1)y'}{xa_2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(a_1x+b_1)y'}{xa_2} + \frac{(x^2a_0+b_0x+c_0)y}{a_2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{a_1 x + b_1}{a_2 x}, P_3(x) = \frac{x^2 a_0 + b_0 x + c_0}{a_2 x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{b_1}{a_2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{c_0}{a_2}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $a_2 x^2 y'' + x(a_1 x + b_1) y' + (x^2 a_0 + b_0 x + c_0) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r) x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1) x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(a_2r^2 - a_2r + b_1r + c_0)\, x^r + ((a_2r^2 + a_2r + b_1r + b_1 + c_0)\, a_1 + a_0(a_1r + b_0))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(a$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$a_2r^2 - a_2r + b_1r + c_0 = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{a_2-b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}}{2a_2}, -\frac{-a_2+b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}}{2a_2}\right\}$$

- Each term must be 0

$$(a_2r^2 + a_2r + b_1r + b_1 + c_0)\, a_1 + a_0(a_1r + b_0) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{a_0(a_1r+b_0)}{a_2r^2+a_2r+b_1r+b_1+c_0}$$

- Each term in the series must be 0, giving the recursion relation

$$((k+r)\,(k+r-1)\, a_2 + b_1k + b_1r + c_0)\, a_k + a_1ka_{k-1} + a_1ra_{k-1} + (-a_1 + b_0)\, a_{k-1} + a_{k-2}a_0 = 0$$

- Shift index using $k->k+2$

$$((k+2+r)\,(k+1+r)\, a_2 + b_1(k+2) + b_1r + c_0)\, a_{k+2} + a_1(k+2)\, a_{k+1} + a_1ra_{k+1} + (-a_1 + b_0)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_1ka_{k+1}+a_1ra_{k+1}+a_ka_0+a_1a_{k+1}+b_0a_{k+1}}{a_2k^2+2a_2kr+a_2r^2+3a_2k+3a_2r+b_1k+b_1r+2a_2+2b_1+c_0}$$

- Recursion relation for $r = \frac{a_2-b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}}{2a_2}$

$$a_{k+2} = -\frac{a_1ka_{k+1}+\frac{a_1\left(a_2-b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}\right)a_{k+1}}{2a_2}+a_ka_0+a_1a_{k+1}+b_0a_{k+}}{a_2k^2+k\left(a_2-b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}\right)+\frac{\left(a_2-b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}\right)^2}{4a_2}+3a_2k+\frac{7a_2}{2}+\frac{b_1}{2}+\frac{3\sqrt{a_2^2-2a_2b_1-4c_0a_2+b}}{2}}$$

- Solution for $r = \frac{a_2-b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}}{2a_2}$

$$\left[y = \sum_{k=0}^{\infty} a_kx^{k+\frac{a_2-b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}}{2a_2}}, a_{k+2} = -\frac{a_1ka_{k+1}+\frac{a_1\left(a_2-b_1+\right.}{}}{a_2k^2+k\left(a_2-b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}\right)+\frac{\left(a_2-b_1+\sqrt{a_2^2-2a_2b_1-\right.}{4a_2}}\right.$$

- Recursion relation for $r = -\frac{-a_2+b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}}{2a_2}$

$$a_{k+2} = -\frac{a_1ka_{k+1}-\frac{a_1\left(-a_2+b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}\right)a_{k+1}}{2a_2}+a_ka_0+a_1a_{k+1}+b_0}{a_2k^2-k\left(-a_2+b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}\right)+\frac{\left(-a_2+b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}\right)^2}{4a_2}+3a_2k+\frac{7a_2}{2}+\frac{b_1}{2}-\frac{3\sqrt{a_2^2-2a_2b_1-4c_0a}}{2}}$$

- Solution for $r = -\frac{-a_2+b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}}{2a_2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{-a_2+b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}}{2a_2}}, a_{k+2} = -\frac{a_1 k a_{k+1} - \frac{a_1\left(-a_2 \cdots\right.}{\cdots}}{a_2k^2 - k\left(-a_2+b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}\right) + \frac{\left(-a_2+b_1+\sqrt{a_2^2-2a \cdots}\right.}{4a_2 \cdots}}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{a_2-b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}}{2a_2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{-a_2+b_1+\sqrt{a_2^2-2a_2b_1-4c_0a_2+b_1^2}}{2a_2}}\right), a_{k+2} = -\frac{\cdots}{a_2k^2+k \cdots}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 150

```
dsolve(a__2*x^2*diff(y(x),x$2)+(a__1*x^2+b__1*x)*diff(y(x),x)+(a__0*x^2+b__0*x+c__0)*y(x)=0,
```

$$y(x) = \mathrm{e}^{-\frac{a_1 x}{2a_2}} x^{-\frac{b_1}{2a_2}} \left(c_1 \,\mathrm{WhittakerM}\left(-\frac{b_1 a_1 - 2a_2 b_0}{2a_2\sqrt{-4a_0 a_2 + a_1^2}}, \frac{\sqrt{a_2^2 + (-2b_1 - 4c_0)\, a_2 + b_1^2}}{2a_2}, \frac{\sqrt{-4a_0 a_2 + a_1^2}\, x}{a_2}\right) + \mathrm{WhittakerW}\left(-\frac{b_1 a_1 - 2a_2 b_0}{2a_2\sqrt{-4a_0 a_2 + a_1^2}}, \frac{\sqrt{a_2^2 + (-2b_1 - 4c_0)\, a_2 + b_1^2}}{2a_2}, \frac{\sqrt{-4a_0 a_2 + a_1^2}\, x}{a_2}\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.538 (sec). Leaf size: 272

```
DSolve[a2*x^2*y''[x]+(a1*x^2+b1*x)*y'[x]+(a0*x^2+b0*x+c0)*y[x]==0,y[x],x,IncludeSingularSolu
```

$$y(x) \to e^{-\frac{x\left(\sqrt{\mathrm{a1}^2-4\mathrm{a0}\mathrm{a2}}+\mathrm{a1}\right)}{2\mathrm{a2}}} x^{\frac{\sqrt{\mathrm{a2}^2-2\mathrm{a2}(\mathrm{b1}+2\mathrm{c0})+\mathrm{b1}^2}+\mathrm{a2}-\mathrm{b1}}{2\mathrm{a2}}} \left(c_1 \,\mathrm{HypergeometricU}\left(\frac{-\frac{2\mathrm{b0}\mathrm{a2}}{\sqrt{\mathrm{a1}^2-4\mathrm{a0}\mathrm{a2}}} + \mathrm{a2} + \frac{\mathrm{a1}\mathrm{b1}}{\sqrt{\mathrm{a1}^2-4\mathrm{a0}\mathrm{a2}}}}{2\mathrm{a2}}\right.\right.$$

$$\left.\left. + c_2 L_{-\frac{-\frac{2\mathrm{b0}\mathrm{a2}}{\sqrt{\mathrm{a1}^2-4\mathrm{a0}\mathrm{a2}}}+\mathrm{a2}+\frac{\mathrm{a1}\mathrm{b1}}{\sqrt{\mathrm{a1}^2-4\mathrm{a0}\mathrm{a2}}}+\sqrt{\mathrm{a2}^2-2(\mathrm{b1}+2\mathrm{c0})\mathrm{a2}+\mathrm{b1}^2}}{2\mathrm{a2}}}^{\frac{\sqrt{\mathrm{a2}^2-2(\mathrm{b1}+2\mathrm{c0})\mathrm{a2}+\mathrm{b1}^2}}{\mathrm{a2}}}\left(\frac{\sqrt{\mathrm{a1}^2-4\mathrm{a0}\mathrm{a2}}\,x}{\mathrm{a2}}\right)\right)$$

29.31 problem 140

Internal problem ID [10964]
Internal file name [OUTPUT/10220_Sunday_December_31_2023_11_10_14_AM_28903513/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 140.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(a\,x^2 + (ab-1)\,x + b\right)y' + a^2bxy = 0}$$

29.31.1 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + \left(abx + a\,x^2 + b - x\right)y' + a^2bxy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= abx + a\,x^2 + b - x \\ C &= a^2bx \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{a^2b^2x^2 - 2a^2b\,x^3 + a^2x^4 + 2a\,b^2x - 2ab\,x^2 - 2a\,x^3 + b^2 - 6bx + 3x^2}{4x^4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= a^2b^2x^2 - 2a^2b\,x^3 + a^2x^4 + 2a\,b^2x - 2ab\,x^2 - 2a\,x^3 + b^2 - 6bx + 3x^2 \\
t &= 4x^4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{a^2b^2x^2 - 2a^2b\,x^3 + a^2x^4 + 2a\,b^2x - 2ab\,x^2 - 2a\,x^3 + b^2 - 6bx + 3x^2}{4x^4}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 140: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{a^2}{4} + \frac{b^2}{4x^4} + \frac{\frac{1}{2}a\,b^2 - \frac{3}{2}b}{x^3} + \frac{\frac{1}{4}a^2b^2 - \frac{1}{2}ab + \frac{3}{4}}{x^2} + \frac{-\frac{1}{2}a^2b - \frac{1}{2}a}{x}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{b}{2x^2} + \frac{2a\,b^2 - 6b}{4bx} + \frac{b\left(\frac{a^2b^2-2ab+3}{2b^2} - \frac{(2a\,b^2-6b)^2}{8b^4}\right)}{2} + \frac{b\left(\frac{-2a^2b-2a}{2b^2} - \frac{(2a\,b^2-6b)\,(a^2b^2-2ab+3)}{4b^4} + \frac{(2a\,b^2-6b)^3}{16b^6}\right)x}{2} + \tag{2B}$$

Using eq. (1B), taking the sum up to $v=2$ the above becomes

$$[\sqrt{r}]_c = \frac{b}{2x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = \frac{b}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c=0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $\frac{1}{2}a\,b^2 - \frac{3}{2}b$. Therefore

$$\begin{aligned}
b &= \left(\frac{1}{2}a\,b^2 - \frac{3}{2}b\right) - (0) \\
&= \frac{1}{2}a\,b^2 - \frac{3}{2}b
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_c &= \frac{b}{2x^2} \\
\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a}+v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}a\,b^2-\frac{3}{2}b}{\frac{b}{2}}+2\right) &&= \frac{\frac{1}{2}a\,b^2-\frac{3}{2}b}{b}+1 \\
\alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a}+v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}a\,b^2-\frac{3}{2}b}{\frac{b}{2}}+2\right) &&= -\frac{\frac{1}{2}a\,b^2-\frac{3}{2}b}{b}+1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty)=0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{a}{2} - \frac{1}{2x} - \frac{ab}{2x} - \frac{13b^2}{4a\,x^4} - \frac{163b^4}{4a^2x^7} - \frac{20b^3}{a^2x^6} - \frac{23b^2}{4a^3x^6} + \frac{11b}{4a^3x^5} + \frac{14b^2}{a^4x^7} + \frac{11b}{2a^4x^6} + \frac{3b}{a^5x^7} - \frac{3}{4a^6x^7} - \frac{b^6}{2x^7} - \frac{b^5}{2x^6} - \frac{b^4}{2x^5} - \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{a}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{a}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{a^2}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{a^2b^2x^2 - 2a^2b\,x^3 + a^2x^4 + 2a\,b^2x - 2ab\,x^2 - 2a\,x^3 + b^2 - 6bx + 3x^2}{4x^4} \\ &= Q + \frac{R}{4x^4} \\ &= \left(\frac{a^2}{4}\right) + \left(\frac{(-2a^2b - 2a)\,x^3 + (a^2b^2 - 2ab + 3)\,x^2 + (2a\,b^2 - 6b)\,x + b^2}{4x^4}\right) \\ &= \frac{a^2}{4} + \frac{(-2a^2b - 2a)\,x^3 + (a^2b^2 - 2ab + 3)\,x^2 + (2a\,b^2 - 6b)\,x + b^2}{4x^4}\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is $-2a^2b - 2a$. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}a^2b - \frac{1}{2}a$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}a^2b - \frac{1}{2}a\right) - (0) \\
&= -\frac{1}{2}a^2b - \frac{1}{2}a
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{a}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}a^2b - \frac{1}{2}a}{\frac{a}{2}} - 0\right) &&= \frac{-\frac{1}{2}a^2b - \frac{1}{2}a}{a} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}a^2b - \frac{1}{2}a}{\frac{a}{2}} - 0\right) &&= -\frac{-\frac{1}{2}a^2b - \frac{1}{2}a}{a}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a^2b^2x^2 - 2a^2b\,x^3 + a^2x^4 + 2a\,b^2x - 2ab\,x^2 - 2a\,x^3 + b^2 - 6bx + 3x^2}{4x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{b}{2x^2}$	$\frac{\frac{1}{2}a\,b^2 - \frac{3}{2}b}{b} + 1$	$-\frac{\frac{1}{2}a\,b^2 - \frac{3}{2}b}{b} + 1$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{a}{2}$	$\frac{-\frac{1}{2}a^2b - \frac{1}{2}a}{a}$	$-\frac{-\frac{1}{2}a^2b - \frac{1}{2}a}{a}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{-\frac{1}{2}a^2b - \frac{1}{2}a}{a}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= -\frac{-\frac{1}{2}a^2b - \frac{1}{2}a}{a} - \left(-\frac{-\frac{1}{2}a^2b - \frac{1}{2}a}{a} - 1\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{b}{2x^2} + \frac{\frac{\frac{1}{2}a\,b^2-\frac{3}{2}b}{b}+1}{x} + (-)\left(\frac{a}{2}\right) \\
&= \frac{b}{2x^2} + \frac{\frac{\frac{1}{2}a\,b^2-\frac{3}{2}b}{b}+1}{x} - \frac{a}{2} \\
&= \frac{(-x+b)(ax+1)}{2x^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{b}{2x^2} + \frac{\frac{\frac{1}{2}a\,b^2-\frac{3}{2}b}{b}+1}{x} - \frac{a}{2}\right)(1) + \left(\left(-\frac{b}{x^3} - \frac{\frac{\frac{1}{2}a\,b^2-\frac{3}{2}b}{b}+1}{x^2}\right) + \left(\frac{b}{2x^2} + \frac{\frac{\frac{1}{2}a\,b^2-\frac{3}{2}b}{b}+1}{x} - \frac{a}{2}\right)^2 - \left(\frac{a^2 b}{\cdots}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x+\frac{1}{a}\right) \mathrm{e}^{\int \left(\frac{b}{2x^2}+\frac{\frac{\frac{1}{2} a\, b^2-\frac{3}{2} b}{b}+1}{x}-\frac{a}{2}\right) dx} \\
&= \left(x+\frac{1}{a}\right) \mathrm{e}^{-\frac{b}{2x}-\frac{ax}{2}+\frac{\ln(x)ab}{2}-\frac{\ln(x)}{2}} \\
&= \frac{x^{\frac{ab}{2}-\frac{1}{2}} (ax+1)\, \mathrm{e}^{-\frac{a\,x^2+b}{2x}}}{a}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{abx+a\,x^2+b-x}{x^2} \, dx} \\
&= z_1 e^{-\frac{ax}{2}+\frac{b}{2x}-\frac{(ab-1)\ln(x)}{2}} \\
&= z_1 \left(x^{-\frac{ab}{2}+\frac{1}{2}} \mathrm{e}^{\frac{-a\,x^2+b}{2x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(ax+1)\, \mathrm{e}^{-ax}}{a}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{abx+a\,x^2+b-x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-ax+\frac{b}{x}-\ln(x)ab+\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{x^{-ab+1} a^2 \mathrm{e}^{\frac{a\,x^2+b}{x}}}{(ax+1)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a}\right) + c_2 \left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a} \left(\int \frac{x^{-ab+1} a^2 \mathrm{e}^{\frac{a\,x^2+b}{x}}}{(ax+1)^2} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (ax+1)\,\mathrm{e}^{-ax}}{a} + c_2 a (ax+1)\,\mathrm{e}^{-ax} \left(\int \frac{x^{-ab+1} \mathrm{e}^{\frac{a\,x^2+b}{x}}}{(ax+1)^2} dx\right) \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 (ax+1)\,\mathrm{e}^{-ax}}{a} + c_2 a (ax+1)\,\mathrm{e}^{-ax} \left(\int \frac{x^{-ab+1} \mathrm{e}^{\frac{a\,x^2+b}{x}}}{(ax+1)^2} dx\right)
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunD  ODE, case  c = 0
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.312 (sec). Leaf size: 199

```
dsolve(x^2*diff(y(x),x$2)+(a*x^2+(a*b-1)*x+b)*diff(y(x),x)+a^2*b*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\operatorname{HeunD}\left(4\sqrt{ab}, -a^2b^2 + 4ab - 8\sqrt{ab} - 4, -8\sqrt{ab}\,(ab-1), a^2b^2 - 4ab - 8\sqrt{ab} + 4, \frac{\sqrt{ab}\,x - b}{\sqrt{ab}\,x + b}\right) e^{\frac{-a\,x^2+b}{x}} c_1 + \operatorname{HeunD}\left(-4\sqrt{ab}, -a^2b^2 + 4ab - 8\sqrt{ab} - 4, -8\sqrt{ab}\,(ab-1), a^2b^2 - 4ab - 8\sqrt{ab} + 4, \frac{\sqrt{ab}\,x - b}{\sqrt{ab}\,x + b}\right) c_2\right) x^{1-\frac{ab}{2}}$$

✓ Solution by Mathematica

Time used: 4.002 (sec). Leaf size: 67

```
DSolve[x^2*y''[x]+(a*x^2+(a*b-1)*x+b)*y'[x]+a^2*b*x*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^{-ax}(ax+1)\left(c_2 \int_1^x \frac{a^2 e^{\frac{b}{K[1]}+aK[1]} K[1]^{1-ab}}{(aK[1]+1)^2} dK[1] + c_1\right)}{a}$$

29.32 problem 141

Internal problem ID [10965]
Internal file name [`OUTPUT/10221_Sunday_December_31_2023_11_10_17_AM_90655406/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 141.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^2y'' - 2x\left(x^2 - a\right) y' + \left(2n\, x^2 + ((-1)^n - 1)\, a\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 81

```
dsolve(x^2*diff(y(x),x$2)-2*x*(x^2-a)*diff(y(x),x)+(2*n*x^2+( (-1)^n-1)*a )*y(x)=0,y(x), sin
```

$$y(x) = x^{-a-\frac{1}{2}} e^{\frac{x^2}{2}} \left(\text{WhittakerM}\left(\frac{a}{2}+\frac{n}{2}+\frac{1}{4}, \frac{\sqrt{1-4a\left(-1\right)^n+4a^2}}{4}, x^2\right) c_1 + \text{WhittakerW}\left(\frac{a}{2}+\frac{n}{2}+\frac{1}{4}, \frac{\sqrt{1-4a\left(-1\right)^n+4a^2}}{4}, x^2\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.646 (sec). Leaf size: 231

```
DSolve[x^2*y''[x]-2*x*(x^2-a)*y'[x]+(2*n*x^2+( (-1)^n-1)*a )*y[x]==0,y[x],x,IncludeSingularS
```

$$y(x) \to i^{-a}(-1)^{\frac{1}{4}\left(1-\sqrt{4a^2-4a(-1)^n+1}\right)}x^{\frac{1}{2}\left(-\sqrt{4a^2-4a(-1)^n+1}-2a+1\right)}\Bigg(c_1 \text{Hypergeometric1F1}\left(\frac{1}{4}\left(-2a-2n-\sqrt{4a^2-4\right.\right.$$

$$-\frac{1}{2}\sqrt{4a^2-4(-1)^n a+1}, x^2\Bigg)$$

$$+c_2 i^{\sqrt{4a^2-4a(-1)^n+1}}x^{\sqrt{4a^2-4a(-1)^n+1}}\text{Hypergeometric1F1}\left(\frac{1}{4}\left(-2a-2n+\sqrt{4a^2-4(-1)^n a+1}+1\right), \frac{1}{2}\left(\sqrt{\right.\right.$$

29.33 problem 142

Internal problem ID [10966]
Internal file name [OUTPUT/10222_Sunday_December_31_2023_11_10_18_AM_21141346/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 142.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$x^2y'' + x(a\,x^2 + bx + c)\,y' + (A\,x^3 + B\,x^2 + Cx + d)\,y = 0$$

29.33.1 Maple step by step solution

Let's solve

$y''x^2 + x(a\,x^2 + bx + c)\,y' + (A\,x^3 + B\,x^2 + Cx + d)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(A\,x^3+B\,x^2+Cx+d)y}{x^2} - \frac{(a\,x^2+bx+c)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(a\,x^2+bx+c)y'}{x} + \frac{(A\,x^3+B\,x^2+Cx+d)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

$$\left[P_2(x) = \frac{a\,x^2+bx+c}{x}, P_3(x) = \frac{A\,x^3+B\,x^2+Cx+d}{x^2}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = c$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = d$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x^2 + x(a\,x^2 + bx + c)\,y' + (A\,x^3 + B\,x^2 + Cx + d)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..3$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(cr + r^2 + d - r)\, x^r + ((cr + r^2 + c + d + r)\, a_1 + a_0(br + C))\, x^{1+r} + ((cr + r^2 + 2c + d + 3r +$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$cr + r^2 + d - r = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{c}{2} + \frac{1}{2} + \frac{\sqrt{c^2-2c-4d+1}}{2}, -\frac{c}{2} - \frac{\sqrt{c^2-2c-4d+1}}{2} + \frac{1}{2}\right\}$$

- The coefficients of each power of x must be 0

  $$[(cr + r^2 + c + d + r)\, a_1 + a_0(br + C) = 0, (cr + r^2 + 2c + d + 3r + 2)\, a_2 + a_1(br + C + b) + a_0($$

- Solve for the dependent coefficient(s)

  $$\left\{a_1 = -\frac{a_0(br+C)}{cr+r^2+c+d+r}, a_2 = -\frac{a_0\left(a\, r^2 c + a\, r^3 - b^2 r^2 + Bcr + B\, r^2 - 2brC + arc + ard + a\, r^2 - b^2 r + Bc + Bd + Br - C^2 - Cb\right)}{c^2r^2 + 2c\, r^3 + r^4 + 3c^2 r + 2crd + 7c\, r^2 + 2r^2 d + 4r^3 + 2c^2 + 3cd + 7cr + d^2 + 4dr + 5r^2 + 2c + 2d + 2r}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + (c + 2r - 1)\, k + r^2 + (c - 1)\, r + d)\, a_k + (a a_{k-2} + b a_{k-1})\, k + (a a_{k-2} + b a_{k-1})\, r + (B - 2a)$$

- Shift index using $k- >k + 3$

  $$\left((k + 3)^2 + (c + 2r - 1)\, (k + 3) + r^2 + (c - 1)\, r + d\right) a_{k+3} + (a a_{k+1} + b a_{k+2})\, (k + 3) + (a a_{k+1} +$$

- Recursion relation that defines series solution to ODE

  $$a_{k+3} = -\frac{aka_{k+1} + ara_{k+1} + bka_{k+2} + bra_{k+2} + Aa_k + Ba_{k+1} + Ca_{k+2} + aa_{k+1} + 2ba_{k+2}}{ck + cr + k^2 + 2kr + r^2 + 3c + d + 5k + 5r + 6}$$

- Recursion relation for $r = -\frac{c}{2} + \frac{1}{2} + \frac{\sqrt{c^2-2c-4d+1}}{2}$

  $$a_{k+3} = -\frac{aka_{k+1} + a\left(-\frac{c}{2} + \frac{1}{2} + \frac{\sqrt{c^2-2c-4d+1}}{2}\right) a_{k+1} + bka_{k+2} + b\left(-\frac{c}{2} + \frac{1}{2} + \frac{\sqrt{c^2-2c-4d+1}}{2}\right) a_{k+2} + Aa_k + Ba_{k+1} + Ca_{k+2} + aa_{k+1} -}{ck + c\left(-\frac{c}{2} + \frac{1}{2} + \frac{\sqrt{c^2-2c-4d+1}}{2}\right) + k^2 + 2k\left(-\frac{c}{2} + \frac{1}{2} + \frac{\sqrt{c^2-2c-4d+1}}{2}\right) + \left(-\frac{c}{2} + \frac{1}{2} + \frac{\sqrt{c^2-2c-4d+1}}{2}\right)^2 + \frac{c}{2} + d + 5k + \frac{17}{2} + \frac{5\sqrt{c^2-}}{}}$$

- Solution for $r = -\frac{c}{2} + \frac{1}{2} + \frac{\sqrt{c^2-2c-4d+1}}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k - \frac{c}{2} + \frac{1}{2} + \frac{\sqrt{c^2-2c-4d+1}}{2}}, a_{k+3} = -\frac{aka_{k+1} + a\left(-\frac{c}{2} + \frac{1}{2} + \frac{\sqrt{c^2-2c-4d+1}}{2}\right) a_{k+1} + bka_{k+2} + b\left(-\frac{c}{2} + \frac{1}{2} + \frac{\sqrt{c^2-2c-}}{2}\right.}{ck + c\left(-\frac{c}{2} + \frac{1}{2} + \frac{\sqrt{c^2-2c-4d+1}}{2}\right) + k^2 + 2k\left(-\frac{c}{2} + \frac{1}{2} + \frac{\sqrt{c^2-2c-4d+1}}{2}\right) + \left(-\frac{c}{2} +\right.}\right.$$

- Recursion relation for $r = -\frac{c}{2} - \frac{\sqrt{c^2-2c-4d+1}}{2} + \frac{1}{2}$

  $$a_{k+3} = -\frac{aka_{k+1} + a\left(-\frac{c}{2} - \frac{\sqrt{c^2-2c-4d+1}}{2} + \frac{1}{2}\right) a_{k+1} + bka_{k+2} + b\left(-\frac{c}{2} - \frac{\sqrt{c^2-2c-4d+1}}{2} + \frac{1}{2}\right) a_{k+2} + Aa_k + Ba_{k+1} + Ca_{k+2} + aa_{k+1} -}{ck + c\left(-\frac{c}{2} - \frac{\sqrt{c^2-2c-4d+1}}{2} + \frac{1}{2}\right) + k^2 + 2k\left(-\frac{c}{2} - \frac{\sqrt{c^2-2c-4d+1}}{2} + \frac{1}{2}\right) + \left(-\frac{c}{2} - \frac{\sqrt{c^2-2c-4d+1}}{2} + \frac{1}{2}\right)^2 + \frac{c}{2} + d + 5k - \frac{5\sqrt{c^2-2c-}}{2}}$$

- Solution for $r = -\frac{c}{2} - \frac{\sqrt{c^2-2c-4d+1}}{2} + \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{c}{2}-\frac{\sqrt{c^2-2c-4d+1}}{2}+\frac{1}{2}}, a_{k+3} = -\frac{aka_{k+1}+a\left(-\frac{c}{2}-\frac{\sqrt{c^2-2c-4d+1}}{2}+\frac{1}{2}\right)a_{k+1}+bka_{k+2}+b\left(-\frac{c}{2}-\frac{\sqrt{c^2-2c-4d+1}}{2}\right.}{ck+c\left(-\frac{c}{2}-\frac{\sqrt{c^2-2c-4d+1}}{2}+\frac{1}{2}\right)+k^2+2k\left(-\frac{c}{2}-\frac{\sqrt{c^2-2c-4d+1}}{2}+\frac{1}{2}\right)+\left(-\frac{c}{2}-\right.}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} e_k x^{k-\frac{c}{2}+\frac{1}{2}+\frac{\sqrt{c^2-2c-4d+1}}{2}}\right) + \left(\sum_{k=0}^{\infty} f_k x^{k-\frac{c}{2}-\frac{\sqrt{c^2-2c-4d+1}}{2}+\frac{1}{2}}\right), e_{k+3} = -\frac{ake_{1+k}+a\left(-\frac{c}{2}+\frac{1}{2}+\frac{\sqrt{c^2}}{}\right.}{ck+c\left(-\frac{c}{2}+\frac{1}{2}+\frac{\sqrt{c^2-2c-4d}}{2}\right.}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
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   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 232

```
dsolve(x^2*diff(y(x),x$2)+x*(a*x^2+b*x+c)*diff(y(x),x)+(A*x^3+B*x^2+C*x+d)*y(x)=0,y(x), sing
```

$$y(x) = x^{-\frac{c}{2}+\frac{1}{2}} e^{\frac{x\left(-a^2x-2ab+2A\right)}{2a}} \left(c_1 x^{\frac{\sqrt{c^2-2c-4d+1}}{2}} \operatorname{HeunB}\left(\sqrt{c^2-2c-4d+1}, \frac{\sqrt{2}\,(-ab+2A)}{a^{\frac{3}{2}}}, -c-\frac{2Ab}{a^2}+\frac{2B}{a}-1+\frac{2A^2}{a^3}, \frac{\sqrt{2}\,(-bc+2C)}{\sqrt{a}}, -\frac{\sqrt{2}\,\sqrt{a}\,x}{2}\right) + c_2 x^{-\frac{\sqrt{c^2-2c-4d+1}}{2}} \operatorname{HeunB}\left(-\sqrt{c^2-2c-4d+1}, \frac{\sqrt{2}\,(-ab+2A)}{a^{\frac{3}{2}}}, -c-\frac{2Ab}{a^2}+\frac{2B}{a}-1+\frac{2A^2}{a^3}, \frac{\sqrt{2}\,(-bc+2C)}{\sqrt{a}}, -\frac{\sqrt{2}\,\sqrt{a}\,x}{2}\right)\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*y''[x]+x*(a*x^2+b*x+c)*y'[x]+(A*x^3+B*x^2+C0*x+d)*y[x]==0,y[x],x,IncludeSingularS
```

Not solved

29.34 problem 143

Internal problem ID [10967]
Internal file name [`OUTPUT/10223_Sunday_December_31_2023_11_10_20_AM_8686326/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 143.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^2y'' + a\,x^n y' - \left(ab\,x^n + ac\,x^{n-1} + b^2x^2 + 2bcx + c^2 - c\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(x^2*diff(y(x),x$2)+a*x^n*diff(y(x),x)-(a*b*x^n+a*c*x^(n-1)+b^2*x^2+2*b*c*x+c^2-c)*y(x
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*y''[x]+a*x^n*y'[x]-(a*b*x^n+a*c*x^(n-1)+b^2*x^2+2*b*c*x+c^2-c)*y[x]==0,y[x],x,Inc
```

Not solved

29.35 problem 144

Internal problem ID [10968]
Internal file name [`OUTPUT/10224_Sunday_December_31_2023_11_10_21_AM_39182112/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 144.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^2y'' + a\,x^n y' + \left(ab\,x^{n+2m} - b^2 x^{4m+2} + am\,x^{n-1} - m^2 - m\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(x^2*diff(y(x),x$2)+a*x^n*diff(y(x),x)+(a*b*x^(n+2*m)-b^2*x^(4*m+2)+a*m*x^(n-1)-m^2-m)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*y''[x]+a*x^n*y'[x]+(a*b*x^(n+2*m)-b^2*x^(4*m+2)+a*m*x^(n-1)-m^2-m)*y[x]==0,y[x],x
```

Not solved

29.36 problem 145

Internal problem ID [10969]
Internal file name [`OUTPUT/10225_Sunday_December_31_2023_11_10_22_AM_99014734/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 145.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(a\,x^n + b)\,y' + b(a\,x^n - 1)\,y = 0}$$

29.36.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + x(a\,x^n + b)\,y' + b(a\,x^n - 1)\,y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{a\,x^n + b}{x} \\ q(x) &= \frac{b(a\,x^n - 1)}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{n(a\,x^n+b)}{x^2}+\frac{b(a\,x^n-1)}{x^2}=0 \tag{5}$$

Solving (5) for n gives

$$n=-b \tag{6}$$

Substituting this value in (3) gives

$$v''(x)+\left(-\frac{2b}{x}+\frac{a\,x^n+b}{x}\right)v'(x)=0$$

$$v''(x)+\frac{(-b+a\,x^n)\,v'(x)}{x}=0 \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x)+\frac{(-b+a\,x^n)\,u(x)}{x}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(-b+a\,x^n)\,u}{x} \end{aligned}$$

Where $f(x)=-\frac{-b+a\,x^n}{x}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{-b+a\,x^n}{x}\,dx \\ \int\frac{1}{u}\,du &= \int-\frac{-b+a\,x^n}{x}\,dx \\ \ln(u) &= -\frac{a\,x^n}{n}+\frac{b\ln(x^n)}{n}+c_1 \\ u &= \mathrm{e}^{-\frac{a\,x^n}{n}+\frac{b\ln(x^n)}{n}+c_1} \\ &= c_1\mathrm{e}^{-\frac{a\,x^n}{n}+\frac{b\ln(x^n)}{n}} \end{aligned}$$

Which simplifies to

$$u(x) = c_1 \mathrm{e}^{-\frac{a\,x^n}{n}} \left(x^n\right)^{\frac{b}{n}}$$

Now that $u(x)$ is known, then

$$v'(x) = u(x)$$

$$v(x) = \int u(x)\,dx + c_2$$

$$= \frac{c_1 \left(x^n\right)^{\frac{b}{n}} x^{-b} \left(\frac{a}{n}\right)^{-\frac{b}{n}-\frac{1}{n}} \left(\frac{n^3 x^{b-n+1} \left(\frac{a}{n}\right)^{\frac{b}{n}+\frac{1}{n}} (a\,x^n+b+n+1) \left(\frac{a\,x^n}{n}\right)^{-\frac{n+b+1}{2n}} \mathrm{e}^{-\frac{a\,x^n}{2n}} \mathrm{WhittakerM}\left(\frac{1+b}{n}-\frac{n+b+1}{2n}, \frac{n+b+1}{2n}+\frac{1}{2}, \frac{a\,x^n}{n}\right)}{(1+b)(n+b+1)(2n+b+1)a}\right.}{n}$$

Hence

$$y = v(x)\,x^n$$

$$= \left(\frac{c_1 \left(x^n\right)^{\frac{b}{n}} x^{-b} \left(\frac{a}{n}\right)^{-\frac{b}{n}-\frac{1}{n}} \left(\frac{n^3 x^{b-n+1} \left(\frac{a}{n}\right)^{\frac{b}{n}+\frac{1}{n}} (a\,x^n+b+n+1) \left(\frac{a\,x^n}{n}\right)^{-\frac{n+b+1}{2n}} \mathrm{e}^{-\frac{a\,x^n}{2n}} \mathrm{WhittakerM}\left(\frac{1+b}{n}-\frac{n+b+1}{2n}, \frac{n+b+1}{2n}+\frac{1}{2}, \frac{a\,x^n}{n}\right)}{(1+b)(n+b+1)(2n+b+1)a}\right.}{n}\right.$$

$$= \frac{x^{-b}\left(\left(\frac{a\,x^n}{n}\right)^{-\frac{n+b+1}{2n}} \left((n+b+1)\,x^{1-n} + ax\right) n^2 c_1 \mathrm{e}^{-\frac{a\,x^n}{2n}} \left(x^n\right)^{\frac{b}{n}} \mathrm{WhittakerM}\left(\frac{b-n+1}{2n}, \frac{2n+b+1}{2n}, \frac{a\,x^n}{n}\right) + (n+b\right.}{(1+b)\,(n}$$

Summary

The solution(s) found are the following

$$y = \left(\frac{c_1 \left(x^n\right)^{\frac{b}{n}} x^{-b} \left(\frac{a}{n}\right)^{-\frac{b}{n}-\frac{1}{n}} \left(\frac{n^3 x^{b-n+1} \left(\frac{a}{n}\right)^{\frac{b}{n}+\frac{1}{n}} (a\,x^n+b+n+1) \left(\frac{a\,x^n}{n}\right)^{-\frac{n+b+1}{2n}} \mathrm{e}^{-\frac{a\,x^n}{2n}} \mathrm{WhittakerM}\left(\frac{1+b}{n}-\frac{n+b+1}{2n}, \frac{n+b+1}{2n}+\frac{1}{2}, \frac{a\,x^n}{n}\right)}{(1+b)(n+b+1)(2n+b+1)a} + \right.}{n} + c_2\right) x^{-b} \tag{1}$$

Verification of solutions

$$y = \left(\frac{c_1 (x^n)^{\frac{b}{n}} x^{-b} \left(\frac{a}{n}\right)^{-\frac{b}{n}-\frac{1}{n}} \left(\frac{n^3 x^{b-n+1} \left(\frac{a}{n}\right)^{\frac{b}{n}+\frac{1}{n}} (a x^n+b+n+1) \left(\frac{a x^n}{n}\right)^{-\frac{n+b+1}{2n}} e^{-\frac{a x^n}{2n}} \operatorname{WhittakerM}\left(\frac{1+b}{n}-\frac{n+b+1}{2n}, \frac{n+b+1}{2n}+\frac{1}{2}, \frac{a x^n}{n}\right)}{(1+b)(n+b+1)(2n+b+1)a} + \right.}{n} + c_2\right) x^{-b}$$

Verified OK.

<u>Maple trace</u> **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- Equivalence, under non-integer power transformations successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 141

```
dsolve(x^2*diff(y(x),x$2)+x*(a*x^n+b)*diff(y(x),x)+b*(a*x^n-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{a x^n}{2n}}\left((b+n+1) x^{-\frac{3n}{2}+\frac{1}{2}-\frac{b}{2}} + a x^{\frac{1}{2}-\frac{b}{2}-\frac{n}{2}}\right) n c_2 \operatorname{WhittakerM}\left(\frac{b-n+1}{2n}, \frac{b+2n+1}{2n}, \frac{a x^n}{n}\right) + x^{-\frac{3n}{2}+\frac{1}{2}-\frac{b}{2}} e^{-\frac{a x^n}{2n}} c_2 (b+n+1)^2 \operatorname{WhittakerM}\left(\frac{b+n+1}{2n}, \frac{b+2n+1}{2n}, \frac{a x^n}{n}\right) + c_1 x^{-b}$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 76

```
DSolve[x^2*y''[x]+x*(a*x^n+b)*y'[x]+b*(a*x^n-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to (-1)^{-\frac{b}{n}} n^{\frac{b}{n}-1} a^{-\frac{b}{n}} (x^n)^{-\frac{b}{n}} \left((b+1) c_1 (-1)^{b/n} \Gamma\left(\frac{b+1}{n}, 0, \frac{a x^n}{n}\right) + c_2 n\right)$$

29.37 problem 146

Internal problem ID [10970]
Internal file name [OUTPUT/10226_Sunday_December_31_2023_11_10_23_AM_15916150/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 146.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^2y'' + x(a\,x^n + b)\,y' + \left(\alpha\,x^{2n} + \beta\,x^n + \gamma\right)y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.187 (sec). Leaf size: 148

```
dsolve(x^2*diff(y(x),x$2)+x*(a*x^n+b)*diff(y(x),x)+(alpha*x^(2*n)+beta*x^n+gamma)*y(x)=0,y(x
```

$$y(x) = x^{\frac{1}{2}-\frac{b}{2}-\frac{n}{2}} \mathrm{e}^{-\frac{a x^n}{2n}} \left(c_1 \operatorname{WhittakerM}\left(-\frac{a(b+n-1)-2\beta}{2\sqrt{a^2-4\alpha}\, n}, \frac{\sqrt{b^2-2b-4\gamma+1}}{2n}, \frac{\sqrt{a^2-4\alpha}\, x^n}{n}\right) + c_2 \operatorname{WhittakerW}\left(-\frac{a(b+n-1)-2\beta}{2\sqrt{a^2-4\alpha}\, n}, \frac{\sqrt{b^2-2b-4\gamma+1}}{2n}, \frac{\sqrt{a^2-4\alpha}\, x^n}{n}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.485 (sec). Leaf size: 420

```
DSolve[x^2*y''[x]+x*(a*x^n+b)*y'[x]+(\[Alpha]*x^(2*n)+\[Beta]*x^n+\[Gamma])*y[x]==0,y[x],x,I
```

$$y(x) \to x^{\frac{1}{2}-\frac{n}{2}} 2^{\frac{1}{2}\left(\frac{\sqrt{n^2(b^2-2b-4\gamma+1)}}{n^2}+1\right)} e^{-\frac{(\sqrt{a^2-4\alpha}+a)x^n}{2n}} (x^n)^{\frac{\sqrt{n^2(b^2-2b-4\gamma+1)}-bn+n^2}{2n^2}} \left(c_1 \text{HypergeometricU}\left(\frac{\left(n^2+\sqrt{n^2\right.}}{}\right.\right.$$

$$\left.+c_2 L^{\frac{\sqrt{n^2(b^2-2b-4\gamma+1)}}{n^2}}_{\frac{-\left(\left(n^2+\sqrt{n^2(b^2-2b-4\gamma+1)}\right)a^2\right)-n(b+n-1)\sqrt{a^2-4\alpha}a+4n^2\alpha+2n\sqrt{a^2-4\alpha}\beta+4\alpha\sqrt{n^2(b^2-2b-4\gamma+1)}}{2n^2(a^2-4\alpha)}}\left(\frac{x^n\sqrt{a^2-4\alpha}}{n}\right)\right)$$

29.38 problem 147

Internal problem ID [10971]
Internal file name [`OUTPUT/10227_Sunday_December_31_2023_11_10_24_AM_96377248/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 147.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$x^2y'' + x(2a\,x^n + b)\,y' + \left(x^{2n}a^2 + a(b+n-1)\,x^n + \alpha\,x^{2m} + \beta\,x^m + \gamma\right)y = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 115

```
dsolve(x^2*diff(y(x),x$2)+x*(2*a*x^n+b)*diff(y(x),x)+(a^2*x^(2*n)+a*(b+n-1)*x^n+alpha*x^(2*m
```

$$y(x) = x^{-\frac{b}{2}} x^{-\frac{m}{2}} \sqrt{x}\, e^{-\frac{a x^n}{n}} \left(c_1 \operatorname{WhittakerM}\left(-\frac{i\beta}{2m\sqrt{\alpha}}, \frac{\sqrt{b^2-2b-4\gamma+1}}{2m}, \frac{2i\sqrt{\alpha}\, x^m}{m}\right) + c_2 \operatorname{WhittakerW}\left(-\frac{i\beta}{2m\sqrt{\alpha}}, \frac{\sqrt{b^2-2b-4\gamma+1}}{2m}, \frac{2i\sqrt{\alpha}\, x^m}{m}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.47 (sec). Leaf size: 291

```
DSolve[x^2*y''[x]+x*(2*a*x^n+b)*y'[x]+(a^2*x^(2*n)+a*(b+n-1)*x^n+\[Alpha]*x^(2*m)+\[Beta]*x^
```

$$y(x) \to x^{\frac{1}{2}-\frac{m}{2}} 2^{\frac{1}{2}\left(\frac{\sqrt{m^2(b^2-2b-4\gamma+1)}}{m^2}+1\right)} (x^n)^{-\frac{b}{2n}} (x^m)^{\frac{1}{2}\left(\frac{\sqrt{m^2(b^2-2b-4\gamma+1)}}{m^2}+1\right)} e^{-\frac{ax^n}{n}+\frac{i\sqrt{\alpha}x^m}{m}} \left(c_1 \text{HypergeometricU}\left(\frac{m^2 \right. \right.$$

$$\left. -\frac{2ix^m\sqrt{\alpha}}{m}\right) + c_2 L^{\frac{\sqrt{m^2(b^2-2b-4\gamma+1)}}{m^2}}_{-\frac{m^2-\frac{i\beta m}{\sqrt{\alpha}}+\sqrt{m^2(b^2-2b-4\gamma+1)}}{2m^2}}\left(-\frac{2ix^m\sqrt{\alpha}}{m}\right)\Bigg)$$

29.39 problem 148

Internal problem ID [10972]
Internal file name [`OUTPUT/10228_Sunday_December_31_2023_11_10_26_AM_19684180/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-4 Equation of form $x^2y'' + f(x)y' + g(x)y = 0$
Problem number: 148.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^2y'' + \left(a\,x^{2+n} + b\,x^2 + c\right)y' + \left(an\,x^{n+1} + x^n ac + bc\right)y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(x^2*diff(y(x),x$2)+(a*x^(n+2)+b*x^2+c)*diff(y(x),x)+(a*n*x^(n+1)+a*c*x^n+b*c)*y(x)=0,
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*y''[x]+(a*x^(n+2)+b*x^2+c)*y'[x]+(a*n*x^(n+1)+a*c*x^n+b*c)*y[x]==0,y[x],x,Include
```

Not solved

30 Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$

30.1 problem 149

Internal problem ID [10973]
Internal file name [`OUTPUT/10229_Sunday_December_31_2023_11_10_27_AM_99537949/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 149.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_Gegenbauer]
```

Unable to solve or complete the solution.

$$\boxed{(-x^2 + 1)\, y'' + n(n-1)\, y = 0}$$

30.1.1 Maple step by step solution

Let's solve

$(-x^2 + 1)\, y'' + (n^2 - n)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{n(n-1)y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{n(n-1)y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = 0, P_3(x) = -\frac{(n-1)n}{x^2-1}\right]$

- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = 0$

- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $y''(x^2-1) - n(n-1)\,y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (-n^2+n)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-1+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(k+r) + a_k(r-1+n+k)\,(r-n+k)\right)u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $$-2a_{k+1}(k+1+r)(k+r)+a_k(r-1+n+k)(r-n+k)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1}=\frac{a_k(r-1+n+k)(r-n+k)}{2(k+1+r)(k+r)}$$

- Recursion relation for $r=0$

  $$a_{k+1}=\frac{a_k(-1+n+k)(-n+k)}{2(k+1)k}$$

- Solution for $r=0$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{a_k(-1+n+k)(-n+k)}{2(k+1)k}\right]$$

- Revert the change of variables $u=1+x$

  $$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k, a_{k+1}=\frac{a_k(-1+n+k)(-n+k)}{2(k+1)k}\right]$$

- Recursion relation for $r=1$

  $$a_{k+1}=\frac{a_k(n+k)(1-n+k)}{2(k+2)(k+1)}$$

- Solution for $r=1$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+1}, a_{k+1}=\frac{a_k(n+k)(1-n+k)}{2(k+2)(k+1)}\right]$$

- Revert the change of variables $u=1+x$

  $$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k+1}, a_{k+1}=\frac{a_k(n+k)(1-n+k)}{2(k+2)(k+1)}\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty}a_k(1+x)^k\right)+\left(\sum_{k=0}^{\infty}b_k(1+x)^{1+k}\right), a_{1+k}=\frac{a_k(k+n-1)(-n+k)}{2(1+k)k}, b_{1+k}=\frac{b_k(k+n)(1-n+k)}{2(k+2)(1+k)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 52

```
dsolve((1-x^2)*diff(y(x),x$2)+n*(n-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = -(-1+x)(1+x)\left(\text{hypergeom}\left(\left[\frac{n}{2}+1, \frac{3}{2}-\frac{n}{2}\right], \left[\frac{3}{2}\right], x^2\right)c_2x + c_1\,\text{hypergeom}\left(\left[-\frac{n}{2}+1, \frac{n}{2}+\frac{1}{2}\right], \left[\frac{1}{2}\right], x^2\right)\right)$$

✓ Solution by Mathematica

Time used: 0.149 (sec). Leaf size: 56

```
DSolve[(1-x^2)*y''[x]+n*(n-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to ic_2x\,\text{Hypergeometric2F1}\left(\frac{1}{2}-\frac{n}{2},\frac{n}{2},\frac{3}{2},x^2\right) + c_1\,\text{Hypergeometric2F1}\left(\frac{n-1}{2},-\frac{n}{2},\frac{1}{2},x^2\right)$$

30.2 problem 150

Internal problem ID [10974]
Internal file name [`OUTPUT/10230_Sunday_December_31_2023_11_10_28_AM_77036271/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 150.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{(-a^2 + x^2)\, y'' + y'b - 6y = 0}$$

30.2.1 Maple step by step solution

Let's solve

$(-a^2 + x^2)\, y'' + y'b - 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{6y}{a^2-x^2} + \frac{by'}{a^2-x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{by'}{a^2-x^2} + \frac{6y}{a^2-x^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{b}{a^2-x^2}, P_3(x) = \frac{6}{a^2-x^2}\right]$

- $(x-a) \cdot P_2(x)$ is analytic at $x = a$

  $\left((x-a) \cdot P_2(x)\right)\Big|_{x=a} = \frac{b}{2a}$

- $(x-a)^2 \cdot P_3(x)$ is analytic at $x = a$

  $\left((x-a)^2 \cdot P_3(x)\right)\Big|_{x=a} = 0$

- $x = a$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = a$

- Multiply by denominators

  $y''(a^2 - x^2) - y'b + 6y = 0$

- Change variables using $x = u + a$ so that the regular singular point is at $u = 0$

  $(-2ua - u^2)\left(\frac{d^2}{du^2}y(u)\right) - b\left(\frac{d}{du}y(u)\right) + 6y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1}$

- Shift index using $k- >k+1$

  $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$-a_0 r(2ar-2a+b)\,u^{r-1}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\left(2a(k+1)+2ar-2a+b\right)-a_k(k+r+2)\left(\right.\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(2ar-2a+b)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{2a-b}{2a}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2(k+1+r)\left(ak+ar+\frac{1}{2}b\right)a_{k+1}-a_k(k+r+2)\left(k+r-3\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{a_k(k+r+2)(k+r-3)}{(k+1+r)(2ak+2ar+b)}$$

- Recursion relation for $r=0$; series terminates at $k=3$

$$a_{k+1}=-\frac{a_k(k+2)(k-3)}{(k+1)(2ak+b)}$$

- Apply recursion relation for $k=0$

$$a_1=\frac{6a_0}{b}$$

- Apply recursion relation for $k=1$

$$a_2=\frac{3a_1}{2a+b}$$

- Express in terms of a_0

$$a_2=\frac{18a_0}{b(2a+b)}$$

- Apply recursion relation for $k=2$

$$a_3=\frac{4a_2}{3(4a+b)}$$

- Express in terms of a_0

$$a_3=\frac{24a_0}{b(2a+b)(4a+b)}$$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

$$y(u)=a_0\cdot\left(1+\frac{6u}{b}+\frac{18u^2}{b(2a+b)}+\frac{24u^3}{b(2a+b)(4a+b)}\right)$$

- Revert the change of variables $u=x-a$

$$\left[y=\frac{a_0\left(-10a^2b-24a^2x+b^3+6b^2x+18b\,x^2+24x^3\right)}{b(2a+b)(4a+b)}\right]$$

- Recursion relation for $r=\frac{2a-b}{2a}$

$$a_{k+1}=-\frac{a_k\left(k+\frac{2a-b}{2a}+2\right)\left(k+\frac{2a-b}{2a}-3\right)}{\left(k+1+\frac{2a-b}{2a}\right)(2ak+2a)}$$

- Solution for $r = \frac{2a-b}{2a}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{2a-b}{2a}}, a_{k+1} = -\frac{a_k \left(k+\frac{2a-b}{2a}+2\right)\left(k+\frac{2a-b}{2a}-3\right)}{\left(k+1+\frac{2a-b}{2a}\right)(2ak+2a)}\right]$$

- Revert the change of variables $u = x - a$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-a)^{k+\frac{2a-b}{2a}}, a_{k+1} = -\frac{a_k \left(k+\frac{2a-b}{2a}+2\right)\left(k+\frac{2a-b}{2a}-3\right)}{\left(k+1+\frac{2a-b}{2a}\right)(2ak+2a)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \frac{c_0(-10a^2b-24a^2x+b^3+6b^2x+18b\,x^2+24x^3)}{b(2a+b)(4a+b)} + \left(\sum_{k=0}^{\infty} d_k (x-a)^{k+\frac{2a-b}{2a}}\right), d_{1+k} = -\frac{d_k \left(k+\frac{2a-b}{2a}+2\right)\left(k+\frac{2a-b}{2a}-3\right.}{\left(k+1+\frac{2a-b}{2a}\right)(2ak+2a)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 76

```
dsolve((x^2-a^2)*diff(y(x),x$2)+b*diff(y(x),x)-6*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{c_1(10a^2b + 24a^2x - b^3 - 6b^2x - 18x^2b - 24x^3)}{24} + c_2(a+x)(a-x)(b-4x)\left(\frac{a+x}{a-x}\right)^{\frac{b}{2a}}$$

✓ Solution by Mathematica

Time used: 13.059 (sec). Leaf size: 1171

```
DSolve[(x^2-a^2)*y''[x]+b*y'[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)$$

$$\rightarrow e^{\frac{b\operatorname{arctanh}\left(\frac{x}{a}\right)}{2a}+\frac{\left(b^5-20a^2b^3+64a^4b+\sqrt{b^2\left(64a^4-20b^2a^2+b^4\right)^2}\right)\text{RootSum}\left[-b^3-6\#1b^2+10a^2b-18\#1^2b-24\#1^3+24a^2\#1\&,\log(x-\#1)\&\right]}{2\left(b^5-20a^2b^3+64a^4b\right)}}(a+x)^{\frac{1}{2}-\frac{\sqrt{b^2\left(64a^4-20b^2a^2+b^4\right)^2}}{4ab\left(32a^3-16ba^2-2b^2a+b^3\right)}}(4x-b)^{\frac{b^5}{2\left(b^5-20a^2b^3+64a^4b\right)}-\frac{10a^2b^3}{b^5-20a^2b^3+64a^4b}+\frac{32a^4b}{b^5-20a^2b^3+64a^4b}-\frac{\sqrt{b^2\left(64a^4-20b^2a^2+b^4\right)^2}}{2\left(b^5-20a^2b^3+64a^4b\right)}}c_2\int_1^x -\frac{e^{-\frac{\left(b^5-20a^2b^3+64a^4b+\sqrt{b^2\left(64a^4-20b^2a^2+b^4\right)^2}\right)\text{RootSum}\left[-b^3-6\#1b^2+10a^2b-18\#1^2b-24\#1^3+24a^2\#1\&,\log(K[1]-\#1)\&\right]}{b^5-20a^2b^3+64a^4b}}(K[1]-a)^{\frac{\sqrt{b^2}}{2a(}}}{}$$

$$-a)^{\frac{1}{2}-\frac{\sqrt{b^2\left(64a^4-20b^2a^2+b^4\right)^2}}{4a(4a-b)b(2a+b)(4a+b)}}$$

$$+e^{\frac{1}{2}\left(\frac{b\operatorname{arctanh}\left(\frac{x}{a}\right)}{a}+\frac{\left(b^5-20a^2b^3+64a^4b+\sqrt{\left(b^5-20a^2b^3+64a^4b\right)^2}\right)\text{RootSum}\left[-b^3-6\#1b^2+10a^2b-18\#1^2b-24\#1^3+24a^2\#1\&,\log(x-\#1)\&\right]}{b^5-20a^2b^3+64a^4b}\right)}(a+x)^{\frac{1}{4}\left(2-\frac{\sqrt{\left(b^5-20a^2b^3+64a^4b\right)^2}}{ab\left(32a^3-16ba^2-2b^2a+b^3\right)}\right)}(4x-b)^{\frac{b^5-20a^2b^3+64a^4b-\sqrt{\left(b^5-20a^2b^3+64a^4b\right)^2}}{2\left(b^5-20a^2b^3+64a^4b\right)}}c_1(x-a)^{\frac{1}{2}-\frac{\sqrt{\left(b^5-20a^2b^3+64a^4b\right)^2}}{4a(4a-b)b(2a+b)(4a+b)}}$$

30.3 problem 151

Internal problem ID [10975]
Internal file name [`OUTPUT/10231_Sunday_December_31_2023_11_10_29_AM_95943088/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2+bx+c)y''+f(x)y'+g(x)y=0$
Problem number: 151.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[_Gegenbauer, [_2nd_order, _linear, `_with_symmetry_[0,F(x)]`]]
```

$$\boxed{(x^2-1)\,y''+y'x+ay=0}$$

30.3.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$(x^2-1)\,y''+y'x+ay=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=\frac{x}{x^2-1}\\ q(x)&=\frac{a}{x^2-1}\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\
&= \int e^{-\left(\int \frac{x}{x^2-1}dx\right)}dx \\
&= \int e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}}\,dx \\
&= \int \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}dx \\
&= \frac{\sqrt{(x-1)(1+x)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{1+x}}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{a}{x^2-1}}{\frac{1}{(x-1)(1+x)}} \\
&= a
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + a y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = a$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + a\, e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + a = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = a$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(a)} \\ &= \pm\sqrt{-a}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-a} \\ \lambda_2 &= -\sqrt{-a}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-a} \\ \lambda_2 &= -\sqrt{-a}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\ y(\tau) &= c_1 e^{(\sqrt{-a})\tau} + c_2 e^{(-\sqrt{-a})\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{\sqrt{-a}\,\tau} + c_2 e^{-\sqrt{-a}\,\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(x + \sqrt{x^2 - 1}\right)^{\frac{\sqrt{-a}\,\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x + \sqrt{x^2 - 1}\right)^{-\frac{\sqrt{-a}\,\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(x + \sqrt{x^2 - 1}\right)^{\frac{\sqrt{-a}\,\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x + \sqrt{x^2 - 1}\right)^{-\frac{\sqrt{-a}\,\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} \tag{1}$$

Verification of solutions

$$y = c_1 \left(x + \sqrt{x^2 - 1}\right)^{\frac{\sqrt{-a}\,\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x + \sqrt{x^2 - 1}\right)^{-\frac{\sqrt{-a}\,\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}}$$

Verified OK.

30.3.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$\left(x^2 - 1\right) y'' + y'x + ay = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x}{x^2 - 1} \\ q(x) &= \frac{a}{x^2 - 1} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{a}{x^2-1}}}{c} \\ \tau'' &= -\frac{ax}{c\sqrt{\frac{a}{x^2-1}}\,(x^2-1)^2} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{ax}{c\sqrt{\frac{a}{x^2-1}}\,(x^2-1)^2} + \frac{x}{x^2-1}\frac{\sqrt{\frac{a}{x^2-1}}}{c}}{\left(\frac{\sqrt{\frac{a}{x^2-1}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \sqrt{\frac{a}{x^2-1}}dx}{c} \\ &= \frac{\sqrt{\frac{a}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{c} \end{aligned}$$

Substituting the above into the solution obtained gives

$$\begin{aligned} y = {} & c_1 \cos\left(\sqrt{a}\sqrt{\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \\ & + c_2 \sin\left(\sqrt{a}\sqrt{\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 \cos\left(\sqrt{a}\,\sqrt{\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \\
& + c_2 \sin\left(\sqrt{a}\,\sqrt{\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 \cos\left(\sqrt{a}\,\sqrt{\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \\
& + c_2 \sin\left(\sqrt{a}\,\sqrt{\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned}
$$

Verified OK.

30.3.3 Solving using Kovacic algorithm

Writing the ode as

$$
\left(x^2-1\right) y'' + y'x + ay = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 - 1 \\
B &= x \\
C &= a
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4a\,x^2 - x^2 + 4a - 2}{4\,(x^2-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4a\,x^2 - x^2 + 4a - 2 \\ t &= 4(x^2-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4a\,x^2 - x^2 + 4a - 2}{4\,(x^2-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 144: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(x-1)^2} + \frac{\frac{1}{16}-\frac{a}{2}}{x-1} - \frac{3}{16(1+x)^2} + \frac{-\frac{1}{16}+\frac{a}{2}}{1+x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4a\,x^2 - x^2 + 4a - 2}{4(x^2-1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -1$. Hence

$$\begin{aligned} E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1,2,3\}$
-1	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(1))} + \frac{1}{(x-(-1))}\right) \\ &= \frac{1}{2x-2} + \frac{1}{2+2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta + \frac{p'}{p} \\
&= \frac{1}{2x-2} + \frac{1}{2+2x}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0
$$

Substituting the values for ϕ and r into the above equation gives

$$
w^2 - \left(\frac{1}{2x-2} + \frac{1}{2+2x}\right) w + \frac{4a\,x^2 + x^2 - 4a}{4\left(x^2-1\right)^2} = 0
$$

Solving for ω gives

$$
\omega = \frac{x + 2\sqrt{-a\left(x^2-1\right)}}{2\left(x-1\right)\left(1+x\right)}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \frac{x+2\sqrt{-a\left(x^2-1\right)}}{2(x-1)(1+x)} dx} \\
&= \left(x^2-1\right)^{\frac{1}{4}} \mathrm{e}^{-\sqrt{a}\,\arctan\left(\frac{\sqrt{a}\,x}{\sqrt{\left(-x^2+1\right)a}}\right)}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2-1}\, dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{4} - \frac{\ln(1+x)}{4}} \\
&= z_1 \left(\frac{1}{\left(x-1\right)^{\frac{1}{4}}\left(1+x\right)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2-1\right)^{\frac{1}{4}} \mathrm{e}^{-\sqrt{a}\,\arctan\left(x\sqrt{-\frac{1}{x^2-1}}\right)}}{\left(x-1\right)^{\frac{1}{4}}\left(1+x\right)^{\frac{1}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{2\sqrt{a}\, \arctan\left(x\sqrt{-\frac{1}{x^2-1}}\right)}}{\sqrt{x^2-1}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2-1)^{\frac{1}{4}}\, \mathrm{e}^{-\sqrt{a}\, \arctan\left(x\sqrt{-\frac{1}{x^2-1}}\right)}}{(x-1)^{\frac{1}{4}}\, (1+x)^{\frac{1}{4}}}\right) \\
&\quad + c_2 \left(\frac{(x^2-1)^{\frac{1}{4}}\, \mathrm{e}^{-\sqrt{a}\, \arctan\left(x\sqrt{-\frac{1}{x^2-1}}\right)}}{(x-1)^{\frac{1}{4}}\, (1+x)^{\frac{1}{4}}} \left(\int \frac{\mathrm{e}^{2\sqrt{a}\, \arctan\left(x\sqrt{-\frac{1}{x^2-1}}\right)}}{\sqrt{x^2-1}} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= \frac{c_1 (x^2-1)^{\frac{1}{4}}\, \mathrm{e}^{-\sqrt{a}\, \arctan\left(x\sqrt{-\frac{1}{x^2-1}}\right)}}{(x-1)^{\frac{1}{4}}\, (1+x)^{\frac{1}{4}}} \\
&\quad + \frac{c_2 (x^2-1)^{\frac{1}{4}}\, \mathrm{e}^{-\sqrt{a}\, \arctan\left(x\sqrt{-\frac{1}{x^2-1}}\right)} \left(\int \frac{\mathrm{e}^{2\sqrt{a}\, \arctan\left(x\sqrt{-\frac{1}{x^2-1}}\right)}}{\sqrt{x^2-1}} dx\right)}{(x-1)^{\frac{1}{4}}\, (1+x)^{\frac{1}{4}}}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2-1)^{\frac{1}{4}} \mathrm{e}^{-\sqrt{a}\,\arctan\left(x\sqrt{-\frac{1}{x^2-1}}\right)}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} + \frac{c_2(x^2-1)^{\frac{1}{4}} \mathrm{e}^{-\sqrt{a}\,\arctan\left(x\sqrt{-\frac{1}{x^2-1}}\right)} \left(\int \frac{\mathrm{e}^{2\sqrt{a}\,\arctan\left(x\sqrt{-\frac{1}{x^2-1}}\right)}}{\sqrt{x^2-1}}dx\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}$$

Verified OK.

30.3.4 Maple step by step solution

Let's solve

$y''(x^2-1)+y'x+ay=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{xy'}{x^2-1}-\frac{ay}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{xy'}{x^2-1}+\frac{ay}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{x}{x^2-1}, P_3(x)=\frac{a}{x^2-1}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{1}{2}$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

$y''(x^2-1)+y'x+ay=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)+ay(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ○ Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ○ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  ○ Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0r(-1+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(2k+1+2r)+a_k(k^2+2kr+r^2+a)\right)u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(-1+2r)=0$

- Values of r that satisfy the indicial equation

$r\in\left\{0,\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$-2\left(k+\frac{1}{2}+r\right)(k+1+r)\,a_{k+1}+a_k\left(k^2+2kr+r^2+a\right)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{a_k\,(k^2+2kr+r^2+a)}{(2k+1+2r)(k+1+r)}$

- Recursion relation for $r=0$

$a_{k+1} = \frac{a_k(k^2+a)}{(2k+1)(k+1)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k^2+a)}{(2k+1)(k+1)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+1} = \frac{a_k(k^2+a)}{(2k+1)(k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k(k^2+a+k+\frac{1}{4})}{(2k+2)(k+\frac{3}{2})}$

- Solution for $r = \frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k(k^2+a+k+\frac{1}{4})}{(2k+2)(k+\frac{3}{2})}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k(k^2+a+k+\frac{1}{4})}{(2k+2)(k+\frac{3}{2})}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} b_k (1+x)^k\right) + \left(\sum_{k=0}^{\infty} c_k (1+x)^{k+\frac{1}{2}}\right), b_{1+k} = \frac{b_k(k^2+a)}{(2k+1)(1+k)}, c_{1+k} = \frac{c_k(k^2+a+k+\frac{1}{4})}{(2k+2)(k+\frac{3}{2})}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve((x^2-1)*diff(y(x),x$2)+x*diff(y(x),x)+a*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(x+\sqrt{x^2-1}\right)^{i\sqrt{a}} + c_2\left(x+\sqrt{x^2-1}\right)^{-i\sqrt{a}}$$

✓ Solution by Mathematica

Time used: 0.144 (sec). Leaf size: 97

```
DSolve[(x^2-1)*y''[x]+x*y'[x]+a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) \to\ & c_1 \cos\left(\frac{1}{2}\sqrt{a}\left(\log\left(1-\frac{x}{\sqrt{x^2-1}}\right)-\log\left(\frac{x}{\sqrt{x^2-1}}+1\right)\right)\right) \\ & - c_2 \sin\left(\frac{1}{2}\sqrt{a}\left(\log\left(1-\frac{x}{\sqrt{x^2-1}}\right)-\log\left(\frac{x}{\sqrt{x^2-1}}+1\right)\right)\right) \end{aligned}$$

30.4 problem 152

Internal problem ID [10976]
Internal file name [OUTPUT/10232_Sunday_December_31_2023_11_10_31_AM_33496255/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2+bx+c)y''+f(x)y'+g(x)y=0$
Problem number: 152.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[_Gegenbauer, [_2nd_order, _linear, `_with_symmetry_[0,F(x)]`]]
```

$$\boxed{\left(-x^2+1\right)y''-y'x+yn^2=0}$$

30.4.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$\left(-x^2+1\right)y''-y'x+yn^2=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x}{x^2-1} \\ q(x) &= \frac{n^2}{-x^2+1} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int \frac{x}{x^2-1} dx\right)} dx \\
&= \int e^{-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2}} dx \\
&= \int \frac{1}{\sqrt{x-1}\,\sqrt{1+x}} dx \\
&= \frac{\sqrt{(x-1)(1+x)}\,\ln\left(x + \sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{1+x}}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{n^2}{-x^2+1}}{\frac{1}{(x-1)(1+x)}} \\
&= -n^2
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) - n^2 y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -n^2$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} - n^2 e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - n^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -n^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-n^2)} \\ &= \pm\sqrt{n^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{n^2} \\ \lambda_2 &= -\sqrt{n^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{n^2} \\ \lambda_2 &= -\sqrt{n^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1\tau} + c_2 e^{\lambda_2\tau} \\ y(\tau) &= c_1 e^{\left(\sqrt{n^2}\right)\tau} + c_2 e^{\left(-\sqrt{n^2}\right)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{\sqrt{n^2}\,\tau} + c_2 e^{-\sqrt{n^2}\,\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(x + \sqrt{x^2 - 1}\right)^n + c_2 \left(x + \sqrt{x^2 - 1}\right)^{-n}$$

Summary
The solution(s) found are the following

$$y = c_1 \left(x + \sqrt{x^2 - 1}\right)^n + c_2 \left(x + \sqrt{x^2 - 1}\right)^{-n} \tag{1}$$

Verification of solutions

$$y = c_1 \left(x + \sqrt{x^2 - 1}\right)^n + c_2 \left(x + \sqrt{x^2 - 1}\right)^{-n}$$

Verified OK.

30.4.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$\left(-x^2 + 1\right) y'' - y'x + yn^2 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x}{x^2 - 1} \\ q(x) &= -\frac{n^2}{x^2 - 1} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-\frac{n^2}{x^2-1}}}{c} \\ \tau'' &= \frac{n^2 x}{c\sqrt{-\frac{n^2}{x^2-1}}\,(x^2-1)^2}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{n^2 x}{c\sqrt{-\frac{n^2}{x^2-1}}\,(x^2-1)^2} + \frac{x}{x^2-1}\frac{\sqrt{-\frac{n^2}{x^2-1}}}{c}}{\left(\frac{\sqrt{-\frac{n^2}{x^2-1}}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \sqrt{-\frac{n^2}{x^2-1}}dx}{c} \\ &= \frac{\sqrt{-\frac{n^2}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$
\begin{aligned}
y = {} & c_1 \cos\left(n\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \\
& + c_2 \sin\left(n\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 \cos\left(n\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \\
& + c_2 \sin\left(n\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 \cos\left(n\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \\
& + c_2 \sin\left(n\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned}
$$

Verified OK.

30.4.3 Solving using Kovacic algorithm

Writing the ode as

$$
\left(-x^2+1\right) y'' - y'x + yn^2 = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= -x^2+1 \\
B &= -x \\
C &= n^2
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4n^2x^2 - 4n^2 - x^2 - 2}{4\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4n^2x^2 - 4n^2 - x^2 - 2 \\ t &= 4\left(x^2 - 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4n^2x^2 - 4n^2 - x^2 - 2}{4\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 146: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(x-1)^2} + \frac{\frac{1}{16} + \frac{n^2}{2}}{x-1} - \frac{3}{16(1+x)^2} + \frac{-\frac{n^2}{2} - \frac{1}{16}}{1+x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4n^2x^2 - 4n^2 - x^2 - 2}{4(x^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=1$. Hence

$$\begin{aligned}E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\ &= \{2\}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1,2,3\}$
-1	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\ &= \frac{1}{2}(2-(1+(1)))\\ &= 0\end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{1}{(x-(1))}+\frac{1}{(x-(-1))}\right)\\ &= \frac{1}{2x-2}+\frac{1}{2+2x}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{2x-2} + \frac{1}{2+2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2x-2} + \frac{1}{2+2x}\right)w + \frac{-4n^2x^2 + 4n^2 + x^2}{4\left(x^2-1\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{x + 2n\sqrt{x^2-1}}{2\left(x-1\right)\left(1+x\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx}\\ &= \mathrm{e}^{\int \frac{x+2n\sqrt{x^2-1}}{2(x-1)(1+x)}dx}\\ &= \left(x^2-1\right)^{\frac{1}{4}}\left(x+\sqrt{x^2-1}\right)^n\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{-x^2+1}\, dx}\\ &= z_1 e^{-\frac{\ln(x-1)}{4} - \frac{\ln(1+x)}{4}}\\ &= z_1\left(\frac{1}{\left(x-1\right)^{\frac{1}{4}}\left(1+x\right)^{\frac{1}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2-1)^{\frac{1}{4}} \left(x+\sqrt{x^2-1}\right)^n}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{-x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\left(x+\sqrt{x^2-1}\right)^{-2n}}{2n}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(x^2-1)^{\frac{1}{4}} \left(x+\sqrt{x^2-1}\right)^n}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}\right) \\ &\quad + c_2 \left(\frac{(x^2-1)^{\frac{1}{4}} \left(x+\sqrt{x^2-1}\right)^n}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}} \left(-\frac{\left(x+\sqrt{x^2-1}\right)^{-2n}}{2n}\right)\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1 (x^2-1)^{\frac{1}{4}} \left(x+\sqrt{x^2-1}\right)^n}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}} - \frac{c_2 (x^2-1)^{\frac{1}{4}} \left(x+\sqrt{x^2-1}\right)^{-n}}{2n (1+x)^{\frac{1}{4}} (x-1)^{\frac{1}{4}}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 (x^2-1)^{\frac{1}{4}} \left(x+\sqrt{x^2-1}\right)^n}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}} - \frac{c_2 (x^2-1)^{\frac{1}{4}} \left(x+\sqrt{x^2-1}\right)^{-n}}{2n (1+x)^{\frac{1}{4}} (x-1)^{\frac{1}{4}}}$$

Verified OK.

30.4.4 Maple step by step solution

Let's solve

$(-x^2+1)\,y''-y'x+yn^2=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{xy'}{x^2-1}+\frac{n^2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{xy'}{x^2-1}-\frac{n^2y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x}{x^2-1}, P_3(x)=-\frac{n^2}{x^2-1}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=\frac{1}{2}$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $y''(x^2-1)+y'x-yn^2=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)-n^2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(k+n+r)\,(k-n+r)\right) u^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + a_k(k+n+r)\,(k-n+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+n+r)(k-n+r)}{(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k(k+n)(k-n)}{(2k+1)(k+1)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+n)(k-n)}{(2k+1)(k+1)}\right]$$

- Revert the change of variables $u = 1+x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k(k+n)(k-n)}{(2k+1)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k \left(k+n+\frac{1}{2}\right) \left(k-n+\frac{1}{2}\right)}{(2k+2) \left(k+\frac{3}{2}\right)}$

- Solution for $r = \frac{1}{2}$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k \left(k+n+\frac{1}{2}\right) \left(k-n+\frac{1}{2}\right)}{(2k+2) \left(k+\frac{3}{2}\right)}\right]$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k \left(k+n+\frac{1}{2}\right) \left(k-n+\frac{1}{2}\right)}{(2k+2) \left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+\frac{1}{2}}\right), a_{1+k} = \frac{a_k (k+n) (k-n)}{(2k+1) (1+k)}, b_{1+k} = \frac{b_k \left(k+n+\frac{1}{2}\right) \left(k-n+\frac{1}{2}\right)}{(2k+2) \left(k+\frac{3}{2}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve((1-x^2)*diff(y(x),x$2)-x*diff(y(x),x)+n^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \left(x + \sqrt{x^2 - 1}\right)^{-n} + c_2 \left(x + \sqrt{x^2 - 1}\right)^{n}$$

✓ Solution by Mathematica

Time used: 0.139 (sec). Leaf size: 91

```
DSolve[(1-x^2)*y''[x]-x*y'[x]+n^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cosh\left(\frac{1}{2}n\left(\log\left(1-\frac{x}{\sqrt{x^2-1}}\right)-\log\left(\frac{x}{\sqrt{x^2-1}}+1\right)\right)\right) - ic_2 \sinh\left(\frac{1}{2}n\left(\log\left(1-\frac{x}{\sqrt{x^2-1}}\right)-\log\left(\frac{x}{\sqrt{x^2-1}}+1\right)\right)\right)$$

30.5 problem 153

Internal problem ID [10977]
Internal file name [OUTPUT/10233_Sunday_December_31_2023_11_10_33_AM_23421686/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 153.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_Gegenbauer]`

Unable to solve or complete the solution.

$$\boxed{(-x^2 + 1)\, y'' - 2y'x + n(n + 1)\, y = 0}$$

30.5.1 Maple step by step solution

Let's solve

$(-x^2 + 1)\, y'' - 2y'x + (n^2 + n)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{n(n+1)y}{x^2-1} - \frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2xy'}{x^2-1} - \frac{n(n+1)y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{n(n+1)}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $((1+x) \cdot P_2(x))\bigg|_{x=-1} = 1$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $((1+x)^2 \cdot P_3(x))\bigg|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $y''(x^2-1) + 2y'x - n(n+1)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) + (-n^2-n)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$-2a_0 r^2 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(r+1+n+k)(r-n+k)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(1+n+k)(-n+k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 15

```
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+n*(n+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{LegendreP}(n, x) + c_2 \operatorname{LegendreQ}(n, x)$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 18

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+n*(n+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \text{LegendreP}(n, x) + c_2 \text{LegendreQ}(n, x)$$

30.6 problem 154

Internal problem ID [10978]
Internal file name [`OUTPUT/10234_Sunday_December_31_2023_11_10_34_AM_21094205/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 154.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_Gegenbauer]`

Unable to solve or complete the solution.

$$\left(-x^2+1\right) y''-2y'x+\nu(\nu+1)\, y=0$$

30.6.1 Maple step by step solution

Let's solve

$\left(-x^2+1\right) y''-2y'x+\left(\nu^2+\nu\right) y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{\nu(\nu+1)y}{x^2-1}-\frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2xy'}{x^2-1}-\frac{\nu(\nu+1)y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

$\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{\nu(\nu+1)}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $$((1+x) \cdot P_2(x)) \bigg|_{x=-1} = 1$$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $$((1+x)^2 \cdot P_3(x)) \bigg|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $$y''(x^2-1) + 2y'x - \nu(\nu+1)\, y = 0$$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $$(u^2 - 2u)\left(\frac{d^2}{du^2} y(u)\right) + (2u-2)\left(\frac{d}{du} y(u)\right) + (-\nu^2 - \nu)\, y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

$$-2a_0 r^2 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(r+1+\nu+k)(r-\nu+k)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(1+\nu+k)(-\nu+k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(1+\nu+k)(-\nu+k)}{2(k+1)^2}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(1+\nu+k)(-\nu+k)}{2(k+1)^2}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(1+\nu+k)(-\nu+k)}{2(k+1)^2}\right]$
- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k(1+\nu+k)(-\nu+k)}{2(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 15

```
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+nu*(nu+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{LegendreP}(\nu, x) + c_2 \operatorname{LegendreQ}(\nu, x)$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 18

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+\[Nu]*(\[Nu]+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to c_1 \text{LegendreP}(\nu, x) + c_2 \text{LegendreQ}(\nu, x)$$

30.7 problem 155

Internal problem ID [10979]
Internal file name [`OUTPUT/10235_Sunday_December_31_2023_11_10_35_AM_79879110/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2+bx+c)y''+f(x)y'+g(x)y=0$
Problem number: 155.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(-x^2+1)\,y''-3y'x+ny(2+n)=0}$$

30.7.1 Solving using Kovacic algorithm

Writing the ode as

$$(-x^2+1)\,y''-3y'x+(n^2+2n)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^2+1\\ B&=-3x\\ C&=n^2+2n\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4n^2x^2 + 8n\,x^2 - 4n^2 + 3x^2 - 8n - 6}{4\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4n^2x^2 + 8n\,x^2 - 4n^2 + 3x^2 - 8n - 6 \\ t &= 4\left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4n^2x^2 + 8n\,x^2 - 4n^2 + 3x^2 - 8n - 6}{4\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 150: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(x-1)^2} + \frac{\frac{9}{16} + \frac{1}{2}n^2 + n}{x-1} - \frac{3}{16(1+x)^2} + \frac{-\frac{9}{16} - \frac{1}{2}n^2 - n}{1+x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4n^2x^2 + 8n\,x^2 - 4n^2 + 3x^2 - 8n - 6}{4(x^2-1)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = 1$. Hence

$$\begin{aligned} E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1,2,3\}$
-1	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \;\; e_2 = 1, \;\; e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2-(1+(1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(1))} + \frac{1}{(x-(-1))}\right) \\ &= \frac{1}{2x-2} + \frac{1}{2+2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}
\phi &= \theta + \frac{p'}{p} \\
&= \frac{1}{2x-2} + \frac{1}{2+2x}
\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2x-2} + \frac{1}{2+2x}\right) w + \frac{-4n^2x^2 - 8n\,x^2 + 4n^2 - 3x^2 + 8n + 4}{4\left(x^2-1\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{2n\sqrt{x^2-1} + 2\sqrt{x^2-1} + x}{2\left(x-1\right)\left(1+x\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= e^{\int \frac{2n\sqrt{x^2-1}+2\sqrt{x^2-1}+x}{2(x-1)(1+x)} dx} \\
&= \left(x^2-1\right)^{\frac{1}{4}} \left(x + \sqrt{x^2-1}\right)^{n+1}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{-x^2+1} \, dx} \\
&= z_1 e^{-\frac{3\ln(x-1)}{4} - \frac{3\ln(1+x)}{4}} \\
&= z_1 \left(\frac{1}{\left(x-1\right)^{\frac{3}{4}}\left(1+x\right)^{\frac{3}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2-1)^{\frac{1}{4}} \left(x+\sqrt{x^2-1}\right)^{n+1}}{(x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{-x^2+1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-\frac{3\ln(x-1)}{2} - \frac{3\ln(1+x)}{2}}}{(y_1)^2} dx \\ &= y_1 \left(-\frac{\left(x+\sqrt{x^2-1}\right)^{-2n-2}}{2+2n} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(x^2-1)^{\frac{1}{4}} \left(x+\sqrt{x^2-1}\right)^{n+1}}{(x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}}} \right) \\ &\quad + c_2 \left(\frac{(x^2-1)^{\frac{1}{4}} \left(x+\sqrt{x^2-1}\right)^{n+1}}{(x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}}} \left(-\frac{\left(x+\sqrt{x^2-1}\right)^{-2n-2}}{2+2n} \right) \right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1 (x^2-1)^{\frac{1}{4}} \left(x+\sqrt{x^2-1}\right)^{n+1}}{(x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}}} - \frac{c_2 (x^2-1)^{\frac{1}{4}} \left(x+\sqrt{x^2-1}\right)^{-n-1}}{(x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}} (2+2n)} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 (x^2-1)^{\frac{1}{4}} \left(x+\sqrt{x^2-1}\right)^{n+1}}{(x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}}} - \frac{c_2 (x^2-1)^{\frac{1}{4}} \left(x+\sqrt{x^2-1}\right)^{-n-1}}{(x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}} (2+2n)}$$

Verified OK.

30.7.2 Maple step by step solution

Let's solve

$(-x^2+1)\,y''-3y'x+(n^2+2n)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{n(2+n)y}{x^2-1}-\frac{3xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3xy'}{x^2-1}-\frac{n(2+n)y}{x^2-1}=0$

☐ Check to see if x_0 is a regular singular point

- ∘ Define functions

  $\left[P_2(x)=\frac{3x}{x^2-1},P_3(x)=-\frac{n(2+n)}{x^2-1}\right]$

- ∘ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{3}{2}$

- ∘ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ∘ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $y''(x^2-1)+3y'x-ny(2+n)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(3u-3)\left(\frac{d}{du}y(u)\right)+(-n^2-2n)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(2k+3+2r) + a_k(r+2+n+k)(r-n+k)\right) u^{k}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(r+2+n+k)(r-n+k) - 2\left(k+\frac{3}{2}+r\right)(k+1+r)\, a_{k+1} = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(r+2+n+k)(r-n+k)}{(2k+3+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k(2+n+k)(-n+k)}{(2k+3)(k+1)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(2+n+k)(-n+k)}{(2k+3)(k+1)}\right]$$

- Revert the change of variables $u = 1+x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k(2+n+k)(-n+k)}{(2k+3)(k+1)}\right]$$

- Recursion relation for $r = -\frac{1}{2}$

  $$a_{k+1} = \frac{a_k\left(\frac{3}{2}+n+k\right)\left(-\frac{1}{2}-n+k\right)}{(2k+2)\left(k+\frac{1}{2}\right)}$$

- Solution for $r = -\frac{1}{2}$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(\frac{3}{2}+n+k\right)\left(-\frac{1}{2}-n+k\right)}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(\frac{3}{2}+n+k\right)\left(-\frac{1}{2}-n+k\right)}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k-\frac{1}{2}}\right), a_{1+k} = \frac{a_k(2+n+k)(-n+k)}{(2k+3)(1+k)}, b_{1+k} = \frac{b_k\left(\frac{3}{2}+n+k\right)\left(-\frac{1}{2}-n+k\right)}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 68

```
dsolve((1-x^2)*diff(y(x),x$2)-3*x*diff(y(x),x)+n*(n+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1\left(-\sqrt{x^2-1}+x\right)\left(x+\sqrt{x^2-1}\right)^{-n-1} - c_2\left(x+\sqrt{x^2-1}\right)^n}{\sqrt{x^2-1}\,\left(-\sqrt{x^2-1}+x\right)}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 42

```
DSolve[(1-x^2)*y''[x]-3*x*y'[x]+n*(n+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 P_{n+\frac{1}{2}}^{\frac{1}{2}}(x) + c_2 Q_{n+\frac{1}{2}}^{\frac{1}{2}}(x)}{\sqrt[4]{x^2-1}}$$

30.8 problem 156

Internal problem ID [10980]
Internal file name [`OUTPUT/10236_Sunday_December_31_2023_11_10_36_AM_51136809/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 156.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_Gegenbauer]`

Unable to solve or complete the solution.

$$\boxed{(x^2 - 1)\, y'' + 2(n + 1)\, xy' - (\nu + n + 1)\,(\nu - n)\, y = 0}$$

30.8.1 Maple step by step solution

Let's solve

$y''(x^2 - 1) + (2xn + 2x)\, y' + (n^2 - \nu^2 + n - \nu)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(n^2-\nu^2+n-\nu)y}{x^2-1} - \frac{2(n+1)xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(n+1)xy'}{x^2-1} + \frac{(n^2-\nu^2+n-\nu)y}{x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{2(n+1)x}{x^2-1}, P_3(x) = \frac{n^2-\nu^2+n-\nu}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $((1+x) \cdot P_2(x)) \bigg|_{x=-1} = n+1$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right) \bigg|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $y''(x^2-1) + 2(n+1)\,xy' + (n^2-\nu^2+n-\nu)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2nu-2n+2u-2)\left(\frac{d}{du}y(u)\right) + (n^2-\nu^2+n-\nu)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$-2a_0 r(r+n)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(k+1+r+n) + a_k(k+n+\nu+r+1)\,(r-\nu+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-2r(r+n) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, -n\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-2a_{k+1}(k+1+r)\,(k+1+r+n) + a_k(k+n+\nu+r+1)\,(r-\nu+n+k) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+n+\nu+r+1)(r-\nu+n+k)}{2(k+1+r)(k+1+r+n)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k(k+n+\nu+1)(-\nu+n+k)}{2(k+1)(k+1+n)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+n+\nu+1)(-\nu+n+k)}{2(k+1)(k+1+n)}\right]$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k(k+n+\nu+1)(-\nu+n+k)}{2(k+1)(k+1+n)}\right]$$

- Recursion relation for $r = -n$

  $$a_{k+1} = \frac{a_k(k+\nu+1)(-\nu+k)}{2(k+1-n)(k+1)}$$

- Solution for $r = -n$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-n}, a_{k+1} = \frac{a_k(k+\nu+1)(-\nu+k)}{2(k+1-n)(k+1)}\right]$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-n}, a_{k+1} = \frac{a_k(k+\nu+1)(-\nu+k)}{2(k+1-n)(k+1)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{-n+k}\right), a_{1+k} = \frac{a_k(k+n+\nu+1)(-\nu+n+k)}{2(1+k)(k+1+n)}, b_{1+k} = \frac{b_k(k+\nu+1)(-\nu+}{2(k+1-n)(1+k}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 27

```
dsolve((x^2-1)*diff(y(x),x$2)+2*(n+1)*x*diff(y(x),x)-(nu+n+1)*(nu-n)*y(x)=0,y(x), singsol=al
```

$$y(x) = (\text{LegendreP}(\nu, n, x)\, c_1 + \text{LegendreQ}(\nu, n, x)\, c_2)\,(x^2-1)^{-\frac{n}{2}}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 32

```
DSolve[(x^2-1)*y''[x]+2*(n+1)*x*y'[x]-(\[Nu]+n+1)*(\[Nu]-n)*y[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to (x^2-1)^{-n/2}\,(c_1 P_\nu^n(x) + c_2 Q_\nu^n(x))$$

30.9 problem 157

Internal problem ID [10981]
Internal file name [`OUTPUT/10237_Sunday_December_31_2023_11_10_37_AM_18961001/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2+bx+c)y''+f(x)y'+g(x)y=0$
Problem number: 157.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_Gegenbauer]`

Unable to solve or complete the solution.

$$\boxed{(x^2-1)\,y''-2(n-1)\,xy'-(\nu-n+1)\,(\nu+n)\,y=0}$$

30.9.1 Maple step by step solution

Let's solve

$y''(x^2-1)+(-2xn+2x)\,y'+(n^2-\nu^2-n-\nu)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(n^2-\nu^2-n-\nu)y}{x^2-1}+\frac{2(n-1)xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2(n-1)xy'}{x^2-1}+\frac{(n^2-\nu^2-n-\nu)y}{x^2-1}=0$

☐ Check to see if x_0 is a regular singular point

  ○ Define functions

$$\left[P_2(x) = -\frac{2x(n-1)}{x^2-1}, P_3(x) = \frac{n^2-\nu^2-n-\nu}{x^2-1}\right]$$

- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $$\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = 1-n$$

- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $$\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $$y''(x^2-1) - 2(n-1)\,xy' + (n^2-\nu^2-n-\nu)\,y = 0$$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (-2un+2n+2u-2)\left(\frac{d}{du}y(u)\right) + (n^2-\nu^2-n-\nu)\,y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

$$-2a_0 r(r-n)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(k+1+r-n) + a_k(k-n+\nu+r+1)\,(r-\nu-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(r-n) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, n\}$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1+r)\,(k+1+r-n) + a_k(k-n+\nu+r+1)\,(r-\nu-n+k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k-n+\nu+r+1)(r-\nu-n+k)}{2(k+1+r)(k+1+r-n)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k-n+\nu+1)(-\nu-n+k)}{2(k+1)(k+1-n)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k-n+\nu+1)(-\nu-n+k)}{2(k+1)(k+1-n)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k(k-n+\nu+1)(-\nu-n+k)}{2(k+1)(k+1-n)}\right]$

- Recursion relation for $r = n$

  $a_{k+1} = \frac{a_k(k+\nu+1)(-\nu+k)}{2(k+1+n)(k+1)}$

- Solution for $r = n$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+n}, a_{k+1} = \frac{a_k(k+\nu+1)(-\nu+k)}{2(k+1+n)(k+1)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+n}, a_{k+1} = \frac{a_k(k+\nu+1)(-\nu+k)}{2(k+1+n)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k+n}\right), a_{1+k} = \frac{a_k(k-n+\nu+1)(-\nu-n+k)}{2(1+k)(k+1-n)}, b_{1+k} = \frac{b_k(k+\nu+1)(-\nu+k}{2(k+1+n)(1+k)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 27

```
dsolve((x^2-1)*diff(y(x),x$2)-2*(n-1)*x*diff(y(x),x)-(nu-n+1)*(nu+n)*y(x)=0,y(x), singsol=al
```

$$y(x) = (\text{LegendreP}\,(\nu, n, x)\,c_1 + \text{LegendreQ}\,(\nu, n, x)\,c_2)\,(x^2-1)^{\frac{n}{2}}$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 32

```
DSolve[(x^2-1)*y''[x]-2*(n-1)*x*y'[x]-(\[Nu]-n+1)*(\[Nu]+n)*y[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to (x^2-1)^{n/2}\,(c_1 P_\nu^n(x) + c_2 Q_\nu^n(x))$$

30.10 problem 158

Internal problem ID [10982]
Internal file name [OUTPUT/10238_Sunday_December_31_2023_11_10_39_AM_64591386/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 158.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$(x^2 - 1)\, y'' + (2a + 1)\, y' - b(2a + b)\, y = 0$$

30.10.1 Maple step by step solution

Let's solve

$y''(x^2 - 1) + (2a + 1)\, y' + (-2ab - b^2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{b(2a+b)y}{x^2-1} - \frac{(2a+1)y'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2a+1)y'}{x^2-1} - \frac{b(2a+b)y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{2a+1}{x^2-1}, P_3(x) = -\frac{b(2a+b)}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $((1+x) \cdot P_2(x))\Big|_{x=-1} = -a - \frac{1}{2}$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $((1+x)^2 \cdot P_3(x))\Big|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $y''(x^2-1) + (2a+1)\,y' - b(2a+b)\,y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (2a+1)\left(\frac{d}{du}y(u)\right) + (-2ab - b^2)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r(3-2r+2a)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(-2k+1-2r+2a)-a_k(2ab+b^2-k^2-2kr-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(3-2r+2a)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{3}{2}+a\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(-2k+1-2r+2a)+\left(r^2+(2k-1)\,r-2ab-b^2+k^2-k\right)a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{(2ab+b^2-k^2-2kr-r^2+k+r)\,a_k}{(k+1+r)\,(-2k+1-2r+2a)}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{(2ab+b^2-k^2+k)\,a_k}{(k+1)\,(-2k+1+2a)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{(2ab+b^2-k^2+k)\,a_k}{(k+1)\,(-2k+1+2a)}\right]$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k, a_{k+1}=\frac{(2ab+b^2-k^2+k)\,a_k}{(k+1)\,(-2k+1+2a)}\right]$$

- Recursion relation for $r=\frac{3}{2}+a$

$$a_{k+1}=\frac{\left(2ab+b^2-k^2-2k\left(\frac{3}{2}+a\right)-\left(\frac{3}{2}+a\right)^2+k+\frac{3}{2}+a\right)a_k}{\left(k+\frac{5}{2}+a\right)(-2k-2)}$$

- Solution for $r=\frac{3}{2}+a$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{3}{2}+a}, a_{k+1}=\frac{\left(2ab+b^2-k^2-2k\left(\frac{3}{2}+a\right)-\left(\frac{3}{2}+a\right)^2+k+\frac{3}{2}+a\right)a_k}{\left(k+\frac{5}{2}+a\right)(-2k-2)}\right]$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k+\frac{3}{2}+a}, a_{k+1}=\frac{\left(2ab+b^2-k^2-2k\left(\frac{3}{2}+a\right)-\left(\frac{3}{2}+a\right)^2+k+\frac{3}{2}+a\right)a_k}{\left(k+\frac{5}{2}+a\right)(-2k-2)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}c_k(1+x)^k\right)+\left(\sum_{k=0}^{\infty}d_k(1+x)^{k+\frac{3}{2}+a}\right), c_{1+k}=\frac{(2ab+b^2-k^2+k)\,c_k}{(1+k)\,(-2k+1+2a)}, d_{1+k}=\frac{\left(2ab+b^2-k^2-2k\left(\frac{3}{2}+\right.\right.}{\left(k+\frac{5}{2}+\right.}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 112

```
dsolve((x^2-1)*diff(y(x),x$2)+(2*a+1)*diff(y(x),x)-b*(2*a+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \,\mathrm{hypergeom}\left(\left[-\frac{1}{2}-\frac{\sqrt{8ab+4b^2+1}}{2}, \frac{\sqrt{8ab+4b^2+1}}{2}-\frac{1}{2}\right], \left[-a-\frac{1}{2}\right], \frac{1}{2}+\frac{x}{2}\right) + c_2 \left(\frac{1}{2}+\frac{x}{2}\right)^{a+\frac{3}{2}} \mathrm{hypergeom}\left(\left[1-\frac{\sqrt{8ab+4b^2+1}}{2}+a, \frac{\sqrt{8ab+4b^2+1}}{2}+1+a\right], \left[\frac{5}{2}+a\right], \frac{1}{2}+\frac{x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.304 (sec). Leaf size: 152

```
DSolve[(x^2-1)*y''[x]+(2*a+1)*y'[x]-b*(2*a+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to 2^{a-\frac{1}{2}} c_2 (x-1)^{\frac{1}{2}-a} \text{Hypergeometric2F1}\left(-a - \frac{1}{2}\sqrt{4b^2+8ab+1}, \frac{1}{2}\sqrt{4b^2+8ab+1} - a, \frac{3}{2} - a, \frac{1}{2} - \frac{x}{2}\right) + c_1 \text{Hypergeometric2F1}\left(\frac{1}{2}\left(-\sqrt{4b^2+8ab+1}-1\right), \frac{1}{2}\left(\sqrt{4b^2+8ab+1}-1\right), a + \frac{1}{2}, \frac{1-x}{2}\right)$$

30.11 problem 159

Internal problem ID [10983]
Internal file name [`OUTPUT/10239_Sunday_December_31_2023_11_10_39_AM_59917378/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 159.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_Gegenbauer]`

Unable to solve or complete the solution.

$$(-x^2 + 1)\, y'' + (2a - 3)\, xy' + (n + 1)\,(n + 2a - 1)\, y = 0$$

30.11.1 Maple step by step solution

Let's solve

$(-x^2 + 1)\, y'' + (2a - 3)\, xy' + ((2 + 2n)\, a + n^2 - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2an+n^2+2a-1)y}{x^2-1} + \frac{x(2a-3)y'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{x(2a-3)y'}{x^2-1} - \frac{(2an+n^2+2a-1)y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$$\left[P_2(x) = -\frac{x(2a-3)}{x^2-1}, P_3(x) = -\frac{2an+n^2+2a-1}{x^2-1}\right]$$

- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

$$\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1} = \frac{3}{2}-a$$

- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -1$

- Multiply by denominators

$$y''(x^2-1) - (2a-3)\,xy' + (-2an-n^2-2a+1)\,y = 0$$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

$$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (-2ua+2a+3u-3)\left(\frac{d}{du}y(u)\right) + (-2an-n^2-2a+1)\,y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1-2r+2a)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(-2k-3-2r+2a) - a_k(k+n+r+1)\,(-k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$r(-1-2r+2a) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{0, -\frac{1}{2}+a\right\}$$
- Each term in the series must be 0, giving the recursion relation
$$a_{k+1}(k+1+r)\,(-2k-3-2r+2a) + a_k(k+n+r+1)\,(k-2a+r+1-n) = 0$$
- Recursion relation that defines series solution to ODE
$$a_{k+1} = \frac{a_k(k+n+r+1)(-k+2a-r-1+n)}{(k+1+r)(-2k-3-2r+2a)}$$
- Recursion relation for $r = 0$
$$a_{k+1} = \frac{a_k(k+n+1)(-k+2a-1+n)}{(k+1)(-2k-3+2a)}$$
- Solution for $r = 0$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+n+1)(-k+2a-1+n)}{(k+1)(-2k-3+2a)}\right]$$
- Revert the change of variables $u = 1+x$
$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k(k+n+1)(-k+2a-1+n)}{(k+1)(-2k-3+2a)}\right]$$
- Recursion relation for $r = -\frac{1}{2}+a$
$$a_{k+1} = \frac{a_k\left(k+n+\frac{1}{2}+a\right)\left(-k+a-\frac{1}{2}+n\right)}{\left(k+\frac{1}{2}+a\right)(-2k-2)}$$
- Solution for $r = -\frac{1}{2}+a$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}+a}, a_{k+1} = \frac{a_k\left(k+n+\frac{1}{2}+a\right)\left(-k+a-\frac{1}{2}+n\right)}{\left(k+\frac{1}{2}+a\right)(-2k-2)}\right]$$
- Revert the change of variables $u = 1+x$
$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-\frac{1}{2}+a}, a_{k+1} = \frac{a_k\left(k+n+\frac{1}{2}+a\right)\left(-k+a-\frac{1}{2}+n\right)}{\left(k+\frac{1}{2}+a\right)(-2k-2)}\right]$$
- Combine solutions and rename parameters
$$\left[y = \left(\sum_{k=0}^{\infty} b_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} c_k(1+x)^{k-\frac{1}{2}+a}\right), b_{1+k} = \frac{b_k(k+n+1)(-k+2a-1+n)}{(1+k)(-2k-3+2a)}, c_{1+k} = \frac{c_k\left(k+n+\frac{1}{2}+a\right)}{\left(k+\frac{1}{2}+a\right)}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 39

```
dsolve((1-x^2)*diff(y(x),x$2)+(2*a-3)*x*diff(y(x),x)+(n+1)*(n+2*a-1)*y(x)=0,y(x), singsol=al
```

$$y(x) = \left(\text{LegendreP}\left(a+n-\frac{1}{2}, a-\frac{1}{2}, x\right) c_1 + \text{LegendreQ}\left(a+n-\frac{1}{2}, a-\frac{1}{2}, x\right) c_2\right) (x^2 - 1)^{\frac{a}{2}-\frac{1}{4}}$$

✓ Solution by Mathematica

Time used: 0.325 (sec). Leaf size: 158

```
DSolve[(1-x^2)*y''[x]+(2*a-3)*y'[x]+(n+1)*(n+2*a-1)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to 2^{\frac{1}{2}-a} c_2 (x-1)^{a-\frac{1}{2}} \text{Hypergeometric2F1}\left(a-\frac{1}{2}\sqrt{4n^2+8a(n+1)-3}-1, a+\frac{1}{2}\sqrt{4n^2+8a(n+1)-3}-1, a+\frac{1}{2}, \frac{1-x}{2}\right) + c_1 \text{Hypergeometric2F1}\left(\frac{1}{2}\left(-\sqrt{4n^2+8a(n+1)-3}-1\right), \frac{1}{2}\left(\sqrt{4n^2+8a(n+1)-3}-1\right), \frac{3}{2}-a, \frac{1-x}{2}\right)$$

30.12 problem 160

Internal problem ID [10984]
Internal file name [OUTPUT/10240_Sunday_December_31_2023_11_10_41_AM_52504833/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 160.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$(-x^2 + 1) y'' + (\beta - \alpha - (\alpha + \beta + 2) x) y' + n(n + \alpha + \beta + 1) y = 0$$

30.12.1 Maple step by step solution

Let's solve

$(-x^2 + 1) y'' + ((-\beta - \alpha - 2) x + \beta - \alpha) y' + n(n + \alpha + \beta + 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{n(n+\alpha+\beta+1)y}{x^2-1} - \frac{(x\alpha+\beta x+\alpha-\beta+2x)y'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x\alpha+\beta x+\alpha-\beta+2x)y'}{x^2-1} - \frac{n(n+\alpha+\beta+1)y}{x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{x\alpha+\beta x+\alpha-\beta+2x}{x^2-1}, P_3(x) = -\frac{n(n+\alpha+\beta+1)}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x=-1$

  $((1+x) \cdot P_2(x))\bigg|_{x=-1} = \beta + 1$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $y''(x^2-1) + (x\alpha + \beta x + \alpha - \beta + 2x)\, y' - n(n+\alpha+\beta+1)\, y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (\alpha u + \beta u - 2\beta + 2u - 2)\left(\frac{d}{du}y(u)\right) + (-\alpha n - \beta n - n^2 - n)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$-2a_0 r(r+\beta)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(k+1+r+\beta) + a_k(k-n+r)\,(k+\alpha+\beta+r+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-2r(r+\beta) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, -\beta\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-2a_{k+1}(k+1+r)\,(k+1+r+\beta) + a_k(k-n+r)\,(k+\alpha+\beta+r+1+n) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k-n+r)(k+\alpha+\beta+r+1+n)}{2(k+1+r)(k+1+r+\beta)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k(k-n)(k+\alpha+\beta+1+n)}{2(k+1)(k+1+\beta)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k-n)(k+\alpha+\beta+1+n)}{2(k+1)(k+1+\beta)}\right]$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k(k-n)(k+\alpha+\beta+1+n)}{2(k+1)(k+1+\beta)}\right]$$

- Recursion relation for $r = -\beta$

  $$a_{k+1} = \frac{a_k(k-n-\beta)(k+\alpha+1+n)}{2(k+1-\beta)(k+1)}$$

- Solution for $r = -\beta$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\beta}, a_{k+1} = \frac{a_k(k-n-\beta)(k+\alpha+1+n)}{2(k+1-\beta)(k+1)}\right]$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-\beta}, a_{k+1} = \frac{a_k(k-n-\beta)(k+\alpha+1+n)}{2(k+1-\beta)(k+1)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k-\beta}\right), a_{1+k} = \frac{a_k(k-n)(k+\alpha+\beta+1+n)}{2(1+k)(k+1+\beta)}, b_{1+k} = \frac{b_k(k-n-\beta)(k+\alpha+1}{2(k+1-\beta)(1+k)}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 61

```
dsolve((1-x^2)*diff(y(x),x$2)+(beta-alpha-(alpha+beta+2)*x)*diff(y(x),x)+n*(n+alpha+beta+1)*
```

$$y(x) = c_1 \operatorname{hypergeom}\left([-n, n+\alpha+\beta+1], [\beta+1], \frac{1}{2}+\frac{x}{2}\right) + c_2\left(\frac{1}{2}+\frac{x}{2}\right)^{-\beta} \operatorname{hypergeom}\left([-n-\beta, n+\alpha+1], [1-\beta], \frac{1}{2}+\frac{x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.28 (sec). Leaf size: 69

```
DSolve[(1-x^2)*y''[x]+(\[Beta]-\[Alpha]-(\[Alpha]+\[Beta]+2)*x)*y'[x]+n*(n+\[Alpha]+\[Beta]+
```

$$y(x) \to 2^{\alpha} c_2 (x-1)^{-\alpha} \text{Hypergeometric2F1}\left(-n-\alpha, n+\beta+1, 1-\alpha, \frac{1-x}{2}\right) + c_1 \text{Hypergeometric2F1}\left(-n, n+\alpha+\beta+1, \alpha+1, \frac{1-x}{2}\right)$$

30.13 problem 161

Internal problem ID [10985]
Internal file name [`OUTPUT/10241_Sunday_December_31_2023_11_10_43_AM_21876183/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2+bx+c)y''+f(x)y'+g(x)y=0$
Problem number: 161.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{(-x^2+1)\,y''+(\alpha-\beta+(\alpha+\beta-2)\,x)\,y'+(n+1)\,(n+\alpha+\beta)\,y=0}$$

30.13.1 Maple step by step solution

Let's solve

$(-x^2+1)\,y''+(\alpha-\beta+(\alpha+\beta-2)\,x)\,y'+(n+1)\,(n+\alpha+\beta)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(n+1)(n+\alpha+\beta)y}{x^2-1}+\frac{(x\alpha+\beta x+\alpha-\beta-2x)y'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x\alpha+\beta x+\alpha-\beta-2x)y'}{x^2-1}-\frac{(n+1)(n+\alpha+\beta)y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

$$\left[P_2(x) = -\frac{x\alpha+\beta x+\alpha-\beta-2x}{x^2-1}, P_3(x) = -\frac{(n+1)(n+\alpha+\beta)}{x^2-1}\right]$$

- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $$\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1} = -\beta+1$$

- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $$\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $$y''(x^2-1)+(-x\alpha-\beta x-\alpha+\beta+2x)\,y'-(n+1)\,(n+\alpha+\beta)\,y=0$$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-\alpha u-\beta u+2\beta+2u-2)\left(\frac{d}{du}y(u)\right)+(-\alpha n-\beta n-n^2-\alpha-\beta-n)\,y($$

- Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

$$2a_0 r(-r+\beta)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)\,(-k-1-r+\beta) - a_k(k+n+r+1)\,(-k+\alpha+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$2r(-r+\beta) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, \beta\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+n+r+1)\,(k+r-\beta-\alpha-n)\, a_k - 2a_{k+1}(k+1+r)\,(k+r-\beta+1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{(k+n+r+1)(-k+\alpha+\beta-r+n)a_k}{2(k+1+r)(-k-1-r+\beta)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{(k+n+1)(-k+\alpha+\beta+n)a_k}{2(k+1)(-k-1+\beta)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{(k+n+1)(-k+\alpha+\beta+n)a_k}{2(k+1)(-k-1+\beta)}\right]$$

- Revert the change of variables $u = 1+x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{(k+n+1)(-k+\alpha+\beta+n)a_k}{2(k+1)(-k-1+\beta)}\right]$$

- Recursion relation for $r = \beta$

  $$a_{k+1} = \frac{(k+n+\beta+1)(-k+\alpha+n)a_k}{2(k+1+\beta)(-k-1)}$$

- Solution for $r = \beta$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\beta}, a_{k+1} = \frac{(k+n+\beta+1)(-k+\alpha+n)a_k}{2(k+1+\beta)(-k-1)}\right]$$

- Revert the change of variables $u = 1+x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+\beta}, a_{k+1} = \frac{(k+n+\beta+1)(-k+\alpha+n)a_k}{2(k+1+\beta)(-k-1)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k+\beta}\right), a_{1+k} = \frac{(k+n+1)(-k+\alpha+\beta+n)a_k}{2(1+k)(-k-1+\beta)}, b_{1+k} = \frac{(k+n+\beta+1)(-k+}{2(k+1+\beta)(-k}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 64

```
dsolve((1-x^2)*diff(y(x),x$2)+(alpha-beta+(alpha+beta-2)*x)*diff(y(x),x)+(n+1)*(n+alpha+beta
```

$$y(x) = c_1 \,\mathrm{hypergeom}\left([n+1,-n-\beta-\alpha],[1-\beta],\frac{1}{2}+\frac{x}{2}\right) + c_2\left(\frac{1}{2}+\frac{x}{2}\right)^{\beta}\mathrm{hypergeom}\left([-n-\alpha,n+\beta+1],[\beta+1],\frac{1}{2}+\frac{x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.225 (sec). Leaf size: 74

```
DSolve[(1-x^2)*y''[x]+(\[Alpha]-\[Beta]+(\[Alpha]+\[Beta]-2)*x)*y'[x]+(n+1)*(n+\[Alpha]+\[Be
```

$$y(x) \to 2^{-\alpha} c_2 (x-1)^{\alpha} \, \text{Hypergeometric2F1}\left(n+\alpha+1, -n-\beta, \alpha+1, \frac{1-x}{2}\right) + c_1 \, \text{Hypergeometric2F1}\left(n+1, -n-\alpha-\beta, 1-\alpha, \frac{1-x}{2}\right)$$

30.14 problem 162

Internal problem ID [10986]
Internal file name [`OUTPUT/10242_Sunday_December_31_2023_11_10_44_AM_62076601/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 162.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{\left(a\,x^2+b\right)y''+axy'+yc=0}$$

30.14.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$\left(a\,x^2+b\right)y''+axy'+yc=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=\frac{ax}{a\,x^2+b}$$
$$q(x)=\frac{c}{a\,x^2+b}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\
&= \int e^{-\left(\int \frac{ax}{a\,x^2+b}dx\right)}dx \\
&= \int e^{-\frac{\ln\left(a\,x^2+b\right)}{2}}dx \\
&= \int \frac{1}{\sqrt{a\,x^2+b}}dx \\
&= \frac{\ln\left(\sqrt{a}\,x + \sqrt{a\,x^2+b}\right)}{\sqrt{a}}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{c}{a\,x^2+b}}{\frac{1}{a\,x^2+b}} \\
&= c
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + cy(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = c$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + c\, e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + c = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = c$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(c)}\\ &= \pm\sqrt{-c}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-c}\\ \lambda_2 &= -\sqrt{-c}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-c}\\ \lambda_2 &= -\sqrt{-c}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1\tau} + c_2 e^{\lambda_2\tau}\\ y(\tau) &= c_1 e^{(\sqrt{-c})\tau} + c_2 e^{(-\sqrt{-c})\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{\sqrt{-c}\,\tau} + c_2 e^{-\sqrt{-c}\,\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(\sqrt{a}\, x + \sqrt{a\, x^2 + b}\right)^{\frac{\sqrt{-c}}{\sqrt{a}}} + c_2 \left(\sqrt{a}\, x + \sqrt{a\, x^2 + b}\right)^{-\frac{\sqrt{-c}}{\sqrt{a}}}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(\sqrt{a}\, x + \sqrt{a\, x^2 + b}\right)^{\frac{\sqrt{-c}}{\sqrt{a}}} + c_2 \left(\sqrt{a}\, x + \sqrt{a\, x^2 + b}\right)^{-\frac{\sqrt{-c}}{\sqrt{a}}} \tag{1}$$

Verification of solutions

$$y = c_1 \left(\sqrt{a}\, x + \sqrt{a\, x^2 + b}\right)^{\frac{\sqrt{-c}}{\sqrt{a}}} + c_2 \left(\sqrt{a}\, x + \sqrt{a\, x^2 + b}\right)^{-\frac{\sqrt{-c}}{\sqrt{a}}}$$

Verified OK.

30.14.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$\left(a\, x^2 + b\right) y'' + axy' + yc = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{ax}{a\, x^2 + b} \\ q(x) &= \frac{c}{a\, x^2 + b} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{c}{a\,x^2+b}}}{c} \\ \tau'' &= -\frac{cax}{c\sqrt{\frac{c}{a\,x^2+b}}\,(a\,x^2+b)^2} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{cax}{c\sqrt{\frac{c}{a\,x^2+b}}\,(a\,x^2+b)^2} + \frac{ax}{a\,x^2+b}\frac{\sqrt{\frac{c}{a\,x^2+b}}}{c}}{\left(\frac{\sqrt{\frac{c}{a\,x^2+b}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \sqrt{\frac{c}{a\,x^2+b}}dx}{c} \\ &= \frac{\sqrt{\frac{c}{a\,x^2+b}}\,\sqrt{a\,x^2+b}\,\ln\left(\sqrt{a}\,x + \sqrt{a\,x^2+b}\right)}{c\sqrt{a}} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\frac{\sqrt{c}\,\ln\left(\sqrt{a}\,x + \sqrt{a\,x^2+b}\right)}{\sqrt{a}}\right) + c_2 \sin\left(\frac{\sqrt{c}\,\ln\left(\sqrt{a}\,x + \sqrt{a\,x^2+b}\right)}{\sqrt{a}}\right)$$

Summary
The solution(s) found are the following

$$y = c_1 \cos\left(\frac{\sqrt{c}\, \ln\left(\sqrt{a}\, x+\sqrt{a\, x^2+b}\right)}{\sqrt{a}}\right)+c_2 \sin\left(\frac{\sqrt{c}\, \ln\left(\sqrt{a}\, x+\sqrt{a\, x^2+b}\right)}{\sqrt{a}}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(\frac{\sqrt{c}\, \ln\left(\sqrt{a}\, x+\sqrt{a\, x^2+b}\right)}{\sqrt{a}}\right)+c_2 \sin\left(\frac{\sqrt{c}\, \ln\left(\sqrt{a}\, x+\sqrt{a\, x^2+b}\right)}{\sqrt{a}}\right)$$

Verified OK.

30.14.3 Solving using Kovacic algorithm

Writing the ode as

$$\left(a\, x^2+b\right) y''+a x y'+y c = 0 \tag{1}$$
$$A y''+B y'+C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= a\, x^2+b \\ B &= a x \\ C &= c \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2 x^2-4 a c\, x^2+2 a b-4 b c}{4\left(a\, x^2+b\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -a^2x^2 - 4ac\,x^2 + 2ab - 4bc \\
t &= 4\left(a\,x^2 + b\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-a^2x^2 - 4ac\,x^2 + 2ab - 4bc}{4\left(a\,x^2 + b\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 158: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4\left(a\,x^2 + b\right)^2$. There is a pole at $x = \frac{\sqrt{-ab}}{a}$ of order 2. There is a pole at

$x = -\frac{\sqrt{-ab}}{a}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Unable to find solution using case two.

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\left(x-\sqrt{-\frac{b}{a}}\right)^2} - \frac{3}{16\left(x+\sqrt{-\frac{b}{a}}\right)^2} + \frac{-ab+8bc}{16\left(-\frac{b}{a}\right)^{\frac{3}{2}} a^2 \left(x-\sqrt{-\frac{b}{a}}\right)} - \frac{-ab+8bc}{16\left(-\frac{b}{a}\right)^{\frac{3}{2}} a^2 \left(x+\sqrt{-\frac{b}{a}}\right)}$$

For the pole at $x = \frac{\sqrt{-ab}}{a}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{-ab}}{a}\right)^2}$ in the partial fractions decomposition of r given above. This shows that $b = 0$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{0, 3, 6, 9, 12\}$$

For the pole at $x = -\frac{\sqrt{-ab}}{a}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{-ab}}{a}\right)^2}$ in the partial fractions decomposition of r given above. This shows that $b = 0$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{0, 3, 6, 9, 12\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx \frac{-a^2 - 4ac}{4a^2 x^2} + \frac{-\frac{(-a^2-4ac)b}{2a^3} + \frac{2ab-4bc}{4a^2}}{x^4} + \frac{\frac{3(-a^2-4ac)b^2}{4a^4} - \frac{(2ab-4bc)b}{2a^3}}{x^6} + \frac{-\frac{(-a^2-4ac)b^3}{a^5} + \frac{3(2ab-4bc)b^2}{4a^4}}{x^8} + \frac{\frac{5(-a^2-4ac)b^4}{4a^6}}{x}$$

The above shows that

$$b = -\frac{1}{4}$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{6\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
$\frac{\sqrt{-ab}}{a}$	2	$\{0, 3, 6, 9, 12\}$
$-\frac{\sqrt{-ab}}{a}$	2	$\{0, 3, 6, 9, 12\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{6\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \dots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 3, \ e_2 = 3, \ e_\infty = 6$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\
&= \frac{4}{12}(6-(3+(3)))\\
&= 0
\end{aligned}
$$

The following rational function is

$$
\begin{aligned}
\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\
&= \frac{4}{12}\left(\frac{3}{\left(x-\left(\frac{\sqrt{-ab}}{a}\right)\right)}+\frac{3}{\left(x-\left(-\frac{\sqrt{-ab}}{a}\right)\right)}\right)\\
&= \frac{2ax}{a\,x^2+b}
\end{aligned}
$$

And

$$
\begin{aligned}
S &= \prod_{c\in\Gamma}(x-c)\\
&= \left(x-\frac{\sqrt{-ab}}{a}\right)\left(x+\frac{\sqrt{-ab}}{a}\right)
\end{aligned}
$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d=0$, then let

$$
p(x) = 1
$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$
\begin{aligned}
P_n &= -p(x)\\
&= -1\\
P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2rP_{i+1} \qquad i = n, n-1, \ldots, 0
\end{aligned} \tag{1A}
$$

The coefficients a_i are solved for from

$$
P_{-1} = 0 \tag{2A}
$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 4$ and $r = \frac{-a^2x^2-4ac\,x^2+2ab-4bc}{4(a\,x^2+b)^2}$).

$$
\begin{aligned}
P_4 &= -p\\
&= -1\\
P_3 &= 2x\\
P_2 &= \frac{-3a^2x^2-4ac\,x^2-4bc}{a^2}\\
P_1 &= \frac{3x(a\,x^2(a+4c)+4bc)}{a^2}\\
P_0 &= -\frac{3(a^2x^2+4ac\,x^2+4bc)^2}{2a^4}\\
P_{-1} &= 0
\end{aligned}
$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$
\begin{aligned}
\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!}\omega^i &= 0\\
\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!}\omega^i &= 0
\end{aligned}
$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$
\frac{1}{16a^4}\left(-\left(4a^2x^4\omega^2-4a^2x^3\omega+8ab\,\omega^2x^2+a^2x^2-4ab\omega x+4ac\,x^2+4b^2\omega^2+4bc\right)^2\right) = 0 \tag{3A}
$$

The solution ω of eq. 3A is found as

$$
\omega = \frac{1}{2a\,x^2+2b}\left(ax-2\sqrt{-\left(a\,x^2+b\right)c}\right)
$$

This ω is used to find a solution to $z'' = rz$.

$$
z_1(x) = e^{\int \omega\,dx} \tag{5A}
$$

Doing the integration gives in eq. (4A) gives

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{ax-2\sqrt{-\left(a\,x^2+b\right)c}}{2a\,x^2+2b}dx} \\
&= \left(a\,x^2+b\right)^{\frac{1}{4}} \mathrm{e}^{\frac{c \arctan\left(\frac{\sqrt{ac}\,x}{\sqrt{-\left(a\,x^2+b\right)c}}\right)}{\sqrt{ac}}}
\end{aligned}
$$

Which simplifies to

$$
z_1(x) = \left(a\,x^2+b\right)^{\frac{1}{4}} \mathrm{e}^{\frac{c \arctan\left(x\sqrt{-\frac{a}{a\,x^2+b}}\right)}{\sqrt{ac}}}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{ax}{a\,x^2+b}\, dx} \\
&= z_1 e^{-\frac{\ln\left(a\,x^2+b\right)}{4}} \\
&= z_1 \left(\frac{1}{\left(a\,x^2+b\right)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{c \arctan\left(x\sqrt{-\frac{a}{a\,x^2+b}}\right)}{\sqrt{ac}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{ax}{a\,x^2+b}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{-\frac{\ln\left(a\,x^2+b\right)}{2}}}{(y_1)^2}\,dx$$

$$= y_1 \left(\int \frac{\mathrm{e}^{-\frac{2c\arctan\left(x\sqrt{-\frac{a}{a\,x^2+b}}\right)}{\sqrt{ac}}}}{\sqrt{a\,x^2+b}}dx\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\mathrm{e}^{\frac{c\arctan\left(x\sqrt{-\frac{a}{a\,x^2+b}}\right)}{\sqrt{ac}}}\right) + c_2 \left(\mathrm{e}^{\frac{c\arctan\left(x\sqrt{-\frac{a}{a\,x^2+b}}\right)}{\sqrt{ac}}}\left(\int \frac{\mathrm{e}^{-\frac{2c\arctan\left(x\sqrt{-\frac{a}{a\,x^2+b}}\right)}{\sqrt{ac}}}}{\sqrt{a\,x^2+b}}dx\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{\frac{c\arctan\left(x\sqrt{-\frac{a}{a\,x^2+b}}\right)}{\sqrt{ac}}} + c_2\mathrm{e}^{\frac{c\arctan\left(x\sqrt{-\frac{a}{a\,x^2+b}}\right)}{\sqrt{ac}}}\left(\int \frac{\mathrm{e}^{-\frac{2c\arctan\left(x\sqrt{-\frac{a}{a\,x^2+b}}\right)}{\sqrt{ac}}}}{\sqrt{a\,x^2+b}}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{\frac{c\arctan\left(x\sqrt{-\frac{a}{a\,x^2+b}}\right)}{\sqrt{ac}}} + c_2\mathrm{e}^{\frac{c\arctan\left(x\sqrt{-\frac{a}{a\,x^2+b}}\right)}{\sqrt{ac}}}\left(\int \frac{\mathrm{e}^{-\frac{2c\arctan\left(x\sqrt{-\frac{a}{a\,x^2+b}}\right)}{\sqrt{ac}}}}{\sqrt{a\,x^2+b}}dx\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve((a*x^2+b)*diff(y(x),x$2)+a*x*diff(y(x),x)+c*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(\sqrt{a}\,x + \sqrt{a\,x^2 + b}\right)^{\frac{i\sqrt{c}}{\sqrt{a}}} + c_2\left(\sqrt{a}\,x + \sqrt{a\,x^2 + b}\right)^{-\frac{i\sqrt{c}}{\sqrt{a}}}$$

✓ Solution by Mathematica

Time used: 0.196 (sec). Leaf size: 74

```
DSolve[(a*x^2+b)*y''[x]+a*x*y'[x]+c*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(\frac{\sqrt{c}\operatorname{arctanh}\left(\frac{\sqrt{a}x}{\sqrt{ax^2+b}}\right)}{\sqrt{a}}\right) + c_2 \sin\left(\frac{\sqrt{c}\operatorname{arctanh}\left(\frac{\sqrt{a}x}{\sqrt{ax^2+b}}\right)}{\sqrt{a}}\right)$$

30.15 problem 163

Internal problem ID [10987]
Internal file name [`OUTPUT/10243_Sunday_December_31_2023_11_13_29_AM_55836174/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 163.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2 + a)\, y'' + 2bxy' + 2(b-1)\, y = 0}$$

30.15.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left((x^2 + a)\, y'' + 2bxy' + (2b-2)\, y\right) dx = 0$$
$$(2bx - 2x)\, y + (x^2 + a)\, y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2x(b-1)}{x^2+a}$$
$$q(x) = \frac{c_1}{x^2+a}$$

Hence the ode is

$$y' + \frac{2x(b-1)\,y}{x^2+a} = \frac{c_1}{x^2+a}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int \frac{2x(b-1)}{x^2+a}dx}$$
$$= \mathrm{e}^{\frac{(2b-2)\ln\left(x^2+a\right)}{2}}$$

Which simplifies to

$$\mu = \left(x^2+a\right)^{b-1}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{c_1}{x^2+a}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2+a\right)^{b-1} y\right) = \left(\left(x^2+a\right)^{b-1}\right)\left(\frac{c_1}{x^2+a}\right)$$
$$\mathrm{d}\left(\left(x^2+a\right)^{b-1} y\right) = \left(c_1\left(x^2+a\right)^{-2+b}\right)\,\mathrm{d}x$$

Integrating gives

$$\left(x^2+a\right)^{b-1} y = \int c_1\left(x^2+a\right)^{-2+b}\,\mathrm{d}x$$
$$\left(x^2+a\right)^{b-1} y = \int c_1\left(x^2+a\right)^{-2+b} dx + c_2$$

Dividing both sides by the integrating factor $\mu = \left(x^2+a\right)^{b-1}$ results in

$$y = \left(x^2+a\right)^{-b+1}\left(\int c_1\left(x^2+a\right)^{-2+b} dx\right) + c_2\left(x^2+a\right)^{-b+1}$$

which simplifies to

$$y = \left(c_1\left(\int \left(x^2+a\right)^{-2+b} dx\right) + c_2\right)\left(x^2+a\right)^{-b+1}$$

Summary
The solution(s) found are the following

$$y = \left(c_1\left(\int \left(x^2+a\right)^{-2+b} dx\right)+c_2\right)\left(x^2+a\right)^{-b+1} \tag{1}$$

Verification of solutions

$$y = \left(c_1\left(\int \left(x^2+a\right)^{-2+b} dx\right)+c_2\right)\left(x^2+a\right)^{-b+1}$$

Verified OK.

30.15.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$\left(x^2+a\right)y''+2bxy'+(2b-2)\,y=0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(x^2+a\right)y''+2bxy'+(2b-2)\,y\right)dx = 0$$
$$(2bx-2x)\,y+\left(x^2+a\right)y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$p(x) = \frac{2x(b-1)}{x^2+a}$$
$$q(x) = \frac{c_1}{x^2+a}$$

Hence the ode is

$$y'+\frac{2x(b-1)\,y}{x^2+a} = \frac{c_1}{x^2+a}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int \frac{2x(b-1)}{x^2+a}dx}$$
$$= \mathrm{e}^{\frac{(2b-2)\ln\left(x^2+a\right)}{2}}$$

Which simplifies to

$$\mu = \left(x^2+a\right)^{b-1}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x^2+a}\right) \\
\frac{d}{dx}\left(\left(x^2+a\right)^{b-1} y\right) &= \left(\left(x^2+a\right)^{b-1}\right)\left(\frac{c_1}{x^2+a}\right) \\
d\left(\left(x^2+a\right)^{b-1} y\right) &= \left(c_1\left(x^2+a\right)^{-2+b}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\left(x^2+a\right)^{b-1} y &= \int c_1\left(x^2+a\right)^{-2+b} dx \\
\left(x^2+a\right)^{b-1} y &= \int c_1\left(x^2+a\right)^{-2+b} dx + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \left(x^2+a\right)^{b-1}$ results in

$$y = \left(x^2+a\right)^{-b+1}\left(\int c_1\left(x^2+a\right)^{-2+b} dx\right) + c_2\left(x^2+a\right)^{-b+1}$$

which simplifies to

$$y = \left(c_1\left(\int \left(x^2+a\right)^{-2+b} dx\right) + c_2\right)\left(x^2+a\right)^{-b+1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(c_1\left(\int \left(x^2+a\right)^{-2+b} dx\right) + c_2\right)\left(x^2+a\right)^{-b+1} \tag{1}$$

<u>Verification of solutions</u>

$$y = \left(c_1\left(\int \left(x^2+a\right)^{-2+b} dx\right) + c_2\right)\left(x^2+a\right)^{-b+1}$$

Verified OK.

30.15.3 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 + a \\ q(x) &= 2bx \\ r(x) &= 2b - 2 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 2b \end{aligned}$$

Therefore (1) becomes

$$2 - (2b) + (2b - 2) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$(2bx - 2x)\,y + \left(x^2 + a\right)y' = c_1$$

We now have a first order ode to solve which is

$$(2bx - 2x)\,y + \left(x^2 + a\right)y' = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2x(b-1)}{x^2+a}$$
$$q(x) = \frac{c_1}{x^2+a}$$

Hence the ode is

$$y' + \frac{2x(b-1)\,y}{x^2+a} = \frac{c_1}{x^2+a}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2x(b-1)}{x^2+a}dx}$$
$$= e^{\frac{(2b-2)\ln\left(x^2+a\right)}{2}}$$

Which simplifies to

$$\mu = \left(x^2+a\right)^{b-1}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{c_1}{x^2+a}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2+a\right)^{b-1} y\right) = \left(\left(x^2+a\right)^{b-1}\right)\left(\frac{c_1}{x^2+a}\right)$$
$$\mathrm{d}\left(\left(x^2+a\right)^{b-1} y\right) = \left(c_1\left(x^2+a\right)^{-2+b}\right)\,\mathrm{d}x$$

Integrating gives

$$\left(x^2+a\right)^{b-1} y = \int c_1\left(x^2+a\right)^{-2+b}\,\mathrm{d}x$$
$$\left(x^2+a\right)^{b-1} y = \int c_1\left(x^2+a\right)^{-2+b} dx + c_2$$

Dividing both sides by the integrating factor $\mu = \left(x^2+a\right)^{b-1}$ results in

$$y = \left(x^2+a\right)^{-b+1}\left(\int c_1\left(x^2+a\right)^{-2+b} dx\right) + c_2\left(x^2+a\right)^{-b+1}$$

which simplifies to

$$y = \left(c_1\left(\int \left(x^2+a\right)^{-2+b} dx\right) + c_2\right)\left(x^2+a\right)^{-b+1}$$

Summary
The solution(s) found are the following

$$y = \left(c_1 \left(\int \left(x^2+a\right)^{-2+b} dx\right) + c_2\right) \left(x^2+a\right)^{-b+1} \tag{1}$$

Verification of solutions

$$y = \left(c_1 \left(\int \left(x^2+a\right)^{-2+b} dx\right) + c_2\right) \left(x^2+a\right)^{-b+1}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form for at least one hypergeometric solution is achieved - returning with
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.047 (sec). Leaf size: 41

```
dsolve((x^2+a)*diff(y(x),x$2)+2*b*x*diff(y(x),x)+2*(b-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \left(\frac{x^2+a}{a}\right)^{-b+1} + c_2 x \,\text{hypergeom}\left(\left[1, b-\frac{1}{2}\right], \left[\frac{3}{2}\right], -\frac{x^2}{a}\right)$$

✓ Solution by Mathematica

Time used: 0.426 (sec). Leaf size: 64

```
DSolve[(x^2+a)*y''[x]+2*b*x*y'[x]+2*(b-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (a+x^2) \left(\frac{c_2 x \left(\frac{a+x^2}{a}\right)^{-b} \text{Hypergeometric2F1}\left(\frac{1}{2}, 2-b, \frac{3}{2}, -\frac{x^2}{a}\right)}{a^2} + c_1 (a+x^2)^{-b} \right)$$

30.16 problem 164

Internal problem ID [10988]
Internal file name [`OUTPUT/10244_Sunday_December_31_2023_11_13_32_AM_79566971/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2+bx+c)y''+f(x)y'+g(x)y=0$
Problem number: 164.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{(-a^2+x^2)\,y''+2bxy'+b(b-1)\,y=0}$$

30.16.1 Maple step by step solution

Let's solve

$(-a^2+x^2)\,y''+2bxy'+(b^2-b)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{b(b-1)y}{a^2-x^2}+\frac{2bxy'}{a^2-x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2bxy'}{a^2-x^2}-\frac{b(b-1)y}{a^2-x^2}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{2bx}{a^2-x^2}, P_3(x) = -\frac{b(b-1)}{a^2-x^2}\right]$

- $(x-a)\cdot P_2(x)$ is analytic at $x=a$

  $$\left((x-a)\cdot P_2(x)\right)\Big|_{x=a} = b$$

- $(x-a)^2\cdot P_3(x)$ is analytic at $x=a$

  $$\left((x-a)^2\cdot P_3(x)\right)\Big|_{x=a} = 0$$

- $x=a$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = a$

- Multiply by denominators

  $$y''(a^2-x^2) - 2bxy' - b(b-1)\,y = 0$$

- Change variables using $x = u+a$ so that the regular singular point is at $u=0$

  $$(-2ua-u^2)\left(\frac{d^2}{du^2}y(u)\right) + (-2ab-2bu)\left(\frac{d}{du}y(u)\right) + (-b^2+b)\,y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

$$-2aa_0r(r-1+b)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2aa_{k+1}(k+1+r)\,(r+k+b)-a_k(r+k+b)\,(r-1+k+b\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2ar(r-1+b)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{0,-b+1\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2(r+k+b)\left(\frac{a_k(r-1+k+b)}{2}+aa_{k+1}(k+1+r)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{a_k(r-1+k+b)}{2a(k+1+r)}$$

- Recursion relation for $r=0$

$$a_{k+1}=-\frac{a_k(-1+k+b)}{2a(k+1)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=-\frac{a_k(-1+k+b)}{2a(k+1)}\right]$$

- Revert the change of variables $u=x-a$

$$\left[y=\sum_{k=0}^{\infty}a_k(x-a)^k, a_{k+1}=-\frac{a_k(-1+k+b)}{2a(k+1)}\right]$$

- Recursion relation for $r=-b+1$

$$a_{k+1}=-\frac{a_k k}{2a(k+2-b)}$$

- Solution for $r=-b+1$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k-b+1}, a_{k+1}=-\frac{a_k k}{2a(k+2-b)}\right]$$

- Revert the change of variables $u=x-a$

$$\left[y=\sum_{k=0}^{\infty}a_k(x-a)^{k-b+1}, a_{k+1}=-\frac{a_k k}{2a(k+2-b)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}c_k(x-a)^k\right)+\left(\sum_{k=0}^{\infty}d_k(x-a)^{k-b+1}\right), c_{1+k}=-\frac{c_k(-1+k+b)}{2a(1+k)}, d_{1+k}=-\frac{d_k k}{2a(k+2-b)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 29

```
dsolve((x^2-a^2)*diff(y(x),x$2)+2*b*x*diff(y(x),x)+b*(b-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(a+x)^{-b+1} + c_2(a-x)^{-b+1}$$

✓ Solution by Mathematica
Time used: 0.727 (sec). Leaf size: 127

```
DSolve[(x^2-a^2)*y''[x]+2*b*x*y'[x]+b*(b-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(x-a)^{\frac{1}{2}-\frac{1}{2}\sqrt{(b-1)^2}}(a+x)^{\frac{1}{2}-\frac{1}{2}\sqrt{(b-1)^2}}(x^2-a^2)^{-b/2}\left(2a\sqrt{(b-1)^2}c_1(x-a)^{\sqrt{(b-1)^2}} - c_2(a+x)^{\sqrt{(b-1)^2}}\right)}{2a\sqrt{(b-1)^2}}$$

30.17 problem 165

Internal problem ID [10989]
Internal file name [`OUTPUT/10245_Sunday_December_31_2023_11_14_36_AM_89762430/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 165.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(a^2 + x^2)\, y'' + 2bxy' + b(b-1)\, y = 0}$$

30.17.1 Solving using Kovacic algorithm

Writing the ode as

$$(a^2 + x^2)\, y'' + 2bxy' + (b^2 - b)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= a^2 + x^2 \\ B &= 2bx \\ C &= b^2 - b \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-b\,a^2(-2+b)}{(a^2+x^2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -b\,a^2(-2+b)$$
$$t = (a^2+x^2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{b\,a^2(-2+b)}{(a^2+x^2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 160: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (a^2 + x^2)^2$. There is a pole at $x = ia$ of order 2. There is a pole at $x = -ia$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{b(-2+b)}{4\left(x - \sqrt{-a^2}\right)^2} + \frac{b(-2+b)}{4\left(x + \sqrt{-a^2}\right)^2} + \frac{b\,a^2(-2+b)}{4\left(-a^2\right)^{\frac{3}{2}}\left(x - \sqrt{-a^2}\right)} - \frac{b\,a^2(-2+b)}{4\left(-a^2\right)^{\frac{3}{2}}\left(x + \sqrt{-a^2}\right)}$$

For the pole at $x = ia$ let b be the coefficient of $\frac{1}{(-ia+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 0$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{0, 2, 4\} \end{aligned}$$

For the pole at $x = -ia$ let b be the coefficient of $\frac{1}{(ia+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 0$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{0, 2, 4\} \end{aligned}$$

Now since the order of r at ∞ is $4 > 2$ then

$$E_\infty = \{0, 2, 4\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
ia	2	$\{0,2,4\}$
$-ia$	2	$\{0,2,4\}$

Order of r at ∞	E_∞
4	$\{0,2,4\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 2,\ e_2 = 2,\ e_\infty = 4$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(4 - (2 + (2))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{2}{(x-(ia))} + \frac{2}{(x-(-ia))}\right) \\ &= \frac{1}{-ia+x} + \frac{1}{ia+x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{-ia+x} + \frac{1}{ia+x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{-ia+x} + \frac{1}{ia+x}\right) w + \frac{(b-1)^2 a^2 + x^2}{(a^2+x^2)^2} = 0$$

Solving for ω gives

$$\omega = \frac{iab - ia + x}{a^2 + x^2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx}\\ &= \mathrm{e}^{\int \frac{iab-ia+x}{a^2+x^2} dx}\\ &= \sqrt{a^2+x^2}\, \mathrm{e}^{i(b-1)\arctan\left(\frac{x}{a}\right)}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2bx}{a^2+x^2}\, dx}\\ &= z_1 e^{-\frac{b\ln\left(a^2+x^2\right)}{2}}\\ &= z_1\left(\left(a^2+x^2\right)^{-\frac{b}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \left(a^2+x^2\right)^{-\frac{b}{2}+\frac{1}{2}} \mathrm{e}^{i(b-1)\arctan\left(\frac{x}{a}\right)}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2bx}{a^2+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-b\ln(a^2+x^2)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{-2i(b-1)\arctan\left(\frac{x}{a}\right)}(ia+x)(ix+a)}{2a(b-1)(a^2+x^2)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(a^2+x^2\right)^{-\frac{b}{2}+\frac{1}{2}} \mathrm{e}^{i(b-1)\arctan\left(\frac{x}{a}\right)}\right) \\
&\quad + c_2\left(\left(a^2+x^2\right)^{-\frac{b}{2}+\frac{1}{2}} \mathrm{e}^{i(b-1)\arctan\left(\frac{x}{a}\right)}\left(\frac{\mathrm{e}^{-2i(b-1)\arctan\left(\frac{x}{a}\right)}(ia+x)(ix+a)}{2a(b-1)(a^2+x^2)}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\left(a^2+x^2\right)^{-\frac{b}{2}+\frac{1}{2}} \mathrm{e}^{i(b-1)\arctan\left(\frac{x}{a}\right)} + \frac{c_2(ia+x)(ix+a)\left(a^2+x^2\right)^{-\frac{1}{2}-\frac{b}{2}} \mathrm{e}^{-i(b-1)\arctan\left(\frac{x}{a}\right)}}{2a(b-1)} \tag{1}$$

Verification of solutions

$$y = c_1\left(a^2+x^2\right)^{-\frac{b}{2}+\frac{1}{2}} \mathrm{e}^{i(b-1)\arctan\left(\frac{x}{a}\right)} + \frac{c_2(ia+x)(ix+a)\left(a^2+x^2\right)^{-\frac{1}{2}-\frac{b}{2}} \mathrm{e}^{-i(b-1)\arctan\left(\frac{x}{a}\right)}}{2a(b-1)}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.14 (sec). Leaf size: 33

```
dsolve((x^2+a^2)*diff(y(x),x$2)+2*b*x*diff(y(x),x)+b*(b-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(-ix+a)^{-b+1} + c_2(ix+a)^{-b+1}$$

✓ Solution by Mathematica
Time used: 0.813 (sec). Leaf size: 101

```
DSolve[(x^2+a^2)*y''[x]+2*b*x*y'[x]+b*(b-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(a^2+x^2)^{\frac{1}{2}-\frac{b}{2}} e^{-i\sqrt{(b-1)^2}\arctan\left(\frac{a}{x}\right)} \left(ic_2 e^{2i\sqrt{(b-1)^2}\arctan\left(\frac{a}{x}\right)} + 2a\sqrt{(b-1)^2}c_1\right)}{2a\sqrt{(b-1)^2}}$$

30.18 problem 166

Internal problem ID [10990]
Internal file name [`OUTPUT/10246_Sunday_December_31_2023_11_14_37_AM_52742929/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2+bx+c)y''+f(x)y'+g(x)y=0$
Problem number: 166.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{\left(a\,x^{2}+b\right)y''+\left(2n+1\right)axy'+yc=0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 93

```
dsolve((a*x^2+b)*diff(y(x),x$2)+(2*n+1)*a*x*diff(y(x),x)+c*y(x)=0,y(x), singsol=all)
```

$$y(x) = (a\,x^2+b)^{-\frac{n}{2}+\frac{1}{4}}\left(c_1\,\mathrm{LegendreP}\left(-\frac{-2\sqrt{a\,n^2-c}+\sqrt{a}}{2\sqrt{a}}, n-\frac{1}{2}, \frac{ax}{\sqrt{-ab}}\right)\right.$$
$$\left.+\,c_2\,\mathrm{LegendreQ}\left(-\frac{-2\sqrt{a\,n^2-c}+\sqrt{a}}{2\sqrt{a}}, n-\frac{1}{2}, \frac{ax}{\sqrt{-ab}}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.149 (sec). Leaf size: 118

```
DSolve[(a*x^2+b)*y''[x]+(2*n+1)*a*x*y'[x]+c*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (ax^2+b)^{\frac{1}{4}-\frac{n}{2}}\left(c_1 P^{n-\frac{1}{2}}_{\frac{\sqrt{an^2-c}}{\sqrt{a}}-\frac{1}{2}}\left(\frac{i\sqrt{a}x}{\sqrt{b}}\right)+c_2 Q^{n-\frac{1}{2}}_{\frac{\sqrt{an^2-c}}{\sqrt{a}}-\frac{1}{2}}\left(\frac{i\sqrt{a}x}{\sqrt{b}}\right)\right)$$

30.19 problem 167

Internal problem ID [10991]
Internal file name [OUTPUT/10247_Sunday_December_31_2023_11_24_06_AM_26071113/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 167.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{\left(-x^{2}+1\right) y''-y' x+\left(2 a \,x^{2}+b \right) y=0}$$

30.19.1 Maple step by step solution

Let's solve

$\left(-x^{2}+1\right) y''-y' x+\left(2 a \,x^{2}+b \right) y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{\left(2 a \,x^{2}+b \right) y}{x^{2}-1}-\frac{x y'}{x^{2}-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{x y'}{x^{2}-1}-\frac{\left(2 a \,x^{2}+b \right) y}{x^{2}-1}=0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

$\left[P_2(x) = \frac{x}{x^2-1}, P_3(x) = -\frac{2a\,x^2+b}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = \frac{1}{2}$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $y''(x^2-1) + y'x + (-2a\,x^2 - b)\,y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (u-1)\left(\frac{d}{du}y(u)\right) + (-2a\,u^2 + 4au - 2a - b)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- >k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,u^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(-1+2r)\,u^{-1+r} + (-a_1(1+r)\,(1+2r) - a_0(-r^2+2a+b))\,u^r + (-a_2(2+r)\,(3+2r) - a$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- The coefficients of each power of u must be 0

$$[-a_1(1+r)\,(1+2r) - a_0(-r^2+2a+b) = 0, -a_2(2+r)\,(3+2r) - a_1(-r^2+2a+b-2r-1) +$$

- Solve for the dependent coefficient(s)

$$\left\{a_1 = -\frac{a_0(-r^2+2a+b)}{2r^2+3r+1}, a_2 = \frac{a_0(r^4+4r^2a-2r^2b+2r^3+4a^2+4ab+8ar+b^2-2br+r^2+2a-b)}{4r^4+20r^3+35r^2+25r+6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(k+\frac{1}{2}+r\right)(k+1+r)\,a_{k+1} + (k^2+2kr+r^2-2a-b)\,a_k - 2a(a_{k-2}-2a_{k-1}) = 0$$

- Shift index using $k->k+2$

$$-2\left(k+\frac{5}{2}+r\right)(k+3+r)\,a_{k+3} + \left((k+2)^2+2(k+2)\,r+r^2-2a-b\right)a_{k+2} - 2a(a_k-2a_{k+1}) =$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = -\frac{-k^2a_{k+2}-2kra_{k+2}-r^2a_{k+2}+2a_ka-4aa_{k+1}+2aa_{k+2}+ba_{k+2}-4ka_{k+2}-4ra_{k+2}-4a_{k+2}}{(2k+5+2r)(k+3+r)}$$

- Recursion relation for $r=0$

$$a_{k+3} = -\frac{-k^2a_{k+2}+2a_ka-4aa_{k+1}+2aa_{k+2}+ba_{k+2}-4ka_{k+2}-4a_{k+2}}{(2k+5)(k+3)}$$

- Solution for $r=0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_ku^k, a_{k+3} = -\frac{-k^2a_{k+2}+2a_ka-4aa_{k+1}+2aa_{k+2}+ba_{k+2}-4ka_{k+2}-4a_{k+2}}{(2k+5)(k+3)}, a_1 = -a_0(2a+b)\,, a_2 =\right.$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k\,, a_{k+3} = -\frac{-k^2a_{k+2}+2a_ka-4aa_{k+1}+2aa_{k+2}+ba_{k+2}-4ka_{k+2}-4a_{k+2}}{(2k+5)(k+3)}, a_1 = -a_0(2a+b)\,, a_2\right.$$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+3} = -\frac{-k^2 a_{k+2}+2a_k a-4a a_{k+1}+2a a_{k+2}+b a_{k+2}-5k a_{k+2}-\frac{25}{4}a_{k+2}}{(2k+6)\left(k+\frac{7}{2}\right)}$

- Solution for $r = \frac{1}{2}$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+3} = -\frac{-k^2 a_{k+2}+2a_k a-4a a_{k+1}+2a a_{k+2}+b a_{k+2}-5k a_{k+2}-\frac{25}{4}a_{k+2}}{(2k+6)\left(k+\frac{7}{2}\right)}, a_1 = -\frac{a_0\left(-\frac{1}{4}+2a+b\right)}{3}, a\right.$

- Revert the change of variables $u = 1 + x$

$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{1}{2}}, a_{k+3} = -\frac{-k^2 a_{k+2}+2a_k a-4a a_{k+1}+2a a_{k+2}+b a_{k+2}-5k a_{k+2}-\frac{25}{4}a_{k+2}}{(2k+6)\left(k+\frac{7}{2}\right)}, a_1 = -\frac{a_0\left(-\frac{1}{4}+2a+b\right.}{3}\right.$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} c_k (1+x)^k\right) + \left(\sum_{k=0}^{\infty} d_k (1+x)^{k+\frac{1}{2}}\right), c_{k+3} = -\frac{-k^2 c_{k+2}+2a c_k-4a c_{1+k}+2a c_{k+2}+b c_{k+2}-4k c_{k+2}-4c}{(2k+5)(k+3)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      Equivalence transformation and function parameters: {x = t}, {kappa = 4*b-4, mu = -8*a
      <- Equivalence to the rational form of Mathieu ODE successful
   <- Mathieu successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.359 (sec). Leaf size: 27

```
dsolve((1-x^2)*diff(y(x),x$2)-x*diff(y(x),x)+(2*a*x^2+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{MathieuC}\left(a+b, -\frac{a}{2}, \arccos(x)\right) + c_2 \operatorname{MathieuS}\left(a+b, -\frac{a}{2}, \arccos(x)\right)$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 34

```
DSolve[(1-x^2)*y''[x]-x*y'[x]+(2*a*x^2+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \text{MathieuC}\left[a+b, -\frac{a}{2}, \arccos(x)\right] + c_2 \text{MathieuS}\left[a+b, -\frac{a}{2}, \arccos(x)\right]$$

30.20 problem 168

Internal problem ID [10992]
Internal file name [`OUTPUT/10248_Sunday_December_31_2023_11_24_07_AM_58330519/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2+bx+c)y''+f(x)y'+g(x)y=0$
Problem number: 168.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$(-x^2+1)\,y''+(ax+b)\,y'+yc=0$$

30.20.1 Maple step by step solution

Let's solve

$(-x^2+1)\,y''+(ax+b)\,y'+yc=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{cy}{x^2-1}+\frac{(ax+b)y'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(ax+b)y'}{x^2-1}-\frac{cy}{x^2-1}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

$\left[P_2(x) = -\frac{ax+b}{x^2-1}, P_3(x) = -\frac{c}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = -\frac{a}{2} + \frac{b}{2}$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $y''(x^2-1) + (-ax-b)\, y' - yc = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (-au+a-b)\left(\frac{d}{du}y(u)\right) - cy(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r(2-2r+a-b)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(-2k-2r+a-b)-a_k(ak+ar-k^2-2kr\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(2-2r+a-b)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{a}{2}-\frac{b}{2}+1\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(-2k-2r+a-b)-a_k(-k^2+(a-2r+1)\,k-r^2+(a+1)\,r+c)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(ak+ar-k^2-2kr-r^2+c+k+r)}{(k+1+r)(-2k-2r+a-b)}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{a_k(ak-k^2+c+k)}{(k+1)(-2k+a-b)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{a_k(ak-k^2+c+k)}{(k+1)(-2k+a-b)}\right]$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k, a_{k+1}=\frac{a_k(ak-k^2+c+k)}{(k+1)(-2k+a-b)}\right]$$

- Recursion relation for $r=\frac{a}{2}-\frac{b}{2}+1$

$$a_{k+1}=\frac{a_k\left(ak+a\left(\frac{a}{2}-\frac{b}{2}+1\right)-k^2-2k\left(\frac{a}{2}-\frac{b}{2}+1\right)-\left(\frac{a}{2}-\frac{b}{2}+1\right)^2+c+k+\frac{a}{2}-\frac{b}{2}+1\right)}{\left(k+2+\frac{a}{2}-\frac{b}{2}\right)(-2k-2)}$$

- Solution for $r=\frac{a}{2}-\frac{b}{2}+1$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{a}{2}-\frac{b}{2}+1}, a_{k+1}=\frac{a_k\left(ak+a\left(\frac{a}{2}-\frac{b}{2}+1\right)-k^2-2k\left(\frac{a}{2}-\frac{b}{2}+1\right)-\left(\frac{a}{2}-\frac{b}{2}+1\right)^2+c+k+\frac{a}{2}-\frac{b}{2}+1\right)}{\left(k+2+\frac{a}{2}-\frac{b}{2}\right)(-2k-2)}\right]$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k+\frac{a}{2}-\frac{b}{2}+1}, a_{k+1}=\frac{a_k\left(ak+a\left(\frac{a}{2}-\frac{b}{2}+1\right)-k^2-2k\left(\frac{a}{2}-\frac{b}{2}+1\right)-\left(\frac{a}{2}-\frac{b}{2}+1\right)^2+c+k+\frac{a}{2}-\frac{b}{2}+1\right)}{\left(k+2+\frac{a}{2}-\frac{b}{2}\right)(-2k-2)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}d_k(1+x)^k\right)+\left(\sum_{k=0}^{\infty}e_k(1+x)^{k+\frac{a}{2}-\frac{b}{2}+1}\right), d_{1+k}=\frac{d_k(ak-k^2+c+k)}{(1+k)(-2k+a-b)}, e_{1+k}=\frac{e_k\left(ak+a\left(\frac{a}{2}-\frac{b}{2}+1\right.\right.}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.079 (sec). Leaf size: 134

```
dsolve((1-x^2)*diff(y(x),x$2)+(a*x+b)*diff(y(x),x)+c*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{hypergeom}\left(\left[-\frac{1}{2}-\frac{a}{2}-\frac{\sqrt{a^2+2a+4c+1}}{2}, -\frac{1}{2}-\frac{a}{2} + \frac{\sqrt{a^2+2a+4c+1}}{2}\right], \left[-\frac{a}{2}+\frac{b}{2}\right], \frac{1}{2}+\frac{x}{2}\right) + c_2\left(\frac{1}{2}+\frac{x}{2}\right)^{1+\frac{a}{2}-\frac{b}{2}} \operatorname{hypergeom}\left(\left[\frac{1}{2} - \frac{\sqrt{a^2+2a+4c+1}}{2} - \frac{b}{2}, \frac{1}{2}+\frac{\sqrt{a^2+2a+4c+1}}{2}-\frac{b}{2}\right], \left[2+\frac{a}{2}-\frac{b}{2}\right], \frac{1}{2}+\frac{x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.317 (sec). Leaf size: 184

```
DSolve[(1-x^2)*y''[x]+(a*x+b)*y'[x]+c*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2^{\frac{1}{2}(-a-b-2)}\Bigg(c_2(x$$

$$-1)^{\frac{1}{2}(a+b+2)}\,\text{Hypergeometric2F1}\left(\frac{1}{2}\left(b-\sqrt{a^2+2a+4c+1}+1\right),\frac{1}{2}\left(b+\sqrt{a^2+2a+4c+1}+1\right),\frac{1}{2}(a+$$

$$+c_1 2^{\frac{1}{2}(a+b+2)}\,\text{Hypergeometric2F1}\left(\frac{1}{2}\left(-a-\sqrt{a^2+2a+4c+1}-1\right),\frac{1}{2}\left(-a+\sqrt{a^2+2a+4c+1}-1\right),\frac{1}{2}\right.$$

30.21 problem 169

Internal problem ID [10993]
Internal file name [OUTPUT/10249_Sunday_December_31_2023_11_24_09_AM_34106090/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2+bx+c)y''+f(x)y'+g(x)y=0$
Problem number: 169.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{\left(a\,x^2+b\right)y''+\left(c\,x^2+d\right)y'+\lambda\left(\left(-a\lambda+c\right)x^2+d-b\lambda\right)y=0}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.281 (sec). Leaf size: 939

```
dsolve((a*x^2+b)*diff(y(x),x$2)+(c*x^2+d)*diff(y(x),x)+lambda*((c-a*lambda)*x^2+d-b*lambda)*
```

$$y(x) = \left(-ax+\sqrt{-ab}\right)^{\frac{2a^2b+\sqrt{4a^2b(ad-bc)\sqrt{-ab}+4a^4b^2-a^3bd^2+2db^2ca^2-b^3c^2a}}{4a^2b}}\Bigg(c_2\Big(ax+\sqrt{-ab}\Big)^{-\frac{-2a^2b+\sqrt{-ab\left(4\sqrt{-ab}\,a^2d-4\sqrt{-ab}\,abc-4a^3b+a^2d^2-2abcd+b^2c^2\right)}}{4a^2b}}\operatorname{HeunC}\Bigg(\frac{(4a\lambda-2c)\sqrt{-\frac{b}{a}}}{a}, -\frac{\sqrt{-ab\left(4\sqrt{-ab}\,a^2d-4\sqrt{-ab}\,abc-4a^3b+a^2d^2-2abcd+b^2c^2\right)}}{2a^2b}, \frac{\sqrt{4a^2b(ad-bc)\sqrt{-ab}+4a^4b^2-a}}{2a^2b} - \frac{bc\lambda}{a^2}+\frac{1}{2}-\frac{d^2}{8ab}-\frac{cd}{4a^2}+\frac{3bc^2}{8a^3}, \frac{ax}{2\sqrt{-ab}}+\frac{1}{2}\Bigg)\,\mathrm{e}^{\frac{-i\pi\sqrt{4a^2b(ad-bc)\sqrt{-ab}+4a^4b^2-a^3bd^2+2db^2ca^2-b^3c^2a}+i\pi\sqrt{-ab\left(4\sqrt{-ab}\,a^2d-4\sqrt{-ab}\,abc-4a^3b+a^2d^2-2abcd+b^2c^2\right)}-4b\left(a^2\left(\frac{d}{\sqrt{b}\sqrt{a}}-\frac{\sqrt{b}\,c}{a^{\frac{3}{2}}}\right.\right.}{8a^2b}} + c_1\Big(ax+\sqrt{-ab}\Big)^{\frac{2a^2b+\sqrt{-ab\left(4\sqrt{-ab}\,a^2d-4\sqrt{-ab}\,abc-4a^3b+a^2d^2-2abcd+b^2c^2\right)}}{4a^2b}}\mathrm{e}^{x\lambda+\frac{\sqrt{-ab}\,\lambda}{a}-\frac{cx}{a}-\frac{\sqrt{-ab}\,c}{2a^2}-\frac{\arctan\left(\frac{\sqrt{a}\,x}{\sqrt{b}}\right)d}{2\sqrt{a}\sqrt{b}}+\frac{\sqrt{b}\arctan\left(\frac{\sqrt{a}\,x}{\sqrt{b}}\right)c}{2a^{\frac{3}{2}}}}\,\mathrm{He} - \frac{bc\lambda}{a^2}+\frac{1}{2}-\frac{d^2}{8ab}-\frac{cd}{4a^2}+\frac{3bc^2}{8a^3}, \frac{ax}{2\sqrt{-ab}}+\frac{1}{2}\Big)\Bigg)$$

✓ Solution by Mathematica

Time used: 2.859 (sec). Leaf size: 74

```
DSolve[(a*x^2+b)*y''[x]+(c*x^2+d)*y'[x]+\[Lambda]*((c-a*\[Lambda])*x^2+d-b*\[Lambda])*y[x]==
```

$$y(x) \to e^{\lambda(-x)}\left(c_2\int_1^x \exp\left(\frac{(bc-ad)\arctan\left(\frac{\sqrt{a}K[1]}{\sqrt{b}}\right)}{a^{3/2}\sqrt{b}}+\left(2\lambda-\frac{c}{a}\right)K[1]\right)dK[1]+c_1\right)$$

30.22 problem 170

Internal problem ID [10994]
Internal file name [`OUTPUT/10250_Sunday_December_31_2023_11_24_11_AM_22104146/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 170.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{\left(a\,x^{2}+b\right) y''+\left(\lambda(a+c)\,x^{2}+(c-a)\,x+2b\lambda\right) y'+\lambda^{2}\left(c\,x^{2}+b\right) y=0}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.422 (sec). Leaf size: 1381

```
dsolve((a*x^2+b)*diff(y(x),x$2)+(lambda*(c+a)*x^2+(c-a)*x+2*b*lambda)*diff(y(x),x)+lambda^2*
```

Expression too large to display

✓ Solution by Mathematica

Time used: 5.408 (sec). Leaf size: 104

```
DSolve[(a*x^2+b)*y''[x]+(\[Lambda]*(c+a)*x^2+(c-a)*x+2*b*\[Lambda])*y'[x]+\[Lambda]^2*(c*x^2
```

$$y(x) \to e^{\lambda(-x)}(\lambda x + 1)\left(c_2 \int_1^x \frac{\exp\left(\frac{(a-c)\lambda\left(\sqrt{a}K[1]-\sqrt{b}\arctan\left(\frac{\sqrt{a}K[1]}{\sqrt{b}}\right)\right)}{a^{3/2}}\right)(aK[1]^2+b)^{\frac{a-c}{2a}}}{(\lambda K[1]+1)^2} dK[1] + c_1\right)$$

30.23 problem 171

Internal problem ID [10995]
Internal file name [OUTPUT/10251_Sunday_December_31_2023_11_24_12_AM_56588853/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 171.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

[_Jacobi]

Unable to solve or complete the solution.

$$x(x-1)\,y'' + ((\alpha+\beta+1)\,x-\gamma)\,y' + \alpha\beta y = 0$$

30.23.1 Maple step by step solution

Let's solve

$(x^2-x)\,y'' + ((\alpha+\beta+1)\,x-\gamma)\,y' + \alpha\beta y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{\alpha\beta y}{x(x-1)} - \frac{(x\alpha+\beta x-\gamma+x)y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x\alpha+\beta x-\gamma+x)y'}{x(x-1)} + \frac{\alpha\beta y}{x(x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{x\alpha+\beta x-\gamma+x}{x(x-1)}, P_3(x) = \frac{\alpha\beta}{x(x-1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \gamma$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x-1)\, y'' + (x\alpha + \beta x - \gamma + x)\, y' + \alpha\beta y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-1+r+\gamma)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k+r+\gamma) + a_k(\beta+k+r)\,(\alpha+k+r)\right) x^{k+}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(-1+r+\gamma)=0$

- Values of r that satisfy the indicial equation

$r \in \{0, -\gamma+1\}$

- Each term in the series must be 0, giving the recursion relation

$-a_{k+1}(k+1+r)\,(k+r+\gamma)+a_k(\beta+k+r)\,(\alpha+k+r)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{a_k(\beta+k+r)(\alpha+k+r)}{(k+1+r)(k+r+\gamma)}$

- Recursion relation for $r=0$

$a_{k+1}=\frac{a_k(\beta+k)(\alpha+k)}{(k+1)(k+\gamma)}$

- Solution for $r=0$

$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{a_k(\beta+k)(\alpha+k)}{(k+1)(k+\gamma)}\right]$

- Recursion relation for $r=-\gamma+1$

$a_{k+1}=\frac{a_k(\beta+k-\gamma+1)(\alpha+k-\gamma+1)}{(k+2-\gamma)(k+1)}$

- Solution for $r=-\gamma+1$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\gamma+1}, a_{k+1}=\frac{a_k(\beta+k-\gamma+1)(\alpha+k-\gamma+1)}{(k+2-\gamma)(k+1)}\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\gamma+1}\right), a_{1+k}=\frac{a_k(\beta+k)(\alpha+k)}{(1+k)(k+\gamma)}, b_{1+k}=\frac{b_k(\beta+k-\gamma+1)(\alpha+k-\gamma+1)}{(k+2-\gamma)(1+k)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 44

```
dsolve(x*(x-1)*diff(y(x),x$2)+((alpha+beta+1)*x-gamma)*diff(y(x),x)+alpha*beta*y(x)=0,y(x),
```

$$y(x) = c_1 \operatorname{hypergeom}([\alpha, \beta], [\gamma], x) + c_2 x^{1-\gamma} \operatorname{hypergeom}([\beta + 1 - \gamma, \alpha + 1 - \gamma], [2 - \gamma], x)$$

✓ Solution by Mathematica

Time used: 0.281 (sec). Leaf size: 49

```
DSolve[x*(x-1)*y''[x]+((\[Alpha]+\[Beta]+1)*x-\[Gamma])*y'[x]+\[Alpha]*\[Beta]*y[x]==0,y[x],
```

$$\begin{aligned} y(x) \to\ & c_1 \operatorname{Hypergeometric2F1}(\alpha, \beta, \gamma, x) \\ & - (-1)^{-\gamma} c_2 x^{1-\gamma} \operatorname{Hypergeometric2F1}(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, x) \end{aligned}$$

30.24 problem 172

Internal problem ID [10996]
Internal file name [`OUTPUT/10252_Sunday_December_31_2023_11_24_13_AM_550928/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2+bx+c)y''+f(x)y'+g(x)y=0$
Problem number: 172.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x(x+a)\,y''+(bx+c)\,y'+yd=0}$$

30.24.1 Maple step by step solution

Let's solve

$x(x+a)\,y''+(bx+c)\,y'+yd=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{dy}{x(x+a)}-\frac{(bx+c)y'}{x(x+a)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(bx+c)y'}{x(x+a)}+\frac{dy}{x(x+a)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{bx+c}{x(x+a)}, P_3(x) = \frac{d}{x(x+a)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{c}{a}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x+a)\, y'' + (bx+c)\, y' + yd = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(ar-a+c)\, x^{r-1} + \left(\sum_{k=0}^{\infty} (a_{k+1}(k+1+r)\,(a(k+1)+ar-a+c) + a_k(bk+br+k^2+2kr+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(ar - a + c) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{a-c}{a}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+1+r)\,(ak+ar+c)\,a_{k+1} + \left(k^2 + (b+2r-1)\,k + r^2 + (b-1)\,r + d\right) a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{(bk+br+k^2+2kr+r^2+d-k-r)\,a_k}{(k+1+r)(ak+ar+c)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{(bk+k^2+d-k)\,a_k}{(k+1)(ak+c)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{(bk+k^2+d-k)\,a_k}{(k+1)(ak+c)}\right]$$

- Recursion relation for $r = \frac{a-c}{a}$

$$a_{k+1} = -\frac{\left(bk+\frac{b(a-c)}{a}+k^2+\frac{2k(a-c)}{a}+\frac{(a-c)^2}{a^2}+d-k-\frac{a-c}{a}\right)a_k}{\left(k+1+\frac{a-c}{a}\right)(ak+a)}$$

- Solution for $r = \frac{a-c}{a}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{a-c}{a}}, a_{k+1} = -\frac{\left(bk+\frac{b(a-c)}{a}+k^2+\frac{2k(a-c)}{a}+\frac{(a-c)^2}{a^2}+d-k-\frac{a-c}{a}\right)a_k}{\left(k+1+\frac{a-c}{a}\right)(ak+a)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} e_k x^k\right) + \left(\sum_{k=0}^{\infty} f_k x^{k+\frac{a-c}{a}}\right), e_{1+k} = -\frac{(bk+k^2+d-k)\,e_k}{(1+k)(ak+c)}, f_{1+k} = -\frac{\left(bk+\frac{b(a-c)}{a}+k^2+\frac{2k(a-c)}{a}+\frac{(a-c)^2}{a^2}\right.}{\left(k+1+\frac{a-c}{a}\right)(ak+a)}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 230

```
dsolve(x*(x+a)*diff(y(x),x$2)+(b*x+c)*diff(y(x),x)+d*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_2(\operatorname{csgn}(a)\, a + a + 2x)^{-\frac{((b-2)\operatorname{csgn}(a)a+ab-2c)\operatorname{csgn}(a)}{2a}} \operatorname{hypergeom}\left(\left[\frac{\operatorname{csgn}(a)\left(\operatorname{csgn}(a)\, a + \sqrt{b^2-2b-4d+1}\, \operatorname{csgn}(a)\, a - ab + 2\right.}{2a}\right.\right.$$

$$-\frac{\operatorname{csgn}(a)\left(\sqrt{b^2-2b-4d+1}\, \operatorname{csgn}(a)\, a - \operatorname{csgn}(a)\, a + ab - 2c\right)}{2a}\Bigg], \left[-\frac{\operatorname{csgn}(a)\left((b-4)\operatorname{csgn}(a)\, a + ab\right.}{2a}\right.$$

$$+ c_1 \operatorname{hypergeom}\left(\left[-\frac{1}{2}+\frac{b}{2}-\frac{\sqrt{b^2-2b-4d+1}}{2}, -\frac{1}{2}+\frac{b}{2} + \frac{\sqrt{b^2-2b-4d+1}}{2}\right], \left[\frac{(b\operatorname{csgn}(a)\, a + ab - 2c)\operatorname{csgn}(a)}{2a}\right], \frac{\operatorname{csgn}(a)(\operatorname{csgn}(a)\, a + a + 2x)}{2a}\right)$$

✓ Solution by Mathematica

Time used: 0.423 (sec). Leaf size: 165

```
DSolve[x*(x+a)*y''[x]+(b*x+c)*y'[x]+d*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 a^{\frac{c}{a}-1} x^{1-\frac{c}{a}} \text{Hypergeometric2F1}\left(\frac{1}{2}\left(b-\frac{2c}{a}+\sqrt{b^2-2b-4d+1}+1\right), \frac{ba-\sqrt{b^2-2b-4d+1}a+a-2c}{2a}, 2-\frac{c}{a}, -\frac{x}{a}\right) + c_1 \text{Hypergeometric2F1}\left(\frac{1}{2}\left(b-\sqrt{b^2-2b-4d+1}-1\right), \frac{1}{2}\left(b+\sqrt{b^2-2b-4d+1}-1\right), \frac{c}{a}, -\frac{x}{a}\right)$$

30.25 problem 173

Internal problem ID [10997]
Internal file name [`OUTPUT/10253_Sunday_December_31_2023_11_24_15_AM_29690673/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 173.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_Jacobi]`

Unable to solve or complete the solution.

$$\boxed{2x(x-1)\,y'' + (2x-1)\,y' + (ax+b)\,y = 0}$$

30.25.1 Maple step by step solution

Let's solve

$(2x^2 - 2x)\,y'' + (2x-1)\,y' + (ax+b)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(ax+b)y}{2x(x-1)} - \frac{(2x-1)y'}{2x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x-1)y'}{2x(x-1)} + \frac{(ax+b)y}{2x(x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x)=\frac{2x-1}{2x(x-1)}, P_3(x)=\frac{ax+b}{2x(x-1)}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{1}{2}$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x(x-1)\,y''+(2x-1)\,y'+(ax+b)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m\cdot y''$ to series expansion for $m=1..2$

  $x^m\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-1+2r)\, x^{-1+r} + (-a_1(1+r)(1+2r) + a_0(2r^2+b))\, x^r + \left(\sum_{k=1}^{\infty} (-a_{k+1}(k+1+r)(2k+1\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term must be 0

$$-a_1(1+r)(1+2r) + a_0(2r^2+b) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-2(k+1+r)\left(k+\frac{1}{2}+r\right)a_{k+1} + 2k^2 a_k + 4kra_k + 2r^2 a_k + a_{k-1}a + a_k b = 0$$

- Shift index using $k->k+1$

$$-2(k+2+r)\left(k+\frac{3}{2}+r\right)a_{k+2} + 2(k+1)^2 a_{k+1} + 4(k+1)\, ra_{k+1} + 2r^2 a_{k+1} + a_k a + a_{k+1} b = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2k^2 a_{k+1} + 4kra_{k+1} + 2r^2 a_{k+1} + a_k a + a_{k+1} b + 4ka_{k+1} + 4ra_{k+1} + 2a_{k+1}}{(k+2+r)(2k+3+2r)}$$

- Recursion relation for $r=0$

$$a_{k+2} = \frac{2k^2 a_{k+1} + a_k a + a_{k+1} b + 4ka_{k+1} + 2a_{k+1}}{(k+2)(2k+3)}$$

- Solution for $r=0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2k^2 a_{k+1} + a_k a + a_{k+1} b + 4ka_{k+1} + 2a_{k+1}}{(k+2)(2k+3)}, a_0 b - a_1 = 0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2} = \frac{2k^2 a_{k+1} + a_k a + a_{k+1} b + 6ka_{k+1} + \frac{9}{2}a_{k+1}}{\left(k+\frac{5}{2}\right)(2k+4)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{2k^2 a_{k+1} + a_k a + a_{k+1} b + 6ka_{k+1} + \frac{9}{2}a_{k+1}}{\left(k+\frac{5}{2}\right)(2k+4)}, -3a_1 + a_0\left(b+\frac{1}{2}\right) = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} c_k x^k\right) + \left(\sum_{k=0}^{\infty} d_k x^{k+\frac{1}{2}}\right), c_{k+2} = \frac{2k^2 c_{1+k} + ac_k + bc_{1+k} + 4kc_{1+k} + 2c_{1+k}}{(k+2)(2k+3)}, bc_0 - c_1 = 0, d_{k+2} = \frac{2k^2 c\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      Equivalence transformation and function parameters: {x = t}, {kappa = -8*b-4, mu = 8*a
      <- Equivalence to the rational form of Mathieu ODE successful
   <- Mathieu successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.297 (sec). Leaf size: 39

```
dsolve(2*x*(x-1)*diff(y(x),x$2)+(2*x-1)*diff(y(x),x)+(a*x+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{MathieuC}\left(-a-2b, \frac{a}{2}, \arccos\left(\sqrt{x}\right)\right) + c_2 \operatorname{MathieuS}\left(-a-2b, \frac{a}{2}, \arccos\left(\sqrt{x}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.258 (sec). Leaf size: 50

```
DSolve[2*x*(x-1)*y''[x]+(2*x-1)*y'[x]+(a*x+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to c_1 \text{MathieuC}\left[-a-2b, \frac{a}{2}, \arccos\left(\sqrt{x}\right)\right] + c_2 \text{MathieuS}\left[-a-2b, \frac{a}{2}, \arccos\left(\sqrt{x}\right)\right]$$

30.26 problem 174

Internal problem ID [10998]
Internal file name [`OUTPUT/10254_Sunday_December_31_2023_11_24_16_AM_3198913/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 174.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{\left(2ax + x^2 + b\right) y'' + (x + a) y' - y m^2 = 0}$$

30.26.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$\left(2ax + x^2 + b\right) y'' + (x + a) y' - y m^2 = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x + a}{2ax + x^2 + b} \\ q(x) &= -\frac{m^2}{2ax + x^2 + b} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{x+a}{2ax+x^2+b} dx\right)} dx \\
&= \int e^{-\frac{\ln\left(2ax+x^2+b\right)}{2}} dx \\
&= \int \frac{1}{\sqrt{2ax+x^2+b}} dx \\
&= \ln\left(x+a+\sqrt{2ax+x^2+b}\right)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{m^2}{2ax+x^2+b}}{\frac{1}{2ax+x^2+b}} \\
&= -m^2
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - m^2 y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -m^2$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} - m^2 \mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - m^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -m^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-m^2)} \\ &= \pm\sqrt{m^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{m^2} \\ \lambda_2 &= -\sqrt{m^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{m^2} \\ \lambda_2 &= -\sqrt{m^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\ y(\tau) &= c_1 e^{\left(\sqrt{m^2}\right)\tau} + c_2 e^{\left(-\sqrt{m^2}\right)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 \mathrm{e}^{\sqrt{m^2}\,\tau} + c_2 \mathrm{e}^{-\sqrt{m^2}\,\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(x + a + \sqrt{2ax + x^2 + b}\right)^m + c_2 \left(x + a + \sqrt{2ax + x^2 + b}\right)^{-m}$$

Summary
The solution(s) found are the following

$$y = c_1 \left(x + a + \sqrt{2ax + x^2 + b}\right)^m + c_2 \left(x + a + \sqrt{2ax + x^2 + b}\right)^{-m} \tag{1}$$

Verification of solutions

$$y = c_1 \left(x + a + \sqrt{2ax + x^2 + b}\right)^m + c_2 \left(x + a + \sqrt{2ax + x^2 + b}\right)^{-m}$$

Verified OK.

30.26.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$\left(2ax + x^2 + b\right) y'' + (x + a) y' - y m^2 = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$p(x) = \frac{x + a}{2ax + x^2 + b}$$
$$q(x) = -\frac{m^2}{2ax + x^2 + b}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$
$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-\frac{m^2}{2ax+x^2+b}}}{c} \\ \tau'' &= \frac{m^2(2a+2x)}{2c\sqrt{-\frac{m^2}{2ax+x^2+b}}\,(2ax+x^2+b)^2}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{m^2(2a+2x)}{2c\sqrt{-\frac{m^2}{2ax+x^2+b}}\,(2ax+x^2+b)^2} + \frac{x+a}{2ax+x^2+b}\frac{\sqrt{-\frac{m^2}{2ax+x^2+b}}}{c}}{\left(\frac{\sqrt{-\frac{m^2}{2ax+x^2+b}}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1\cos(c\tau) + c_2\sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \sqrt{-\frac{m^2}{2ax+x^2+b}}dx}{c} \\ &= \frac{\sqrt{-\frac{m^2}{2ax+x^2+b}}\,\sqrt{2ax+x^2+b}\,\ln\left(x+a+\sqrt{2ax+x^2+b}\right)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cosh\left(m \ln\left(x + a + \sqrt{2ax + x^2 + b}\right)\right) + i c_2 \sinh\left(m \ln\left(x + a + \sqrt{2ax + x^2 + b}\right)\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & c_1 \cosh\left(m \ln\left(x + a + \sqrt{2ax + x^2 + b}\right)\right) \\ & + i c_2 \sinh\left(m \ln\left(x + a + \sqrt{2ax + x^2 + b}\right)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 \cosh\left(m \ln\left(x + a + \sqrt{2ax + x^2 + b}\right)\right) + i c_2 \sinh\left(m \ln\left(x + a + \sqrt{2ax + x^2 + b}\right)\right)$$

Verified OK.

30.26.3 Solving using Kovacic algorithm

Writing the ode as

$$\left(2ax + x^2 + b\right) y'' + (x + a)\, y' - y m^2 = 0 \tag{1}$$

$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2ax + x^2 + b \\ B &= x + a \\ C &= -m^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8m^2ax + 4m^2x^2 + 4b\,m^2 - 3a^2 - 2ax - x^2 + 2b}{4\left(2ax + x^2 + b\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8m^2ax + 4m^2x^2 + 4b\,m^2 - 3a^2 - 2ax - x^2 + 2b \\ t &= 4\left(2ax + x^2 + b\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8m^2ax + 4m^2x^2 + 4b\,m^2 - 3a^2 - 2ax - x^2 + 2b}{4\left(2ax + x^2 + b\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 166: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2ax + x^2 + b)^2$. There is a pole at $x = -a + \sqrt{a^2 - b}$ of order 2. There is a pole at $x = -a - \sqrt{a^2 - b}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned} r &= \frac{4\left(-a+\sqrt{a^2-b}\right)^2 m^2 + 8\left(-a+\sqrt{a^2-b}\right) a\, m^2 + 4b\, m^2 - \left(-a+\sqrt{a^2-b}\right)^2 - 2\left(-a+\sqrt{a^2-b}\right) a - 3a}{16\left(a^2-b\right)\left(x+a-\sqrt{a^2-b}\right)^2} \\ &+ \frac{4\left(-a-\sqrt{a^2-b}\right)^2 m^2 + 8\left(-a-\sqrt{a^2-b}\right) a\, m^2 + 4b\, m^2 - \left(-a-\sqrt{a^2-b}\right)^2 - 2\left(-a-\sqrt{a^2-b}\right) a -}{16\left(a^2-b\right)\left(x+a+\sqrt{a^2-b}\right)^2} \\ &+ \frac{4\left(-a+\sqrt{a^2-b}\right)^2 m^2 + 8\left(-a+\sqrt{a^2-b}\right) a\, m^2 + 8a^2m^2 - 4b\, m^2 - \left(-a+\sqrt{a^2-b}\right)^2 - 2\left(-a+\sqrt{a^2}\right.}{16\left(a^2-b\right)^{\frac{3}{2}}\left(x+a-\sqrt{a^2-b}\right)} \\ &- \frac{4\left(-a-\sqrt{a^2-b}\right)^2 m^2 + 8\left(-a-\sqrt{a^2-b}\right) a\, m^2 + 8a^2m^2 - 4b\, m^2 - \left(-a-\sqrt{a^2-b}\right)^2 - 2\left(-a-\sqrt{a^2}\right.}{16\left(a^2-b\right)^{\frac{3}{2}}\left(x+a+\sqrt{a^2-b}\right)} \end{aligned}$$

For the pole at $x = -a + \sqrt{a^2 - b}$ let b be the coefficient of $\frac{1}{\left(x+a-\sqrt{a^2-b}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -a - \sqrt{a^2 - b}$ let b be the coefficient of $\frac{1}{\left(x+a+\sqrt{a^2-b}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{8m^2ax + 4m^2x^2 + 4b\,m^2 - 3a^2 - 2ax - x^2 + 2b}{4\left(2ax + x^2 + b\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
$-a + \sqrt{a^2 - b}$	2	$\{1, 2, 3\}$
$-a - \sqrt{a^2 - b}$	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{1, 2, 3\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{\left(x - \left(-a + \sqrt{a^2-b}\right)\right)} + \frac{1}{\left(x - \left(-a - \sqrt{a^2-b}\right)\right)}\right) \\ &= \frac{1}{2x + 2a - 2\sqrt{a^2-b}} + \frac{1}{2x + 2a + 2\sqrt{a^2-b}} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{2x + 2a - 2\sqrt{a^2 - b}} + \frac{1}{2x + 2a + 2\sqrt{a^2 - b}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2x + 2a - 2\sqrt{a^2 - b}} + \frac{1}{2x + 2a + 2\sqrt{a^2 - b}}\right) w + \frac{\left(-4m^2 + 1\right) x^2 + \left(-8m^2 + 2\right) a x - 4b\, m^2 + a^2}{4 \left(x + a - \sqrt{a^2 - b}\right)^2 \left(x + a + \sqrt{a^2 - b}\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{2m\sqrt{2ax + x^2 + b} + a + x}{4ax + 2x^2 + 2b}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx}\\ &= \mathrm{e}^{\int \frac{2m\sqrt{2ax + x^2 + b} + a + x}{4ax + 2x^2 + 2b} dx}\\ &= \left(2ax + x^2 + b\right)^{\frac{1}{4}} \left(x + a + \sqrt{2ax + x^2 + b}\right)^m\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x+a}{2ax+x^2+b}\,dx} \\
&= z_1 e^{-\frac{\ln\left(2ax+x^2+b\right)}{4}} \\
&= z_1\left(\frac{1}{\left(2ax+x^2+b\right)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x+a+\sqrt{2ax+x^2+b}\right)^m
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x+a}{2ax+x^2+b}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln\left(2ax+x^2+b\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{\left(x+a+\sqrt{2ax+x^2+b}\right)^{-2m}}{2m}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x+a+\sqrt{2ax+x^2+b}\right)^m\right) \\
&\quad + c_2\left(\left(x+a+\sqrt{2ax+x^2+b}\right)^m\left(-\frac{\left(x+a+\sqrt{2ax+x^2+b}\right)^{-2m}}{2m}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 \left(x + a + \sqrt{2ax + x^2 + b}\right)^m - \frac{c_2 \left(x + a + \sqrt{2ax + x^2 + b}\right)^{-m}}{2m} \tag{1}$$

Verification of solutions

$$y = c_1 \left(x + a + \sqrt{2ax + x^2 + b}\right)^m - \frac{c_2 \left(x + a + \sqrt{2ax + x^2 + b}\right)^{-m}}{2m}$$

Verified OK.

30.26.4 Maple step by step solution

Let's solve

$(2ax + x^2 + b)\, y'' + (x + a)\, y' - y m^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{m^2 y}{2ax+x^2+b} - \frac{(x+a)y'}{2ax+x^2+b}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+a)y'}{2ax+x^2+b} - \frac{m^2 y}{2ax+x^2+b} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{x+a}{2ax+x^2+b}, P_3(x) = -\frac{m^2}{2ax+x^2+b}\right]$

- ◦ $\left(x + a + \sqrt{a^2 - b}\right) \cdot P_2(x)$ is analytic at $x = -a - \sqrt{a^2 - b}$

  $\left(\left(x + a + \sqrt{a^2 - b}\right) \cdot P_2(x)\right)\Big|_{x=-a-\sqrt{a^2-b}} = 0$

- ◦ $\left(x + a + \sqrt{a^2 - b}\right)^2 \cdot P_3(x)$ is analytic at $x = -a - \sqrt{a^2 - b}$

  $\left(\left(x + a + \sqrt{a^2 - b}\right)^2 \cdot P_3(x)\right)\Big|_{x=-a-\sqrt{a^2-b}} = 0$

- ◦ $x = -a - \sqrt{a^2 - b}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -a - \sqrt{a^2 - b}$

- Multiply by denominators

  $(2ax + x^2 + b)\, y'' + (x + a)\, y' - y m^2 = 0$

- Change variables using $x = u - a - \sqrt{a^2 - b}$ so that the regular singular point is at $u = 0$

  $\left(u^2 - 2u\sqrt{a^2 - b}\right)\left(\frac{d^2}{du^2}y(u)\right) + \left(u - \sqrt{a^2 - b}\right)\left(\frac{d}{du}y(u)\right) - m^2 y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k + r)\, u^{k+r-1+m}$

  - Shift index using $k- > k + 1 - m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, u^{k+r-2+m}$

  - Shift index using $k- > k + 2 - m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k + 2 - m + r)\,(k + 1 - m + r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-\sqrt{a^2 - b}\, a_0 r(-1 + 2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-\sqrt{a^2 - b}\, a_{k+1}(k + 1 + r)\,(2k + 1 + 2r) + a_k(k + m + r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-\sqrt{a^2 - b}\, r(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}\left(k + r + \frac{1}{2}\right)(k + 1 + r)\sqrt{a^2 - b} + a_k(k + m + r)\,(k - m + r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k (k^2 + 2kr - m^2 + r^2)}{\sqrt{a^2 - b}\,(2k^2 + 4kr + 2r^2 + 3k + 3r + 1)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k^2-m^2)}{\sqrt{a^2-b}\,(2k^2+3k+1)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k^2-m^2)}{\sqrt{a^2-b}\,(2k^2+3k+1)}\right]$

- Revert the change of variables $u = x + a + \sqrt{a^2-b}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x+a+\sqrt{a^2-b}\right)^k, a_{k+1} = \frac{a_k(k^2-m^2)}{\sqrt{a^2-b}\,(2k^2+3k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k(k^2-m^2+k+\frac{1}{4})}{\sqrt{a^2-b}\,(2k^2+5k+3)}$

- Solution for $r = \frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k(k^2-m^2+k+\frac{1}{4})}{\sqrt{a^2-b}\,(2k^2+5k+3)}\right]$

- Revert the change of variables $u = x + a + \sqrt{a^2-b}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x+a+\sqrt{a^2-b}\right)^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k(k^2-m^2+k+\frac{1}{4})}{\sqrt{a^2-b}\,(2k^2+5k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} c_k \left(x+a+\sqrt{a^2-b}\right)^k\right) + \left(\sum_{k=0}^{\infty} d_k \left(x+a+\sqrt{a^2-b}\right)^{k+\frac{1}{2}}\right), c_{1+k} = \frac{c_k(k^2-m^2)}{\sqrt{a^2-b}\,(2k^2+3k+1)}, d_{1-}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve((x^2+2*a*x+b)*diff(y(x),x$2)+(x+a)*diff(y(x),x)-m^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(a+x+\sqrt{2ax+x^2+b}\right)^{-m} + c_2\left(a+x+\sqrt{2ax+x^2+b}\right)^{m}$$

✓ Solution by Mathematica

Time used: 0.301 (sec). Leaf size: 63

```
DSolve[(x^2+2*a*x+b)*y''[x]+(x+a)*y'[x]-m^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cosh\left(m\log\left(\sqrt{2ax+b+x^2}-a-x\right)\right) - ic_2 \sinh\left(m\log\left(\sqrt{2ax+b+x^2}-a-x\right)\right)$$

30.27 problem 175

Internal problem ID [10999]
Internal file name [`OUTPUT/10255_Sunday_December_31_2023_11_24_17_AM_39314145/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2+bx+c)y''+f(x)y'+g(x)y=0$
Problem number: 175.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{\left(a\,x^{2}+bx+c\right)y''+\left(dx+k\right)y'+\left(-2a+d\right)y=0}$$

30.27.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(a\,x^{2}+bx+c\right)y''+\left(dx+k\right)y'+\left(-2a+d\right)y\right)dx=0$$
$$\left(-2ax+dx-b+k\right)y+\left(a\,x^{2}+bx+c\right)y'=c_{1}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{(2a-d)\,x+b-k}{a\,x^2+bx+c}\\
q(x) &= \frac{c_1}{a\,x^2+bx+c}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{((2a-d)\,x+b-k)\,y}{a\,x^2+bx+c} = \frac{c_1}{a\,x^2+bx+c}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{(2a-d)x+b-k}{a\,x^2+bx+c}dx}\\
&= \mathrm{e}^{-\frac{(2a-d)\ln\left(a\,x^2+bx+c\right)}{2a}-\frac{2\left(b-k-\frac{(2a-d)b}{2a}\right)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}
\end{aligned}
$$

Which simplifies to

$$
\mu = \left(a\,x^2+bx+c\right)^{\frac{-2a+d}{2a}}\,\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{a\,x^2+bx+c}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(a\,x^2+bx+c\right)^{\frac{-2a+d}{2a}}\,\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}y\right) &= \left(\left(a\,x^2+bx+c\right)^{\frac{-2a+d}{2a}}\,\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}\right)\left(\frac{}{a\,x^2}\right.\\
\mathrm{d}\left(\left(a\,x^2+bx+c\right)^{\frac{-2a+d}{2a}}\,\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}y\right) &= \left(c_1\left(a\,x^2+bx+c\right)^{\frac{-4a+d}{2a}}\,\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\left(a\,x^2+bx+c\right)^{\frac{-2a+d}{2a}}\,\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}y &= \int c_1\left(a\,x^2+bx+c\right)^{\frac{-4a+d}{2a}}\,\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}\,\mathrm{d}x\\
\left(a\,x^2+bx+c\right)^{\frac{-2a+d}{2a}}\,\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}y &= \int c_1\left(a\,x^2+bx+c\right)^{\frac{-4a+d}{2a}}\,\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}dx+c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \left(a\,x^2+bx+c\right)^{\frac{-2a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}$ results in

$$y = \left(a\,x^2+bx+c\right)^{\frac{2a-d}{2a}} \mathrm{e}^{-\frac{2\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\left(ak-\frac{bd}{2}\right)}{\sqrt{4ac-b^2}\,a}} \left(\int c_1 \left(a\,x^2+bx+c\right)^{\frac{-4a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}} dx\right) + c_2$$

which simplifies to

$$y = \left(a\,x^2+bx+c\right)^{\frac{2a-d}{2a}} \mathrm{e}^{-\frac{2\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\left(ak-\frac{bd}{2}\right)}{\sqrt{4ac-b^2}\,a}} \left(c_1 \left(\int \left(a\,x^2+bx+c\right)^{\frac{-4a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}} dx\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = \left(a\,x^2+bx+c\right)^{\frac{2a-d}{2a}} \mathrm{e}^{-\frac{2\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\left(ak-\frac{bd}{2}\right)}{\sqrt{4ac-b^2}\,a}} \left(c_1 \left(\int \left(a\,x^2+bx+c\right)^{\frac{-4a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}} dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = \left(a\,x^2+bx+c\right)^{\frac{2a-d}{2a}} \mathrm{e}^{-\frac{2\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\left(ak-\frac{bd}{2}\right)}{\sqrt{4ac-b^2}\,a}} \left(c_1 \left(\int \left(a\,x^2+bx+c\right)^{\frac{-4a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}} dx\right) + c_2\right)$$

Verified OK.

30.27.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$\left(a\,x^{2}+b x +c \right) y''+\left(d x +k \right) y'+\left(-2 a +d \right) y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(a\,x^{2}+b x +c \right) y''+\left(d x +k \right) y'+\left(-2 a +d \right) y\right)d x = 0$$
$$\left(-2 a x +d x -b +k \right) y+\left(a\,x^{2}+b x +c \right) y' = c_{1}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{\left(2 a -d \right) x +b -k}{a\,x^{2}+b x +c}$$
$$q(x) = \frac{c_{1}}{a\,x^{2}+b x +c}$$

Hence the ode is

$$y'-\frac{\left(\left(2 a -d \right) x +b -k \right) y}{a\,x^{2}+b x +c} = \frac{c_{1}}{a\,x^{2}+b x +c}$$

The integrating factor μ is

$$\mu = {\mathrm e}^{\int -\frac{\left(2 a -d \right) x +b -k}{a\,x^{2}+b x +c}d x}$$
$$= {\mathrm e}^{-\frac{\left(2 a -d \right) \ln\left(a\,x^{2}+b x +c \right)}{2 a}-\frac{2 \left(b -k -\frac{\left(2 a -d \right) b}{2 a}\right) \arctan\left(\frac{2 a x +b}{\sqrt{4 a c -b^{2}}}\right)}{\sqrt{4 a c -b^{2}}}}$$

Which simplifies to

$$\mu = \left(a\,x^{2}+b x +c \right)^{\frac{-2 a +d}{2 a}} {\mathrm e}^{\frac{\left(2 a k -b d \right) \arctan\left(\frac{2 a x +b}{\sqrt{4 a c -b^{2}}}\right)}{a \sqrt{4 a c -b^{2}}}}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{c_1}{a\,x^2 + bx + c}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(a\,x^2 + bx + c\right)^{\frac{-2a+d}{2a}}\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}} y\right) = \left(\left(a\,x^2 + bx + c\right)^{\frac{-2a+d}{2a}}\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}\right)\left(\frac{}{a\,x^2}\right.$$

$$\mathrm{d}\left(\left(a\,x^2 + bx + c\right)^{\frac{-2a+d}{2a}}\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}} y\right) = \left(c_1\left(a\,x^2 + bx + c\right)^{\frac{-4a+d}{2a}}\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}\right)\mathrm{d}x$$

Integrating gives

$$\left(a\,x^2 + bx + c\right)^{\frac{-2a+d}{2a}}\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}} y = \int c_1\left(a\,x^2 + bx + c\right)^{\frac{-4a+d}{2a}}\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}\mathrm{d}x$$

$$\left(a\,x^2 + bx + c\right)^{\frac{-2a+d}{2a}}\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}} y = \int c_1\left(a\,x^2 + bx + c\right)^{\frac{-4a+d}{2a}}\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}dx + c_2$$

Dividing both sides by the integrating factor $\mu = \left(a\,x^2 + bx + c\right)^{\frac{-2a+d}{2a}}\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}$ results in

$$y = \left(a\,x^2 + bx + c\right)^{\frac{2a-d}{2a}}\mathrm{e}^{-\frac{2\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\left(ak-\frac{bd}{2}\right)}{\sqrt{4ac-b^2}\,a}}\left(\int c_1\left(a\,x^2 + bx + c\right)^{\frac{-4a+d}{2a}}\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}dx\right) + c_2$$

which simplifies to

$$y = \left(a\,x^2 + bx + c\right)^{\frac{2a-d}{2a}}\mathrm{e}^{-\frac{2\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\left(ak-\frac{bd}{2}\right)}{\sqrt{4ac-b^2}\,a}}\left(c_1\left(\int\left(a\,x^2 + bx + c\right)^{\frac{-4a+d}{2a}}\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}dx\right) + \right.$$

<u>Summary</u>

The solution(s) found are the following

$$y = \left(a\,x^2 + bx + c\right)^{\frac{2a-d}{2a}}\mathrm{e}^{-\frac{2\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\left(ak-\frac{bd}{2}\right)}{\sqrt{4ac-b^2}\,a}}\left(c_1\left(\int\left(a\,x^2+bx+c\right)^{\frac{-4a+d}{2a}}\mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = \left(a\,x^2+bx+c\right)^{\frac{2a-d}{2a}} e^{-\frac{2\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\left(ak-\frac{bd}{2}\right)}{\sqrt{4ac-b^2}\,a}} \left(c_1\left(\int \left(a\,x^2+bx+c\right)^{\frac{-4a+d}{2a}} e^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}} dx\right) + c_2\right)$$

Verified OK.

30.27.3 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)$$

is exact if

$$p''(x)-q'(x)+r(x)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x)&=a\,x^2+bx+c\\ q(x)&=dx+k\\ r(x)&=-2a+d\\ s(x)&=0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x)&=2a\\ q'(x)&=d \end{aligned}$$

Therefore (1) becomes

$$2a-(d)+(-2a+d)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+(q(x)-p'(x))\,y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$(-2ax + dx - b + k)\, y + \left(a\, x^2 + bx + c\right) y' = c_1$$

We now have a first order ode to solve which is

$$(-2ax + dx - b + k)\, y + \left(a\, x^2 + bx + c\right) y' = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{(2a-d)\, x + b - k}{a\, x^2 + bx + c} \\ q(x) &= \frac{c_1}{a\, x^2 + bx + c} \end{aligned}$$

Hence the ode is

$$y' - \frac{((2a-d)\, x + b - k)\, y}{a\, x^2 + bx + c} = \frac{c_1}{a\, x^2 + bx + c}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{(2a-d)x+b-k}{a\, x^2+bx+c}dx} \\ &= \mathrm{e}^{-\frac{(2a-d)\ln\left(a\, x^2+bx+c\right)}{2a} - \frac{2\left(b-k-\frac{(2a-d)b}{2a}\right)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}} \end{aligned}$$

Which simplifies to

$$\mu = \left(a\, x^2 + bx + c\right)^{\frac{-2a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{a\, x^2 + bx + c}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(a\, x^2 + bx + c\right)^{\frac{-2a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}} y\right) &= \left(\left(a\, x^2 + bx + c\right)^{\frac{-2a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}\right)\left(\frac{}{a\, x^2}\right. \\ \mathrm{d}\left(\left(a\, x^2 + bx + c\right)^{\frac{-2a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}} y\right) &= \left(c_1\left(a\, x^2 + bx + c\right)^{\frac{-4a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\left(a\,x^{2}+bx+c\right)^{\frac{-2a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^{2}}}\right)}{a\sqrt{4ac-b^{2}}}} y = \int c_{1}\left(a\,x^{2}+bx+c\right)^{\frac{-4a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^{2}}}\right)}{a\sqrt{4ac-b^{2}}}} \mathrm{d}x$$

$$\left(a\,x^{2}+bx+c\right)^{\frac{-2a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^{2}}}\right)}{a\sqrt{4ac-b^{2}}}} y = \int c_{1}\left(a\,x^{2}+bx+c\right)^{\frac{-4a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^{2}}}\right)}{a\sqrt{4ac-b^{2}}}} dx + c_{2}$$

Dividing both sides by the integrating factor $\mu = \left(a\,x^{2}+bx+c\right)^{\frac{-2a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^{2}}}\right)}{a\sqrt{4ac-b^{2}}}}$ results in

$$y = \left(a\,x^{2}+bx+c\right)^{\frac{2a-d}{2a}} \mathrm{e}^{-\frac{2\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^{2}}}\right)\left(ak-\frac{bd}{2}\right)}{\sqrt{4ac-b^{2}}\,a}} \left(\int c_{1}\left(a\,x^{2}+bx+c\right)^{\frac{-4a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^{2}}}\right)}{a\sqrt{4ac-b^{2}}}} dx\right) + c_{2}$$

which simplifies to

$$y = \left(a\,x^{2}+bx+c\right)^{\frac{2a-d}{2a}} \mathrm{e}^{-\frac{2\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^{2}}}\right)\left(ak-\frac{bd}{2}\right)}{\sqrt{4ac-b^{2}}\,a}} \left(c_{1}\left(\int \left(a\,x^{2}+bx+c\right)^{\frac{-4a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^{2}}}\right)}{a\sqrt{4ac-b^{2}}}} dx\right) + \cdots$$

Summary

The solution(s) found are the following

$$y = \left(a\,x^{2}+bx+c\right)^{\frac{2a-d}{2a}} \mathrm{e}^{-\frac{2\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^{2}}}\right)\left(ak-\frac{bd}{2}\right)}{\sqrt{4ac-b^{2}}\,a}} \left(c_{1}\left(\int \left(a\,x^{2}+bx+c\right)^{\frac{-4a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^{2}}}\right)}{a\sqrt{4ac-b^{2}}}} dx\right) + c_{2}\right) \tag{1}$$

Verification of solutions

$$y = \left(a\,x^{2}+bx+c\right)^{\frac{2a-d}{2a}} \mathrm{e}^{-\frac{2\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^{2}}}\right)\left(ak-\frac{bd}{2}\right)}{\sqrt{4ac-b^{2}}\,a}} \left(c_{1}\left(\int \left(a\,x^{2}+bx+c\right)^{\frac{-4a+d}{2a}} \mathrm{e}^{\frac{(2ak-bd)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^{2}}}\right)}{a\sqrt{4ac-b^{2}}}} dx\right) + c_{2}\right)$$

Verified OK.

30.27.4 Maple step by step solution

Let's solve

$(a\,x^2+bx+c)\,y''+(dx+k)\,y'+(-2a+d)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(2a-d)y}{a\,x^2+bx+c}-\frac{(dx+k)y'}{a\,x^2+bx+c}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(dx+k)y'}{a\,x^2+bx+c}-\frac{(2a-d)y}{a\,x^2+bx+c}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{dx+k}{a\,x^2+bx+c},P_3(x)=-\frac{2a-d}{a\,x^2+bx+c}\right]$

  - $\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)\cdot P_2(x)$ is analytic at $x=\frac{-b+\sqrt{-4ac+b^2}}{2a}$

    $\left(\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{-b+\sqrt{-4ac+b^2}}{2a}}=0$

  - $\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{-b+\sqrt{-4ac+b^2}}{2a}$

    $\left(\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{-b+\sqrt{-4ac+b^2}}{2a}}=0$

  - $x=\frac{-b+\sqrt{-4ac+b^2}}{2a}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=\frac{-b+\sqrt{-4ac+b^2}}{2a}$

- Multiply by denominators

  $(a\,x^2+bx+c)\,y''+(dx+k)\,y'+(-2a+d)\,y=0$

- Change variables using $x=u+\frac{-b+\sqrt{-4ac+b^2}}{2a}$ so that the regular singular point is at $u=0$

  $\left(a\,u^2+u\sqrt{-4ac+b^2}\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(du-\frac{db}{2a}+\frac{d\sqrt{-4ac+b^2}}{2a}+k\right)\left(\frac{d}{du}y(u)\right)+(-2a+d)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$
  - Shift index using $k- >k+1-m$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$
  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$$
  - Shift index using $k- >k+2-m$

    $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$\frac{a_0 r\left(2\sqrt{-4ac+b^2}\, ar-2\sqrt{-4ac+b^2}\, a+\sqrt{-4ac+b^2}\, d+2ak-bd\right)u^{-1+r}}{2a} + \left(\sum_{k=0}^{\infty}\left(\frac{a_{k+1}(k+r+1)\left(2\sqrt{-4ac+b^2}\, a(k+1)+2\sqrt{-4ac+b^2}\right.}{2a}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$\frac{r\left(2\sqrt{-4ac+b^2}\, ar-2\sqrt{-4ac+b^2}\, a+\sqrt{-4ac+b^2}\, d+2ak-bd\right)}{2a} = 0$$
- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{2\sqrt{-4ac+b^2}\, a-\sqrt{-4ac+b^2}\, d-2ak+bd}{2\sqrt{-4ac+b^2}\, a}\right\}$$
- Each term in the series must be 0, giving the recursion relation

  $$\frac{\left(a_{k+1}\left((k+r)a+\frac{d}{2}\right)\sqrt{-4ac+b^2}+a_k(k+r-2)a^2+(a_k d+k a_{k+1})a-\frac{bda_{k+1}}{2}\right)(k+r+1)}{a} = 0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{2aa_k(ak+ar-2a+d)}{2\sqrt{-4ac+b^2}\, ak+2\sqrt{-4ac+b^2}\, ar+\sqrt{-4ac+b^2}\, d+2ak-bd}$$
- Recursion relation for $r = 0$

  $$a_{k+1} = -\frac{2aa_k(ak-2a+d)}{2\sqrt{-4ac+b^2}\, ak+\sqrt{-4ac+b^2}\, d+2ak-bd}$$
- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = -\frac{2aa_k(ak-2a+d)}{2\sqrt{-4ac+b^2}\, ak+\sqrt{-4ac+b^2}\, d+2ak-bd}\right]$$
- Revert the change of variables $u = x - \frac{-b+\sqrt{-4ac+b^2}}{2a}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^k, a_{k+1}=-\frac{2aa_k(ak-2a+d)}{2\sqrt{-4ac+b^2}\,ak+\sqrt{-4ac+b^2}\,d+2ak-bd}\right]$$

- Recursion relation for $r=\frac{2\sqrt{-4ac+b^2}\,a-\sqrt{-4ac+b^2}\,d-2ak+bd}{2\sqrt{-4ac+b^2}\,a}$

$$a_{k+1}=-\frac{2aa_k\left(ak+\frac{2\sqrt{-4ac+b^2}\,a-\sqrt{-4ac+b^2}\,d-2ak+bd}{2\sqrt{-4ac+b^2}}-2a+d\right)}{2\sqrt{-4ac+b^2}\,ak+2\sqrt{-4ac+b^2}\,a}$$

- Solution for $r=\frac{2\sqrt{-4ac+b^2}\,a-\sqrt{-4ac+b^2}\,d-2ak+bd}{2\sqrt{-4ac+b^2}\,a}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{2\sqrt{-4ac+b^2}\,a-\sqrt{-4ac+b^2}\,d-2ak+bd}{2\sqrt{-4ac+b^2}\,a}}, a_{k+1}=-\frac{2aa_k\left(ak+\frac{2\sqrt{-4ac+b^2}\,a-\sqrt{-4ac+b^2}\,d-2ak+bd}{2\sqrt{-4ac+b^2}}-2a+d\right)}{2\sqrt{-4ac+b^2}\,ak+2\sqrt{-4ac+b^2}\,a}\right]$$

- Revert the change of variables $u=x-\frac{-b+\sqrt{-4ac+b^2}}{2a}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^{k+\frac{2\sqrt{-4ac+b^2}\,a-\sqrt{-4ac+b^2}\,d-2ak+bd}{2\sqrt{-4ac+b^2}\,a}}, a_{k+1}=-\frac{2aa_k\left(ak+\frac{2\sqrt{-4ac+b^2}\,a-\sqrt{-4ac+b^2}\,d-\right.}{2\sqrt{-4ac+b^2}}}{2\sqrt{-4ac+b^2}\,ak+2\sqrt{-4ac-}}\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{m=0}^{\infty} e_m \left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^m\right)+\left(\sum_{m=0}^{\infty} f_m \left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^{m+\frac{2\sqrt{-4ac+b^2}\,a-\sqrt{-4ac+b^2}\,d-2ak+bd}{2\sqrt{-4ac+b^2}\,a}}\right)\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form is not straightforward to achieve - returning hypergeometric solution
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 1412

```
dsolve((a*x^2+b*x+c)*diff(y(x),x$2)+(d*x+k)*diff(y(x),x)+(d-2*a)*y(x)=0,y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 15.225 (sec). Leaf size: 164

```
DSolve[(a*x^2+b*x+c)*y''[x]+(d*x+k)*y'[x]+(d-2*a)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to (x(ax+b)+c)^{1-\frac{d}{2a}} \exp\left(\frac{(bd-2ak)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}\right)\left(c_2\int_1^x \exp\left(\frac{(d-4a)\log(c+K[1](b+aK[1])) - \frac{2(bd-2}{}}{2a}\right) + c_1\right)$$

30.28 problem 176

Internal problem ID [11000]
Internal file name [`OUTPUT/10256_Sunday_December_31_2023_11_33_06_AM_29844681/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 176.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(a\,x^2+bx+c\right)y''+\left(kx+d\right)y'-yk=0}$$

30.28.1 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay''+By'+Cy=F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y=Bv$$

This results in

$$\begin{aligned} y' &= B'v+v'B \\ y'' &= B''v+B'v'+v''B+v'B' \\ &= v''B+2v'+B'+B''v \end{aligned}$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= a\,x^2 + bx + c \\
B &= kx + d \\
C &= -k \\
F &= 0
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(a\,x^2 + bx + c\right)(0) + (kx + d)(k) + (-k)(kx + d) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
\left(a\,x^2 + bx + c\right)(kx + d)\, v'' + \left(2k\left(a\,x^2 + bx + c\right) + (kx + d)^2\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
\left(a\,x^2 + bx + c\right)(kx + d)\, u'(x) + 2\left(k\left(a + \frac{k}{2}\right) x^2 + k(b + d)\, x + ck + \frac{d^2}{2}\right) u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{u(2ak\,x^2 + k^2x^2 + 2kxb + 2dkx + 2ck + d^2)}{\left(a\,x^2 + bx + c\right)(kx + d)}
\end{aligned}
$$

Where $f(x) = -\frac{2ak\,x^2+k^2x^2+2kxb+2dkx+2ck+d^2}{(a\,x^2+bx+c)(kx+d)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{2ak\,x^2+k^2x^2+2kxb+2dkx+2ck+d^2}{(a\,x^2+bx+c)\,(kx+d)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2ak\,x^2+k^2x^2+2kxb+2dkx+2ck+d^2}{(a\,x^2+bx+c)\,(kx+d)}\,dx \\
\ln(u) &= -\frac{k\ln(a\,x^2+bx+c)}{2a} - \frac{2\left(d-\frac{kb}{2a}\right)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}} - 2\ln(kx+d) + c_1 \\
u &= \mathrm{e}^{-\frac{k\ln\left(a\,x^2+bx+c\right)}{2a}-\frac{2\left(d-\frac{kb}{2a}\right)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}-2\ln(kx+d)+c_1} \\
&= c_1\mathrm{e}^{-\frac{k\ln\left(a\,x^2+bx+c\right)}{2a}-\frac{2\left(d-\frac{kb}{2a}\right)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}-2\ln(kx+d)}
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1\mathrm{e}^{-\frac{k\ln\left(a\,x^2+bx+c\right)}{2a}-\frac{2\left(d-\frac{kb}{2a}\right)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}-2\ln(kx+d)}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int c_1\mathrm{e}^{-\frac{k\ln\left(a\,x^2+bx+c\right)}{2a}-\frac{2\left(d-\frac{kb}{2a}\right)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}-2\ln(kx+d)}\,\mathrm{d}x \\
&= \int c_1\mathrm{e}^{-\frac{k\ln\left(a\,x^2+bx+c\right)}{2a}-\frac{2\left(d-\frac{kb}{2a}\right)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}-2\ln(kx+d)}dx + c_2
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (kx+d)\left(\int c_1\mathrm{e}^{-\frac{k\ln\left(a\,x^2+bx+c\right)}{2a}-\frac{2\left(d-\frac{kb}{2a}\right)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}-2\ln(kx+d)}dx + c_2\right) \\
&= (kx+d)\left(c_1\left(\int \frac{(a\,x^2+bx+c)^{-\frac{k}{2a}}\,\mathrm{e}^{-\frac{2\left(ad-\frac{bk}{2}\right)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}\,a}}}{(kx+d)^2}dx\right) + c_2\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = (kx+d)\left(c_1\left(\int \frac{(a\,x^2+bx+c)^{-\frac{k}{2a}}\,\mathrm{e}^{-\frac{2\left(ad-\frac{bk}{2}\right)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}\,a}}}{(kx+d)^2}dx\right) + c_2\right) \tag{1}
$$

Verification of solutions

$$y = (kx+d)\left(c_1\left(\int \frac{(a\,x^2+bx+c)^{-\frac{k}{2a}}\,\mathrm{e}^{-\frac{2\left(ad-\frac{bk}{2}\right)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}\,a}}}{(kx+d)^2}dx\right)+c_2\right)$$

Verified OK.

30.28.2 Maple step by step solution

Let's solve

$(a\,x^2+bx+c)\,y''+(kx+d)\,y'-yk=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{ky}{a\,x^2+bx+c}-\frac{(kx+d)y'}{a\,x^2+bx+c}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(kx+d)y'}{a\,x^2+bx+c}-\frac{ky}{a\,x^2+bx+c}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{kx+d}{a\,x^2+bx+c},P_3(x)=-\frac{k}{a\,x^2+bx+c}\right]$

  ◦ $\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)\cdot P_2(x)$ is analytic at $x=\frac{-b+\sqrt{-4ac+b^2}}{2a}$

  $\left(\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{-b+\sqrt{-4ac+b^2}}{2a}}=0$

  ◦ $\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{-b+\sqrt{-4ac+b^2}}{2a}$

  $\left(\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{-b+\sqrt{-4ac+b^2}}{2a}}=0$

  ◦ $x=\frac{-b+\sqrt{-4ac+b^2}}{2a}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=\frac{-b+\sqrt{-4ac+b^2}}{2a}$

- Multiply by denominators

  $(a\,x^2+bx+c)\,y''+(kx+d)\,y'-yk=0$

- Change variables using $x = u + \frac{-b+\sqrt{-4ac+b^2}}{2a}$ so that the regular singular point is at $u = 0$

$$\left(a\,u^2 + u\sqrt{-4ac+b^2}\right)\left(\frac{d^2}{du^2}y(u)\right) + \left(ku - \frac{kb}{2a} + \frac{k\sqrt{-4ac+b^2}}{2a} + d\right)\left(\frac{d}{du}y(u)\right) - ky(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

  - Shift index using $k- > k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

  - Shift index using $k- > k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$\frac{a_0 r\left(2\sqrt{-4ac+b^2}\,ar - 2\sqrt{-4ac+b^2}\,a + k\sqrt{-4ac+b^2} + 2ad - bk\right)u^{r-1}}{2a} + \left(\sum_{k=0}^{\infty}\left(\frac{a_{k+1}(k+1+r)\left(2\sqrt{-4ac+b^2}\,a(k+1) + 2\sqrt{-4ac+b^2}\,ar\right.}{2a}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\frac{r\left(2\sqrt{-4ac+b^2}\,ar - 2\sqrt{-4ac+b^2}\,a + k\sqrt{-4ac+b^2} + 2ad - bk\right)}{2a} = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{2\sqrt{-4ac+b^2}\,a - k\sqrt{-4ac+b^2} - 2ad + bk}{2\sqrt{-4ac+b^2}\,a}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$\frac{2a_{k+1}\left((k+r)a + \frac{k}{2}\right)(k+1+r)\sqrt{-4ac+b^2} + 2a_k(k+r)(k+r-1)a^2 + (2d(k+1+r)a_{k+1} + 2ka_k(k+r-1))a - a_{k+1}bk(k+1+r)}{2a} = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{2aa_k\left(a\,k^2 + 2akr + a\,r^2 - ak - ar + kk + kr - k\right)}{2\sqrt{-4ac+b^2}\,a\,k^2 + 4\sqrt{-4ac+b^2}\,akr + 2\sqrt{-4ac+b^2}\,a\,r^2 + 2\sqrt{-4ac+b^2}\,ak + 2\sqrt{-4ac+b^2}\,ar + \sqrt{-4ac+b^2}\,kk + \sqrt{-4ac+b^2}\,kr}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$$a_{k+1} = -\frac{2 a a_k \left(a\,k^2 - ak + kk - k\right)}{2\sqrt{-4ac+b^2}\,a\,k^2 + 2\sqrt{-4ac+b^2}\,ak + \sqrt{-4ac+b^2}\,kk + 2adk - bkk + k\sqrt{-4ac+b^2} + 2ad - bk}$$

- Apply recursion relation for $k = 0$

$$a_1 = \frac{2 a a_0 k}{k\sqrt{-4ac+b^2} + 2ad - bk}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot \left(1 + \frac{2aku}{k\sqrt{-4ac+b^2} + 2ad - bk}\right)$$

- Revert the change of variables $u = x - \frac{-b+\sqrt{-4ac+b^2}}{2a}$

$$\left[y = \frac{2 a_0 a (kx+d)}{k\sqrt{-4ac+b^2} + 2ad - bk}\right]$$

- Recursion relation for $r = \frac{2\sqrt{-4ac+b^2}\,a - k\sqrt{-4ac+b^2} - 2ad + bk}{2\sqrt{-4ac+b^2}\,a}$

$$a_{k+1} = -\frac{2 a a_k \left(a\,k^2 + \frac{k\left(2\sqrt{-4ac+b^2}\,a - k\sqrt{-4ac+b^2} - 2ad + bk\right)}{\sqrt{-4ac+b^2}} + \frac{\left(2\sqrt{-4ac+b^2}\,a - \cdots\right.}{4a\left(\cdots\right.}\right.}{2\sqrt{-4ac+b^2}\,a\,k^2 + 2k\left(2\sqrt{-4ac+b^2}\,a - k\sqrt{-4ac+b^2} - 2ad + bk\right) + \frac{\left(2\sqrt{-4ac+b^2}\,a - k\sqrt{-4ac+b^2} - 2ad + bk\right)^2}{2\sqrt{-4ac+b^2}\,a} + 2\sqrt{-4ac+b^2}\,ak + \cdots}$$

- Solution for $r = \frac{2\sqrt{-4ac+b^2}\,a - k\sqrt{-4ac+b^2} - 2ad + bk}{2\sqrt{-4ac+b^2}\,a}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k + \frac{2\sqrt{-4ac+b^2}\,a - k\sqrt{-4ac+b^2} - 2ad + bk}{2\sqrt{-4ac+b^2}\,a}}, a_{k+1} = -\frac{2 a a_k \left(\cdots\right.}{2\sqrt{-4ac+b^2}\,a\,k^2 + 2k\left(2\sqrt{-4ac+b^2}\,a - k\sqrt{-4ac+b^2} - 2ad + \cdots\right.}\right.$$

- Revert the change of variables $u = x - \frac{-b+\sqrt{-4ac+b^2}}{2a}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^{k + \frac{2\sqrt{-4ac+b^2}\,a - k\sqrt{-4ac+b^2} - 2ad + bk}{2\sqrt{-4ac+b^2}\,a}}, a_{k+1} = -\frac{\cdots}{2\sqrt{-4ac+b^2}\,a\,k^2 + 2k\left(2\sqrt{-4ac+b^2}\,a - \cdots\right.}\right.$$

- Combine solutions and rename parameters

$$\left[y = \frac{2 e_0 a (kx+d)}{k\sqrt{-4ac+b^2} + 2ad - bk} + \left(\sum_{m=0}^{\infty} f_m \left(x - \frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^{m + \frac{2\sqrt{-4ac+b^2}\,a - k\sqrt{-4ac+b^2} - 2ad + bk}{2\sqrt{-4ac+b^2}\,a}}\right), f_{m+1} = -\frac{\cdots}{2\sqrt{\cdots}}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 315

```
dsolve((a*x^2+b*x+c)*diff(y(x),x$2)+(k*x+d)*diff(y(x),x)-k*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(kx+d)$$

$$+c_2\left(2\sqrt{\frac{-4ac+b^2}{a^2}}\,x\,a^2+\sqrt{\frac{-4ac+b^2}{a^2}}\,ba-4ac+b^2\right)^{\frac{a\left(a-\frac{k}{2}\right)\sqrt{\frac{-4ac+b^2}{a^2}}+ad-\frac{kb}{2}}{\sqrt{\frac{-4ac+b^2}{a^2}}\,a^2}}\ \text{hypergeom}\left(\left[-\frac{k\sqrt{\frac{-4ac+\cdots}{a^2}}}{2a^2}\right.\right.$$

✓ Solution by Mathematica

Time used: 4.256 (sec). Leaf size: 107

```
DSolve[(a*x^2+b*x+c)*y''[x]+(k*x+d)*y'[x]-k*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(d+kx)\left(c_2\int_1^x \frac{\exp\left(\frac{(bk-2ad)\arctan\left(\frac{b+2aK[1]}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}\right)(c+K[1](b+aK[1]))^{-\frac{k}{2a}}}{(d+kK[1])^2}dK[1]+c_1\right)}{d}$$

30.29 problem 177

Internal problem ID [11001]
Internal file name [`OUTPUT/10257_Sunday_December_31_2023_11_33_11_AM_17031722/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 177.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__change__of__variable__on__x__method__1", "second__order__change__of__variable__on__x__method__2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{\left(a\,x^{2}+2bx+c\right)y''+\left(ax+b\right)y'+yd=0}$$

30.29.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$\left(a\,x^{2}+2bx+c\right)y''+\left(ax+b\right)y'+yd=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{ax+b}{a\,x^{2}+2bx+c} \\ q(x) &= \frac{d}{a\,x^{2}+2bx+c} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{ax+b}{a\,x^2+2bx+c}dx\right)}dx \\
&= \int e^{-\frac{\ln\left(a\,x^2+2bx+c\right)}{2}}dx \\
&= \int \frac{1}{\sqrt{a\,x^2+2bx+c}}dx \\
&= \frac{\ln\left(\frac{ax+b}{\sqrt{a}} + \sqrt{a\,x^2+2bx+c}\right)}{\sqrt{a}}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{d}{a\,x^2+2bx+c}}{\frac{1}{a\,x^2+2bx+c}} \\
&= d
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + d y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = d$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + d\, e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + d = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = d$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(d)}\\ &= \pm\sqrt{-d}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-d}\\ \lambda_2 &= -\sqrt{-d}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-d}\\ \lambda_2 &= -\sqrt{-d}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1\tau} + c_2 e^{\lambda_2\tau}\\ y(\tau) &= c_1 e^{(\sqrt{-d})\tau} + c_2 e^{(-\sqrt{-d})\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{\sqrt{-d}\,\tau} + c_2 e^{-\sqrt{-d}\,\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(\frac{ax+b}{\sqrt{a}} + \sqrt{a\,x^2+2bx+c}\right)^{\frac{\sqrt{-d}}{\sqrt{a}}} + c_2 \left(\frac{ax+b}{\sqrt{a}} + \sqrt{a\,x^2+2bx+c}\right)^{-\frac{\sqrt{-d}}{\sqrt{a}}}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(\frac{ax+b}{\sqrt{a}} + \sqrt{a\,x^2+2bx+c}\right)^{\frac{\sqrt{-d}}{\sqrt{a}}} + c_2 \left(\frac{ax+b}{\sqrt{a}} + \sqrt{a\,x^2+2bx+c}\right)^{-\frac{\sqrt{-d}}{\sqrt{a}}} \tag{1}$$

Verification of solutions

$$y = c_1 \left(\frac{ax+b}{\sqrt{a}} + \sqrt{a\,x^2+2bx+c}\right)^{\frac{\sqrt{-d}}{\sqrt{a}}} + c_2 \left(\frac{ax+b}{\sqrt{a}} + \sqrt{a\,x^2+2bx+c}\right)^{-\frac{\sqrt{-d}}{\sqrt{a}}}$$

Verified OK.

30.29.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$\left(a\,x^2+2bx+c\right) y'' + (ax+b)\, y' + yd = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{ax+b}{a\,x^2+2bx+c}$$
$$q(x) = \frac{d}{a\,x^2+2bx+c}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{d}{a\,x^2+2bx+c}}}{c} \\ \tau'' &= -\frac{d(2ax+2b)}{2c\sqrt{\frac{d}{a\,x^2+2bx+c}}\,(a\,x^2+2bx+c)^2} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{d(2ax+2b)}{2c\sqrt{\frac{d}{a\,x^2+2bx+c}}\,(a\,x^2+2bx+c)^2}+\frac{ax+b}{a\,x^2+2bx+c}\frac{\sqrt{\frac{d}{a\,x^2+2bx+c}}}{c}}{\left(\frac{\sqrt{\frac{d}{a\,x^2+2bx+c}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)''+p_1 y(\tau)'+q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau)+c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1\cos(c\tau)+c_2\sin(c\tau)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \sqrt{\frac{d}{a\,x^2+2bx+c}}dx}{c} \\ &= \frac{\sqrt{\frac{d}{a\,x^2+2bx+c}}\,\sqrt{a\,x^2+2bx+c}\,\ln\left(\frac{\sqrt{a\,x^2+2bx+c}\,\sqrt{a}+ax+b}{\sqrt{a}}\right)}{c\sqrt{a}} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\frac{\sqrt{d}\left(2\ln\left(\sqrt{a\,x^2+2bx+c}\,\sqrt{a}+ax+b\right)-\ln\left(a\right)\right)}{2\sqrt{a}}\right) + c_2 \sin\left(\frac{\sqrt{d}\left(2\ln\left(\sqrt{a\,x^2+2bx+c}\,\sqrt{a}+ax+b\right)-\ln\left(a\right)\right)}{2\sqrt{a}}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\frac{\sqrt{d}\left(2\ln\left(\sqrt{a\,x^2+2bx+c}\,\sqrt{a}+ax+b\right)-\ln\left(a\right)\right)}{2\sqrt{a}}\right) + c_2 \sin\left(\frac{\sqrt{d}\left(2\ln\left(\sqrt{a\,x^2+2bx+c}\,\sqrt{a}+ax+b\right)-\ln\left(a\right)\right)}{2\sqrt{a}}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(\frac{\sqrt{d}\left(2\ln\left(\sqrt{a\,x^2+2bx+c}\,\sqrt{a}+ax+b\right)-\ln\left(a\right)\right)}{2\sqrt{a}}\right) + c_2 \sin\left(\frac{\sqrt{d}\left(2\ln\left(\sqrt{a\,x^2+2bx+c}\,\sqrt{a}+ax+b\right)-\ln\left(a\right)\right)}{2\sqrt{a}}\right)$$

Verified OK.

30.29.3 Solving using Kovacic algorithm

Writing the ode as

$$\left(a\,x^2+2bx+c\right)y''+\left(ax+b\right)y'+yd=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= a\,x^2+2bx+c \\ B &= ax+b \\ C &= d \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2x^2 - 4ad\,x^2 - 2abx - 8bdx + 2ac - 3b^2 - 4cd}{4\left(a\,x^2 + 2bx + c\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -a^2x^2 - 4ad\,x^2 - 2abx - 8bdx + 2ac - 3b^2 - 4cd \\ t &= 4\left(a\,x^2 + 2bx + c\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-a^2x^2 - 4ad\,x^2 - 2abx - 8bdx + 2ac - 3b^2 - 4cd}{4\left(a\,x^2 + 2bx + c\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 170: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(a\,x^2 + 2bx + c)^2$. There is a pole at $x = -\frac{b-\sqrt{-ac+b^2}}{a}$ of order 2. There is a pole at $x = -\frac{b+\sqrt{-ac+b^2}}{a}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned} r &= \frac{-\left(-b+\sqrt{-ac+b^2}\right)^2 - \frac{4\left(-b+\sqrt{-ac+b^2}\right)^2 d}{a} - 2\left(-b+\sqrt{-ac+b^2}\right) b - \frac{8\left(-b+\sqrt{-ac+b^2}\right) bd}{a} + 2ac - 3b^2 - 4cd}{16\left(-ac+b^2\right)\left(x - \frac{-b+\sqrt{-ac+b^2}}{a}\right)^2} \\ &+ \frac{-\left(b+\sqrt{-ac+b^2}\right)^2 - \frac{4\left(b+\sqrt{-ac+b^2}\right)^2 d}{a} + 2\left(b+\sqrt{-ac+b^2}\right) b + \frac{8\left(b+\sqrt{-ac+b^2}\right) bd}{a} + 2ac - 3b^2 - 4cd}{16\left(-ac+b^2\right)\left(x + \frac{b+\sqrt{-ac+b^2}}{a}\right)^2} \\ &+ \frac{-\left(-b+\sqrt{-ac+b^2}\right)^2 a - 4\left(-b+\sqrt{-ac+b^2}\right)^2 d - 2\left(-b+\sqrt{-ac+b^2}\right) ab - 8\left(-b+\sqrt{-ac+b^2}\right) bd}{16\left(-ac+b^2\right)^{\frac{3}{2}}\left(x - \frac{-b+\sqrt{-ac+b^2}}{a}\right)} \\ &- \frac{-\left(b+\sqrt{-ac+b^2}\right)^2 a - 4\left(b+\sqrt{-ac+b^2}\right)^2 d + 2\left(b+\sqrt{-ac+b^2}\right) ab + 8\left(b+\sqrt{-ac+b^2}\right) bd - 2a^2c}{16\left(-ac+b^2\right)^{\frac{3}{2}}\left(x + \frac{b+\sqrt{-ac+b^2}}{a}\right)} \end{aligned}$$

For the pole at $x = -\frac{b-\sqrt{-ac+b^2}}{a}$ let b be the coefficient of $\frac{1}{\left(x + \frac{b-\sqrt{-ac+b^2}}{a}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -\frac{b+\sqrt{-ac+b^2}}{a}$ let b be the coefficient of $\frac{1}{\left(x+\frac{b+\sqrt{-ac+b^2}}{a}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-a^2x^2 - 4ad\,x^2 - 2abx - 8bdx + 2ac - 3b^2 - 4cd}{4\left(a\,x^2 + 2bx + c\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
$-\frac{b-\sqrt{-ac+b^2}}{a}$	2	$\{1, 2, 3\}$
$-\frac{b+\sqrt{-ac+b^2}}{a}$	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{1}{\left(x-\left(-\frac{b-\sqrt{-ac+b^2}}{a}\right)\right)}+\frac{1}{\left(x-\left(-\frac{b+\sqrt{-ac+b^2}}{a}\right)\right)}\right)\\ &= \frac{1}{2x+\frac{2\left(b-\sqrt{-ac+b^2}\right)}{a}}+\frac{1}{2x+\frac{2\left(b+\sqrt{-ac+b^2}\right)}{a}}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p'''+3\theta p''+\left(3\theta^2+3\theta'-4r\right)p'+\left(\theta''+3\theta\theta'+\theta^3-4r\theta-2r'\right)p=0 \tag{1A}$$

Since $d=0$, then letting

$$p=1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0=0$$

And solving for p gives

$$p=1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta+\frac{p'}{p}\\ &= \frac{1}{2x+\frac{2\left(b-\sqrt{-ac+b^2}\right)}{a}}+\frac{1}{2x+\frac{2\left(b+\sqrt{-ac+b^2}\right)}{a}}\end{aligned}$$

Let ω be the solution of

$$\omega^2-\phi\omega+\left(\frac{1}{2}\phi'+\frac{1}{2}\phi^2-r\right)=0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2-\left(\frac{1}{2x+\frac{2\left(b-\sqrt{-ac+b^2}\right)}{a}}+\frac{1}{2x+\frac{2\left(b+\sqrt{-ac+b^2}\right)}{a}}\right)w+\frac{\left(a^2x^2+\left(4d\,x^2+2bx\right)a+8bdx+b^2+4cd\right)a^2}{4\left(ax+b-\sqrt{-ac+b^2}\right)^2\left(ax+\sqrt{-ac+b^2}+b\right)^2}=0$$

Solving for ω gives

$$\omega = \frac{ax + 2\sqrt{-d\left(a\,x^{2}+2bx+c\right)} + b}{2a\,x^{2}+4bx+2c}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{ax+2\sqrt{-d\left(a\,x^{2}+2bx+c\right)}+b}{2a\,x^{2}+4bx+2c}dx} \\
&= \left(a\,x^{2}+2bx+c\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{d \arctan\left(\frac{\sqrt{ad}\,(ax+b)}{a\sqrt{-\frac{d\left(ax+b-\sqrt{-ac+b^{2}}\right)\left(ax+\sqrt{-ac+b^{2}}+b\right)}{a}}}\right)}{\sqrt{ad}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{ax+b}{a\,x^{2}+2bx+c}\,dx} \\
&= z_1 e^{-\frac{\ln\left(a\,x^{2}+2bx+c\right)}{4}} \\
&= z_1\left(\frac{1}{\left(a\,x^{2}+2bx+c\right)^{\frac{1}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{d \arctan\left((ax+b)\sqrt{-\frac{1}{\left(ax+b-\sqrt{-ac+b^{2}}\right)\left(ax+\sqrt{-ac+b^{2}}+b\right)}}\right)}{\sqrt{ad}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^{2}}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{ax+b}{a\,x^2+2bx+c}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln\left(a\,x^2+2bx+c\right)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{2d\arctan\left((ax+b)\sqrt{-\frac{1}{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}}\right)}{\sqrt{ad}}}}{\sqrt{a\,x^2+2bx+c}}\,dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{d\arctan\left((ax+b)\sqrt{-\frac{1}{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}}\right)}{\sqrt{ad}}} \right) \\
&\quad + c_2 \left(\mathrm{e}^{-\frac{d\arctan\left((ax+b)\sqrt{-\frac{1}{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}}\right)}{\sqrt{ad}}} \right) \left(\int \frac{\mathrm{e}^{\frac{2d\arctan\left((ax+b)\sqrt{-\frac{1}{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}}\right)}{\sqrt{ad}}}}{\sqrt{a\,x^2+2bx+c}}\,dx \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \mathrm{e}^{-\frac{d\arctan\left((ax+b)\sqrt{-\frac{1}{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}}\right)}{\sqrt{ad}}} \\
&\quad + c_2 \mathrm{e}^{-\frac{d\arctan\left((ax+b)\sqrt{-\frac{1}{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}}\right)}{\sqrt{ad}}} \left(\int \frac{\mathrm{e}^{\frac{2d\arctan\left((ax+b)\sqrt{-\frac{1}{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}}\right)}{\sqrt{ad}}}}{\sqrt{a\,x^2+2bx+c}}\,dx \right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = c_1 e^{-\frac{d \arctan\left((ax+b)\sqrt{-\frac{1}{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}}\right)}{\sqrt{ad}}} + c_2 e^{-\frac{d \arctan\left((ax+b)\sqrt{-\frac{1}{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}}\right)}{\sqrt{ad}}} \left(\int \frac{e^{\frac{2d \arctan\left((ax+b)\sqrt{-\frac{1}{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}}\right)}{\sqrt{ad}}}}{\sqrt{a\,x^2+2bx+c}}dx\right)$$

Verified OK.

30.29.4 Maple step by step solution

Let's solve

$(a\,x^2+2bx+c)\,y''+(ax+b)\,y'+yd=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{dy}{a\,x^2+2bx+c}-\frac{(ax+b)y'}{a\,x^2+2bx+c}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(ax+b)y'}{a\,x^2+2bx+c}+\frac{dy}{a\,x^2+2bx+c}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{ax+b}{a\,x^2+2bx+c}, P_3(x)=\frac{d}{a\,x^2+2bx+c}\right]$

  - $\left(x-\frac{-b+\sqrt{-ac+b^2}}{a}\right)\cdot P_2(x)$ is analytic at $x=\frac{-b+\sqrt{-ac+b^2}}{a}$

    $\left(\left(x-\frac{-b+\sqrt{-ac+b^2}}{a}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{-b+\sqrt{-ac+b^2}}{a}}=0$

  - $\left(x-\frac{-b+\sqrt{-ac+b^2}}{a}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{-b+\sqrt{-ac+b^2}}{a}$

    $\left(\left(x-\frac{-b+\sqrt{-ac+b^2}}{a}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{-b+\sqrt{-ac+b^2}}{a}}=0$

  - $x=\frac{-b+\sqrt{-ac+b^2}}{a}$ is a regular singular point

    Check to see if x_0 is a regular singular point

$x_0 = \frac{-b+\sqrt{-ac+b^2}}{a}$

- Multiply by denominators

$(a\,x^2+2bx+c)\,y''+(ax+b)\,y'+yd=0$

- Change variables using $x=u+\frac{-b+\sqrt{-ac+b^2}}{a}$ so that the regular singular point is at $u=0$

$\left(a\,u^2+2u\sqrt{-ac+b^2}\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(au+\sqrt{-ac+b^2}\right)\left(\frac{d}{du}y(u)\right)+dy(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  ○ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $\sqrt{-ac+b^2}\,a_0r(-1+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(\sqrt{-ac+b^2}\,a_{k+1}(k+1+r)\,(2k+1+2r)+a_k(a\,k^2+2a\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$\sqrt{-ac+b^2}\,r(-1+2r)=0$

- Values of r that satisfy the indicial equation

$r\in\left\{0,\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$2a_{k+1}(k+1+r)\left(k+r+\frac{1}{2}\right)\sqrt{-ac+b^2}+a_k(a\,k^2+2akr+a\,r^2+d)=0$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k (a\,k^2+2akr+a\,r^2+d)}{\sqrt{-ac+b^2}\,(2k^2+4kr+2r^2+3k+3r+1)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k (a\,k^2+d)}{\sqrt{-ac+b^2}\,(2k^2+3k+1)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = -\frac{a_k (a\,k^2+d)}{\sqrt{-ac+b^2}\,(2k^2+3k+1)}\right]$$

- Revert the change of variables $u = x - \frac{-b+\sqrt{-ac+b^2}}{a}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{-b+\sqrt{-ac+b^2}}{a}\right)^k, a_{k+1} = -\frac{a_k (a\,k^2+d)}{\sqrt{-ac+b^2}\,(2k^2+3k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{a_k (a\,k^2+ak+\frac{1}{4}a+d)}{\sqrt{-ac+b^2}\,(2k^2+5k+3)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k (a\,k^2+ak+\frac{1}{4}a+d)}{\sqrt{-ac+b^2}\,(2k^2+5k+3)}\right]$$

- Revert the change of variables $u = x - \frac{-b+\sqrt{-ac+b^2}}{a}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{-b+\sqrt{-ac+b^2}}{a}\right)^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k (a\,k^2+ak+\frac{1}{4}a+d)}{\sqrt{-ac+b^2}\,(2k^2+5k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} e_k \left(x - \frac{-b+\sqrt{-ac+b^2}}{a}\right)^k\right) + \left(\sum_{k=0}^{\infty} f_k \left(x - \frac{-b+\sqrt{-ac+b^2}}{a}\right)^{k+\frac{1}{2}}\right), e_{1+k} = -\frac{e_k (a\,k^2+d)}{\sqrt{-ac+b^2}\,(2k^2+3k+1)},\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 81

```
dsolve((a*x^2+2*b*x+c)*diff(y(x),x$2)+(a*x+b)*diff(y(x),x)+d*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \left(\frac{\sqrt{a\,x^2+2bx+c}\,\sqrt{a}+ax+b}{\sqrt{a}}\right)^{\frac{i\sqrt{d}}{\sqrt{a}}} + c_2 \left(\frac{\sqrt{a\,x^2+2bx+c}\,\sqrt{a}+ax+b}{\sqrt{a}}\right)^{-\frac{i\sqrt{d}}{\sqrt{a}}}$$

✓ Solution by Mathematica

Time used: 0.404 (sec). Leaf size: 93

```
DSolve[(a*x^2+2*b*x+c)*y''[x]+(a*x+b)*y'[x]+d*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to c_1 \cos\left(\frac{\sqrt{d}\log\left(-\sqrt{a}\sqrt{ax^2+2bx+c}+ax+b\right)}{\sqrt{a}}\right) - c_2 \sin\left(\frac{\sqrt{d}\log\left(-\sqrt{a}\sqrt{ax^2+2bx+c}+ax+b\right)}{\sqrt{a}}\right)$$

30.30 problem 178

Internal problem ID [11002]
Internal file name [`OUTPUT/10258_Sunday_December_31_2023_11_33_16_AM_13214284/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 178.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(a\,x^2 + 2bx + c)\,y'' + 3(ax + b)\,y' + yd = 0}$$

30.30.1 Solving using Kovacic algorithm

Writing the ode as

$$(a\,x^2 + 2bx + c)\,y'' + (3ax + 3b)\,y' + yd = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= a\,x^2 + 2bx + c \\ B &= 3ax + 3b \\ C &= d \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3a^2x^2 - 4ad\,x^2 + 6abx - 8bdx + 6ac - 3b^2 - 4cd}{4\,(a\,x^2 + 2bx + c)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3a^2x^2 - 4ad\,x^2 + 6abx - 8bdx + 6ac - 3b^2 - 4cd \\ t &= 4\left(a\,x^2 + 2bx + c\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3a^2x^2 - 4ad\,x^2 + 6abx - 8bdx + 6ac - 3b^2 - 4cd}{4\,(a\,x^2 + 2bx + c)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 172: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(a\,x^2 + 2bx + c)^2$. There is a pole at $x = -\frac{b-\sqrt{-ac+b^2}}{a}$ of order 2. There is a pole at $x = -\frac{b+\sqrt{-ac+b^2}}{a}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned} r &= \frac{3\left(-b+\sqrt{-ac+b^2}\right)^2 - \frac{4\left(-b+\sqrt{-ac+b^2}\right)^2 d}{a} + 6\left(-b+\sqrt{-ac+b^2}\right) b - \frac{8\left(-b+\sqrt{-ac+b^2}\right) bd}{a} + 6ac - 3b^2 - 4cd}{16\left(-ac+b^2\right)\left(x - \frac{-b+\sqrt{-ac+b^2}}{a}\right)^2} \\ &+ \frac{3\left(b+\sqrt{-ac+b^2}\right)^2 - \frac{4\left(b+\sqrt{-ac+b^2}\right)^2 d}{a} - 6\left(b+\sqrt{-ac+b^2}\right) b + \frac{8\left(b+\sqrt{-ac+b^2}\right) bd}{a} + 6ac - 3b^2 - 4cd}{16\left(-ac+b^2\right)\left(x + \frac{b+\sqrt{-ac+b^2}}{a}\right)^2} \\ &+ \frac{3\left(-b+\sqrt{-ac+b^2}\right)^2 a - 4\left(-b+\sqrt{-ac+b^2}\right)^2 d + 6\left(-b+\sqrt{-ac+b^2}\right) ab - 8\left(-b+\sqrt{-ac+b^2}\right) bd -}{16\left(-ac+b^2\right)^{\frac{3}{2}}\left(x - \frac{-b+\sqrt{-ac+b^2}}{a}\right)} \\ &- \frac{3\left(b+\sqrt{-ac+b^2}\right)^2 a - 4\left(b+\sqrt{-ac+b^2}\right)^2 d - 6\left(b+\sqrt{-ac+b^2}\right) ab + 8\left(b+\sqrt{-ac+b^2}\right) bd - 6a^2c -}{16\left(-ac+b^2\right)^{\frac{3}{2}}\left(x + \frac{b+\sqrt{-ac+b^2}}{a}\right)} \end{aligned}$$

For the pole at $x = -\frac{b-\sqrt{-ac+b^2}}{a}$ let b be the coefficient of $\frac{1}{\left(x+\frac{b-\sqrt{-ac+b^2}}{a}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -\frac{b+\sqrt{-ac+b^2}}{a}$ let b be the coefficient of $\frac{1}{\left(x+\frac{b+\sqrt{-ac+b^2}}{a}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3a^2x^2 - 4ad\,x^2 + 6abx - 8bdx + 6ac - 3b^2 - 4cd}{4\,(a\,x^2 + 2bx + c)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{-2, 2, 6\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
$-\frac{b-\sqrt{-ac+b^2}}{a}$	2	$\{1, 2, 3\}$
$-\frac{b+\sqrt{-ac+b^2}}{a}$	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{-2, 2, 6\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{1}{\left(x-\left(-\frac{b-\sqrt{-ac+b^2}}{a}\right)\right)}+\frac{1}{\left(x-\left(-\frac{b+\sqrt{-ac+b^2}}{a}\right)\right)}\right)\\ &= \frac{1}{2x+\frac{2\left(b-\sqrt{-ac+b^2}\right)}{a}}+\frac{1}{2x+\frac{2\left(b+\sqrt{-ac+b^2}\right)}{a}}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p'''+3\theta p''+\left(3\theta^2+3\theta'-4r\right)p'+\left(\theta''+3\theta\theta'+\theta^3-4r\theta-2r'\right)p=0 \tag{1A}$$

Since $d=0$, then letting

$$p=1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0=0$$

And solving for p gives

$$p=1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta+\frac{p'}{p}\\ &= \frac{1}{2x+\frac{2\left(b-\sqrt{-ac+b^2}\right)}{a}}+\frac{1}{2x+\frac{2\left(b+\sqrt{-ac+b^2}\right)}{a}}\end{aligned}$$

Let ω be the solution of

$$\omega^2-\phi\omega+\left(\frac{1}{2}\phi'+\frac{1}{2}\phi^2-r\right)=0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2-\left(\frac{1}{2x+\frac{2\left(b-\sqrt{-ac+b^2}\right)}{a}}+\frac{1}{2x+\frac{2\left(b+\sqrt{-ac+b^2}\right)}{a}}\right)w\\ -\frac{3a^2\left(a^2x^2+\left(-\frac{4}{3}d\,x^2+2bx+\frac{4}{3}c\right)a-\frac{8bdx}{3}-\frac{b^2}{3}-\frac{4cd}{3}\right)}{4\left(ax+b-\sqrt{-ac+b^2}\right)^2\left(ax+\sqrt{-ac+b^2}+b\right)^2}=0$$

Solving for ω gives

$$\omega = \frac{ax + 2\sqrt{a^2x^2 - ad\,x^2 + 2abx - 2bdx + ac - cd} + b}{2a\,x^2 + 4bx + 2c}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= e^{\int \omega \,dx} \\
&= \mathrm{e}^{\int \frac{ax+2\sqrt{a^2x^2-ad\,x^2+2abx-2bdx+ac-cd}+b}{2a\,x^2+4bx+2c}dx} \\
&= \left(a\,x^2 + 2bx + c\right)^{\frac{1}{4}} \left(\frac{\sqrt{\frac{(a-d)\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}{a}}\,\sqrt{a\,(a-d)} + (a-d)\,(ax+b)}{\sqrt{a\,(a-d)}}\right)^{\frac{a-d}{\sqrt{a(a-d)}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3ax+3b}{a\,x^2+2bx+c}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(a\,x^2+2bx+c\right)}{4}} \\
&= z_1 \left(\frac{1}{\left(a\,x^2+2bx+c\right)^{\frac{3}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(\frac{(a-d)\left(ax+\sqrt{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}+b\right)}{\sqrt{a(a-d)}}\right)^{\frac{a-d}{\sqrt{a(a-d)}}}}{\sqrt{a\,x^2+2bx+c}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3ax+3b}{a\,x^2+2bx+c}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln\left(a\,x^2+2bx+c\right)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\left(\frac{(a-d)\left(ax+\sqrt{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}+b\right)}{\sqrt{a(a-d)}}\right)^{-\frac{2(a-d)}{\sqrt{a(a-d)}}}}{\sqrt{a\,x^2+2bx+c}}\,dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(\frac{(a-d)\left(ax+\sqrt{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}+b\right)}{\sqrt{a(a-d)}}\right)^{\frac{a-d}{\sqrt{a(a-d)}}}}{\sqrt{a\,x^2+2bx+c}} \right) \\
&\quad + c_2 \left(\frac{\left(\frac{(a-d)\left(ax+\sqrt{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}+b\right)}{\sqrt{a(a-d)}}\right)^{\frac{a-d}{\sqrt{a(a-d)}}}}{\sqrt{a\,x^2+2bx+c}} \right) \left(\int \frac{\left(\frac{(a-d)\left(ax+\sqrt{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac\right.}\right.}{\sqrt{a(a-d)}}\right.}{\sqrt{a\,x^2+2bx+c}} \right.
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \left(\frac{(a-d)\left(ax+\sqrt{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}+b\right)}{\sqrt{a(a-d)}}\right)^{\frac{a-d}{\sqrt{a(a-d)}}}}{\sqrt{a\,x^2+2bx+c}} + \frac{c_2 \left(\frac{(a-d)\left(ax+\sqrt{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}+b\right)}{\sqrt{a(a-d)}}\right)^{\frac{a-d}{\sqrt{a(a-d)}}} \left(\int \frac{\left(\frac{(a-d)\left(ax+\sqrt{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}+b\right)}{\sqrt{a(a-d)}}\right. \cdots}{\sqrt{a\,x^2+2bx+c}} \cdots\right.}{\sqrt{a\,x^2+2bx+c}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \left(\frac{(a-d)\left(ax+\sqrt{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}+b\right)}{\sqrt{a(a-d)}}\right)^{\frac{a-d}{\sqrt{a(a-d)}}}}{\sqrt{a\,x^2+2bx+c}} + \frac{c_2 \left(\frac{(a-d)\left(ax+\sqrt{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}+b\right)}{\sqrt{a(a-d)}}\right)^{\frac{a-d}{\sqrt{a(a-d)}}} \left(\int \frac{\left(\frac{(a-d)\left(ax+\sqrt{\left(ax+b-\sqrt{-ac+b^2}\right)\left(ax+\sqrt{-ac+b^2}+b\right)}+b\right)}{\sqrt{a(a-d)}}\right. \cdots}{\sqrt{a\,x^2+2bx+c}} \cdots\right.}{\sqrt{a\,x^2+2bx+c}}$$

Verified OK.

30.30.2 Maple step by step solution

Let's solve

$(a\,x^2+2bx+c)\,y''+(3ax+3b)\,y'+yd=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{dy}{a\,x^2+2bx+c}-\frac{3(ax+b)y'}{a\,x^2+2bx+c}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3(ax+b)y'}{a\,x^2+2bx+c}+\frac{dy}{a\,x^2+2bx+c}=0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3(ax+b)}{a\,x^2+2bx+c}, P_3(x) = \frac{d}{a\,x^2+2bx+c}\right]$

- $\left(x - \frac{-b+\sqrt{-ac+b^2}}{a}\right) \cdot P_2(x)$ is analytic at $x = \frac{-b+\sqrt{-ac+b^2}}{a}$

  $\left(\left(x - \frac{-b+\sqrt{-ac+b^2}}{a}\right) \cdot P_2(x)\right)\bigg|_{x=\frac{-b+\sqrt{-ac+b^2}}{a}} = 0$

- $\left(x - \frac{-b+\sqrt{-ac+b^2}}{a}\right)^2 \cdot P_3(x)$ is analytic at $x = \frac{-b+\sqrt{-ac+b^2}}{a}$

  $\left(\left(x - \frac{-b+\sqrt{-ac+b^2}}{a}\right)^2 \cdot P_3(x)\right)\bigg|_{x=\frac{-b+\sqrt{-ac+b^2}}{a}} = 0$

- $x = \frac{-b+\sqrt{-ac+b^2}}{a}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = \frac{-b+\sqrt{-ac+b^2}}{a}$

- Multiply by denominators

  $(a\,x^2 + 2bx + c)\,y'' + (3ax + 3b)\,y' + yd = 0$

- Change variables using $x = u + \frac{-b+\sqrt{-ac+b^2}}{a}$ so that the regular singular point is at $u = 0$

  $\left(a\,u^2 + 2u\sqrt{-ac+b^2}\right)\left(\frac{d^2}{du^2}y(u)\right) + \left(3au + 3\sqrt{-ac+b^2}\right)\left(\frac{d}{du}y(u)\right) + dy(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m} \left(k+2-m+r \right) \left(k+1-m+r \right) u^{k+r}$$

  Rewrite ODE with series expansions

  $$\sqrt{-a c+b^{2}}\, a_{0} \left(1+2 r \right) r \,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(\sqrt{-a c+b^{2}}\, a_{k+1} \left(2 k+3+2 r \right) \left(k+1+r \right)+a_{k} \left(a \,k^{2}+2 a k\right.\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$\sqrt{-a c+b^{2}}\, \left(1+2 r \right) r =0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2 a_{k+1} \left(k+1+r \right) \left(k+r+\frac{3}{2}\right) \sqrt{-a c+b^{2}}+\left(a \,k^{2}+2 a \left(r+1\right) k+a \,r^{2}+2 a r+d \right) a_{k}=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1}=-\frac{a_{k} \left(a \,k^{2}+2 a k r+a \,r^{2}+2 a k+2 a r+d \right)}{\sqrt{-a c+b^{2}}\, \left(2 k^{2}+4 k r+2 r^{2}+5 k+5 r+3\right)}$$

- Recursion relation for $r =0$

  $$a_{k+1}=-\frac{a_{k} \left(a \,k^{2}+2 a k+d \right)}{\sqrt{-a c+b^{2}}\, \left(2 k^{2}+5 k+3\right)}$$

- Solution for $r =0$

  $$\left[y \left(u \right)=\sum_{k=0}^{\infty}a_{k} u^{k}, a_{k+1}=-\frac{a_{k} \left(a \,k^{2}+2 a k+d \right)}{\sqrt{-a c+b^{2}}\, \left(2 k^{2}+5 k+3\right)}\right]$$

- Revert the change of variables $u =x -\frac{-b+\sqrt{-a c+b^{2}}}{a}$

  $$\left[y=\sum_{k=0}^{\infty}a_{k} \left(x -\frac{-b+\sqrt{-a c+b^{2}}}{a}\right)^{k}, a_{k+1}=-\frac{a_{k} \left(a \,k^{2}+2 a k+d \right)}{\sqrt{-a c+b^{2}}\, \left(2 k^{2}+5 k+3\right)}\right]$$

- Recursion relation for $r =-\frac{1}{2}$

  $$a_{k+1}=-\frac{a_{k} \left(a \,k^{2}+a k-\frac{3}{4} a+d \right)}{\sqrt{-a c+b^{2}}\, \left(2 k^{2}+3 k+1\right)}$$

- Solution for $r =-\frac{1}{2}$

  $$\left[y \left(u \right)=\sum_{k=0}^{\infty}a_{k} u^{k -\frac{1}{2}}, a_{k+1}=-\frac{a_{k} \left(a \,k^{2}+a k-\frac{3}{4} a+d \right)}{\sqrt{-a c+b^{2}}\, \left(2 k^{2}+3 k+1\right)}\right]$$

- Revert the change of variables $u =x -\frac{-b+\sqrt{-a c+b^{2}}}{a}$

  $$\left[y=\sum_{k=0}^{\infty}a_{k} \left(x -\frac{-b+\sqrt{-a c+b^{2}}}{a}\right)^{k -\frac{1}{2}}, a_{k+1}=-\frac{a_{k} \left(a \,k^{2}+a k-\frac{3}{4} a+d \right)}{\sqrt{-a c+b^{2}}\, \left(2 k^{2}+3 k+1\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} e_k \left(x - \frac{-b+\sqrt{-ac+b^2}}{a}\right)^k\right) + \left(\sum_{k=0}^{\infty} f_k \left(x - \frac{-b+\sqrt{-ac+b^2}}{a}\right)^{k-\frac{1}{2}}\right), e_{1+k} = -\frac{e_k\left(a\,k^2+2ak+d\right)}{\sqrt{-ac+b^2}\,(2k^2+5k+3)},\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.219 (sec). Leaf size: 88

```
dsolve((a*x^2+2*b*x+c)*diff(y(x),x$2)+3*(a*x+b)*diff(y(x),x)+d*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2\left(\sqrt{a\,(a\,x^2+2bx+c)}+ax+b\right)^{-\frac{\sqrt{-d+a}}{\sqrt{a}}} + c_1\left(\sqrt{a\,(a\,x^2+2bx+c)}+ax+b\right)^{\frac{\sqrt{-d+a}}{\sqrt{a}}}}{\sqrt{a\,x^2+2bx+c}}$$

✓ Solution by Mathematica

Time used: 0.171 (sec). Leaf size: 152

```
DSolve[(a*x^2+2*b*x+c)*y''[x]+3*(a*x+b)*y'[x]+d*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{c_1 P^{\frac{1}{2}}_{\frac{\sqrt{a-d}}{\sqrt{a}}-\frac{1}{2}}\left(\frac{\sqrt{-b^2-ac}(b+ax)}{a\sqrt{c^2-\frac{b^4}{a^2}}}\right) + c_2 Q^{\frac{1}{2}}_{\frac{\sqrt{a-d}}{\sqrt{a}}-\frac{1}{2}}\left(\frac{\sqrt{-b^2-ac}(b+ax)}{a\sqrt{c^2-\frac{b^4}{a^2}}}\right)}{\sqrt[4]{x(ax+2b)+c}}$$

30.31 problem 179

Internal problem ID [11003]
Internal file name [`OUTPUT/10259_Sunday_December_31_2023_11_33_17_AM_58603201/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2+bx+c)y''+f(x)y'+g(x)y=0$
Problem number: 179.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{(a_2x^2+b_2x+c_2)\,y''+(b_1x+c_1)\,y'+c_0y=0}$$

30.31.1 Maple step by step solution

Let's solve

$(a_2x^2+b_2x+c_2)\,y''+(b_1x+c_1)\,y'+c_0y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{c_0y}{a_2x^2+b_2x+c_2}-\frac{(b_1x+c_1)y'}{a_2x^2+b_2x+c_2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(b_1x+c_1)y'}{a_2x^2+b_2x+c_2}+\frac{c_0y}{a_2x^2+b_2x+c_2}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

$$\left[P_2(x) = \frac{b_1 x + c_1}{a_2 x^2 + b_2 x + c_2}, P_3(x) = \frac{c_0}{a_2 x^2 + b_2 x + c_2}\right]$$

- $\left(x - \frac{-b_2 + \sqrt{-4c_2 a_2 + b_2^2}}{2a_2}\right) \cdot P_2(x)$ is analytic at $x = \frac{-b_2 + \sqrt{-4c_2 a_2 + b_2^2}}{2a_2}$

  $$\left(\left(x - \frac{-b_2 + \sqrt{-4c_2 a_2 + b_2^2}}{2a_2}\right) \cdot P_2(x)\right)\bigg|_{x = \frac{-b_2 + \sqrt{-4c_2 a_2 + b_2^2}}{2a_2}} = 0$$

- $\left(x - \frac{-b_2 + \sqrt{-4c_2 a_2 + b_2^2}}{2a_2}\right)^2 \cdot P_3(x)$ is analytic at $x = \frac{-b_2 + \sqrt{-4c_2 a_2 + b_2^2}}{2a_2}$

  $$\left(\left(x - \frac{-b_2 + \sqrt{-4c_2 a_2 + b_2^2}}{2a_2}\right)^2 \cdot P_3(x)\right)\bigg|_{x = \frac{-b_2 + \sqrt{-4c_2 a_2 + b_2^2}}{2a_2}} = 0$$

- $x = \frac{-b_2 + \sqrt{-4c_2 a_2 + b_2^2}}{2a_2}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $$x_0 = \frac{-b_2 + \sqrt{-4c_2 a_2 + b_2^2}}{2a_2}$$

- Multiply by denominators

  $$\left(a_2 x^2 + b_2 x + c_2\right) y'' + \left(b_1 x + c_1\right) y' + c_0 y = 0$$

- Change variables using $x = u + \frac{-b_2 + \sqrt{-4c_2 a_2 + b_2^2}}{2a_2}$ so that the regular singular point is at $u = 0$

  $$\left(a_2 u^2 + u\sqrt{-4c_2 a_2 + b_2^2}\right)\left(\frac{d^2}{du^2} y(u)\right) + \left(b_1 u - \frac{b_1 b_2}{2a_2} + \frac{b_1 \sqrt{-4c_2 a_2 + b_2^2}}{2a_2} + c_1\right)\left(\frac{d}{du} y(u)\right) + c_0 y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion for $m = 1..2$

$$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right) = \sum_{k =0}^{\infty} a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- Shift index using $k->k +2-m$

$$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right) = \sum_{k =-2+m}^{\infty} a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$\frac{a_{0} r \left(2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2} r -2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2}+\sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, b_{1}+2 c_{1} a_{2}-b_{1} b_{2}\right) u^{-1+r}}{2 a_{2}}+\left(\sum_{k =0}^{\infty} \left(\frac{a_{k +1} \left(k +1+r \right) \left(2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2}\right.}{}\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$\frac{r \left(2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2} r -2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2}+\sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, b_{1}+2 c_{1} a_{2}-b_{1} b_{2}\right)}{2 a_{2}} = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2}-\sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, b_{1}-2 c_{1} a_{2}+b_{1} b_{2}}{2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2}}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$\frac{2 \left(\left(k +r \right) a_{2}+\frac{b_{1}}{2}\right) a_{k +1} \left(k +1+r \right) \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}+2 a_{k} \left(k +r \right) \left(k +r -1\right) a_{2}^{2}+\left(2 c_{1} \left(k +1+r \right) a_{k +1}+2 a_{k} \left(b_{1} k +b_{1} r +c_{0}\right)\right) a_{2}-b_{1} b_{2} a_{k +1} \left(k +1+r \right)}{2 a_{2}}$$

- Recursion relation that defines series solution to ODE

$$a_{k +1} = -\frac{2 a_{2} a_{k} \left(a_{2} k^{2}+2 a_{2} k r +a_{2} r^{2}-a_{2} k -a_{2} r +b_{1}\right.}{2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2} k^{2}+4 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2} k r +2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2} r^{2}+2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2} k +2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2} r +\sqrt{-4 c_{2} a_{2}+}}$$

- Recursion relation for $r =0$

$$a_{k +1} = -\frac{2 a_{2} a_{k} \left(a_{2} k^{2}-a_{2} k +b_{1} k +c_{0}\right)}{2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2} k^{2}+2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2} k +\sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, b_{1} k +2 c_{1} a_{2} k -b_{1} b_{2} k +\sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, b_{1}+2 c_{1} a_{2}-b_{1} b_{2}}$$

- Solution for $r =0$

$$\left[y \left(u \right) = \sum_{k =0}^{\infty} a_{k} u^{k}, a_{k +1} = -\frac{2 a_{2} a_{k} \left(a_{2} k^{2}-a_{2} k +b_{1} k +c_{0}\right)}{2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2} k^{2}+2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2} k +\sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, b_{1} k +2 c_{1} a_{2} k -b_{1} b_{2} k +\sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, b_{1}}\right.$$

- Revert the change of variables $u = x -\frac{-b_{2}+\sqrt{-4 c_{2} a_{2}+b_{2}^{2}}}{2 a_{2}}$

$$\left[y = \sum_{k =0}^{\infty} a_{k} \left(x -\frac{-b_{2}+\sqrt{-4 c_{2} a_{2}+b_{2}^{2}}}{2 a_{2}}\right)^{k}, a_{k +1} = -\frac{2 a_{2} a_{k} \left(a_{2} k^{2}-a_{2} k +b_{1} k +c_{0}\right)}{2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2} k^{2}+2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2} k +\sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, b_{1} k +2 c_{1} a_{2} k}\right.$$

- Recursion relation for $r = \frac{2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2}-\sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, b_{1}-2 c_{1} a_{2}+b_{1} b_{2}}{2 \sqrt{-4 c_{2} a_{2}+b_{2}^{2}}\, a_{2}}$

$$a_{k+1} = -\frac{2a_2 a_k \left(a_2 k^2 + \frac{k\left(2\sqrt{-4c_2a_2+b_2^2}\, a_2 - \sqrt{-4c_2a_2+b_2^2}\, b_1 - 2c \cdots}{\sqrt{-4c_2a_2+b_2^2}} \cdots\right.}{2\sqrt{-4c_2a_2+b_2^2}\, a_2 k^2 + 2k\left(2\sqrt{-4c_2a_2+b_2^2}\, a_2 - \sqrt{-4c_2a_2+b_2^2}\, b_1 - 2c_1 a_2 + b_1 b_2\right) + \frac{\left(2\sqrt{-4c_2a_2+b_2^2}\, a_2 - \sqrt{-4c_2a_2+b_2^2}\, b_1 - 2c_1 a \cdots\right.}{2\sqrt{-4c_2a_2+b_2^2}\, a_2} \cdots}$$

- Solution for $r = \frac{2\sqrt{-4c_2a_2+b_2^2}\, a_2 - \sqrt{-4c_2a_2+b_2^2}\, b_1 - 2c_1 a_2 + b_1 b_2}{2\sqrt{-4c_2a_2+b_2^2}\, a_2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k + \frac{2\sqrt{-4c_2a_2+b_2^2}\, a_2 - \sqrt{-4c_2a_2+b_2^2}\, b_1 - 2c_1 a_2 + b_1 b_2}{2\sqrt{-4c_2a_2+b_2^2}\, a_2}}, a_{k+1} = -\frac{\cdots}{2\sqrt{-4c_2a_2+b_2^2}\, a_2 k^2 + 2k\left(2\sqrt{-4c_2a_2+b_2^2}\, a_2 - \cdots\right.} \cdots\right.$$

- Revert the change of variables $u = x - \frac{-b_2 + \sqrt{-4c_2a_2+b_2^2}}{2a_2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{-b_2 + \sqrt{-4c_2a_2+b_2^2}}{2a_2}\right)^{k + \frac{2\sqrt{-4c_2a_2+b_2^2}\, a_2 - \sqrt{-4c_2a_2+b_2^2}\, b_1 - 2c_1 a_2 + b_1 b_2}{2\sqrt{-4c_2a_2+b_2^2}\, a_2}}, a_{k+1} = -\frac{\cdots}{2\sqrt{-4c_2a_2+b_2^2}\, a_2 k^2 + \cdots} \cdots\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x - \frac{-b_2 + \sqrt{-4c_2a_2+b_2^2}}{2a_2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x - \frac{-b_2 + \sqrt{-4c_2a_2+b_2^2}}{2a_2}\right)^{k + \frac{2\sqrt{-4c_2a_2+b_2^2}\, a_2 - \sqrt{-4c_2a_2+b_2^2} \cdots}{2\sqrt{-4c_2a_2+b_2^2}\, a_2 \cdots}} \cdots\right.\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 482

```
dsolve((a__2*x^2+b__2*x+c__2)*diff(y(x),x$2)+(b__1*x+c__1)*diff(y(x),x)+c__0*y(x)=0,y(x), si
```

$$y(x) = c_3 \,\mathrm{hypergeom}\left(\left[\frac{-a_2+b_1+\sqrt{a_2^2+(-2b_1-4c_0)\,a_2+b_1^2}}{2a_2},\right.\right.$$
$$\left.-\frac{a_2-b_1+\sqrt{a_2^2+(-2b_1-4c_0)\,a_2+b_1^2}}{2a_2}\right],\left[\frac{b_1\sqrt{\frac{-4c_2a_2+b_2^2}{a_2^2}}\,a_2-2a_2c_1+b_1b_2}{2a_2^2\sqrt{\frac{-4c_2a_2+b_2^2}{a_2^2}}}\right],\frac{(-2x\,a_2^2-b_2a_2)\sqrt{\frac{-4c_2}{a}}}{8c_2a_2-}$$
$$+c_4\left(2\sqrt{\frac{-4c_2a_2+b_2^2}{a_2^2}}\,x\,a_2^2+\sqrt{\frac{-4c_2a_2+b_2^2}{a_2^2}}\,b_2a_2-4c_2a_2+b_2^2\right)^{\frac{a_2\left(a_2-\frac{b_1}{2}\right)\sqrt{\frac{-4c_2a_2+b_2^2}{a_2^2}}+a_2c_1-\frac{b_1b_2}{2}}{\sqrt{\frac{-4c_2a_2+b_2^2}{a_2^2}}\,a_2^2}}\mathrm{hypergeom}$$

✓ Solution by Mathematica

Time used: 6.771 (sec). Leaf size: 498

```
DSolve[(a2*x^2+b2*x+c2)*y''[x]+(b1*x+c1)*y'[x]+c0*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x)$$

$$\rightarrow c_1 \text{Hypergeometric2F1}\left(-\frac{\text{a2}-\text{b1}+\sqrt{(\text{a2}-\text{b1})^2-4\text{a2c0}}}{2\text{a2}}, \frac{-\text{a2}+\text{b1}+\sqrt{(\text{a2}-\text{b1})^2-4\text{a2c0}}}{2\text{a2}}, \frac{\text{b1}\Big(\text{b2}$$

$$-c_2 2^{\frac{\frac{\text{b1b2}}{\sqrt{\text{b2}^2-4\text{a2c2}}}+\text{b1}}{2\text{a2}}-\frac{\text{c1}}{\sqrt{\text{b2}^2-4\text{a2c2}}}-1} \exp\left(-\frac{i\pi\left(\text{b1}\left(\sqrt{\text{b2}^2-4\text{a2c2}}+\text{b2}\right)-2\text{a2c1}\right)}{2\text{a2}\sqrt{\text{b2}^2-4\text{a2c2}}}\right)\left(\frac{\sqrt{\text{b2}^2-4\text{a2c2}}+2\text{a}}{\sqrt{\text{b2}^2-4\text{a2c}}}\right.$$

$$\left.-\frac{\frac{\text{b2b1}}{\sqrt{\text{b2}^2-4\text{a2c2}}}+\text{b1}+\text{a2}\left(-\frac{2\text{c1}}{\sqrt{\text{b2}^2-4\text{a2c2}}}-4\right)}{2\text{a2}}, \frac{\text{b2}+2\text{a2}x+\sqrt{\text{b2}^2-4\text{a2c2}}}{2\sqrt{\text{b2}^2-4\text{a2c2}}}\right)$$

30.32 problem 180

Internal problem ID [11004]
Internal file name [`OUTPUT/10260_Sunday_December_31_2023_11_33_20_AM_7107991/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2+bx+c)y''+f(x)y'+g(x)y=0$
Problem number: 180.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\left(a\,x^2+bx+c\right)y''-\left(-k^2+x^2\right)y'+(x+k)\,y=0$$

30.32.1 Maple step by step solution

Let's solve

$\left(a\,x^2+bx+c\right)y''+\left(k^2-x^2\right)y'+(x+k)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x+k)y}{a\,x^2+bx+c}-\frac{(k^2-x^2)y'}{a\,x^2+bx+c}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(k^2-x^2)y'}{a\,x^2+bx+c}+\frac{(x+k)y}{a\,x^2+bx+c}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$$\left[P_2(x)=\frac{k^2-x^2}{a\,x^2+bx+c}, P_3(x)=\frac{x+k}{a\,x^2+bx+c}\right]$$

- $\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)\cdot P_2(x)$ is analytic at $x=\frac{-b+\sqrt{-4ac+b^2}}{2a}$

  $$\left(\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{-b+\sqrt{-4ac+b^2}}{2a}}=0$$

- $\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{-b+\sqrt{-4ac+b^2}}{2a}$

  $$\left(\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{-b+\sqrt{-4ac+b^2}}{2a}}=0$$

- $x=\frac{-b+\sqrt{-4ac+b^2}}{2a}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=\frac{-b+\sqrt{-4ac+b^2}}{2a}$

- Multiply by denominators

  $$(a\,x^2+bx+c)\,y''+(k^2-x^2)\,y'+(x+k)\,y=0$$

- Change variables using $x=u+\frac{-b+\sqrt{-4ac+b^2}}{2a}$ so that the regular singular point is at $u=0$

  $$\left(a\,u^2+u\sqrt{-4ac+b^2}\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(k^2-u^2+\frac{ub}{a}-\frac{u\sqrt{-4ac+b^2}}{a}-\frac{b^2}{2a^2}+\frac{b\sqrt{-4ac+b^2}}{2a^2}+\frac{c}{a}\right)\left(\frac{d}{du}y(u)\right)$$

- Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$\frac{a_0 r\left(2a^2\sqrt{-4ac+b^2}\,r+2a^2k^2-2a^2\sqrt{-4ac+b^2}+b\sqrt{-4ac+b^2}+2ac-b^2\right)u^{-1+r}}{2a^2} + \left(\frac{a_1(1+r)\left(2a^2\sqrt{-4ac+b^2}\,r+2a^2k^2+b\sqrt{-4ac+b^2}+\right.}{2a^2}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\frac{r\left(2a^2\sqrt{-4ac+b^2}\,r+2a^2k^2-2a^2\sqrt{-4ac+b^2}+b\sqrt{-4ac+b^2}+2ac-b^2\right)}{2a^2} = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{-2a^2k^2+2a^2\sqrt{-4ac+b^2}-b\sqrt{-4ac+b^2}-2ac+b^2}{2a^2\sqrt{-4ac+b^2}}\right\}$$

- Each term must be 0

$$\frac{a_1(1+r)\left(2a^2\sqrt{-4ac+b^2}\,r+2a^2k^2+b\sqrt{-4ac+b^2}+2ac-b^2\right)}{2a^2} - \frac{a_0\left(-2a^2r^2+2a^2r+2\sqrt{-4ac+b^2}\,r-2ak-2br-\sqrt{-4ac+b^2}+b\right)}{2a} = 0$$

- Each term in the series must be 0, giving the recursion relation

$$\frac{\left(2a_{k+1}(k+1+r)(k+r)a^2-2\left(k+r-\frac{1}{2}\right)a_k a+b a_{k+1}(k+1+r)\right)\sqrt{-4ac+b^2}+2a_k(k+r)(k+r-1)a^3+\left(2k^2(k+1+r)a_{k+1}+2a_k k-2k a_{k-1}-\right.}{2a^2}$$

- Shift index using $k->k+1$

$$\frac{\left(2a_{k+2}(k+2+r)(k+1+r)a^2-2\left(k+\frac{1}{2}+r\right)a_{k+1}a+b a_{k+2}(k+2+r)\right)\sqrt{-4ac+b^2}+2a_{k+1}(k+1+r)(k+r)a^3+\left(2k^2(k+2+r)a_{k+2}+2a_{k+1}k-\right.}{2a^2}$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a\left(-2a^2k^2a_{k+1}-4a^2kra_{k+1}-2a^2r^2a_{k+1}-2a^2ka_{k+1}-2a^2ra_{k+1}+2\sqrt{-4ac+b^2}\,ka_{k+1}+2\sqrt{-4ac+b^2}\,ra_{k+1}-\right.}{4\sqrt{-4ac+b^2}\,a^2kr-2b^2-b^2k-b^2r+4a^2k^2+4ac+4a^2\sqrt{-4ac+b^2}+2b\sqrt{-4ac+b^2}+2a^2k^2k+2a^2k^2r+2ack+2acr+2\sqrt{-4ac+b^2}}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{a\left(-2a^2k^2a_{k+1}-2a^2ka_{k+1}+2\sqrt{-4ac+b^2}\,ka_{k+1}-2aka_{k+1}+2aka_k-2bka_{k+1}+\sqrt{-4ac+b^2}\,a_{k+1}-2a_ka-ba_{k+1}\right)}{-2b^2-b^2k+4a^2k^2+4ac+4a^2\sqrt{-4ac+b^2}+2b\sqrt{-4ac+b^2}+2a^2k^2k+2ack+2\sqrt{-4ac+b^2}\,a^2k^2+6a^2\sqrt{-4ac+b^2}\,k+\sqrt{-4ac+b^2}}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = \frac{a\left(-2a^2k^2a_{k+1}-2a^2ka_{k+1}+2\sqrt{-4ac+b^2}\,ka_{k+1}-2aka_{k+1}+2aka_k-2bka_{k+1}+\sqrt{-4ac+b^2}\right.}{-2b^2-b^2k+4a^2k^2+4ac+4a^2\sqrt{-4ac+b^2}+2b\sqrt{-4ac+b^2}+2a^2k^2k+2ack+2\sqrt{-4ac+b^2}\,a^2k^2+6a^2\sqrt{}}\right.$$

- Revert the change of variables $u = x - \frac{-b+\sqrt{-4ac+b^2}}{2a}$

$$\left[y = \sum_{k=0}^{\infty} a_k\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^k, a_{k+2} = \frac{a\left(-2a^2k^2a_{k+1}-2a^2ka_{k+1}+2\sqrt{-4ac+b^2}\,ka_{k+1}-2aka_{k+1}+2aka_k-2b\right.}{-2b^2-b^2k+4a^2k^2+4ac+4a^2\sqrt{-4ac+b^2}+2b\sqrt{-4ac+b^2}+2a^2k^2k+2ack+2\sqrt{-}}\right.$$

- Recursion relation for $r = \frac{-2a^2k^2+2a^2\sqrt{-4ac+b^2}-b\sqrt{-4ac+b^2}-2ac+b^2}{2a^2\sqrt{-4ac+b^2}}$

$$a_{k+2} = \frac{a\left(-2a^2k^2a_{k+1}-\frac{2k\left(-2a^2k^2+2a^2\sqrt{-4ac+b^2}-b\sqrt{-4ac+b^2}-2ac+b^2\right)a_{k+1}}{\sqrt{-4ac+b^2}}-\frac{\left(-2a^2k^2+2a^2\sqrt{-4ac+b^2}-b\sqrt{-4ac+b^2}-2ac+b^2\right)^2a_{k-}}{2a^2\left(-4ac+b^2\right)}\right.}{2k\left(-2a^2k^2+2a^2\sqrt{-4ac+b^2}-b\sqrt{-4ac+b^2}-2ac+b^2\right)+b^2-b^2k-\frac{b^2\left(-2a^2k^2+2a^2\sqrt{-4ac+b^2}-b\sqrt{-4ac+b^2}-2ac+b^2\right)}{2a^2\sqrt{-4ac+b^2}}-2a^2k}$$

- Solution for $r = \frac{-2a^2k^2+2a^2\sqrt{-4ac+b^2}-b\sqrt{-4ac+b^2}-2ac+b^2}{2a^2\sqrt{-4ac+b^2}}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{-2a^2k^2+2a^2\sqrt{-4ac+b^2}-b\sqrt{-4ac+b^2}-2ac+b^2}{2a^2\sqrt{-4ac+b^2}}}, a_{k+2} = \frac{a\left(-2a^2k^2a_{k+1}-\frac{2k\left(-2a^2k^2+2a^2\sqrt{-4ac+b^2}-b\sqrt{-4a}\right.}{\sqrt{-4ac+b^2}}\right.}{2k\left(-2a^2k^2+2a^2\sqrt{-4ac+b^2}-b\sqrt{-4ac+b^2}-2ac\right.}\right.$$

- Revert the change of variables $u = x - \frac{-b+\sqrt{-4ac+b^2}}{2a}$

$$\left[y = \sum_{k=0}^{\infty} a_k\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^{k+\frac{-2a^2k^2+2a^2\sqrt{-4ac+b^2}-b\sqrt{-4ac+b^2}-2ac+b^2}{2a^2\sqrt{-4ac+b^2}}}, a_{k+2} = \frac{a\left(-2a^2k^2a_{k+1}-\frac{2k\left(-2a^2k^2+2\right.}{}\right.}{2k\left(-2a^2k^2+2a^2\sqrt{-4ac+b}\right.}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{m=0}^{\infty} d_m\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^m\right) + \left(\sum_{m=0}^{\infty} e_m\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^{m+\frac{-2a^2k^2+2a^2\sqrt{-4ac+b^2}-b\sqrt{-4ac+b^2}-}{2a^2\sqrt{-4ac+b^2}}}\right.\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 1535

```
dsolve((a*x^2+b*x+c)*diff(y(x),x$2)-(x^2-k^2)*diff(y(x),x)+(x+k)*y(x)=0,y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 2.442 (sec). Leaf size: 119

```
DSolve[(a*x^2+b*x+c)*y''[x]-(x^2-k^2)*y'[x]+(x+k)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{(k-x)\left(c_2\int_1^x \frac{\exp\left(\frac{\frac{\left(b^2-2a\left(ak^2+c\right)\right)\arctan\left(\frac{b+2aK[1]}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}+aK[1]}{a^2}\right)(c+K[1](b+aK[1]))^{-\frac{b}{2a^2}}}{(k-K[1])^2}dK[1]+c_1\right)}{k}$$

30.33 problem 181

Internal problem ID [11005]
Internal file name [`OUTPUT/10261_Sunday_December_31_2023_11_33_25_AM_57004415/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-5 Equation of form $(ax^2 + bx + c)y'' + f(x)y' + g(x)y = 0$
Problem number: 181.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{(a\,x^2 + bx + c)\,y'' + (k^3 + x^3)\,y' - (k^2 - kx + x^2)\,y = 0}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
   No special function solution was found.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.437 (sec). Leaf size: 246

```
dsolve((a*x^2+b*x+c)*diff(y(x),x$2)+(x^3+k^3)*diff(y(x),x)-(x^2-k*x+k^2)*y(x)=0,y(x), singso
```

$$y(x) = (x + k) \left(\left(\left(\int \frac{\left(2ax + b - \sqrt{-4ac + b^2}\right)^{-\frac{k^3}{\sqrt{-4ac+b^2}}} \left(\frac{-2ax-b+\sqrt{-4ac+b^2}}{2ax+\sqrt{-4ac+b^2}+b}\right)^{-\frac{3bc}{2a^2\sqrt{-4ac+b^2}}} \left(\frac{2ax+\sqrt{-4ac+b^2}+b}{-2ax-b+\sqrt{-4ac+b^2}}\right)^{-\frac{b^3}{2a^3\sqrt{-4}}}}{(x+k)^2} \right. \right. \right.$$

$$\left. + c_1 \right)$$

✓ Solution by Mathematica

Time used: 3.224 (sec). Leaf size: 137

```
DSolve[(a*x^2+b*x+c)*y''[x]+(x^3+k^3)*y'[x]-(x^2-k*x+k^2)*y[x]==0,y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{(k+x)\left(c_2 \int_1^x \frac{\exp\left(\frac{\left(b^3-3acb-2a^3k^3\right)\arctan\left(\frac{b+2aK[1]}{\sqrt{4ac-b^2}}\right)}{a^3\sqrt{4ac-b^2}} - \frac{K[1](aK[1]-2b)}{2a^2}\right)(c+K[1](b+aK[1]))^{-\frac{b^2-ac}{2a^3}}}{(k+K[1])^2} dK[1] + c_1\right)}{k}$$

31 Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$

31.1 problem 182

Internal problem ID [11006]
Internal file name [OUTPUT/10262_Sunday_December_31_2023_11_33_28_AM_16378108/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 182.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y'' + (ax + b)\,y = 0}$$

31.1.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(a + \frac{b}{x}\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= 2\sqrt{b} \\
n &= \sqrt{1-4a} \\
\gamma &= -\frac{1}{2}
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\sqrt{1-4a}, \frac{2\sqrt{b}}{\sqrt{x}}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\sqrt{1-4a}, \frac{2\sqrt{b}}{\sqrt{x}}\right)
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\sqrt{1-4a}, \frac{2\sqrt{b}}{\sqrt{x}}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\sqrt{1-4a}, \frac{2\sqrt{b}}{\sqrt{x}}\right) \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\sqrt{1-4a}, \frac{2\sqrt{b}}{\sqrt{x}}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\sqrt{1-4a}, \frac{2\sqrt{b}}{\sqrt{x}}\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 49

```
dsolve(x^3*diff(y(x),x$2)+(a*x+b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\text{BesselJ}\left(-\sqrt{-4a+1}, \frac{2\sqrt{b}}{\sqrt{x}}\right) c_1 + \text{BesselY}\left(-\sqrt{-4a+1}, \frac{2\sqrt{b}}{\sqrt{x}}\right) c_2\right) \sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.128 (sec). Leaf size: 101

```
DSolve[x^3*y''[x]+(a*x+b)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 \operatorname{Gamma}\left(1-\sqrt{1-4a}\right) \text{BesselJ}\left(-\sqrt{1-4a}, 2\sqrt{b}\sqrt{\frac{1}{x}}\right) + c_2 \operatorname{Gamma}\left(\sqrt{1-4a}+1\right) \text{BesselJ}\left(\sqrt{1-4a}\right.}{\sqrt{b}\sqrt{\frac{1}{x}}}$$

31.2 problem 183

Internal problem ID [11007]
Internal file name [`OUTPUT/10263_Sunday_December_31_2023_11_33_29_AM_57178898/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 183.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^3y'' + \left(a\,x^2 + b\right)y' + ycx = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 120

```
dsolve(x^3*diff(y(x),x$2)+(a*x^2+b)*diff(y(x),x)+c*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^{-\frac{\sqrt{a^2-2a-4c+1}}{2}-\frac{a}{2}+\frac{1}{2}}\left(\text{KummerM}\left(-\frac{1}{4}+\frac{\sqrt{a^2-2a-4c+1}}{4}+\frac{a}{4}, 1+\frac{\sqrt{a^2-2a-4c+1}}{2}, \frac{b}{2x^2}\right)c_1 + \text{KummerU}\left(-\frac{1}{4}+\frac{\sqrt{a^2-2a-4c+1}}{4}+\frac{a}{4}, 1+\frac{\sqrt{a^2-2a-4c+1}}{2}, \frac{b}{2x^2}\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.646 (sec). Leaf size: 308

```
DSolve[x^3*y''[x]+(a*x^2+b)*y'[x]+c*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -(-1)^{\frac{1}{4}\left(-\sqrt{a^2-2a-4c+1}+a+3\right)} 2^{\frac{1}{4}\left(-\sqrt{a^2-2a-4c+1}-a+1\right)} b^{\frac{1}{4}\left(-\sqrt{a^2-2a-4c+1}+a-1\right)} \left(\frac{1}{x}\right)^{\frac{1}{2}\left(-\sqrt{a^2-2a-4c+1}+a-1\right)} \left(c_2 i^{\sqrt{a}} \right.$$

$$\left. + c_1 2^{\frac{1}{2}\sqrt{a^2-2a-4c+1}} \text{Hypergeometric1F1}\left(\frac{1}{4}\left(a-\sqrt{a^2-2a-4c+1}-1\right), 1 - \frac{1}{2}\sqrt{a^2-2a-4c+1}, \frac{b}{2x^2}\right)\right)$$

31.3 problem 184

Internal problem ID [11008]
Internal file name [OUTPUT/10264_Sunday_December_31_2023_11_33_31_AM_3795090/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 184.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^3y'' + \left(a\,x^2 + bx\right)y' + yb = 0}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 108

```
dsolve(x^3*diff(y(x),x$2)+(a*x^2+b*x)*diff(y(x),x)+b*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x\left(\Gamma\left(a, -\frac{b}{x}\right) - \Gamma(a)\right)(-1)^{-a}(a-2)\, b^{-a+1} + c_1\left(\Gamma\left(a, -\frac{b}{x}\right) - \Gamma(a)\right)(-1)^{-a} b^{-a+2} + b\, x^{-a+1} c_1 \mathrm{e}^{\frac{b}{x}} + c_2(a}{x}$$

✓ Solution by Mathematica

Time used: 2.653 (sec). Leaf size: 62

```
DSolve[x^3*y''[x]+(a*x^2+b*x)*y'[x]+b*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{((a-2)x+b)\left(c_2\int_1^x \frac{e^{\frac{b}{K[1]}}K[1]^{2-a}}{(b+(a-2)K[1])^2}dK[1]+c_1\right)}{x(a+b-2)}$$

31.4 problem 185

Internal problem ID [11009]
Internal file name [`OUTPUT/10265_Sunday_December_31_2023_11_33_32_AM_61588335/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 185.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x^3y'' + \left(a\,x^2 + bx\right)y' + yc = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 146

```
dsolve(x^3*diff(y(x),x$2)+(a*x^2+b*x)*diff(y(x),x)+c*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^{-a}\left(-cxc_2(ab-c)(b-c)\operatorname{KummerU}\left(\frac{(a+1)b-c}{b},a,\frac{b}{x}\right)+\left(c_1xb(ab-c)\operatorname{KummerM}\left(\frac{(a+1)b-c}{b},a,\frac{b}{x}\right)-(bc\right.\right.}{b^2c}$$

✓ Solution by Mathematica

Time used: 0.435 (sec). Leaf size: 62

```
DSolve[x^3*y''[x]+(a*x^2+b*x)*y'[x]+c*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_1\,\text{Hypergeometric1F1}\left(-\frac{c}{b},2-a,\frac{b}{x}\right)-(-1)^a c_2 b^{a-1}\left(\frac{1}{x}\right)^{a-1}\text{Hypergeometric1F1}\left(a-\frac{b+c}{b},a,\frac{b}{x}\right)$$

31.5 problem 186

Internal problem ID [11010]
Internal file name [OUTPUT/10266_Sunday_December_31_2023_11_33_32_AM_61507542/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 186.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^3y'' + \left(a\,x^2 + bx\right)y' + (cx + d)\,y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 132

```
dsolve(x^3*diff(y(x),x$2)+(a*x^2+b*x)*diff(y(x),x)+(c*x+d)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^{-\frac{\sqrt{a^2-2a-4c+1}}{2}-\frac{a}{2}+\frac{1}{2}}\left(\operatorname{KummerM}\left(\frac{\sqrt{a^2-2a-4c+1}\,b+b(a-1)-2d}{2b}, 1+\sqrt{a^2-2a-4c+1}, \frac{b}{x}\right)c_1 + \operatorname{KummerU}\left(\frac{\sqrt{a^2-2a-4c+1}\,b+b(a-1)-2d}{2b}, 1+\sqrt{a^2-2a-4c+1}, \frac{b}{x}\right)c_2\right)$$

✓ <u>Solution by Mathematica</u>

Time used: 0.641 (sec). Leaf size: 255

```
DSolve[x^3*y''[x]+(a*x^2+b*x)*y'[x]+(c*x+d)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow$$

$$-i^{-\sqrt{a^2-2a-4c+1}+a+1} b^{\frac{1}{2}\left(-\sqrt{a^2-2a-4c+1}+a-1\right)} \left(\frac{1}{x}\right)^{\frac{1}{2}\left(-\sqrt{a^2-2a-4c+1}+a-1\right)} \left(c_2 i^{2\sqrt{a^2-2a-4c+1}} b^{\sqrt{a^2-2a-4c+1}} \left(\frac{1}{x}\right)^{\sqrt{}}\right.$$

$$\left. +1, \frac{b}{x}\right) + c_1 \text{Hypergeometric1F1}\left(\frac{1}{2}\left(a - \frac{2d}{b} - \sqrt{a^2-2a-4c+1} - 1\right), 1\right.$$

$$\left. - \sqrt{a^2-2a-4c+1}, \frac{b}{x}\right)\Bigg)$$

31.6 problem 187

Internal problem ID [11011]
Internal file name [OUTPUT/10267_Sunday_December_31_2023_11_33_33_AM_40075043/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 187.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^3y'' + \left(a\,x^3 + abx - x^2 + b\right)y' + a^2bxy = 0}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
   No special function solution was found.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.454 (sec). Leaf size: 49

```
dsolve(x^3*diff(y(x),x$2)+(a*x^3-x^2+a*b*x+b)*diff(y(x),x)+a^2*b*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-ax}\left(c_2\left(\int \frac{x\,\mathrm{e}^{\frac{2a\,x^3+2abx+b}{2x^2}}}{(ax+1)^2}dx\right)+c_1\right)(ax+1)$$

✓ Solution by Mathematica

Time used: 1.347 (sec). Leaf size: 70

```
DSolve[x^3*y''[x]+(a*x^3-x^2+a*b*x+b)*y'[x]+a^2*b*x*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^{-ax}(ax+1)\left(c_2 \int_1^x \frac{a^2 e^{aK[1]+\frac{2aK[1]b+b}{2K[1]^2}}K[1]}{(aK[1]+1)^2}dK[1]+c_1\right)}{a}$$

31.7 problem 188

Internal problem ID [11012]
Internal file name [`OUTPUT/10268_Sunday_December_31_2023_11_33_34_AM_35096271/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 188.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x^3y'' + x(a\,x^n + b)\,y' - \left(a\,x^n - x^{n-1}ab + b\right)y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(x^3*diff(y(x),x$2)+x*(a*x^n+b)*diff(y(x),x)-(a*x^n-a*b*x^(n-1)+b)*y(x)=0,y(x), singso
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^3*y''[x]+x*(a*x^n+b)*y'[x]-(a*x^n-a*b*x^(n-1)+b)*y[x]==0,y[x],x,IncludeSingularSolu
```

Not solved

31.8 problem 189

Internal problem ID [11013]
Internal file name [`OUTPUT/10269_Sunday_December_31_2023_11_33_34_AM_92961714/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 189.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x\left(a\,x^2+b\right)y''+2\left(a\,x^2+b\right)y'-2yax=0}$$

31.8.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x\left(a\,x^2+b\right)y''+\left(2a\,x^2+2b\right)y'-2yax\right)dx=0$$
$$\left(-a\,x^2+b\right)y+\left(a\,x^3+bx\right)y'=c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{a\,x^2 - b}{x\,(a\,x^2 + b)} \\
q(x) &= \frac{c_1}{x\,(a\,x^2 + b)}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{(a\,x^2 - b)\,y}{x\,(a\,x^2 + b)} = \frac{c_1}{x\,(a\,x^2 + b)}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{a\,x^2 - b}{x\left(a\,x^2 + b\right)}dx} \\
&= \mathrm{e}^{\ln(x) - \ln\left(a\,x^2 + b\right)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{x}{a\,x^2 + b}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x\,(a\,x^2 + b)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xy}{a\,x^2 + b}\right) &= \left(\frac{x}{a\,x^2 + b}\right)\left(\frac{c_1}{x\,(a\,x^2 + b)}\right) \\
\mathrm{d}\left(\frac{xy}{a\,x^2 + b}\right) &= \left(\frac{c_1}{(a\,x^2 + b)^2}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{xy}{a\,x^2 + b} &= \int \frac{c_1}{(a\,x^2 + b)^2}\,\mathrm{d}x \\
\frac{xy}{a\,x^2 + b} &= c_1\left(\frac{x}{2b\,(a\,x^2 + b)} + \frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{2b\sqrt{ab}}\right) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{x}{a\,x^2 + b}$ results in

$$
y = \frac{(a\,x^2 + b)\,c_1\left(\frac{x}{2b(a\,x^2 + b)} + \frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{2b\sqrt{ab}}\right)}{x} + \frac{c_2(a\,x^2 + b)}{x}
$$

which simplifies to

$$y = \frac{c_1}{2b} + \frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)(a\,x^2+b)\,c_1}{2b\sqrt{ab}\,x} + \frac{c_2(a\,x^2+b)}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{2b} + \frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)(a\,x^2+b)\,c_1}{2b\sqrt{ab}\,x} + \frac{c_2(a\,x^2+b)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{2b} + \frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)(a\,x^2+b)\,c_1}{2b\sqrt{ab}\,x} + \frac{c_2(a\,x^2+b)}{x}$$

Verified OK.

31.8.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x\left(a\,x^2+b\right)y'' + \left(2a\,x^2+2b\right)y' - 2yax = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x\left(a\,x^2+b\right)y'' + \left(2a\,x^2+2b\right)y' - 2yax\right)dx = 0$$
$$\left(-a\,x^2+b\right)y + \left(a\,x^3+bx\right)y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{a\,x^2-b}{x\,(a\,x^2+b)}$$
$$q(x) = \frac{c_1}{x\,(a\,x^2+b)}$$

Hence the ode is

$$y' - \frac{(a\,x^2 - b)\,y}{x\,(a\,x^2 + b)} = \frac{c_1}{x\,(a\,x^2 + b)}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{a\,x^2 - b}{x\left(a\,x^2 + b\right)} dx} \\ &= \mathrm{e}^{\ln(x) - \ln\left(a\,x^2 + b\right)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{x}{a\,x^2 + b}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x\,(a\,x^2 + b)}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xy}{a\,x^2 + b}\right) &= \left(\frac{x}{a\,x^2 + b}\right)\left(\frac{c_1}{x\,(a\,x^2 + b)}\right) \\ \mathrm{d}\!\left(\frac{xy}{a\,x^2 + b}\right) &= \left(\frac{c_1}{(a\,x^2 + b)^2}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{xy}{a\,x^2 + b} &= \int \frac{c_1}{(a\,x^2 + b)^2}\,\mathrm{d}x \\ \frac{xy}{a\,x^2 + b} &= c_1\left(\frac{x}{2b\,(a\,x^2 + b)} + \frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{2b\sqrt{ab}}\right) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{x}{a\,x^2 + b}$ results in

$$y = \frac{(a\,x^2 + b)\,c_1\left(\frac{x}{2b(a\,x^2 + b)} + \frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{2b\sqrt{ab}}\right)}{x} + \frac{c_2(a\,x^2 + b)}{x}$$

which simplifies to

$$y = \frac{c_1}{2b} + \frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)(a\,x^2 + b)\,c_1}{2b\sqrt{ab}\,x} + \frac{c_2(a\,x^2 + b)}{x}$$

Summary
The solution(s) found are the following

$$
y=\frac{c_{1}}{2b}+\frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)\left(a\,x^{2}+b\right)c_{1}}{2b\sqrt{ab}\,x}+\frac{c_{2}\left(a\,x^{2}+b\right)}{x} \tag{1}
$$

Verification of solutions

$$
y=\frac{c_{1}}{2b}+\frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)\left(a\,x^{2}+b\right)c_{1}}{2b\sqrt{ab}\,x}+\frac{c_{2}\left(a\,x^{2}+b\right)}{x}
$$

Verified OK.

31.8.3 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
x\left(a\,x^{2}+b\right)y''+\left(2a\,x^{2}+2b\right)y'-2yax&=0 \tag{1}\\
Ay''+By'+Cy&=0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A&=x\left(a\,x^{2}+b\right)\\
B&=2a\,x^{2}+2b \tag{3}\\
C&=-2ax
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x)=ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x)=rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r&=\frac{s}{t} \tag{5}\\
&=\frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}}
\end{align*}
$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$
r=\frac{2a}{a\,x^{2}+b} \tag{6}
$$

Comparing the above to (5) shows that

$$s = 2a$$
$$t = a\,x^2 + b$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2a}{a\,x^2 + b}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 176: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = a\,x^2 + b$. There is a pole at $x = \frac{\sqrt{-ab}}{a}$ of order 1. There is a pole at

$x = -\frac{\sqrt{-ab}}{a}$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = \frac{\sqrt{-ab}}{a}$ of order 1 then

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= 1\\ \alpha_c^- &= 1\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2a}{a\,x^2 + b}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2a}{a\,x^2 + b}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{\sqrt{-ab}}{a}$	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{+} = 2$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x - \frac{\sqrt{-ab}}{a}} + (0) \\
&= \frac{1}{x - \frac{\sqrt{-ab}}{a}} \\
&= -\frac{a}{-ax + \sqrt{-ab}}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x - \frac{\sqrt{-ab}}{a}}\right)(1) + \left(\left(-\frac{1}{\left(x - \frac{\sqrt{-ab}}{a}\right)^2}\right) + \left(\frac{1}{x - \frac{\sqrt{-ab}}{a}}\right)^2 - \left(\frac{2a}{a\,x^2 + b}\right)\right) = 0$$

$$-\frac{2a\left((-x - a_0)\sqrt{-ab} + axa_0 - b\right)}{\left(ax - \sqrt{-ab}\right)\left(a\,x^2 + b\right)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{\sqrt{-ab}}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{\sqrt{-ab}}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{\sqrt{-ab}}{a}\right) \mathrm{e}^{\int \frac{1}{x - \frac{\sqrt{-ab}}{a}} dx} \\
&= \left(x + \frac{\sqrt{-ab}}{a}\right) x - \frac{\sqrt{-ab}}{a} \\
&= \frac{a\,x^2 + b}{a}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2a\,x^2+2b}{x\left(a\,x^2+b\right)} \, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{a\,x^2 + b}{ax}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2a\,x^2+2b}{x\left(a\,x^2+b\right)}dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{a^2\left(\frac{x}{a\,x^2+b}+\frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}\right)}{2b}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{a\,x^2+b}{ax}\right) + c_2\left(\frac{a\,x^2+b}{ax}\left(\frac{a^2\left(\frac{x}{a\,x^2+b}+\frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}\right)}{2b}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(a\,x^2+b)}{ax} + \frac{c_2 a\left(\sqrt{ab}\,x + \arctan\left(\frac{ax}{\sqrt{ab}}\right)(a\,x^2+b)\right)}{2xb\sqrt{ab}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(a\,x^2+b)}{ax} + \frac{c_2 a\left(\sqrt{ab}\,x + \arctan\left(\frac{ax}{\sqrt{ab}}\right)(a\,x^2+b)\right)}{2xb\sqrt{ab}}
$$

Verified OK.

31.8.4 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x\left(a\,x^2 + b\right) \\ q(x) &= 2a\,x^2 + 2b \\ r(x) &= -2ax \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 6ax \\ q'(x) &= 4ax \end{aligned}$$

Therefore (1) becomes

$$6ax - (4ax) + (-2ax) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$x\left(a\,x^2 + b\right)y' + \left(-a\,x^2 + b\right)y = c_1$$

We now have a first order ode to solve which is

$$x\left(a\,x^2 + b\right)y' + \left(-a\,x^2 + b\right)y = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{a\,x^2-b}{x\,(a\,x^2+b)} \\
q(x) &= \frac{c_1}{x\,(a\,x^2+b)}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{(a\,x^2-b)\,y}{x\,(a\,x^2+b)} = \frac{c_1}{x\,(a\,x^2+b)}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{a\,x^2-b}{x\left(a\,x^2+b\right)}dx} \\
&= \mathrm{e}^{\ln(x)-\ln\left(a\,x^2+b\right)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{x}{a\,x^2+b}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x\,(a\,x^2+b)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xy}{a\,x^2+b}\right) &= \left(\frac{x}{a\,x^2+b}\right)\left(\frac{c_1}{x\,(a\,x^2+b)}\right) \\
\mathrm{d}\left(\frac{xy}{a\,x^2+b}\right) &= \left(\frac{c_1}{(a\,x^2+b)^2}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{xy}{a\,x^2+b} &= \int \frac{c_1}{(a\,x^2+b)^2}\,\mathrm{d}x \\
\frac{xy}{a\,x^2+b} &= c_1\left(\frac{x}{2b\,(a\,x^2+b)} + \frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{2b\sqrt{ab}}\right) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{x}{a\,x^2+b}$ results in

$$
y = \frac{(a\,x^2+b)\,c_1\left(\frac{x}{2b(a\,x^2+b)} + \frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{2b\sqrt{ab}}\right)}{x} + \frac{c_2(a\,x^2+b)}{x}
$$

which simplifies to

$$y = \frac{c_1}{2b} + \frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)(a\,x^2+b)\,c_1}{2b\sqrt{ab}\,x} + \frac{c_2(a\,x^2+b)}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{2b} + \frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)(a\,x^2+b)\,c_1}{2b\sqrt{ab}\,x} + \frac{c_2(a\,x^2+b)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{2b} + \frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)(a\,x^2+b)\,c_1}{2b\sqrt{ab}\,x} + \frac{c_2(a\,x^2+b)}{x}$$

Verified OK.

31.8.5 Maple step by step solution

Let's solve

$x(a\,x^2+b)\,y'' + (2a\,x^2+2b)\,y' - 2yax = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} + \frac{2ay}{a\,x^2+b}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} - \frac{2ay}{a\,x^2+b} = 0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = -\frac{2a}{a\,x^2+b}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $(x \cdot P_2(x))\Big|_{x=0} = 2$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(a\,x^2+b)\,y''+(2a\,x^2+2b)\,y'-2yax=0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k->k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $ba_0r(1+r)\,x^{-1+r}+ba_1(1+r)\,(2+r)\,x^r+\left(\sum_{k=1}^{\infty}(ba_{k+1}(k+r+1)\,(k+r+2)+aa_{k-1}(k+r+1\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$br(1+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $ba_1(1+r)(2+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+1)(ba_{k+1}(k+r+2)+aa_{k-1}(k-2+r))=0$

- Shift index using $k->k+1$

  $(k+r+2)(ba_{k+2}(k+3+r)+aa_k(k+r-1))=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{aa_k(k+r-1)}{b(k+3+r)}$

- Recursion relation for $r=-1$; series terminates at $k=2$

  $a_{k+2}=-\frac{aa_k(k-2)}{b(k+2)}$

- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{aa_k(k-2)}{b(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

  $a_{k+2}=-\frac{aa_k(k-1)}{b(k+3)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{aa_k(k-1)}{b(k+3)}, 2ba_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} c_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} d_k x^k\right), c_{k+2}=-\frac{ac_k(k-2)}{b(k+2)}, 0=0, d_{k+2}=-\frac{ad_k(k-1)}{b(k+3)}, 2bd_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 46

```
dsolve(x*(a*x^2+b)*diff(y(x),x$2)+2*(a*x^2+b)*diff(y(x),x)-2*a*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(a\,x^2+b)\,c_2 \arctan\left(\frac{\sqrt{ab}\,x}{b}\right) + \sqrt{ab}\,c_2 x + c_1(a\,x^2+b)}{x}$$

✓ Solution by Mathematica

Time used: 0.16 (sec). Leaf size: 78

```
DSolve[x*(a*x^2+b)*y''[x]+2*(a*x^2+b)*y'[x]-2*a*x*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{c_2(ax^2+b)\arctan\left(\frac{\sqrt{a}x}{\sqrt{b}}\right) + \sqrt{a}\sqrt{b}(2abc_1x^2+2b^2c_1+c_2x)}{2\sqrt{a}b^{3/2}x}$$

31.9 problem 190

Internal problem ID [11014]
Internal file name [`OUTPUT/10270_Sunday_December_31_2023_11_33_36_AM_47036669/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 190.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x(x^2+a)\,y''+(b\,x^2+c)\,y'+sxy=0}$$

31.9.1 Maple step by step solution

Let's solve

$x(x^2+a)\,y''+(b\,x^2+c)\,y'+sxy=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(b\,x^2+c)y'}{x(x^2+a)}-\frac{sy}{x^2+a}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(b\,x^2+c)y'}{x(x^2+a)}+\frac{sy}{x^2+a}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{b\,x^2+c}{x(x^2+a)}, P_3(x) = \frac{s}{x^2+a}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{c}{a}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x^2+a)\,y'' + (b\,x^2+c)\,y' + sxy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ◦ Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - ◦ Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(ar-a+c)\, x^{r-1} + a_1(1+r)(ar+c)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(a(k+1)+ar-a+c)+a\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(ar-a+c) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{a-c}{a}\right\}$$

- Each term must be 0

$$a_1(1+r)(ar+c) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2+(b+2r-3)\, k+r^2+(b-3)\, r-b+s+2\right) a_{k-1} + a_{k+1}(k+r+1)(ak+ar+c) = 0$$

- Shift index using $k- >k+1$

$$\left((k+1)^2+(b+2r-3)(k+1)+r^2+(b-3)\, r-b+s+2\right) a_k + a_{k+2}(k+2+r)(a(k+1)+a$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{(bk+br+k^2+2kr+r^2-k-r+s)\, a_k}{(k+2+r)(ak+ar+a+c)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{(bk+k^2-k+s)\, a_k}{(k+2)(ak+a+c)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{(bk+k^2-k+s)\, a_k}{(k+2)(ak+a+c)}, a_1 c = 0\right]$$

- Recursion relation for $r = \frac{a-c}{a}$

$$a_{k+2} = -\frac{\left(bk+\frac{b(a-c)}{a}+k^2+\frac{2k(a-c)}{a}+\frac{(a-c)^2}{a^2}-k-\frac{a-c}{a}+s\right) a_k}{\left(k+2+\frac{a-c}{a}\right)(ak+2a)}$$

- Solution for $r = \frac{a-c}{a}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{a-c}{a}}, a_{k+2} = -\frac{\left(bk+\frac{b(a-c)}{a}+k^2+\frac{2k(a-c)}{a}+\frac{(a-c)^2}{a^2}-k-\frac{a-c}{a}+s\right) a_k}{\left(k+2+\frac{a-c}{a}\right)(ak+2a)}, a_1\left(1+\frac{a-c}{a}\right) a = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} d_k x^k\right) + \left(\sum_{k=0}^{\infty} e_k x^{k+\frac{a-c}{a}}\right), d_{k+2} = -\frac{(bk+k^2-k+s)\, d_k}{(k+2)\,(ak+a+c)}, d_1 c = 0, e_{k+2} = -\frac{\left(bk+\frac{b(a-c)}{a}+k^2+\frac{2k(a-}{a}\right.}{(k+2+\frac{a}{}}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.157 (sec). Leaf size: 175

```
dsolve(x*(x^2+a)*diff(y(x),x$2)+(b*x^2+c)*diff(y(x),x)+s*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(x^2 + a\right)^{\frac{(-b+2)a+c}{2a}} \left(x^{\frac{a-c}{a}} \operatorname{hypergeom}\left(\left[-\frac{b}{4}+\frac{5}{4}-\frac{\sqrt{b^2-2b-4s+1}}{4}, -\frac{b}{4}+\frac{5}{4}+\frac{\sqrt{b^2-2b-4s+1}}{4}\right], \left[\frac{3a-c}{2a}\right], -\frac{x^2}{a}\right) c_1 + \operatorname{hypergeom}\left(\left[-\frac{b}{4}+\frac{3}{4}+\frac{c}{2a}+\frac{\sqrt{b^2-2b-4s+1}}{4}, -\frac{\sqrt{b^2-2b-4s+1}}{4}-\frac{b}{4}+\frac{3}{4}+\frac{c}{2a}\right], \left[\frac{1}{2}+\frac{c}{2a}\right], -\frac{x^2}{a}\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.967 (sec). Leaf size: 185

```
DSolve[x*(x^2+a)*y''[x]+(b*x^2+c)*y'[x]+s*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 a^{\frac{1}{2}\left(\frac{c}{a}-1\right)} x^{1-\frac{c}{a}} \operatorname{Hypergeometric2F1}\left(\frac{a\left(b+\sqrt{b^2-2b-4s+1}+1\right)-2c}{4a}, \frac{ba-\sqrt{b^2-2b-4s+1}a+a}{4a}\right. \ldots$$

$$\left. -\frac{c}{2a}, -\frac{x^2}{a}\right) + c_1 \operatorname{Hypergeometric2F1}\left(\frac{1}{4}\left(b-\sqrt{b^2-2b-4s+1}-1\right), \frac{1}{4}\left(b+\sqrt{b^2-2b-4s+1}-1\right), \frac{a+c}{2a}, -\frac{x^2}{a}\right)$$

31.10 problem 191

Internal problem ID [11015]
Internal file name [`OUTPUT/10271_Sunday_December_31_2023_11_33_37_AM_1689924/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 191.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$x^2(ax+b)\,y'' + \left(c\,x^2 + (a\lambda + 2b)\,x + b\lambda\right)y' + \lambda(-2a+c)\,y = 0$$

31.10.1 Maple step by step solution

Let's solve

$x^2(ax+b)\,y'' + (c\,x^2 + (a\lambda + 2b)\,x + b\lambda)\,y' + \lambda(-2a+c)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{\lambda(2a-c)y}{x^2(ax+b)} - \frac{(a\lambda x+c\,x^2+b\lambda+2bx)y'}{x^2(ax+b)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(a\lambda x+c\,x^2+b\lambda+2bx)y'}{x^2(ax+b)} - \frac{\lambda(2a-c)y}{x^2(ax+b)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x)=\frac{a\lambda x+c\,x^2+b\lambda+2bx}{x^2(ax+b)}, P_3(x)=-\frac{\lambda(2a-c)}{x^2(ax+b)}\right]$

- $\left(x+\frac{b}{a}\right)\cdot P_2(x)$ is analytic at $x=-\frac{b}{a}$

  $\left(\left(x+\frac{b}{a}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{b}{a}}=\frac{\left(\frac{c\,b^2}{a^2}-\frac{2b^2}{a}\right)a}{b^2}$

- $\left(x+\frac{b}{a}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{b}{a}$

  $\left(\left(x+\frac{b}{a}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{b}{a}}=0$

- $x=-\frac{b}{a}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{b}{a}$

- Multiply by denominators

  $x^2(ax+b)\,y''+\left(a\lambda x+c\,x^2+b\lambda+2bx\right)y'-\lambda(2a-c)\,y=0$

- Change variables using $x=u-\frac{b}{a}$ so that the regular singular point is at $u=0$

  $\left(a\,u^3-2u^2b+\frac{u\,b^2}{a}\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(a\lambda u+c\,u^2-\frac{2cub}{a}+\frac{c\,b^2}{a^2}+2bu-\frac{2b^2}{a}\right)\left(\frac{d}{du}y(u)\right)+(-2a\lambda+c\lambda$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$\frac{a_0 r\, b^2 (ar-3a+c) u^{r-1}}{a^2} + \left(\frac{a_1(1+r) b^2 (ar-2a+c)}{a^2} + \frac{a_0 (ar-2a+c)(a\lambda-2br)}{a}\right) u^r + \left(\sum_{k=1}^{\infty} \left(\frac{a_{k+1}(k+1+r) b^2 (a(k+1)+ar-3a+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\frac{r\, b^2 (ar-3a+c)}{a^2} = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{3a-c}{a}\right\}$$

- Each term must be 0

$$\frac{a_1(1+r) b^2 (ar-2a+c)}{a^2} + \frac{a_0 (ar-2a+c)(a\lambda-2br)}{a} = 0$$

- Each term in the series must be 0, giving the recursion relation

$$\frac{((k+r-2)a+c)\left((k a_{k-1} + \lambda a_k + r a_{k-1} - a_{k-1}) a^2 - 2 a_k (k+r) b a + a_{k+1}(k+1+r) b^2\right)}{a^2} = 0$$

- Shift index using $k- >k+1$

$$\frac{((k+r-1)a+c)\left(((k+1) a_k + \lambda a_{k+1} + a_k r - a_k) a^2 - 2 a_{k+1}(k+1+r) b a + a_{k+2}(k+2+r) b^2\right)}{a^2} = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a(a k a_k + a\lambda a_{k+1} + a r a_k - 2bk a_{k+1} - 2br a_{k+1} - 2b a_{k+1})}{(k+2+r) b^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a(a k a_k + a\lambda a_{k+1} - 2bk a_{k+1} - 2b a_{k+1})}{(k+2) b^2}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{a(a k a_k + a\lambda a_{k+1} - 2bk a_{k+1} - 2b a_{k+1})}{(k+2) b^2}, \frac{a_1 b^2 (-2a+c)}{a^2} + a_0 (-2a+c)\lambda = 0\right]$$

- Revert the change of variables $u = x + \frac{b}{a}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{b}{a}\right)^k, a_{k+2} = -\frac{a(a k a_k + a\lambda a_{k+1} - 2bk a_{k+1} - 2b a_{k+1})}{(k+2) b^2}, \frac{a_1 b^2 (-2a+c)}{a^2} + a_0 (-2a+c)\lambda = 0\right]$$

- Recursion relation for $r = \frac{3a-c}{a}$

$$a_{k+2} = -\frac{a\left(a k a_k + a\lambda a_{k+1} + (3a-c) a_k - 2bk a_{k+1} - \frac{2b(3a-c) a_{k+1}}{a} - 2b a_{k+1}\right)}{\left(k+2+\frac{3a-c}{a}\right) b^2}$$

- Solution for $r = \frac{3a-c}{a}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3a-c}{a}}, a_{k+2} = -\frac{a\left(a k a_k + a\lambda a_{k+1} + (3a-c) a_k - 2bk a_{k+1} - \frac{2b(3a-c) a_{k+1}}{a} - 2b a_{k+1}\right)}{\left(k+2+\frac{3a-c}{a}\right) b^2}, \frac{a_1 \left(1+\frac{3a-c}{a}\right) b^2}{a} + a\right.$$

- Revert the change of variables $u = x + \frac{b}{a}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{b}{a}\right)^{k+\frac{3a-c}{a}}, a_{k+2} = -\frac{a\left(a k a_k + a\lambda a_{k+1} + (3a-c) a_k - 2bk a_{k+1} - \frac{2b(3a-c) a_{k+1}}{a} - 2b a_{k+1}\right)}{\left(k+2+\frac{3a-c}{a}\right) b^2}, \frac{a_1 \left(1+\frac{3a-c}{a}\right) b^2}{a}\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} d_k \left(x+\frac{b}{a}\right)^k\right)+\left(\sum_{k=0}^{\infty} e_k \left(x+\frac{b}{a}\right)^{k+\frac{3a-c}{a}}\right), d_{k+2}=-\frac{a(akd_k+a\lambda d_{1+k}-2bkd_{1+k}-2bd_{1+k})}{(k+2)b^2}, \frac{d_1 b^2(-2a}{a^2}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a = 0, e <> 0
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.469 (sec). Leaf size: 169

```
dsolve(x^2*(a*x+b)*diff(y(x),x$2)+(c*x^2+(2*b+a*lambda)*x+b*lambda)*diff(y(x),x)+lambda*(c-2
```

$$y(x) = \frac{(ax+b)^{\frac{3a-c}{a}} \left(c_1 x^{\frac{-3a+c}{a}} \operatorname{HeunC}\left(\frac{\lambda a}{b}, 1-\frac{c}{a}, 3-\frac{c}{a}, 0, -\frac{\lambda a}{b}+\frac{c\lambda}{2b}+\frac{5}{2}-\frac{2c}{a}+\frac{c^2}{2a^2}, -\frac{b}{ax}\right) x^2 + c_2 \operatorname{HeunC}\left(\frac{\lambda a}{b}, \frac{c}{a}\right.\right.}{x^2}$$

✓ Solution by Mathematica

Time used: 1.48 (sec). Leaf size: 55

```
DSolve[x^2*(a*x+b)*y''[x]+(c*x^2+(2*b+a*\[Lambda])*x+b*\[Lambda])*y'[x]+\[Lambda]*(c-2*a)*y[
```

$$y(x) \to e^{\frac{\lambda}{x}} \left(c_2 \int_1^x \frac{e^{-\frac{\lambda}{K[1]}} (b+aK[1])^{2-\frac{c}{a}}}{K[1]^2} dK[1] + c_1\right)$$

31.11 problem 192

Internal problem ID [11016]
Internal file name [`OUTPUT/10272_Sunday_December_31_2023_11_33_38_AM_27467238/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 192.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(ax+b)\,y'' - 2x(ax+2b)\,y' + 2(ax+3b)\,y = 0}$$

31.11.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{-2a\,x^2 - 4bx}{a\,x^3 + b\,x^2} \\ q(x) &= \frac{2ax + 6b}{a\,x^3 + b\,x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{2ax+6b}{a\,x^3+b\,x^2} - \frac{\left(\frac{-2a\,x^2-4bx}{a\,x^3+b\,x^2}\right)'}{2} - \frac{\left(\frac{-2a\,x^2-4bx}{a\,x^3+b\,x^2}\right)^2}{4} \\
&= \frac{2ax+6b}{a\,x^3+b\,x^2} - \frac{\left(\frac{-4ax-4b}{a\,x^3+b\,x^2} - \frac{(-2a\,x^2-4bx)\,(3a\,x^2+2bx)}{(a\,x^3+b\,x^2)^2}\right)}{2} - \frac{\left(\frac{(-2a\,x^2-4bx)^2}{(a\,x^3+b\,x^2)^2}\right)}{4} \\
&= \frac{2ax+6b}{a\,x^3+b\,x^2} - \left(\frac{-4ax-4b}{2a\,x^3+2b\,x^2} - \frac{(-2a\,x^2-4bx)\,(3a\,x^2+2bx)}{2\,(a\,x^3+b\,x^2)^2}\right) - \frac{(-2a\,x^2-4bx)^2}{4\,(a\,x^3+b\,x^2)^2} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\,z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2}dx\right)} \\
&= e^{-\int \frac{\frac{-2a\,x^2-4bx}{a\,x^3+b\,x^2}}{2}} \\
&= \frac{x^2}{ax+b}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = \frac{v(x)\,x^2}{ax+b} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
x^4 v''(x) = 0
$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$
v(x) = c_1 x + c_2
$$

Now that $v(x)$ is known, then

$$
\begin{aligned}
y &= v(x)\,z(x) \\
&= (c_1 x + c_2)\,(z(x))
\end{aligned} \tag{7}
$$

But from (5)

$$z(x) = \frac{x^2}{ax+b}$$

Hence (7) becomes

$$y = \frac{(c_1x+c_2)\,x^2}{ax+b}$$

Summary

The solution(s) found are the following

$$y = \frac{(c_1x+c_2)\,x^2}{ax+b} \tag{1}$$

Verification of solutions

$$y = \frac{(c_1x+c_2)\,x^2}{ax+b}$$

Verified OK.

31.11.2 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2(ax+b)\,y'' + \left(-2a\,x^2-4bx\right)y' + (2ax+6b)\,y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{-2ax-4b}{(ax+b)\,x}$$
$$q(x) = \frac{2ax+6b}{x^2\,(ax+b)}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x}+p\right)v'(x) + \left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(-2ax-4b)}{(ax+b)\,x^2} + \frac{2ax+6b}{x^2\,(ax+b)} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{4}{x} + \frac{-2ax-4b}{(ax+b)\,x}\right) v'(x) &= 0 \\ v''(x) + \frac{2av'(x)}{ax+b} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2au(x)}{ax+b} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2au}{ax+b} \end{aligned}$$

Where $f(x) = -\frac{2a}{ax+b}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2a}{ax+b}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{2a}{ax+b}\,dx \\ \ln(u) &= -2\ln(ax+b) + c_1 \\ u &= \mathrm{e}^{-2\ln(ax+b)+c_1} \\ &= \frac{c_1}{(ax+b)^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{(ax+b)\,a} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{(ax+b)\,a} + c_2\right)x^2 \\
&= \left(-\frac{c_1}{(ax+b)\,a} + c_2\right)x^2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{(ax+b)\,a} + c_2\right)x^2 \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{(ax+b)\,a} + c_2\right)x^2
$$

Verified OK.

31.11.3 Solving using Kovacic algorithm

Writing the ode as

$$
x^2(ax+b)\,y'' + \left(-2a\,x^2 - 4bx\right)y' + (2ax+6b)\,y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2(ax+b) \\
B &= -2a\,x^2 - 4bx \\
C &= 2ax+6b
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 180: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2a\,x^2-4bx}{x^2(ax+b)}\,dx} \\
&= z_1 e^{2\ln(x)-\ln(ax+b)} \\
&= z_1\left(\frac{x^2}{ax+b}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2}{ax+b}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2a\,x^2-4bx}{x^2(ax+b)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)-2\ln(ax+b)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2}{ax+b}\right) + c_2\left(\frac{x^2}{ax+b}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^2}{ax+b} + \frac{c_2 x^3}{ax+b} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^2}{ax+b} + \frac{c_2 x^3}{ax+b}
$$

Verified OK.

31.11.4 Maple step by step solution

Let's solve

$x^2(ax+b)\,y'' + (-2a\,x^2 - 4bx)\,y' + (2ax+6b)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(ax+3b)y}{x^2(ax+b)} + \frac{2(ax+2b)y'}{x(ax+b)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(ax+2b)y'}{x(ax+b)} + \frac{2(ax+3b)y}{x^2(ax+b)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(ax+2b)}{(ax+b)x}, P_3(x) = \frac{2(ax+3b)}{x^2(ax+b)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -4$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^{2} \cdot P_{3}(x)\right)\bigg|_{x=0} = 6$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^{2}(ax+b)\,y'' - 2x(ax+2b)\,y' + (2ax+6b)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $b a_0(-2+r)\,(-3+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(b a_k(k+r-2)\,(k+r-3) + a a_{k-1}(k+r-2)\,(k+r-3)\right) x^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$b(-2+r)(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)(k+r-3)(a a_{k-1} + a_k b) = 0$

- Shift index using $k- > k+1$

  $(k+r-1)(k+r-2)(a_k a + b a_{k+1}) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a a_k}{b}$

- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{a a_k}{b}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a a_k}{b}\right]$

- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a a_k}{b}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = -\frac{a a_k}{b}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} c_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} d_k x^{k+3}\right), c_{1+k} = -\frac{a c_k}{b}, d_{1+k} = -\frac{a d_k}{b}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(x^2*(a*x+b)*diff(y(x),x$2)-2*x*(a*x+2*b)*diff(y(x),x)+2*(a*x+3*b)*y(x)=0,y(x), singso
```

$$y(x) = \frac{x^2(c_2x + c_1)}{ax + b}$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 23

```
DSolve[x^2*(a*x+b)*y''[x]-2*x*(a*x+2*b)*y'[x]+2*(a*x+3*b)*y[x]==0,y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{x^2(c_2x + c_1)}{ax + b}$$

31.12 problem 193

Internal problem ID [11017]
Internal file name [`OUTPUT/10273_Sunday_December_31_2023_11_39_39_AM_16279652/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 193.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$x^2(ax+b)\,y'' + \left(a(2-m-n)\,x^2 - b(m+n)\,x\right)y' + \left(am(n-1)\,x + bn(m+1)\right)y = 0$$

31.12.1 Solving using Kovacic algorithm

Writing the ode as

$$x^2(ax+b)\,y'' - x(a(m+n-2)\,x + (m+n)\,b)\,y' + (((ax+b)\,n - ax)\,m + nb)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(ax+b) \\ B &= -x(a(m+n-2)\,x + (m+n)\,b) \\ C &= ((ax+b)\,n - ax)\,m + nb \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{m^2 - 2nm + n^2 + 2m - 2n}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= m^2 - 2nm + n^2 + 2m - 2n \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{m^2 - 2nm + n^2 + 2m - 2n}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 182: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{\frac{1}{4}m^2 - \frac{1}{2}nm + \frac{1}{4}n^2 + \frac{1}{2}m - \frac{1}{2}n}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{(m-n+2)(m-n)}{4}$. Hence

$$
\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \left\{2, 2 - 2\sqrt{(-m - 1 + n)^2}, 2 + 2\sqrt{(-m - 1 + n)^2}\right\}
\end{aligned}
$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{m^2 - 2nm + n^2 + 2m - 2n}{4x^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{1}{4}$. Hence

$$
\begin{aligned}
E_\infty &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{2\}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\left\{2, 2-2\sqrt{(-m-1+n)^2}, 2+2\sqrt{(-m-1+n)^2}\right\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 2,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2-(2)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{2}{(x-(0))}\right) \\ &= \frac{1}{x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{x} - \frac{(m-n+2)(m-n)}{4x^2} = 0$$

Solving for ω gives

$$\omega = -\frac{m-n}{2x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= e^{\int -\frac{m-n}{2x} dx} \\ &= x^{-\frac{m}{2}+\frac{n}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x(a(m+n-2)x+(m+n)b)}{x^2(ax+b)} \, dx} \\ &= z_1 e^{\frac{(m+n)\ln(x)}{2} - \ln(ax+b)} \\ &= z_1 \left(\frac{x^{\frac{m}{2}+\frac{n}{2}}}{ax+b}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^n}{ax+b}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-x(a(m+n-2)x+(m+n)b)}{x^2(ax+b)}dx}}{(y_1)^2} dx$$

$$= y_1 \int \frac{e^{-2\ln(ax+b)+(m+n)\ln(x)}}{(y_1)^2} dx$$

$$= y_1 \left(\frac{x^{m+1-n}}{m+1-n}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x^n}{ax+b}\right) + c_2 \left(\frac{x^n}{ax+b}\left(\frac{x^{m+1-n}}{m+1-n}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^n}{ax+b} + \frac{c_2 x^{m+1}}{(m+1-n)(ax+b)} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^n}{ax+b} + \frac{c_2 x^{m+1}}{(m+1-n)(ax+b)}$$

Verified OK.

31.12.2 Maple step by step solution

Let's solve

$x^2(ax+b)y'' - x(a(m+n-2)x+(m+n)b)y' + (((ax+b)n-ax)m+nb)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(axmn-axm+nbm+nb)y}{x^2(ax+b)} + \frac{(axm+anx-2ax+bm+nb)y'}{(ax+b)x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{(axm+anx-2ax+bm+nb)y'}{(ax+b)x} + \frac{(axmn-axm+nbm+nb)y}{x^2(ax+b)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{axm+anx-2ax+bm+nb}{(ax+b)x}, P_3(x) = \frac{axmn-axm+nbm+nb}{x^2(ax+b)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -\frac{bm+nb}{b}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{nbm+nb}{b}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(ax+b)\,y'' - (axm+anx-2ax+bm+nb)\,y'x + (axmn-axm+nbm+nb)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$b a_0(-n+r)(-m+r-1)\, x^r + \left(\sum_{k=1}^{\infty} (b a_k(k-n+r)(k-m+r-1) + a a_{k-1}(k-n+r)(k-n$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$b(-n+r)(-m+r-1) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{n, m+1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k-n+r)(k-m+r-1)(a a_{k-1} + b a_k) = 0$$

- Shift index using $k- >k+1$

$$(k-n+r+1)(k-m+r)(a a_k + b a_{k+1}) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a a_k}{b}$$

- Recursion relation for $r = n$

$$a_{k+1} = -\frac{a a_k}{b}$$

- Solution for $r = n$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+n}, a_{k+1} = -\frac{a a_k}{b}\right]$$

- Recursion relation for $r = m+1$

$$a_{k+1} = -\frac{a a_k}{b}$$

- Solution for $r = m+1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+m+1}, a_{k+1} = -\frac{a a_k}{b}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} c_k x^{k+n}\right) + \left(\sum_{k=0}^{\infty} d_k x^{k+m+1}\right), c_{1+k} = -\frac{a c_k}{b}, d_{1+k} = -\frac{a d_k}{b}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 25

```
dsolve(x^2*(a*x+b)*diff(y(x),x$2)+(a*(2-n-m)*x^2-b*(n+m)*x)*diff(y(x),x)+(a*m*(n-1)*x+b*n*(m
```

$$y(x) = \frac{c_1 x^n + c_2 x^{1+m}}{ax + b}$$

✓ Solution by Mathematica

Time used: 0.403 (sec). Leaf size: 82

```
DSolve[x^2*(a*x+b)*y''[x]+(a*(2-n-m)*x^2-b*(n+m)*x)*y'[x]+(a*m*(n-1)*x+b*n*(m+1))*y[x]==0,y[
```

$$y(x) \to \frac{x^{\frac{1}{2}\left(-\sqrt{(m-n+1)^2}+m+n+1\right)}\left(c_2 x^{\sqrt{(m-n+1)^2}} + c_1\sqrt{(m-n+1)^2}\right)}{\sqrt{(m-n+1)^2}(ax+b)}$$

31.13 problem 194

Internal problem ID [11018]
Internal file name [OUTPUT/10274_Sunday_December_31_2023_11_39_42_AM_20266088/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 194.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$x^2(x + a_2)\, y'' + x(b_1x + a_1)\, y' + (b_0x + a_0)\, y = 0$$

31.13.1 Maple step by step solution

Let's solve

$x^2(x + a_2)\, y'' + x(b_1x + a_1)\, y' + (b_0x + a_0)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(b_0x+a_0)y}{x^2(x+a_2)} - \frac{(b_1x+a_1)y'}{x(x+a_2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(b_1x+a_1)y'}{x(x+a_2)} + \frac{(b_0x+a_0)y}{x^2(x+a_2)} = 0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x) = \frac{b_1 x + a_1}{x(x+a_2)}, P_3(x) = \frac{b_0 x + a_0}{x^2(x+a_2)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{a_1}{a_2}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{a_0}{a_2}$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x+a_2)\, y'' + x(b_1 x + a_1)\, y' + (b_0 x + a_0)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- ◦ Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- ◦ Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- ◦ Shift index using $k- >k + 2 - m$

$$x^{m}\cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_{0}(a_{2}r^{2}+a_{1}r-a_{2}r+a_{0})\,x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k}(a_{2}k^{2}+2a_{2}kr+a_{2}r^{2}+a_{1}k+a_{1}r-a_{2}k-a_{2}r+a_{0})+a\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$a_{2}r^{2}+a_{1}r-a_{2}r+a_{0}=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{-\frac{-a_{2}+a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}}{2a_{2}},\frac{a_{2}-a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}}{2a_{2}}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^{2}+(2r+b_{1}-3)\,k+r^{2}+(b_{1}-3)\,r+b_{0}-b_{1}+2)\,a_{k-1}+(a_{2}k^{2}+(2a_{2}r+a_{1}-a_{2})\,k+a_{2}r^{2}+$$

- Shift index using $k->k+1$

$$\left((k+1)^{2}+(2r+b_{1}-3)\,(k+1)+r^{2}+(b_{1}-3)\,r+b_{0}-b_{1}+2\right)a_{k}+\left(a_{2}(k+1)^{2}+(2a_{2}r+a_{1}\right.$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{(b_{1}k+b_{1}r+k^{2}+2kr+r^{2}+b_{0}-k-r)\,a_{k}}{a_{2}k^{2}+2a_{2}kr+a_{2}r^{2}+a_{1}k+a_{1}r+a_{2}k+a_{2}r+a_{0}+a_{1}}$$

- Recursion relation for $r=-\frac{-a_{2}+a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}}{2a_{2}}$

$$a_{k+1}=-\frac{\left(b_{1}k-\frac{b_{1}\left(-a_{2}+a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}\right)}{2a_{2}}+k^{2}-\frac{k\left(-a_{2}+a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}\right)}{a_{2}}+\frac{\left(-a_{2}+a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a\right.}{4a_{2}^{2}}\right.}{a_{2}k^{2}-k\left(-a_{2}+a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}\right)+\frac{\left(-a_{2}+a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}\right)^{2}}{4a_{2}}+a_{1}k-\frac{a_{1}\left(-a_{2}+a_{1}+\sqrt{-4a_{0}a_{2}+a\right.}{2a_{2}}}$$

- Solution for $r=-\frac{-a_{2}+a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}}{2a_{2}}$

$$\left[y=\sum_{k=0}^{\infty}a_{k}x^{k-\frac{-a_{2}+a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}}{2a_{2}}},a_{k+1}=-\frac{\left(b_{1}k-\frac{b_{1}\left(-a_{2}+a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}\right)}{2a_{2}}+k^{2}-\frac{k\left(-a_{2}+a_{1}\right.}{}\right.}{a_{2}k^{2}-k\left(-a_{2}+a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}\right)+\frac{\left(-a_{2}+a_{1}+\sqrt{-}\right.}{}}\right.$$

- Recursion relation for $r=\frac{a_{2}-a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}}{2a_{2}}$

$$a_{k+1}=-\frac{\left(b_{1}k+\frac{b_{1}\left(a_{2}-a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}\right)}{2a_{2}}+k^{2}+\frac{k\left(a_{2}-a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}\right)}{a_{2}}+\frac{\left(a_{2}-a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}\right.}{4a_{2}^{2}}\right.}{a_{2}k^{2}+k\left(a_{2}-a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}\right)+\frac{\left(a_{2}-a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a_{1}a_{2}+a_{2}^{2}}\right)^{2}}{4a_{2}}+a_{1}k+\frac{a_{1}\left(a_{2}-a_{1}+\sqrt{-4a_{0}a_{2}+a_{1}^{2}-2a\right.}{2a_{2}}}$$

- Solution for $r = \frac{a_2-a_1+\sqrt{-4a_0a_2+a_1^2-2a_1a_2+a_2^2}}{2a_2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{a_2-a_1+\sqrt{-4a_0a_2+a_1^2-2a_1a_2+a_2^2}}{2a_2}}, a_{k+1} = -\frac{\left(b_1 k+\frac{b_1\left(a_2-a_1+\sqrt{-4a_0a_2+a_1^2-2a_1a_2+a_2^2}\right)}{2a_2}+k^2+\frac{k\left(a_2-a_1+\sqrt{\right.}}{}\right.}{a_2k^2+k\left(a_2-a_1+\sqrt{-4a_0a_2+a_1^2-2a_1a_2+a_2^2}\right)+\frac{\left(a_2-a_1+\sqrt{-4a_0\right.}}{}}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{-a_2+a_1+\sqrt{-4a_0a_2+a_1^2-2a_1a_2+a_2^2}}{2a_2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{a_2-a_1+\sqrt{-4a_0a_2+a_1^2-2a_1a_2+a_2^2}}{2a_2}}\right), a_{1+k} = -\frac{\left(\right.}{a_2k}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 317

```
dsolve(x^2*(x+a__2)*diff(y(x),x$2)+x*(b__1*x+a__1)*diff(y(x),x)+(b__0*x+a__0)*y(x)=0,y(x), s
```

$$y(x) = c_1 x^{\frac{a_2-a_1+\sqrt{a_2^2+(-4a_0-2a_1)a_2+a_1^2}}{2a_2}} \text{hypergeom}\left(\left[\frac{a_2b_1-a_1+\sqrt{a_2^2+(-4a_0-2a_1)\,a_2+a_1^2}-\sqrt{b_1^2-4b_0-2b_1+}}{2a_2}\right.\right.$$

$$\left.-\frac{x}{a_2}\right)$$

$$+c_2 x^{-\frac{-a_2+a_1+\sqrt{a_2^2+(-4a_0-2a_1)a_2+a_1^2}}{2a_2}} \text{hypergeom}\left(\left[-\frac{-\sqrt{b_1^2-4b_0-2b_1+1}\,a_2-a_2b_1+\sqrt{a_2^2+(-4a_0-2a_1)}}{2a_2}\right.\right.$$

$$-\frac{\sqrt{b_1^2-4b_0-2b_1+1}\,a_2-a_2b_1+\sqrt{a_2^2+(-4a_0-2a_1)\,a_2+a_1^2}+a_1}{2a_2}\Bigg], \left[\frac{a_2-\sqrt{a_2^2+(-4a_0-2a_1)\,a_2}}{a_2}\right.$$

$$\left.-\frac{x}{a_2}\right)$$

✓ Solution by Mathematica

Time used: 1.236 (sec). Leaf size: 384

```
DSolve[x^2*(x+a2)*y''[x]+x*(b1*x+a1)*y'[x]+(b0*x+a0)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \text{a2}^{-\frac{\sqrt{-4\text{a0a2}+\text{a1}^2-2\text{a1a2}+\text{a2}^2}-\text{a1}+\text{a2}}{2\text{a2}}} x^{-\frac{\sqrt{-4\text{a0a2}+\text{a1}^2-2\text{a1a2}+\text{a2}^2}+\text{a1}-\text{a2}}{2\text{a2}}} \left(c_2 x^{\frac{\sqrt{-4\text{a0a2}+\text{a1}^2-2\text{a1a2}+\text{a2}^2}}{\text{a2}}} \text{Hypergeometric2F1}\right.$$

$$\left.-\frac{x}{\text{a2}}\right)$$

$$+c_1 \text{a2}^{\frac{\sqrt{-4\text{a0a2}+\text{a1}^2-2\text{a1a2}+\text{a2}^2}}{\text{a2}}} \text{Hypergeometric2F1}\left(-\frac{\text{a1}-\text{a2b1}+\sqrt{\text{a1}^2-2\text{a2a1}+\text{a2}(\text{a2}-4\text{a0})}+\text{a2}\sqrt{(}}{2\text{a2}}\right.$$

$$-\frac{\text{a1}-\text{a2}\left(\text{b1}+\sqrt{(\text{b1}-1)^2-4\text{b0}}\right)+\sqrt{\text{a1}^2-2\text{a2a1}+\text{a2}(\text{a2}-4\text{a0})}}{2\text{a2}}, 1$$

$$\left.\left.-\frac{\sqrt{\text{a1}^2-2\text{a2a1}+\text{a2}^2-4\text{a0a2}}}{\text{a2}}, -\frac{x}{\text{a2}}\right)\right)$$

31.14 problem 195

Internal problem ID [11019]
Internal file name [OUTPUT/10275_Sunday_December_31_2023_11_39_43_AM_40293410/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 195.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\left(a\,x^3+b\,x^2+cx\right)y''+\left(\alpha\,x^2+\beta x+2c\right)y'+\left(\beta-2b\right)y=0$$

31.14.1 Maple step by step solution

Let's solve

$y''x(a\,x^2+bx+c)+\left(\alpha\,x^2+\beta x+2c\right)y'+\left(\beta-2b\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(-\beta+2b)y}{x(a\,x^2+bx+c)}-\frac{(\alpha\,x^2+\beta x+2c)y'}{x(a\,x^2+bx+c)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(\alpha\,x^2+\beta x+2c)y'}{x(a\,x^2+bx+c)}-\frac{(-\beta+2b)y}{x(a\,x^2+bx+c)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$$\left[P_2(x)=\frac{\alpha x^2+\beta x+2c}{x(a x^2+bx+c)}, P_3(x)=-\frac{-\beta+2b}{x(a x^2+bx+c)}\right]$$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $$\left(x\cdot P_2(x)\right)\Big|_{x=0}=2$$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $$\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$$

- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $$y''x(a x^2+bx+c)+(\alpha x^2+\beta x+2c)\,y'+(\beta-2b)\,y=0$$

- Assume series solution for y

  $$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y'$ to series expansion for $m=0..2$

  $$x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m\cdot y''$ to series expansion for $m=1..3$

  $$x^m\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$ca_0r(r+1)\,x^{r-1}+\left(ca_1(r+1)\,(2+r)+a_0(r+1)\,(br-2b+\beta)\right)x^r+\left(\sum_{k=1}^{\infty}\left(ca_{k+1}(k+r+1)\,(k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$cr(r+1) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- Each term must be 0

$$ca_1(r+1)(2+r) + a_0(r+1)(br - 2b + \beta) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$ca_{k+1}(k+r+1)(k+2+r) + ((k+r-2)b+\beta)(k+r+1)a_k + ((k+r-2)a+\alpha)(k+r-1$$

- Shift index using $k- >k+1$

$$ca_{k+2}(k+2+r)(k+3+r) + ((k+r-1)b+\beta)(k+2+r)a_{k+1} + ((k+r-1)a+\alpha)(k+r)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a\,k^2 a_k + 2akra_k + a\,r^2 a_k + b\,k^2 a_{k+1} + 2bkra_{k+1} + b\,r^2 a_{k+1} - aka_k - ara_k + a_k\alpha k + a_k\alpha r + bka_{k+1} + bra_{k+1} + \beta ka_{k+1} + \beta ra_{k+}}{c(k+2+r)(k+3+r)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{a\,k^2 a_k + b\,k^2 a_{k+1} - 3aka_k + a_k\alpha k - bka_{k+1} + \beta ka_{k+1} + 2aa_k - a_k\alpha - 2ba_{k+1} + \beta a_{k+1}}{c(k+1)(k+2)}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a\,k^2 a_k + b\,k^2 a_{k+1} - 3aka_k + a_k\alpha k - bka_{k+1} + \beta ka_{k+1} + 2aa_k - a_k\alpha - 2ba_{k+1} + \beta a_{k+1}}{c(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a\,k^2 a_k + b\,k^2 a_{k+1} - aka_k + a_k\alpha k + bka_{k+1} + \beta ka_{k+1} - 2ba_{k+1} + 2\beta a_{k+1}}{c(k+2)(k+3)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a\,k^2 a_k + b\,k^2 a_{k+1} - aka_k + a_k\alpha k + bka_{k+1} + \beta ka_{k+1} - 2ba_{k+1} + 2\beta a_{k+1}}{c(k+2)(k+3)}, 2ca_1 + a_0(\beta - 2b) =\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} d_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} e_k x^k\right), d_{k+2} = -\frac{a\,k^2 d_k + b\,k^2 d_{1+k} - 3akd_k + \alpha kd_k - bkd_{1+k} + \beta kd_{1+k} + 2ad_k - \alpha d_k - 2bd_{1+}}{c(1+k)(k+2)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 9.625 (sec). Leaf size: 1505

```
dsolve((a*x^3+b*x^2+c*x)*diff(y(x),x$2)+(alpha*x^2+beta*x+2*c)*diff(y(x),x)+(beta-2*b)*y(x)=
```

Expression too large to display

✓ Solution by Mathematica

Time used: 4.64 (sec). Leaf size: 139

```
DSolve[(a*x^3+b*x^2+c*x)*y''[x]+(\[Alpha]*x^2+\[Beta]*x+2*c)*y'[x]+(\[Beta]-2*b)*y[x]==0,y[x
```

$$y(x) \to \frac{(2ax+2b-\beta-\alpha x)\left(c_2\int_1^x \frac{\exp\left(\frac{(b\alpha+2a(b-\beta))\arctan\left(\frac{b+2aK[1]}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}\right)(c+K[1](b+aK[1]))^{1-\frac{\alpha}{2a}}}{(-2b+\beta+(\alpha-2a)K[1])^2}dK[1]+c_1\right)}{x(2b-\beta)}$$

31.15 problem 196

Internal problem ID [11020]
Internal file name [`OUTPUT/10276_Sunday_December_31_2023_11_39_46_AM_38210555/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 196.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\left(a\,x^3+b\,x^2+cx\right)y''+\left(\alpha\,x^2+\beta x+2c\right)y'-\left(x\alpha+2b-\beta\right)y=0$$

31.15.1 Maple step by step solution

Let's solve

$y''x(a\,x^2+bx+c)+(\alpha\,x^2+\beta x+2c)\,y'+(-x\alpha-2b+\beta)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(x\alpha+2b-\beta)y}{x(a\,x^2+bx+c)}-\frac{(\alpha\,x^2+\beta x+2c)y'}{x(a\,x^2+bx+c)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(\alpha\,x^2+\beta x+2c)y'}{x(a\,x^2+bx+c)}-\frac{(x\alpha+2b-\beta)y}{x(a\,x^2+bx+c)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x) = \frac{\alpha x^2+\beta x+2c}{x(a x^2+bx+c)}, P_3(x) = -\frac{x\alpha+2b-\beta}{x(a x^2+bx+c)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 2$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x(a x^2 + bx + c) + (\alpha x^2 + \beta x + 2c) y' + (-x\alpha - 2b + \beta) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r) x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1) x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$c a_0 r(r+1)\,x^{r-1} + (c a_1(r+1)\,(2+r) + a_0(r+1)\,(br-2b+\beta))\,x^r + \left(\sum_{k=1}^{\infty} (c a_{k+1}(k+r+1)\,(k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$cr(r+1) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- Each term must be 0

$$c a_1(r+1)\,(2+r) + a_0(r+1)\,(br-2b+\beta) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$c a_{k+1}(k+r+1)\,(k+2+r) + ((k+r-2)\,b+\beta)\,(k+r+1)\,a_k + (k+r-2)\,((k+r-1)\,a+c$$

- Shift index using $k->k+1$

$$c a_{k+2}(k+2+r)\,(k+3+r) + ((k+r-1)\,b+\beta)\,(k+2+r)\,a_{k+1} + (k+r-1)\,((k+r)\,a+\alpha)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a\,k^2 a_k + 2akra_k + a\,r^2 a_k + b\,k^2 a_{k+1} + 2bkra_{k+1} + b\,r^2 a_{k+1} - aka_k - ara_k + a_k\alpha k + a_k\alpha r + bka_{k+1} + bra_{k+1} + \beta ka_{k+1} + \beta ra_{k}}{c(k+2+r)(k+3+r)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{a\,k^2 a_k + b\,k^2 a_{k+1} - 3aka_k + a_k\alpha k - bka_{k+1} + \beta ka_{k+1} + 2aa_k - 2a_k\alpha - 2ba_{k+1} + \beta a_{k+1}}{c(k+1)(k+2)}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a\,k^2 a_k + b\,k^2 a_{k+1} - 3aka_k + a_k\alpha k - bka_{k+1} + \beta ka_{k+1} + 2aa_k - 2a_k\alpha - 2ba_{k+1} + \beta a_{k+1}}{c(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a\,k^2 a_k + b\,k^2 a_{k+1} - aka_k + a_k\alpha k + bka_{k+1} + \beta ka_{k+1} - a_k\alpha - 2ba_{k+1} + 2\beta a_{k+1}}{c(k+2)(k+3)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a\,k^2 a_k + b\,k^2 a_{k+1} - aka_k + a_k\alpha k + bka_{k+1} + \beta ka_{k+1} - a_k\alpha - 2ba_{k+1} + 2\beta a_{k+1}}{c(k+2)(k+3)}, 2ca_1 + a_0(\beta - 2b\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} d_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} e_k x^k\right), d_{k+2} = -\frac{a\,k^2 d_k + b\,k^2 d_{1+k} - 3akd_k + \alpha kd_k - bkd_{1+k} + \beta kd_{1+k} + 2ad_k - 2\alpha d_k - 2bd_1}{c(1+k)(k+2)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 1.062 (sec). Leaf size: 1505

```
dsolve((a*x^3+b*x^2+c*x)*diff(y(x),x$2)+(alpha*x^2+beta*x+2*c)*diff(y(x),x)-(alpha*x+2*b-bet
```

Expression too large to display

✓ Solution by Mathematica

Time used: 6.092 (sec). Leaf size: 224

```
DSolve[(a*x^3+b*x^2+c*x)*y''[x]+(\[Alpha]*x^2+\[Beta]*x+2*c)*y'[x]-(\[Alpha]*x+2*b-\[Beta])*
```

$y(x)$

$$\rightarrow \frac{(b(2ax-3\beta-2\alpha x)-a\alpha x^2-2a\beta x+2b^2+\beta^2+\alpha c+\alpha^2x^2+2\alpha\beta x)\left(c_2\int_1^x \frac{\exp\left(\frac{(b\alpha+2a(b-\beta))\arctan\left(\frac{b\cdot}{\sqrt{\cdot}}\right.}{a\sqrt{4ac-b^2}}\right.}{(2b^2-3\beta b+2(a-\alpha)K[1]b+\beta^2+\alpha^2\cdot}\right.}{x\,(2b^2+\beta^2-3\beta b+\alpha c)}$$

31.16 problem 197

Internal problem ID [11021]
Internal file name [`OUTPUT/10277_Sunday_December_31_2023_11_39_47_AM_81148635/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 197.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\left(a\,x^3+b\,x^2+cx\right)y''+\left(-2a\,x^2-(1+b)\,x+k\right)y'+2(ax+1)\,y=0$$

31.16.1 Maple step by step solution

Let's solve

$y''x(a\,x^2+bx+c)+\left(-2a\,x^2+(-1-b)\,x+k\right)y'+(2ax+2)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2(ax+1)y}{x(a\,x^2+bx+c)}+\frac{(2a\,x^2+bx-k+x)y'}{x(a\,x^2+bx+c)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(2a\,x^2+bx-k+x)y'}{x(a\,x^2+bx+c)}+\frac{2(ax+1)y}{x(a\,x^2+bx+c)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

$$\left[P_2(x) = -\frac{2a\,x^2+bx-k+x}{x(a\,x^2+bx+c)}, P_3(x) = \frac{2(ax+1)}{x(a\,x^2+bx+c)}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\bigg|_{x=0} = \frac{k}{c}$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x(a\,x^2+bx+c) + (-2a\,x^2-bx+k-x)\,y' + (2ax+2)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(cr-c+k)\, x^{r-1} + (a_1(1+r)(cr+k) + a_0(r-2)(br-1))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(c(k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(cr-c+k) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{c-k}{c}\right\}$$

- Each term must be 0

$$a_1(1+r)(cr+k) + a_0(r-2)(br-1) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)((k+r)c+k) + (k+r-2)\, a_k(-1+(k+r)b) + a a_{k-1}(k+r-2)(k-3+r) =$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2+r)((k+1+r)c+k) + (k+r-1)\, a_{k+1}(-1+(k+1+r)b) + a a_k(k+r-1)(k+$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a\,k^2 a_k + 2akra_k + a\,r^2 a_k + b\,k^2 a_{k+1} + 2bkra_{k+1} + b\,r^2 a_{k+1} - 3aka_k - 3ara_k + 2a_k a - ba_{k+1} - ka_{k+1} - ra_{k+1} + a_{k+1}}{(k+2+r)(ck+cr+c+k)}$$

- Recursion relation for $r=0$; series terminates at $k=1$

$$a_{k+2} = -\frac{a\,k^2 a_k + b\,k^2 a_{k+1} - 3aka_k + 2a_k a - ba_{k+1} - ka_{k+1} + a_{k+1}}{(k+2)(ck+c+k)}$$

- Solution for $r=0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a\,k^2 a_k + b\,k^2 a_{k+1} - 3aka_k + 2a_k a - ba_{k+1} - ka_{k+1} + a_{k+1}}{(k+2)(ck+c+k)}, a_1 k + 2a_0 = 0\right]$$

- Recursion relation for $r = \frac{c-k}{c}$

$$a_{k+2} = -\frac{a\,k^2 a_k + \frac{2ak(c-k)a_k}{c} + \frac{a(c-k)^2 a_k}{c^2} + b\,k^2 a_{k+1} + \frac{2bk(c-k)a_{k+1}}{c} + \frac{b(c-k)^2 a_{k+1}}{c^2} - 3aka_k - \frac{3a(c-k)a_k}{c} + 2a_k a - ba_{k+1} - ka_{k+1} - \frac{(}{}}{\left(k+2+\frac{c-k}{c}\right)(ck+2c)}$$

- Solution for $r = \frac{c-k}{c}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{c-k}{c}}, a_{k+2} = -\frac{a\,k^2 a_k + \frac{2ak(c-k)a_k}{c} + \frac{a(c-k)^2 a_k}{c^2} + b\,k^2 a_{k+1} + \frac{2bk(c-k)a_{k+1}}{c} + \frac{b(c-k)^2 a_{k+1}}{c^2} - 3aka_k - \frac{3a(c-k)a_k}{c}}{\left(k+2+\frac{c-k}{c}\right)(ck+2c)}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{m=0}^{\infty} d_m x^m\right) + \left(\sum_{m=0}^{\infty} e_m x^{m+\frac{c-k}{c}}\right), d_{m+2} = -\frac{a\,m^2 d_m + b\,m^2 d_{m+1} - 3amd_m + 2ad_m - bd_{m+1} - md_{m+1} + d_{m+1}}{(m+2)(cm+c+k)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.437 (sec). Leaf size: 2278

```
dsolve((a*x^3+b*x^2+c*x)*diff(y(x),x$2)+(-2*a*x^2-(b+1)*x+k)*diff(y(x),x)+2*(a*x+1)*y(x)=0,y
```

Expression too large to display

✓ Solution by Mathematica

Time used: 10.126 (sec). Leaf size: 186

```
DSolve[(a*x^3+b*x^2+c*x)*y''[x]+(-2*a*x^2-(b+1)*x+k)*y'[x]+2*(a*x+1)*y[x]==0,y[x],x,IncludeS
```

$$y(x) \to -\frac{(-kx(ax+2)-(b-1)x^2+c(k-2x)+k^2)\left(c_2\int_1^x \frac{\exp\left(\frac{(2c+bk)\arctan\left(\frac{b+2aK[1]}{\sqrt{4ac-b^2}}\right)}{c\sqrt{4ac-b^2}}\right)K[1]^{-\frac{k}{c}}(c+K[1](b+aK[1]))^{\frac{k}{2c}+1}}{(k^2-K[1](aK[1]+2)k-(b-1)K[1]^2+c(k-2K[1]))^2}\right.}{ak+b-c(k-2)-k^2+2k-1}$$

31.17 problem 198

Internal problem ID [11022]
Internal file name [`OUTPUT/10278_Sunday_December_31_2023_11_39_49_AM_273934/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 198.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$(a\,x^3 + b\,x^2 + cx)\,y'' + (n\,x^2 + mx + k)\,y' + (k-1)\,((-ak+n)\,x + m - bk)\,y = 0$$

✗ Solution by Maple

```
dsolve((a*x^3+b*x^2+c*x)*diff(y(x),x$2)+(n*x^2+m*x+k)*diff(y(x),x)+(k-1)*((n-a*k)*x+m-b*k)*y
```

No solution found

✓ Solution by Mathematica

Time used: 148.451 (sec). Leaf size: 570

```
DSolve[(a*x^3+b*x^2+c*x)*y''[x]+(n*x^2+m*x+k)*y'[x]+(k-1)*((n-a*k)*x+m-b*k)*y[x]==0,y[x],x,I
```

$y(x) \to$

$$-\frac{2^{-\frac{k}{c}}\left(-\frac{ax}{\sqrt{b^2-4ac}+b}\right)^{1-\frac{k}{c}}\left(ac_1x\left(-2^{\frac{k}{c}}\right)\left(-\frac{ax}{\sqrt{b^2-4ac}+b}\right)^{\frac{k}{c}-1}\text{HeunG}\left[\frac{b-\sqrt{b^2-4ac}}{\sqrt{b^2-4ac}+b},-\frac{2(k-1)(bk-m)}{\sqrt{b^2-4ac}+b},\frac{1}{2}\left(-\sqrt{\frac{(-2ak+\dots}{a\dots}}\right.\right.\right.}{\dots}$$

31.18 problem 199

Internal problem ID [11023]
Internal file name [`OUTPUT/10279_Sunday_December_31_2023_11_39_52_AM_61793738/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 199.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$(a\,x^3 + b\,x^2 + cx)\,y'' + ((m-a)\,x^2 + (2cm-1)\,x - c)\,y' + (-2mx+1)\,y = 0$$

✗ Solution by Maple

```
dsolve((a*x^3+b*x^2+c*x)*diff(y(x),x$2)+((m-a)*x^2+(2*c*m-1)*x-c)*diff(y(x),x)+(-2*m*x+1)*y(
```

No solution found

✓ Solution by Mathematica
Time used: 17.694 (sec). Leaf size: 192

```
DSolve[(a*x^3+b*x^2+c*x)*y''[x]+((m-a)*x^2+(2*c*m-1)*x-c)*y'[x]+(-2*m*x+1)*y[x]==0,y[x],x,In
```

$$y(x) \to \frac{(x(ax+2b+mx-1)+c(2b+4mx-1)+4c^2m)\left(c_2\int_1^x \frac{\exp\left(\frac{(bm-2a(b+2cm-1))\arctan\left(\frac{b+2aK[1]}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}}\right)K[1](c+K[1](b\cdots}{(4mc^2+(2b+4mK[1]-1)c+K[1](2b+aK[1]+mK[1]\cdots}\cdots\right.}{a+2b(c+1)+4c^2m+4cm-c+m-1}$$

31.19 problem 200

Internal problem ID [11024]
Internal file name [`OUTPUT/10280_Sunday_December_31_2023_03_56_49_PM_14438536/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 200.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\left(a\,x^3 + b\,x^2 + cx\right) y'' + \left(n\,x^2 + mx + k\right) y' + \left(-2(a+n)\,x + 1\right) y = 0$$

✗ Solution by Maple

```
dsolve((a*x^3+b*x^2+c*x)*diff(y(x),x$2)+(n*x^2+m*x+k)*diff(y(x),x)+(-2*(a+n)*x+1)*y(x)=0,y(x
```

No solution found

✓ Solution by Mathematica

Time used: 121.038 (sec). Leaf size: 552

```
DSolve[(a*x^3+b*x^2+c*x)*y''[x]+(n*x^2+m*x+k)*y'[x]+(-2*(a+n)*x+1)*y[x]==0,y[x],x,IncludeSin
```

$y(x) \to$

$$-\frac{2^{-\frac{k}{c}}\left(-\frac{ax}{\sqrt{b^2-4ac}+b}\right)^{1-\frac{k}{c}}\left(ac_1x\left(-2^{\frac{k}{c}}\right)\left(-\frac{ax}{\sqrt{b^2-4ac}+b}\right)^{\frac{k}{c}-1}\text{HeunG}\left[\frac{b-\sqrt{b^2-4ac}}{\sqrt{b^2-4ac}+b},\frac{2}{\sqrt{b^2-4ac}+b},\frac{1}{2}\left(-\sqrt{\frac{(3a+n)^2}{a^2}}+\right.\right.\right.}{}$$

31.20 problem 201

Internal problem ID [11025]
Internal file name [OUTPUT/10281_Sunday_December_31_2023_03_57_10_PM_26080740/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 201.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\left(a\,x^3 + x^2 + b\right) y'' + a^2x\left(x^2 - b\right) y' - a^3bxy = 0$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
   No special function solution was found.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.562 (sec). Leaf size: 79

```
dsolve((a*x^3+x^2+b)*diff(y(x),x$2)+a^2*x*(x^2-b)*diff(y(x),x)-a^3*b*x*y(x)=0,y(x), singsol=
```

$$y(x) = \mathrm{e}^{-ax}\left(c_2\left(\int \mathrm{e}^{a\left(\int \frac{a^2x^4+2a\,x^3+\left(a^2b+2\right)x^2+4abx+2b}{\left(a\,x^3+x^2+b\right)(ax+2)}dx\right)}dx\right)+c_1\right)(ax+2)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(a*x^3+x^2+b)*y''[x]+a^2*x*(x^2-b)*y'[x]-a^3*b*x*y[x]==0,y[x],x,IncludeSingularSoluti
```

Timed out

31.21 problem 202

Internal problem ID [11026]
Internal file name [`OUTPUT/10282_Sunday_December_31_2023_03_57_16_PM_16783930/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 202.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{2y''x\left(a\,x^2+bx+c\right)+\left(a\,x^2-c\right)y'+\lambda\,x^2y=0}$$

31.21.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$\left(2a\,x^3+2b\,x^2+2cx\right)y''+\left(a\,x^2-c\right)y'+\lambda\,x^2y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{a\,x^2-c}{2a\,x^3+2b\,x^2+2cx} \\ q(x) &= \frac{\lambda x}{2a\,x^2+2bx+2c} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{a\,x^2-c}{2a\,x^3+2b\,x^2+2cx}dx\right)} dx \\
&= \int e^{\frac{\ln(x)}{2}-\frac{\ln\left(a\,x^2+bx+c\right)}{2}} dx \\
&= \int \frac{\sqrt{x}}{\sqrt{a\,x^2+bx+c}} dx \\
&= -\frac{\left(b+\sqrt{-4ac+b^2}\right)\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}\sqrt{\frac{-2ax+\sqrt{-4ac+b^2}-b}{\sqrt{-4ac+b^2}}}\sqrt{-\frac{ax}{b+\sqrt{-4ac+b^2}}}\left(2\sqrt{-4ac+b^2}\ \mathrm{EllipticE}\left(\right.\right.}{}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{\lambda x}{2a\,x^2+2bx+2c}}{\frac{x}{a\,x^2+bx+c}} \\
&= \frac{\lambda}{2}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{\lambda y(\tau)}{2} &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = \frac{\lambda}{2}$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + \frac{\lambda\, e^{\lambda\tau}}{2} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + \frac{\lambda}{2} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \frac{\lambda}{2}$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\left(\frac{\lambda}{2}\right)}\\
&= \pm\frac{\sqrt{-2\lambda}}{2}
\end{aligned}$$

Hence

$$\lambda_1 = +\frac{\sqrt{-2\lambda}}{2}$$
$$\lambda_2 = -\frac{\sqrt{-2\lambda}}{2}$$

Which simplifies to

$$\lambda_1 = \frac{\sqrt{2}\,\sqrt{-\lambda}}{2}$$
$$\lambda_2 = -\frac{\sqrt{2}\,\sqrt{-\lambda}}{2}$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{\left(\frac{\sqrt{2}\,\sqrt{-\lambda}}{2}\right)\tau} + c_2 e^{\left(-\frac{\sqrt{2}\,\sqrt{-\lambda}}{2}\right)\tau}$$

Or

$$
y(\tau) = c_1 \mathrm{e}^{\frac{\sqrt{2}\,\sqrt{-\lambda}\,\tau}{2}} + c_2 \mathrm{e}^{-\frac{\sqrt{2}\,\sqrt{-\lambda}\,\tau}{2}}
$$

The above solution is now transformed back to y using (6) which results in

$$
y = c_1 \mathrm{e}^{-\frac{\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\mathrm{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\frac{\sqrt{2b+2\sqrt{-4ac+b^2}}}{2\left(-4ac+b^2\right)^{\frac{1}{4}}}\right)}{2}+\sqrt{-4ac+b^2}\,\mathrm{EllipticE}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\frac{\sqrt{2b \cdots}}{2(\cdots}\right.\right. \cdots}{2\sqrt{a\,x^2+bx+c}\,\sqrt{x}\,a^2 \cdots}}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \mathrm{e}^{-\frac{\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\mathrm{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\frac{\sqrt{2b+2\sqrt{-4ac+b^2}}}{2\left(-4ac+b^2\right)^{\frac{1}{4}}}\right)}{2}+\sqrt{-4ac+b^2}\,\mathrm{EllipticE}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\frac{\sqrt{2b+2 \cdots}}{2(-4a \cdots}\right.\right. \cdots}{2\sqrt{a\,x^2+bx+c}\,\sqrt{x}\,a^2 \cdots}} \\
&\quad + c_2 \mathrm{e}^{\frac{\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\mathrm{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\frac{\sqrt{2b+2\sqrt{-4ac+b^2}}}{2\left(-4ac+b^2\right)^{\frac{1}{4}}}\right)}{2}+\sqrt{-4ac+b^2}\,\mathrm{EllipticE}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\frac{\sqrt{2b+ \cdots}}{2(- \cdots}\right.\right. \cdots}{2\sqrt{a\,x^2+bx+c}\,\sqrt{x}\,a^2 \cdots}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 \mathrm{e}^{-\frac{\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\mathrm{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\frac{\sqrt{2b+2\sqrt{-4ac+b^2}}}{2\left(-4ac+b^2\right)^{\frac{1}{4}}}\right)}{2}+\sqrt{-4ac+b^2}\,\mathrm{EllipticE}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\frac{\sqrt{2b+2 \cdots}}{2(-4a \cdots}\right.\right. \cdots}{2\sqrt{a\,x^2+bx+c}\,\sqrt{x}\,a^2 \cdots}} \\
&\quad + c_2 \mathrm{e}^{\frac{\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\mathrm{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\frac{\sqrt{2b+2\sqrt{-4ac+b^2}}}{2\left(-4ac+b^2\right)^{\frac{1}{4}}}\right)}{2}+\sqrt{-4ac+b^2}\,\mathrm{EllipticE}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\frac{\sqrt{2b+ \cdots}}{2(- \cdots}\right.\right. \cdots}{2\sqrt{a\,x^2+bx+c}\,\sqrt{x}\,a^2 \cdots}}
\end{aligned}
$$

Verified OK.

31.21.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$\left(2a\,x^{3}+2b\,x^{2}+2cx\right) y''+\left(a\,x^{2}-c\right) y'+\lambda\,x^{2} y = 0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{a\,x^{2}-c}{2x\left(a\,x^{2}+bx+c\right)}\\ q(x) &= \frac{\lambda x}{2a\,x^{2}+2bx+2c} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^{2}}{d\tau^{2}}y(\tau)+p_{1}\left(\frac{d}{d\tau}y(\tau)\right)+q_{1}y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_{1}(\tau) = \frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'\left(x\right)^{2}} \tag{4}$$

$$q_{1}(\tau) = \frac{q(x)}{\tau'\left(x\right)^{2}} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{\sqrt{\frac{\lambda x}{2a\,x^{2}+2bx+2c}}}{c} \\ \tau'' &= \frac{\frac{\lambda}{2a\,x^{2}+2bx+2c}-\frac{\lambda x(4ax+2b)}{\left(2a\,x^{2}+2bx+2c\right)^{2}}}{2c\sqrt{\frac{\lambda x}{2a\,x^{2}+2bx+2c}}} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_{1}(\tau) &= \frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'\left(x\right)^{2}}\\ &= \frac{\frac{\frac{\lambda}{2a\,x^{2}+2bx+2c}-\frac{\lambda x(4ax+2b)}{\left(2a\,x^{2}+2bx+2c\right)^{2}}}{2c\sqrt{\frac{\lambda x}{2a\,x^{2}+2bx+2c}}}+\frac{a\,x^{2}-c}{2x(a\,x^{2}+bx+c)}\frac{\sqrt{\frac{\lambda x}{2a\,x^{2}+2bx+2c}}}{c}}{\left(\frac{\sqrt{\frac{\lambda x}{2a\,x^{2}+2bx+2c}}}{c}\right)^{2}}\\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c} \sqrt{q}\, dx \\ &= \frac{\int \sqrt{\frac{\lambda x}{2a\,x^2+2bx+2c}} dx}{c} \\ &= -\frac{\sqrt{\frac{\lambda x}{a\,x^2+bx+c}} \left(a\,x^2+bx+c\right) \left(b+\sqrt{-4ac+b^2}\right) \sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}} \sqrt{\frac{-2ax+\sqrt{-4ac+b^2}-b}{\sqrt{-4ac+b^2}}} \sqrt{-\frac{2ax}{b+\sqrt{-4ac+b^2}}} \Bigg(}{} \end{aligned}$$

Substituting the above into the solution obtained gives

$$\begin{aligned} y &= c_1 \cos\left(\frac{\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}} \left(\frac{\left(b-\sqrt{-4ac+b^2}\right) \operatorname{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}, \frac{\sqrt{2b+2\sqrt{-4ac+b^2}}}{2\left(-4ac+b^2\right)^{\frac{1}{4}}}\right)}{2} + \sqrt{-4ac+b^2}\, \operatorname{EllipticE}\right.}{2\sqrt{a\,x^2}}\right. \\ &\quad - c_2 \sin\left(\frac{\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}} \left(\frac{\left(b-\sqrt{-4ac+b^2}\right) \operatorname{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}, \frac{\sqrt{2b+2\sqrt{-4ac+b^2}}}{2\left(-4ac+b^2\right)^{\frac{1}{4}}}\right)}{2} + \sqrt{-4ac+b^2}\, \operatorname{Elliptic}\right.}{2\sqrt{a\,x}}\right. \end{aligned}$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos\left(\frac{\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\operatorname{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}, \frac{\sqrt{2b+2\sqrt{-4ac+b^2}}}{2\left(-4ac+b^2\right)^{\frac{1}{4}}}\right)}{2} + \sqrt{-4ac+b^2}\ \operatorname{EllipticE}\cdots}{2\sqrt{a\,x^2 \cdots}}\right) - c_2 \sin\left(\frac{\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\operatorname{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}, \frac{\sqrt{2b+2\sqrt{-4ac+b^2}}}{2\left(-4ac+b^2\right)^{\frac{1}{4}}}\right)}{2} + \sqrt{-4ac+b^2}\ \operatorname{Elliptic}\cdots}{2\sqrt{a\,x \cdots}}\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 \cos\left(\frac{\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\operatorname{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}, \frac{\sqrt{2b+2\sqrt{-4ac+b^2}}}{2\left(-4ac+b^2\right)^{\frac{1}{4}}}\right)}{2} + \sqrt{-4ac+b^2}\ \operatorname{EllipticE}\cdots}{2\sqrt{a\,x^2 \cdots}}\right) - c_2 \sin\left(\frac{\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\operatorname{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}, \frac{\sqrt{2b+2\sqrt{-4ac+b^2}}}{2\left(-4ac+b^2\right)^{\frac{1}{4}}}\right)}{2} + \sqrt{-4ac+b^2}\ \operatorname{Elliptic}\cdots}{2\sqrt{a\,x \cdots}}\right)
$$

Warning, solution could not be verified

31.21.3 Solving using Kovacic algorithm

Writing the ode as

$$\left(2a\,x^3+2b\,x^2+2cx\right)y''+\left(a\,x^2-c\right)y'+\lambda\,x^2y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2a\,x^3+2b\,x^2+2cx\\B&=a\,x^2-c\\C&=\lambda\,x^2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-8a\lambda\,x^5-3a^2x^4-8b\lambda\,x^4-8c\lambda\,x^3+14ac\,x^2+8bcx+5c^2}{16\left(a\,x^3+b\,x^2+cx\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-8a\lambda\,x^5-3a^2x^4-8b\lambda\,x^4-8c\lambda\,x^3+14ac\,x^2+8bcx+5c^2\\t&=16\left(a\,x^3+b\,x^2+cx\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-8a\lambda\,x^5-3a^2x^4-8b\lambda\,x^4-8c\lambda\,x^3+14ac\,x^2+8bcx+5c^2}{16\left(a\,x^3+b\,x^2+cx\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y=z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 188: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 5 \\
&= 1
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(a\,x^3 + b\,x^2 + cx)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{b-\sqrt{-4ac+b^2}}{2a}$ of order 2. There is a pole at $x = -\frac{b+\sqrt{-4ac+b^2}}{2a}$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [2]
$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{\frac{\left(-b+\sqrt{-4ac+b^2}\right)^3 b}{4a} - \frac{\left(-b+\sqrt{-4ac+b^2}\right)^3 c\lambda}{a^2} - 2\left(-b+\sqrt{-4ac+b^2}\right)^2 c + \frac{\left(-b+\sqrt{-4ac+b^2}\right)^2 b^2}{a} - \frac{2\left(-b+\sqrt{-4ac+b^2}\right)^2 bc\lambda}{a^2} \cdots}{16c\left(-4ac+b^2\right)\left(x - \frac{-b+\sqrt{-4ac \cdots}}{2a}\right.}$$

$$+ \frac{-\frac{\left(b+\sqrt{-4ac+b^2}\right)^3 b}{4a} + \frac{\left(b+\sqrt{-4ac+b^2}\right)^3 c\lambda}{a^2} - 2\left(b+\sqrt{-4ac+b^2}\right)^2 c + \frac{\left(b+\sqrt{-4ac+b^2}\right)^2 b^2}{a} - \frac{2\left(b+\sqrt{-4ac+b^2}\right)^2 bc\lambda}{a^2} + 3 \cdots}{16c\left(-4ac+b^2\right)\left(x + \frac{b+\sqrt{-4ac+b^2}}{2a}\right.}$$

$$+ \frac{\frac{\left(-b+\sqrt{-4ac+b^2}\right)^3 b}{2} - \frac{2\left(-b+\sqrt{-4ac+b^2}\right)^3 c\lambda}{a} - 2a\left(-b+\sqrt{-4ac+b^2}\right)^2 c + \frac{7\left(-b+\sqrt{-4ac+b^2}\right)^2 b^2}{4} - \frac{5\left(-b+\sqrt{-4ac+b^2}\right. \cdots}{a}}{8c\left(-4ac+b^2 \cdots\right.}$$

$$- \frac{-\frac{\left(b+\sqrt{-4ac+b^2}\right)^3 b}{2} + \frac{2\left(b+\sqrt{-4ac+b^2}\right)^3 c\lambda}{a} - 2\left(b+\sqrt{-4ac+b^2}\right)^2 ac + \frac{7\left(b+\sqrt{-4ac+b^2}\right)^2 b^2}{4} - \frac{5\left(b+\sqrt{-4ac+b^2}\right)^2 bc\lambda}{a} \cdots}{8c\left(-4ac+b^2\right)^{\frac{3}{2}}\left(\cdots\right.}$$

$$+ \frac{5}{16x^2} - \frac{b}{8cx}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

For the pole at $x = -\frac{b-\sqrt{-4ac+b^2}}{2a}$ let b be the coefficient of $\frac{1}{\left(x+\frac{b-\sqrt{-4ac+b^2}}{2a}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -\frac{b+\sqrt{-4ac+b^2}}{2a}$ let b be the coefficient of $\frac{1}{\left(x+\frac{b+\sqrt{-4ac+b^2}}{2a}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$
$-\frac{b-\sqrt{-4ac+b^2}}{2a}$	2	$\{1, 2, 3\}$
$-\frac{b+\sqrt{-4ac+b^2}}{2a}$	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1,\ e_2 = 1,\ e_3 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (-1 + (1) + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))} + \frac{1}{\left(x - \left(-\frac{b-\sqrt{-4ac+b^2}}{2a}\right)\right)} + \frac{1}{\left(x - \left(-\frac{b+\sqrt{-4ac+b^2}}{2a}\right)\right)}\right) \\ &= -\frac{1}{2x} + \frac{1}{2x + \frac{b-\sqrt{-4ac+b^2}}{a}} + \frac{1}{2x + \frac{b+\sqrt{-4ac+b^2}}{a}} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= -\frac{1}{2x} + \frac{1}{2x + \frac{b-\sqrt{-4ac+b^2}}{a}} + \frac{1}{2x + \frac{b+\sqrt{-4ac+b^2}}{a}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(-\frac{1}{2x} + \frac{1}{2x + \frac{b-\sqrt{-4ac+b^2}}{a}} + \frac{1}{2x + \frac{b+\sqrt{-4ac+b^2}}{a}}\right) w + \frac{\left(8a\lambda\, x^5 + (a^2 + 8b\lambda)\, x^4 + 8c\lambda\, x^3 - 2ac\, x^2 + c^2\right) a^2}{x^2 \left(2ax + b - \sqrt{-4ac+b^2}\right)^2 \left(2ax + \sqrt{-4ac+b^2} + b\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{a\,x^2 + 2x\sqrt{2}\,\sqrt{-\lambda x\,(a\,x^2 + bx + c)} - c}{4x\,(a\,x^2 + bx + c)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{a\,x^2 + 2x\sqrt{2}\,\sqrt{-\lambda x\left(a\,x^2+bx+c\right)} - c}{4x\left(a\,x^2+bx+c\right)} dx} \\ &= \frac{(a\,x^2 + bx + c)^{\frac{1}{4}}\, \mathrm{e}^{\frac{\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}\,\sqrt{2}\,\sqrt{-\frac{ax}{b+\sqrt{-4ac+b^2}}}\,\lambda\sqrt{\frac{-2ax+\sqrt{-4ac+b^2}-b}{\sqrt{-4ac+b^2}}}\,\left(b+\sqrt{-4ac+b^2}\right)\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\mathrm{EllipticF}\left(\sqrt{\frac{2a \cdots}{}}\right.}{\cdots}\right.}{2\sqrt{-\lambda x\left(a\,x^2+bx \cdots\right)}}}}{x^{\frac{1}{4}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{a\,x^2-c}{2a\,x^3+2b\,x^2+2cx}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{4}-\frac{\ln\left(a\,x^2+bx+c\right)}{4}} \\
&= z_1\left(\frac{x^{\frac{1}{4}}}{(a\,x^2+bx+c)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{\sqrt{2}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\mathrm{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\frac{\left(-\left(4ac-b^2\right)^3\right)^{\frac{1}{4}}\sqrt{2b+2\sqrt{-4ac+b^2}}}{8ac-2b^2}\right)}{2}+\sqrt{-4ac+b^2}\,\mathrm{EllipticE}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\frac{\left(-\left(4ac-b^2\right)^{\cdots}\right.}{\cdots}\right.\right.}{a^2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{a\,x^2-c}{2a\,x^3+2b\,x^2+2cx}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}-\frac{\ln\left(a\,x^2+bx+c\right)}{2}}}{(y_1)^2}\,dx
\end{aligned}
$$

$$= y_1 \left(\int \frac{\sqrt{x}\, \mathrm{e}^{-\frac{2\sqrt{2}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right) \mathrm{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\, \frac{\left(-\left(4ac-b^2\right)^3\right)^{\frac{1}{4}} \sqrt{2b+2\sqrt{-4ac+b^2}}}{8ac-2b^2}\right)}{2} + \sqrt{-4ac+b^2}\, \mathrm{EllipticE}\left(\sqrt{\frac{2ax+\sqrt{-4a \cdots}}{b+\sqrt{-4a \cdots}}} \cdots \right)\right)}{a^2}}}{\sqrt{a\, x^2+bx+c}} \cdots \right.$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\mathrm{e}^{\frac{\sqrt{2}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right) \mathrm{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\, \frac{\left(-\left(4ac-b^2\right)^3\right)^{\frac{1}{4}} \sqrt{2b+2\sqrt{-4ac+b^2}}}{8ac-2b^2}\right)}{2} + \sqrt{-4ac+b^2}\, \mathrm{EllipticE}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\, \frac{\left(-\left(4 \cdots\right.\right.}{\cdots} \cdots \right)\right)}{a^2}} \cdots \right.$$

$$+ c_2 \left(\mathrm{e}^{\frac{\sqrt{2}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right) \mathrm{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\, \frac{\left(-\left(4ac-b^2\right)^3\right)^{\frac{1}{4}} \sqrt{2b+2\sqrt{-4ac+b^2}}}{8ac-2b^2}\right)}{2} + \sqrt{-4ac+b^2}\, \mathrm{EllipticE}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}},\, \frac{\left(\cdots\right.}{\cdots} \cdots \right)\right)}{a^2}} \cdots \right.$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{\sqrt{2}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\mathrm{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}, \frac{\left(-\left(4ac-b^2\right)^3\right)^{\frac{1}{4}}\sqrt{2b+2\sqrt{-4ac+b^2}}}{8ac-2b^2}\right)}{2}+\sqrt{-4ac+b^2}\;\mathrm{EllipticE}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}, \frac{\left(-\left(4ac-b\right.\right.}{}\right.\right.}{a^2}} + c_2 \mathrm{e}^{\frac{\sqrt{2}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\mathrm{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}, \frac{\left(-\left(4ac-b^2\right)^3\right)^{\frac{1}{4}}\sqrt{2b+2\sqrt{-4ac+b^2}}}{8ac-2b^2}\right)}{2}+\sqrt{-4ac+b^2}\;\mathrm{EllipticE}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}, \frac{\left(-\left(4a\right.\right.}{}\right.\right.}{a^2}} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{\sqrt{2}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\mathrm{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}, \frac{\left(-\left(4ac-b^2\right)^3\right)^{\frac{1}{4}}\sqrt{2b+2\sqrt{-4ac+b^2}}}{8ac-2b^2}\right)}{2}+\sqrt{-4ac+b^2}\,\mathrm{EllipticE}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}, \frac{\left(-\left(4ac-b\right.\right.}{}\right.}{a^2}} + c_2 \mathrm{e}^{\frac{\sqrt{2}\left(\frac{\left(b-\sqrt{-4ac+b^2}\right)\mathrm{EllipticF}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}, \frac{\left(-\left(4ac-b^2\right)^3\right)^{\frac{1}{4}}\sqrt{2b+2\sqrt{-4ac+b^2}}}{8ac-2b^2}\right)}{2}+\sqrt{-4ac+b^2}\,\mathrm{EllipticE}\left(\sqrt{\frac{2ax+\sqrt{-4ac+b^2}+b}{b+\sqrt{-4ac+b^2}}}, \frac{\left(-\left(4a\right.\right.}{}\right.}{a^2}}$$

Verified OK.

31.21.4 Maple step by step solution

Let's solve

$(2a\,x^3+2b\,x^2+2cx)\,y''+(a\,x^2-c)\,y'+\lambda\,x^2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{\lambda xy}{2(a\,x^2+bx+c)}-\frac{(a\,x^2-c)y'}{2x(a\,x^2+bx+c)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(a\,x^2-c)y'}{2x(a\,x^2+bx+c)}+\frac{\lambda xy}{2(a\,x^2+bx+c)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{a\,x^2-c}{2x(a\,x^2+bx+c)}, P_3(x)=\frac{\lambda x}{2(a\,x^2+bx+c)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$(x \cdot P_2(x))\Big|_{x=0} = -\frac{1}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x(a\,x^2 + bx + c) + (a\,x^2 - c)\,y' + \lambda\,x^2 y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^2 \cdot y$ to series expansion

  $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+2}$

- Shift index using $k- > k - 2$

  $x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k- > k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-2+m}$

- Shift index using $k- > k + 2 - m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\, x^{k+r}$

  Rewrite ODE with series expansions

$$ca_0r(-3+2r)\,x^{-1+r}+(ca_1(1+r)\,(-1+2r)+2a_0r(-1+r)\,b)\,x^r+(ca_2(2+r)\,(1+2r)+2a_1($$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$cr(-3+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{3}{2}\right\}$$

- The coefficients of each power of x must be 0

$$[ca_1(1+r)\,(-1+2r)+2a_0r(-1+r)\,b=0, ca_2(2+r)\,(1+2r)+2a_1(1+r)\,rb+aa_0r(-1+2r)$$

- Solve for the dependent coefficient(s)

$$\left\{a_1=-\frac{2a_0r(-1+r)b}{c(2r^2+r-1)}, a_2=-\frac{a_0r\,(4ac\,r^2-4b^2r^2-4acr+4b^2r+ac)}{c^2\,(4r^3+8r^2-r-2)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$ca_{k+1}(k+1+r)\,(2k+2r-1)+2a_k(k+r)\,(k+r-1)\,b+aa_{k-1}(k+r-1)\,(2k+2r-3)+\lambda a$$

- Shift index using $k->k+2$

$$ca_{k+3}(k+3+r)\,(2k+3+2r)+2a_{k+2}(k+r+2)\,(k+1+r)\,b+aa_{k+1}(k+1+r)\,(2k+1+2r$$

- Recursion relation that defines series solution to ODE

$$a_{k+3}=-\frac{2a\,k^2a_{k+1}+4akra_{k+1}+2a\,r^2a_{k+1}+2b\,k^2a_{k+2}+4bkra_{k+2}+2b\,r^2a_{k+2}+3aka_{k+1}+3ara_{k+1}+6bka_{k+2}+6bra_{k+2}+aa_{k+1}+}{c(k+3+r)(2k+3+2r)}$$

- Recursion relation for $r=0$

$$a_{k+3}=-\frac{2a\,k^2a_{k+1}+2b\,k^2a_{k+2}+3aka_{k+1}+6bka_{k+2}+aa_{k+1}+4ba_{k+2}+\lambda a_k}{c(k+3)(2k+3)}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty}a_kx^k, a_{k+3}=-\frac{2a\,k^2a_{k+1}+2b\,k^2a_{k+2}+3aka_{k+1}+6bka_{k+2}+aa_{k+1}+4ba_{k+2}+\lambda a_k}{c(k+3)(2k+3)}, a_1=0, a_2=0\right]$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+3}=-\frac{2a\,k^2a_{k+1}+2b\,k^2a_{k+2}+9aka_{k+1}+12bka_{k+2}+10aa_{k+1}+\frac{35}{2}ba_{k+2}+\lambda a_k}{c(k+\frac{9}{2})(2k+6)}$$

- Solution for $r=\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{3}{2}}, a_{k+3}=-\frac{2a\,k^2a_{k+1}+2b\,k^2a_{k+2}+9aka_{k+1}+12bka_{k+2}+10aa_{k+1}+\frac{35}{2}ba_{k+2}+\lambda a_k}{c(k+\frac{9}{2})(2k+6)}, a_1=-\frac{3a_0b}{10c}, a_2=\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}d_kx^k\right)+\left(\sum_{k=0}^{\infty}e_kx^{k+\frac{3}{2}}\right), d_{k+3}=-\frac{2a\,k^2d_{1+k}+2b\,k^2d_{k+2}+3akd_{1+k}+6bkd_{k+2}+ad_{1+k}+4bd_{k+2}+\lambda d_k}{c(k+3)(2k+3)}, d_1\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   Solution is available but has integrals. Trying a simpler solution using Kovacics algorit
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
      Solution has integrals. Trying a special function solution free of integrals...
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
      No special function solution was found.
   <- Kovacics algorithm successful
   Solution via Kovacic is not simpler. Returning default solution
   <- linear_1 successful`
```

✓ Solution by Maple

Time used: 7.125 (sec). Leaf size: 65

```
dsolve(2*x*(a*x^2+b*x+c)*diff(y(x),x$2)+(a*x^2-c)*diff(y(x),x)+lambda*x^2*y(x)=0,y(x), sings
```

$$y(x) = c_1 e^{\frac{i\sqrt{2}\,\sqrt{\lambda}\left(\int \frac{\sqrt{x}}{\sqrt{a\,x^2+bx+c}}dx\right)}{2}} + c_2 e^{-\frac{i\sqrt{2}\,\sqrt{\lambda}\left(\int \frac{\sqrt{x}}{\sqrt{a\,x^2+bx+c}}dx\right)}{2}}$$

✓ Solution by Mathematica

Time used: 144.69 (sec). Leaf size: 501

```
DSolve[2*x*(a*x^2+b*x+c)*y''[x]+(a*x^2-c)*y'[x]+\[Lambda]*x^2*y[x]==0,y[x],x,IncludeSingular
```

$$y(x) \to c_1 \cosh\left(\frac{\sqrt{\lambda}\left(\sqrt{b^2-4ac}-b\right)\sqrt{\sqrt{b^2-4ac}+2ax+b}\sqrt{\frac{2ax}{b-\sqrt{b^2-4ac}}+1}\left(E\left(i\operatorname{arcsinh}\left(\frac{\sqrt{2}\sqrt{a}\sqrt{x}}{\sqrt{b+\sqrt{b^2-4ac}}}\right)|\frac{b+\sqrt{b^2-}}{b-\sqrt{b^2-}}\right.\right.}{2a^{3/2}\sqrt{x(ax+b)+c}}\right.$$
$$+ic_2 \sinh\left(\frac{\sqrt{\lambda}\left(\sqrt{b^2-4ac}-b\right)\sqrt{\sqrt{b^2-4ac}+2ax+b}\sqrt{\frac{2ax}{b-\sqrt{b^2-4ac}}+1}\left(E\left(i\operatorname{arcsinh}\left(\frac{\sqrt{2}\sqrt{a}\sqrt{x}}{\sqrt{b+\sqrt{b^2-4ac}}}\right)|\frac{b+\sqrt{}}{b-\sqrt{}}\right.\right.}{2a^{3/2}\sqrt{x(ax+b)+c}}\right.$$

31.22 problem 203

Internal problem ID [11027]
Internal file name [`OUTPUT/10283_Sunday_December_31_2023_03_59_55_PM_84761429/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 203.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$x\left(a\,x^2 + bx + 1\right) y'' + \left(\alpha\,x^2 + \beta x + \gamma\right) y' + (xn + m)\, y = 0$$

✗ Solution by Maple

```
dsolve(x*(a*x^2+b*x+1)*diff(y(x),x$2)+(alpha*x^2+beta*x+gamma)*diff(y(x),x)+(n*x+m)*y(x)=0,y
```

No solution found

✓ Solution by Mathematica
Time used: 108.623 (sec). Leaf size: 524

```
DSolve[x*(a*x^2+b*x+1)*y''[x]+(\[Alpha]*x^2+\[Beta]*x+\[Gamma])*y'[x]+(n*x+m)*y[x]==0,y[x],x
```

$$y(x) \to -\frac{2^{-\gamma}\left(-\frac{ax}{\sqrt{b^2-4a}+b}\right)^{1-\gamma}\left(a(-2^{\gamma})\,c_1x\left(-\frac{ax}{\sqrt{b^2-4a}+b}\right)^{\gamma-1}\text{HeunG}\left[\frac{b-\sqrt{b^2-4a}}{\sqrt{b^2-4a}+b}, \frac{2m}{\sqrt{b^2-4a}+b}, \frac{1}{2}\left(-\sqrt{\frac{a^2+\alpha^2-2a(\alpha+2n)}{a^2}}\right.\right.\right.}{}$$

31.23 problem 204

Internal problem ID [11028]
Internal file name [`OUTPUT/10284_Sunday_December_31_2023_03_59_58_PM_34416455/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 204.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$x(x-1)(x-a)\,y'' + \left((\alpha+\beta+1)\,x^2 - (\alpha+\beta+1+a(\gamma+d)-a)\,x + a\gamma\right)y' + (\alpha\beta x - q)\,y = 0$$

31.23.1 Maple step by step solution

Let's solve

$-y''x(x-1)(a-x) + ((\alpha+\beta+1)\,x^2 + ((-d-\gamma+1)\,a-\beta-\alpha-1)\,x + a\gamma)\,y' + (\alpha\beta x - q)\,y$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(\alpha\beta x-q)y}{x(x-1)(a-x)} - \frac{(adx+a\gamma x-\alpha x^2-\beta x^2-a\gamma-ax+x\alpha+\beta x-x^2+x)y'}{x(x-1)(a-x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(adx+a\gamma x-\alpha x^2-\beta x^2-a\gamma-ax+x\alpha+\beta x-x^2+x)y'}{x(x-1)(a-x)} - \frac{(\alpha\beta x-q)y}{x(x-1)(a-x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{adx+a\gamma x-\alpha x^2-\beta x^2-a\gamma-ax+x\alpha+\beta x-x^2+x}{x(x-1)(a-x)}, P_3(x) = -\frac{\alpha\beta x-q}{x(x-1)(a-x)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \gamma$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x(x-1)(a-x) + (adx + a\gamma x - \alpha x^2 - \beta x^2 - a\gamma - ax + x\alpha + \beta x - x^2 + x)y' + y(-\alpha\beta x +$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- > k + 2 - m$

$$x^{m} \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-aa_{0}r(-1+r+\gamma)\, x^{-1+r} + \left(-aa_{1}(1+r)(r+\gamma) + a_{0}(adr + a\gamma r + a\,r^{2} - 2ar + \alpha r + \beta r + r^{2} + \right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-ar(-1+r+\gamma) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, -\gamma+1\}$$

- Each term must be 0

  $$-aa_{1}(1+r)(r+\gamma) + a_{0}(adr + a\gamma r + a\,r^{2} - 2ar + \alpha r + \beta r + r^{2} + q) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$-aa_{k+1}(k+1+r)(k+r+\gamma) + a_{k}((k+r)(k+d+r+\gamma-2)\,a + k^{2} + (2r+\beta+\alpha)\,k + r^{2} + ($$

- Shift index using $k->k+1$

  $$-aa_{k+2}(k+2+r)(k+1+r+\gamma) + a_{k+1}\left((k+1+r)(k-1+d+r+\gamma)\,a + (k+1)^{2} + (2r+\right.$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{adka_{k+1} + adra_{k+1} + a\gamma ka_{k+1} + a\gamma ra_{k+1} + a\,k^{2}a_{k+1} + 2akra_{k+1} + a\,r^{2}a_{k+1} + ada_{k+1} + a\gamma a_{k+1} - a_{k}\alpha\beta - a_{k}\alpha k + \alpha ka_{k+1} - a_{k}\alpha}{}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = \frac{adka_{k+1} + a\gamma ka_{k+1} + a\,k^{2}a_{k+1} + ada_{k+1} + a\gamma a_{k+1} - a_{k}\alpha\beta - a_{k}\alpha k + \alpha ka_{k+1} - a_{k}\beta k + \beta ka_{k+1} - k^{2}a_{k} + k^{2}a_{k+1} - aa_{k+1} + \alpha a_{k+1}}{a(k+2)(k+1+\gamma)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_{k}x^{k}, a_{k+2} = \frac{adka_{k+1} + a\gamma ka_{k+1} + a\,k^{2}a_{k+1} + ada_{k+1} + a\gamma a_{k+1} - a_{k}\alpha\beta - a_{k}\alpha k + \alpha ka_{k+1} - a_{k}\beta k + \beta ka_{k+1} - k^{2}a_{k} + k^{2}a}{a(k+2)(k+1+\gamma)}\right.$$

- Recursion relation for $r = -\gamma+1$

  $$a_{k+2} = \frac{-(-\gamma+1)^{2}a_{k} + (-\gamma+1)^{2}a_{k+1} + 2(-\gamma+1)a_{k+1} + adka_{k+1} + k^{2}a_{k+1} + \alpha a_{k+1} + \beta a_{k+1} + 2ka_{k+1} + qa_{k+1} - k^{2}a_{k} - aa_{k+1} + a\gamma ka_{k}}{}$$

- Solution for $r = -\gamma+1$

  $$\left[y = \sum_{k=0}^{\infty} a_{k}x^{-\gamma+k+1}, a_{k+2} = \frac{-(-\gamma+1)^{2}a_{k} + (-\gamma+1)^{2}a_{k+1} + 2(-\gamma+1)a_{k+1} + adka_{k+1} + k^{2}a_{k+1} + \alpha a_{k+1} + \beta a_{k+1} + 2ka_{k+1} + }{}\right.$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} b_{k}x^{k}\right) + \left(\sum_{k=0}^{\infty} c_{k}x^{-\gamma+k+1}\right), b_{k+2} = \frac{adkb_{1+k} + a\gamma kb_{1+k} + a\,k^{2}b_{1+k} + adb_{1+k} + a\gamma b_{1+k} - \alpha\beta b_{k} - \alpha kb_{k} + \alpha kb_{1-}}{a(}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0, g <
```

✓ Solution by Maple

Time used: 0.578 (sec). Leaf size: 82

```
dsolve(x*(x-1)*(x-a)*diff(y(x),x$2)+((alpha+beta+1)*x^2-(alpha+beta+1+a*(gamma+d)-a)*x+a*gam
```

$$y(x) = c_1 \operatorname{HeunG}\left(a, q, \alpha, \beta, \gamma, \frac{a(d-1)}{a-1}, x\right) + c_2 x^{1-\gamma} \operatorname{HeunG}\left(a, q - (-1+\gamma)\left(a(d-1)+\alpha+\beta-\gamma+1\right), \beta+1-\gamma, \alpha+1-\gamma, 2-\gamma, \frac{a(d-1)}{a-1}, x\right)$$

✓ Solution by Mathematica

Time used: 2.215 (sec). Leaf size: 85

```
DSolve[x*(x-1)*(x-a)*y''[x]+((\[Alpha]+\[Beta]+1)*x^2-(\[Alpha]+\[Beta]+1+a*(\[Gamma]+d)-a)*
```

$$y(x) \to c_2 x^{1-\gamma}\text{HeunG}\left[a, q-(\gamma-1)(a(d-1)+\alpha+\beta-\gamma+1), \alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, \frac{a(d-1)}{a-1}, x\right] + c_1\text{HeunG}\left[a, q, \alpha, \beta, \gamma, \frac{a(d-1)}{a-1}, x\right]$$

31.24 problem 205

Internal problem ID [11029]
Internal file name [OUTPUT/10285_Sunday_December_31_2023_04_00_00_PM_67352977/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 205.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{(a\,x^3 + b\,x^2 + cx + d)\,y'' - (-\lambda^2 + x^2)\,y' + (\lambda + x)\,y = 0}$$

31.24.1 Maple step by step solution

Let's solve

$(a\,x^3 + b\,x^2 + cx + d)\,y'' + (\lambda^2 - x^2)\,y' + (\lambda + x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(\lambda+x)y}{a\,x^3+b\,x^2+cx+d} - \frac{(\lambda^2-x^2)y'}{a\,x^3+b\,x^2+cx+d}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(\lambda^2-x^2)y'}{a\,x^3+b\,x^2+cx+d} + \frac{(\lambda+x)y}{a\,x^3+b\,x^2+cx+d} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{\lambda^2-x^2}{a\,x^3+b\,x^2+cx+d}, P_3(x) = \frac{\lambda+x}{a\,x^3+b\,x^2+cx+d}\right]$

- $\left(x - \frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} + \frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a\right.}\right.$

  $\left(\left(x - \frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} + \frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\right.}\right.\right.$

- $\left(x - \frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} + \frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a\right.}\right.$

  $\left(\left(x - \frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} + \frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b\right.}\right.\right.$

- $x = \frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} - \frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-1\right.}$

  Check to see if x_0 is a regular singular point

  $x_0 = \frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} - \frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-\right.}$

- Multiply by denominators

  $(a\,x^3+b\,x^2+cx+d)\,y''+(\lambda^2-x^2)\,y'+(\lambda+x)\,y=0$

- Change variables using $x = u + \frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} - \frac{}{3a\left(12\sqrt{3}\,\sqrt{\right.}}$

  $\left(\frac{d}{2} - \frac{8uc\,b^2}{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{2}{3}}} + \frac{\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}}{18a} + \frac{u\,b^2}{3a}\right.$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$0 = 0$$

- Values of r that satisfy the indicial equation

$$r = r$$

- The coefficients of each power of u must be 0

$$\left[\frac{a_0 r\left(240b^2ca-48b^4-48a\,b^4-432a^3c^2-\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{4}{3}}-12\left(12\sqrt{3}\,\sqrt{27a^2d^2\cdots\right.\right.}{\cdots}\right.$$

- Each term in the series must be 0, giving the recursion relation

$$-\frac{54\left(\left(-\frac{a_{k-1}(k+r-1)(k-2+r)a^3}{54}+\left(\frac{(k+1+r)\left(ck+cr-\lambda^2\right)a_{k+1}}{54}+\frac{ka_{k-1}}{54}-\frac{a_k\lambda}{54}+\frac{ra_{k-1}}{54}-\frac{a_{k-1}}{27}\right)a^2+\left(-\frac{(k+1+r)\left((k+r)b^2+2c\right)a_{k+1}}{162}-\cdots\right.\right.\right.}{\cdots}$$

- Shift index using $k- >k+1$

$$-\frac{54\left(\left(-\frac{a_k(k+r)(k+r-1)a^3}{54}+\left(\frac{(k+2+r)\left(c(k+1)+cr-\lambda^2\right)a_{k+2}}{54}+\frac{(k+1)a_k}{54}-\frac{a_{k+1}\lambda}{54}+\frac{a_k r}{54}-\frac{a_k}{27}\right)a^2+\left(-\frac{(k+2+r)\left((k+1+r)b^2+2c\right)a_{k+2}}{162}-\right.\right.\right.}{}$$

- Recursion relation that defines series solution to ODE
- Recursion relation for $r = r$
- Solution for $r = r$
- Revert the change of variables $u = x - \frac{\left(12\sqrt{3}\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} + -$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
   No special function solution was found.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.5 (sec). Leaf size: 84

```
dsolve((a*x^3+b*x^2+c*x+d)*diff(y(x),x$2)-(x^2-lambda^2)*diff(y(x),x)+(x+lambda)*y(x)=0,y(x)
```

$$y(x) = (\lambda - x)\left(\left(\int e^{\int \frac{(1-2a)x^3+(-2b-\lambda)x^2+\left(-\lambda^2-2c\right)x+\lambda^3-2d}{\left(a\,x^3+x^2b+cx+d\right)(-\lambda+x)}dx}dx\right)c_2 - c_1\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(a*x^3+b*x^2+c*x+d)*y''[x]-(x^2-\[Lambda]^2)*y'[x]+(x+\[Lambda])*y[x]==0,y[x],x,Inclu
```

Timed out

31.25 problem 206

Internal problem ID [11030]
Internal file name [`OUTPUT/10286_Sunday_December_31_2023_04_01_02_PM_82853361/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 206.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{2\left(a\,x^3 + b\,x^2 + cx + d\right) y'' + \left(3a\,x^2 + 2bx + c\right) y' + y\lambda = 0}$$

31.25.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$y''\left(2a\,x^3 + 2b\,x^2 + 2cx + 2d\right) + \left(3a\,x^2 + 2bx + c\right) y' + y\lambda = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3a\,x^2 + 2bx + c}{2a\,x^3 + 2b\,x^2 + 2cx + 2d} \\ q(x) &= \frac{\lambda}{2a\,x^3 + 2b\,x^2 + 2cx + 2d} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{3a\,x^2+2bx+c}{2a\,x^3+2b\,x^2+2cx+2d}dx\right)}dx \\
&= \int e^{-\frac{\ln\left(a\,x^3+b\,x^2+cx+d\right)}{2}}\,dx \\
&= \int \frac{1}{\sqrt{a\,x^3+b\,x^2+cx+d}}dx \\
&= \text{Expression too large to display}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{\lambda}{2a\,x^3+2b\,x^2+2cx+2d}}{\frac{1}{a\,x^3+b\,x^2+cx+d}} \\
&= \frac{\lambda}{2}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{\lambda y(\tau)}{2} &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = \frac{\lambda}{2}$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} + \frac{\lambda\, \mathrm{e}^{\lambda\tau}}{2} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + \frac{\lambda}{2} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \frac{\lambda}{2}$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\left(\frac{\lambda}{2}\right)} \\
&= \pm\frac{\sqrt{-2\lambda}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +\frac{\sqrt{-2\lambda}}{2} \\
\lambda_2 &= -\frac{\sqrt{-2\lambda}}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= \frac{\sqrt{2}\,\sqrt{-\lambda}}{2} \\
\lambda_2 &= -\frac{\sqrt{2}\,\sqrt{-\lambda}}{2}
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\
y(\tau) &= c_1 e^{\left(\frac{\sqrt{2}\,\sqrt{-\lambda}}{2}\right)\tau} + c_2 e^{\left(-\frac{\sqrt{2}\,\sqrt{-\lambda}}{2}\right)\tau}
\end{aligned}$$

Or

$$y(\tau) = c_1 \mathrm{e}^{\frac{\sqrt{2}\,\sqrt{-\lambda}\,\tau}{2}} + c_2 \mathrm{e}^{-\frac{\sqrt{2}\,\sqrt{-\lambda}\,\tau}{2}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \text{Expression too large to display}$$

Summary

The solution(s) found are the following

$$\text{Expression too large to display} \tag{1}$$

Verification of solutions

$$\text{Expression too large to display}$$

Warning, solution could not be verified

31.25.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$y''(2a\,x^3 + 2b\,x^2 + 2cx + 2d) + (3a\,x^2 + 2bx + c)\,y' + y\lambda = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3a\,x^2 + 2bx + c}{2a\,x^3 + 2b\,x^2 + 2cx + 2d} \\ q(x) &= \frac{\lambda}{2a\,x^3 + 2b\,x^2 + 2cx + 2d} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\tau' = \frac{1}{c}\sqrt{q} = \frac{\sqrt{\frac{\lambda}{2a\,x^3+2b\,x^2+2cx+2d}}}{c} \tag{6}$$
$$\tau'' = -\frac{\lambda(6a\,x^2+4bx+2c)}{2c\sqrt{\frac{\lambda}{2a\,x^3+2b\,x^2+2cx+2d}}\,(2a\,x^3+2b\,x^2+2cx+2d)^2}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\lambda(6a\,x^2+4bx+2c)}{2c\sqrt{\frac{\lambda}{2a\,x^3+2b\,x^2+2cx+2d}}\,(2a\,x^3+2b\,x^2+2cx+2d)^2}+\frac{3a\,x^2+2bx+c}{2a\,x^3+2b\,x^2+2cx+2d}\frac{\sqrt{\frac{\lambda}{2a\,x^3+2b\,x^2+2cx+2d}}}{c}}{\left(\frac{\sqrt{\frac{\lambda}{2a\,x^3+2b\,x^2+2cx+2d}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$y(\tau)''+p_1y(\tau)'+q_1y(\tau)=0$$
$$\frac{d^2}{d\tau^2}y(\tau)+c^2y(\tau)=0 \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau)=c_1\cos(c\tau)+c_2\sin(c\tau)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \sqrt{\frac{\lambda}{2a\,x^3+2b\,x^2+2cx+2d}}dx}{c} \\ &= \text{Expression too large to display} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \text{Expression too large to display}$$

Summary

The solution(s) found are the following

$$\text{Expression too large to display} \tag{1}$$

Verification of solutions

$$\text{Expression too large to display}$$

Warning, solution could not be verified

31.25.3 Maple step by step solution

Let's solve

$y''(2a\,x^3+2b\,x^2+2cx+2d)+(3a\,x^2+2bx+c)\,y'+y\lambda=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{\lambda y}{2(a\,x^3+b\,x^2+cx+d)}-\frac{(3a\,x^2+2bx+c)\,y'}{2(a\,x^3+b\,x^2+cx+d)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3a\,x^2+2bx+c)\,y'}{2(a\,x^3+b\,x^2+cx+d)}+\frac{\lambda y}{2(a\,x^3+b\,x^2+cx+d)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{3a\,x^2+2bx+c}{2(a\,x^3+b\,x^2+cx+d)},P_3(x)=\frac{\lambda}{2(a\,x^3+b\,x^2+cx+d)}\right]$

- ○ $\left(x-\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}+\frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a \cdots\right.}\right.$

  $\left(\left(x-\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}+\frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2} \cdots\right.}\right.\right.$

- ○ $\left(x-\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}+\frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a \cdots\right.}\right.$

$$\left(\left(x-\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}+\frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b\right.}\right.\right.$$

- $x=\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}-\frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-1\right.}$

  Check to see if x_0 is a regular singular point

  $x_0=\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}-\frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-\right.}$

- Multiply by denominators

  $y''(2a\,x^3+2b\,x^2+2cx+2d)+(3a\,x^2+2bx+c)\,y'+y\lambda=0$

- Change variables using $x=u+\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}-\frac{}{3a\left(12\sqrt{3}\,\sqrt{\right.}}$

  $\left(d+\frac{\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}}{9a}+\frac{16b^6}{27a^2\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)}-\right.$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_ku^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=0..3$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$$\frac{a_0 r\left(\left(12\sqrt{3}\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{4}{3}}-12ca\left(12\sqrt{3}\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-10\right.}{12a\left(12\sqrt{3}\sqrt{27a^2d^2-18abcd+4a\,c\right.}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $0 = 0$

- Values of r that satisfy the indicial equation

  $r = r$

- The coefficients of each power of u must be 0

  $$\left[\frac{a_0 r\left(\left(12\sqrt{3}\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{4}{3}}-12ca\left(12\sqrt{3}\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-\right.}{12a\left(12\sqrt{3}\sqrt{27a^2d^2-18abcd+4a\,c^3\right.}\right.$$

- Each term in the series must be 0, giving the recursion relation

  $$-\frac{108\left(\frac{\left(-(k+r-1)\left(-\frac{1}{2}+k+r\right)a_{k-1}a^2+\left((k+1+r)c\left(k+r+\frac{1}{2}\right)a_{k+1}-\frac{\lambda a_k}{2}\right)a-\frac{(k+1+r)a_{k+1}b^2\left(k+r+\frac{1}{2}\right)}{3}\right)\left(12\sqrt{3}\sqrt{27a^2d^2+\left(-18abc+4b^3\right)d+\right.}{54}\right.}{}$$

- Shift index using $k- >k+1$

  $$-\frac{108\left(\frac{\left(-(k+r)\left(k+r+\frac{1}{2}\right)a_k a^2+\left((k+2+r)c\left(k+\frac{3}{2}+r\right)a_{k+2}-\frac{\lambda a_{k+1}}{2}\right)a-\frac{(k+2+r)a_{k+2}b^2\left(k+\frac{3}{2}+r\right)}{3}\right)\left(12\sqrt{3}\sqrt{27a^2d^2+\left(-18abc+4b^3\right)d+4a\,c^3\right.}{54}\right.}{}$$

- Recursion relation that defines series solution to ODE
- Recursion relation for $r = r$
- Solution for $r = r$
- Revert the change of variables $u = x-\frac{\left(12\sqrt{3}\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}+-$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   Solution is available but has integrals. Trying a simpler solution using Kovacics algorit
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
      Solution has integrals. Trying a special function solution free of integrals...
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
      No special function solution was found.
   <- Kovacics algorithm successful
   Solution via Kovacic is not simpler. Returning default solution
   <- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.453 (sec). Leaf size: 67

```
dsolve(2*(a*x^3+b*x^2+c*x+d)*diff(y(x),x$2)+(3*a*x^2+2*b*x+c)*diff(y(x),x)+lambda*y(x)=0,y(x
```

$$y(x) = c_1 e^{\frac{i\sqrt{2}\,\sqrt{\lambda}\left(\int \frac{1}{\sqrt{a\,x^3+x^2 b+c x+d}}dx\right)}{2}} + c_2 e^{-\frac{i\sqrt{2}\,\sqrt{\lambda}\left(\int \frac{1}{\sqrt{a\,x^3+x^2 b+c x+d}}dx\right)}{2}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[2*(a*x^3+b*x^2+c*x+d)*y''[x]+(3*a*x^2+2*b*x+c)*y'[x]+lambda*y[x]==0,y[x],x,IncludeSin
```

Timed out

31.26 problem 207

Internal problem ID [11031]
Internal file name [`OUTPUT/10287_Sunday_December_31_2023_04_01_34_PM_76722025/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 207.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$2\left(a\,x^3+b\,x^2+cx+d\right)y''+3\left(3a\,x^2+2bx+c\right)y'+\left(6ax+2b+\lambda\right)y=0$$

31.26.1 Maple step by step solution

Let's solve

$y''(2a\,x^3+2b\,x^2+2cx+2d)+(9a\,x^2+6bx+3c)\,y'+(6ax+2b+\lambda)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(6ax+2b+\lambda)y}{2(a\,x^3+b\,x^2+cx+d)}-\frac{3(3a\,x^2+2bx+c)y'}{2(a\,x^3+b\,x^2+cx+d)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3(3a\,x^2+2bx+c)y'}{2(a\,x^3+b\,x^2+cx+d)}+\frac{(6ax+2b+\lambda)y}{2(a\,x^3+b\,x^2+cx+d)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x)=\frac{3(3a\,x^2+2bx+c)}{2(a\,x^3+b\,x^2+cx+d)}, P_3(x)=\frac{6ax+2b+\lambda}{2(a\,x^3+b\,x^2+cx+d)}\right]$

- $\left(x-\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}+\frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a\right.}\right.$

  $\left(\left(x-\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}+\frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\right.}\right.\right.$

- $\left(x-\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}+\frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a\right.}\right.$

  $\left(\left(x-\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}+\frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b\right.}\right.\right.$

- $x=\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}-\frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-1\right.}$

  Check to see if x_0 is a regular singular point

  $x_0=\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}-\frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-\right.}$

- Multiply by denominators

  $y''(2a\,x^3+2b\,x^2+2cx+2d)+(9a\,x^2+6bx+3c)\,y'+(6ax+2b+\lambda)\,y=0$

- Change variables using $x=u+\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}-\frac{}{3a\left(12\sqrt{3}\,\sqrt{\right.}}$

  $\left(d+\frac{\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}}{9a}+\frac{16b^6}{27a^2\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)}-\right.$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$\frac{a_0 r\left(\left(\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{4}{3}}-12ca\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-10\right.\right.\right.}{12a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3\right.}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$0 = 0$$

- Values of r that satisfy the indicial equation

$$r = r$$

- The coefficients of each power of u must be 0

$$\left[\frac{a_0 r\left(\left(\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{4}{3}}-12ca\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-\right.\right.\right.}{12a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3-\right.}\right.$$

- Each term in the series must be 0, giving the recursion relation

$$-\frac{108\left(\frac{\left(-(k+1+r)a_{k-1}\left(k+r+\frac{1}{2}\right)a^2+\left((k+1+r)c\left(k+r+\frac{3}{2}\right)a_{k+1}-\frac{a_k\lambda}{2}\right)a-\frac{(k+1+r)a_{k+1}b^2\left(k+r+\frac{3}{2}\right)}{3}\right)\left(12\sqrt{3}\sqrt{27a^2d^2+\left(-18abc+4b^3\right)d+4a}\right.}{54}\right.}{}$$

- Shift index using $k->k+1$

$$-\frac{108\left(\frac{\left(-(k+r+2)a_k\left(k+r+\frac{3}{2}\right)a^2+\left((k+r+2)c\left(k+\frac{5}{2}+r\right)a_{k+2}-\frac{a_{k+1}\lambda}{2}\right)a-\frac{(k+r+2)a_{k+2}b^2\left(k+\frac{5}{2}+r\right)}{3}\right)\left(12\sqrt{3}\sqrt{27a^2d^2+\left(-18abc+4b^3\right)d+4a}\right.}{54}\right.}{}$$

- Recursion relation that defines series solution to ODE
- Recursion relation for $r = r$
- Solution for $r = r$
- Revert the change of variables $u = x - \frac{\left(12\sqrt{3}\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} + -$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
   No special function solution was found.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.531 (sec). Leaf size: 101

```
dsolve(2*(a*x^3+b*x^2+c*x+d)*diff(y(x),x$2)+3*(3*a*x^2+2*b*x+c)*diff(y(x),x)+(6*a*x+2*b+lamb
```

$$y(x) = \frac{c_1 \mathrm{e}^{\frac{\sqrt{2}\,\sqrt{-\frac{\lambda}{a}}\left(\int \frac{1}{\sqrt{\frac{a\,x^3+x^2b+cx+d}{a}}}dx\right)}{2}} + c_2 \mathrm{e}^{-\frac{\sqrt{2}\,\sqrt{-\frac{\lambda}{a}}\left(\int \frac{1}{\sqrt{\frac{a\,x^3+x^2b+cx+d}{a}}}dx\right)}{2}}}{\sqrt{a\,x^3+x^2b+cx+d}}$$

✓ Solution by Mathematica

Time used: 135.727 (sec). Leaf size: 3202

```
DSolve[2*(a*x^3+b*x^2+c*x+d)*y''[x]+3*(3*a*x^2+2*b*x+c)*y'[x]+(6*a*x+2*b+\[Lambda])*y[x]==0,
```

Too large to display

31.27 problem 208

Internal problem ID [11032]
Internal file name [`OUTPUT/10288_Sunday_December_31_2023_04_01_43_PM_42673453/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 208.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\left(a\,x^3+b\,x^2+cx+d\right)y''+\left(\alpha\,x^2+\left(\alpha\gamma+\beta\right)x+\beta\lambda\right)y'-\left(x\alpha+\beta\right)y=0$$

31.27.1 Maple step by step solution

Let's solve

$(a\,x^3+b\,x^2+cx+d)\,y''+(\alpha\,x^2+(\alpha\gamma+\beta)\,x+\beta\lambda)\,y'+(-x\alpha-\beta)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(x\alpha+\beta)y}{a\,x^3+b\,x^2+cx+d}-\frac{(\alpha\gamma x+\alpha\,x^2+\beta\lambda+\beta x)y'}{a\,x^3+b\,x^2+cx+d}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(\alpha\gamma x+\alpha\,x^2+\beta\lambda+\beta x)y'}{a\,x^3+b\,x^2+cx+d}-\frac{(x\alpha+\beta)y}{a\,x^3+b\,x^2+cx+d}=0$

☐ Check to see if x_0 is a regular singular point

  ∘ Define functions

$\left[P_2(x) = \frac{\alpha\gamma x+\alpha\,x^2+\beta\lambda+\beta x}{a\,x^3+b\,x^2+cx+d}, P_3(x) = -\frac{x\alpha+\beta}{a\,x^3+b\,x^2+cx+d}\right]$

- $\left(x - \frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} + \frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a\right.}\right.$

  $\left(\left(x - \frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} + \frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\right.}\right.\right.$

- $\left(x - \frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} + \frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a\right.}\right.$

  $\left(\left(x - \frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} + \frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b\right.}\right.\right.$

- $x = \frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} - \frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-1\right.}$

  Check to see if x_0 is a regular singular point

  $x_0 = \frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} - \frac{2(3ac-b^2)}{3a\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-\right.}$

- Multiply by denominators

  $(a\,x^3+b\,x^2+cx+d)\,y''+(\alpha\gamma x+\alpha\,x^2+\beta\lambda+\beta x)\,y'+(-x\alpha-\beta)\,y=0$

- Change variables using $x = u + \frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a} - \frac{}{3a\left(12\sqrt{3}\,\sqrt{}\right.}$

  $\left(\frac{d}{2} + a\,u^3 + \frac{\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}}{18a} + \frac{8b^6}{27a^2\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-\right.}\right.$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$0 = 0$$

- Values of r that satisfy the indicial equation

$$r = r$$

- The coefficients of each power of u must be 0

$$\left[\frac{a_0 r\left(-48\beta a\, b^3-48a\, b^4-432a^3c^2-3\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{4}{3}}a-648a^3\alpha d\gamma-48a\alpha\, b^3\gamma\right.}{}\right.$$

- Each term in the series must be 0, giving the recursion relation

$$-\frac{54\left(\left(-\frac{a_{k-1}(k+r-1)(k-2+r)a^3}{2}+\left(-\frac{(k+1+r)(\beta\lambda-ck-cr)a_{k+1}}{2}+\frac{\left(\left(-\gamma a_k-a_{k-1}\right)\alpha-a_k\beta\right)k}{2}+\frac{\left(\left(-\gamma a_k-a_{k-1}\right)\alpha-a_k\beta\right)r}{2}+\alpha a_{k-1}+\frac{a_k\beta}{2}\right)a^{\cdots}\right.\right.}{\cdots}$$

- Shift index using $k->k+1$

$$-\frac{54\left(\left(-\frac{a_k(k+r)(k+r-1)a^3}{2}+\left(-\frac{(k+2+r)(\beta\lambda-c(k+1)-cr)a_{k+2}}{2}+\frac{\left(\left(-\gamma a_{k+1}-a_k\right)\alpha-a_{k+1}\beta\right)(k+1)}{2}+\frac{\left(\left(-\gamma a_{k+1}-a_k\right)\alpha-a_{k+1}\beta\right)r}{2}+a_k\alpha+\cdots\right.\right.\right.}{\cdots}$$

- Recursion relation that defines series solution to ODE
- Recursion relation for $r = r$
- Solution for $r = r$
- Revert the change of variables $u = x-\frac{\left(12\sqrt{3}\,\sqrt{27a^2d^2-18abcd+4a\,c^3+4b^3d-c^2b^2}\,a-108d\,a^2+36abc-8b^3\right)^{\frac{1}{3}}}{6a}+\frac{\cdots}{3\cdots}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✗ Solution by Maple

```
dsolve((a*x^3+b*x^2+c*x+d)*diff(y(x),x$2)+(alpha*x^2+(alpha*gamma+beta)*x+beta*lambda)*diff(
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(a*x^3+b*x^2+c*x+d)*y''[x]+(\[Alpha]*x^2+(\[Alpha]*\[Gamma]+\[Beta])*x+\[Beta]*\[Lamb
```

Timed out

31.28 problem 209

Internal problem ID [11033]
Internal file name [`OUTPUT/10289_Sunday_December_31_2023_04_04_24_PM_7610360/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 209.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{\left(a\,x^3 + b\,x^2 + cx + d\right) y'' + \left(\lambda^3 + x^3\right) y' - \left(\lambda^2 - \lambda x + x^2\right) y = 0}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
   No special function solution was found.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.532 (sec). Leaf size: 76

```
dsolve((a*x^3+b*x^2+c*x+d)*diff(y(x),x$2)+(x^3+lambda^3)*diff(y(x),x)-(x^2-lambda*x+lambda^2
```

$$y(x) = (x+\lambda)\left(\left(\int e^{-\left(\int \frac{x^4+(2a+\lambda)x^3+2x^2b+\left(\lambda^3+2c\right)x+\lambda^4+2d}{\left(a\,x^3+x^2b+cx+d\right)(x+\lambda)}dx\right)}dx\right)c_2+c_1\right)$$

✓ Solution by Mathematica

Time used: 1.343 (sec). Leaf size: 240

```
DSolve[(a*x^3+b*x^2+c*x+d)*y''[x]+(x^3+\[Lambda]^3)*y'[x]-(x^2-\[Lambda]*x+\[Lambda]^2)*y[x]
```

$$y(x) \to \frac{c_2(\lambda + x)\int_1^x \exp\left(-\frac{\lambda+K[1]+2a\log(\lambda+K[1])+\text{RootSum}\left[-a\lambda^3+b\lambda^2+3a\#1\lambda^2-3a\#1^2\lambda-c\lambda-2b\#1\lambda+a\#1^3+b\#1^2+d+c\right.}{}\right.}{} + \frac{c_1(\lambda+x)}{\lambda}$$

31.29 problem 210

Internal problem ID [11034]
Internal file name [OUTPUT/10290_Sunday_December_31_2023_04_06_32_PM_17535480/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-6 Equation of form $(a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 210.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$2x\left(a\,x^2 + bx + c\right)y'' + \left(a(2-k)\,x^2 + b(-k+1)\,x - ck\right)y' + \lambda\,x^{1+k}y = 0$$

✗ Solution by Maple

```
dsolve(2*x(a*x^2+b*x+c)*diff(y(x),x$2)+(a*(2-k)*x^2+b*(1-k)*x-c*k)*diff(y(x),x)+(lambda*x^(k
```

No solution found

✓ Solution by Mathematica

Time used: 157.344 (sec). Leaf size: 790

```
DSolve[2*x(a*x^2+b*x+c)*y''[x]+(a*(2-k)*x^2+b*(1-k)*x-c*k)*y'[x]+(\[Lambda]*x^(k+1))*y[x]==0
```

$$y(x) \to \frac{\sqrt{2}\sqrt{c_1}\tan\left(\frac{\sqrt{2}x\sqrt{\frac{-\sqrt{b^2-4ac}+2ax+b}{b-\sqrt{b^2-4ac}}}\sqrt{\frac{\sqrt{b^2-4ac}+2ax+b}{\sqrt{b^2-4ac}+b}}\,\text{AppellF1}\left(\frac{k+2}{2},\frac{1}{2},\frac{1}{2},\frac{k+4}{2},-\frac{2ax}{b+\sqrt{b^2-4ac}},\frac{2ax}{\sqrt{b^2-4ac}-b}\right)}{(k+2)\sqrt{\frac{x^{-k}(x(ax+b)+c)}{\lambda}}}-c_2\right)}{\sqrt{-1-\tan^2\left(\frac{\sqrt{2}x\sqrt{\frac{-\sqrt{b^2-4ac}+2ax+b}{b-\sqrt{b^2-4ac}}}\sqrt{\frac{\sqrt{b^2-4ac}+2ax+b}{\sqrt{b^2-4ac}+b}}\,\text{AppellF1}\left(\frac{k+2}{2},\frac{1}{2},\frac{1}{2},\frac{k+4}{2},-\frac{2ax}{b+\sqrt{b^2-4ac}},\frac{2ax}{\sqrt{b^2-4ac}-b}\right)}{(k+2)\sqrt{\frac{x^{-k}(x(ax+b)+c)}{\lambda}}}-c_2\right)}}$$

$$y(x) \to -\frac{\sqrt{2}\sqrt{c_1}\tan\left(\frac{\sqrt{2}x\sqrt{\frac{-\sqrt{b^2-4ac}+2ax+b}{b-\sqrt{b^2-4ac}}}\sqrt{\frac{\sqrt{b^2-4ac}+2ax+b}{\sqrt{b^2-4ac}+b}}\,\text{AppellF1}\left(\frac{k+2}{2},\frac{1}{2},\frac{1}{2},\frac{k+4}{2},-\frac{2ax}{b+\sqrt{b^2-4ac}},\frac{2ax}{\sqrt{b^2-4ac}-b}\right)}{(k+2)\sqrt{\frac{x^{-k}(x(ax+b)+c)}{\lambda}}}+c_2\right)}{\sqrt{-1-\tan^2\left(\frac{\sqrt{2}x\sqrt{\frac{-\sqrt{b^2-4ac}+2ax+b}{b-\sqrt{b^2-4ac}}}\sqrt{\frac{\sqrt{b^2-4ac}+2ax+b}{\sqrt{b^2-4ac}+b}}\,\text{AppellF1}\left(\frac{k+2}{2},\frac{1}{2},\frac{1}{2},\frac{k+4}{2},-\frac{2ax}{b+\sqrt{b^2-4ac}},\frac{2ax}{\sqrt{b^2-4ac}-b}\right)}{(k+2)\sqrt{\frac{x^{-k}(x(ax+b)+c)}{\lambda}}}+c_2\right)}}$$

32 Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$

32.1 problem 211

Internal problem ID [11035]
Internal file name [OUTPUT/10291_Wednesday_January_24_2024_10_06_22_PM_16279652/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 211.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^4y'' + ay = 0}$$

32.1.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \frac{ay}{x^2} = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= \sqrt{a} \\
n &= \frac{1}{2} \\
\gamma &= -1
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{a}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{a}}{x}}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{a}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{a}}{x}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{a}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{a}}{x}}}
$$

Verified OK.

32.1.2 Solving using Kovacic algorithm

Writing the ode as

$$
x^4 y'' + ay = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^4 \\
B &= 0 \\
C &= a
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a}{x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -a \\ t &= x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{a}{x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 195: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_2^v \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{a}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i\sqrt{a}}{x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{i\sqrt{a}}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = i\sqrt{a}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{i\sqrt{a}}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{0}{i\sqrt{a}} + 2\right) &= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{0}{i\sqrt{a}} + 2\right) &= 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{a}{x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{i\sqrt{a}}{x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{i\sqrt{a}}{x^2} + \frac{1}{x} + (-)\,(0) \\
&= -\frac{i\sqrt{a}}{x^2} + \frac{1}{x} \\
&= \frac{-i\sqrt{a} + x}{x^2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{i\sqrt{a}}{x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{2i\sqrt{a}}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{i\sqrt{a}}{x^2} + \frac{1}{x}\right)^2 - \left(-\frac{a}{x^4}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{i\sqrt{a}}{x^2} + \frac{1}{x}\right) dx} \\
&= x\,\mathrm{e}^{\frac{i\sqrt{a}}{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x\,\mathrm{e}^{\frac{i\sqrt{a}}{x}} \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{\frac{i\sqrt{a}}{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= x\,\mathrm{e}^{\frac{i\sqrt{a}}{x}} \int \frac{1}{x^2 \mathrm{e}^{\frac{2i\sqrt{a}}{x}}}\,dx \\ &= x\,\mathrm{e}^{\frac{i\sqrt{a}}{x}} \left(-\frac{i\mathrm{e}^{-\frac{2i\sqrt{a}}{x}}}{2\sqrt{a}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x\,\mathrm{e}^{\frac{i\sqrt{a}}{x}}\right) + c_2\left(x\,\mathrm{e}^{\frac{i\sqrt{a}}{x}}\left(-\frac{i\mathrm{e}^{-\frac{2i\sqrt{a}}{x}}}{2\sqrt{a}}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{\frac{i\sqrt{a}}{x}} - \frac{i c_2 x\,\mathrm{e}^{-\frac{i\sqrt{a}}{x}}}{2\sqrt{a}} \tag{1}$$

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{\frac{i\sqrt{a}}{x}} - \frac{i c_2 x \,\mathrm{e}^{-\frac{i\sqrt{a}}{x}}}{2\sqrt{a}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(x^4*diff(y(x),x$2)+a*y(x)=0,y(x), singsol=all)
```

$$y(x) = x\left(c_1 \sinh\left(\frac{\sqrt{-a}}{x}\right) + c_2 \cosh\left(\frac{\sqrt{-a}}{x}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.199 (sec). Leaf size: 52

```
DSolve[x^4*y''[x]+a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x e^{\frac{i\sqrt{a}}{x}} - \frac{i c_2 x e^{-\frac{i\sqrt{a}}{x}}}{2\sqrt{a}}$$

32.2 problem 212

Internal problem ID [11036]
Internal file name [`OUTPUT/10292_Wednesday_January_24_2024_10_06_23_PM_69199443/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 212.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^4y'' + \left(a\,x^2 + bx + c\right)y = 0}$$

32.2.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(a + \frac{b}{x} + \frac{c}{x^2}\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1\,\mathrm{BesselJ}\left(n, \beta\,x^{\gamma}\right) + c_2\,\mathrm{BesselY}\left(n, \beta\,x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = \sqrt{1-4a}$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\sqrt{1-4a}, 2\sqrt{x}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\sqrt{1-4a}, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\sqrt{1-4a}, 2\sqrt{x}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\sqrt{1-4a}, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\sqrt{1-4a}, 2\sqrt{x}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\sqrt{1-4a}, 2\sqrt{x}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 63

```
dsolve(x^4*diff(y(x),x$2)+(a*x^2+b*x+c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x\left(c_1 \operatorname{WhittakerM}\left(-\frac{ib}{2\sqrt{c}}, \frac{\sqrt{-4a+1}}{2}, \frac{2i\sqrt{c}}{x}\right) + c_2 \operatorname{WhittakerW}\left(-\frac{ib}{2\sqrt{c}}, \frac{\sqrt{-4a+1}}{2}, \frac{2i\sqrt{c}}{x}\right)\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^4*y''[x]+(a*x^2+b*x+c)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

32.3 problem 213

Internal problem ID [11037]
Internal file name [OUTPUT/10293_Wednesday_January_24_2024_10_06_24_PM_85874191/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 213.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^4y'' - (a+b)\,x^2y' + ((a+b)\,x + ab)\,y = 0}$$

32.3.1 Solving using Kovacic algorithm

Writing the ode as

$$x^4y'' - (a+b)\,x^2y' + (a(x+b) + bx)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= -x^2(a+b) \\ C &= a(x+b) + bx \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{a^2 - 2ab + b^2}{4x^4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= a^2 - 2ab + b^2 \\
t &= 4x^4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{a^2 - 2ab + b^2}{4x^4}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 196: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{\frac{1}{4}a^2 - \frac{1}{2}ab + \frac{1}{4}b^2}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{\sqrt{a^2 - 2ab + b^2}}{2x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{\sqrt{a^2 - 2ab + b^2}}{2x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = \frac{\sqrt{a^2 - 2ab + b^2}}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{\sqrt{a^2 - 2ab + b^2}}{2x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{\sqrt{a^2-2ab+b^2}}{2}} + 2\right) &&= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{\sqrt{a^2-2ab+b^2}}{2}} + 2\right) &&= 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a^2 - 2ab + b^2}{4x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{\sqrt{a^2-2ab+b^2}}{2x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{\sqrt{a^2-2ab+b^2}}{2x^2} + \frac{1}{x} + (-)(0) \\ &= -\frac{\sqrt{a^2-2ab+b^2}}{2x^2} + \frac{1}{x} \\ &= \frac{-\sqrt{(a-b)^2} + 2x}{2x^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{\sqrt{a^2-2ab+b^2}}{2x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{\sqrt{a^2-2ab+b^2}}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{\sqrt{a^2-2ab+b^2}}{2x^2} + \frac{1}{x}\right)^2 - \left(\frac{a^2 -}{}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{\sqrt{a^2-2ab+b^2}}{2x^2}+\frac{1}{x}\right) dx} \\
&= x\,\mathrm{e}^{\frac{\operatorname{csgn}(a-b)(a-b)}{2x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2(a+b)}{x^4}\, dx} \\
&= z_1 e^{-\frac{\frac{a}{2}+\frac{b}{2}}{x}} \\
&= z_1 \left(\mathrm{e}^{-\frac{a+b}{2x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\,\mathrm{e}^{\frac{\operatorname{csgn}(a-b)(a-b)-b-a}{2x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2(a+b)}{x^4}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{-a-b}{x}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{-\frac{\operatorname{csgn}(a-b)(a-b)}{x}} \operatorname{csgn}(a-b)}{a-b}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{\frac{\operatorname{csgn}(a-b)(a-b)-b-a}{2x}}\right) + c_2 \left(x\,\mathrm{e}^{\frac{\operatorname{csgn}(a-b)(a-b)-b-a}{2x}} \left(\frac{\mathrm{e}^{-\frac{\operatorname{csgn}(a-b)(a-b)}{x}} \operatorname{csgn}(a-b)}{a-b}\right)\right)
\end{aligned}
$$

Simplifying the solution $y = c_1 x \,\mathrm{e}^{\frac{\mathrm{csgn}(a-b)(a-b)-b-a}{2x}} + \frac{c_2 x \,\mathrm{e}^{\frac{(b-a)\,\mathrm{csgn}(a-b)-b-a}{2x}}}{\sqrt{(a-b)^2}}$ to $y = c_1 x \,\mathrm{e}^{-\frac{b}{x}} + \frac{c_2 x \,\mathrm{e}^{-\frac{a}{x}}}{\sqrt{(a-b)^2}}$

Summary
The solution(s) found are the following

$$y = c_1 x \,\mathrm{e}^{-\frac{b}{x}} + \frac{c_2 x \,\mathrm{e}^{-\frac{a}{x}}}{\sqrt{(a-b)^2}} \tag{1}$$

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{-\frac{b}{x}} + \frac{c_2 x \,\mathrm{e}^{-\frac{a}{x}}}{\sqrt{(a-b)^2}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 25

```
dsolve(x^4*diff(y(x),x$2)-(a+b)*x^2*diff(y(x),x)+((a+b)*x+a*b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x\left(\mathrm{e}^{-\frac{a}{x}} c_1 + \mathrm{e}^{-\frac{b}{x}} c_2\right)$$

✓ Solution by Mathematica

Time used: 0.214 (sec). Leaf size: 37

```
DSolve[x^4*y''[x]-(a+b)*x^2*y'[x]+((a+b)*x+a*b)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{c_2 x e^{-\frac{a}{x}}}{a-b} + c_1 x e^{-\frac{b}{x}}$$

32.4 problem 214

Internal problem ID [11038]
Internal file name [`OUTPUT/10294_Wednesday_January_24_2024_10_06_25_PM_38663670/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 214.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^4 y'' + 2x^2(x + a)\, y' + yb = 0}$$

32.4.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^4 y'' + 2x^2(x + a)\, y' + yb = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2x + 2a}{x^2} \\ q(x) &= \frac{b}{x^4} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{2x+2a}{x^2} dx\right)} dx \\
&= \int e^{\frac{2a}{x} - 2\ln(x)}\, dx \\
&= \int \frac{\mathrm{e}^{\frac{2a}{x}}}{x^2} dx \\
&= -\frac{\mathrm{e}^{\frac{2a}{x}}}{2a}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{b}{x^4}}{\frac{\mathrm{e}^{\frac{4a}{x}}}{x^4}} \\
&= b\,\mathrm{e}^{-\frac{4a}{x}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) + b\,\mathrm{e}^{-\frac{4a}{x}} y(\tau) = 0$$

But in terms of τ

$$b\,\mathrm{e}^{-\frac{4a}{x}} = \frac{b}{4a^2\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{by(\tau)}{4a^2\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)a^2\tau^2 + by(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4a^2\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + b\,\tau^r = 0$$

Simplifying gives

$$4a^2r(r-1)\,\tau^r + 0\,\tau^r + b\,\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4a^2r(r-1) + 0 + b = 0$$

Or

$$4a^2r^2 - 4a^2r + b = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{-a + \sqrt{a^2-b}}{2a}$$
$$r_2 = \frac{a + \sqrt{a^2-b}}{2a}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{-\frac{-a+\sqrt{a^2-b}}{2a}} + c_2\tau^{\frac{a+\sqrt{a^2-b}}{2a}}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(-\frac{e^{\frac{2a}{x}}}{2a}\right)^{-\frac{-a+\sqrt{a^2-b}}{2a}} + c_2 \left(-\frac{e^{\frac{2a}{x}}}{2a}\right)^{\frac{a+\sqrt{a^2-b}}{2a}}$$

Summary
The solution(s) found are the following

$$y = c_1 \left(-\frac{e^{\frac{2a}{x}}}{2a}\right)^{-\frac{-a+\sqrt{a^2-b}}{2a}} + c_2 \left(-\frac{e^{\frac{2a}{x}}}{2a}\right)^{\frac{a+\sqrt{a^2-b}}{2a}} \tag{1}$$

Verification of solutions

$$y = c_1 \left(-\frac{e^{\frac{2a}{x}}}{2a}\right)^{-\frac{-a+\sqrt{a^2-b}}{2a}} + c_2 \left(-\frac{e^{\frac{2a}{x}}}{2a}\right)^{\frac{a+\sqrt{a^2-b}}{2a}}$$

Verified OK.

32.4.2 Solving using Kovacic algorithm

Writing the ode as

$$x^4 y'' + 2x^2(x+a)\, y' + yb = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= 2x^2(x+a) \\ C &= b \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2 - b}{x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2 - b \\ t &= x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2 - b}{x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 197: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_2^v \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\alpha_c^+ = \frac{1}{2}\left(\frac{b}{a} + v\right)$$
$$\alpha_c^- = \frac{1}{2}\left(-\frac{b}{a} + v\right)$$

The partial fraction decomposition of r is

$$r = \frac{a^2 - b}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{\sqrt{a^2 - b}}{x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{\sqrt{a^2 - b}}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = \sqrt{a^2 - b}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{\sqrt{a^2-b}}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a}+v\right) &&= \frac{1}{2}\left(\frac{0}{\sqrt{a^2-b}}+2\right) &&= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a}+v\right) &&= \frac{1}{2}\left(-\frac{0}{\sqrt{a^2-b}}+2\right) &&= 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a^2-b}{x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{\sqrt{a^2-b}}{x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{\sqrt{a^2 - b}}{x^2} + \frac{1}{x} + (-)(0) \\
&= -\frac{\sqrt{a^2 - b}}{x^2} + \frac{1}{x} \\
&= \frac{-\sqrt{a^2 - b} + x}{x^2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{\sqrt{a^2 - b}}{x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{2\sqrt{a^2 - b}}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{\sqrt{a^2 - b}}{x^2} + \frac{1}{x}\right)^2 - \left(\frac{a^2 - b}{x^4}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{\sqrt{a^2 - b}}{x^2} + \frac{1}{x}\right) dx} \\
&= x\, \mathrm{e}^{\frac{\sqrt{a^2 - b}}{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2(x+a)}{x^4}\,dx} \\
&= z_1 e^{\frac{a}{x}-\ln(x)} \\
&= z_1\left(\frac{e^{\frac{a}{x}}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = e^{\frac{a+\sqrt{a^2-b}}{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2(x+a)}{x^4}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{2a}{x}-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{e^{-\frac{2\sqrt{a^2-b}}{x}}}{2\sqrt{a^2-b}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{\frac{a+\sqrt{a^2-b}}{x}}\right) + c_2\left(e^{\frac{a+\sqrt{a^2-b}}{x}}\left(\frac{e^{-\frac{2\sqrt{a^2-b}}{x}}}{2\sqrt{a^2-b}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{\frac{a+\sqrt{a^2-b}}{x}} + \frac{c_2 e^{\frac{a-\sqrt{a^2-b}}{x}}}{2\sqrt{a^2-b}} \tag{1}
$$

Verification of solutions

$$y = c_1 e^{\frac{a+\sqrt{a^2-b}}{x}} + \frac{c_2 e^{\frac{a-\sqrt{a^2-b}}{x}}}{2\sqrt{a^2-b}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(x^4*diff(y(x),x$2)+2*x^2*(x+a)*diff(y(x),x)+b*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{a-\sqrt{a^2-b}}{x}} + c_2 e^{\frac{a+\sqrt{a^2-b}}{x}}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 51

```
DSolve[x^4*y''[x]+2*x^2*(x+a)*y'[x]+b*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{a-\sqrt{a^2-b}}{x}} \left(c_1 e^{\frac{2\sqrt{a^2-b}}{x}} + c_2\right)$$

32.5 problem 215

Internal problem ID [11039]
Internal file name [`OUTPUT/10295_Wednesday_January_24_2024_10_06_25_PM_81868881/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 215.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x^4 y'' + a\,x^n y' - \left(a\,x^{n-1} + ab\,x^{n-2} + b^2\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(x^4*diff(y(x),x$2)+a*x^n*diff(y(x),x)-(a*x^(n-1)+a*b*x^(n-2)+b^2)*y(x)=0,y(x), singso
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^4*y''[x]+a*x^n*y'[x]-(a*x^(n-1)+a*b*x^(n-2)+b^2)*y[x]==0,y[x],x,IncludeSingularSolu
```

Not solved

32.6 problem 216

Internal problem ID [11040]
Internal file name [`OUTPUT/10296_Wednesday_January_24_2024_10_06_26_PM_32998932/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 216.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x-a)^2 y'' + yb = 0}$$

32.6.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \frac{by}{x^2} = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= \sqrt{b} \\
n &= \frac{1}{2} \\
\gamma &= -1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}}{x}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}}{x}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}}{x}}}$$

Verified OK.

32.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' x^2 (a - x)^2 + y b = 0 \tag{1}$$

$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= x^2 (a - x)^2 \\
B &= 0 \\
C &= b
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-b}{\left(ax - x^2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -b \\ t &= \left(ax - x^2\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{b}{\left(ax - x^2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 198: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (ax - x^2)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = a$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{b}{a^2 x^2} - \frac{b}{a^2 (x-a)^2} + \frac{2b}{a^3 (x-a)} - \frac{2b}{a^3 x}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{b}{a^2}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{a^2 - 4b}}{2a} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{a^2 - 4b}}{2a}
\end{aligned}
$$

For the pole at $x = a$ let b be the coefficient of $\frac{1}{(x-a)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{b}{a^2}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{a^2 - 4b}}{2a} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{a^2 - 4b}}{2a}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{b}{\left(ax - x^2\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}+\frac{\sqrt{a^2-4b}}{2a}$	$\frac{1}{2}-\frac{\sqrt{a^2-4b}}{2a}$
a	2	0	$\frac{1}{2}+\frac{\sqrt{a^2-4b}}{2a}$	$\frac{1}{2}-\frac{\sqrt{a^2-4b}}{2a}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2}-\frac{\sqrt{a^2-4b}}{2a}}{x} + \frac{\frac{1}{2}+\frac{\sqrt{a^2-4b}}{2a}}{x-a} + (-)\,(0) \\
&= \frac{\frac{1}{2}-\frac{\sqrt{a^2-4b}}{2a}}{x} + \frac{\frac{1}{2}+\frac{\sqrt{a^2-4b}}{2a}}{x-a} \\
&= \frac{\sqrt{a^2-4b} - a + 2x}{2x\,(x-a)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{a^2-4b}}{2a}}{x} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-4b}}{2a}}{x-a}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{a^2-4b}}{2a}}{x^2} - \frac{\frac{1}{2} + \frac{\sqrt{a^2-4b}}{2a}}{(x-a)^2}\right) + \left(\frac{\frac{1}{2} - \frac{\sqrt{a^2-4b}}{2a}}{x} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-4b}}{2a}}{x-a}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{\frac{1}{2} - \frac{\sqrt{a^2-4b}}{2a}}{x} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-4b}}{2a}}{x-a}\right) dx} \\
&= x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}
\end{aligned}$$

Which simplifies to

$$y_1 = x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \int \frac{1}{x^{-\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{a}}}\, dx \\
&= x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}\, dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \right) \\
&\quad + c_2 \left(x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}\, dx \right) \right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \\
&\quad + c_2 x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}\, dx \right)
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y &= c_1 x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \\
&\quad + c_2 x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}\, dx \right)
\end{aligned}
$$

Verified OK.

32.6.3 Maple step by step solution

Let's solve

$y''x^2(a-x)^2 + yb = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{by}{x^2(a-x)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{by}{x^2(a-x)^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{b}{x^2(a-x)^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{b}{a^2}$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x^2(a-x)^2 + yb = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $$x^m\cdot y''=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(a^2r^2-a^2r+b)\,x^r+((a^2r^2+a^2r+b)\,a_1-2a_0r(-1+r)\,a)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(a^2k^2+2a^2kr+a\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$a^2r^2-a^2r+b=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{\frac{\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}}{a},\frac{\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}}{a}\right\}$$

- Each term must be 0

  $$(a^2r^2+a^2r+b)\,a_1-2a_0r(-1+r)\,a=0$$

- Solve for the dependent coefficient(s)

  $$a_1=\frac{2a_0r(-1+r)a}{a^2r^2+a^2r+b}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r)(k+r-1)\,a^2-2a_{k-1}(k+r-1)(k-2+r)\,a+k^2a_{k-2}+a_{k-2}(2r-5)\,k+r^2a_{k-2}+a$$

- Shift index using $k->k+2$

  $$a_{k+2}(k+2+r)(k+1+r)\,a^2-2a_{k+1}(k+1+r)(k+r)\,a+(k+2)^2\,a_k+a_k(2r-5)(k+2)+$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=\frac{2a\,k^2a_{k+1}+4akra_{k+1}+2a\,r^2a_{k+1}+2aka_{k+1}+2ara_{k+1}-k^2a_k-2kra_k-r^2a_k+a_kk+a_kr}{a^2k^2+2a^2kr+a^2r^2+3a^2k+3a^2r+2a^2+b}$$

- Recursion relation for $r=\frac{\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}}{a}$

  $$a_{k+2}=\frac{2a\,k^2a_{k+1}+4k\left(\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)a_{k+1}+\frac{2\left(\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)^2a_{k+1}}{a}+2aka_{k+1}+2\left(\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)a_{k+1}-k^2a_k-\frac{2k\left(\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)a_k}{a}-}{a^2k^2+2ak\left(\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)+\left(\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)^2+3a^2k+3a\left(\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)+2a^2+b}$$

- Solution for $r=\frac{\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}}{a}$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}}{a}},a_{k+2}=\frac{2a\,k^2a_{k+1}+4k\left(\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)a_{k+1}+\frac{2\left(\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)^2a_{k+1}}{a}+2aka_{k+1}+2\left(\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)}{a^2k^2+2ak\left(\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)+\left(\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)}\right.$$

- Recursion relation for $r=\frac{\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}}{a}$

$$a_{k+2} = \frac{2a\,k^2 a_{k+1}+4k\left(\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}\right)a_{k+1}+\frac{2\left(\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}\right)^2 a_{k+1}}{a}+2ak a_{k+1}+2\left(\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}\right)a_{k+1}-k^2 a_k-\frac{2k\left(\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}\right)a_k}{a} -}{a^2k^2+2ak\left(\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}\right)+\left(\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}\right)^2+3a^2k+3a\left(\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}\right)+2a^2+b}$$

- Solution for $r = \frac{\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}}{a}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}}{a}}, a_{k+2} = \frac{2a\,k^2 a_{k+1}+4k\left(\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}\right)a_{k+1}+\frac{2\left(\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}\right)^2 a_{k+1}}{a}+2ak a_{k+1}+2\left(\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}\right)}{a^2k^2+2ak\left(\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}\right)+\left(\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}\right)}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} c_k x^{k+\frac{\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}}{a}}\right)+\left(\sum_{k=0}^{\infty} d_k x^{k+\frac{\frac{a}{2}+\frac{\sqrt{a^2-4b}}{2}}{a}}\right), c_{k+2} = \frac{2a\,k^2 c_{1+k}+4k\left(\frac{a}{2}-\frac{\sqrt{a^2-4b}}{2}\right)c_{1+k}+\frac{2\left(\frac{a}{2}-\frac{\sqrt{a^2}}{2}\right.}{a}}{a^2k^2}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 67

```
dsolve(x^2*(x-a)^2*diff(y(x),x$2)+b*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x(a-x)}\left(\left(\frac{x}{a-x}\right)^{\frac{\sqrt{a^2-4b}}{2a}} c_2+\left(\frac{a-x}{x}\right)^{\frac{\sqrt{a^2-4b}}{2a}} c_1\right)$$

✓ Solution by Mathematica

Time used: 0.487 (sec). Leaf size: 121

```
DSolve[x^2*(x-a)^2*y''[x]+b*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^{\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{4b}{a^2}}}(x-a)^{\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{4b}{a^2}}}\left(ac_1\sqrt{1-\frac{4b}{a^2}}x^{\sqrt{1-\frac{4b}{a^2}}}+c_2(x-a)^{\sqrt{1-\frac{4b}{a^2}}}\right)}{a\sqrt{1-\frac{4b}{a^2}}}$$

32.7 problem 217

Internal problem ID [11041]
Internal file name [`OUTPUT/10297_Wednesday_January_24_2024_10_06_27_PM_84057678/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 217.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x^2(x-a)^2 y'' + yb = c\,x^2(x-a)^2}$$

32.7.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \frac{by}{x^2} = c(a-x)^2 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= \sqrt{b} \\
n &= \frac{1}{2} \\
\gamma &= -1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}}{x}}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}}{x}}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \frac{\sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}}{x}}} \\
y_2 &= -\frac{\sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}}{x}}}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\sqrt{x}\,\sqrt{2}\,\sin\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}} & -\frac{\sqrt{x}\,\sqrt{2}\,\cos\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}} \\ \frac{d}{dx}\left(\frac{\sqrt{x}\,\sqrt{2}\,\sin\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{x}\,\sqrt{2}\,\cos\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\sqrt{x}\,\sqrt{2}\,\sin\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}} & -\frac{\sqrt{x}\,\sqrt{2}\,\cos\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}} \\ \frac{\sqrt{2}\,\sin\left(\frac{\sqrt{b}}{x}\right)}{2\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}} + \frac{\sqrt{2}\,\sin\left(\frac{\sqrt{b}}{x}\right)\sqrt{b}}{2x^{\frac{3}{2}}\sqrt{\pi}\,\left(\frac{\sqrt{b}}{x}\right)^{\frac{3}{2}}} - \frac{\sqrt{2}\,\sqrt{b}\,\cos\left(\frac{\sqrt{b}}{x}\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}} & -\frac{\sqrt{2}\,\cos\left(\frac{\sqrt{b}}{x}\right)}{2\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}} - \frac{\sqrt{2}\,\cos\left(\frac{\sqrt{b}}{x}\right)\sqrt{b}}{2x^{\frac{3}{2}}\sqrt{\pi}\,\left(\frac{\sqrt{b}}{x}\right)^{\frac{3}{2}}} - \frac{\sqrt{2}\,\sqrt{b}\,\sin\left(\frac{\sqrt{b}}{x}\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\frac{\sqrt{x}\,\sqrt{2}\,\sin\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}}\right)\left(-\frac{\sqrt{2}\,\cos\left(\frac{\sqrt{b}}{x}\right)}{2\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}} - \frac{\sqrt{2}\,\cos\left(\frac{\sqrt{b}}{x}\right)\sqrt{b}}{2x^{\frac{3}{2}}\sqrt{\pi}\,\left(\frac{\sqrt{b}}{x}\right)^{\frac{3}{2}}} - \frac{\sqrt{2}\,\sqrt{b}\,\sin\left(\frac{\sqrt{b}}{x}\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}}\right) \\ & - \left(-\frac{\sqrt{x}\,\sqrt{2}\,\cos\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}}\right)\left(\frac{\sqrt{2}\,\sin\left(\frac{\sqrt{b}}{x}\right)}{2\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}} + \frac{\sqrt{2}\,\sin\left(\frac{\sqrt{b}}{x}\right)\sqrt{b}}{2x^{\frac{3}{2}}\sqrt{\pi}\,\left(\frac{\sqrt{b}}{x}\right)^{\frac{3}{2}}} - \frac{\sqrt{2}\,\sqrt{b}\,\cos\left(\frac{\sqrt{b}}{x}\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}}\right) \end{aligned}$$

Which simplifies to

$$W = -\frac{2\left(\sin\left(\frac{\sqrt{b}}{x}\right)^2 + \cos\left(\frac{\sqrt{b}}{x}\right)^2\right)}{\pi}$$

Which simplifies to

$$W = -\frac{2}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{\sqrt{x}\,\sqrt{2}\,\cos\left(\frac{\sqrt{b}}{x}\right)c(a-x)^2}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}}}{-\frac{2x^2}{\pi}} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(\frac{\sqrt{b}}{x}\right)c(a-x)^2}{2x^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{x}}} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(\frac{\sqrt{b}}{\alpha}\right)c(a-\alpha)^2}{2\alpha^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{\alpha}}} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\sqrt{x}\,\sqrt{2}\,\sin\left(\frac{\sqrt{b}}{x}\right)c(a-x)^2}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}}}{-\frac{2x^2}{\pi}} dx$$

Which simplifies to

$$u_2 = \int -\frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(\frac{\sqrt{b}}{x}\right)c(a-x)^2}{2x^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{x}}} dx$$

Hence

$$u_2 = \int_0^x -\frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(\frac{\sqrt{b}}{\alpha}\right)c(a-\alpha)^2}{2\alpha^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{\alpha}}} d\alpha$$

Which simplifies to

$$u_1 = -\frac{\sqrt{2}\,\sqrt{\pi}\,c\left(\int_0^x \frac{\cos\left(\frac{\sqrt{b}}{\alpha}\right)(a-\alpha)^2}{\alpha^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{\alpha}}}d\alpha\right)}{2}$$

$$u_2 = -\frac{\sqrt{2}\,\sqrt{\pi}\,c\left(\int_0^x \frac{\sin\left(\frac{\sqrt{b}}{\alpha}\right)(a-\alpha)^2}{\alpha^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{\alpha}}}d\alpha\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{c\left(\int_0^x \frac{\cos\left(\frac{\sqrt{b}}{\alpha}\right)(a-\alpha)^2}{\alpha^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{\alpha}}}d\alpha\right)\sqrt{x}\,\sin\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\frac{\sqrt{b}}{x}}} + \frac{c\left(\int_0^x \frac{\sin\left(\frac{\sqrt{b}}{\alpha}\right)(a-\alpha)^2}{\alpha^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{\alpha}}}d\alpha\right)\sqrt{x}\,\cos\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\frac{\sqrt{b}}{x}}}$$

Which simplifies to

$$y_p(x) = \frac{c\sqrt{x}\left(-\left(\int_0^x \frac{\cos\left(\frac{\sqrt{b}}{\alpha}\right)(a-\alpha)^2}{\alpha^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{\alpha}}}d\alpha\right)\sin\left(\frac{\sqrt{b}}{x}\right) + \left(\int_0^x \frac{\sin\left(\frac{\sqrt{b}}{\alpha}\right)(a-\alpha)^2}{\alpha^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{\alpha}}}d\alpha\right)\cos\left(\frac{\sqrt{b}}{x}\right)\right)}{\sqrt{\frac{\sqrt{b}}{x}}}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1\sqrt{x}\,\sqrt{2}\,\sin\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}} - \frac{c_2\sqrt{x}\,\sqrt{2}\,\cos\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}}\right) \\
&\quad + \left(\frac{c\sqrt{x}\left(-\left(\int_0^x \frac{\cos\left(\frac{\sqrt{b}}{\alpha}\right)(a-\alpha)^2}{\alpha^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{\alpha}}}d\alpha\right)\sin\left(\frac{\sqrt{b}}{x}\right) + \left(\int_0^x \frac{\sin\left(\frac{\sqrt{b}}{\alpha}\right)(a-\alpha)^2}{\alpha^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{\alpha}}}d\alpha\right)\cos\left(\frac{\sqrt{b}}{x}\right)\right)}{\sqrt{\frac{\sqrt{b}}{x}}}\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{align*}
y &= \frac{c_1\sqrt{x}\,\sqrt{2}\,\sin\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}}-\frac{c_2\sqrt{x}\,\sqrt{2}\,\cos\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}}\\
&+\frac{c\sqrt{x}\left(-\left(\int_0^x\frac{\cos\left(\frac{\sqrt{b}}{\alpha}\right)(a-\alpha)^2}{\alpha^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{\alpha}}}d\alpha\right)\sin\left(\frac{\sqrt{b}}{x}\right)+\left(\int_0^x\frac{\sin\left(\frac{\sqrt{b}}{\alpha}\right)(a-\alpha)^2}{\alpha^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{\alpha}}}d\alpha\right)\cos\left(\frac{\sqrt{b}}{x}\right)\right)}{\sqrt{\frac{\sqrt{b}}{x}}} \tag{1}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= \frac{c_1\sqrt{x}\,\sqrt{2}\,\sin\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}}-\frac{c_2\sqrt{x}\,\sqrt{2}\,\cos\left(\frac{\sqrt{b}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}}{x}}}\\
&+\frac{c\sqrt{x}\left(-\left(\int_0^x\frac{\cos\left(\frac{\sqrt{b}}{\alpha}\right)(a-\alpha)^2}{\alpha^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{\alpha}}}d\alpha\right)\sin\left(\frac{\sqrt{b}}{x}\right)+\left(\int_0^x\frac{\sin\left(\frac{\sqrt{b}}{\alpha}\right)(a-\alpha)^2}{\alpha^{\frac{3}{2}}\sqrt{\frac{\sqrt{b}}{\alpha}}}d\alpha\right)\cos\left(\frac{\sqrt{b}}{x}\right)\right)}{\sqrt{\frac{\sqrt{b}}{x}}}
\end{align*}
$$

Verified OK.

32.7.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y''x^2(a-x)^2+yb &= 0 \tag{1}\\
Ay''+By'+Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= x^2(a-x)^2\\
B &= 0 \tag{3}\\
C &= b
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x)=ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x)=rz(x) \tag{4}
$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-b}{(ax - x^2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -b$$
$$t = (ax - x^2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{b}{(ax - x^2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 200: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (ax - x^2)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = a$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{b}{a^2 x^2} - \frac{b}{a^2 (x-a)^2} + \frac{2b}{a^3 (x-a)} - \frac{2b}{a^3 x}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{b}{a^2}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{a^2-4b}}{2a} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{a^2-4b}}{2a}
\end{aligned}
$$

For the pole at $x = a$ let b be the coefficient of $\frac{1}{(x-a)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{b}{a^2}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{a^2-4b}}{2a} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{a^2-4b}}{2a}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{b}{(ax - x^2)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + \frac{\sqrt{a^2-4b}}{2a}$	$\frac{1}{2} - \frac{\sqrt{a^2-4b}}{2a}$
a	2	0	$\frac{1}{2} + \frac{\sqrt{a^2-4b}}{2a}$	$\frac{1}{2} - \frac{\sqrt{a^2-4b}}{2a}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2} - \frac{\sqrt{a^2-4b}}{2a}}{x} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-4b}}{2a}}{x - a} + (-)\,(0) \\ &= \frac{\frac{1}{2} - \frac{\sqrt{a^2-4b}}{2a}}{x} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-4b}}{2a}}{x - a} \\ &= \frac{\sqrt{a^2 - 4b} - a + 2x}{2x\,(x - a)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{a^2-4b}}{2a}}{x} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-4b}}{2a}}{x-a}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{a^2-4b}}{2a}}{x^2} - \frac{\frac{1}{2} + \frac{\sqrt{a^2-4b}}{2a}}{(x-a)^2}\right) + \left(\frac{\frac{1}{2} - \frac{\sqrt{a^2-4b}}{2a}}{x} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-4b}}{2a}}{x-a}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{\frac{1}{2} - \frac{\sqrt{a^2-4b}}{2a}}{x} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-4b}}{2a}}{x-a}\right) dx} \\
&= x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}
\end{aligned}$$

Which simplifies to

$$y_1 = x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \int \frac{1}{x^{-\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{a}}} \, dx \\
&= x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}} \, dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \right) \\
&\quad + c_2 \left(x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}} \, dx \right) \right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' x^2 (a-x)^2 + yb = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
\begin{aligned}
y_h &= c_1 x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \\
&\quad + c_2 x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}} \, dx \right)
\end{aligned}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \\
y_2 &= x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}} dx\right)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} & x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}} dx \right. \\ \frac{d}{dx}\left(x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}\right) & \frac{d}{dx}\left(x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}} d \right.\right. \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \\ -\frac{x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} \left(-a+\sqrt{a^2-4b}\right)(x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}}{2ax} + \frac{x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(a+\sqrt{a^2-4b}\right)}{2a(x-a)} - \frac{x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} \left(-a+\sqrt{a^2-\right.}{} \end{vmatrix}
$$

Therefore

$$W = \left(x^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}\right)\left(-\frac{x^{-\frac{-a+\sqrt{a^2-4b}}{2a}}\left(-a+\sqrt{a^2-4b}\right)(x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}\left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}}(x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}dx\right)}{2ax} + \frac{x^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}\left(a+\sqrt{a^2-4b}\right)\left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}}(x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}dx\right)}{2a(x-a)} + x^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}x^{\frac{-a+\sqrt{a^2-4b}}{a}}(x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}\right) - \left(x^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}\left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}}(x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}dx\right)\right)\left(-\frac{x^{-\frac{-a+\sqrt{a^2-4b}}{2a}}\left(-a+\sqrt{a^2-4b}\right)(x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}}{2ax} + \frac{x^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}\left(a+\sqrt{a^2-4b}\right)}{2a(x-a)}\right)$$

Which simplifies to

$$W = (x-a)^{-\frac{a+\sqrt{a^2-4b}}{a}}x^{\frac{-a+\sqrt{a^2-4b}}{a}}x^{-\frac{-a+\sqrt{a^2-4b}}{a}}(x-a)^{\frac{a+\sqrt{a^2-4b}}{a}}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}\left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}}(x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}dx\right)c\,x^2(a-x)^2}{x^2(a-x)^2}dx$$

Which simplifies to

$$u_1 = -\int x^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}\left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}}(x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}dx\right)c\,dx$$

Hence

$$u_1 = -\left(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (\alpha - a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int \alpha^{\frac{-a+\sqrt{a^2-4b}}{a}} (\alpha - a)^{\frac{-a-\sqrt{a^2-4b}}{a}} d\alpha\right) c d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} c\, x^2 (a-x)^2}{x^2 (a-x)^2} dx$$

Which simplifies to

$$u_2 = \int x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} c dx$$

Hence

$$u_2 = \int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (\alpha - a)^{\frac{a+\sqrt{a^2-4b}}{2a}} c d\alpha$$

Which simplifies to

$$u_1 = -\left(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (\alpha - a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int \alpha^{\frac{-a+\sqrt{a^2-4b}}{a}} (\alpha - a)^{\frac{-a-\sqrt{a^2-4b}}{a}} d\alpha\right) d\alpha\right) c$$

$$u_2 = c\left(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (\alpha - a)^{\frac{a+\sqrt{a^2-4b}}{2a}} d\alpha\right)$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = -\Bigg(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (\alpha \\ - a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int \alpha^{\frac{-a+\sqrt{a^2-4b}}{a}} (\alpha - a)^{\frac{-a-\sqrt{a^2-4b}}{a}} d\alpha\right) d\alpha\Bigg) c\, x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x \\ - a)^{\frac{a+\sqrt{a^2-4b}}{2a}} + c\left(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (\alpha - a)^{\frac{a+\sqrt{a^2-4b}}{2a}} d\alpha\right) x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x \\ - a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x - a)^{\frac{-a-\sqrt{a^2-4b}}{a}} dx\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x - a)^{\frac{a+\sqrt{a^2-4b}}{2a}} c\left(\left(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (\alpha-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} d\alpha\right)\left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}} dx\right) - \left(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (\alpha - a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int \alpha^{\frac{-a+\sqrt{a^2-4b}}{a}} (\alpha - a)^{\frac{-a-\sqrt{a^2-4b}}{a}} d\alpha\right) d\alpha\right)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} + c_2 x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}} dx\right)\right) \\ &+ \left(x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x - a)^{\frac{a+\sqrt{a^2-4b}}{2a}} c\left(\left(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (\alpha-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} d\alpha\right)\left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}} dx\right) - \left(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (\alpha - a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int \alpha^{\frac{-a+\sqrt{a^2-4b}}{a}} (\alpha - a)^{\frac{-a-\sqrt{a^2-4b}}{a}} d\alpha\right) d\alpha\right)\right)\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} y &= x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(c_1 + c_2\left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}} dx\right)\right) \\ &+ x^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (x - a)^{\frac{a+\sqrt{a^2-4b}}{2a}} c\left(\left(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (\alpha-a)^{\frac{a+\sqrt{a^2-4b}}{2a}} d\alpha\right)\left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}} (x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}} dx\right) - \left(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}} (\alpha - a)^{\frac{a+\sqrt{a^2-4b}}{2a}} \left(\int \alpha^{\frac{-a+\sqrt{a^2-4b}}{a}} (\alpha - a)^{\frac{-a-\sqrt{a^2-4b}}{a}} d\alpha\right) d\alpha\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}\left(c_1+c_2\left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}}(x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}dx\right)\right) + x^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}c\left(\left(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(\alpha-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}d\alpha\right)\left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}}(x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}dx\right) - \left(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(\alpha-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}\left(\int \alpha^{\frac{-a+\sqrt{a^2-4b}}{a}}(\alpha-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}d\alpha\right)d\alpha\right)\right) \tag{1}$$

Verification of solutions

$$y = x^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}\left(c_1+c_2\left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}}(x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}dx\right)\right) + x^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(x-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}c\left(\left(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(\alpha-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}d\alpha\right)\left(\int x^{\frac{-a+\sqrt{a^2-4b}}{a}}(x-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}dx\right) - \left(\int_0^x \alpha^{-\frac{-a+\sqrt{a^2-4b}}{2a}}(\alpha-a)^{\frac{a+\sqrt{a^2-4b}}{2a}}\left(\int \alpha^{\frac{-a+\sqrt{a^2-4b}}{a}}(\alpha-a)^{\frac{-a-\sqrt{a^2-4b}}{a}}d\alpha\right)d\alpha\right)\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 219

```
dsolve(x^2*(x-a)^2*diff(y(x),x$2)+b*y(x)=c*x^2*(x-a)^2,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{x(a-x)}\left(\left(\frac{x}{a-x}\right)^{\frac{\sqrt{a^2-4b}}{2a}} c_1\sqrt{a^2-4b} + \left(\frac{a-x}{x}\right)^{\frac{\sqrt{a^2-4b}}{2a}} c_2\sqrt{a^2-4b} + \left(\frac{x}{a-x}\right)^{\frac{\sqrt{a^2-4b}}{2a}}\left(\int \sqrt{x(a-x)}\left(\frac{x}{a-x}\right)^{-\sqrt{}}\right.\right.}{\sqrt{a^2-4b}}$$

✓ Solution by Mathematica

Time used: 0.958 (sec). Leaf size: 371

```
DSolve[x^2*(x-a)^2*y''[x]+b*y[x]==c*x^2*(x-a)^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{acx^2(a-x)\left(1-\frac{x}{a}\right)^{-\frac{1}{2}\sqrt{1-\frac{4b}{a^2}}-\frac{1}{2}}\left(\left(\sqrt{1-\frac{4b}{a^2}}-3\right)\left(1-\frac{x}{a}\right)^{\sqrt{1-\frac{4b}{a^2}}} \text{Hypergeometric2F1}\left(\frac{1}{2}\sqrt{1-\frac{4b}{a^2}}-\frac{1}{2}, \frac{1}{2}\right.\right.}{}$$

$$+ c_1 x^{\frac{1}{2}\sqrt{1-\frac{4b}{a^2}}+\frac{1}{2}}(x-a)^{\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{4b}{a^2}}} + \frac{c_2 x^{\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{4b}{a^2}}}(x-a)^{\frac{1}{2}\sqrt{1-\frac{4b}{a^2}}+\frac{1}{2}}}{a\sqrt{1-\frac{4b}{a^2}}}$$

32.8 problem 218

Internal problem ID [11042]
Internal file name [`OUTPUT/10298_Wednesday_January_24_2024_10_06_33_PM_44604537/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 218.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{a\,x^2(x-1)^2\,y'' + \left(b\,x^2 + cx + d\right)y = 0}$$

32.8.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(\frac{b}{a} + \frac{c}{xa} + \frac{d}{x^2a}\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1\,\mathrm{BesselJ}\,(n, \beta\,x^{\gamma}) + c_2\,\mathrm{BesselY}\,(n, \beta\,x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= 2 \\
n &= \frac{\sqrt{a\,(a-4b)}}{a} \\
\gamma &= \frac{1}{2}
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\frac{\sqrt{a\,(a-4b)}}{a}, 2\sqrt{x}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\frac{\sqrt{a\,(a-4b)}}{a}, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\frac{\sqrt{a\,(a-4b)}}{a}, 2\sqrt{x}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\frac{\sqrt{a\,(a-4b)}}{a}, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\frac{\sqrt{a\,(a-4b)}}{a}, 2\sqrt{x}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\frac{\sqrt{a\,(a-4b)}}{a}, 2\sqrt{x}\right)$$

Verified OK.

32.8.2 Maple step by step solution

Let's solve

$a\,x^2(x-1)^2\,y'' + (b\,x^2+cx+d)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(b\,x^2+cx+d)\,y}{a\,x^2(x-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(b\,x^2+cx+d)\,y}{a\,x^2(x-1)^2} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x)=0, P_3(x)=\frac{b\,x^2+cx+d}{a\,x^2(x-1)^2}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=0$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=\frac{d}{a}$

- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $a\,x^2(x-1)^2\,y''+(b\,x^2+cx+d)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m\cdot y''$ to series expansion for $m=2..4$

  $x^m\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(a\,r^2-ar+d)\,x^r+((a\,r^2+ar+d)\,a_1-a_0(2a\,r^2-2ar-c))\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(a\,k^2+2akr+\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $a\,r^2-ar+d=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{\frac{a+\sqrt{a^2-4ad}}{2a},-\frac{-a+\sqrt{a^2-4ad}}{2a}\right\}$

- Each term must be 0

  $(a\,r^2+ar+d)\,a_1-a_0(2a\,r^2-2ar-c)=0$

- Solve for the dependent coefficient(s)

  $a_1=\frac{a_0(2a\,r^2-2ar-c)}{a\,r^2+ar+d}$

- Each term in the series must be 0, giving the recursion relation

  $((a_k+a_{k-2}-2a_{k-1})\,k^2+((2a_k+2a_{k-2}-4a_{k-1})\,r-a_k-5a_{k-2}+6a_{k-1})\,k+(a_k+a_{k-2}-2a_{k-}$

- Shift index using $k->k+2$

  $\left((a_{k+2}+a_k-2a_{k+1})\,(k+2)^2+((2a_{k+2}+2a_k-4a_{k+1})\,r-a_{k+2}-5a_k+6a_{k+1})\,(k+2)+(a_{k+2}\right.$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a\,k^2a_k-2a\,k^2a_{k+1}+2akra_k-4akra_{k+1}+a\,r^2a_k-2a\,r^2a_{k+1}-aka_k-2aka_{k+1}-ara_k-2ara_{k+1}+a_kb+ca_{k+1}}{a\,k^2+2akr+a\,r^2+3ak+3ar+2a+d}$

- Recursion relation for $r=\frac{a+\sqrt{a^2-4ad}}{2a}$

  $a_{k+2}=-\frac{a\,k^2a_k-2a\,k^2a_{k+1}+k\left(a+\sqrt{a^2-4ad}\right)a_k-2k\left(a+\sqrt{a^2-4ad}\right)a_{k+1}+\frac{\left(a+\sqrt{a^2-4ad}\right)^2a_k}{4a}-\frac{\left(a+\sqrt{a^2-4ad}\right)^2a_{k+1}}{2a}-aka_k-2ak}{a\,k^2+k\left(a+\sqrt{a^2-4ad}\right)+\frac{\left(a+\sqrt{a^2-4ad}\right)^2}{4a}+3ak+\frac{7a}{2}+\frac{3\sqrt{a^2-4ad}}{2}}$

- Solution for $r=\frac{a+\sqrt{a^2-4ad}}{2a}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{a+\sqrt{a^2-4ad}}{2a}},a_{k+2}=-\frac{a\,k^2a_k-2a\,k^2a_{k+1}+k\left(a+\sqrt{a^2-4ad}\right)a_k-2k\left(a+\sqrt{a^2-4ad}\right)a_{k+1}+\frac{\left(a+\sqrt{a^2-4ad}\right)^2a_k}{4a}}{a\,k^2+k\left(a+\sqrt{a^2-4ad}\right)+\frac{(a-}{}}\right.$

- Recursion relation for $r=-\frac{-a+\sqrt{a^2-4ad}}{2a}$

  $a_{k+2}=-\frac{a\,k^2a_k-2a\,k^2a_{k+1}-k\left(-a+\sqrt{a^2-4ad}\right)a_k+2k\left(-a+\sqrt{a^2-4ad}\right)a_{k+1}+\frac{\left(-a+\sqrt{a^2-4ad}\right)^2a_k}{4a}-\frac{\left(-a+\sqrt{a^2-4ad}\right)^2a_{k+1}}{2a}-aka}{a\,k^2-k\left(-a+\sqrt{a^2-4ad}\right)+\frac{\left(-a+\sqrt{a^2-4ad}\right)^2}{4a}+3ak+\frac{7a}{2}-\frac{3\sqrt{c}}{}}$

- Solution for $r=-\frac{-a+\sqrt{a^2-4ad}}{2a}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k-\frac{-a+\sqrt{a^2-4ad}}{2a}},a_{k+2}=-\frac{a\,k^2a_k-2a\,k^2a_{k+1}-k\left(-a+\sqrt{a^2-4ad}\right)a_k+2k\left(-a+\sqrt{a^2-4ad}\right)a_{k+1}+\frac{\left(-a+\sqrt{a^2-4}\right.}{4a}}{a\,k^2-k\left(-a+\sqrt{a^2-4ad}\right.}\right.$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} e_k x^{k+\frac{a+\sqrt{a^2-4ad}}{2a}}\right) + \left(\sum_{k=0}^{\infty} f_k x^{k-\frac{-a+\sqrt{a^2-4ad}}{2a}}\right), e_{k+2} = -\frac{a\,k^2 e_k - 2a\,k^2 e_{1+k} + k\left(a+\sqrt{a^2-4ad}\right) e_k - 2k}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 267

```
dsolve(a*x^2*(x-1)^2*diff(y(x),x$2)+(b*x^2+c*x+d)*y(x)=0,y(x), singsol=all)
```

$$y(x) = (-1 + x)^{-\frac{\sqrt{a-4b-4c-4d}-\sqrt{a}}{2\sqrt{a}}} \left(c_1 x^{\frac{\sqrt{a}+\sqrt{a-4d}}{2\sqrt{a}}} \operatorname{hypergeom}\left(\left[\frac{-\sqrt{a-4b-4c-4d}+\sqrt{a}+\sqrt{a-4d}+\sqrt{a-4b}}{2\sqrt{a}}, -\frac{\sqrt{}}{} \right.\right.\right.$$
$$+ c_2 x^{-\frac{-\sqrt{a}+\sqrt{a-4d}}{2\sqrt{a}}} \operatorname{hypergeom}\left(\left[\frac{-\sqrt{a-4b-4c-4d}+\sqrt{a}-\sqrt{a-4d}+\sqrt{a-4b}}{2\sqrt{a}}, -\frac{\sqrt{a-4b-4c-4d}}{}\right.\right.$$

✓ Solution by Mathematica

Time used: 135.53 (sec). Leaf size: 413606

```
DSolve[a*x^2*(x-1)^2*y''[x]+(b*x^2+c*x+d)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

32.9 problem 219

Internal problem ID [11043]
Internal file name [`OUTPUT/10299_Wednesday_January_24_2024_10_06_52_PM_19035987/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 219.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$x^2(x^2 + a)\, y'' + (b\,x^2 + c)\, xy' + yd = 0$$

32.9.1 Maple step by step solution

Let's solve

$x^2(x^2 + a)\, y'' + (b\,x^2 + c)\, xy' + yd = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{dy}{x^2(x^2+a)} - \frac{(b\,x^2+c)y'}{x(x^2+a)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(b\,x^2+c)y'}{x(x^2+a)} + \frac{dy}{x^2(x^2+a)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{b\,x^2+c}{x(x^2+a)}, P_3(x) = \frac{d}{x^2(x^2+a)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{c}{a}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{d}{a}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+a)\,y'' + (b\,x^2+c)\,xy' + yd = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(a\,r^2-ar+cr+d)\,x^r + a_1(a\,r^2+ar+cr+c+d)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(a\,k^2+2akr+a\,r^2-ak-\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$a\,r^2 - ar + cr + d = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{-a+c+\sqrt{a^2-2ac-4ad+c^2}}{2a}, \frac{a-c+\sqrt{a^2-2ac-4ad+c^2}}{2a}\right\}$$

- Each term must be 0

$$a_1(a\,r^2 + ar + cr + c + d) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k-2}(k-2+r)\,(k-3+r+b) + a_k(a\,k^2 + (2ar - a + c)\,k + a\,r^2 + (c-a)\,r + d) = 0$$

- Shift index using $k- >k+2$

$$a_k(k+r)\,(k+r-1+b) + a_{k+2}\left(a(k+2)^2 + (2ar-a+c)\,(k+2) + a\,r^2 + (c-a)\,r + d\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r)(k+r-1+b)}{a\,k^2+2akr+a\,r^2+3ak+3ar+ck+cr+2a+2c+d}$$

- Recursion relation for $r = -\frac{-a+c+\sqrt{a^2-2ac-4ad+c^2}}{2a}$

$$a_{k+2} = -\frac{a_k\left(k-\frac{-a+c+\sqrt{a^2-2ac-4ad+c^2}}{2a}\right)\left(k-\frac{-a+c+\sqrt{a^2-2ac-4ad+c^2}}{2a}-1+b\right)}{a\,k^2-k\left(-a+c+\sqrt{a^2-2ac-4ad+c^2}\right)+\frac{\left(-a+c+\sqrt{a^2-2ac-4ad+c^2}\right)^2}{4a}+3ak+\frac{7a}{2}+\frac{c}{2}-\frac{3\sqrt{a^2-2ac-4ad+c^2}}{2}+ck-\frac{c\left(-a+c+\sqrt{a^2\right.}}{}}$$

- Solution for $r = -\frac{-a+c+\sqrt{a^2-2ac-4ad+c^2}}{2a}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{-a+c+\sqrt{a^2-2ac-4ad+c^2}}{2a}}, a_{k+2} = -\frac{a_k\left(k-\frac{-a+c+\sqrt{a^2-2ac-4ad+c^2}}{2a}\right)\left(k-\right.}{a\,k^2-k\left(-a+c+\sqrt{a^2-2ac-4ad+c^2}\right)+\frac{\left(-a+c+\sqrt{a^2-2ac-4ad+c^2}\right)^2}{4a}+3ak}\right.$$

- Recursion relation for $r = \frac{a-c+\sqrt{a^2-2ac-4ad+c^2}}{2a}$

$$a_{k+2} = -\frac{a_k\left(k+\frac{a-c+\sqrt{a^2-2ac-4ad+c^2}}{2a}\right)\left(k+\frac{a-c+\sqrt{a^2-2ac-4ad+c^2}}{2a}-1+b\right)}{a\,k^2+k\left(a-c+\sqrt{a^2-2ac-4ad+c^2}\right)+\frac{\left(a-c+\sqrt{a^2-2ac-4ad+c^2}\right)^2}{4a}+3ak+\frac{7a}{2}+\frac{c}{2}+\frac{3\sqrt{a^2-2ac-4ad+c^2}}{2}+ck+\frac{c\left(a-c+\sqrt{a^2-2ac\right.}}{2a}}$$

- Solution for $r = \frac{a-c+\sqrt{a^2-2ac-4ad+c^2}}{2a}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{a-c+\sqrt{a^2-2ac-4ad+c^2}}{2a}}, a_{k+2} = -\frac{a_k\left(k+\frac{a-c+\sqrt{a^2-2ac-4ad+c^2}}{2a}\right)\left(k+\frac{a-c}{}\right.}{a\,k^2+k\left(a-c+\sqrt{a^2-2ac-4ad+c^2}\right)+\frac{\left(a-c+\sqrt{a^2-2ac-4ad+c^2}\right)^2}{4a}+3ak+\frac{7a}{2}}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} e_k x^{k-\frac{-a+c+\sqrt{a^2-2ac-4ad+c^2}}{2a}}\right) + \left(\sum_{k=0}^{\infty} f_k x^{k+\frac{a-c+\sqrt{a^2-2ac-4ad+c^2}}{2a}}\right), e_{k+2} = -\frac{}{a k^2 - k\left(-a+c+\sqrt{a^2-\right.}}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 286

```
dsolve(x^2*(x^2+a)*diff(y(x),x$2)+(b*x^2+c)*x*diff(y(x),x)+d*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(x^2 + a\right)^{\frac{(-b+2)a+c}{2a}} \left(c_2 x^{-\frac{-a+c+\sqrt{a^2+(-2c-4d)a+c^2}}{2a}} \operatorname{hypergeom}\left(\left[-\frac{-3a-c+\sqrt{a^2+(-2c-4d)a+c^2}}{4a}, \frac{-\sqrt{a^2+ \cdots}}{\cdots}\right], \ldots, -\frac{x^2}{a}\right) + c_1 x^{\frac{a-c+\sqrt{a^2+(-2c-4d)a+c^2}}{2a}} \operatorname{hypergeom}\left(\left[\frac{3a+c+\sqrt{a^2+(-2c-4d)a+c^2}}{4a}, \frac{\sqrt{a^2+(-2c-4d)a+c^2}+ \cdots}{4a}\right], \ldots, -\frac{x^2}{a}\right)\right)$$

✓ Solution by Mathematica

Time used: 2.385 (sec). Leaf size: 336

```
DSolve[x^2*(x^2+a)*y''[x]+(b*x^2+c)*x*y'[x]+d*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to a^{-\frac{\sqrt{a^2-2a(c+2d)+c^2}+a-c}{4a}} x^{-\frac{\sqrt{a^2-2a(c+2d)+c^2}-a+c}{2a}} \left(c_2 x^{\frac{\sqrt{a^2-2a(c+2d)+c^2}}{a}} \text{Hypergeometric2F1}\left(-\frac{-2ba+a+c-\sqrt{a \cdots}}{4 \cdots}, \ldots + 1, -\frac{x^2}{a}\right) + c_1 a^{\frac{\sqrt{a^2-2a(c+2d)+c^2}}{2a}} \text{Hypergeometric2F1}\left(-\frac{-a+c+\sqrt{a^2-2(c+2d)a+c^2}}{4a}, -\frac{-2ba+a+c+\sqrt{a^2-2(c+2d)a+c^2}}{4a}, 1-\frac{\sqrt{a^2-2(c+2d)a+c^2}}{2a}, -\frac{x^2}{a}\right)\right)$$

32.10 problem 220

Internal problem ID [11044]
Internal file name [OUTPUT/10300_Wednesday_January_24_2024_10_06_53_PM_49965746/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

`[_Halm]`

$$\boxed{\left(x^2+1\right)^2 y''+ay=0}$$

32.10.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y''+\frac{ay}{x^2}=0 \tag{1}$$

Bessel ode has the form

$$x^2y''+y'x+\left(-n^2+x^2\right)y=0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y''+(1-2\alpha)\,xy'+\left(\beta^2\gamma^2x^{2\gamma}-n^2\gamma^2+\alpha^2\right)y=0 \tag{3}$$

With the standard solution

$$y=x^{\alpha}(c_1\,\text{BesselJ}\left(n,\beta\,x^{\gamma}\right)+c_2\,\text{BesselY}\left(n,\beta\,x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = \sqrt{a}$$
$$n = \frac{1}{2}$$
$$\gamma = -1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{a}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{a}}{x}}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{a}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{a}}{x}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{a}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{a}}{x}}}$$

Verified OK.

32.10.2 Solving using Kovacic algorithm

Writing the ode as

$$\left(x^2+1\right)^2 y'' + ay = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= \left(x^2+1\right)^2 \\ B &= 0 \\ C &= a \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -a \\ t &= (x^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{a}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 203: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{a}{4(x-i)^2} + \frac{a}{4(x+i)^2} + \frac{ia}{4x-4i} - \frac{ia}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{a}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{a+1}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{a+1}}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{a}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{a+1}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{a+1}}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{a}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{1}{2}+\frac{\sqrt{a+1}}{2}$	$\frac{1}{2}-\frac{\sqrt{a+1}}{2}$
$-i$	2	0	$\frac{1}{2}+\frac{\sqrt{a+1}}{2}$	$\frac{1}{2}-\frac{\sqrt{a+1}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2}-\frac{\sqrt{a+1}}{2}}{x-i} + \frac{\frac{1}{2}+\frac{\sqrt{a+1}}{2}}{x+i} + (-)\,(0) \\
&= \frac{\frac{1}{2}-\frac{\sqrt{a+1}}{2}}{x-i} + \frac{\frac{1}{2}+\frac{\sqrt{a+1}}{2}}{x+i} \\
&= \frac{-i\sqrt{a+1}+x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{a+1}}{2}}{x-i} + \frac{\frac{1}{2} + \frac{\sqrt{a+1}}{2}}{x+i}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{a+1}}{2}}{(x-i)^2} - \frac{\frac{1}{2} + \frac{\sqrt{a+1}}{2}}{(x+i)^2}\right) + \left(\frac{\frac{1}{2} - \frac{\sqrt{a+1}}{2}}{x-i} + \frac{\frac{1}{2} + \frac{\sqrt{a+1}}{2}}{x+i}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{\frac{1}{2} - \frac{\sqrt{a+1}}{2}}{x-i} + \frac{\frac{1}{2} + \frac{\sqrt{a+1}}{2}}{x+i}\right) dx} \\ &= \sqrt{x^2+1}\, \mathrm{e}^{-i\sqrt{a+1}\, \arctan(x)} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \sqrt{x^2+1}\, \mathrm{e}^{-i\sqrt{a+1}\, \arctan(x)} \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x^2+1}\, \mathrm{e}^{-i\sqrt{a+1}\, \arctan(x)}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \sqrt{x^2+1}\, \mathrm{e}^{-i\sqrt{a+1}\, \arctan(x)} \int \frac{1}{(x^2+1)\, \mathrm{e}^{-2i\sqrt{a+1}\, \arctan(x)}}\, dx \\
&= \sqrt{x^2+1}\, \mathrm{e}^{-i\sqrt{a+1}\, \arctan(x)} \left(-\frac{i\mathrm{e}^{2i\sqrt{a+1}\, \arctan(x)}}{2\sqrt{a+1}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x^2+1}\, \mathrm{e}^{-i\sqrt{a+1}\, \arctan(x)}\right) + c_2\left(\sqrt{x^2+1}\, \mathrm{e}^{-i\sqrt{a+1}\, \arctan(x)} \left(-\frac{i\mathrm{e}^{2i\sqrt{a+1}\, \arctan(x)}}{2\sqrt{a+1}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x^2+1}\, \mathrm{e}^{-i\sqrt{a+1}\, \arctan(x)} - \frac{i c_2\sqrt{x^2+1}\, \mathrm{e}^{i\sqrt{a+1}\, \arctan(x)}}{2\sqrt{a+1}} \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x^2+1}\, \mathrm{e}^{-i\sqrt{a+1}\, \arctan(x)} - \frac{i c_2\sqrt{x^2+1}\, \mathrm{e}^{i\sqrt{a+1}\, \arctan(x)}}{2\sqrt{a+1}}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve((x^2+1)^2*diff(y(x),x$2)+a*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\left(\frac{x+i}{-x+i}\right)^{-\frac{\sqrt{a+1}}{2}} c_2 + \left(\frac{x+i}{-x+i}\right)^{\frac{\sqrt{a+1}}{2}} c_1\right)\sqrt{x^2+1}$$

✓ Solution by Mathematica

Time used: 0.215 (sec). Leaf size: 83

```
DSolve[(x^2+1)^2*y''[x]+a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\sqrt{x^2+1}e^{i\sqrt{a+1}\arctan(x)}\left(\frac{ic_2(1-ix)^{\sqrt{a+1}}(1+ix)^{-\sqrt{a+1}}}{\sqrt{a+1}} + 2c_1\right)$$

32.11 problem 221

Internal problem ID [11045]
Internal file name [`OUTPUT/10301_Wednesday_January_24_2024_10_06_53_PM_54231854/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 221.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{\left(x^2-1\right)^2 y''+ay=0}$$

32.11.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y''+\frac{ay}{x^2}=0 \tag{1}$$

Bessel ode has the form

$$x^2y''+y'x+\left(-n^2+x^2\right)y=0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y''+(1-2\alpha)\,xy'+\left(\beta^2\gamma^2x^{2\gamma}-n^2\gamma^2+\alpha^2\right)y=0 \tag{3}$$

With the standard solution

$$y=x^{\alpha}(c_1\,\mathrm{BesselJ}\left(n,\beta\,x^{\gamma}\right)+c_2\,\mathrm{BesselY}\left(n,\beta\,x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= \frac{1}{2}\\ \beta &= \sqrt{a}\\ n &= \frac{1}{2}\\ \gamma &= -1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{x}\,\sqrt{2}\,\sin\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{a}}{x}}} - \frac{c_2\sqrt{x}\,\sqrt{2}\,\cos\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{a}}{x}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{x}\,\sqrt{2}\,\sin\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{a}}{x}}} - \frac{c_2\sqrt{x}\,\sqrt{2}\,\cos\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{a}}{x}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{x}\,\sqrt{2}\,\sin\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{a}}{x}}} - \frac{c_2\sqrt{x}\,\sqrt{2}\,\cos\left(\frac{\sqrt{a}}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{a}}{x}}}$$

Verified OK.

32.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y''\left(x^4-2x^2+1\right)+ay = 0 \tag{1}$$

$$Ay''+By'+Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= x^4-2x^2+1\\ B &= 0\\ C &= a\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -a \\ t &= (x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{a}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 204: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{a}{4x - 4} - \frac{a}{4(x-1)^2} - \frac{a}{4(1+x)^2} - \frac{a}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{a}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{-a+1}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{-a+1}}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{a}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{-a+1}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{-a+1}}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{a}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}+\frac{\sqrt{-a+1}}{2}$	$\frac{1}{2}-\frac{\sqrt{-a+1}}{2}$
-1	2	0	$\frac{1}{2}+\frac{\sqrt{-a+1}}{2}$	$\frac{1}{2}-\frac{\sqrt{-a+1}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2}-\frac{\sqrt{-a+1}}{2}}{x-1} + \frac{\frac{1}{2}+\frac{\sqrt{-a+1}}{2}}{1+x} + (-)(0) \\
&= \frac{\frac{1}{2}-\frac{\sqrt{-a+1}}{2}}{x-1} + \frac{\frac{1}{2}+\frac{\sqrt{-a+1}}{2}}{1+x} \\
&= \frac{-\sqrt{-a+1}+x}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{-a+1}}{2}}{x-1} + \frac{\frac{1}{2} + \frac{\sqrt{-a+1}}{2}}{1+x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{-a+1}}{2}}{\left(x-1\right)^2} - \frac{\frac{1}{2} + \frac{\sqrt{-a+1}}{2}}{\left(1+x\right)^2}\right) + \left(\frac{\frac{1}{2} - \frac{\sqrt{-a+1}}{2}}{x-1} + \frac{\frac{1}{2} + \frac{\sqrt{-a+1}}{2}}{1+x}\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \,dx} \\ &= \mathrm{e}^{\int \left(\frac{\frac{1}{2} - \frac{\sqrt{-a+1}}{2}}{x-1} + \frac{\frac{1}{2} + \frac{\sqrt{-a+1}}{2}}{1+x}\right) dx} \\ &= (1+x)^{\frac{1}{2} + \frac{\sqrt{-a+1}}{2}} (x-1)^{\frac{1}{2} - \frac{\sqrt{-a+1}}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= (1+x)^{\frac{1}{2} + \frac{\sqrt{-a+1}}{2}} (x-1)^{\frac{1}{2} - \frac{\sqrt{-a+1}}{2}} \end{aligned}$$

Which simplifies to

$$y_1 = (1+x)^{\frac{1}{2} + \frac{\sqrt{-a+1}}{2}} (x-1)^{\frac{1}{2} - \frac{\sqrt{-a+1}}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \,dx}}{y_1^2} \,dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= (1+x)^{\frac{1}{2}+\frac{\sqrt{-a+1}}{2}} (x-1)^{\frac{1}{2}-\frac{\sqrt{-a+1}}{2}} \int \frac{1}{(1+x)^{1+\sqrt{-a+1}} (x-1)^{1-\sqrt{-a+1}}}\, dx \\
&= (1+x)^{\frac{1}{2}+\frac{\sqrt{-a+1}}{2}} (x-1)^{\frac{1}{2}-\frac{\sqrt{-a+1}}{2}} \left(\int (1+x)^{-1-\sqrt{-a+1}} (x-1)^{-1+\sqrt{-a+1}}\, dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left((1+x)^{\frac{1}{2}+\frac{\sqrt{-a+1}}{2}} (x-1)^{\frac{1}{2}-\frac{\sqrt{-a+1}}{2}}\right) \\
&\quad + c_2 \left((1+x)^{\frac{1}{2}+\frac{\sqrt{-a+1}}{2}} (x-1)^{\frac{1}{2}-\frac{\sqrt{-a+1}}{2}} \left(\int (1+x)^{-1-\sqrt{-a+1}} (x-1)^{-1+\sqrt{-a+1}}\, dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 (1+x)^{\frac{1}{2}+\frac{\sqrt{-a+1}}{2}} (x-1)^{\frac{1}{2}-\frac{\sqrt{-a+1}}{2}} \\
&\quad + c_2 (1+x)^{\frac{1}{2}+\frac{\sqrt{-a+1}}{2}} (x-1)^{\frac{1}{2}-\frac{\sqrt{-a+1}}{2}} \left(\int (1+x)^{-1-\sqrt{-a+1}} (x-1)^{-1+\sqrt{-a+1}}\, dx\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 (1+x)^{\frac{1}{2}+\frac{\sqrt{-a+1}}{2}} (x-1)^{\frac{1}{2}-\frac{\sqrt{-a+1}}{2}} \\
&\quad + c_2 (1+x)^{\frac{1}{2}+\frac{\sqrt{-a+1}}{2}} (x-1)^{\frac{1}{2}-\frac{\sqrt{-a+1}}{2}} \left(\int (1+x)^{-1-\sqrt{-a+1}} (x-1)^{-1+\sqrt{-a+1}}\, dx\right)
\end{aligned}
$$

Verified OK.

32.11.3 Maple step by step solution

Let's solve

$y''(x^4-2x^2+1)+ay=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{ay}{x^4-2x^2+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{ay}{x^4-2x^2+1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=0, P_3(x)=\frac{a}{x^4-2x^2+1}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left.((1+x)\cdot P_2(x))\right|_{x=-1}=0$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left.((1+x)^2\cdot P_3(x))\right|_{x=-1}=\frac{a}{4}$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $y''(x^4-2x^2+1)+ay=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^4-4u^3+4u^2)\left(\frac{d^2}{du^2}y(u)\right)+ay(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^{m} \cdot \left(\frac{d^{2}}{d u^{2}}y(u)\right)$ to series expansion for $m = 2..4$

  $$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}}y(u)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\left(k+r-1\right)u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\left(k+1-m+r\right)u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}(4r^{2}+a-4r)\,u^{r}+\left(\left(4r^{2}+a+4r\right)a_{1}-4a_{0}r(-1+r)\right)u^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}(4k^{2}+8kr+4r^{2}+a-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4r^{2}+a-4r = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{2}-\frac{\sqrt{-a+1}}{2}, \frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right\}$$

- Each term must be 0

  $$\left(4r^{2}+a+4r\right)a_{1}-4a_{0}r(-1+r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_{1} = \frac{4a_{0}r(-1+r)}{4r^{2}+a+4r}$$

- Each term in the series must be 0, giving the recursion relation

  $$\left(4a_{k}+a_{k-2}-4a_{k-1}\right)k^{2}+\left(\left(8a_{k}+2a_{k-2}-8a_{k-1}\right)r-4a_{k}-5a_{k-2}+12a_{k-1}\right)k+\left(4a_{k}+a_{k-2}-\right.$$

- Shift index using $k->k+2$

  $$\left(4a_{k+2}+a_{k}-4a_{k+1}\right)\left(k+2\right)^{2}+\left(\left(8a_{k+2}+2a_{k}-8a_{k+1}\right)r-4a_{k+2}-5a_{k}+12a_{k+1}\right)\left(k+2\right)+\left(4a\right.$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{k^{2}a_{k}-4k^{2}a_{k+1}+2kra_{k}-8kra_{k+1}+r^{2}a_{k}-4r^{2}a_{k+1}-ka_{k}-4ka_{k+1}-ra_{k}-4ra_{k+1}}{4k^{2}+8kr+4r^{2}+a+12k+12r+8}$$

- Recursion relation for $r = \frac{1}{2}-\frac{\sqrt{-a+1}}{2}$

  $$a_{k+2} = -\frac{k^{2}a_{k}-4k^{2}a_{k+1}+2k\left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right)a_{k}-8k\left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right)a_{k+1}+\left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right)^{2}a_{k}-4\left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right)^{2}a_{k+1}-ka_{k}-4ka_{k+1}-\left(\right.}{4k^{2}+8k\left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right)+4\left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right)^{2}+a+12k+14-6\sqrt{-a+1}}$$

- Solution for $r = \frac{1}{2}-\frac{\sqrt{-a+1}}{2}$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_{k}u^{k+\frac{1}{2}-\frac{\sqrt{-a+1}}{2}}, a_{k+2} = -\frac{k^{2}a_{k}-4k^{2}a_{k+1}+2k\left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right)a_{k}-8k\left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right)a_{k+1}+\left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right)^{2}a_{k}-4}{4k^{2}+8k\left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right)+4\left(\frac{1}{2}-\frac{\sqrt{-a}}{2}\right.}\right.$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{1}{2}-\frac{\sqrt{-a+1}}{2}}, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 2k\left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right)a_k - 8k\left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right)a_{k+1} + \left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right)^2 a}{4k^2 + 8k\left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right) + 4\left(\frac{1}{2}-\right.}\right.$$

- Recursion relation for $r = \frac{1}{2} + \frac{\sqrt{-a+1}}{2}$

$$a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 2k\left(\frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right)a_k - 8k\left(\frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right)a_{k+1} + \left(\frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right)^2 a_k - 4\left(\frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right)^2 a_{k+1} - k a_k - 4k a_{k+1} - (}{4k^2 + 8k\left(\frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right) + 4\left(\frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right)^2 + a + 12k + 14 + 6\sqrt{-a+1}}$$

- Solution for $r = \frac{1}{2} + \frac{\sqrt{-a+1}}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}+\frac{\sqrt{-a+1}}{2}}, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 2k\left(\frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right)a_k - 8k\left(\frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right)a_{k+1} + \left(\frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right)^2 a_k - 4}{4k^2 + 8k\left(\frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right) + 4\left(\frac{1}{2}+\frac{\sqrt{-a}}{2}\right.}\right.$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{1}{2}+\frac{\sqrt{-a+1}}{2}}, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 2k\left(\frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right)a_k - 8k\left(\frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right)a_{k+1} + \left(\frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right)^2 a}{4k^2 + 8k\left(\frac{1}{2}+\frac{\sqrt{-a+1}}{2}\right) + 4\left(\frac{1}{2}+\right.}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+\frac{1}{2}-\frac{\sqrt{-a+1}}{2}}\right) + \left(\sum_{k=0}^{\infty} c_k (1+x)^{k+\frac{1}{2}+\frac{\sqrt{-a+1}}{2}}\right), b_{k+2} = -\frac{k^2 b_k - 4k^2 b_{1+k} + 2k\left(\frac{1}{2}-\frac{\sqrt{-a+1}}{2}\right)}{}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve((x^2-1)^2*diff(y(x),x$2)+a*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x^2-1}\left(\left(\frac{-1+x}{1+x}\right)^{\frac{\sqrt{-a+1}}{2}} c_1 + \left(\frac{-1+x}{1+x}\right)^{-\frac{\sqrt{-a+1}}{2}} c_2\right)$$

✓ Solution by Mathematica

Time used: 0.249 (sec). Leaf size: 88

```
DSolve[(x^2-1)^2*y''[x]+a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(1-x^2)^{\frac{1}{2}-\frac{\sqrt{1-a}}{2}}\left(2\sqrt{1-a}c_1(1-x)^{\sqrt{1-a}} + c_2(x+1)^{\sqrt{1-a}}\right)}{2\sqrt{1-a}}$$

32.12 problem 222 A

Internal problem ID [11046]
Internal file name [`OUTPUT/10302_Wednesday_January_24_2024_10_06_54_PM_92506048/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 222 A.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{(a^2 + x^2)^2 y'' + yb^2 = 0}$$

32.12.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \frac{b^2y}{x^2} = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + (-n^2 + x^2) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha) xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = b$$
$$n = \frac{1}{2}$$
$$\gamma = -1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{b}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{b}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{b}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{b}{x}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{b}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{b}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{b}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{b}{x}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{b}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{b}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{b}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{b}{x}}}$$

Verified OK.

32.12.2 Solving using Kovacic algorithm

Writing the ode as

$$\left(a^2 + x^2\right)^2 y'' + y b^2 = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= \left(a^2 + x^2\right)^2 \\ B &= 0 \\ C &= b^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-b^2}{(a^2 + x^2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -b^2 \\ t &= (a^2 + x^2)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{b^2}{(a^2 + x^2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 206: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (a^2 + x^2)^2$. There is a pole at $x = ia$ of order 2. There is a pole at $x = -ia$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{b^2}{4a^2 \left(x - \sqrt{-a^2}\right)^2} + \frac{b^2}{4a^2 \left(x + \sqrt{-a^2}\right)^2} + \frac{b^2}{4 \left(-a^2\right)^{\frac{3}{2}} \left(x - \sqrt{-a^2}\right)} - \frac{b^2}{4 \left(-a^2\right)^{\frac{3}{2}} \left(x + \sqrt{-a^2}\right)}$$

For the pole at $x = ia$ let b be the coefficient of $\frac{1}{(-ia+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 0$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{0, 2, 4\} \end{aligned}$$

For the pole at $x = -ia$ let b be the coefficient of $\frac{1}{(ia+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 0$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{0, 2, 4\} \end{aligned}$$

Now since the order of r at ∞ is $4 > 2$ then

$$E_\infty = \{0, 2, 4\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
ia	2	$\{0, 2, 4\}$
$-ia$	2	$\{0, 2, 4\}$

Order of r at ∞	E_∞
4	$\{0, 2, 4\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 2, \; e_2 = 2, \; e_\infty = 4$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{1}{2}(4 - (2 + (2))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2} \sum_{c \in \Gamma} \frac{e_c}{x - c} \\ &= \frac{1}{2}\left(\frac{2}{(x - (ia))} + \frac{2}{(x - (-ia))}\right) \\ &= \frac{1}{-ia + x} + \frac{1}{ia + x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{-ia+x} + \frac{1}{ia+x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{-ia+x} + \frac{1}{ia+x}\right) w + \frac{a^2+b^2+x^2}{\left(a^2+x^2\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{x + \sqrt{-a^2-b^2}}{a^2+x^2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= e^{\int \frac{x+\sqrt{-a^2-b^2}}{a^2+x^2} dx} \\ &= \sqrt{a^2+x^2}\, e^{\frac{\sqrt{-a^2-b^2}\, \arctan\left(\frac{x}{a}\right)}{a}}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \sqrt{a^2+x^2}\, e^{\frac{\sqrt{-a^2-b^2}\, \arctan\left(\frac{x}{a}\right)}{a}}\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{a^2+x^2}\, e^{\frac{\sqrt{-a^2-b^2}\, \arctan\left(\frac{x}{a}\right)}{a}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \sqrt{a^2+x^2}\,e^{\frac{\sqrt{-a^2-b^2}\,\arctan\left(\frac{x}{a}\right)}{a}} \int \frac{1}{(a^2+x^2)\,e^{\frac{2\sqrt{-a^2-b^2}\,\arctan\left(\frac{x}{a}\right)}{a}}}\,dx \\
&= \sqrt{a^2+x^2}\,e^{\frac{\sqrt{-a^2-b^2}\,\arctan\left(\frac{x}{a}\right)}{a}} \left(-\frac{e^{-\frac{2\sqrt{-a^2-b^2}\,\arctan\left(\frac{x}{a}\right)}{a}}}{2\sqrt{-a^2-b^2}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\sqrt{a^2+x^2}\,e^{\frac{\sqrt{-a^2-b^2}\,\arctan\left(\frac{x}{a}\right)}{a}}\right) \\
&\quad + c_2\left(\sqrt{a^2+x^2}\,e^{\frac{\sqrt{-a^2-b^2}\,\arctan\left(\frac{x}{a}\right)}{a}} \left(-\frac{e^{-\frac{2\sqrt{-a^2-b^2}\,\arctan\left(\frac{x}{a}\right)}{a}}}{2\sqrt{-a^2-b^2}}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{a^2+x^2}\,e^{\frac{\sqrt{-a^2-b^2}\,\arctan\left(\frac{x}{a}\right)}{a}} - \frac{c_2\sqrt{a^2+x^2}\,e^{-\frac{\sqrt{-a^2-b^2}\,\arctan\left(\frac{x}{a}\right)}{a}}}{2\sqrt{-a^2-b^2}} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{a^2+x^2}\,e^{\frac{\sqrt{-a^2-b^2}\,\arctan\left(\frac{x}{a}\right)}{a}} - \frac{c_2\sqrt{a^2+x^2}\,e^{-\frac{\sqrt{-a^2-b^2}\,\arctan\left(\frac{x}{a}\right)}{a}}}{2\sqrt{-a^2-b^2}}$$

Verified OK.

32.12.3 Maple step by step solution

Let's solve

$\left(a^2+x^2\right)^2 y''+y b^2=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y b^2}{\left(a^2+x^2\right)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y b^2}{\left(a^2+x^2\right)^2}=0$

- Multiply by denominators of the ODE

  $\left(a^2+x^2\right)^2 y''+y b^2=0$

- Make a change of variables

  $t=\ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y'=\left(\frac{d}{dt}y(t)\right)t'(x)$

  ◦ Compute derivative

    $y'=\frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y''=\left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2+t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y''=\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(a^2+x^2\right)^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)+y(t)\,b^2=0$

- Simplify

  $\frac{\left(a^2+x^2\right)^2\left(\frac{d^2}{dt^2}y(t)-\frac{d}{dt}y(t)\right)}{x^2}+y(t)\,b^2=0$

- Isolate 2nd derivative

$$\frac{d^2}{dt^2}y(t) = -\frac{b^2x^2y(t)}{(a^2+x^2)^2} + \frac{d}{dt}y(t)$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^2}{dt^2}y(t) + \frac{b^2x^2y(t)}{(a^2+x^2)^2} - \frac{d}{dt}y(t) = 0$$

- Characteristic polynomial of ODE

$$r^2 + \frac{b^2x^2}{(a^2+x^2)^2} - r = 0$$

- Factor the characteristic polynomial

$$\frac{a^4r^2+2a^2r^2x^2+r^2x^4-a^4r-2a^2r\,x^2-r\,x^4+b^2x^2}{(a^2+x^2)^2} = 0$$

- Roots of the characteristic polynomial

$$r = \left(\frac{\frac{a^2}{2}+\frac{x^2}{2}+\frac{\sqrt{a^4+2a^2x^2-4b^2x^2+x^4}}{2}}{a^2+x^2}, \frac{\frac{a^2}{2}+\frac{x^2}{2}-\frac{\sqrt{a^4+2a^2x^2-4b^2x^2+x^4}}{2}}{a^2+x^2}\right)$$

- 1st solution of the ODE

$$y_1(t) = e^{\frac{\left(\frac{a^2}{2}+\frac{x^2}{2}+\frac{\sqrt{a^4+2a^2x^2-4b^2x^2+x^4}}{2}\right)t}{a^2+x^2}}$$

- 2nd solution of the ODE

$$y_2(t) = e^{\frac{\left(\frac{a^2}{2}+\frac{x^2}{2}-\frac{\sqrt{a^4+2a^2x^2-4b^2x^2+x^4}}{2}\right)t}{a^2+x^2}}$$

- General solution of the ODE

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

- Substitute in solutions

$$y(t) = c_1 e^{\frac{\left(\frac{a^2}{2}+\frac{x^2}{2}+\frac{\sqrt{a^4+2a^2x^2-4b^2x^2+x^4}}{2}\right)t}{a^2+x^2}} + c_2 e^{\frac{\left(\frac{a^2}{2}+\frac{x^2}{2}-\frac{\sqrt{a^4+2a^2x^2-4b^2x^2+x^4}}{2}\right)t}{a^2+x^2}}$$

- Change variables back using $t = \ln(x)$

$$y = c_1 e^{\frac{\left(\frac{a^2}{2}+\frac{x^2}{2}+\frac{\sqrt{a^4+2a^2x^2-4b^2x^2+x^4}}{2}\right)\ln(x)}{a^2+x^2}} + c_2 e^{\frac{\left(\frac{a^2}{2}+\frac{x^2}{2}-\frac{\sqrt{a^4+2a^2x^2-4b^2x^2+x^4}}{2}\right)\ln(x)}{a^2+x^2}}$$

- Simplify

$$y = c_1 x^{\frac{a^2+x^2+\sqrt{a^4+2a^2x^2-4b^2x^2+x^4}}{2a^2+2x^2}} + x^{\frac{a^2+x^2-\sqrt{a^4+2a^2x^2-4b^2x^2+x^4}}{2a^2+2x^2}} c_2$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 83

```
dsolve((x^2+a^2)^2*diff(y(x),x$2)+b^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\left(\frac{ix-a}{ix+a}\right)^{\frac{\sqrt{a^2+b^2}}{2a}} c_1 + \left(\frac{ix-a}{ix+a}\right)^{-\frac{\sqrt{a^2+b^2}}{2a}} c_2\right)\sqrt{a^2+x^2}$$

✓ Solution by Mathematica

Time used: 0.489 (sec). Leaf size: 97

```
DSolve[(x^2+a^2)^2*y''[x]+b^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\sqrt{a^2+x^2}e^{-i\sqrt{\frac{b^2}{a^2}+1}\arctan\left(\frac{a}{x}\right)}\left(\frac{ic_2e^{2i\sqrt{\frac{b^2}{a^2}+1}\arctan\left(\frac{a}{x}\right)}}{a\sqrt{\frac{b^2}{a^2}+1}} + 2c_1\right)$$

32.13 problem 222 B

Internal problem ID [11047]
Internal file name [`OUTPUT/10303_Wednesday_January_24_2024_10_06_55_PM_99934314/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 222 B.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(-a^2 + x^2\right)^2 y'' + yb^2 = 0}$$

32.13.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \frac{b^2y}{x^2} = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= \frac{1}{2}\\ \beta &= b\\ n &= \frac{1}{2}\\ \gamma &= -1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{x}\,\sqrt{2}\,\sin\left(\frac{b}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{b}{x}}} - \frac{c_2\sqrt{x}\,\sqrt{2}\,\cos\left(\frac{b}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{b}{x}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{x}\,\sqrt{2}\,\sin\left(\frac{b}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{b}{x}}} - \frac{c_2\sqrt{x}\,\sqrt{2}\,\cos\left(\frac{b}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{b}{x}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{x}\,\sqrt{2}\,\sin\left(\frac{b}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{b}{x}}} - \frac{c_2\sqrt{x}\,\sqrt{2}\,\cos\left(\frac{b}{x}\right)}{\sqrt{\pi}\,\sqrt{\frac{b}{x}}}$$

Verified OK.

32.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y''(a-x)^2\,(x+a)^2 + yb^2 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= (a-x)^2\,(x+a)^2\\ B &= 0\\ C &= b^2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-b^2}{(a^2 - x^2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -b^2 \\ t &= (a^2 - x^2)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{b^2}{(a^2 - x^2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 208: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (a^2 - x^2)^2$. There is a pole at $x = a$ of order 2. There is a pole at $x = -a$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{b^2}{4a^2 (x+a)^2} - \frac{b^2}{4a^3 (x+a)} - \frac{b^2}{4a^2 (x-a)^2} + \frac{b^2}{4a^3 (x-a)}$$

For the pole at $x = a$ let b be the coefficient of $\frac{1}{(x-a)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{b^2}{4a^2}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{a^2 - b^2}}{2a} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{a^2 - b^2}}{2a} \end{aligned}$$

For the pole at $x = -a$ let b be the coefficient of $\frac{1}{(x+a)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{b^2}{4a^2}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{a^2 - b^2}}{2a} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{a^2 - b^2}}{2a} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{b^2}{(a^2 - x^2)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
a	2	0	$\frac{1}{2} + \frac{\sqrt{a^2-b^2}}{2a}$	$\frac{1}{2} - \frac{\sqrt{a^2-b^2}}{2a}$
$-a$	2	0	$\frac{1}{2} + \frac{\sqrt{a^2-b^2}}{2a}$	$\frac{1}{2} - \frac{\sqrt{a^2-b^2}}{2a}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{\sqrt{a^2-b^2}}{2a}}{x - a} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-b^2}}{2a}}{x + a} + (-)(0) \\
&= \frac{\frac{1}{2} - \frac{\sqrt{a^2-b^2}}{2a}}{x - a} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-b^2}}{2a}}{x + a} \\
&= \frac{\sqrt{a^2 - b^2} - x}{a^2 - x^2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{a^2-b^2}}{2a}}{x - a} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-b^2}}{2a}}{x + a}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{a^2-b^2}}{2a}}{(x - a)^2} - \frac{\frac{1}{2} + \frac{\sqrt{a^2-b^2}}{2a}}{(x + a)^2}\right) + \left(\frac{\frac{1}{2} - \frac{\sqrt{a^2-b^2}}{2a}}{x - a} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-b^2}}{2a}}{x + a}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{\frac{1}{2} - \frac{\sqrt{a^2-b^2}}{2a}}{x-a} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-b^2}}{2a}}{x+a}\right) dx} \\
&= (x - a)^{-\frac{-a+\sqrt{a^2-b^2}}{2a}} (x + a)^{\frac{a+\sqrt{a^2-b^2}}{2a}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= (x - a)^{-\frac{-a+\sqrt{a^2-b^2}}{2a}} (x + a)^{\frac{a+\sqrt{a^2-b^2}}{2a}}
\end{aligned}
$$

Which simplifies to

$$y_1 = (x-a)^{-\frac{-a+\sqrt{a^2-b^2}}{2a}} (x+a)^{\frac{a+\sqrt{a^2-b^2}}{2a}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= (x-a)^{-\frac{-a+\sqrt{a^2-b^2}}{2a}} (x+a)^{\frac{a+\sqrt{a^2-b^2}}{2a}} \int \frac{1}{(x-a)^{-\frac{-a+\sqrt{a^2-b^2}}{a}} (x+a)^{\frac{a+\sqrt{a^2-b^2}}{a}}}\, dx \\
&= (x-a)^{-\frac{-a+\sqrt{a^2-b^2}}{2a}} (x+a)^{\frac{a+\sqrt{a^2-b^2}}{2a}} \left(\int (x-a)^{\frac{-a+\sqrt{a^2-b^2}}{a}} (x+a)^{\frac{-a-\sqrt{a^2-b^2}}{a}}\, dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x-a)^{-\frac{-a+\sqrt{a^2-b^2}}{2a}} (x+a)^{\frac{a+\sqrt{a^2-b^2}}{2a}}\right) \\
&\quad + c_2\left((x-a)^{-\frac{-a+\sqrt{a^2-b^2}}{2a}} (x+a)^{\frac{a+\sqrt{a^2-b^2}}{2a}} \left(\int (x-a)^{\frac{-a+\sqrt{a^2-b^2}}{a}} (x+a)^{\frac{-a-\sqrt{a^2-b^2}}{a}}\, dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 (x-a)^{-\frac{-a+\sqrt{a^2-b^2}}{2a}} (x+a)^{\frac{a+\sqrt{a^2-b^2}}{2a}} \\
&\quad + c_2 (x-a)^{-\frac{-a+\sqrt{a^2-b^2}}{2a}} (x+a)^{\frac{a+\sqrt{a^2-b^2}}{2a}} \left(\int (x-a)^{\frac{-a+\sqrt{a^2-b^2}}{a}} (x+a)^{\frac{-a-\sqrt{a^2-b^2}}{a}}\, dx\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 (x-a)^{-\frac{-a+\sqrt{a^2-b^2}}{2a}} (x+a)^{\frac{a+\sqrt{a^2-b^2}}{2a}} \\
&\quad + c_2 (x-a)^{-\frac{-a+\sqrt{a^2-b^2}}{2a}} (x+a)^{\frac{a+\sqrt{a^2-b^2}}{2a}} \left(\int (x-a)^{\frac{-a+\sqrt{a^2-b^2}}{a}} (x+a)^{\frac{-a-\sqrt{a^2-b^2}}{a}}\, dx\right)
\end{aligned}$$

Verified OK.

32.13.3 Maple step by step solution

Let's solve

$y''(a-x)^2(x+a)^2+yb^2=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{b^2y}{(a-x)^2(x+a)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{b^2y}{(a-x)^2(x+a)^2}=0$

- Multiply by denominators of the ODE

  $y''(a-x)^2(x+a)^2+yb^2=0$

- Make a change of variables

  $t=\ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y'=\left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y'=\frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y''=\left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2+t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y''=\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)(a-x)^2(x+a)^2+y(t)\,b^2=0$

- Simplify

  $\frac{\left(\frac{d^2}{dt^2}y(t)-\frac{d}{dt}y(t)\right)(a-x)^2(x+a)^2}{x^2}+y(t)\,b^2=0$

- Isolate 2nd derivative

$$\frac{d^2}{dt^2}y(t) = -\frac{b^2x^2y(t)}{(a-x)^2(x+a)^2} + \frac{d}{dt}y(t)$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^2}{dt^2}y(t) + \frac{b^2x^2y(t)}{(a-x)^2(x+a)^2} - \frac{d}{dt}y(t) = 0$$

- Characteristic polynomial of ODE

$$r^2 + \frac{b^2x^2}{(a-x)^2(x+a)^2} - r = 0$$

- Factor the characteristic polynomial

$$\frac{a^4r^2-2a^2r^2x^2+r^2x^4-a^4r+2a^2r\,x^2-r\,x^4+b^2x^2}{(a-x)^2(x+a)^2} = 0$$

- Roots of the characteristic polynomial

$$r = \left(\frac{\frac{a^2}{2}-\frac{x^2}{2}+\frac{\sqrt{a^4-2a^2x^2-4b^2x^2+x^4}}{2}}{(x+a)(a-x)}, \frac{\frac{a^2}{2}-\frac{x^2}{2}-\frac{\sqrt{a^4-2a^2x^2-4b^2x^2+x^4}}{2}}{(x+a)(a-x)}\right)$$

- 1st solution of the ODE

$$y_1(t) = \mathrm{e}^{\frac{\left(\frac{a^2}{2}-\frac{x^2}{2}+\frac{\sqrt{a^4-2a^2x^2-4b^2x^2+x^4}}{2}\right)t}{(x+a)(a-x)}}$$

- 2nd solution of the ODE

$$y_2(t) = \mathrm{e}^{\frac{\left(\frac{a^2}{2}-\frac{x^2}{2}-\frac{\sqrt{a^4-2a^2x^2-4b^2x^2+x^4}}{2}\right)t}{(x+a)(a-x)}}$$

- General solution of the ODE

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

- Substitute in solutions

$$y(t) = c_1\mathrm{e}^{\frac{\left(\frac{a^2}{2}-\frac{x^2}{2}+\frac{\sqrt{a^4-2a^2x^2-4b^2x^2+x^4}}{2}\right)t}{(x+a)(a-x)}} + c_2\mathrm{e}^{\frac{\left(\frac{a^2}{2}-\frac{x^2}{2}-\frac{\sqrt{a^4-2a^2x^2-4b^2x^2+x^4}}{2}\right)t}{(x+a)(a-x)}}$$

- Change variables back using $t = \ln(x)$

$$y = c_1\mathrm{e}^{\frac{\left(\frac{a^2}{2}-\frac{x^2}{2}+\frac{\sqrt{a^4-2a^2x^2-4b^2x^2+x^4}}{2}\right)\ln(x)}{(x+a)(a-x)}} + c_2\mathrm{e}^{\frac{\left(\frac{a^2}{2}-\frac{x^2}{2}-\frac{\sqrt{a^4-2a^2x^2-4b^2x^2+x^4}}{2}\right)\ln(x)}{(x+a)(a-x)}}$$

- Simplify

$$y = c_1x^{\frac{a^2-x^2+\sqrt{a^4-2a^2x^2-4b^2x^2+x^4}}{2a^2-2x^2}} + x^{\frac{a^2-x^2-\sqrt{a^4-2a^2x^2-4b^2x^2+x^4}}{2a^2-2x^2}}c_2$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 77

```
dsolve((x^2-a^2)^2*diff(y(x),x$2)+b^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{a^2-x^2}\left(\left(\frac{a-x}{a+x}\right)^{-\frac{\sqrt{a^2-b^2}}{2a}} c_2 + \left(\frac{a-x}{a+x}\right)^{\frac{\sqrt{a^2-b^2}}{2a}} c_1\right)$$

✓ Solution by Mathematica

Time used: 0.529 (sec). Leaf size: 142

```
DSolve[(x^2-a^2)^2*y''[x]+b^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(x-a)^{\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{b^2}{a^2}}}(a+x)^{\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{b^2}{a^2}}}\left(2ac_1\sqrt{1-\frac{b^2}{a^2}}(x-a)^{\sqrt{1-\frac{b^2}{a^2}}} - c_2(a+x)^{\sqrt{1-\frac{b^2}{a^2}}}\right)}{2a\sqrt{1-\frac{b^2}{a^2}}}$$

32.14 problem 223

Internal problem ID [11048]
Internal file name [`OUTPUT/10304_Wednesday_January_24_2024_10_06_55_PM_42614464/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 223.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[_Halm]`

$$4\left(x^2+1\right)^2 y'' + \left(a\,x^2 + a - 3\right) y = 0$$

32.14.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \left(\frac{a}{4} + \frac{a}{4x^2} - \frac{3}{4x^2}\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= \frac{1}{2}\\ \beta &= \frac{\sqrt{-3+a}}{2}\\ n &= \frac{\sqrt{-a+1}}{2}\\ \gamma &= -1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\frac{\sqrt{-a+1}}{2}, \frac{\sqrt{-3+a}}{2x}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\frac{\sqrt{-a+1}}{2}, \frac{\sqrt{-3+a}}{2x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\frac{\sqrt{-a+1}}{2}, \frac{\sqrt{-3+a}}{2x}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\frac{\sqrt{-a+1}}{2}, \frac{\sqrt{-3+a}}{2x}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\frac{\sqrt{-a+1}}{2}, \frac{\sqrt{-3+a}}{2x}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\frac{\sqrt{-a+1}}{2}, \frac{\sqrt{-3+a}}{2x}\right)$$

Verified OK.

32.14.2 Solving using Kovacic algorithm

Writing the ode as

$$4\left(x^2+1\right)^2 y'' + \left(a\,x^2 + a - 3\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 4\left(x^2+1\right)^2\\ B &= 0\\ C &= a\,x^2 + a - 3\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a\,x^2 - a + 3}{4\,(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -a\,x^2 - a + 3 \\ t &= 4\big(x^2+1\big)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-a\,x^2 - a + 3}{4\,(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 210: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(x-i)^2} - \frac{3}{16(x+i)^2} + \frac{i\left(\frac{a}{8} - \frac{3}{16}\right)}{x-i} - \frac{i\left(\frac{a}{8} - \frac{3}{16}\right)}{x+i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-a\,x^2 - a + 3}{4\left(x^2+1\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\ &= \{2\}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
i	2	$\{1,2,3\}$
$-i$	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\ &= \frac{1}{2}(2-(1+(1)))\\ &= 0\end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{1}{(x-(i))}+\frac{1}{(x-(-i))}\right)\\ &= \frac{1}{2x-2i}+\frac{1}{2x+2i}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0=0$$

And solving for p gives

$$p=1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta+\frac{p'}{p}\\ &= \frac{1}{2x-2i}+\frac{1}{2x+2i}\end{aligned}$$

Let ω be the solution of

$$\omega^2-\phi\omega+\left(\frac{1}{2}\phi'+\frac{1}{2}\phi^2-r\right)=0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2-\left(\frac{1}{2x-2i}+\frac{1}{2x+2i}\right)w+\frac{a\,x^2+a-1}{4\left(-x+i\right)^2\left(x+i\right)^2}=0$$

Solving for ω gives

$$\omega=\frac{x+\sqrt{-\left(x^2+1\right)\left(a-1\right)}}{2x^2+2}$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\,dx}\\ &= \mathrm{e}^{\int \frac{x+\sqrt{-\left(x^2+1\right)\left(a-1\right)}}{2x^2+2}dx}\\ &= \left(x^2+1\right)^{\frac{1}{4}}\mathrm{e}^{-\frac{\sqrt{a-1}\,\arctan\left(\frac{\sqrt{a-1}\,x}{\sqrt{x^2+1}\,\sqrt{-a+1}}\right)}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \left(x^2+1\right)^{\frac{1}{4}}\mathrm{e}^{-\frac{i\sqrt{a-1}\,\operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right)}{2}}\end{aligned}$$

Which simplifies to

$$y_1 = \left(x^2+1\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{i\sqrt{a-1}\,\operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \left(x^2+1\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{i\sqrt{a-1}\,\operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right)}{2}} \int \frac{1}{\sqrt{x^2+1}\,\mathrm{e}^{-i\sqrt{a-1}\,\operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right)}}\,dx \\
&= \left(x^2+1\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{i\sqrt{a-1}\,\operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right)}{2}} \left(-\frac{i\mathrm{e}^{i\sqrt{a-1}\,\operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right)}}{\sqrt{a-1}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2+1\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{i\sqrt{a-1}\,\operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right)}{2}}\right) \\
&\quad + c_2\left(\left(x^2+1\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{i\sqrt{a-1}\,\operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right)}{2}} \left(-\frac{i\mathrm{e}^{i\sqrt{a-1}\,\operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right)}}{\sqrt{a-1}}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\left(x^2+1\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{i\sqrt{a-1}\,\operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right)}{2}} - \frac{ic_2\left(x^2+1\right)^{\frac{1}{4}} \mathrm{e}^{\frac{i\sqrt{a-1}\,\operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right)}{2}}}{\sqrt{a-1}} \tag{1}$$

Verification of solutions

$$y = c_1 (x^2+1)^{\frac{1}{4}} e^{-\frac{i\sqrt{a-1}\,\operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right)}{2}} - \frac{i c_2 (x^2+1)^{\frac{1}{4}} e^{\frac{i\sqrt{a-1}\,\operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right)}{2}}}{\sqrt{a-1}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve(4*(x^2+1)^2*diff(y(x),x$2)+(a*x^2+a-3)*y(x)=0,y(x), singsol=all)
```

$$y(x) = (x^2+1)^{\frac{1}{4}} \left(\left(x+\sqrt{x^2+1}\right)^{-\frac{\sqrt{-a+1}}{2}} c_2 + \left(x+\sqrt{x^2+1}\right)^{\frac{\sqrt{-a+1}}{2}} c_1 \right)$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 70

```
DSolve[4*(x^2+1)^2*y''[x]+(a*x^2+a-3)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x^2+1} \left(c_1 P_{\frac{1}{2}\left(\sqrt{1-a}-1\right)}^{\frac{1}{2}}(ix) + c_2 Q_{\frac{1}{2}\left(\sqrt{1-a}-1\right)}^{\frac{1}{2}}(ix) \right)$$

32.15 problem 224

Internal problem ID [11049]
Internal file name [`OUTPUT/10305_Wednesday_January_24_2024_10_06_58_PM_32395236/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 224.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{\left(a\,x^{2}+b\right)^{2} y''+2ax\left(a\,x^{2}+b\right) y'+yc=0}$$

32.15.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$\left(a\,x^{2}+b\right)^{2} y''+\left(2x^{3}a^{2}+2abx\right) y'+yc=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2ax}{a\,x^{2}+b}\\ q(x) &= \frac{c}{\left(a\,x^{2}+b\right)^{2}} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{2ax}{a\,x^2+b}dx\right)}dx \\
&= \int e^{-\ln\left(a\,x^2+b\right)}\,dx \\
&= \int \frac{1}{a\,x^2+b}dx \\
&= \frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{c}{\left(a\,x^2+b\right)^2}}{\frac{1}{\left(a\,x^2+b\right)^2}} \\
&= c
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + cy(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau)+By'(\tau)+Cy(\tau)=0$$

Where in the above $A=1, B=0, C=c$. Let the solution be $y(\tau)=e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau}+c\,\mathrm{e}^{\lambda\tau}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2+c=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=0, C=c$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{0}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{0^2-(4)(1)(c)}\\&=\pm\sqrt{-c}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=+\sqrt{-c}\\\lambda_2&=-\sqrt{-c}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=\sqrt{-c}\\\lambda_2&=-\sqrt{-c}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau)&=c_1e^{\lambda_1\tau}+c_2e^{\lambda_2\tau}\\y(\tau)&=c_1e^{(\sqrt{-c})\tau}+c_2e^{(-\sqrt{-c})\tau}\end{aligned}$$

Or

$$y(\tau)=c_1\mathrm{e}^{\sqrt{-c}\,\tau}+c_2\mathrm{e}^{-\sqrt{-c}\,\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \mathrm{e}^{\frac{\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}} + c_2 \mathrm{e}^{-\frac{\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}} + c_2 \mathrm{e}^{-\frac{\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}} + c_2 \mathrm{e}^{-\frac{\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}}$$

Verified OK.

32.15.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$\left(a\, x^{2}+b\right)^{2} y''+\left(2 x^{3} a^{2}+2 a b x\right) y'+y c = 0 \tag{1}$$

Becomes

$$y''+p(x)\, y'+q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2 a x}{a\, x^{2}+b}$$
$$q(x) = \frac{c}{\left(a\, x^{2}+b\right)^{2}}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^{2}}{d\tau^{2}}y(\tau)+p_{1}\left(\frac{d}{d\tau}y(\tau)\right)+q_{1}y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_{1}(\tau) = \frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'\left(x\right)^{2}} \tag{4}$$

$$q_{1}(\tau) = \frac{q(x)}{\tau'\left(x\right)^{2}} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{\frac{c}{(a\,x^2+b)^2}}}{c} \\
\tau'' &= -\frac{2cax}{c\sqrt{\frac{c}{(a\,x^2+b)^2}}\,(a\,x^2+b)^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2cax}{c\sqrt{\frac{c}{(a\,x^2+b)^2}}\,(a\,x^2+b)^3} + \frac{2ax}{a\,x^2+b}\,\frac{\sqrt{\frac{c}{(a\,x^2+b)^2}}}{c}}{\left(\frac{\sqrt{\frac{c}{(a\,x^2+b)^2}}}{c}\right)^2} \\
&= 0
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{c}{(a\,x^2+b)^2}}dx}{c} \\
&= \frac{\sqrt{\frac{c}{(a\,x^2+b)^2}}\,(a\,x^2+b)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{c\sqrt{ab}}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\frac{\sqrt{c}\, \arctan\left(\frac{\sqrt{a}\, x}{\sqrt{b}}\right)}{\sqrt{a}\, \sqrt{b}}\right) + c_2 \sin\left(\frac{\sqrt{c}\, \arctan\left(\frac{\sqrt{a}\, x}{\sqrt{b}}\right)}{\sqrt{a}\, \sqrt{b}}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\frac{\sqrt{c}\, \arctan\left(\frac{\sqrt{a}\, x}{\sqrt{b}}\right)}{\sqrt{a}\, \sqrt{b}}\right) + c_2 \sin\left(\frac{\sqrt{c}\, \arctan\left(\frac{\sqrt{a}\, x}{\sqrt{b}}\right)}{\sqrt{a}\, \sqrt{b}}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(\frac{\sqrt{c}\, \arctan\left(\frac{\sqrt{a}\, x}{\sqrt{b}}\right)}{\sqrt{a}\, \sqrt{b}}\right) + c_2 \sin\left(\frac{\sqrt{c}\, \arctan\left(\frac{\sqrt{a}\, x}{\sqrt{b}}\right)}{\sqrt{a}\, \sqrt{b}}\right)$$

Verified OK.

32.15.3 Solving using Kovacic algorithm

Writing the ode as

$$\left(a\, x^2 + b\right)^2 y'' + \left(2x^3 a^2 + 2abx\right) y' + yc = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= \left(a\, x^2 + b\right)^2 \\ B &= 2x^3 a^2 + 2abx \\ C &= c \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{ab - c}{(a\,x^2 + b)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= ab - c \\ t &= (a\,x^2 + b)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{ab - c}{(a\,x^2 + b)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 211: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (a\,x^2+b)^2$. There is a pole at $x = \frac{\sqrt{-ab}}{a}$ of order 2. There is a pole at $x = -\frac{\sqrt{-ab}}{a}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned} r = &-\frac{ab-c}{4ab\left(x-\sqrt{-\frac{b}{a}}\right)^2}-\frac{ab-c}{4ab\left(x+\sqrt{-\frac{b}{a}}\right)^2} \\ &+\frac{-ab+c}{4\left(-\frac{b}{a}\right)^{\frac{3}{2}}a^2\left(x-\sqrt{-\frac{b}{a}}\right)}-\frac{-ab+c}{4\left(-\frac{b}{a}\right)^{\frac{3}{2}}a^2\left(x+\sqrt{-\frac{b}{a}}\right)} \end{aligned}$$

For the pole at $x = \frac{\sqrt{-ab}}{a}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{-ab}}{a}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 0$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{0, 2, 4\} \end{aligned}$$

For the pole at $x = -\frac{\sqrt{-ab}}{a}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{-ab}}{a}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 0$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{0, 2, 4\} \end{aligned}$$

Now since the order of r at ∞ is $4 > 2$ then

$$E_\infty = \{0, 2, 4\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
$\frac{\sqrt{-ab}}{a}$	2	$\{0,2,4\}$
$-\frac{\sqrt{-ab}}{a}$	2	$\{0,2,4\}$

Order of r at ∞	E_∞
4	$\{0,2,4\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 2,\ e_2 = 2,\ e_\infty = 4$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(4 - (2 + (2))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{2}{\left(x - \left(\frac{\sqrt{-ab}}{a}\right)\right)} + \frac{2}{\left(x - \left(-\frac{\sqrt{-ab}}{a}\right)\right)}\right) \\ &= \frac{1}{x - \frac{\sqrt{-ab}}{a}} + \frac{1}{x + \frac{\sqrt{-ab}}{a}} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x - \frac{\sqrt{-ab}}{a}} + \frac{1}{x + \frac{\sqrt{-ab}}{a}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x - \frac{\sqrt{-ab}}{a}} + \frac{1}{x + \frac{\sqrt{-ab}}{a}}\right) w + \frac{a^2x^2 + c}{\left(a\,x^2 + b\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{ax + \sqrt{-c}}{a\,x^2 + b}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \frac{ax+\sqrt{-c}}{a\,x^2+b}dx} \\ &= \sqrt{a\,x^2 + b}\,\mathrm{e}^{\frac{\sqrt{-c}\,\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^3a^2+2abx}{\left(a\,x^2+b\right)^2}\,dx} \\ &= z_1 e^{-\frac{\ln\left(a\,x^2+b\right)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{a\,x^2 + b}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3 a^2 + 2abx}{\left(a x^2 + b\right)^2} \, dx}}{\left(y_1\right)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln\left(a x^2 + b\right)}}{\left(y_1\right)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-\frac{2\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}}}{2\sqrt{-c}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{\frac{\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}}\right) + c_2 \left(\mathrm{e}^{\frac{\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}} \left(-\frac{\mathrm{e}^{-\frac{2\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}}}{2\sqrt{-c}}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}} - \frac{c_2 \mathrm{e}^{-\frac{\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}}}{2\sqrt{-c}} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}} - \frac{c_2 \mathrm{e}^{-\frac{\sqrt{-c}\, \arctan\left(\frac{a x}{\sqrt{a b}}\right)}{\sqrt{a b}}}}{2\sqrt{-c}}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
dsolve((a*x^2+b)^2*diff(y(x),x$2)+2*a*x*(a*x^2+b)*diff(y(x),x)+c*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin\left(\frac{\sqrt{c}\,\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}\right) + c_2 \cos\left(\frac{\sqrt{c}\,\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}\right)$$

✓ Solution by Mathematica

Time used: 2.133 (sec). Leaf size: 72

```
DSolve[(a*x^2+b)^2*y''[x]+2*a*x*(a*x^2+b)*y'[x]+c*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to c_1 \cos\left(\frac{\sqrt{c}\arctan\left(\frac{\sqrt{a}x}{\sqrt{b}}\right)}{\sqrt{a}\sqrt{b}}\right) + c_2 \sin\left(\frac{\sqrt{c}\arctan\left(\frac{\sqrt{a}x}{\sqrt{b}}\right)}{\sqrt{a}\sqrt{b}}\right)$$

32.16 problem 225

Internal problem ID [11050]
Internal file name [`OUTPUT/10306_Wednesday_January_24_2024_10_07_00_PM_65258785/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 225.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\left(x^2-1\right)^2 y'' + 2x\left(x^2-1\right) y' - \left(\nu(\nu+1)\left(x^2-1\right)+n^2\right) y = 0$$

32.16.1 Maple step by step solution

Let's solve

$y''(x^4 - 2x^2 + 1) + (2x^3 - 2x) y' + (-x^2\nu^2 - x^2\nu - n^2 + \nu^2 + \nu) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2\nu^2+x^2\nu+n^2-\nu^2-\nu)y}{x^4-2x^2+1} - \frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2xy'}{x^2-1} - \frac{(x^2\nu^2+x^2\nu+n^2-\nu^2-\nu)y}{x^4-2x^2+1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$$\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{x^2\nu^2+x^2\nu+n^2-\nu^2-\nu}{x^4-2x^2+1}\right]$$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

$$\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = 1$$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

$$\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = -\frac{n^2}{4}$$

- $x = -1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -1$

- Multiply by denominators

$$y''(x^2-1)\left(x^4-2x^2+1\right)+2y'x\left(x^4-2x^2+1\right)-\left(x^2\nu^2+x^2\nu+n^2-\nu^2-\nu\right)\left(x^2-1\right)y = 0$$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

$$\left(u^6-6u^5+12u^4-8u^3\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(2u^5-10u^4+16u^3-8u^2\right)\left(\frac{d}{du}y(u)\right)+\left(-\nu^2u^4+4\nu^2u^3-\right.$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 1..4$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 2..5$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 3..6$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0(n+2r)(-n+2r)\, u^{1+r} + \left(-2a_1(2+n+2r)(2-n+2r) + a_0(-n^2-4\nu^2+12r^2-4\nu+4$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2(n+2r)(-n+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{n}{2}, \frac{n}{2}\right\}$$

- The coefficients of each power of u must be 0

$$[-2a_1(2+n+2r)(2-n+2r) + a_0(-n^2-4\nu^2+12r^2-4\nu+4r) = 0, -2a_2(4+n+2r)(4-n$$

- Solve for the dependent coefficient(s)

$$\left\{a_1 = \frac{a_0(n^2+4\nu^2-12r^2+4\nu-4r)}{2(n^2-4r^2-8r-4)}, a_2 = \frac{a_0(n^4-12n^2r^2+16\nu^4-64\nu^2r^2+96r^4-24n^2r+32\nu^3-64\nu^2r-64\nu r^2+256r^3-16n^2-16}{4(n^4-8n^2r^2+16r^4-24n^2r+96r^3-20n^2+208r^2+192r+64)}\right.$$

- Each term in the series must be 0, giving the recursion relation

$$(a_{k-4}-6a_{k-3}+12a_{k-2}-8a_{k-1})\, k^2 + (2(a_{k-4}-6a_{k-3}+12a_{k-2}-8a_{k-1})\, r - 7a_{k-4} + 32a_{k-3} -$$

- Shift index using $k->k+4$

$$(a_k-6a_{k+1}+12a_{k+2}-8a_{k+3})(k+4)^2 + (2(a_k-6a_{k+1}+12a_{k+2}-8a_{k+3})\, r - 7a_k + 32a_{k+1} -$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = \frac{k^2a_k-6k^2a_{k+1}+12k^2a_{k+2}+2kra_k-12kra_{k+1}+24kra_{k+2}-n^2a_{k+2}-a_k\nu^2+4\nu^2a_{k+1}-4\nu^2a_{k+2}+r^2a_k-6r^2a_{k+1}+12r^2a_{k+}}{2(4k^2+8kr-n^2+4r^2+24k+24}$$

- Recursion relation for $r = -\frac{n}{2}$

$$a_{k+3} = \frac{k^2a_k-6k^2a_{k+1}+12k^2a_{k+2}-kna_k+6kna_{k+1}-12kna_{k+2}+\frac{1}{4}a_kn^2-\frac{3}{2}n^2a_{k+1}+2n^2a_{k+2}-a_k\nu^2+4\nu^2a_{k+1}-4\nu^2a_{k+2}+ka_k-}{2(4k^2-4kn+24k-12n+36)}$$

- Solution for $r = -\frac{n}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{n}{2}}, a_{k+3} = \frac{k^2a_k-6k^2a_{k+1}+12k^2a_{k+2}-kna_k+6kna_{k+1}-12kna_{k+2}+\frac{1}{4}a_kn^2-\frac{3}{2}n^2a_{k+1}+2n^2a_{k+2}-a_k\nu^2}{2}\right.$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-\frac{n}{2}}, a_{k+3} = \frac{k^2a_k-6k^2a_{k+1}+12k^2a_{k+2}-kna_k+6kna_{k+1}-12kna_{k+2}+\frac{1}{4}a_kn^2-\frac{3}{2}n^2a_{k+1}+2n^2a_{k+2}-}{}\right.$$

- Recursion relation for $r = \frac{n}{2}$

$$a_{k+3} = \frac{k^2 a_k - 6k^2 a_{k+1} + 12k^2 a_{k+2} + kn a_k - 6kn a_{k+1} + 12kn a_{k+2} + \frac{1}{4} a_k n^2 - \frac{3}{2} n^2 a_{k+1} + 2n^2 a_{k+2} - a_k \nu^2 + 4\nu^2 a_{k+1} - 4\nu^2 a_{k+2} + k a_k -}{2(4k^2 + 4kn + 24k + 12n + 36)}$$

- Solution for $r = \frac{n}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{n}{2}}, a_{k+3} = \frac{k^2 a_k - 6k^2 a_{k+1} + 12k^2 a_{k+2} + kn a_k - 6kn a_{k+1} + 12kn a_{k+2} + \frac{1}{4} a_k n^2 - \frac{3}{2} n^2 a_{k+1} + 2n^2 a_{k+2} - a_k \nu^2}{2}\right.$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{n}{2}}, a_{k+3} = \frac{k^2 a_k - 6k^2 a_{k+1} + 12k^2 a_{k+2} + kn a_k - 6kn a_{k+1} + 12kn a_{k+2} + \frac{1}{4} a_k n^2 - \frac{3}{2} n^2 a_{k+1} + 2n^2 a_{k+2} -}{}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k-\frac{n}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+\frac{n}{2}}\right), a_{k+3} = \frac{k^2 a_k - 6k^2 a_{1+k} + 12k^2 a_{k+2} - kn a_k + 6kn a_{1+k} - 12kn a}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 17

```
dsolve((x^2-1)^2*diff(y(x),x$2)+2*x*(x^2-1)*diff(y(x),x)-(nu*(nu+1)*(x^2-1)+n^2)*y(x)=0,y(x)
```

$$y(x) = c_1 \operatorname{LegendreP}(\nu, n, x) + c_2 \operatorname{LegendreQ}(\nu, n, x)$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 20

```
DSolve[(x^2-1)^2*y''[x]+2*x*(x^2-1)*y'[x]-(\[Nu]*(\[Nu]+1)*(x^2-1)+n^2)*y[x]==0,y[x],x,Inclu
```

$$y(x) \to c_1 P_{\nu}^{n}(x) + c_2 Q_{\nu}^{n}(x)$$

32.17 problem 226

Internal problem ID [11051]
Internal file name [`OUTPUT/10307_Wednesday_January_24_2024_10_07_00_PM_26256630/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 226.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\left(-x^2+1\right)^2 y'' - 2x\left(-x^2+1\right) y' + \left(\nu(\nu+1)\left(-x^2+1\right) - \mu^2\right) y = 0$$

32.17.1 Maple step by step solution

Let's solve

$y''(x^4 - 2x^2 + 1) + (2x^3 - 2x)\, y' + (-x^2\nu^2 - x^2\nu - \mu^2 + \nu^2 + \nu)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2\nu^2+x^2\nu+\mu^2-\nu^2-\nu)y}{x^4-2x^2+1} - \frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2xy'}{x^2-1} - \frac{(x^2\nu^2+x^2\nu+\mu^2-\nu^2-\nu)y}{x^4-2x^2+1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$$\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{x^2\nu^2+x^2\nu+\mu^2-\nu^2-\nu}{x^4-2x^2+1}\right]$$

- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $$\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = 1$$

- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $$\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = -\frac{\mu^2}{4}$$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $$y''(x^2-1)\left(x^4-2x^2+1\right)+2y'x\left(x^4-2x^2+1\right)-\left(x^2\nu^2+x^2\nu+\mu^2-\nu^2-\nu\right)\left(x^2-1\right)y = 0$$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $$\left(u^6-6u^5+12u^4-8u^3\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(2u^5-10u^4+16u^3-8u^2\right)\left(\frac{d}{du}y(u)\right)+\left(-\nu^2u^4+4\nu^2u^3-\right.$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=1..4$

    $$u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

  - Shift index using $k->k-m$

    $$u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=2..5$

    $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

    $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=3..6$

$$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right) = \sum_{k =0}^{\infty} a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- Shift index using $k->k +2-m$

  $$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right) = \sum_{k =-2+m}^{\infty} a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

  Rewrite ODE with series expansions

  $$-2 a_{0} \left(\mu +2 r \right) \left(-\mu +2 r \right) u^{1+r}+\left(-2 a_{1} \left(2+\mu +2 r \right) \left(2-\mu +2 r \right)+a_{0} \left(-\mu^{2}-4 \nu^{2}+12 r^{2}-4 \nu +\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$-2 \left(\mu +2 r \right) \left(-\mu +2 r \right) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{\mu}{2}, \frac{\mu}{2}\right\}$$

- The coefficients of each power of u must be 0

  $$\left[-2 a_{1} \left(2+\mu +2 r \right) \left(2-\mu +2 r \right)+a_{0} \left(-\mu^{2}-4 \nu^{2}+12 r^{2}-4 \nu +4 r \right) = 0, -2 a_{2} \left(4+\mu +2 r \right) \left(4-\right.\right.$$

- Solve for the dependent coefficient(s)

  $$\left\{a_{1} = \frac{a_{0} \left(\mu^{2}+4 \nu^{2}-12 r^{2}+4 \nu -4 r \right)}{2 \left(\mu^{2}-4 r^{2}-8 r -4\right)}, a_{2} = \frac{a_{0} \left(\mu^{4}-12 \mu^{2} r^{2}+16 \nu^{4}-64 \nu^{2} r^{2}+96 r^{4}-24 \mu^{2} r +32 \nu^{3}-64 \nu^{2} r -64 \nu \,r^{2}+256 r^{3}-16 \mu^{2}-16\right.}{4 \left(\mu^{4}-8 \mu^{2} r^{2}+16 r^{4}-24 \mu^{2} r +96 r^{3}-20 \mu^{2}+208 r^{2}+192 r +64\right)}\right.$$

- Each term in the series must be 0, giving the recursion relation

  $$\left(a_{k -4}-6 a_{k -3}+12 a_{k -2}-8 a_{k -1}\right) k^{2}+\left(2 \left(a_{k -4}-6 a_{k -3}+12 a_{k -2}-8 a_{k -1}\right) r -7 a_{k -4}+32 a_{k -3}-\right.$$

- Shift index using $k->k +4$

  $$\left(a_{k}-6 a_{k +1}+12 a_{k +2}-8 a_{k +3}\right) \left(k +4\right)^{2}+\left(2 \left(a_{k}-6 a_{k +1}+12 a_{k +2}-8 a_{k +3}\right) r -7 a_{k}+32 a_{k +1}-\right.$$

- Recursion relation that defines series solution to ODE

  $$a_{k +3} = \frac{k^{2} a_{k}-6 k^{2} a_{k +1}+12 k^{2} a_{k +2}+2 k r a_{k}-12 k r a_{k +1}+24 k r a_{k +2}-\mu^{2} a_{k +2}-a_{k} \nu^{2}+4 \nu^{2} a_{k +1}-4 \nu^{2} a_{k +2}+r^{2} a_{k}-6 r^{2} a_{k +1}+12 r^{2} a_{k +}}{2 \left(4 k^{2}+8 k r -\mu^{2}+4 r^{2}+24 k +24\right.}$$

- Recursion relation for $r = -\frac{\mu}{2}$

  $$a_{k +3} = \frac{k^{2} a_{k}-6 k^{2} a_{k +1}+12 k^{2} a_{k +2}-k \mu a_{k}+6 k \mu a_{k +1}-12 k \mu a_{k +2}+\frac{1}{4} a_{k} \mu^{2}-\frac{3}{2} \mu^{2} a_{k +1}+2 \mu^{2} a_{k +2}-a_{k} \nu^{2}+4 \nu^{2} a_{k +1}-4 \nu^{2} a_{k +2}+k a_{k}-}{2 \left(4 k^{2}-4 k \mu +24 k -12 \mu +36\right)}$$

- Solution for $r = -\frac{\mu}{2}$

  $$\left[y \left(u \right) = \sum_{k =0}^{\infty} a_{k} u^{k -\frac{\mu}{2}}, a_{k +3} = \frac{k^{2} a_{k}-6 k^{2} a_{k +1}+12 k^{2} a_{k +2}-k \mu a_{k}+6 k \mu a_{k +1}-12 k \mu a_{k +2}+\frac{1}{4} a_{k} \mu^{2}-\frac{3}{2} \mu^{2} a_{k +1}+2 \mu^{2} a_{k +2}-a_{k} \nu^{2}}{2}\right.$$

- Revert the change of variables $u = 1+x$

  $$\left[y = \sum_{k =0}^{\infty} a_{k} \left(1+x \right)^{k -\frac{\mu}{2}}, a_{k +3} = \frac{k^{2} a_{k}-6 k^{2} a_{k +1}+12 k^{2} a_{k +2}-k \mu a_{k}+6 k \mu a_{k +1}-12 k \mu a_{k +2}+\frac{1}{4} a_{k} \mu^{2}-\frac{3}{2} \mu^{2} a_{k +1}+2 \mu^{2} a_{k +2}-}{}\right.$$

- Recursion relation for $r = \frac{\mu}{2}$

$$a_{k+3} = \frac{k^2a_k - 6k^2a_{k+1} + 12k^2a_{k+2} + k\mu a_k - 6k\mu a_{k+1} + 12k\mu a_{k+2} + \frac{1}{4}a_k\mu^2 - \frac{3}{2}\mu^2a_{k+1} + 2\mu^2a_{k+2} - a_k\nu^2 + 4\nu^2a_{k+1} - 4\nu^2a_{k+2} + ka_k -}{2(4k^2+4k\mu+24k+12\mu+36)}$$

- Solution for $r = \frac{\mu}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{\mu}{2}}, a_{k+3} = \frac{k^2a_k - 6k^2a_{k+1} + 12k^2a_{k+2} + k\mu a_k - 6k\mu a_{k+1} + 12k\mu a_{k+2} + \frac{1}{4}a_k\mu^2 - \frac{3}{2}\mu^2a_{k+1} + 2\mu^2a_{k+2} - a_k\nu^2}{2}\right.$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+\frac{\mu}{2}}, a_{k+3} = \frac{k^2a_k - 6k^2a_{k+1} + 12k^2a_{k+2} + k\mu a_k - 6k\mu a_{k+1} + 12k\mu a_{k+2} + \frac{1}{4}a_k\mu^2 - \frac{3}{2}\mu^2a_{k+1} + 2\mu^2a_{k+2} -}{}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^{k-\frac{\mu}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k+\frac{\mu}{2}}\right), a_{k+3} = \frac{k^2a_k - 6k^2a_{1+k} + 12k^2a_{k+2} - k\mu a_k + 6k\mu a_{1+k} - 12k\mu a}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 17

```
dsolve((1-x^2)^2*diff(y(x),x$2)-2*x*(1-x^2)*diff(y(x),x)+(nu*(nu+1)*(1-x^2)-mu^2)*y(x)=0,y(x
```

$$y(x) = c_1 \operatorname{LegendreP}(\nu, \mu, x) + c_2 \operatorname{LegendreQ}(\nu, \mu, x)$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 20

```
DSolve[(1-x^2)^2*y''[x]-2*x*(1-x^2)*y'[x]+(\[Nu]*(\[Nu]+1)*(1-x^2)-\[Mu]^2)*y[x]==0,y[x],x,I
```

$$y(x) \to c_1 P_\nu^\mu(x) + c_2 Q_\nu^\mu(x)$$

32.18 problem 227

Internal problem ID [11052]
Internal file name [OUTPUT/10308_Wednesday_January_24_2024_10_07_01_PM_84430845/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 227.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$a\left(x^2-1\right)^2 y'' + bx\left(x^2-1\right) y' + \left(c\,x^2+dx+e\right) y = 0$$

32.18.1 Maple step by step solution

Let's solve

$y''a(x-1)^2\,(1+x)^2 + b(x^3-x)\,y' + (c\,x^2+dx+e)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(c\,x^2+dx+e)y}{a(x-1)^2(1+x)^2} - \frac{bxy'}{a(x-1)(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{bxy'}{a(x-1)(1+x)} + \frac{(c\,x^2+dx+e)y}{a(x-1)^2(1+x)^2} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{bx}{a(x-1)(1+x)}, P_3(x) = \frac{c\,x^2+dx+e}{a(x-1)^2(1+x)^2}\right]$

- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1} = \frac{b}{2a}$

- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1} = \frac{e-d+c}{4a}$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $y''a(x-1)^2(1+x)^2 + bxy'(x-1)(1+x) + (c\,x^2+dx+e)\,y = 0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(a\,u^4 - 4a\,u^3 + 4a\,u^2)\left(\frac{d^2}{du^2}y(u)\right) + (b\,u^3 - 3b\,u^2 + 2bu)\left(\frac{d}{du}y(u)\right) + (c\,u^2 - 2cu + du + c - d + e)$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

  $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..4$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(4a\,r^2-4ar+2br+c-d+e)\,u^r + \left((4a\,r^2+4ar+2br+2b+c-d+e)\,a_1 - a_0(4a\,r^2-4ar$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4a\,r^2-4ar+2br+c-d+e=0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{-2a+b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}}{4a}, \frac{2a-b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}}{4a}\right\}$$

- Each term must be 0

  $$(4a\,r^2+4ar+2br+2b+c-d+e)\,a_1 - a_0(4a\,r^2-4ar+3br+2c-d)=0$$

- Solve for the dependent coefficient(s)

  $$a_1 = \frac{a_0(4a\,r^2-4ar+3br+2c-d)}{4a\,r^2+4ar+2br+2b+c-d+e}$$

- Each term in the series must be 0, giving the recursion relation

  $$((-4a_{k-1}+4a_k+a_{k-2})\,k^2 + ((-8a_{k-1}+8a_k+2a_{k-2})\,r+12a_{k-1}-4a_k-5a_{k-2})\,k + (-4a_{k-1}+$$

- Shift index using $k->k+2$

  $$((-4a_{k+1}+4a_{k+2}+a_k)\,(k+2)^2 + ((-8a_{k+1}+8a_{k+2}+2a_k)\,r+12a_{k+1}-4a_{k+2}-5a_k)\,(k+2)$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a\,k^2a_k-4a\,k^2a_{k+1}+2akra_k-8akra_{k+1}+a\,r^2a_k-4a\,r^2a_{k+1}-aka_k-4aka_{k+1}-ara_k-4ara_{k+1}+a_kbk-3bka_{k+1}+a_kbr-}{4a\,k^2+8akr+4a\,r^2+12ak+12ar+2bk+2br+8a+4b+c-d+e}$$

- Recursion relation for $r = -\frac{-2a+b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}}{4a}$

  $$a_{k+2} = -\frac{a\,k^2a_k-4a\,k^2a_{k+1}-\frac{k\left(-2a+b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}\right)a_k}{2}+2k\left(-2a+b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}\right)a_{k+1}+\frac{(-2}{}}{}$$

- Solution for $r = -\frac{-2a+b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}}{4a}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{-2a+b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}}{4a}}, a_{k+2} = -\frac{a\,k^2 a_k - 4a\,k^2 a_{k+1} - \frac{k\left(-2a+b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}\right.}{2}}{}\right.$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-\frac{-2a+b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}}{4a}}, a_{k+2} = -\frac{a\,k^2 a_k - 4a\,k^2 a_{k+1} - \frac{k\left(-2a+b+\sqrt{4a^2-4ab-4ac+4ad-4\right.}{2}}{}\right.$$

- Recursion relation for $r = \frac{2a-b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}}{4a}$

$$a_{k+2} = -\frac{a\,k^2 a_k - 4a\,k^2 a_{k+1} + \frac{k\left(2a-b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}\right)a_k}{2} - 2k\left(2a-b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}\right)a_{k+1} + \frac{\left(2a-b+\right.}{}}{}$$

- Solution for $r = \frac{2a-b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}}{4a}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{2a-b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}}{4a}}, a_{k+2} = -\frac{a\,k^2 a_k - 4a\,k^2 a_{k+1} + \frac{k\left(2a-b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}\right)}{2}}{}\right.$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{2a-b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}}{4a}}, a_{k+2} = -\frac{a\,k^2 a_k - 4a\,k^2 a_{k+1} + \frac{k\left(2a-b+\sqrt{4a^2-4ab-4ac+4ad-4ae\right.}{2}}{}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} f_k (1+x)^{k-\frac{-2a+b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}}{4a}}\right) + \left(\sum_{k=0}^{\infty} g_k (1+x)^{k+\frac{2a-b+\sqrt{4a^2-4ab-4ac+4ad-4ae+b^2}}{4a}}\right)\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 562

```
dsolve(a*(x^2-1)^2*diff(y(x),x$2)+b*x*(x^2-1)*diff(y(x),x)+(c*x^2+d*x+e)*y(x)=0,y(x), singso
```

$$y(x) = \frac{(x^2-1)^{-\frac{b}{4a}}\sqrt{2+2x}\left(c_1 \operatorname{hypergeom}\left(\left[-\frac{-\sqrt{4a^2+(-4b-4c-4d-4e)a+b^2}+2\sqrt{a^2+(-2b-4c)a+b^2}+\sqrt{4a^2+(-4b-4c+4d-4e)}}{4a}\right.\right.\right.}{}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[a*(x^2-1)^2*y''[x]+b*x*(x^2-1)*y'[x]+(c*x^2+d*x+e)*y[x]==0,y[x],x,IncludeSingularSolu
```

Timed out

32.19 problem 228

Internal problem ID [11053]
Internal file name [`OUTPUT/10309_Wednesday_January_24_2024_10_07_01_PM_39341230/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 228.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(a\,x^2+b\right)^2 y''+(2ax+c)\left(a\,x^2+b\right)y'+yk=0}$$

32.19.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$\left(a\,x^2+b\right)^2 y''+(2ax+c)\left(a\,x^2+b\right)y'+yk=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=\frac{2ax+c}{a\,x^2+b}\\ q(x)&=\frac{k}{\left(a\,x^2+b\right)^2}\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{2ax+c}{a\,x^2+b}dx\right)}dx \\
&= \int e^{-\ln\left(a\,x^2+b\right)-\frac{c\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}dx \\
&= \int \frac{\mathrm{e}^{-\frac{c\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}}{a\,x^2+b}dx \\
&= -\frac{\mathrm{e}^{-\frac{c\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}}{c}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{k}{\left(a\,x^2+b\right)^2}}{\frac{\mathrm{e}^{-\frac{2c\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}}{\left(a\,x^2+b\right)^2}} \\
&= k\,\mathrm{e}^{\frac{2c\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) + k\,\mathrm{e}^{\frac{2c\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}} y(\tau) = 0$$

But in terms of τ

$$k\,\mathrm{e}^{\frac{2c\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}} = \frac{k}{c^2\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{k y(\tau)}{c^2\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)c^2\tau^2 + k y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$c^2\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + k\,\tau^r = 0$$

Simplifying gives

$$c^2 r(r-1)\,\tau^r + 0\,\tau^r + k\,\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$c^2 r(r-1) + 0 + k = 0$$

Or

$$c^2 r^2 - c^2 r + k = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{-c + \sqrt{c^2 - 4k}}{2c}$$
$$r_2 = \frac{c + \sqrt{c^2 - 4k}}{2c}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{-\frac{-c+\sqrt{c^2-4k}}{2c}} + c_2 \tau^{\frac{c+\sqrt{c^2-4k}}{2c}}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(-\frac{e^{-\frac{c \arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}}{c}\right)^{-\frac{-c+\sqrt{c^2-4k}}{2c}} + c_2 \left(-\frac{e^{-\frac{c \arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}}{c}\right)^{\frac{c+\sqrt{c^2-4k}}{2c}}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(-\frac{e^{-\frac{c \arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}}{c}\right)^{-\frac{-c+\sqrt{c^2-4k}}{2c}} + c_2 \left(-\frac{e^{-\frac{c \arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}}{c}\right)^{\frac{c+\sqrt{c^2-4k}}{2c}} \tag{1}$$

Verification of solutions

$$y = c_1 \left(-\frac{e^{-\frac{c \arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}}{c}\right)^{-\frac{-c+\sqrt{c^2-4k}}{2c}} + c_2 \left(-\frac{e^{-\frac{c \arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}}{c}\right)^{\frac{c+\sqrt{c^2-4k}}{2c}}$$

Verified OK.

32.19.2 Solving using Kovacic algorithm

Writing the ode as

$$\left(a\,x^2+b\right)^2 y'' + (2ax+c)\left(a\,x^2+b\right) y' + yk = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= \left(a\,x^2+b\right)^2 \\ B &= (2ax+c)\left(a\,x^2+b\right) \\ C &= k \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4ab + c^2 - 4k}{4\left(a\,x^2 + b\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4ab + c^2 - 4k \\ t &= 4\left(a\,x^2 + b\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4ab + c^2 - 4k}{4\left(a\,x^2 + b\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 215: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(a\,x^2 + b)^2$. There is a pole at $x = \frac{\sqrt{-ab}}{a}$ of order 2. There is a pole at $x = -\frac{\sqrt{-ab}}{a}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned} r = &-\frac{4ab + c^2 - 4k}{16ab\left(x - \sqrt{-\frac{b}{a}}\right)^2} - \frac{4ab + c^2 - 4k}{16ab\left(x + \sqrt{-\frac{b}{a}}\right)^2} \\ &+ \frac{-4ab - c^2 + 4k}{16\left(-\frac{b}{a}\right)^{\frac{3}{2}} a^2 \left(x - \sqrt{-\frac{b}{a}}\right)} - \frac{-4ab - c^2 + 4k}{16\left(-\frac{b}{a}\right)^{\frac{3}{2}} a^2 \left(x + \sqrt{-\frac{b}{a}}\right)} \end{aligned}$$

For the pole at $x = \frac{\sqrt{-ab}}{a}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{-ab}}{a}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 0$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{0, 2, 4\} \end{aligned}$$

For the pole at $x = -\frac{\sqrt{-ab}}{a}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{-ab}}{a}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 0$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{0, 2, 4\} \end{aligned}$$

Now since the order of r at ∞ is $4 > 2$ then

$$E_\infty = \{0, 2, 4\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
$\frac{\sqrt{-ab}}{a}$	2	$\{0, 2, 4\}$
$-\frac{\sqrt{-ab}}{a}$	2	$\{0, 2, 4\}$

Order of r at ∞	E_∞
4	$\{0, 2, 4\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 2, \; e_2 = 2, \; e_\infty = 4$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(4 - (2 + (2))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\
&= \frac{1}{2}\left(\frac{2}{\left(x-\left(\frac{\sqrt{-ab}}{a}\right)\right)}+\frac{2}{\left(x-\left(-\frac{\sqrt{-ab}}{a}\right)\right)}\right) \\
&= \frac{1}{x-\frac{\sqrt{-ab}}{a}}+\frac{1}{x+\frac{\sqrt{-ab}}{a}}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$
p'''+3\theta p''+\left(3\theta^2+3\theta'-4r\right)p'+\left(\theta''+3\theta\theta'+\theta^3-4r\theta-2r'\right)p=0 \tag{1A}
$$

Since $d=0$, then letting

$$
p=1 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$
0=0
$$

And solving for p gives

$$
p=1
$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta+\frac{p'}{p} \\
&= \frac{1}{x-\frac{\sqrt{-ab}}{a}}+\frac{1}{x+\frac{\sqrt{-ab}}{a}}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2-\phi\omega+\left(\frac{1}{2}\phi'+\frac{1}{2}\phi^2-r\right)=0
$$

Substituting the values for ϕ and r into the above equation gives

$$
w^2-\left(\frac{1}{x-\frac{\sqrt{-ab}}{a}}+\frac{1}{x+\frac{\sqrt{-ab}}{a}}\right)w+\frac{4a^2x^2-c^2+4k}{4\left(a\,x^2+b\right)^2}=0
$$

Solving for ω gives

$$
\omega=\frac{2ax+\sqrt{c^2-4k}}{2a\,x^2+2b}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{2ax+\sqrt{c^2-4k}}{2a\,x^2+2b}dx} \\
&= \sqrt{a\,x^2+b}\,\mathrm{e}^{\frac{\sqrt{c^2-4k}\,\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{2\sqrt{ab}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{(2ax+c)\left(a\,x^2+b\right)}{\left(a\,x^2+b\right)^2}\,dx} \\
&= z_1 e^{-\frac{\ln\left(a\,x^2+b\right)}{2}-\frac{c\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{2\sqrt{ab}}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{c\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{2\sqrt{ab}}}}{\sqrt{a\,x^2+b}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)\left(-c+\sqrt{c^2-4k}\right)}{2\sqrt{ab}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{(2ax+c)\left(a\,x^2+b\right)}{\left(a\,x^2+b\right)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln\left(a\,x^2+b\right)-\frac{c\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{-\frac{\sqrt{c^2-4k}\,\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}}{\sqrt{c^2-4k}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{\frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)\left(-c+\sqrt{c^2-4k}\right)}{2\sqrt{ab}}} \right) + c_2 \left(e^{\frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)\left(-c+\sqrt{c^2-4k}\right)}{2\sqrt{ab}}} \left(-\frac{e^{-\frac{\sqrt{c^2-4k}\,\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{\sqrt{ab}}}}{\sqrt{c^2-4k}} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{\frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)\left(-c+\sqrt{c^2-4k}\right)}{2\sqrt{ab}}} - \frac{c_2 e^{-\frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)\left(c+\sqrt{c^2-4k}\right)}{2\sqrt{ab}}}}{\sqrt{c^2-4k}} \tag{1}
$$

Verification of solutions

$$
y = c_1 e^{\frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)\left(-c+\sqrt{c^2-4k}\right)}{2\sqrt{ab}}} - \frac{c_2 e^{-\frac{\arctan\left(\frac{ax}{\sqrt{ab}}\right)\left(c+\sqrt{c^2-4k}\right)}{2\sqrt{ab}}}}{\sqrt{c^2-4k}}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 162

```
dsolve((a*x^2+b)^2*diff(y(x),x$2)+(2*a*x+c)*(a*x^2+b)*diff(y(x),x)+k*y(x)=0,y(x), singsol=al
```

$$y(x) = c_1 \left(\frac{-i\sqrt{ab}+ax}{i\sqrt{ab}+ax}\right)^{\frac{i\sqrt{ab}\,c\sqrt{-ab}+a^2\sqrt{\frac{c^2-4k}{a^2}}\,b}{4ab\sqrt{-ab}}} + c_2 \left(\frac{-i\sqrt{ab}+ax}{i\sqrt{ab}+ax}\right)^{\frac{i\sqrt{ab}\,c\sqrt{-ab}-a^2\sqrt{\frac{c^2-4k}{a^2}}\,b}{4ab\sqrt{-ab}}}$$

✓ Solution by Mathematica

Time used: 2.188 (sec). Leaf size: 91

```
DSolve[(a*x^2+b)^2*y''[x]+(2*a*x+c)*(a*x^2+b)*y'[x]+k*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to e^{-\frac{\left(\sqrt{c^2-4k}+c\right)\arctan\left(\frac{\sqrt{a}x}{\sqrt{b}}\right)}{2\sqrt{a}\sqrt{b}}}\left(c_2 e^{\frac{\sqrt{c^2-4k}\arctan\left(\frac{\sqrt{a}x}{\sqrt{b}}\right)}{\sqrt{a}\sqrt{b}}}+c_1\right)$$

32.20 problem 229

Internal problem ID [11054]
Internal file name [OUTPUT/10310_Wednesday_January_24_2024_10_07_02_PM_22005237/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 229.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(a\,x^2+b\right)^2 y''+\left(a\,x^2+b\right)\left(c\,x^2+d\right) y'+2(-ad+bc)\,xy=0}$$

32.20.1 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$\left(a\,x^2+b\right)^2 y''+\left(a\,x^2+b\right)\left(c\,x^2+d\right) y'+(-2adx+2bcx)\,y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=r(x) \tag{2}$$

Where

$$p(x)=\frac{c\,x^2+d}{a\,x^2+b}$$
$$q(x)=-\frac{2(ad-bc)\,x}{\left(a\,x^2+b\right)^2}$$
$$r(x)=0$$

The Lagrange adjoint ode is given by

$$\xi'' - (\xi\, p)' + \xi q = 0$$
$$\xi'' - \left(\frac{(c\,x^2+d)\,\xi(x)}{a\,x^2+b}\right)' + \left(-\frac{2(ad-bc)\,x\xi(x)}{\left(a\,x^2+b\right)^2}\right) = 0$$
$$\xi''(x) - \frac{(c\,x^2+d)\,\xi'(x)}{a\,x^2+b} + \left(-\frac{2cx}{a\,x^2+b} + \frac{2(c\,x^2+d)\,ax}{\left(a\,x^2+b\right)^2} - \frac{2(ad-bc)\,x}{\left(a\,x^2+b\right)^2}\right)\xi(x) = 0$$

Which is solved for $\xi(x)$. This is second order ode with missing dependent variable $\xi(x)$. Let

$$p(x) = \xi'(x)$$

Then

$$p'(x) = \xi''(x)$$

Hence the ode becomes

$$p'(x)\left(a\,x^2+b\right) + \left(-c\,x^2-d\right)p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned}
p' &= F(x,p)\\
&= f(x)g(p)\\
&= \frac{p(c\,x^2+d)}{a\,x^2+b}
\end{aligned}$$

Where $f(x) = \frac{c\,x^2+d}{a\,x^2+b}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= \frac{c\,x^2+d}{a\,x^2+b}\,dx\\
\int \frac{1}{p}\,dp &= \int \frac{c\,x^2+d}{a\,x^2+b}\,dx\\
\ln(p) &= \frac{cx}{a} + \frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}} + c_1\\
p &= \mathrm{e}^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}+c_1}\\
&= c_1 \mathrm{e}^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}}
\end{aligned}$$

Since $p=\xi'(x)$ then the new first order ode to solve is

$$\xi'(x)=c_1 \mathrm{e}^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}}$$

Integrating both sides gives

$$\begin{aligned}
\xi(x) &= \int c_1 \mathrm{e}^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}}\,\mathrm{d}x \\
&= \int c_1 \mathrm{e}^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx + c_2
\end{aligned}$$

The original ode (2) now reduces to first order ode

$$\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)} \\
y' + y\left(\frac{c\,x^2+d}{a\,x^2+b} - \frac{c_3 \mathrm{e}^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}}}{\int c_3 \mathrm{e}^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx + c_2}\right) &= 0
\end{aligned}$$

Which is now a first order ode. This is now solved for y. In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{y\left(\mathrm{e}^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 a\,x^2 - \left(\int c_3 \mathrm{e}^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx\right) c\,x^2 - c\,x^2 c_2 + \mathrm{e}^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 b}{(a\,x^2+b)\left(\int c_3 \mathrm{e}^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx + c_2\right)}
\end{aligned}$$

Where $f(x) = \frac{e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 a x^2 - \left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx\right) c x^2 - c x^2 c_2 + e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 b - \left(\int c_3 e^{\dots}\right.}{(a x^2+b)\left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx + c_2\right)}$

and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\,dy = \frac{e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 a x^2 - \left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx\right) c x^2 - c x^2 c_2 + e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c \dots}{(a x^2+b)\left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx + c_2\right)}$$

$$\int \frac{1}{y}\,dy = \int \frac{e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 a x^2 - \left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx\right) c x^2 - c x^2 c_2 + e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{a\dots}}\right)}{a\sqrt{ab}}} \dots}{(a x^2+b)\left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx + c_2\right)}$$

$$\ln(y) = \int \frac{e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 a x^2 - \left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx\right) c x^2 - c x^2 c_2 + e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{a\dots}}\right)}{a\sqrt{ab}}} \dots}{(a x^2+b)\left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx + c_2\right)}$$

$$y = e^{\int \frac{e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 a x^2 - \left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx\right) c x^2 - c x^2 c_2 + e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 b - \left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-\dots}{\dots}}\right.}{\left(a x^2+b\right)\left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx + c_2\right)}}$$

$$= c_3 e^{\int \frac{e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 a x^2 - \left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx\right) c x^2 - c x^2 c_2 + e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 b - \left(\int c_3 e^{\frac{cx}{a}+\frac{(\dots}{\dots}}\right.}{\left(a x^2+b\right)\left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx + c_2\right)}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \int \frac{e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 a x^2 - \left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx\right) c x^2 - c x^2 c_2 + e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 b - \left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan}{a\sqrt{ab}}}\right.}{\left(a x^2+b\right)\left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx + c_2\right)} = c_3 e$$

Summary

The solution(s) found are the following

$$y = \int \frac{e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 a x^2 - \left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx\right) c x^2 - c x^2 c_2 + e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 b - \left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan}{a\sqrt{ab}}}\right.}{\left(a x^2+b\right)\left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx + c_2\right)} = c_3 e \tag{1}$$

Verification of solutions

$$y = \int \frac{e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 a x^2 - \left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx\right) c x^2 - c x^2 c_2 + e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} c_3 b - \left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan}{a\sqrt{ab}}}\right.}{\left(a x^2+b\right)\left(\int c_3 e^{\frac{cx}{a}+\frac{(ad-bc)\arctan\left(\frac{ax}{\sqrt{ab}}\right)}{a\sqrt{ab}}} dx + c_2\right)} = c_3 e$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.312 (sec). Leaf size: 866

```
dsolve((a*x^2+b)^2*diff(y(x),x$2)+(a*x^2+b)*(c*x^2+d)*diff(y(x),x)+2*(b*c-a*d)*x*y(x)=0,y(x)
```

$$y(x) = \left(-ax+\sqrt{-ab}\right)^{\frac{2a^2b+\sqrt{-ab\left(4\sqrt{-ab}\,a^2d-4\sqrt{-ab}\,abc-4a^3b+a^2d^2-2abcd+b^2c^2\right)}}{4a^2b}} \left(c_1\left(ax\right.\right.$$

$$\left.+\sqrt{-ab}\right)^{\frac{2a^2b+\sqrt{4a^2b(ad-bc)\sqrt{-ab}+4a^4b^2-a^3b\,d^2+2d\,b^2c\,a^2-b^3c^2a}}{4a^2b}} \mathrm{e}^{\frac{\sqrt{-ab}\,c}{2a^2}-\frac{\arctan\left(\frac{\sqrt{a}\,x}{\sqrt{b}}\right)d}{2\sqrt{a}\,\sqrt{b}}+\frac{\sqrt{b}\,\arctan\left(\frac{\sqrt{a}\,x}{\sqrt{b}}\right)c}{2a^{\frac{3}{2}}}} \operatorname{HeunC}\left(\frac{2\sqrt{-\frac{b}{a}}\,c}{a},\right.$$

$$\left.-\frac{d^2}{8ab}-\frac{cd}{4a^2}+\frac{3b\,c^2}{8a^3}, \frac{ax}{2\sqrt{-ab}}+\frac{1}{2}\right)$$

$$+ c_2\left(ax+\sqrt{-ab}\right)^{-\frac{-2a^2b+\sqrt{4a^2b(ad-bc)\sqrt{-ab}+4a^4b^2-a^3b\,d^2+2d\,b^2c\,a^2-b^3c^2a}}{4a^2b}} \operatorname{HeunC}\left(\frac{2\sqrt{-\frac{b}{a}}\,c}{a},\right.$$

$$-\frac{\sqrt{4a^2b\,(ad-bc)\,\sqrt{-ab}+4a^4b^2-a^3b\,d^2+2d\,b^2c\,a^2-b^3c^2a}}{2a^2b}, \frac{\sqrt{-ab\,(4\sqrt{-ab}\,a^2d-4\sqrt{-ab}\,abc-4a^3}}{2a^2b}$$

$$-\frac{d^2}{8ab}-\frac{cd}{4a^2}+\frac{3b\,c^2}{8a^3}, \frac{ax}{2\sqrt{-ab}}$$

$$\left.+\frac{1}{2}\right) \mathrm{e}^{\frac{i\pi\sqrt{4a^2b(ad-bc)\sqrt{-ab}+4a^4b^2-a^3b\,d^2+2d\,b^2c\,a^2-b^3c^2a}-i\pi\sqrt{-ab\left(4\sqrt{-ab}\,a^2d-4\sqrt{-ab}\,abc-4a^3b+a^2d^2-2abcd+b^2c^2\right)}-4\left(a^2\left(\frac{d}{\sqrt{b}\,\sqrt{a}}-\frac{\sqrt{b}\,c}{a^{\frac{3}{2}}}\right)a\right.}{8a^2b}}$$

✓ Solution by Mathematica

Time used: 0.165 (sec). Leaf size: 104

```
DSolve[(a*x^2+b)^2*y''[x]+(a*x^2+b)*(c*x^2+d)*y'[x]+2*(b*c-a*d)*x*y[x]==0,y[x],x,IncludeSing
```

$$y(x) \to \exp\left(\frac{\arctan\left(\frac{\sqrt{a}x}{\sqrt{b}}\right)(bc-ad)}{a^{3/2}\sqrt{b}} - \frac{cx}{a}\right)\left(\int_1^x \exp\left(\frac{(ad-bc)\arctan\left(\frac{\sqrt{a}K[1]}{\sqrt{b}}\right)}{a^{3/2}\sqrt{b}} + \frac{cK[1]}{a}\right)c_1dK[1] + c_2\right)$$

32.21 problem 230

Internal problem ID [11055]
Internal file name [`OUTPUT/10311_Wednesday_January_24_2024_10_07_03_PM_50427788/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 230.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{\left(x^2+a\right)^2 y''+b\,x^n\left(x^2+a\right) y'-\left(b\,x^{n+1}+a\right) y=0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve((x^2+a)^2*diff(y(x),x$2)+b*x^n*(x^2+a)*diff(y(x),x)-(b*x^(n+1)+a)*y(x)=0,y(x), singso
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(x^2+a)^2*y''[x]+b*x^n*(x^2+a)*y'[x]-(b*x^(n+1)+a)*y[x]==0,y[x],x,IncludeSingularSolu
```

Not solved

32.22 problem 231

Internal problem ID [11056]
Internal file name [`OUTPUT/10312_Wednesday_January_24_2024_10_07_04_PM_36399781/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\left(x^2+a\right)^2 y''+b\,x^n\left(x^2+a\right) y'-m\left(b\,x^{n+1}+(m-1)\,x^2+a\right) y=0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
```

✗ Solution by Maple

```
dsolve((x^2+a)^2*diff(y(x),x$2)+b*x^n*(x^2+a)*diff(y(x),x)-m*(b*x^(n+1)+(m-1)*x^2+a)*y(x)=0,
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(x^2+a)^2*y''[x]+b*x^n*(x^2+a)*y'[x]-m*(b*x^(n+1)+(m-1)*x^2+a)*y[x]==0,y[x],x,Include
```

Not solved

32.23 problem 232

Internal problem ID [11057]
Internal file name [`OUTPUT/10313_Wednesday_January_24_2024_10_07_04_PM_71850249/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 232.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-a)^2 (x-b)^2 y'' - yc = 0}$$

32.23.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' - \frac{cy}{x^2} = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= \sqrt{-c} \\
n &= \frac{1}{2} \\
\gamma &= -1
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{-c}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{-c}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{-c}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{-c}}{x}}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{-c}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{-c}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{-c}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{-c}}{x}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{-c}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{-c}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{-c}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{-c}}{x}}}
$$

Verified OK.

32.23.2 Solving using Kovacic algorithm

Writing the ode as

$$
y''(a-x)^2 (-x+b)^2 - yc = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= (a-x)^2 (-x+b)^2 \\
B &= 0 \\
C &= -c
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{c}{(ab - ax - bx + x^2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= c \\ t &= (ab - ax - bx + x^2)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{c}{(ab - ax - bx + x^2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 216: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 0\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (ab - ax - bx + x^2)^2$. There is a pole at $x = a$ of order 2. There is a pole at $x = b$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{c}{(a-b)^2 (x-b)^2} + \frac{c}{(a-b)^2 (x-a)^2} + \frac{2c}{(a-b)^3 (x-b)} - \frac{2c}{(a-b)^3 (x-a)}$$

For the pole at $x = a$ let b be the coefficient of $\frac{1}{(x-a)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{c}{(a-b)^2}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{a^2 - 2ab + b^2 + 4c}}{2a - 2b}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{a^2 - 2ab + b^2 + 4c}}{2(a-b)}\end{aligned}$$

For the pole at $x = b$ let b be the coefficient of $\frac{1}{(x-b)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{c}{(a-b)^2}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{a^2 - 2ab + b^2 + 4c}}{2a - 2b}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{a^2 - 2ab + b^2 + 4c}}{2(a-b)}\end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= 0\\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{c}{\left(ab - ax - bx + x^2\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
a	2	0	$\frac{1}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}$	$\frac{1}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2(a-b)}$
b	2	0	$\frac{1}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}$	$\frac{1}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2(a-b)}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right)\\
&= 1 - (1)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\omega = \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty}$$
$$= \frac{\frac{1}{2} - \frac{\sqrt{a^2-2ab+b^2+4c}}{2(a-b)}}{x-a} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}}{x-b} + (-)(0)$$
$$= \frac{\frac{1}{2} - \frac{\sqrt{a^2-2ab+b^2+4c}}{2(a-b)}}{x-a} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}}{x-b}$$
$$= \frac{-a - \sqrt{a^2 - 2ab + b^2 + 4c} + 2x - b}{2(x-a)(x-b)}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{a^2-2ab+b^2+4c}}{2(a-b)}}{x-a} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}}{x-b}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{a^2-2ab+b^2+4c}}{2(a-b)}}{(x-a)^2} - \frac{\frac{1}{2} + \frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}}{(x-b)^2}\right) + \right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$z_1(x) = pe^{\int \omega \, dx}$$
$$= e^{\int \left(\frac{\frac{1}{2} - \frac{\sqrt{a^2-2ab+b^2+4c}}{2(a-b)}}{x-a} + \frac{\frac{1}{2} + \frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}}{x-b}\right) dx}$$
$$= e^{\frac{(-\ln(x-a)+\ln(x-b))\sqrt{a^2-2ab+b^2+4c}+(\ln(x-a)+\ln(x-b))(a-b)}{2a-2b}}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$y_1 = z_1$$
$$= \sqrt{(a-x)(-x+b)} \left(\frac{-x+b}{a-x}\right)^{\frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}}$$

Which simplifies to

$$y_1 = \sqrt{(a-x)(-x+b)}\left(\frac{-x+b}{a-x}\right)^{\frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \sqrt{(a-x)(-x+b)}\left(\frac{-x+b}{a-x}\right)^{\frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}} \int \frac{1}{(a-x)(-x+b)\left(\frac{-x+b}{a-x}\right)^{\frac{2\sqrt{a^2-2ab+b^2+4c}}{2a-2b}}}\, dx \\
&= \sqrt{(a-x)(-x+b)}\left(\frac{-x+b}{a-x}\right)^{\frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}} \left(\frac{\left(\frac{-x+b}{a-x}\right)^{-\frac{\sqrt{a^2-2ab+b^2+4c}}{a-b}}}{\sqrt{a^2-2ab+b^2+4c}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{(a-x)(-x+b)}\left(\frac{-x+b}{a-x}\right)^{\frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}}\right) \\
&\quad + c_2\left(\sqrt{(a-x)(-x+b)}\left(\frac{-x+b}{a-x}\right)^{\frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}} \left(\frac{\left(\frac{-x+b}{a-x}\right)^{-\frac{\sqrt{a^2-2ab+b^2+4c}}{a-b}}}{\sqrt{a^2-2ab+b^2+4c}}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1\sqrt{(a-x)(-x+b)}\left(\frac{-x+b}{a-x}\right)^{\frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}} \\
&\quad + \frac{c_2\sqrt{(a-x)(-x+b)}\left(\frac{-x+b}{a-x}\right)^{-\frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}}}{\sqrt{a^2-2ab+b^2+4c}}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{(a-x)(-x+b)} \left(\frac{-x+b}{a-x}\right)^{\frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}} + \frac{c_2 \sqrt{(a-x)(-x+b)} \left(\frac{-x+b}{a-x}\right)^{-\frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}}}{\sqrt{a^2-2ab+b^2+4c}}$$

Verified OK.

32.23.3 Maple step by step solution

Let's solve

$y''(a-x)^2(-x+b)^2 - yc = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{cy}{(a-x)^2(-x+b)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{cy}{(a-x)^2(-x+b)^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{c}{(a-x)^2(-x+b)^2}\right]$

  - $(x-a) \cdot P_2(x)$ is analytic at $x = a$

    $\left((x-a) \cdot P_2(x)\right)\Big|_{x=a} = 0$

  - $(x-a)^2 \cdot P_3(x)$ is analytic at $x = a$

    $\left((x-a)^2 \cdot P_3(x)\right)\Big|_{x=a} = -\frac{c}{(b-a)^2}$

  - $x = a$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = a$

- Multiply by denominators

  $y''(a-x)^2(-x+b)^2 - yc = 0$

- Change variables using $x=u+a$ so that the regular singular point is at $u=0$

$$\left(a^{2}u^{2}-2ab\,u^{2}+2a\,u^{3}+b^{2}u^{2}-2b\,u^{3}+u^{4}\right)\left(\frac{d^{2}}{du^{2}}y(u)\right)-cy(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=2..4$

$$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_{0}(a^{2}r^{2}-2ab\,r^{2}+b^{2}r^{2}-a^{2}r+2abr-b^{2}r-c)\,u^{r}+\left((a^{2}r^{2}-2ab\,r^{2}+b^{2}r^{2}+a^{2}r-2abr+b^{2}r-$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$a^{2}r^{2}-2ab\,r^{2}+b^{2}r^{2}-a^{2}r+2abr-b^{2}r-c=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{\frac{-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^{2}-2ab+b^{2}+4c}}{2}}{a-b},\frac{-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^{2}-2ab+b^{2}+4c}}{2}}{a-b}\right\}$$

- Each term must be 0

$$(a^{2}r^{2}-2ab\,r^{2}+b^{2}r^{2}+a^{2}r-2abr+b^{2}r-c)\,a_{1}+2a_{0}r(-1+r)\,(a-b)=0$$

- Solve for the dependent coefficient(s)

$$a_{1}=-\frac{2a_{0}r(ra-rb-a+b)}{a^{2}r^{2}-2ab\,r^{2}+b^{2}r^{2}+a^{2}r-2abr+b^{2}r-c}$$

- Each term in the series must be 0, giving the recursion relation

$$\left((a-b)^{2}a_{k}+2aa_{k-1}-2ba_{k-1}+a_{k-2}\right)k^{2}+\left(\left(2(a-b)^{2}a_{k}+4aa_{k-1}-4ba_{k-1}+2a_{k-2}\right)r-(a-$$

- Shift index using $k->k+2$

$$\left((a-b)^{2}a_{k+2}+2aa_{k+1}-2ba_{k+1}+a_{k}\right)(k+2)^{2}+\left(\left(2(a-b)^{2}a_{k+2}+4aa_{k+1}-4ba_{k+1}+2a_{k}\right)r-$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2a\,k^{2}a_{k+1}+4akra_{k+1}+2a\,r^{2}a_{k+1}-2b\,k^{2}a_{k+1}-4bkra_{k+1}-2b\,r^{2}a_{k+1}+2aka_{k+1}+2ara_{k+1}-2bka_{k+1}-2bra_{k+1}+k^{2}a_{k}+}{a^{2}k^{2}+2a^{2}kr+a^{2}r^{2}-2ab\,k^{2}-4abkr-2ab\,r^{2}+b^{2}k^{2}+2b^{2}kr+b^{2}r^{2}+3a^{2}k+3a^{2}r-6abk-6abr+3b^{2}k+3b^{2}r+2a^{2}-}$$

- Recursion relation for $r = \frac{-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2}}{a-b}$

$$a_{k+2} = -\frac{2a\,k^2 a_{k+1}+\frac{4ak\left(-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)a_{k+1}}{a-b}+\frac{2a\left(-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)^2 a_{k+1}}{(a-b)^2}-2b\,k^2 a_{k+1}-\frac{4bk\left(-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-\cdots}}{}\right.}{a-\cdots}}{a^2k^2+\frac{2a^2k\left(-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)}{a-b}+\frac{a^2\left(-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)^2}{(a-b)^2}-2ab\,k^2-\frac{4abk\left(-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2\cdots}}{2}\right.}{a-b}}$$

- Solution for $r = \frac{-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2}}{a-b}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2}}{a-b}}, a_{k+2} = -\frac{2a\,k^2 a_{k+1}+\frac{4ak\left(-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)a_{k+1}}{a-b}+\frac{2a\left(-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-\cdots}}{}\right.}{(a\cdots}}{a^2k^2+\frac{2a^2k\left(-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)}{a-b}+\frac{a^2\left(-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a\cdots}}{}\right.}{(a-\cdots}}\right.$$

- Revert the change of variables $u = x-a$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-a)^{k+\frac{-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2}}{a-b}}, a_{k+2} = -\frac{2a\,k^2 a_{k+1}+\frac{4ak\left(-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)a_{k+1}}{a-b}+\frac{2a\left(-\frac{b}{2}+\frac{a}{2}-\cdots\right.}{}}{a^2k^2+\frac{2a^2k\left(-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)}{a-b}+\frac{a^2\left(-\frac{b}{2}+\frac{a}{2}\cdots\right.}{}}\right.$$

- Recursion relation for $r = \frac{-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2}}{a-b}$

$$a_{k+2} = -\frac{2a\,k^2 a_{k+1}+\frac{4ak\left(-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)a_{k+1}}{a-b}+\frac{2a\left(-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)^2 a_{k+1}}{(a-b)^2}-2b\,k^2 a_{k+1}-\frac{4bk\left(-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-\cdots}}{}\right.}{a-\cdots}}{a^2k^2+\frac{2a^2k\left(-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)}{a-b}+\frac{a^2\left(-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)^2}{(a-b)^2}-2ab\,k^2-\frac{4abk\left(-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2\cdots}}{2}\right.}{a-b}}$$

- Solution for $r = \frac{-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2}}{a-b}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2}}{a-b}}, a_{k+2} = -\frac{2a\,k^2 a_{k+1}+\frac{4ak\left(-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)a_{k+1}}{a-b}+\frac{2a\left(-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-\cdots}}{}\right.}{(a\cdots}}{a^2k^2+\frac{2a^2k\left(-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)}{a-b}+\frac{a^2\left(-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a\cdots}}{}\right.}{(a-\cdots}}\right.$$

- Revert the change of variables $u = x-a$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-a)^{k+\frac{-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2}}{a-b}}, a_{k+2} = -\frac{2a\,k^2 a_{k+1}+\frac{4ak\left(-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)a_{k+1}}{a-b}+\frac{2a\left(-\frac{b}{2}+\frac{a}{2}+\cdots\right.}{}}{a^2k^2+\frac{2a^2k\left(-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2}\right)}{a-b}+\frac{a^2\left(-\frac{b}{2}+\frac{a}{2}\cdots\right.}{}}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} d_k (x-a)^{k+\frac{-\frac{b}{2}+\frac{a}{2}-\frac{\sqrt{a^2-2ab+b^2+4c}}{2}}{a-b}}\right) + \left(\sum_{k=0}^{\infty} e_k (x-a)^{k+\frac{-\frac{b}{2}+\frac{a}{2}+\frac{\sqrt{a^2-2ab+b^2+4c}}{2}}{a-b}}\right), d_{k+2} = -\frac{2a}{}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 104

```
dsolve((x-a)^2*(x-b)^2*diff(y(x),x$2)-c*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{(a-x)(b-x)}\left(\left(\frac{a-x}{b-x}\right)^{\frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}} c_1 + \left(\frac{a-x}{b-x}\right)^{-\frac{\sqrt{a^2-2ab+b^2+4c}}{2a-2b}} c_2\right)$$

✓ Solution by Mathematica

Time used: 1.085 (sec). Leaf size: 141

```
DSolve[(x-a)^2*(x-b)^2*y''[x]-c*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x-a)^{\frac{1}{2}\left(1-\sqrt{\frac{4c}{(a-b)^2}+1}\right)}(x-b)^{\frac{1}{2}\left(1-\sqrt{\frac{4c}{(a-b)^2}+1}\right)}\left(c_1(x-a)^{\sqrt{\frac{4c}{(a-b)^2}+1}} - \frac{c_2(x-b)^{\sqrt{\frac{4c}{(a-b)^2}+1}}}{(a-b)\sqrt{\frac{4c}{(a-b)^2}+1}}\right)$$

32.24 problem 233

Internal problem ID [11058]
Internal file name [`OUTPUT/10314_Wednesday_January_24_2024_10_07_05_PM_75631971/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 233.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__change__of__variable__on__x__method__2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-a)^2 (x-b)^2 y'' + (x-a)(x-b)(2x+\lambda) y' + \mu y = 0}$$

32.24.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$y''(a-x)^2 (-x+b)^2 + (2x+\lambda) y'(a-x)(-x+b) + \mu y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2x+\lambda}{(a-x)(-x+b)} \\ q(x) &= \frac{\mu}{(a-x)^2 (-x+b)^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\
&= \int e^{-\left(\int \frac{2x+\lambda}{(a-x)(-x+b)}dx\right)}dx \\
&= \int e^{\frac{(-2a-\lambda)\ln(x-a)+2\left(b+\frac{\lambda}{2}\right)\ln(x-b)}{a-b}}dx \\
&= \int (x-a)^{\frac{-2a-\lambda}{a-b}}(x-b)^{\frac{2b+\lambda}{a-b}}dx \\
&= -\frac{(x-b)^{1+\frac{2b+\lambda}{a-b}}(x-a)^{1-\frac{2a+\lambda}{a-b}}}{a+b+\lambda}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{\mu}{(a-x)^2(-x+b)^2}}{(x-b)^{\frac{4b+2\lambda}{a-b}}(x-a)^{\frac{-4a-2\lambda}{a-b}}} \\
&= \mu(x-a)^{\frac{2a+2b+2\lambda}{a-b}}(x-b)^{\frac{-2a-2b-2\lambda}{a-b}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) + \mu(x-a)^{\frac{2a+2b+2\lambda}{a-b}}(x-b)^{\frac{-2a-2b-2\lambda}{a-b}}y(\tau) = 0$$

But in terms of τ

$$\mu(x-a)^{\frac{2a+2b+2\lambda}{a-b}} (x-b)^{\frac{-2a-2b-2\lambda}{a-b}} = \frac{\mu}{(a+b+\lambda)^2 \tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{\mu y(\tau)}{(a+b+\lambda)^2 \tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)(a+b+\lambda)^2 \tau^2 + \mu y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$(a+b+\lambda)^2 \tau^2 (r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \mu\tau^r = 0$$

Simplifying gives

$$(a+b+\lambda)^2 r(r-1)\tau^r + 0\,\tau^r + \mu\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$(a+b+\lambda)^2 r(r-1) + 0 + \mu = 0$$

Or

$$(a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2) r^2 + (-a^2-2ab-2a\lambda-b^2-2b\lambda-\lambda^2) r + \mu = 0 \quad (1)$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{-a-b-\lambda+\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2(a+b+\lambda)}$$
$$r_2 = \frac{a+b+\lambda+\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2a+2b+2\lambda}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{-\frac{-a-b-\lambda+\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2(a+b+\lambda)}} + c_2 \tau^{\frac{a+b+\lambda+\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2a+2b+2\lambda}}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(-\frac{(x-b)^{\frac{a+b+\lambda}{a-b}} (x-a)^{\frac{-a-b-\lambda}{a-b}}}{a+b+\lambda}\right)^{\frac{a+b+\lambda-\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}{2a+2b+2\lambda}} + c_2 \left(-\frac{(x-b)^{\frac{a+b+\lambda}{a-b}} (x-a)^{\frac{-a-b-\lambda}{a-b}}}{a+b+\lambda}\right)^{a}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & c_1 \left(-\frac{(x-b)^{\frac{a+b+\lambda}{a-b}} (x-a)^{\frac{-a-b-\lambda}{a-b}}}{a+b+\lambda}\right)^{\frac{a+b+\lambda-\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}{2a+2b+2\lambda}} \\ & + c_2 \left(-\frac{(x-b)^{\frac{a+b+\lambda}{a-b}} (x-a)^{\frac{-a-b-\lambda}{a-b}}}{a+b+\lambda}\right)^{\frac{a+b+\lambda+\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}{2a+2b+2\lambda}} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = {} & c_1 \left(-\frac{(x-b)^{\frac{a+b+\lambda}{a-b}} (x-a)^{\frac{-a-b-\lambda}{a-b}}}{a+b+\lambda}\right)^{\frac{a+b+\lambda-\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}{2a+2b+2\lambda}} \\ & + c_2 \left(-\frac{(x-b)^{\frac{a+b+\lambda}{a-b}} (x-a)^{\frac{-a-b-\lambda}{a-b}}}{a+b+\lambda}\right)^{\frac{a+b+\lambda+\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}{2a+2b+2\lambda}} \end{aligned}$$

Verified OK.

32.24.2 Solving using Kovacic algorithm

Writing the ode as

$$y''(a-x)^2 (-x+b)^2 + (2x+\lambda) y'(a-x)(-x+b) + \mu y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (a-x)^2 (-x+b)^2 \\ B &= (2x+\lambda)(a-x)(-x+b) \\ C &= \mu \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4ab + 2a\lambda + 2b\lambda + \lambda^2 - 4\mu}{4\,(ab - ax - bx + x^2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4ab + 2a\lambda + 2b\lambda + \lambda^2 - 4\mu \\ t &= 4\big(ab - ax - bx + x^2\big)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4ab + 2a\lambda + 2b\lambda + \lambda^2 - 4\mu}{4\,(ab - ax - bx + x^2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 218: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(ab - ax - bx + x^2)^2$. There is a pole at $x = a$ of order 2. There is a pole at $x = b$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned} r = &\frac{4ab + 2a\lambda + 2b\lambda + \lambda^2 - 4\mu}{4 (a - b)^2 (x - a)^2} - \frac{4ab + 2a\lambda + 2b\lambda + \lambda^2 - 4\mu}{2 (a - b)^3 (x - a)} \\ &+ \frac{4ab + 2a\lambda + 2b\lambda + \lambda^2 - 4\mu}{4 (a - b)^2 (x - b)^2} - \frac{-4ab - 2a\lambda - 2b\lambda - \lambda^2 + 4\mu}{2 (a - b)^3 (x - b)} \end{aligned}$$

For the pole at $x = a$ let b be the coefficient of $\frac{1}{(x-a)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{\lambda^2+(2b+2a)\lambda+4ab-4\mu}{4(a-b)^2}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{a^2 + 2ab + 2a\lambda + b^2 + 2b\lambda + \lambda^2 - 4\mu}}{2a - 2b} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{a^2 + 2ab + 2a\lambda + b^2 + 2b\lambda + \lambda^2 - 4\mu}}{2 (a - b)} \end{aligned}$$

For the pole at $x = b$ let b be the coefficient of $\frac{1}{(x-b)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{\lambda^2+(2b+2a)\lambda+4ab-4\mu}{4(a-b)^2}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{a^2 + 2ab + 2a\lambda + b^2 + 2b\lambda + \lambda^2 - 4\mu}}{2a - 2b} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{a^2 + 2ab + 2a\lambda + b^2 + 2b\lambda + \lambda^2 - 4\mu}}{2 (a - b)} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4ab + 2a\lambda + 2b\lambda + \lambda^2 - 4\mu}{4\left(ab - ax - bx + x^2\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
a	2	0	$\frac{1}{2} + \frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2a-2b}$	$\frac{1}{2} - \frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2(a-b)}$
b	2	0	$\frac{1}{2} + \frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2a-2b}$	$\frac{1}{2} - \frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2(a-b)}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{1}{2}-\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2(a-b)}}{x-a} + \frac{\frac{1}{2}+\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2a-2b}}{x-b} + (-)(0) \\
&= \frac{\frac{1}{2}-\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2(a-b)}}{x-a} + \frac{\frac{1}{2}+\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2a-2b}}{x-b} \\
&= \frac{-a-\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}+2x-b}{2(x-a)(x-b)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{\frac{1}{2}-\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2(a-b)}}{x-a} + \frac{\frac{1}{2}+\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2a-2b}}{x-b}\right)(0) + \left(\left(-\frac{\frac{1}{2}-\frac{\sqrt{a^2+2ab+2a\lambda+b^2+}}{2(a-b)}}{(x-a)^2}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{\frac{1}{2}-\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2(a-b)}}{x-a} + \frac{\frac{1}{2}+\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2a-2b}}{x-b}\right) dx} \\
&= \mathrm{e}^{\frac{(-\ln(x-a)+\ln(x-b))\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}+(\ln(x-a)+\ln(x-b))(a-b)}{2a-2b}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{(2x+\lambda)(a-x)(-x+b)}{(a-x)^2(-x+b)^2}\, dx} \\
&= z_1 e^{-\frac{(-2b-\lambda)\ln(x-b)}{2(a-b)} - \frac{(2a+\lambda)\ln(x-a)}{2(a-b)}} \\
&= z_1 \left((x-b)^{\frac{2b+\lambda}{2a-2b}} (x-a)^{\frac{-2a-\lambda}{2a-2b}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \sqrt{(a-x)(-x+b)}\left(\frac{-x+b}{a-x}\right)^{\frac{\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}{2a-2b}} (x-b)^{\frac{2b+\lambda}{2a-2b}} (x-a)^{\frac{-2a-\lambda}{2a-2b}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{(2x+\lambda)(a-x)(-x+b)}{(a-x)^2(-x+b)^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{(-2a-\lambda)\ln(x-a)+2\left(b+\frac{\lambda}{2}\right)\ln(x-b)}{a-b}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\left(\frac{-x+b}{a-x}\right)^{-\frac{\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}{a-b}}}{\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{(a-x)(-x+b)}\left(\frac{-x+b}{a-x}\right)^{\frac{\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}{2a-2b}} (x-b)^{\frac{2b+\lambda}{2a-2b}} (x-a)^{\frac{-2a-\lambda}{2a-2b}}\right) \\
&\quad + c_2 \left(\sqrt{(a-x)(-x+b)}\left(\frac{-x+b}{a-x}\right)^{\frac{\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}{2a-2b}} (x-b)^{\frac{2b+\lambda}{2a-2b}} (x-a)^{\frac{-2a-\lambda}{2a-2b}} \left(\frac{\left(\frac{-x+b}{a-x}\right)^{-\frac{\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}{a-b}}}{\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \sqrt{(a-x)(-x+b)}\left(\frac{-x+b}{a-x}\right)^{\frac{\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}{2a-2b}} (x-b)^{\frac{2b+\lambda}{2a-2b}} (x-a)^{\frac{-2a-\lambda}{2a-2b}} + \frac{c_2 \sqrt{(a-x)(-x+b)}\,(x-a)^{\frac{-2a-\lambda}{2a-2b}} (x-b)^{\frac{2b+\lambda}{2a-2b}} \left(\frac{-x+b}{a-x}\right)^{-\frac{\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}{2a-2b}}}{\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}} \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{(a-x)(-x+b)}\left(\frac{-x+b}{a-x}\right)^{\frac{\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}{2a-2b}} (x-b)^{\frac{2b+\lambda}{2a-2b}} (x-a)^{\frac{-2a-\lambda}{2a-2b}} + \frac{c_2 \sqrt{(a-x)(-x+b)}\,(x-a)^{\frac{-2a-\lambda}{2a-2b}} (x-b)^{\frac{2b+\lambda}{2a-2b}} \left(\frac{-x+b}{a-x}\right)^{-\frac{\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}{2a-2b}}}{\sqrt{\lambda^2+(2b+2a)\lambda+a^2+2ab+b^2-4\mu}}$$

Verified OK.

32.24.3 Maple step by step solution

Let's solve

$y''(a-x)^2(-x+b)^2 + (2x+\lambda)\,y'(a-x)(-x+b) + \mu y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2x+\lambda)y'}{(a-x)(-x+b)} - \frac{\mu y}{(a-x)^2(-x+b)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x+\lambda)y'}{(a-x)(-x+b)} + \frac{\mu y}{(a-x)^2(-x+b)^2} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x+\lambda}{(a-x)(-x+b)}, P_3(x) = \frac{\mu}{(a-x)^2(-x+b)^2}\right]$

  - $(x-a)\cdot P_2(x)$ is analytic at $x=a$

    $\left((x-a)\cdot P_2(x)\right)\Big|_{x=a} = -\frac{2a+\lambda}{b-a}$

- $(x-a)^2 \cdot P_3(x)$ is analytic at $x=a$

  $\left((x-a)^2 \cdot P_3(x)\right)\Big|_{x=a} = \frac{\mu}{(b-a)^2}$
- $x=a$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = a$

- Multiply by denominators

  $y''(a-x)^2(-x+b)^2 + (2x+\lambda)y'(a-x)(-x+b) + \mu y = 0$
- Change variables using $x=u+a$ so that the regular singular point is at $u=0$

  $(a^2u^2 - 2ab\,u^2 + 2a\,u^3 + b^2u^2 - 2b\,u^3 + u^4)\left(\frac{d^2}{du^2}y(u)\right) + (2a^2u - 2abu + a\lambda u + 4a\,u^2 - b\lambda u - 2b$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..4$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0(a^2r^2 - 2ab\,r^2 + b^2r^2 + a^2r + a\lambda r - b^2r - b\lambda r + \mu)\,u^r + ((a^2r^2 - 2ab\,r^2 + b^2r^2 + 3a^2r - 4abr$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $a^2r^2 - 2ab\,r^2 + b^2r^2 + a^2r + a\lambda r - b^2r - b\lambda r + \mu = 0$

- Values of r that satisfy the indicial equation

$$r \in \left\{ \frac{-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}-\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}}{a-b}, \frac{-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}+\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}}{a-b} \right\}$$

- Each term must be 0

$$(a^2r^2-2ab\,r^2+b^2r^2+3a^2r-4abr+a\lambda r+b^2r-b\lambda r+2a^2-2ab+a\lambda-b\lambda+\mu)\,a_1+a_0r(2ar$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{a_0r(2ar-2rb+2a+\lambda)}{a^2r^2-2ab\,r^2+b^2r^2+3a^2r-4abr+a\lambda r+b^2r-b\lambda r+2a^2-2ab+a\lambda-b\lambda+\mu}$$

- Each term in the series must be 0, giving the recursion relation

$$\left((a-b)^2 a_k+2aa_{k-1}-2ba_{k-1}+a_{k-2}\right)k^2+\left(\left(2(a-b)^2 a_k+4aa_{k-1}-4ba_{k-1}+2a_{k-2}\right)r+(a-$$

- Shift index using $k->k+2$

$$\left((a-b)^2 a_{k+2}+2aa_{k+1}-2ba_{k+1}+a_k\right)(k+2)^2+\left(\left(2(a-b)^2 a_{k+2}+4aa_{k+1}-4ba_{k+1}+2a_k\right)r-$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2a\,k^2a_{k+1}+4akra_{k+1}+2a\,r^2a_{k+1}-2b\,k^2a_{k+1}-4bkra_{k+1}-2b\,r^2a_{k+1}+6aka_{k+1}+6ara_{k+1}-4bka_{k+1}-4bra_{k+1}+k^2a_k+}{a^2k^2+2a^2kr+a^2r^2-2ab\,k^2-4abkr-2ab\,r^2+b^2k^2+2b^2kr+b^2r^2+5a^2k+5a^2r-8abk-8abr+ak\lambda+a\lambda r+}$$

- Recursion relation for $r = \frac{-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}-\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}}{a-b}$

$$a_{k+2} = -\frac{2a\,k^2a_{k+1}+\frac{4ak\left(-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}-\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}\right)a_{k+1}}{a-b}+\frac{2a\left(-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}-\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}\right.}{(a-b)^2}}{\mu+2b^2+6a^2+2a\lambda-2b\lambda-8ab-2ab\,k^2+ak\lambda-bk\lambda-8abk+\frac{a^2\left(-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}-\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}\right)^2}{(a-b)^2}+\frac{b^2\left(-\frac{a}{2}-\frac{b}{2}-\right.}{}}$$

- Solution for $r = \frac{-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}-\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}}{a-b}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}-\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}}{a-b}}, a_{k+2} = -\frac{2a\,k^2a_{k+1}+\frac{4ak\left(-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}-\frac{\sqrt{a^2+2ab+2a\lambda+\right.}{2}}{a-b}}{\mu+2b^2+6a^2+2a\lambda-2b\lambda-8ab-2ab\,k^2+ak\lambda-bk\lambda-8}\right.$$

- Revert the change of variables $u = x-a$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-a)^{k+\frac{-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}-\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}}{a-b}}, a_{k+2} = -\frac{2a\,k^2 a_{k+1} + \frac{4ak\left(-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}-\frac{\sqrt{a^2+2ab+2}}{}\right.}{a-}}{\mu+2b^2+6a^2+2a\lambda-2b\lambda-8ab-2ab\,k^2+ak\lambda-bk}\right.$$

- Recursion relation for $r = \frac{-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}+\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}}{a-b}$

$$a_{k+2} = -\frac{2a\,k^2 a_{k+1} + \frac{4ak\left(-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}+\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}\right)a_{k+1}}{a-b} + \frac{2a\left(-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}+\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}\right.}{(a-b)^2}}{\mu+2b^2+6a^2+2a\lambda-2b\lambda-8ab-2ab\,k^2+ak\lambda-bk\lambda-8abk+\frac{a^2\left(-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}+\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}\right)^2}{(a-b)^2}+\frac{b^2\left(-\frac{a}{2}-\frac{b}{2}-\right.}{}}$$

- Solution for $r = \frac{-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}+\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}}{a-b}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}+\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}}{a-b}}, a_{k+2} = -\frac{2a\,k^2 a_{k+1} + \frac{4ak\left(-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}+\frac{\sqrt{a^2+2ab+2a\lambda+}}{2}\right.}{a-b}}{\mu+2b^2+6a^2+2a\lambda-2b\lambda-8ab-2ab\,k^2+ak\lambda-bk\lambda-8}\right.$$

- Revert the change of variables $u = x - a$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-a)^{k+\frac{-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}+\frac{\sqrt{a^2+2ab+2a\lambda+b^2+2b\lambda+\lambda^2-4\mu}}{2}}{a-b}}, a_{k+2} = -\frac{2a\,k^2 a_{k+1} + \frac{4ak\left(-\frac{a}{2}-\frac{b}{2}-\frac{\lambda}{2}+\frac{\sqrt{a^2+2ab+2}}{}\right.}{a-}}{\mu+2b^2+6a^2+2a\lambda-2b\lambda-8ab-2ab\,k^2+ak\lambda-bk}\right.$$

- Combine solutions and rename parameters

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 143

```
dsolve((x-a)^2*(x-b)^2*diff(y(x),x$2)+(x-a)*(x-b)*(2*x+lambda)*diff(y(x),x)+mu*y(x)=0,y(x),
```

$$y(x) = \left(\left(\frac{a-x}{b-x}\right)^{-\frac{\sqrt{\lambda^2+(2a+2b)\lambda+a^2+2ab+b^2-4\mu}}{2a-2b}} c_2 + \left(\frac{a-x}{b-x}\right)^{\frac{\sqrt{\lambda^2+(2a+2b)\lambda+a^2+2ab+b^2-4\mu}}{2a-2b}} c_1\right)\left(\frac{b-x}{a-x}\right)^{\frac{a+b+\lambda}{2a-2b}}$$

✓ Solution by Mathematica

Time used: 2.299 (sec). Leaf size: 152

```
DSolve[(x-a)^2*(x-b)^2*y''[x]+(x-a)*(x-b)*(2*x+\[Lambda])*y'[x]+mu*y[x]==0,y[x],x,IncludeSin
```

$$y(x) \to e^{-\frac{(a+b+\lambda)(\log(x-a)-\log(x-b))}{a-b}}\left(c_1\exp\left(\frac{\left(\sqrt{\mu}\sqrt{\frac{(a+b+\lambda)^2}{\mu}-4}+a+b+\lambda\right)(\log(x-a)-\log(x-b))}{2(a-b)}\right) + c_2\exp\left(\frac{\left(-\sqrt{\mu}\sqrt{\frac{(a+b+\lambda)^2}{\mu}-4}+a+b+\lambda\right)(\log(x-a)-\log(x-b))}{2(a-b)}\right)\right)$$

32.25 problem 234

Internal problem ID [11059]
Internal file name [`OUTPUT/10315_Wednesday_January_24_2024_10_07_07_PM_94785127/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 234.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{\left(a\,x^2+bx+c\right)^2 y''+Ay=0}$$

32.25.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y''+\frac{Ay}{x^2}=0 \tag{1}$$

Bessel ode has the form

$$x^2y''+y'x+\left(-n^2+x^2\right)y=0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y''+(1-2\alpha)\,xy'+\left(\beta^2\gamma^2x^{2\gamma}-n^2\gamma^2+\alpha^2\right)y=0 \tag{3}$$

With the standard solution

$$y=x^{\alpha}(c_1\,\mathrm{BesselJ}\left(n,\beta\,x^{\gamma}\right)+c_2\,\mathrm{BesselY}\left(n,\beta\,x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= \frac{1}{2}\\ \beta &= \sqrt{A}\\ n &= \frac{1}{2}\\ \gamma &= -1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{A}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{A}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{A}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{A}}{x}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{A}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{A}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{A}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{A}}{x}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \sqrt{x}\, \sqrt{2}\, \sin\left(\frac{\sqrt{A}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{A}}{x}}} - \frac{c_2 \sqrt{x}\, \sqrt{2}\, \cos\left(\frac{\sqrt{A}}{x}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{A}}{x}}}$$

Verified OK.

32.25.2 Solving using Kovacic algorithm

Writing the ode as

$$\left(a\, x^2 + bx + c\right)^2 y'' + Ay = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= \left(a\, x^2 + bx + c\right)^2\\ B &= 0\\ C &= A\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-A}{(a\,x^2 + bx + c)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -A \\ t &= (a\,x^2 + bx + c)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{A}{(a\,x^2 + bx + c)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 220: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = \left(a\,x^2 + bx + c\right)^2$. There is a pole at $x = -\frac{b-\sqrt{-4ac+b^2}}{2a}$ of order 2. There is a pole at $x = -\frac{b+\sqrt{-4ac+b^2}}{2a}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned} r = &-\frac{A}{\left(-4ac+b^2\right)\left(x - \frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^2} - \frac{A}{\left(-4ac+b^2\right)\left(x + \frac{b+\sqrt{-4ac+b^2}}{2a}\right)^2} \\ &+ \frac{2aA}{\left(-4ac+b^2\right)^{\frac{3}{2}}\left(x - \frac{-b+\sqrt{-4ac+b^2}}{2a}\right)} - \frac{2aA}{\left(-4ac+b^2\right)^{\frac{3}{2}}\left(x + \frac{b+\sqrt{-4ac+b^2}}{2a}\right)} \end{aligned}$$

For the pole at $x = -\frac{b-\sqrt{-4ac+b^2}}{2a}$ let b be the coefficient of $\frac{1}{\left(x+\frac{b-\sqrt{-4ac+b^2}}{2a}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{A}{4ac-b^2}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2} \end{aligned}$$

For the pole at $x = -\frac{b+\sqrt{-4ac+b^2}}{2a}$ let b be the coefficient of $\frac{1}{\left(x+\frac{b+\sqrt{-4ac+b^2}}{2a}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{A}{4ac-b^2}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{A}{(a\,x^2 + bx + c)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{b-\sqrt{-4ac+b^2}}{2a}$	2	0	$\frac{1}{2} + \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}$	$\frac{1}{2} - \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}$
$-\frac{b+\sqrt{-4ac+b^2}}{2a}$	2	0	$\frac{1}{2} + \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}$	$\frac{1}{2} - \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}}{x + \frac{b-\sqrt{-4ac+b^2}}{2a}} + \frac{\frac{1}{2} + \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}}{x + \frac{b+\sqrt{-4ac+b^2}}{2a}} + (-)(0) \\
&= \frac{\frac{1}{2} - \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}}{x + \frac{b-\sqrt{-4ac+b^2}}{2a}} + \frac{\frac{1}{2} + \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}}{x + \frac{b+\sqrt{-4ac+b^2}}{2a}} \\
&= \frac{2ax - \sqrt{-4ac+b^2}\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}} + b}{2a\,x^2 + 2bx + 2c}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}}{x + \frac{b-\sqrt{-4ac+b^2}}{2a}} + \frac{\frac{1}{2} + \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}}{x + \frac{b+\sqrt{-4ac+b^2}}{2a}} \right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}}{\left(x + \frac{b-\sqrt{-4ac+b^2}}{2a}\right)^2} - \frac{\frac{1}{2} + \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}}{\left(x + \frac{b+\sqrt{-4ac+b^2}}{2a}\right)^2} \right) + \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{\frac{1}{2} - \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}}{x + \frac{b-\sqrt{-4ac+b^2}}{2a}} + \frac{\frac{1}{2} + \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}}{x + \frac{b+\sqrt{-4ac+b^2}}{2a}} \right) dx} \\
&= \left(\frac{2ax + \sqrt{-4ac+b^2} + b}{a} \right)^{\frac{1}{2} + \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \sqrt{\frac{2ax + b - \sqrt{-4ac+b^2}}{a}} \left(\frac{2ax + b - \sqrt{-4ac+b^2}}{a} \right)^{-\frac{\sqrt{4a}}{}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \left(\frac{2ax + b - \sqrt{-4ac + b^2}}{a}\right)^{\frac{1}{2} - \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \left(\frac{2ax + \sqrt{-4ac + b^2} + b}{a}\right)^{\frac{1}{2} + \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \end{aligned}$$

Which simplifies to

$$y_1 = \left(\frac{2ax + b - \sqrt{-4ac + b^2}}{a}\right)^{\frac{1}{2} - \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \left(\frac{2ax + \sqrt{-4ac + b^2} + b}{a}\right)^{\frac{1}{2} + \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \left(\frac{2ax + b - \sqrt{-4ac + b^2}}{a}\right)^{\frac{1}{2} - \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \left(\frac{2ax + \sqrt{-4ac + b^2} + b}{a}\right)^{\frac{1}{2} + \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \int \frac{}{\left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{1}} \\ &= \left(\frac{2ax + b - \sqrt{-4ac + b^2}}{a}\right)^{\frac{1}{2} - \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \left(\frac{2ax + \sqrt{-4ac + b^2} + b}{a}\right)^{\frac{1}{2} + \frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \left(\int \left(\frac{2ax + b - \sqrt{-}}{a}\right.\right. \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{\frac{1}{2}-\frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{\frac{1}{2}+\frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \right) \\
&\quad + c_2 \left(\left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{\frac{1}{2}-\frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{\frac{1}{2}+\frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \left(\int \left(\frac{2ax+b-}{} \right.\right.\right.
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = c_1 &\left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{\frac{1}{2}-\frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{\frac{1}{2}+\frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \\
- c_2 &\left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{\frac{1}{2}+\frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \sqrt{\frac{2ax+b-\sqrt{-4ac+b^2}}{a}} \left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{-\frac{\sqrt{\frac{4ac-}{4a}}}{2}} \\
&\quad - \frac{\left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{-\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}} \left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}}{4a\left(a\,x^2+bx+c\right)} dx \Bigg)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 \left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{\frac{1}{2}-\frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{\frac{1}{2}+\frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \\
& - c_2 \left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{\frac{1}{2}+\frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \sqrt{\frac{2ax+b-\sqrt{-4ac+b^2}}{a}} \left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{-\frac{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}{2}} \left(\int \right. \\
& \left. -\frac{\left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{-\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}} \left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{\sqrt{\frac{4ac-b^2+4A}{4ac-b^2}}}}{4a\left(a\,x^2+bx+c\right)} dx\right)
\end{aligned}
$$

Verified OK.

32.25.3 Maple step by step solution

Let's solve

$\left(a\,x^2+bx+c\right)^2 y''+Ay=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{Ay}{\left(a\,x^2+bx+c\right)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{Ay}{\left(a\,x^2+bx+c\right)^2}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=0, P_3(x)=\frac{A}{\left(a\,x^2+bx+c\right)^2}\right]$

- ◦ $\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)\cdot P_2(x)$ is analytic at $x=\frac{-b+\sqrt{-4ac+b^2}}{2a}$

  $\left(\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{-b+\sqrt{-4ac+b^2}}{2a}}=0$

- ◦ $\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{-b+\sqrt{-4ac+b^2}}{2a}$

$$\left(\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^2\cdot P_3(x)\right)\Bigg|_{x=\frac{-b+\sqrt{-4ac+b^2}}{2a}}=0$$

- $x=\frac{-b+\sqrt{-4ac+b^2}}{2a}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=\frac{-b+\sqrt{-4ac+b^2}}{2a}$

- Multiply by denominators

  $(a\,x^2+bx+c)^2\,y''+Ay=0$

- Change variables using $x=u+\frac{-b+\sqrt{-4ac+b^2}}{2a}$ so that the regular singular point is at $u=0$

  $(a^2u^4+2a\,u^3\sqrt{-4ac+b^2}-4ac\,u^2+b^2u^2)\left(\frac{d^2}{du^2}y(u)\right)+Ay(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_ku^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=2..4$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-4ac\,r^2+b^2r^2+4acr-b^2r+A)\,u^r+((-4ac\,r^2+b^2r^2-4acr+b^2r+A)\,a_1+2a_0r(-1+r)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-4ac\,r^2+b^2r^2+4acr-b^2r+A=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{-4ac+b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}}{2(4ac-b^2)},\frac{4ac-b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}}{2(4ac-b^2)}\right\}$

- Each term must be 0

  $(-4ac\,r^2+b^2r^2-4acr+b^2r+A)\,a_1+2a_0r(-1+r)\,a\sqrt{-4ac+b^2}=0$

- Solve for the dependent coefficient(s)

  $a_1=-\frac{2a_0r(-1+r)a\sqrt{-4ac+b^2}}{-4ac\,r^2+b^2r^2-4acr+b^2r+A}$

- Each term in the series must be 0, giving the recursion relation

$$2a_{k-1}(k+r-1)(k-2+r)a\sqrt{-4ac+b^2}+a_{k-2}(k-2+r)(k-3+r)a^2-4ca_k(k+r)(k+r$$

- Shift index using $k->k+2$

$$2a_{k+1}(k+1+r)(k+r)a\sqrt{-4ac+b^2}+a_k(k+r)(k+r-1)a^2-4ca_{k+2}(k+2+r)(k+1+r$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a\left(2\sqrt{-4ac+b^2}\,k^2a_{k+1}+4\sqrt{-4ac+b^2}\,kra_{k+1}+2\sqrt{-4ac+b^2}\,r^2a_{k+1}+a\,k^2a_k+2akra_k+a\,r^2a_k+2\sqrt{-4ac+b^2}\,ka_{k+1}+2\sqrt{-}\right.}{-4ac\,k^2-8ackr-4ac\,r^2+b^2k^2+2b^2kr+b^2r^2-12ack-12acr+3b^2k+3b^2r-8ac+2b^2+A}$$

- Recursion relation for $r=-\frac{-4ac+b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}}{2(4ac-b^2)}$

$$a_{k+2}=-\frac{a\left(2\sqrt{-4ac+b^2}\,k^2a_{k+1}-\frac{2\sqrt{-4ac+b^2}\,k\left(-4ac+b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}\right)a_{k+1}}{4ac-b^2}+\frac{\sqrt{-4ac+b^2}\,\left(-4ac+b^2+\sqrt{16c^2a^2}\right.}{2(4ac}\right.}{-4ac\,k^2+\frac{4ack\left(-4ac+b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}\right)}{4ac-b^2}-\frac{ac\left(-4ac+b^2+\sqrt{16c^2a^2-8b^2ca}\right.}{\left(4ac-b^2\right)}}$$

- Solution for $r=-\frac{-4ac+b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}}{2(4ac-b^2)}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k-\frac{-4ac+b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}}{2\left(4ac-b^2\right)}},a_{k+2}=-\frac{a\left(2\sqrt{-4ac+b^2}\,k^2a_{k+1}-\frac{2\sqrt{-4ac+b^2}\,k\left(-4ac+b^2+\right.}{}\right.}{-4ac\,k^2+\frac{4ack\left(-4a\right.}{}}\right.$$

- Revert the change of variables $u=x-\frac{-b+\sqrt{-4ac+b^2}}{2a}$

$$\left[y=\sum_{k=0}^{\infty}a_k\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^{k-\frac{-4ac+b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}}{2\left(4ac-b^2\right)}},a_{k+2}=-\frac{a\left(2\sqrt{-4ac+b^2}\,k^2a_{k+1}-\frac{2\sqrt{-}}{}\right.}{}\right.$$

- Recursion relation for $r=\frac{4ac-b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}}{2(4ac-b^2)}$

$$a_{k+2}=-\frac{a\left(2\sqrt{-4ac+b^2}\,k^2a_{k+1}+\frac{2\sqrt{-4ac+b^2}\,k\left(4ac-b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}\right)a_{k+1}}{4ac-b^2}+\frac{\sqrt{-4ac+b^2}\,\left(4ac-b^2+\sqrt{16c^2a^2-8b}\right.}{2\left(4ac-\right.}\right.}{-4ac\,k^2-\frac{4ack\left(4ac-b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}\right)}{4ac-b^2}-\frac{ac\left(4ac-b^2+\sqrt{16c^2a^2-8b^2ca+b^4}\right.}{\left(4ac-b^2\right)^2}}$$

- Solution for $r=\frac{4ac-b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}}{2(4ac-b^2)}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{4ac-b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}}{2\left(4ac-b^2\right)}},a_{k+2}=-\frac{a\left(2\sqrt{-4ac+b^2}\,k^2a_{k+1}+\frac{2\sqrt{-4ac+b^2}\,k\left(4ac-b^2+\sqrt{16}\right.}{}\right.}{-4ac\,k^2-\frac{4ack\left(4ac-b\right.}{}}\right.$$

- Revert the change of variables $u=x-\frac{-b+\sqrt{-4ac+b^2}}{2a}$

$$\left[y=\sum_{k=0}^{\infty}a_k\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^{k+\frac{4ac-b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}}{2\left(4ac-b^2\right)}},a_{k+2}=-\frac{a\left(2\sqrt{-4ac+b^2}\,k^2a_{k+1}+\frac{2\sqrt{-4}}{}\right.}{}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} d_k \left(x - \frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^{k-\frac{-4ac+b^2+\sqrt{16c^2a^2-8b^2ca+b^4+16Aac-4A\,b^2}}{2\left(4ac-b^2\right)}}\right) + \left(\sum_{k=0}^{\infty} e_k \left(x - \frac{-b+\sqrt{-4ac+b^2}}{2a}\right.\right.\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 178

```
dsolve((a*x^2+b*x+c)^2*diff(y(x),x$2)+A*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\left(\frac{-b+i\sqrt{4ac-b^2}-2ax}{i\sqrt{4ac-b^2}+2ax+b}\right)^{-\frac{a\sqrt{\frac{-4ac+b^2-4A}{a^2}}}{2\sqrt{-4ac+b^2}}} c_2 + \left(\frac{-b+i\sqrt{4ac-b^2}-2ax}{i\sqrt{4ac-b^2}+2ax+b}\right)^{\frac{a\sqrt{\frac{-4ac+b^2-4A}{a^2}}}{2\sqrt{-4ac+b^2}}} c_1\right)\sqrt{a\,x^2+bx+c}$$

✓ Solution by Mathematica

Time used: 2.154 (sec). Leaf size: 199

```
DSolve[(a*x^2+b*x+c)^2*y''[x]+A*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x(ax+b)+c}\exp\left(-\frac{\sqrt{4ac-b^2}\sqrt{1-\frac{4A}{b^2-4ac}}\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{b^2-4ac}}\right)\left(c_1\exp\left(\frac{2\sqrt{4ac-b^2}\sqrt{1-\frac{4A}{b^2-4ac}}\,\text{a}}{\sqrt{b^2-4a}}\right.\right.$$
$$\left.+\frac{c_2}{\sqrt{b^2-4ac}\sqrt{1-\frac{4A}{b^2-4ac}}}\right)$$

32.26 problem 235

Internal problem ID [11060]
Internal file name [`OUTPUT/10316_Wednesday_January_24_2024_10_07_08_PM_15066980/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 235.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{(x^2-1)^2 y'' + 2x(x^2-1) y' + \left((x^2-1)\left(a^2x^2-\lambda\right)-m^2\right) y = 0}$$

32.26.1 Maple step by step solution

Let's solve

$y''(x^4-2x^2+1) + (2x^3-2x) y' + (a^2x^4 + (-a^2-\lambda) x^2 - m^2 + \lambda) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(a^2x^4-a^2x^2-\lambda x^2-m^2+\lambda)y}{x^4-2x^2+1} - \frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2xy'}{x^2-1} + \frac{(a^2x^4-a^2x^2-\lambda x^2-m^2+\lambda)y}{x^4-2x^2+1} = 0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

$$\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=\frac{a^2x^4-a^2x^2-\lambda x^2-m^2+\lambda}{x^4-2x^2+1}\right]$$

- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $$\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=1$$

- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $$\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=-\frac{m^2}{4}$$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $$y''(x^2-1)(x^4-2x^2+1)+2y'x(x^4-2x^2+1)+(a^2x^4-a^2x^2-\lambda x^2-m^2+\lambda)(x^2-1)y=0$$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $$(u^6-6u^5+12u^4-8u^3)\left(\frac{d^2}{du^2}y(u)\right)+(2u^5-10u^4+16u^3-8u^2)\left(\frac{d}{du}y(u)\right)+(a^2u^6-6a^2u^5+13$$

- Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty}a_ku^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=1..6$

  $$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_ku^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=2..5$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=3..6$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0(m+2r)(-m+2r)\, u^{1+r} + (-2a_1(2+m+2r)(2-m+2r) + a_0(4a^2 - m^2 + 12r^2 - 4\lambda +$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2(m+2r)(-m+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{m}{2}, \frac{m}{2}\right\}$$

- The coefficients of each power of u must be 0

$$[-2a_1(2+m+2r)(2-m+2r) + a_0(4a^2 - m^2 + 12r^2 - 4\lambda + 4r) = 0, -2a_2(4+m+2r)(4-$$

- Solve for the dependent coefficient(s)

$$\left\{a_1 = -\frac{a_0(4a^2 - m^2 + 12r^2 - 4\lambda + 4r)}{2(m^2 - 4r^2 - 8r - 4)}, a_2 = \frac{a_0(16a^4 + 16a^2m^2 + m^4 - 12m^2r^2 + 96r^4 - 32\lambda a^2 - 64a^2r - 64\lambda r^2 - 24m^2r + 256r^3 - 32}{4(m^4 - 8m^2r^2 + 16r^4 - 24m^2r + 96r^3 - 20m^2 + 208r^2}\right.$$

- Each term in the series must be 0, giving the recursion relation

$$(12a_{k-2} - 8a_{k-1} + a_{k-4} - 6a_{k-3})\, k^2 + (2(12a_{k-2} - 8a_{k-1} + a_{k-4} - 6a_{k-3})\, r - 44a_{k-2} + 16a_{k-1} -$$

- Shift index using $k- > k+6$

$$(12a_{k+4} - 8a_{k+5} + a_{k+2} - 6a_{k+3})(k+6)^2 + (2(12a_{k+4} - 8a_{k+5} + a_{k+2} - 6a_{k+3})\, r - 44a_{k+4} + 16a$$

- Recursion relation that defines series solution to ODE

$$a_{k+5} = \frac{a_k a^2 - 6a^2 a_{k+1} + 13a^2 a_{k+2} - 12a^2 a_{k+3} + 4a^2 a_{k+4} + k^2 a_{k+2} - 6k^2 a_{k+3} + 12k^2 a_{k+4} + 2kra_{k+2} - 12kra_{k+3} + 24kra_{k+4} - m^2 a_{k}}{2(4k^2}$$

- Recursion relation for $r = -\frac{m}{2}$

$$a_{k+5} = \frac{a_k a^2 + 6a_{k+2} - 66a_{k+3} + \frac{1}{4}m^2 a_{k+2} - \frac{3}{2}m^2 a_{k+3} - \frac{5}{2}ma_{k+2} + 20ma_{k+3} - 50ma_{k+4} + 208a_{k+4} - 6a^2 a_{k+1} + 13a^2 a_{k+2} - 12a^2 a_{k+}}{2(}$$

- Solution for $r = -\frac{m}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{m}{2}}, a_{k+5} = \frac{a_k a^2 + 6a_{k+2} - 66a_{k+3} + \frac{1}{4}m^2 a_{k+2} - \frac{3}{2}m^2 a_{k+3} - \frac{5}{2}ma_{k+2} + 20ma_{k+3} - 50ma_{k+4} + 208a_{k+4} - 6a}{}\right.$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-\frac{m}{2}}, a_{k+5} = \frac{a_k a^2 + 6a_{k+2} - 66a_{k+3} + \frac{1}{4}m^2 a_{k+2} - \frac{3}{2}m^2 a_{k+3} - \frac{5}{2}ma_{k+2} + 20ma_{k+3} - 50ma_{k+4} + 208a_{k+}}{}\right.$$

- Recursion relation for $r = \frac{m}{2}$

$a_{k+5} = \frac{a_k a^2+6a_{k+2}-66a_{k+3}+\frac{1}{4}m^2a_{k+2}-\frac{3}{2}m^2a_{k+3}+\frac{5}{2}ma_{k+2}-20ma_{k+3}+50ma_{k+4}+208a_{k+4}-6a^2a_{k+1}+13a^2a_{k+2}-12a^2a_{k+}}{2(4}$

- Solution for $r = \frac{m}{2}$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{m}{2}}, a_{k+5} = \frac{a_k a^2+6a_{k+2}-66a_{k+3}+\frac{1}{4}m^2a_{k+2}-\frac{3}{2}m^2a_{k+3}+\frac{5}{2}ma_{k+2}-20ma_{k+3}+50ma_{k+4}+208a_{k+4}-6a}{}\right.$

- Revert the change of variables $u = 1+x$

$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{m}{2}}, a_{k+5} = \frac{a_k a^2+6a_{k+2}-66a_{k+3}+\frac{1}{4}m^2a_{k+2}-\frac{3}{2}m^2a_{k+3}+\frac{5}{2}ma_{k+2}-20ma_{k+3}+50ma_{k+4}+208a_{k+}}{}\right.$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} b_k (1+x)^{k-\frac{m}{2}}\right) + \left(\sum_{k=0}^{\infty} c_k (1+x)^{k+\frac{m}{2}}\right), b_{k+5} = \frac{-6a^2b_{1+k}+13a^2b_{k+2}-12a^2b_{k+3}+4a^2b_{4+k}+k^2b_{k+2}}{}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.5 (sec). Leaf size: 64

```
dsolve((x^2-1)^2*diff(y(x),x$2)+2*x*(x^2-1)*diff(y(x),x)+( (x^2-1)*(a^2*x^2-lambda)-m^2)*y(x
```

$$y(x) = \left(\operatorname{HeunC}\left(0, \frac{1}{2}, m, \frac{a^2}{4}, \frac{1}{4} + \frac{m^2}{4} - \frac{\lambda}{4}, x^2\right) c_2 x + \operatorname{HeunC}\left(0, -\frac{1}{2}, m, \frac{a^2}{4}, \frac{1}{4} + \frac{m^2}{4} - \frac{\lambda}{4}, x^2\right) c_1\right) \left(x^2-1\right)^{\frac{m}{2}}$$

✓ Solution by Mathematica

Time used: 0.602 (sec). Leaf size: 234

```
DSolve[(x^2-1)^2*y''[x]+2*x*(x^2-1)*y'[x]+( (x^2-1)*(a^2*x^2-\[Lambda])-m^2)*y[x]==0,y[x],x,
```

$$y(x) \to e^{i\sqrt{a^2}x}\left(\frac{x+1}{x-1}\right)^{\frac{\sqrt{m^2}}{2}}\left(c_2(x-1)^{\sqrt{m^2}}\text{HeunC}\left[-\left(\sqrt{m^2}+1\right)\left(\sqrt{m^2}+2i\sqrt{a^2}\right)-a^2+\lambda,-4i\sqrt{a^2}\left(\sqrt{m^2}+1\right),\sqrt{m^2}+1,\sqrt{m^2}+1,-4i\sqrt{a^2},\frac{1-x}{2}\right]+c_1\text{HeunC}\left[2i\sqrt{a^2}\left(\sqrt{m^2}-1\right)-a^2+\lambda,-4i\sqrt{a^2},1-\sqrt{m^2},\sqrt{m^2}+1,-4i\sqrt{a^2},\frac{1-x}{2}\right]\right)$$

32.27 problem 236

Internal problem ID [11061]
Internal file name [`OUTPUT/10317_Wednesday_January_24_2024_10_07_09_PM_18207104/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 236.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{\left(x^{2}+1\right)^{2} y''+2x\left(x^{2}+1\right) y'+\left(\left(x^{2}+1\right) \left(a^{2}x^{2}-\lambda\right)+m^{2}\right) y=0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.5 (sec). Leaf size: 68

```
dsolve((x^2+1)^2*diff(y(x),x$2)+2*x*(x^2+1)*diff(y(x),x)+( (x^2+1)*(a^2*x^2-lambda)+m^2)*y(x
```

$$y(x) = \left(\operatorname{HeunC}\left(0, \frac{1}{2}, m, -\frac{a^2}{4}, \frac{1}{4}+\frac{m^2}{4}-\frac{\lambda}{4}, -x^2\right) c_2 x + \operatorname{HeunC}\left(0, -\frac{1}{2}, m, -\frac{a^2}{4}, \frac{1}{4}+\frac{m^2}{4}-\frac{\lambda}{4}, -x^2\right) c_1\right) \left(x^2+1\right)^{\frac{m}{2}}$$

✓ Solution by Mathematica

Time used: 0.605 (sec). Leaf size: 124

```
DSolve[(x^2+1)^2*y''[x]+2*x*(x^2+1)*y'[x]+( (x^2+1)*(a^2*x^2-\[Lambda])+m^2)*y[x]==0,y[x],x,
```

$$y(x) \to \left(x^2+1\right)^{\frac{\sqrt{m^2}}{2}} \left(c_2 x \text{HeunC}\left[\frac{1}{4}\left(\lambda-m^2-3\sqrt{m^2}-2\right),-\frac{a^2}{4},\frac{3}{2},\sqrt{m^2}+1,0,-x^2\right] + c_1 \text{HeunC}\left[\frac{1}{4}\left(\lambda-m^2-\sqrt{m^2}\right),-\frac{a^2}{4},\frac{1}{2},\sqrt{m^2}+1,0,-x^2\right]\right)$$

32.28 problem 237

Internal problem ID [11062]
Internal file name [`OUTPUT/10318_Wednesday_January_24_2024_10_07_09_PM_44762735/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-7 Equation of form $(a_4x^4 + a_3x^3 + a_2x^2x + a_1x + a_0)y'' + f(x)y' + g(x)y = 0$
Problem number: 237.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(a\,x^{2}+bx+c\right)^{2} y''+\left(2ax+k\right)\left(a\,x^{2}+bx+c\right) y'+ym=0}$$

32.28.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$\left(a\,x^{2}+bx+c\right)^{2} y''+\left(2ax+k\right)\left(a\,x^{2}+bx+c\right) y'+ym=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=\frac{2ax+k}{a\,x^{2}+bx+c}\\ q(x)&=\frac{m}{\left(a\,x^{2}+bx+c\right)^{2}}\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{2ax+k}{a\,x^2+bx+c}dx\right)}dx \\
&= \int e^{-\ln\left(a\,x^2+bx+c\right)+\frac{2(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}dx \\
&= \int \frac{\mathrm{e}^{\frac{2(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{a\,x^2+bx+c}dx \\
&= \frac{\mathrm{e}^{\frac{2(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{b-k}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{m}{\left(a\,x^2+bx+c\right)^2}}{\frac{\mathrm{e}^{\frac{4(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{\left(a\,x^2+bx+c\right)^2}} \\
&= m\,\mathrm{e}^{-\frac{4(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) + m\,\mathrm{e}^{-\frac{4(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}y(\tau) = 0$$

But in terms of τ

$$m\,\mathrm{e}^{-\frac{4(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}} = \frac{m}{(b-k)^2\,\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{my(\tau)}{(b-k)^2\,\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)(b-k)^2\,\tau^2 + my(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$(b-k)^2\,\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + m\,\tau^r = 0$$

Simplifying gives

$$(b-k)^2\,r(r-1)\,\tau^r + 0\,\tau^r + m\,\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$(b-k)^2\,r(r-1) + 0 + m = 0$$

Or

$$\left(b^2 - 2bk + k^2\right)r^2 + \left(-b^2 + 2bk - k^2\right)r + m = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{-b + k + \sqrt{b^2 - 2bk + k^2 - 4m}}{2\,(b-k)}$$

$$r_2 = \frac{b - k + \sqrt{b^2 - 2bk + k^2 - 4m}}{2b - 2k}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{-\frac{-b+k+\sqrt{b^2-2bk+k^2-4m}}{2(b-k)}} + c_2 \tau^{\frac{b-k+\sqrt{b^2-2bk+k^2-4m}}{2b-2k}}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(\frac{e^{\frac{2(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{b-k}\right)^{\frac{b-k-\sqrt{b^2-2bk+k^2-4m}}{2b-2k}} + c_2 \left(\frac{e^{\frac{2(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{b-k}\right)^{\frac{b-k+\sqrt{b^2-2bk+k^2-4m}}{2b-2k}}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & c_1 \left(\frac{e^{\frac{2(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{b-k}\right)^{\frac{b-k-\sqrt{b^2-2bk+k^2-4m}}{2b-2k}} \\ & + c_2 \left(\frac{e^{\frac{2(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{b-k}\right)^{\frac{b-k+\sqrt{b^2-2bk+k^2-4m}}{2b-2k}} \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 \left(\frac{e^{\frac{2(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{b-k}\right)^{\frac{b-k-\sqrt{b^2-2bk+k^2-4m}}{2b-2k}} + c_2 \left(\frac{e^{\frac{2(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{b-k}\right)^{\frac{b-k+\sqrt{b^2-2bk+k^2-4m}}{2b-2k}}$$

Verified OK.

32.28.2 Solving using Kovacic algorithm

Writing the ode as

$$\left(a\,x^{2}+bx+c\right)^{2} y''+\left(2ax+k\right)\left(a\,x^{2}+bx+c\right) y'+ym=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=\left(a\,x^{2}+bx+c\right)^{2}\\ B&=\left(a\,x^{2}+bx+c\right)\left(2ax+k\right)\\ C&=m \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{4ac-2bk+k^{2}-4m}{4\left(a\,x^{2}+bx+c\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=4ac-2bk+k^{2}-4m\\ t&=4\left(a\,x^{2}+bx+c\right)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{4ac-2bk+k^{2}-4m}{4\left(a\,x^{2}+bx+c\right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y=z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 223: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(a\,x^2 + bx + c)^2$. There is a pole at $x = -\frac{b-\sqrt{-4ac+b^2}}{2a}$ of order 2. There is a pole at $x = -\frac{b+\sqrt{-4ac+b^2}}{2a}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r=\frac{4ac-2bk+k^2-4m}{4\left(-4ac+b^2\right)\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^2}+\frac{4ac-2bk+k^2-4m}{4\left(-4ac+b^2\right)\left(x+\frac{b+\sqrt{-4ac+b^2}}{2a}\right)^2}$$
$$-\frac{\left(4ac-2bk+k^2-4m\right)a}{2\left(-4ac+b^2\right)^{\frac{3}{2}}\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)}+\frac{\left(4ac-2bk+k^2-4m\right)a}{2\left(-4ac+b^2\right)^{\frac{3}{2}}\left(x+\frac{b+\sqrt{-4ac+b^2}}{2a}\right)}$$

For the pole at $x=-\frac{b-\sqrt{-4ac+b^2}}{2a}$ let b be the coefficient of $\frac{1}{\left(x+\frac{b-\sqrt{-4ac+b^2}}{2a}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{-4ac+2bk-k^2+4m}{16ac-4b^2}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}+\frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}\\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}-\frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}
\end{aligned}$$

For the pole at $x=-\frac{b+\sqrt{-4ac+b^2}}{2a}$ let b be the coefficient of $\frac{1}{\left(x+\frac{b+\sqrt{-4ac+b^2}}{2a}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{-4ac+2bk-k^2+4m}{16ac-4b^2}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}+\frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}\\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}-\frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}
\end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= 0\\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{4ac-2bk+k^2-4m}{4\left(a\,x^2+bx+c\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{b-\sqrt{-4ac+b^2}}{2a}$	2	0	$\frac{1}{2}+\frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}$	$\frac{1}{2}-\frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}$
$-\frac{b+\sqrt{-4ac+b^2}}{2a}$	2	0	$\frac{1}{2}+\frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}$	$\frac{1}{2}-\frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2}-\frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}}{x+\frac{b-\sqrt{-4ac+b^2}}{2a}} + \frac{\frac{1}{2}+\frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}}{x+\frac{b+\sqrt{-4ac+b^2}}{2a}} + (-)(0) \\ &= \frac{\frac{1}{2}-\frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}}{x+\frac{b-\sqrt{-4ac+b^2}}{2a}} + \frac{\frac{1}{2}+\frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}}{x+\frac{b+\sqrt{-4ac+b^2}}{2a}} \\ &= \frac{2ax - \sqrt{-4ac+b^2}\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}} + b}{2a\,x^2 + 2bx + 2c} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}}{x + \frac{b-\sqrt{-4ac+b^2}}{2a}} + \frac{\frac{1}{2} + \frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}}{x + \frac{b+\sqrt{-4ac+b^2}}{2a}}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}}{\left(x + \frac{b-\sqrt{-4ac+b^2}}{2a}\right)^2} - \frac{\frac{1}{2} + \frac{\sqrt{-\frac{b^2-2bk+k^2-4}{4ac-b^2}}}{2}}{\left(x + \frac{b+\sqrt{-4ac+b^2}}{2a}\right.}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{\frac{1}{2} - \frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}}{x + \frac{b-\sqrt{-4ac+b^2}}{2a}} + \frac{\frac{1}{2} + \frac{\sqrt{-\frac{b^2-2bk+k^2-4m}{4ac-b^2}}}{2}}{x + \frac{b+\sqrt{-4ac+b^2}}{2a}}\right) dx} \\
&= \left(\frac{2ax + \sqrt{-4ac+b^2} + b}{a}\right)^{\frac{1}{2} + \frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} \left(\frac{2ax + b - \sqrt{-4ac+b^2}}{a}\right)^{\frac{1}{2} - \frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\left(a\,x^2+bx+c\right)(2ax+k)}{\left(a\,x^2+bx+c\right)^2} \, dx} \\
&= z_1 e^{-\frac{\ln\left(a\,x^2+bx+c\right)}{2} - \frac{(-b+k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{\sqrt{a\,x^2+bx+c}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{\frac{1}{2}+\frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} \left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{\frac{1}{2}-\frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} \mathrm{e}^{\frac{(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{\sqrt{a\,x^2+bx+c}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{(a\,x^2+bx+c)(2ax+k)}{(a\,x^2+bx+c)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(a\,x^2+bx+c)+\frac{2(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{-1-\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}} \left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{-1+\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{\frac{1}{2}+\frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} \left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{\frac{1}{2}-\frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} \mathrm{e}^{\frac{(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{\sqrt{a\,x^2+bx+c}}\right) \\
&\quad + c_2 \left(\frac{\left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{\frac{1}{2}+\frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} \left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{\frac{1}{2}-\frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} \mathrm{e}^{\frac{(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{\sqrt{a\,x^2+bx+c}} \left(\int \left(2\right.\right.\right.
\end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1 \left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{\frac{1}{2}+\frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} \left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{\frac{1}{2}-\frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} e^{\frac{(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{\sqrt{a\,x^2+bx+c}} \\
&\quad - \frac{c_2 \left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{\frac{1}{2}+\frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} \left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{\frac{1}{2}-\frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} e^{\frac{(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}} a^2 \left(\int -\frac{(2a \cdots}{\cdots}\right.}{\sqrt{a\,x^2+bx+c}}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1 \left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{\frac{1}{2}+\frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} \left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{\frac{1}{2}-\frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} e^{\frac{(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}}}{\sqrt{a\,x^2+bx+c}} \\
&\quad - \frac{c_2 \left(\frac{2ax+\sqrt{-4ac+b^2}+b}{a}\right)^{\frac{1}{2}+\frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} \left(\frac{2ax+b-\sqrt{-4ac+b^2}}{a}\right)^{\frac{1}{2}-\frac{\sqrt{\frac{-b^2+2bk-k^2+4m}{4ac-b^2}}}{2}} e^{\frac{(b-k)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}} a^2 \left(\int -\frac{(2a \cdots}{\cdots}\right.}{\sqrt{a\,x^2+bx+c}}
\end{aligned}
$$

Verified OK.

32.28.3 Maple step by step solution

Let's solve

$$\left(a\,x^2+bx+c\right)^2 y''+(2ax+k)\left(a\,x^2+bx+c\right)y'+ym=0$$

- Highest derivative means the order of the ODE is 2

  $$y''$$

- Isolate 2nd derivative

  $$y''=-\frac{(2ax+k)y'}{a\,x^2+bx+c}-\frac{my}{\left(a\,x^2+bx+c\right)^2}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $$y''+\frac{(2ax+k)y'}{a\,x^2+bx+c}+\frac{my}{\left(a\,x^2+bx+c\right)^2}=0$$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$$\left[P_2(x) = \frac{2ax+k}{a\,x^2+bx+c}, P_3(x) = \frac{m}{(a\,x^2+bx+c)^2}\right]$$

- $\left(x - \frac{-b+\sqrt{-4ac+b^2}}{2a}\right) \cdot P_2(x)$ is analytic at $x = \frac{-b+\sqrt{-4ac+b^2}}{2a}$

  $$\left(\left(x - \frac{-b+\sqrt{-4ac+b^2}}{2a}\right) \cdot P_2(x)\right)\bigg|_{x=\frac{-b+\sqrt{-4ac+b^2}}{2a}} = 0$$

- $\left(x - \frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^2 \cdot P_3(x)$ is analytic at $x = \frac{-b+\sqrt{-4ac+b^2}}{2a}$

  $$\left(\left(x - \frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^2 \cdot P_3(x)\right)\bigg|_{x=\frac{-b+\sqrt{-4ac+b^2}}{2a}} = 0$$

- $x = \frac{-b+\sqrt{-4ac+b^2}}{2a}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = \frac{-b+\sqrt{-4ac+b^2}}{2a}$

- Multiply by denominators

  $$(a\,x^2+bx+c)^2\,y'' + (2ax+k)\,(a\,x^2+bx+c)\,y' + ym = 0$$

- Change variables using $x = u + \frac{-b+\sqrt{-4ac+b^2}}{2a}$ so that the regular singular point is at $u = 0$

  $$(a^2u^4 + 2a\,u^3\sqrt{-4ac+b^2} - 4ac\,u^2 + b^2u^2)\left(\frac{d^2}{du^2}y(u)\right) + (2a^2u^3 - a\,u^2b + 3a\,u^2\sqrt{-4ac+b^2} + a\,u$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..4$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0\left(4ac\,r^2-b^2r^2+\sqrt{-4ac+b^2}\,br-\sqrt{-4ac+b^2}\,kr-m\right)u^r+\left(-a_1\left(4ac+8acr+4ac\,r^2-b^2\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-4ac\,r^2+b^2r^2-\sqrt{-4ac+b^2}\,br+\sqrt{-4ac+b^2}\,kr+m=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{-\frac{b\sqrt{-4ac+b^2}-k\sqrt{-4ac+b^2}-\sqrt{-4b^2ca+8kabc-4k^2ac+b^4-2k\,b^3+k^2b^2+16acm-4b^2m}}{2(4ac-b^2)},-\frac{b\sqrt{-4ac+b^2}-k\sqrt{-4ac+b^2}+\sqrt{-}}{}\right.$$

- Each term must be 0

$$-a_1\left(4ac+8acr+4ac\,r^2-b^2-2b^2r-b^2r^2+b\sqrt{-4ac+b^2}+\sqrt{-4ac+b^2}\,br-k\sqrt{-4ac+b^2}-\right.$$

- Solve for the dependent coefficient(s)

$$a_1=\frac{a_0ar\left(2\sqrt{-4ac+b^2}\,r+\sqrt{-4ac+b^2}-b+k\right)}{4ac+8acr+4ac\,r^2-b^2-2b^2r-b^2r^2+b\sqrt{-4ac+b^2}+\sqrt{-4ac+b^2}\,br-k\sqrt{-4ac+b^2}-\sqrt{-4ac+b^2}\,kr-m}$$

- Each term in the series must be 0, giving the recursion relation

$$\left(2\left(k+r-\tfrac{1}{2}\right)a_{k-1}(k+r-1)\,a-a_k(k+r)\,(b-k)\right)\sqrt{-4ac+b^2}+a^2a_{k-2}(k-2+r)\,(k+r-1$$

- Shift index using $k->k+2$

$$\left(2\left(k+\tfrac{3}{2}+r\right)a_{k+1}(k+r+1)\,a-a_{k+2}(k+2+r)\,(b-k)\right)\sqrt{-4ac+b^2}+a^2a_k(k+r)\,(k+r+1$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=\frac{a\left(2\sqrt{-4ac+b^2}\,k^2a_{k+1}+4\sqrt{-4ac+b^2}\,kra_{k+1}+2\sqrt{-4ac+b^2}\,r^2a_{k+1}+a\,k^2a_k+2akra_k+a\,r^2a_k+5\sqrt{-4ac+b^2}\,ka_{k+1}+5\sqrt{-4a}\right.}{4ac\,k^2+4ac\,r^2-2b^2kr+16ack+16acr+\sqrt{-4ac+b^2}\,bk+\sqrt{-4ac+b^2}\,br-b^2k^2-b^2r^2-4b^2k-4b^2r+16ac-}$$

- Recursion relation for $r=-\frac{b\sqrt{-4ac+b^2}-k\sqrt{-4ac+b^2}-\sqrt{-4b^2ca+8kabc-4k^2ac+b^4-2k\,b^3+k^2b^2+16acm-4b^2m}}{2(4ac-b^2)}$

$$a_{k+2}=\frac{a\left(2\sqrt{-4ac+b^2}\,k^2a_{k+1}-\frac{2\sqrt{-4ac+b^2}\,k\left(b\sqrt{-4ac+b^2}-k\sqrt{-4ac+b^2}-\sqrt{-4b^2ca+8kabc-4k^2ac+b^4-2k\,b^3+k^2b^2+16acm-4b^2m}\right)a_{k+1}}{4ac-b^2}\right.}{4ac\,k^2+\frac{ac\left(b\sqrt{-4ac+b^2}-k\sqrt{-4ac+b^2}-\sqrt{-4b^2ca+8kabc-4k^2ac+b^4-2k\,b^3+k^2b^2+16acm-4b^2m}\right)^2}{\left(4ac-b^2\right)^2}+\frac{b^2k\left(b\sqrt{}\right.}{}}$$

- Solution for $r=-\frac{b\sqrt{-4ac+b^2}-k\sqrt{-4ac+b^2}-\sqrt{-4b^2ca+8kabc-4k^2ac+b^4-2k\,b^3+k^2b^2+16acm-4b^2m}}{2(4ac-b^2)}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-\frac{b\sqrt{-4ac+b^2}-k\sqrt{-4ac+b^2}-\sqrt{-4b^2ca+8kabc-4k^2ac+b^4-2k\,b^3+k^2b^2+16acm-4b^2m}}{2\left(4ac-b^2\right)}},a_{k+2}=\frac{a\left(2\sqrt{-4ac+b^2}\,k\right.}{}\right.$$

- Revert the change of variables $u=x-\frac{-b+\sqrt{-4ac+b^2}}{2a}$

$$\left[y=\sum_{k=0}^{\infty}a_k\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^{k-\frac{b\sqrt{-4ac+b^2}-k\sqrt{-4ac+b^2}-\sqrt{-4b^2ca+8kabc-4k^2ac+b^4-2k\,b^3+k^2b^2+16acm-4b^2m}}{2\left(4ac-b^2\right)}},a_{k+2}=\right.$$

- Recursion relation for $r = -\frac{b\sqrt{-4ac+b^2}-k\sqrt{-4ac+b^2}+\sqrt{-4b^2ca+8kabc-4k^2ac+b^4-2k\,b^3+k^2b^2+16acm-4b^2m}}{2(4ac-b^2)}$

$$a_{k+2} = \frac{a\left(2\sqrt{-4ac+b^2}\,k^2a_{k+1}-\frac{2\sqrt{-4ac+b^2}\,k\left(b\sqrt{-4ac+b^2}-k\sqrt{-4ac+b^2}+\sqrt{-4b^2ca+8kabc-4k^2ac+b^4-2k\,b^3+k^2b^2+16acm-4b^2m}\right)a_{k+1}}{4ac-b^2}\right.}{4ac\,k^2+\frac{ac\left(b\sqrt{-4ac+b^2}-k\sqrt{-4ac+b^2}+\sqrt{-4b^2ca+8kabc-4k^2ac+b^4-2k\,b^3+k^2b^2+16acm-4b^2m}\right)^2}{\left(4ac-b^2\right)^2}+\frac{b^2k\left(b\sqrt{\right.}}{}}$$

- Solution for $r = -\frac{b\sqrt{-4ac+b^2}-k\sqrt{-4ac+b^2}+\sqrt{-4b^2ca+8kabc-4k^2ac+b^4-2k\,b^3+k^2b^2+16acm-4b^2m}}{2(4ac-b^2)}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{b\sqrt{-4ac+b^2}-k\sqrt{-4ac+b^2}+\sqrt{-4b^2ca+8kabc-4k^2ac+b^4-2k\,b^3+k^2b^2+16acm-4b^2m}}{2\left(4ac-b^2\right)}}, a_{k+2} = \frac{a\left(2\sqrt{-4ac+b^2}\,k^2\right.}{}\right.$$

- Revert the change of variables $u = x - \frac{-b+\sqrt{-4ac+b^2}}{2a}$

$$\left[y = \sum_{k=0}^{\infty} a_k\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2a}\right)^{k-\frac{b\sqrt{-4ac+b^2}-k\sqrt{-4ac+b^2}+\sqrt{-4b^2ca+8kabc-4k^2ac+b^4-2k\,b^3+k^2b^2+16acm-4b^2m}}{2\left(4ac-b^2\right)}}, a_{k+2} =\right.$$

- Combine solutions and rename parameters

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 274

```
dsolve((a*x^2+b*x+c)^2*diff(y(x),x$2)+(2*a*x+k)*(a*x^2+b*x+c)*diff(y(x),x)+m*y(x)=0,y(x), si
```

$$y(x) = \left(\frac{-2ax-b+\sqrt{-4ac+b^2}}{2ax+\sqrt{-4ac+b^2}+b}\right)^{-\frac{k}{2\sqrt{-4ac+b^2}}}\left(\frac{2ax+\sqrt{-4ac+b^2}+b}{-2ax-b+\sqrt{-4ac+b^2}}\right)^{-\frac{b}{2\sqrt{-4ac+b^2}}}\left(c_1\left(\frac{-b+i\sqrt{4ac-b^2}-}{i\sqrt{4ac-b^2}+2ax}\right.\right.$$

$$\left.+c_2\left(\frac{-b+i\sqrt{4ac-b^2}-2ax}{i\sqrt{4ac-b^2}+2ax+b}\right)^{-\frac{a\sqrt{\frac{b^2-2kb+k^2-4m}{a^2}}}{2\sqrt{-4ac+b^2}}}\right)$$

✓ Solution by Mathematica

Time used: 2.382 (sec). Leaf size: 157

```
DSolve[(a*x^2+b*x+c)^2*y''[x]+(2*a*x+k)*(a*x^2+b*x+c)*y'[x]+m*y[x]==0,y[x],x,IncludeSingular
```

$$y(x) \to c_1 \exp\left(\frac{\left(-\sqrt{m}\sqrt{\frac{b^2-2bk+k^2-4m}{m}}+b-k\right)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}\right) + c_2 \exp\left(\frac{\left(\sqrt{m}\sqrt{\frac{b^2-2bk+k^2-4m}{m}}+b-k\right)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}\right)$$

33 Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.

33.1 problem 238

Internal problem ID [11063]
Internal file name [OUTPUT/10319_Wednesday_January_24_2024_10_07_11_PM_18886836/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 238.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^6y'' - x^5y' + ay = 0}$$

33.1.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' - y'x + \frac{ay}{x^4} = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha) xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 1$$
$$\beta = \frac{\sqrt{a}}{2}$$
$$n = \frac{1}{2}$$
$$\gamma = -2$$

Substituting all the above into (4) gives the solution as

$$y = \frac{2c_1 x \sin\left(\frac{\sqrt{a}}{2x^2}\right)}{\sqrt{\pi}\sqrt{\frac{\sqrt{a}}{x^2}}} - \frac{2c_2 x \cos\left(\frac{\sqrt{a}}{2x^2}\right)}{\sqrt{\pi}\sqrt{\frac{\sqrt{a}}{x^2}}}$$

Summary

The solution(s) found are the following

$$y = \frac{2c_1 x \sin\left(\frac{\sqrt{a}}{2x^2}\right)}{\sqrt{\pi}\sqrt{\frac{\sqrt{a}}{x^2}}} - \frac{2c_2 x \cos\left(\frac{\sqrt{a}}{2x^2}\right)}{\sqrt{\pi}\sqrt{\frac{\sqrt{a}}{x^2}}} \tag{1}$$

Verification of solutions

$$y = \frac{2c_1 x \sin\left(\frac{\sqrt{a}}{2x^2}\right)}{\sqrt{\pi}\sqrt{\frac{\sqrt{a}}{x^2}}} - \frac{2c_2 x \cos\left(\frac{\sqrt{a}}{2x^2}\right)}{\sqrt{\pi}\sqrt{\frac{\sqrt{a}}{x^2}}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(x^6*diff(y(x),x$2)-x^5*diff(y(x),x)+a*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^2 \left(c_1 \sinh\left(\frac{\sqrt{-a}}{2x^2}\right) + c_2 \cosh\left(\frac{\sqrt{-a}}{2x^2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.239 (sec). Leaf size: 58

```
DSolve[x^6*y''[x]-x^5*y'[x]+a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} x^2 e^{-\frac{i\sqrt{a}}{2x^2}} \left(2c_1 e^{\frac{i\sqrt{a}}{x^2}} - \frac{ic_2}{\sqrt{a}}\right)$$

33.2 problem 239

Internal problem ID [11064]
Internal file name [`OUTPUT/10320_Wednesday_January_24_2024_10_07_12_PM_79551424/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 239.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^6y'' + \left(3x^2+a\right)x^3y' + yb = 0}$$

33.2.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^6y'' + \left(3x^2+a\right)x^3y' + yb = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{3x^2+a}{x^3}$$
$$q(x) = \frac{b}{x^6}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\
&= \int e^{-\left(\int \frac{3x^2+a}{x^3}dx\right)}dx \\
&= \int e^{\frac{a}{2x^2}-3\ln(x)}\,dx \\
&= \int \frac{e^{\frac{a}{2x^2}}}{x^3}dx \\
&= -\frac{e^{\frac{a}{2x^2}}}{a} \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{b}{x^6}}{\frac{e^{\frac{a}{x^2}}}{x^6}} \\
&= b\,e^{-\frac{a}{x^2}} \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + b\,e^{-\frac{a}{x^2}}\,y(\tau) = 0$$

But in terms of τ

$$b\,e^{-\frac{a}{x^2}} = \frac{b}{a^2\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{by(\tau)}{a^2\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)a^2\tau^2 + by(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$a^2\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + b\tau^r = 0$$

Simplifying gives

$$a^2r(r-1)\,\tau^r + 0\,\tau^r + b\,\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$a^2r(r-1) + 0 + b = 0$$

Or

$$a^2r^2 - a^2r + b = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{-a + \sqrt{a^2 - 4b}}{2a}$$
$$r_2 = \frac{a + \sqrt{a^2 - 4b}}{2a}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{-\frac{-a+\sqrt{a^2-4b}}{2a}} + c_2\tau^{\frac{a+\sqrt{a^2-4b}}{2a}}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1\left(-\frac{e^{\frac{a}{2x^2}}}{a}\right)^{-\frac{-a+\sqrt{a^2-4b}}{2a}} + c_2\left(-\frac{e^{\frac{a}{2x^2}}}{a}\right)^{\frac{a+\sqrt{a^2-4b}}{2a}}$$

Summary

The solution(s) found are the following

$$y = c_1\left(-\frac{e^{\frac{a}{2x^2}}}{a}\right)^{-\frac{-a+\sqrt{a^2-4b}}{2a}} + c_2\left(-\frac{e^{\frac{a}{2x^2}}}{a}\right)^{\frac{a+\sqrt{a^2-4b}}{2a}} \tag{1}$$

Verification of solutions

$$y = c_1\left(-\frac{e^{\frac{a}{2x^2}}}{a}\right)^{-\frac{-a+\sqrt{a^2-4b}}{2a}} + c_2\left(-\frac{e^{\frac{a}{2x^2}}}{a}\right)^{\frac{a+\sqrt{a^2-4b}}{2a}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
dsolve(x^6*diff(y(x),x$2)+(3*x^2+a)*x^3*diff(y(x),x)+b*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{-a+\sqrt{a^2-4b}}{4x^2}} + c_2 \mathrm{e}^{\frac{a+\sqrt{a^2-4b}}{4x^2}}$$

✓ Solution by Mathematica

Time used: 0.066 (sec). Leaf size: 56

```
DSolve[x^6*y''[x]+(3*x^2+a)*x^3*y'[x]+b*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{a-\sqrt{a^2-4b}}{4x^2}} \left(c_1 e^{\frac{\sqrt{a^2-4b}}{2x^2}} + c_2\right)$$

33.3 problem 241

Internal problem ID [11065]
Internal file name [OUTPUT/10321_Wednesday_January_24_2024_10_07_12_PM_56094527/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 241.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$x^n y'' + c(ax+b)^{n-4} y = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✗ Solution by Maple

```
dsolve(x^n*diff(y(x),x$2)+c*(a*x+b)^(n-4)*y(x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^n*y''[x]+c*(a*x+b)^(n-4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

33.4 problem 242

Internal problem ID [11066]
Internal file name [`OUTPUT/10322_Wednesday_January_24_2024_10_07_12_PM_98034743/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 242.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x^n y'' + axy' - \left(b^2 x^n + 2b\, x^{n-1} + abx + a\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
```

✗ Solution by Maple

```
dsolve(x^n*diff(y(x),x$2)+a*x*diff(y(x),x)-(b^2*x^n+2*b*x^(n-1)+a*b*x+a)*y(x)=0,y(x), singso
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^n*y''[x]+a*x*y'[x]-(b^2*x^n+2*b*x^(n-1)+a*b*x+a)*y[x]==0,y[x],x,IncludeSingularSolu
```

Not solved

33.5 problem 243

Internal problem ID [11067]
Internal file name [`OUTPUT/10323_Wednesday_January_24_2024_10_07_13_PM_55005500/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 243.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^n y'' + (ax+b)\, y' - ay = 0}$$

33.5.1 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^n \\ B &= ax + b \\ C &= -a \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (x^n)\,(0) + (ax + b)\,(a) + (-a)\,(ax + b) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x^n(ax + b)\, v'' + \left(2a\, x^n + (ax + b)^2\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$x^n(ax + b)\, u'(x) + \left(2a\, x^n + (ax + b)^2\right) u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u(a^2x^2 + 2abx + 2a\, x^n + b^2)\, x^{-n}}{ax + b} \end{aligned}$$

Where $f(x) = -\frac{(a^2x^2+2abx+2a\,x^n+b^2)\,x^{-n}}{ax+b}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{(a^2x^2+2abx+2a\,x^n+b^2)\,x^{-n}}{ax+b}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{(a^2x^2+2abx+2a\,x^n+b^2)\,x^{-n}}{ax+b}\,dx \\
\ln(u) &= \left(\frac{a\,x^2}{n-2}+\frac{bx}{n-1}\right)\mathrm{e}^{-n\ln(x)} - 2\ln(ax+b) + c_1 \\
u &= \mathrm{e}^{\left(\frac{a\,x^2}{n-2}+\frac{bx}{n-1}\right)\mathrm{e}^{-n\ln(x)} - 2\ln(ax+b) + c_1} \\
&= c_1 \mathrm{e}^{\left(\frac{a\,x^2}{n-2}+\frac{bx}{n-1}\right)\mathrm{e}^{-n\ln(x)} - 2\ln(ax+b)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 \mathrm{e}^{\frac{x^2 a\,x^{-n}}{n-2}}\,\mathrm{e}^{\frac{x^{-n} bx}{n-1}}}{(ax+b)^2}$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= \frac{c_1 \mathrm{e}^{\frac{x^2 a\,x^{-n}}{n-2}}\,\mathrm{e}^{\frac{x^{-n} bx}{n-1}}}{(ax+b)^2}
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{c_1 \mathrm{e}^{\frac{x^2 a\,x^{-n}}{n-2}}\,\mathrm{e}^{\frac{x^{-n} bx}{n-1}}}{(ax+b)^2}\,\mathrm{d}x \\
&= \int \frac{c_1 \mathrm{e}^{\frac{x^2 a\,x^{-n}}{n-2}}\,\mathrm{e}^{\frac{x^{-n} bx}{n-1}}}{(ax+b)^2}dx + c_2
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y(x) &= Bv \\
&= (ax+b)\left(\int \frac{c_1 \mathrm{e}^{\frac{x^2 a\,x^{-n}}{n-2}}\,\mathrm{e}^{\frac{x^{-n} bx}{n-1}}}{(ax+b)^2}dx + c_2\right) \\
&= (ax+b)\left(c_1\left(\int \frac{\mathrm{e}^{\frac{(a(n-1)x+b(n-2))x^{1-n}}{(n-2)(n-1)}}}{(ax+b)^2}dx\right) + c_2\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = (ax+b)\left(c_1\left(\int \frac{e^{\frac{(a(n-1)x+b(n-2))x^{1-n}}{(n-2)(n-1)}}}{(ax+b)^2}dx\right)+c_2\right) \tag{1}$$

Verification of solutions

$$y = (ax+b)\left(c_1\left(\int \frac{e^{\frac{(a(n-1)x+b(n-2))x^{1-n}}{(n-2)(n-1)}}}{(ax+b)^2}dx\right)+c_2\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   <- linear symmetries successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 56

```
dsolve(x^n*diff(y(x),x$2)+(a*x+b)*diff(y(x),x)-a*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\left(c_1\left(\int \frac{e^{\frac{x^{-n+1}(ax(n-1)+b(n-2))}{(n-2)(n-1)}}}{(ax+b)^2}dx\right)+c_2\right)(ax+b)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^n*y''[x]+(a*x+b)*y'[x]-a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

33.6 problem 244

Internal problem ID [11068]
Internal file name [`OUTPUT/10324_Wednesday_January_24_2024_10_07_14_PM_89762227/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 244.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^{n} y'' + \left(a\,x^{n-1} + bx\right) y' + (a-1)\,y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.782 (sec). Leaf size: 137

```
dsolve(x^n*diff(y(x),x$2)+(a*x^(n-1)+b*x)*diff(y(x),x)+(a-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^{-\frac{a}{2}-\frac{1}{2}+\frac{n}{2}} e^{\frac{b\,x^{2-n}}{-4+2n}} \left(\text{WhittakerM}\left(\frac{(-b+2)\,a-2+b(n-1)}{2b\,(n-2)}, \frac{a-1}{-4+2n}, \frac{b\,x^{2-n}}{n-2}\right) c_1 + \text{WhittakerW}\left(\frac{(-b+2)\,a-2+b(n-1)}{2b\,(n-2)}, \frac{a-1}{-4+2n}, \frac{b\,x^{2-n}}{n-2}\right) c_2\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^n*y''[x]+(a*x^(n-1)+b*x)*y'[x]+(a-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

Not solved

33.7 problem 245

Internal problem ID [11069]
Internal file name [OUTPUT/10325_Wednesday_January_24_2024_10_07_15_PM_54178152/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 245.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$x^{n}y'' + \left(2x^{n-1} + a\,x^{2} + bx\right)y' + yb = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✓ Solution by Maple

Time used: 0.454 (sec). Leaf size: 76

```
dsolve(x^n*diff(y(x),x$2)+(2*x^(n-1)+a*x^2+b*x)*diff(y(x),x)+b*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(ax+b)\left(c_2\left(\int \frac{e^{\frac{b(n-3)x^{2-n}+(n-2)\left(a\,x^{3-n}-2(n-3)\ln(x)\right)}{(n-3)(n-2)}}x^2}{(ax+b)^2}dx\right)+c_1\right)}{x}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^n*y''[x]+(2*x^(n-1)+a*x^2+b*x)*y'[x]+b*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

Not solved

33.8 problem 246

Internal problem ID [11070]
Internal file name [`OUTPUT/10326_Wednesday_January_24_2024_10_07_15_PM_36620420/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 246.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x^n y'' + (a\,x^n + b)\,y' + c((a-c)\,x^n + b)\,y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(x^n*diff(y(x),x$2)+(a*x^n+b)*diff(y(x),x)+c*((a-c)*x^n+b)*y(x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^n*y''[x]+(a*x^n+b)*y'[x]+c*((a-c)*x^n+b)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

Not solved

33.9 problem 247

Internal problem ID [11071]
Internal file name [OUTPUT/10327_Wednesday_January_24_2024_10_07_16_PM_37914138/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 247.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^n y'' + \left(a\,x^n - x^{n-1} + abx + b\right) y' + a^2 bxy = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(x^n*diff(y(x),x$2)+(a*x^n-x^(n-1)+a*b*x+b)*diff(y(x),x)+a^2*b*x*y(x)=0,y(x), singsol=
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^n*y''[x]+(a*x^n-x^(n-1)+a*b*x+b)*y'[x]+a^2*b*x*y[x]==0,y[x],x,IncludeSingularSoluti
```

Not solved

33.10 problem 248

Internal problem ID [11072]
Internal file name [OUTPUT/10328_Wednesday_January_24_2024_10_07_16_PM_18572700/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 248.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^n y'' + \left(a\, x^{m+n} + 1\right) y' + a\, x^m \left(1 + x^{n-1} m\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(x^n*diff(y(x),x$2)+(a*x^(n+m)+1)*diff(y(x),x)+a*x^m*(1+m*x^(n-1))*y(x)=0,y(x), singso
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^n*y''[x]+(a*x^(n+m)+1)*y'[x]+a*x^m*(1+m*x^(n-1))*y[x]==0,y[x],x,IncludeSingularSolu
```

Not solved

33.11 problem 249

Internal problem ID [11073]
Internal file name [`OUTPUT/10329_Wednesday_January_24_2024_10_07_17_PM_72162890/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 249.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{(a\,x^n + b)\,y'' + (x^n c + d)\,y' + \lambda((-a\lambda + c)\,x^n + d - b\lambda)\,y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve((a*x^n+b)*diff(y(x),x$2)+(c*x^n+d)*diff(y(x),x)+lambda*((c-a*lambda)*x^n+d-b*lambda)*
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(a*x^n+b)*y''[x]+(c*x^n+d)*y'[x]+\[Lambda]*((c-a*\[Lambda])*x^n+d-b*\[Lambda])*y[x]==
```

Not solved

33.12 problem 250

Internal problem ID [11074]
Internal file name [`OUTPUT/10330_Wednesday_January_24_2024_10_07_18_PM_82271614/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 250.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$
\boxed{\left(a\,x^{n}+bx+c\right)y''-an(n-1)\,x^{n-2}y=0}
$$

33.12.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\int \left(\left(a\,x^{n}+bx+c\right)y''-an(n-1)\,x^{n-2}y\right)dx=0
$$

$$
-\frac{\left(x^{n}na+bx\right)y}{x}-\left(-a\,x^{n}-bx-c\right)y'=c_1
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y'+p(x)y=q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{x^n n a + b x}{(a\,x^n + b x + c)\,x} \\
q(x) &= \frac{c_1}{a\,x^n + b x + c}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{(x^n n a + b x)\,y}{(a\,x^n + b x + c)\,x} = \frac{c_1}{a\,x^n + b x + c}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{x^n n a + b x}{(a\,x^n + b x + c)x} dx} \\
&= \frac{1}{a\,\mathrm{e}^{n \ln(x)} + b x + c}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{1}{a\,x^n + b x + c}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{a\,x^n + b x + c}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{a\,x^n + b x + c}\right) &= \left(\frac{1}{a\,x^n + b x + c}\right)\left(\frac{c_1}{a\,x^n + b x + c}\right) \\
\mathrm{d}\left(\frac{y}{a\,x^n + b x + c}\right) &= \left(\frac{c_1}{(a\,x^n + b x + c)^2}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{a\,x^n + b x + c} &= \int \frac{c_1}{(a\,x^n + b x + c)^2}\,\mathrm{d}x \\
\frac{y}{a\,x^n + b x + c} &= \frac{x c_1}{(n b x - b x + c n)\,(a\,\mathrm{e}^{n \ln(x)} + b x + c)} + \left(\int \frac{n(n b x - b x + c n - c)}{(n b x - b x + c n)^2\,(a\,\mathrm{e}^{n \ln(x)} + b x + c)} dx\right) c_1 +
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{a\,x^n + b x + c}$ results in

$$
y = (a\,x^n + b x + c)\left(\frac{x c_1}{(n b x - b x + c n)\,(a\,\mathrm{e}^{n \ln(x)} + b x + c)} + \left(\int \frac{n(n b x - b x + c n - c)}{(n b x - b x + c n)^2\,(a\,\mathrm{e}^{n \ln(x)} + b x + c)} dx\right)\right.
$$

which simplifies to

$$
y = (a\,x^n + b x + c)\left(c_1\left(\frac{x}{(a\,x^n + b x + c)\,(n b x - b x + c n)} + n\left(\int \frac{(n-1)\,(b x + c)}{(x\,(n-1)\,b + c n)^2\,(a\,x^n + b x + c)} dx\right)\right.\right.
$$

Summary
The solution(s) found are the following

$$y = (a\,x^n + bx + c)\left(c_1\left(\frac{x}{(a\,x^n + bx + c)\,(nbx - bx + cn)} + n\left(\int \frac{(n-1)\,(bx + c)}{(x\,(n-1)\,b + cn)^2\,(a\,x^n + bx + c)}dx\right)\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = (a\,x^n + bx + c)\left(c_1\left(\frac{x}{(a\,x^n + bx + c)\,(nbx - bx + cn)} + n\left(\int \frac{(n-1)\,(bx + c)}{(x\,(n-1)\,b + cn)^2\,(a\,x^n + bx + c)}dx\right)\right) + c_2\right)$$

Verified OK.

33.12.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(a\,x^n + bx + c)\,y'' - an(n-1)\,x^{n-2}y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left((a\,x^n + bx + c)\,y'' - an(n-1)\,x^{n-2}y\right)dx = 0$$

$$-\frac{(x^n na + bx)\,y}{x} - (-a\,x^n - bx - c)\,y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{x^n na + bx}{(a\,x^n + bx + c)\,x}$$
$$q(x) = \frac{c_1}{a\,x^n + bx + c}$$

Hence the ode is

$$y' - \frac{(x^n na + bx)\,y}{(a\,x^n + bx + c)\,x} = \frac{c_1}{a\,x^n + bx + c}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{x^n n a + b x}{(a\,x^n + b x + c) x}dx} \\
&= \frac{1}{a\,e^{n\ln(x)} + b x + c}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{1}{a\,x^n + b x + c}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{a\,x^n + b x + c}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{a\,x^n + b x + c}\right) &= \left(\frac{1}{a\,x^n + b x + c}\right)\left(\frac{c_1}{a\,x^n + b x + c}\right) \\
\mathrm{d}\left(\frac{y}{a\,x^n + b x + c}\right) &= \left(\frac{c_1}{(a\,x^n + b x + c)^2}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{a\,x^n + b x + c} &= \int \frac{c_1}{(a\,x^n + b x + c)^2}\,\mathrm{d}x \\
\frac{y}{a\,x^n + b x + c} &= \frac{x c_1}{(n b x - b x + c n)\,(a\,e^{n\ln(x)} + b x + c)} + \left(\int \frac{n(n b x - b x + c n - c)}{(n b x - b x + c n)^2\,(a\,e^{n\ln(x)} + b x + c)}dx\right) c_1 +
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{a\,x^n + b x + c}$ results in

$$
y = (a\,x^n + b x + c)\left(\frac{x c_1}{(n b x - b x + c n)\,(a\,e^{n\ln(x)} + b x + c)} + \left(\int \frac{n(n b x - b x + c n - c)}{(n b x - b x + c n)^2\,(a\,e^{n\ln(x)} + b x + c)}dx\right)\right.
$$

which simplifies to

$$
y = (a\,x^n + b x + c)\left(c_1\left(\frac{x}{(a\,x^n + b x + c)\,(n b x - b x + c n)} + n\left(\int \frac{(n-1)\,(b x + c)}{(x\,(n-1)\,b + c n)^2\,(a\,x^n + b x + c)}dx\right)\right)\right.
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = (a\,x^n + b x + c)\Bigg(c_1\Bigg(&\frac{x}{(a\,x^n + b x + c)\,(n b x - b x + c n)} \\
&+ n\left(\int \frac{(n-1)\,(b x + c)}{(x\,(n-1)\,b + c n)^2\,(a\,x^n + b x + c)}dx\right)\Bigg) + c_2\Bigg)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = (a\,x^n + bx + c)\left(c_1\left(\frac{x}{(a\,x^n + bx + c)\,(nbx - bx + cn)} + n\left(\int \frac{(n-1)\,(bx + c)}{(x\,(n-1)\,b + cn)^2\,(a\,x^n + bx + c)}dx\right)\right) + c_2\right)$$

Verified OK.

33.12.3 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= a\,x^n + bx + c \\ q(x) &= 0 \\ r(x) &= -an(n-1)\,x^{n-2} \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= \frac{a\,x^n n^2}{x^2} - \frac{an\,x^n}{x^2} \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$\frac{a\,x^n n^2}{x^2} - \frac{an\,x^n}{x^2} - (0) + \left(-an(n-1)\,x^{n-2}\right) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$(a\,x^n+bx+c)\,y'+\left(-\frac{an\,x^n}{x}-b\right)y=c_1$$

We now have a first order ode to solve which is

$$(a\,x^n+bx+c)\,y'+\left(-\frac{an\,x^n}{x}-b\right)y=c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}
p(x)&=-\frac{x^n na+bx}{(a\,x^n+bx+c)\,x}\\
q(x)&=\frac{c_1}{a\,x^n+bx+c}
\end{aligned}$$

Hence the ode is

$$y'-\frac{(x^n na+bx)\,y}{(a\,x^n+bx+c)\,x}=\frac{c_1}{a\,x^n+bx+c}$$

The integrating factor μ is

$$\begin{aligned}
\mu&=\mathrm{e}^{\int -\frac{x^n na+bx}{(a\,x^n+bx+c)x}dx}\\
&=\frac{1}{a\,\mathrm{e}^{n\ln(x)}+bx+c}
\end{aligned}$$

Which simplifies to

$$\mu=\frac{1}{a\,x^n+bx+c}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=(\mu)\left(\frac{c_1}{a\,x^n+bx+c}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{a\,x^n+bx+c}\right)&=\left(\frac{1}{a\,x^n+bx+c}\right)\left(\frac{c_1}{a\,x^n+bx+c}\right)\\
\mathrm{d}\left(\frac{y}{a\,x^n+bx+c}\right)&=\left(\frac{c_1}{(a\,x^n+bx+c)^2}\right)\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\frac{y}{a\,x^{n}+bx+c} = \int \frac{c_1}{\left(a\,x^{n}+bx+c\right)^{2}}\, \mathrm{d}x$$

$$\frac{y}{a\,x^{n}+bx+c} = \frac{x c_1}{\left(nbx-bx+cn\right)\left(a\,\mathrm{e}^{n\ln(x)}+bx+c\right)} + \left(\int \frac{n(nbx-bx+cn-c)}{\left(nbx-bx+cn\right)^{2}\left(a\,\mathrm{e}^{n\ln(x)}+bx+c\right)}dx\right) c_1 +$$

Dividing both sides by the integrating factor $\mu = \frac{1}{a\,x^{n}+bx+c}$ results in

$$y = \left(a\,x^{n}+bx+c\right)\left(\frac{x c_1}{\left(nbx-bx+cn\right)\left(a\,\mathrm{e}^{n\ln(x)}+bx+c\right)} + \left(\int \frac{n(nbx-bx+cn-c)}{\left(nbx-bx+cn\right)^{2}\left(a\,\mathrm{e}^{n\ln(x)}+bx+c\right)}dx\right)\right.$$

which simplifies to

$$y = \left(a\,x^{n}+bx+c\right)\left(c_1\left(\frac{x}{\left(a\,x^{n}+bx+c\right)\left(nbx-bx+cn\right)} + n\left(\int \frac{(n-1)\left(bx+c\right)}{\left(x\left(n-1\right)b+cn\right)^{2}\left(a\,x^{n}+bx+c\right)}dx\right)\right.\right.$$

Summary

The solution(s) found are the following

$$y = \left(a\,x^{n}+bx+c\right)\left(c_1\left(\frac{x}{\left(a\,x^{n}+bx+c\right)\left(nbx-bx+cn\right)} + n\left(\int \frac{(n-1)\left(bx+c\right)}{\left(x\left(n-1\right)b+cn\right)^{2}\left(a\,x^{n}+bx+c\right)}dx\right)\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = \left(a\,x^{n}+bx+c\right)\left(c_1\left(\frac{x}{\left(a\,x^{n}+bx+c\right)\left(nbx-bx+cn\right)} + n\left(\int \frac{(n-1)\left(bx+c\right)}{\left(x\left(n-1\right)b+cn\right)^{2}\left(a\,x^{n}+bx+c\right)}dx\right)\right) + c_2\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve((a*x^n+b*x+c)*diff(y(x),x$2)=a*n*(n-1)*x^(n-2)*y(x),y(x), singsol=all)
```

$$y(x) = \left(\left(\int \frac{1}{\left(a\,x^{n}+bx+c\right)^{2}}dx\right)c_{1}+c_{2}\right)\left(a\,x^{n}+bx+c\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(a*x^n+b*x+c)*y''[x]==a*n*(n-1)*x^(n-2)*y[x],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

33.13 problem 251

Internal problem ID [11075]
Internal file name [`OUTPUT/10331_Wednesday_January_24_2024_10_07_29_PM_64080441/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 251.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$x(x^n+1)\,y''+((a-b)\,x^n+a-n)\,y'+b(-a+1)\,x^{n-1}y=0$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- Equivalence, under non-integer power transformations successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 65

```
dsolve(x*(x^n+1)*diff(y(x),x$2)+((a-b)*x^n+a-n)*diff(y(x),x)+b*(1-a)*x^(n-1)*y(x)=0,y(x), si
```

$$y(x) = \left(x^{-a+n+1} c_2 \,\mathrm{hypergeom}\left(\left[\frac{b+n}{n}, \frac{-a+n+1}{n}\right], \left[\frac{2n-a+1}{n}\right], -x^n\right) + c_1\right) (x^n + 1)^{\frac{b}{n}}$$

✓ Solution by Mathematica

Time used: 0.164 (sec). Leaf size: 69

```
DSolve[x*(x^n+1)*y''[x]+((a-b)*x^n+a-n)*y'[x]+b*(1-a)*x^(n-1)*y[x]==0,y[x],x,IncludeSingular
```

$$y(x) \to c_2 (x^n)^{\frac{-a+n+1}{n}} \,\mathrm{Hypergeometric2F1}\left(1, \frac{-a-b+n+1}{n}, \frac{-a+2n+1}{n}, -x^n\right) + c_1 (x^n+1)^{b/n}$$

33.14 problem 252

Internal problem ID [11076]
Internal file name [`OUTPUT/10332_Wednesday_January_24_2024_10_07_30_PM_78114833/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 252.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_x_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{x\left(x^{2n}+a\right)y''+\left(x^{2n}+a-an\right)y'-b^{2}x^{-1+2n}y=0}$$

33.14.1 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x\left(x^{2n}+a\right)y''+\left(x^{2n}+a-an\right)y'-b^{2}x^{-1+2n}y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=\frac{x^{2n}+a-an}{x\left(x^{2n}+a\right)}$$
$$q(x)=-\frac{b^{2}x^{2n-2}}{x^{2n}+a}$$

Applying change of variables $\tau=g(x)$ to (2) results

$$\frac{d^{2}}{d\tau^{2}}y(\tau)+p_{1}\left(\frac{d}{d\tau}y(\tau)\right)+q_{1}y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{-\frac{b^2 x^{2n-2}}{x^{2n}+a}}}{c} \\
\tau'' &= \frac{-\frac{b^2 x^{2n-2}(2n-2)}{x(x^{2n}+a)} + \frac{2b^2 x^{2n-2} x^{2n} n}{(x^{2n}+a)^2 x}}{2c\sqrt{-\frac{b^2 x^{2n-2}}{x^{2n}+a}}}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{-\frac{b^2 x^{2n-2}(2n-2)}{x\left(x^{2n}+a\right)} + \frac{2b^2 x^{2n-2} x^{2n} n}{\left(x^{2n}+a\right)^2 x}}{2c\sqrt{-\frac{b^2 x^{2n-2}}{x^{2n}+a}}} + \frac{x^{2n}+a-an}{x(x^{2n}+a)} \frac{\sqrt{-\frac{b^2 x^{2n-2}}{x^{2n}+a}}}{c}}{\left(\frac{\sqrt{-\frac{b^2 x^{2n-2}}{x^{2n}+a}}}{c}\right)^2} \\
&= 0
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\tau = \int \frac{1}{c}\sqrt{q}\, dx$$
$$= \frac{\int \sqrt{-\frac{b^2 x^{2n-2}}{x^{2n}+a}}dx}{c}$$
$$= \frac{\int \sqrt{-\frac{b^2 x^{2n-2}}{x^{2n}+a}}dx}{c}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cosh\left(\frac{b \operatorname{arcsinh}\left(\frac{x^n}{\sqrt{a}}\right)}{n}\right) + i c_2 \sinh\left(\frac{b \operatorname{arcsinh}\left(\frac{x^n}{\sqrt{a}}\right)}{n}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cosh\left(\frac{b \operatorname{arcsinh}\left(\frac{x^n}{\sqrt{a}}\right)}{n}\right) + i c_2 \sinh\left(\frac{b \operatorname{arcsinh}\left(\frac{x^n}{\sqrt{a}}\right)}{n}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cosh\left(\frac{b \operatorname{arcsinh}\left(\frac{x^n}{\sqrt{a}}\right)}{n}\right) + i c_2 \sinh\left(\frac{b \operatorname{arcsinh}\left(\frac{x^n}{\sqrt{a}}\right)}{n}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
   Solution is available but has integrals. Trying a simpler solution using Kovacics algorit
   Solution via Kovacic is not simpler. Returning default solution
   <- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 61

```
dsolve(x*(x^(2*n)+a)*diff(y(x),x$2)+(x^(2*n)+a-a*n)*diff(y(x),x)-b^2*x^(2*n-1)*y(x)=0,y(x),
```

$$y(x) = c_1 e^{ib\left(\int x^{n-1}\sqrt{-\frac{1}{x^{2n}+a}}dx\right)} + c_2 e^{-ib\left(\int x^{n-1}\sqrt{-\frac{1}{x^{2n}+a}}dx\right)}$$

✓ Solution by Mathematica

Time used: 0.458 (sec). Leaf size: 47

```
DSolve[x*(x^(2*n)+a)*y''[x]+(x^(2*n)+a-a*n)*y'[x]-b^2*x^(2*n-1)*y[x]==0,y[x],x,IncludeSingul
```

$$y(x) \to c_1 \cosh\left(\frac{b\operatorname{arcsinh}\left(\frac{x^n}{\sqrt{a}}\right)}{n}\right) + ic_2 \sinh\left(\frac{b\operatorname{arcsinh}\left(\frac{x^n}{\sqrt{a}}\right)}{n}\right)$$

33.15 problem 253

Internal problem ID [11077]
Internal file name [OUTPUT/10333_Wednesday_January_24_2024_10_16_49_PM_16279652/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 253.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$x^2\left(x^{2n}a^2-1\right)y''+x\left(a^2(n+1)\,x^{2n}+n-1\right)y'-\nu(\nu+1)\,a^2n^2x^{2n}y=0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.157 (sec). Leaf size: 23

```
dsolve(x^2*(a^2*x^(2*n)-1)*diff(y(x),x$2)+x*(a^2*(n+1)*x^(2*n)+n-1)*diff(y(x),x)-nu*(nu+1)*a
```

$$y(x) = c_1 \operatorname{LegendreP}\left(\nu, a\, x^n\right) + c_2 \operatorname{LegendreQ}\left(\nu, a\, x^n\right)$$

✓ Solution by Mathematica

Time used: 0.21 (sec). Leaf size: 79

```
DSolve[x^2*(a^2*x^(2*n)-1)*y''[x]+x*(a^2*(n+1)*x^(2*n)+n-1)*y'[x]-\[Nu]*(\[Nu]+1)*a^2*n^2*x^
```

$$y(x) \to iac_2\sqrt{x^{2n}}\,\text{Hypergeometric2F1}\left(\frac{1}{2}-\frac{\nu}{2}, \frac{\nu}{2}+1, \frac{3}{2}, a^2x^{2n}\right) + c_1\,\text{Hypergeometric2F1}\left(-\frac{\nu}{2}, \frac{\nu+1}{2}, \frac{1}{2}, a^2x^{2n}\right)$$

33.16 problem 254

Internal problem ID [11078]
Internal file name [OUTPUT/10334_Wednesday_January_24_2024_10_16_50_PM_90181001/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 254.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^2\left(x^{2n}a^2-1\right)y''+x(ap\,x^n+q)\,y'+(ar\,x^n+s)\,y=0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0, g <
```

✓ Solution by Maple

Time used: 1.734 (sec). Leaf size: 273

```
dsolve(x^2*(a^2*x^(2*n)-1)*diff(y(x),x$2)+x*(a*p*x^n+q)*diff(y(x),x)+(a*r*x^n+s)*y(x)=0,y(x)
```

$$y(x) = x^{\frac{q}{2}+\frac{1}{2}}\left(c_1 x^{\frac{\sqrt{q^2+2q+4s+1}}{2}} \operatorname{HeunG}\left(-1, \frac{pq+\sqrt{q^2+2q+4s+1}\,p+p+2r}{2n^2}, \frac{\sqrt{q^2+2q+4s+1}+q-1}{2n}, \frac{\sqrt{q^2 \cdots}}{\cdots}, -\frac{p-q}{2n}, -a\,x^n\right) + c_2 x^{-\frac{\sqrt{q^2+2q+4s+1}}{2}} \operatorname{HeunG}\left(-1, \frac{-\sqrt{q^2+2q+4s+1}\,p+(q+1)\,p+2r}{2n^2}, -\frac{\sqrt{q^2+2q+4s+1}-q-1}{2n}, -\frac{\sqrt{q^2+2q+4s+1}-q+1}{2n}, \frac{n-\sqrt{q^2+2q+4s+1}}{n}, -\frac{p-q}{2n}, -a\,x^n\right)\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*(a^2*x^(2*n)-1)*y''[x]+x*(a*p*x^n+q)*y'[x]+(a*r*x^n+s)*y[x]==0,y[x],x,IncludeSing
```

Not solved

33.17 problem 255

Internal problem ID [11079]
Internal file name [`OUTPUT/10335_Wednesday_January_24_2024_10_17_11_PM_46271332/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 255.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^n+a)^2 y''-b\,x^{n-2}((b-1)\,x^n+(n-1)\,a)\,y=0}$$

33.17.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y''+\left(-\frac{b^2x^{2n}}{x^2}-\frac{x^n abn}{x^2}+\frac{x^{2n}b}{x^2}+\frac{ab\,x^n}{x^2}\right)y=0 \tag{1}$$

Bessel ode has the form

$$x^2y''+y'x+\left(-n^2+x^2\right)y=0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y''+(1-2\alpha)\,xy'+\left(\beta^2\gamma^2x^{2\gamma}-n^2\gamma^2+\alpha^2\right)y=0 \tag{3}$$

With the standard solution

$$y=x^\alpha(c_1\,\text{BesselJ}\left(n,\beta\,x^\gamma\right)+c_2\,\text{BesselY}\left(n,\beta\,x^\gamma\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= \frac{2\ln(x)}{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)} \\
n &= -\frac{\ln(x)}{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)} \\
\gamma &= \frac{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}{2\ln(x)}
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(-\frac{\ln(x)}{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}, \frac{2\ln(x)\,x^{\frac{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}{2\ln(x)}}}{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}\right)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\,\mathrm{BesselJ}\left(-\frac{\ln(x)}{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}, \frac{2\ln(x)\,x^{\frac{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}{2\ln(x)}}}{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}\right) \\
&\quad + c_2\sqrt{x}\,\mathrm{BesselY}\left(-\frac{\ln(x)}{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}, \frac{2\ln(x)\,x^{\frac{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}{2\ln(x)}}}{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(-\frac{\ln(x)}{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}, \frac{2\ln(x)\,x^{\frac{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}{2\ln(x)}}}{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(-\frac{\ln(x)}{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}, \frac{2\ln(x)\,x^{\frac{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}{2\ln(x)}}}{\ln\left(-\frac{b\left(b\,\mathrm{e}^{n\ln(x)}+an-\mathrm{e}^{n\ln(x)}-a\right)}{x^2}\right)+n\ln(x)}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
      Solution has integrals. Trying a special function solution free of integrals...
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            <- heuristic approach successful
         <- hypergeometric successful
      <- special function solution successful
         -> Trying to convert hypergeometric functions to elementary form...
         <- elementary form could result into a too large expression - returning special fun
      <- Kovacics algorithm successful
<- Equivalence, under non-integer power transformations successful`
```

✓ Solution by Maple

Time used: 0.391 (sec). Leaf size: 75

```
dsolve((x^n+a)^2*diff(y(x),x$2)-b*x^(n-2)*( (b-1)*x^n+a*(n-1))*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2\left(ax + x^{n+1}\right) \operatorname{hypergeom}\left(\left[1, \frac{n-2b+1}{n}\right], \left[1+\frac{1}{n}\right], -\frac{x^n}{a}\right) + \left(\frac{x^n+a}{a}\right)^{\frac{2b}{n}} a c_1\right) (x^n+a)^{-\frac{b}{n}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(x^n+a)^2*y''[x]-b*x^(n-2)*( (b-1)*x^n+a*(n-1))*y[x]==0,y[x],x,IncludeSingularSolutio
```

Not solved

33.18 problem 256

Internal problem ID [11080]
Internal file name [`OUTPUT/10336_Wednesday_January_24_2024_10_17_58_PM_43209986/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 256.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(a\,x^n+b)^2\,y''+(a\,x^n+b)\,(x^n c+d)\,y'+n(-ad+bc)\,x^{n-1}y=0}$$

33.18.1 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$y''\left(x^{2n}a^2+2ab\,x^n+b^2\right)+\left(x^{2n}ac+da\,x^n+bc\,x^n+bd\right)y'+n(-ad+bc)\,x^{n-1}y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=r(x) \tag{2}$$

Where

$$\begin{aligned}p(x)&=\frac{x^{2n}ac+da\,x^n+bc\,x^n+bd}{x^{2n}a^2+2ab\,x^n+b^2}\\ q(x)&=\frac{n\,x^{n-1}(-ad+bc)}{x^{2n}a^2+2ab\,x^n+b^2}\\ r(x)&=0\end{aligned}$$

The Lagrange adjoint ode is given by

$$\xi'' - \left(\frac{(x^{2n}ac + da\,x^n + b}{x^{2n}a^2 + 2ab}\right.$$

$$\xi''(x) - \frac{(x^{2n}ac + da\,x^n + bc\,x^n + bd)\,\xi'(x)}{x^{2n}a^2 + 2ab\,x^n + b^2} + \left(\frac{(x^{2n}ac + da\,x^n + bc\,x^n + bd)\left(\frac{2x^{2n}n\,a^2}{x} + \frac{2abn\,x^n}{x}\right)}{(x^{2n}a^2 + 2ab\,x^n + b^2)^2} - \frac{\frac{2x^{2n}nac}{x}}{x^{2n}a}\right.$$

Which is solved for $\xi(x)$. The ODE is

$$x\left(x^{4n}a^4 + 4b\,x^{3n}a^3 + 6x^{2n}a^2b^2 + 4a\,b^3x^n + b^4\right)\xi''(x) - x\left(\left(3a^2bd + 3b^2ca\right)x^{2n} + \left(a^3d + 3a^2bc\right)x^{3n} + x^{4n}a^3c\right.$$

Or

$$x\left(6x^{2n}\xi''(x)\,a^2b^2 - 3x^{2n}\xi'(x)\,a^2bd - 3x^{2n}\xi'(x)\,a\,b^2c + 4x^n\xi''(x)\,a\,b^3 - 3x^n\xi'(x)\,a\,b^2d - x^n\xi'(x)\,b^3c + 4x^{3n}\xi\right.$$

For $x \neq 0$ the above simplifies to

$$\xi''(x)\left(x^{4n}a^4 + 4b\,x^{3n}a^3 + 6x^{2n}a^2b^2 + 4a\,b^3x^n + b^4\right) - \left(\left(3a^2bd + 3b^2ca\right)x^{2n} + \left(a^3d + 3a^2bc\right)x^{3n} + x^{4n}a^3c + \right.$$

This is second order ode with missing dependent variable $\xi(x)$. Let

$$p(x) = \xi'(x)$$

Then

$$p'(x) = \xi''(x)$$

Hence the ode becomes

$$x\left(x^{4n}a^4 + 4b\,x^{3n}a^3 + 6x^{2n}a^2b^2 + 4a\,b^3x^n + b^4\right)p'(x) - x\left(\left(3a^2bd + 3b^2ca\right)x^{2n} + \left(a^3d + 3a^2bc\right)x^{3n} + x^{4n}a^3\right.$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= \frac{p(x^{4n}a^3c + d\,x^{3n}a^3 + 3bc\,x^{3n}a^2 + 3x^{2n}a^2bd + 3x^{2n}a\,b^2c + 3a\,b^2x^nd + b^3c\,x^n + b^3d)}{x^{4n}a^4 + 4b\,x^{3n}a^3 + 6x^{2n}a^2b^2 + 4a\,b^3x^n + b^4} \end{aligned}$$

Where $f(x) = \frac{x^{4n}a^3c+d\,x^{3n}a^3+3bc\,x^{3n}a^2+3x^{2n}a^2bd+3x^{2n}a\,b^2c+3a\,b^2x^n d+b^3c\,x^n+b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}$ and $g(p) = p$. Integrating both sides gives

$$\frac{1}{p}\,dp = \frac{x^{4n}a^3c+d\,x^{3n}a^3+3bc\,x^{3n}a^2+3x^{2n}a^2bd+3x^{2n}a\,b^2c+3a\,b^2x^n d+b^3c\,x^n+b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}\,dx$$

$$\int\frac{1}{p}\,dp = \int\frac{x^{4n}a^3c+d\,x^{3n}a^3+3bc\,x^{3n}a^2+3x^{2n}a^2bd+3x^{2n}a\,b^2c+3a\,b^2x^n d+b^3c\,x^n+b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}\,dx$$

$$\ln(p) = \int\frac{x^{4n}a^3c+d\,x^{3n}a^3+3bc\,x^{3n}a^2+3x^{2n}a^2bd+3x^{2n}a\,b^2c+3a\,b^2x^n d+b^3c\,x^n+b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}dx + c_1$$

$$p = \mathrm{e}^{\int\frac{x^{4n}a^3c+d\,x^{3n}a^3+3bc\,x^{3n}a^2+3x^{2n}a^2bd+3x^{2n}a\,b^2c+3a\,b^2x^n d+b^3c\,x^n+b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}dx+c_1}$$

$$= c_1\mathrm{e}^{\int\frac{x^{4n}a^3c+d\,x^{3n}a^3+3bc\,x^{3n}a^2+3x^{2n}a^2bd+3x^{2n}a\,b^2c+3a\,b^2x^n d+b^3c\,x^n+b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}dx}$$

Since $p = \xi'(x)$ then the new first order ode to solve is

$$\xi'(x) = c_1\mathrm{e}^{\int\frac{x^{4n}a^3c+d\,x^{3n}a^3+3bc\,x^{3n}a^2+3x^{2n}a^2bd+3x^{2n}a\,b^2c+3a\,b^2x^n d+b^3c\,x^n+b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}dx}$$

Writing the ode as

$$\xi'(x) = c_1\mathrm{e}^{\int\frac{x^{4n}a^3c+d\,x^{3n}a^3+3bc\,x^{3n}a^2+3x^{2n}a^2bd+3x^{2n}a\,b^2c+3a\,b^2x^n d+b^3c\,x^n+b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}dx}$$
$$\xi'(x) = \omega(x,\xi)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_\xi - \xi_x) - \omega^2\xi_\xi - \omega_x\xi - \omega_\xi\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + \xi a_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + \xi b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + c_1\mathrm{e}^{\int\frac{x^{4n}a^3c+d\,x^{3n}a^3+3bc\,x^{3n}a^2+3x^{2n}a^2bd+3x^{2n}a\,b^2c+3a\,b^2x^n d+b^3c\,x^n+b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}dx}(b_3 - a_2) \tag{5E}$$
$$- c_1^2\mathrm{e}^{\int\frac{2x^{4n}a^3c+2d\,x^{3n}a^3+6bc\,x^{3n}a^2+6x^{2n}a^2bd+6x^{2n}a\,b^2c+6a\,b^2x^n d+2b^3c\,x^n+2b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}dx}a_3$$
$$- \frac{c_1(x^{4n}a^3c+d\,x^{3n}a^3+3bc\,x^{3n}a^2+3x^{2n}a^2bd+3x^{2n}a\,b^2c+3a\,b^2x^n d+b^3c\,x^n+b^3d)\,\mathrm{e}^{\int\frac{x^{4n}a^3c+d\,x^{3n}a^3+3bc\,x^{3n}a}{x^{4n}a^4+4b}}}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}$$
$$= 0$$

Putting the above in normal form gives

$$\text{Expression too large to display}$$

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, \xi\}$ in them.

$$\left\{x, \xi, x^{n}, x^{2n}, x^{3n}, x^{4n}, \int \frac{x^{4n}a^{3}c + d\,x^{3n}a^{3} + 3bc\,x^{3n}a^{2} + 3x^{2n}a^{2}bd + 3x^{2n}a\,b^{2}c + 3a\,b^{2}x^{n}d + b^{3}c\,x^{n} + b^{3}d}{x^{4n}a^{4} + 4b\,x^{3n}a^{3} + 6x^{2n}a^{2}b^{2} + 4a\,b^{3}x^{n} + b^{4}} d$$

The following substitution is now made to be able to collect on all terms with $\{x, \xi\}$ in them

$$\left\{x = v_{1}, \xi = v_{2}, x^{n} = v_{3}, x^{2n} = v_{4}, x^{3n} = v_{5}, x^{4n} = v_{6}, \int \frac{x^{4n}a^{3}c + d\,x^{3n}a^{3} + 3bc\,x^{3n}a^{2} + 3x^{2n}a^{2}bd + 3x^{2n}a\,b^{2}}{x^{4n}a^{4} + 4b\,x^{3n}a^{3} + 6x^{2n}a^{2}b^{2} + 4a}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a^{4}c_{1}^{2}a_{3}v_{3}^{4}v_{8} - a^{3}cc_{1}a_{2}v_{1}v_{3}^{4}v_{9} - a^{3}cc_{1}a_{3}v_{2}v_{3}^{4}v_{9} - a^{4}c_{1}a_{2}v_{3}^{4}v_{9} \\
&+ a^{4}c_{1}b_{3}v_{3}^{4}v_{9} - 4a^{3}b\,c_{1}^{2}a_{3}v_{3}^{3}v_{8} - a^{3}cc_{1}a_{1}v_{3}^{4}v_{9} - a^{3}c_{1}da_{2}v_{1}v_{3}^{3}v_{9} \\
&- a^{3}c_{1}da_{3}v_{2}v_{3}^{3}v_{9} - 3a^{2}bcc_{1}a_{2}v_{1}v_{3}^{3}v_{9} - 3a^{2}bcc_{1}a_{3}v_{2}v_{3}^{3}v_{9} - 4a^{3}bc_{1}a_{2}v_{3}^{3}v_{9} \\
&+ 4a^{3}bc_{1}b_{3}v_{3}^{3}v_{9} - a^{3}c_{1}da_{1}v_{3}^{3}v_{9} - 6a^{2}b^{2}c_{1}^{2}a_{3}v_{3}^{2}v_{8} - 3a^{2}bcc_{1}a_{1}v_{3}^{3}v_{9} \\
&- 3a^{2}bc_{1}da_{2}v_{1}v_{3}^{2}v_{9} - 3a^{2}bc_{1}da_{3}v_{2}v_{3}^{2}v_{9} - 3a\,b^{2}cc_{1}a_{2}v_{1}v_{3}^{2}v_{9} \\
&- 3a\,b^{2}cc_{1}a_{3}v_{2}v_{3}^{2}v_{9} + v_{3}^{4}a^{4}b_{2} - 6a^{2}b^{2}c_{1}a_{2}v_{3}^{2}v_{9} + 6a^{2}b^{2}c_{1}b_{3}v_{3}^{2}v_{9} \\
&- 3a^{2}bc_{1}da_{1}v_{3}^{2}v_{9} - 4a\,b^{3}c_{1}^{2}a_{3}v_{3}v_{8} - 3a\,b^{2}cc_{1}a_{1}v_{3}^{2}v_{9} - 3a\,b^{2}c_{1}da_{2}v_{1}v_{3}v_{9} \\
&- 3a\,b^{2}c_{1}da_{3}v_{2}v_{3}v_{9} - b^{3}cc_{1}a_{2}v_{1}v_{3}v_{9} - b^{3}cc_{1}a_{3}v_{2}v_{3}v_{9} + 4v_{3}^{3}a^{3}bb_{2} \\
&- 4a\,b^{3}c_{1}a_{2}v_{3}v_{9} + 4a\,b^{3}c_{1}b_{3}v_{3}v_{9} - 3a\,b^{2}c_{1}da_{1}v_{3}v_{9} - b^{4}c_{1}^{2}a_{3}v_{8} \\
&- b^{3}cc_{1}a_{1}v_{3}v_{9} - b^{3}c_{1}da_{2}v_{1}v_{9} - b^{3}c_{1}da_{3}v_{2}v_{9} + 6v_{3}^{2}a^{2}b^{2}b_{2} \\
&- b^{4}c_{1}a_{2}v_{9} + b^{4}c_{1}b_{3}v_{9} - b^{3}c_{1}da_{1}v_{9} + 4v_{3}a\,b^{3}b_{2} + b^{4}b_{2} = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}, v_{7}, v_{8}, v_{9}\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-b^4c_1^2a_3v_8+b^4b_2+\left(-a^3c_1da_2-3a^2bcc_1a_2\right)v_1v_3^3v_9\\
&+\left(-3a^2bc_1da_2-3a\,b^2cc_1a_2\right)v_1v_3^2v_9+\left(-3a\,b^2c_1da_2-b^3cc_1a_2\right)v_1v_3v_9\\
&+\left(-a^3c_1da_3-3a^2bcc_1a_3\right)v_2v_3^3v_9+\left(-b^4c_1a_2+b^4c_1b_3-b^3c_1da_1\right)v_9\\
&+\left(-3a^2bc_1da_3-3a\,b^2cc_1a_3\right)v_2v_3^2v_9\\
&+\left(-3a\,b^2c_1da_3-b^3cc_1a_3\right)v_2v_3v_9+v_3^4a^4b_2+6v_3^2a^2b^2b_2\\
&+\left(-4a\,b^3c_1a_2+4a\,b^3c_1b_3-3a\,b^2c_1da_1-b^3cc_1a_1\right)v_3v_9\\
&+\left(-a^4c_1a_2+a^4c_1b_3-a^3cc_1a_1\right)v_3^4v_9\\
&+\left(-4a^3bc_1a_2+4a^3bc_1b_3-a^3c_1da_1-3a^2bcc_1a_1\right)v_3^3v_9\\
&+\left(-6a^2b^2c_1a_2+6a^2b^2c_1b_3-3a^2bc_1da_1-3a\,b^2cc_1a_1\right)v_3^2v_9\\
&+4v_3^3a^3bb_2+4v_3a\,b^3b_2-a^4c_1^2a_3v_3^4v_8-a^3cc_1a_2v_1v_3^4v_9\\
&-a^3cc_1a_3v_2v_3^4v_9-4a^3b\,c_1^2a_3v_3^3v_8-6a^2b^2c_1^2a_3v_3^2v_8\\
&-4a\,b^3c_1^2a_3v_3v_8-b^3c_1da_2v_1v_9-b^3c_1da_3v_2v_9=0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a^4 b_2 &= 0\\
b^4 b_2 &= 0\\
-c_1^2 a^4 a_3 &= 0\\
-c_1^2 b^4 a_3 &= 0\\
4a\, b^3 b_2 &= 0\\
6a^2 b^2 b_2 &= 0\\
4a^3 b b_2 &= 0\\
-c_1 a^3 c a_2 &= 0\\
-c_1 a^3 c a_3 &= 0\\
-c_1 b^3 d a_3 &= 0\\
-4c_1^2 a\, b^3 a_3 &= 0\\
-6c_1^2 a^2 b^2 a_3 &= 0\\
-4c_1^2 a^3 b a_3 &= 0\\
-b^3 d a_2 c_1 &= 0\\
-3a\, b^2 c_1 d a_2 - b^3 c c_1 a_2 &= 0\\
-3a^2 b c_1 d a_2 - 3a\, b^2 c c_1 a_2 &= 0\\
-a^3 c_1 d a_2 - 3a^2 b c c_1 a_2 &= 0\\
-3a\, b^2 c_1 d a_3 - b^3 c c_1 a_3 &= 0\\
-3a^2 b c_1 d a_3 - 3a\, b^2 c c_1 a_3 &= 0\\
-a^3 c_1 d a_3 - 3a^2 b c c_1 a_3 &= 0\\
-a^4 c_1 a_2 + a^4 c_1 b_3 - a^3 c c_1 a_1 &= 0\\
-b^4 c_1 a_2 + b^4 c_1 b_3 - b^3 c_1 d a_1 &= 0\\
-4a\, b^3 c_1 a_2 + 4a\, b^3 c_1 b_3 - 3a\, b^2 c_1 d a_1 - b^3 c c_1 a_1 &= 0\\
-6a^2 b^2 c_1 a_2 + 6a^2 b^2 c_1 b_3 - 3a^2 b c_1 d a_1 - 3a\, b^2 c c_1 a_1 &= 0\\
-4a^3 b c_1 a_2 + 4a^3 b c_1 b_3 - a^3 c_1 d a_1 - 3a^2 b c c_1 a_1 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 0 \\ \eta &= 1 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, \xi) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{d\xi}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial \xi}\right) S(x, \xi) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{1} dy \end{aligned}$$

Which results in

$$S = \xi$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, \xi) S_\xi}{R_x + \omega(x, \xi) R_\xi} \tag{2}$$

Where in the above R_x, R_ξ, S_x, S_ξ are all partial derivatives and $\omega(x,\xi)$ is the right hand side of the original ode given by

$$\omega(x,\xi) = c_1 \mathrm{e}^{\int \frac{x^{4n}a^3c+d\,x^{3n}a^3+3bc\,x^{3n}a^2+3x^{2n}a^2bd+3x^{2n}a\,b^2c+3a\,b^2x^n d+b^3c\,x^n+b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}dx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_\xi &= 0 \\ S_x &= 0 \\ S_\xi &= 1 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = c_1 \mathrm{e}^{\int \frac{(3a^2bd+3b^2ca)x^{2n}+(a^3d+3a^2bc)x^{3n}+x^{4n}a^3c+(3a\,b^2d+b^3c)x^n+b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}dx} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, ξ in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = c_1 \mathrm{e}^{\int \frac{(3a^2bd+3b^2ca)R^{2n}+(a^3d+3a^2bc)R^{3n}+R^{4n}a^3c+(3a\,b^2d+b^3c)R^n+b^3d}{R^{4n}a^4+4b\,R^{3n}a^3+6R^{2n}a^2b^2+4a\,b^3R^n+b^4}dR}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int c_1 \mathrm{e}^{\int \frac{3R^{2n}a^2bd+3R^{2n}a\,b^2c+R^{3n}a^3d+3R^{3n}a^2bc+R^{4n}a^3c+3R^n a\,b^2d+R^n b^3c+b^3d}{R^{4n}a^4+4b\,R^{3n}a^3+6R^{2n}a^2b^2+4a\,b^3R^n+b^4}dR}dR + c_2 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, ξ coordinates. This results in

$$\xi(x) = \int c_1 \mathrm{e}^{\int \frac{x^{4n}a^3c+d\,x^{3n}a^3+3bc\,x^{3n}a^2+3x^{2n}a^2bd+3x^{2n}a\,b^2c+3a\,b^2x^n d+b^3c\,x^n+b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}dx}dx + c_2$$

Which simplifies to

$$\xi(x) - c_1\left(\int \mathrm{e}^{\int \frac{(3a^2bd+3b^2ca)x^{2n}+(a^3d+3a^2bc)x^{3n}+x^{4n}a^3c+(3a\,b^2d+b^3c)x^n+b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}dx}dx\right) - c_2 = 0$$

Which gives

$$\xi(x) = c_1\left(\int \mathrm{e}^{\int \frac{(3a^2bd+3b^2ca)x^{2n}+(a^3d+3a^2bc)x^{3n}+x^{4n}a^3c+(3a\,b^2d+b^3c)x^n+b^3d}{x^{4n}a^4+4b\,x^{3n}a^3+6x^{2n}a^2b^2+4a\,b^3x^n+b^4}dx}dx\right) + c_2$$

The original ode (2) now reduces to first order ode

$$\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x$$

$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right.$$

$$y' + y\left(\frac{x^{2n}ac + da\, x^n + bc\, x^n + bd}{x^{2n}a^2 + 2ab\, x^n + b^2} - \frac{c_3 e^{\int \frac{\left(3a^2bd+3b^2ca\right)x^{2n}+\left(a^3d+3a^2bc\right)x^{3n}+x^{4n}a^3c+\left(3a\, b^2d+b^3c\right)x^n+b^3d}{x^{4n}a^4+4b\, x^{3n}a^3+6x^{2n}a^2b^2+4a\, b^3x^n+b^4}dx}}{c_3\left(\int e^{\int \frac{\left(3a^2bd+3b^2ca\right)x^{2n}+\left(a^3d+3a^2bc\right)x^{3n}+x^{4n}a^3c+\left(3a\, b^2d+b^3c\right)x^n+b^3d}{x^{4n}a^4+4b\, x^{3n}a^3+6x^{2n}a^2b^2+4a\, b^3x^n+b^4}dx}dx\right) + c_2}\right.$$

Which is now a first order ode. This is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-c_3(x^{2n}ac + (ad+bc)\, x^n + bd)\left(\int e^{\int \frac{x^nc+d}{a\, x^n+b}dx}dx\right) + c_3(x^{2n}a^2 + 2ab\, x^n + b^2)\, e^{\int \frac{x^nc+d}{a\, x^n+b}dx} - (x^{2n}ac +}{\left(c_3\left(\int e^{\int \frac{x^nc+d}{a\, x^n+b}dx}dx\right) + c_2\right)(x^{2n}a^2 + 2ab\, x^n + b^2)}$$

$$q(x) = 0$$

Hence the ode is

$$y' - \frac{\left(-c_3(x^{2n}ac + (ad+bc)\, x^n + bd)\left(\int e^{\int \frac{x^nc+d}{a\, x^n+b}dx}dx\right) + c_3(x^{2n}a^2 + 2ab\, x^n + b^2)\, e^{\int \frac{x^nc+d}{a\, x^n+b}dx} - (x^{2n}ac + (a\right.}{\left(c_3\left(\int e^{\int \frac{x^nc+d}{a\, x^n+b}dx}dx\right) + c_2\right)(x^{2n}a^2 + 2ab\, x^n + b^2)}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-c_3\left(x^{2n}ac+(ad+bc)x^n+bd\right)\left(\int e^{\int \frac{x^nc+d}{a\, x^n+b}dx}dx\right)+c_3\left(x^{2n}a^2+2ab\, x^n+b^2\right)e^{\int \frac{x^nc+d}{a\, x^n+b}dx}-\left(x^{2n}ac+(ad+bc)x^n+bd\right)c_2}{\left(c_3\left(\int e^{\int \frac{x^nc+d}{a\, x^n+b}dx}dx\right)+c_2\right)\left(x^{2n}a^2+2ab\, x^n+b^2\right)}dx}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$

$$\frac{d}{dx}\left(e^{\int -\frac{-c_3\left(x^{2n}ac+(ad+bc)x^n+bd\right)\left(\int e^{\int \frac{x^nc+d}{a\, x^n+b}dx}dx\right)+c_3\left(x^{2n}a^2+2ab\, x^n+b^2\right)e^{\int \frac{x^nc+d}{a\, x^n+b}dx}-\left(x^{2n}ac+(ad+bc)x^n+bd\right)c_2}{\left(c_3\left(\int e^{\int \frac{x^nc+d}{a\, x^n+b}dx}dx\right)+c_2\right)\left(x^{2n}a^2+2ab\, x^n+b^2\right)}dx} y\right) = 0$$

Integrating gives

$$e^{\int -\frac{-c_3\left(x^{2n}ac+(ad+bc)x^n+bd\right)\left(\int e^{\int \frac{x^n c+d}{a\,x^n+b}dx}dx\right)+c_3\left(x^{2n}a^2+2ab\,x^n+b^2\right)e^{\int \frac{x^n c+d}{a\,x^n+b}dx}-\left(x^{2n}ac+(ad+bc)x^n+bd\right)c_2}{\left(c_3\left(\int e^{\int \frac{x^n c+d}{a\,x^n+b}dx}dx\right)+c_2\right)\left(x^{2n}a^2+2ab\,x^n+b^2\right)}dx}\,y = c_3$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{-c_3\left(x^{2n}ac+(ad+bc)x^n+bd\right)\left(\int e^{\int \frac{x^n c+d}{a\,x^n+b}dx}dx\right)+c_3\left(x^{2n}a^2+2ab\,x^n+b^2\right)e^{\int \frac{x^n c+d}{a\,x^n+b}dx}-\left(x^{2n}ac+(ad+bc)x^n+bd\right)c_2}{\left(c_3\left(\int e^{\int \frac{x^n c+d}{a\,x^n+b}dx}dx\right)+c_2\right)\left(x^{2n}a^2+2ab\,x^n+b^2\right)}dx}$ results in

$$y = c_3 e^{\int \frac{-c_3\left(x^{2n}ac+(ad+bc)x^n+bd\right)\left(\int e^{\int \frac{x^n c+d}{a\,x^n+b}dx}dx\right)+c_3\left(x^{2n}a^2+2ab\,x^n+b^2\right)e^{\int \frac{x^n c+d}{a\,x^n+b}dx}-\left(x^{2n}ac+(ad+bc)x^n+bd\right)c_2}{\left(c_3\left(\int e^{\int \frac{x^n c+d}{a\,x^n+b}dx}dx\right)+c_2\right)\left(x^{2n}a^2+2ab\,x^n+b^2\right)}dx}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_3 e^{\int \frac{-c_3\left(x^{2n}ac+(ad+bc)x^n+bd\right)\left(\int e^{\int \frac{x^n c+d}{a\,x^n+b}dx}dx\right)+c_3\left(x^{2n}a^2+2ab\,x^n+b^2\right)e^{\int \frac{x^n c+d}{a\,x^n+b}dx}-\left(x^{2n}ac+(ad+bc)x^n+bd\right)c_2}{\left(c_3\left(\int e^{\int \frac{x^n c+d}{a\,x^n+b}dx}dx\right)+c_2\right)\left(x^{2n}a^2+2ab\,x^n+b^2\right)}dx}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_3 e^{\int \frac{-c_3\left(x^{2n}ac+(ad+bc)x^n+bd\right)\left(\int e^{\int \frac{x^n c+d}{a\,x^n+b}dx}dx\right)+c_3\left(x^{2n}a^2+2ab\,x^n+b^2\right)e^{\int \frac{x^n c+d}{a\,x^n+b}dx}-\left(x^{2n}ac+(ad+bc)x^n+bd\right)c_2}{\left(c_3\left(\int e^{\int \frac{x^n c+d}{a\,x^n+b}dx}dx\right)+c_2\right)\left(x^{2n}a^2+2ab\,x^n+b^2\right)}dx} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_3 e^{\int \frac{-c_3\left(x^{2n}ac+(ad+bc)x^n+bd\right)\left(\int e^{\int \frac{x^n c+d}{a\,x^n+b}dx}dx\right)+c_3\left(x^{2n}a^2+2ab\,x^n+b^2\right)e^{\int \frac{x^n c+d}{a\,x^n+b}dx}-\left(x^{2n}ac+(ad+bc)x^n+bd\right)c_2}{\left(c_3\left(\int e^{\int \frac{x^n c+d}{a\,x^n+b}dx}dx\right)+c_2\right)\left(x^{2n}a^2+2ab\,x^n+b^2\right)}dx}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
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      -> Bessel
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         -> hyper3: Equivalence to 1F1 under a power @ Moebius
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   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✓ Solution by Maple

Time used: 0.406 (sec). Leaf size: 53

```
dsolve((a*x^n+b)^2*diff(y(x),x$2)+(a*x^n+b)*(c*x^n+d)*diff(y(x),x)+n*(b*c-a*d)*x^(n-1)*y(x)=
```

$$y(x) = \mathrm{e}^{-\left(\int \frac{c\,x^n+d}{a\,x^n+b}dx\right)}\left(c_1+\left(\int \mathrm{e}^{\int \frac{c\,x^n+d}{a\,x^n+b}dx}dx\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.923 (sec). Leaf size: 106

```
DSolve[(a*x^n+b)^2*y''[x]+(a*x^n+b)*(c*x^n+d)*y'[x]+n*(b*c-a*d)*x^(n-1)*y[x]==0,y[x],x,Inclu
```

$$y(x) \to \exp\left(-\frac{x\left((ad-bc)\,\text{Hypergeometric2F1}\left(1,\frac{1}{n},1+\frac{1}{n},-\frac{ax^n}{b}\right)+bc\right)}{ab}\right)\left(\int_1^x \exp\left(\frac{\left(bc+(ad-bc)\,\text{Hyper}\right.}{}\right.\right.$$

$$\left.+c_2\right)$$

33.19 problem 257

Internal problem ID [11081]
Internal file name [`OUTPUT/10337_Wednesday_January_24_2024_10_17_59_PM_53634759/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 257.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$(x^n + a)^2 y'' + b\,x^m(x^n + a)\,y' - x^{n-2}\big(b\,x^{m+1} + an - a\big)\,y = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
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      -> Bessel
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   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve((x^n+a)^2*diff(y(x),x$2)+b*x^m*(x^n+a)*diff(y(x),x)-x^(n-2)*(b*x^(m+1)+a*n-a)*y(x)=0,
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(x^n+a)^2*y''[x]+b*x^m*(x^n+a)*y'[x]-x^(n-2)*(b*x^(m+1)+a*n-a)*y[x]==0,y[x],x,Include
```

Not solved

33.20 problem 258

Internal problem ID [11082]
Internal file name [OUTPUT/10338_Wednesday_January_24_2024_10_18_00_PM_40001740/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 258.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$(a\,x^n+b)^2\,y''+c\,x^m(a\,x^n+b)\,y'+\left(c\,x^m-an\,x^{n-1}-1\right)y=0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
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   -> Bessel
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   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
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   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
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         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve((a*x^n+b)^2*diff(y(x),x$2)+c*x^m*(a*x^n+b)*diff(y(x),x)+(c*x^m-a*n*x^(n-1)-1)*y(x)=0,
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(a*x^n+b)^2*y''[x]+c*x^m*(a*x^n+b)*y'[x]+(c*x^m-a*n*x^(n-1)-1)*y[x]==0,y[x],x,Include
```

Not solved

33.21 problem 259

Internal problem ID [11083]
Internal file name [`OUTPUT/10339_Wednesday_January_24_2024_10_18_01_PM_37857849/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 259.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^2(a\,x^n+b)^2\,y''+(n+1)\,x\left(x^{2n}a^2-b^2\right)y'+yc=0}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- Equivalence, under non-integer power transformations successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 127

```
dsolve(x^2*(a*x^n+b)^2*diff(y(x),x$2)+(n+1)*x*(a^2*x^(2*n)-b^2)*diff(y(x),x)+c*y(x)=0,y(x),
```

$$y(x) = \sqrt{a\,x^{2n}+b\,x^{n}}\,(a\,x^{n}+b)^{\frac{-n-1}{n}}\,x\left(\left(\frac{x^{n}}{a\,x^{n}+b}\right)^{-\frac{\sqrt{\frac{(n+2)^{2}b^{2}-4c}{n^{2}a^{2}}}\,a}{2b}}c_{2}+\left(\frac{x^{n}}{a\,x^{n}+b}\right)^{\frac{\sqrt{\frac{(n+2)^{2}b^{2}-4c}{n^{2}a^{2}}}\,a}{2b}}c_{1}\right)$$

✓ Solution by Mathematica

Time used: 0.407 (sec). Leaf size: 149

```
DSolve[x^2*(a*x^n+b)^2*y''[x]+(n+1)*x*(a^2*x^(2*n)-b^2)*y'[x]+c*y[x]==0,y[x],x,IncludeSingul
```

$$y(x) \to c_1\exp\left(\frac{\left(b(n+2)-\sqrt{c}\sqrt{\frac{b^2(n+2)^2-4c}{c}}\right)\left(-\log\left(ax^n+b\right)-\log(b)+n\log(x)-\log(n)\right)}{2bn}\right)+c_2\exp\left(\frac{\left(\sqrt{c}\sqrt{\frac{b^2(n+2)^2-4c}{c}}+b(n+2)\right)\left(-\log\left(ax^n+b\right)-\log(b)+n\log(x)-\log(n)\right)}{2bn}\right)$$

33.22 problem 260

Internal problem ID [11084]
Internal file name [`OUTPUT/10340_Wednesday_January_24_2024_10_18_01_PM_17636971/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 260.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$(a\,x^{n+1}+b\,x^n+c)^2\,y''+(\alpha\,x^n+\beta\,x^{n-1}+\gamma)\,y'+\left(n(-an-a+\alpha)\,x^{n-1}+(n-1)\,(-nb+\beta)\,x^{n-2}\right)y =$$

✗ Solution by Maple

```
dsolve((a*x^(n+1)+b*x^n+c)^2*diff(y(x),x$2)+(alpha*x^n+beta*x^(n-1)+gamma)*diff(y(x),x)+(n*(
```

No solution found

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(a*x^(n+1)+b*x^n+c)^2*y''[x]+(\[Alpha]*x^n+\[Beta]*x^(n-1)+\[Gamma])*y'[x]+(n*(\[Alph
```

Not solved

33.23 problem 261

Internal problem ID [11085]
Internal file name [OUTPUT/10341_Wednesday_January_24_2024_10_18_04_PM_71204093/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 261.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(a\,x^n + b\,x^m + c)\,y'' + (\lambda - x)\,y' + y = 0}$$

33.23.1 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B+2v'B'+B''v)+B(B'v+v'B)+CBv&=0\\
ABv''+\left(2AB'+B^2\right)v'+\left(AB''+BB'+CB\right)v&=0 \qquad (1)
\end{aligned}
$$

If the term $AB''+BB'+CB$ is zero, then this method works and can be used to solve

$$
ABv''+\left(2AB'+B^2\right)v'=0
$$

By Using $u=v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu'+\left(2AB'+B^2\right)u=0
$$

The above ode is first order ode which is solved for u. Now a new ode $v'=u$ is solved for v as first order ode. Then the final solution is obtain from $y=Bv$.

This method works only if the term $AB''+BB'+CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A&=a\,x^n+b\,x^m+c\\
B&=\lambda-x\\
C&=1\\
F&=0
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB''+BB'+CB&=\left(a\,x^n+b\,x^m+c\right)(0)+(\lambda-x)(-1)+(1)(\lambda-x)\\
&=0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
\left(a\,x^n+b\,x^m+c\right)(\lambda-x)\,v''+\left(-2a\,x^n-2b\,x^m-2c+(\lambda-x)^2\right)v'=0
$$

Now by applying $v'=u$ the above becomes

$$
\left(a\,x^n+b\,x^m+c\right)(\lambda-x)\,u'(x)-2\left(a\,x^n+b\,x^m-\frac{x^2}{2}+\lambda x-\frac{\lambda^2}{2}+c\right)u(x)=0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u'&=F(x,u)\\
&=f(x)g(u)\\
&=\frac{u(2b\,x^m+2a\,x^n-\lambda^2+2\lambda x-x^2+2c)}{x^m b\lambda-x^m bx+x^n a\lambda-ax\,x^n+c\lambda-cx}
\end{aligned}
$$

Where $f(x) = \frac{2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x-x^{2}+2c}{x^{m}b\lambda-x^{m}bx+x^{n}a\lambda-ax\,x^{n}+c\lambda-cx}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x-x^{2}+2c}{x^{m}b\lambda-x^{m}bx+x^{n}a\lambda-ax\,x^{n}+c\lambda-cx}\,dx\\
\int \frac{1}{u}\,du &= \int \frac{2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x-x^{2}+2c}{x^{m}b\lambda-x^{m}bx+x^{n}a\lambda-ax\,x^{n}+c\lambda-cx}\,dx\\
\ln\left(u\right) &= \int \frac{2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x-x^{2}+2c}{x^{m}b\lambda-x^{m}bx+x^{n}a\lambda-ax\,x^{n}+c\lambda-cx}dx+c_{1}\\
u &= \mathrm{e}^{\int \frac{2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x-x^{2}+2c}{x^{m}b\lambda-x^{m}bx+x^{n}a\lambda-ax\,x^{n}+c\lambda-cx}dx+c_{1}}\\
&= c_{1}\mathrm{e}^{\int \frac{2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x-x^{2}+2c}{x^{m}b\lambda-x^{m}bx+x^{n}a\lambda-ax\,x^{n}+c\lambda-cx}dx}
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u\\
&= c_{1}\mathrm{e}^{\int \frac{2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x-x^{2}+2c}{x^{m}b\lambda-x^{m}bx+x^{n}a\lambda-ax\,x^{n}+c\lambda-cx}dx}
\end{aligned}
$$

Which is now solved for v. Writing the ode as

$$
\begin{aligned}
v'(x) &= c_{1}\mathrm{e}^{\int \frac{2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x-x^{2}+2c}{x^{m}b\lambda-x^{m}bx+x^{n}a\lambda-ax\,x^{n}+c\lambda-cx}dx}\\
v'(x) &= \omega(x,v)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_v-\xi_x)-\omega^2\xi_v-\omega_x\xi-\omega_v\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = va_{3}+xa_{2}+a_{1} \tag{1E}
$$

$$
\eta = vb_{3}+xb_{2}+b_{1} \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_{1}, a_{2}, a_{3}, b_{1}, b_{2}, b_{3}\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_{2}+c_{1}\mathrm{e}^{\int \frac{2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x-x^{2}+2c}{x^{m}b\lambda-x^{m}bx+x^{n}a\lambda-ax\,x^{n}+c\lambda-cx}dx}(b_{3}-a_{2})-c_{1}^{2}\mathrm{e}^{\int \frac{4b\,x^{m}+4a\,x^{n}-2\lambda^{2}+4\lambda x-2x^{2}+4c}{x^{m}b\lambda-b\,x^{m+1}+x^{n}a\lambda-a\,x^{n+1}+c\lambda-cx}dx}a_{3}\\
&-\frac{c_{1}\left(2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x-x^{2}+2c\right)\mathrm{e}^{\int \frac{2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x-x^{2}+2c}{x^{m}b\lambda-x^{m}bx+x^{n}a\lambda-ax\,x^{n}+c\lambda-cx}dx}(va_{3}+xa_{2}+a_{1})}{x^{m}b\lambda-x^{m}bx+x^{n}a\lambda-ax\,x^{n}+c\lambda-cx}\\
&=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Looking at the above PDE shows the following are all the terms with $\{v, x\}$ in them.

$$\left\{v, x, x^{m}, x^{n}, \int \frac{2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x-x^{2}+2c}{x^{m} b\lambda -x^{m} b x+x^{n} a\lambda -a x\,x^{n}+c\lambda -c x}d x, \mathrm{e}^{2\left(\int \frac{2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x-x^{2}+2c}{x^{m} b\lambda -x^{m} b x+x^{n} a\lambda -a x\,x^{n}+c\lambda -c x}d x\right)}, \mathrm{e}^{\int \frac{2b\,x^{m}+2a\,x}{x^{m} b\lambda -x^{m} b x-}}\right.$$

The following substitution is now made to be able to collect on all terms with $\{v, x\}$ in them

$$\left\{v = v_{1}, x = v_{2}, x^{m} = v_{3}, x^{n} = v_{4}, \int \frac{2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x-x^{2}+2c}{x^{m} b\lambda -x^{m} b x+x^{n} a\lambda -a x\,x^{n}+c\lambda -c x}d x = v_{5}, \mathrm{e}^{2\left(\int \frac{2b\,x^{m}+2a\,x^{n}-\lambda^{2}+2\lambda x}{x^{m} b\lambda -x^{m} b x+x^{n} a\lambda -a x}\right.}\right.$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a\,c_{1}^{2} \lambda a_{3} v_{4} v_{6}+a\,c_{1}^{2} a_{3} v_{2} v_{4} v_{6}-b\,c_{1}^{2} \lambda a_{3} v_{3} v_{6}+b\,c_{1}^{2} a_{3} v_{2} v_{3} v_{6}-a c_{1} \lambda a_{2} v_{4} v_{7}\\
&+a c_{1} \lambda b_{3} v_{4} v_{7}-a c_{1} a_{2} v_{2} v_{4} v_{7}-2 a c_{1} a_{3} v_{1} v_{4} v_{7}-a c_{1} b_{3} v_{2} v_{4} v_{7}-b c_{1} \lambda a_{2} v_{3} v_{7}\\
&+b c_{1} \lambda b_{3} v_{3} v_{7}-b c_{1} a_{2} v_{2} v_{3} v_{7}-2 b c_{1} a_{3} v_{1} v_{3} v_{7}-b c_{1} b_{3} v_{2} v_{3} v_{7}-c\,c_{1}^{2} \lambda a_{3} v_{6}\\
&+c\,c_{1}^{2} a_{3} v_{2} v_{6}+c_{1} \lambda^{2} a_{2} v_{2} v_{7}+c_{1} \lambda^{2} a_{3} v_{1} v_{7}-2 c_{1} \lambda a_{2} v_{2}^{2} v_{7}-2 c_{1} \lambda a_{3} v_{1} v_{2} v_{7}\\
&+c_{1} a_{2} v_{2}^{3} v_{7}+c_{1} a_{3} v_{1} v_{2}^{2} v_{7}-2 a c_{1} a_{1} v_{4} v_{7}-2 b c_{1} a_{1} v_{3} v_{7}-c c_{1} \lambda a_{2} v_{7}+c c_{1} \lambda b_{3} v_{7}\\
&-c c_{1} a_{2} v_{2} v_{7}-2 c c_{1} a_{3} v_{1} v_{7}-c c_{1} b_{3} v_{2} v_{7}+c_{1} \lambda^{2} a_{1} v_{7}-2 c_{1} \lambda a_{1} v_{2} v_{7}+c_{1} a_{1} v_{2}^{2} v_{7}\\
&+v_{4} a \lambda b_{2}-v_{4} a v_{2} b_{2}+v_{3} b \lambda b_{2}-v_{3} b v_{2} b_{2}-2 c c_{1} a_{1} v_{7}+c \lambda b_{2}-c v_{2} b_{2} = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}, v_{7}\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-bc_1a_2 - bc_1b_3)\, v_2v_3v_7 + (-ac_1a_2 - ac_1b_3)\, v_2v_4v_7 + c\lambda b_2 \\
&\; - a\, c_1^2\lambda a_3v_4v_6 + a\, c_1^2a_3v_2v_4v_6 - b\, c_1^2\lambda a_3v_3v_6 + b\, c_1^2a_3v_2v_3v_6 \\
&\; + c_1a_3v_1v_2^2v_7 - 2ac_1a_3v_1v_4v_7 - 2bc_1a_3v_1v_3v_7 - 2c_1\lambda a_3v_1v_2v_7 \\
&\; - c\, c_1^2\lambda a_3v_6 + c\, c_1^2a_3v_2v_6 - cv_2b_2 + \left(c_1\lambda^2a_3 - 2cc_1a_3\right) v_1v_7 \\
&\; + (-2c_1\lambda a_2 + c_1a_1)\, v_2^2v_7 + \left(c_1\lambda^2a_2 - cc_1a_2 - cc_1b_3 - 2c_1\lambda a_1\right) v_2v_7 \\
&\; + (-bc_1\lambda a_2 + bc_1\lambda b_3 - 2bc_1a_1)\, v_3v_7 + (-ac_1\lambda a_2 + ac_1\lambda b_3 - 2ac_1a_1)\, v_4v_7 \\
&\; + v_3b\lambda b_2 - v_3bv_2b_2 + v_4a\lambda b_2 - v_4av_2b_2 + c_1a_2v_2^3v_7 \\
&\; + \left(-cc_1\lambda a_2 + cc_1\lambda b_3 + c_1\lambda^2a_1 - 2cc_1a_1\right) v_7 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
c_1 a_2 &= 0 \\
c_1 a_3 &= 0 \\
c_1^2 a a_3 &= 0 \\
c_1^2 b a_3 &= 0 \\
a \lambda b_2 &= 0 \\
b \lambda b_2 &= 0 \\
c\, c_1^2 a_3 &= 0 \\
c \lambda b_2 &= 0 \\
-a b_2 &= 0 \\
-b b_2 &= 0 \\
-b_2 c &= 0 \\
-2 c_1 a a_3 &= 0 \\
-2 c_1 b a_3 &= 0 \\
-2 c_1 \lambda a_3 &= 0 \\
-c_1^2 a \lambda a_3 &= 0 \\
-c_1^2 b \lambda a_3 &= 0 \\
-c_1^2 c \lambda a_3 &= 0 \\
-a c_1 a_2 - a c_1 b_3 &= 0 \\
-b c_1 a_2 - b c_1 b_3 &= 0 \\
c_1 \lambda^2 a_3 - 2 c c_1 a_3 &= 0 \\
-2 c_1 \lambda a_2 + c_1 a_1 &= 0 \\
-a c_1 \lambda a_2 + a c_1 \lambda b_3 - 2 a c_1 a_1 &= 0 \\
-b c_1 \lambda a_2 + b c_1 \lambda b_3 - 2 b c_1 a_1 &= 0 \\
c_1 \lambda^2 a_2 - c c_1 a_2 - c c_1 b_3 - 2 c_1 \lambda a_1 &= 0 \\
-c c_1 \lambda a_2 + c c_1 \lambda b_3 + c_1 \lambda^2 a_1 - 2 c c_1 a_1 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= 1
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, v) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dv}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial v}\right) S(x, v) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{1} dy
\end{aligned}
$$

Which results in

$$
S = v
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, v) S_v}{R_x + \omega(x, v) R_v} \tag{2}
$$

Where in the above R_x, R_v, S_x, S_v are all partial derivatives and $\omega(x,v)$ is the right hand side of the original ode given by

$$\omega(x,v) = c_1 \mathrm{e}^{\int \frac{2b\,x^m+2a\,x^n-\lambda^2+2\lambda x-x^2+2c}{x^m b\lambda - x^m bx + x^n a\lambda - ax\,x^n + c\lambda - cx}dx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_v &= 0 \\ S_x &= 0 \\ S_v &= 1 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = c_1 \mathrm{e}^{\int \frac{-2b\,x^m-2a\,x^n+\lambda^2-2\lambda x+x^2-2c}{(a\,x^n+b\,x^m+c)(-\lambda+x)}dx} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, v in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = c_1 \mathrm{e}^{\int \frac{-2b\,R^m-2a\,R^n+\lambda^2-2\lambda R+R^2-2c}{(a\,R^n+b\,R^m+c)(-\lambda+R)}dR}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int c_1 \mathrm{e}^{-\left(\int \frac{2b\,R^m+2a\,R^n-R^2+2\lambda R-\lambda^2+2c}{(a\,R^n+b\,R^m+c)(-\lambda+R)}dR\right)}dR + c_2 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, v coordinates. This results in

$$v(x) = \int c_1 \mathrm{e}^{-\left(\int \frac{2b\,x^m+2a\,x^n-\lambda^2+2\lambda x-x^2+2c}{(a\,x^n+b\,x^m+c)(-\lambda+x)}dx\right)}dx + c_2$$

Which simplifies to

$$v(x) - c_1\left(\int \mathrm{e}^{\int \frac{-2b\,x^m-2a\,x^n+\lambda^2-2\lambda x+x^2-2c}{(a\,x^n+b\,x^m+c)(-\lambda+x)}dx}dx\right) - c_2 = 0$$

Which gives

$$v(x) = c_1\left(\int \mathrm{e}^{\int \frac{-2b\,x^m-2a\,x^n+\lambda^2-2\lambda x+x^2-2c}{(a\,x^n+b\,x^m+c)(-\lambda+x)}dx}dx\right) + c_2$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (\lambda - x)\left(c_1\left(\int \mathrm{e}^{\int \frac{-2b\,x^m - 2a\,x^n + \lambda^2 - 2\lambda x + x^2 - 2c}{(a\,x^n + b\,x^m + c)(-\lambda + x)}dx}dx\right) + c_2\right) \\ &= (\lambda - x)\left(c_1\left(\int \mathrm{e}^{\int \frac{-2b\,x^m - 2a\,x^n + \lambda^2 - 2\lambda x + x^2 - 2c}{(a\,x^n + b\,x^m + c)(-\lambda + x)}dx}dx\right) + c_2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = (\lambda - x)\left(c_1\left(\int \mathrm{e}^{\int \frac{-2b\,x^m - 2a\,x^n + \lambda^2 - 2\lambda x + x^2 - 2c}{(a\,x^n + b\,x^m + c)(-\lambda + x)}dx}dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = (\lambda - x)\left(c_1\left(\int \mathrm{e}^{\int \frac{-2b\,x^m - 2a\,x^n + \lambda^2 - 2\lambda x + x^2 - 2c}{(a\,x^n + b\,x^m + c)(-\lambda + x)}dx}dx\right) + c_2\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   <- linear symmetries successful`
```

✓ Solution by Maple

Time used: 0.781 (sec). Leaf size: 68

```
dsolve((a*x^n+b*x^m+c)*diff(y(x),x$2)+(lambda-x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = -\left(\left(\int e^{\int \frac{-2a\,x^{n}-2b\,x^{m}-2c+x^{2}-2x\lambda+\lambda^{2}}{(a\,x^{n}+b\,x^{m}+c)(-\lambda+x)}dx}dx\right)c_1+c_2\right)(\lambda-x)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(a*x^n+b*x^m+c)*y''[x]+(\[Lambda]-x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions ->
```

Not solved

33.24 problem 262

Internal problem ID [11086]
Internal file name [`OUTPUT/10342_Wednesday_January_24_2024_10_18_09_PM_19808305/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 262.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{(a\,x^n + b\,x^m + c)\,y'' + \left(\lambda^2 - x^2\right)y' + (\lambda + x)\,y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   <- linear symmetries successful`
```

✓ Solution by Maple

Time used: 1.0 (sec). Leaf size: 76

```
dsolve((a*x^n+b*x^m+c)*diff(y(x),x$2)+(lambda^2-x^2)*diff(y(x),x)+(x+lambda)*y(x)=0,y(x), si
```

$$y(x) = -\left(\left(\int e^{\int \frac{\lambda^3 - x\lambda^2 - x^2\lambda + x^3 - 2a\,x^n - 2b\,x^m - 2c}{(a\,x^n + b\,x^m + c)(-\lambda + x)}dx}dx\right)c_1 + c_2\right)(\lambda - x)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(a*x^n+b*x^m+c)*y''[x]+(\[Lambda]^2-x^2)*y'[x]+(x+\[Lambda])*y[x]==0,y[x],x,IncludeSi
```

Not solved

33.25 problem 263

Internal problem ID [11087]
Internal file name [`OUTPUT/10343_Wednesday_January_24_2024_10_18_09_PM_83448012/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 263.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{2(a\,x^{n}+b\,x^{m}+c)\,y''+an\,x^{n-1}bm\,x^{m-1}y'+yd=0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying an equivalence, under non-integer power transformations,
   to LODEs admitting Liouvillian solutions.
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
```

✗ Solution by Maple

```
dsolve(2*(a*x^n+b*x^m+c)*diff(y(x),x$2)+(a*n*x^(n-1)*b*m*x^(m-1))*diff(y(x),x)+d*y(x)=0,y(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[2*(a*x^n+b*x^m+c)*y''[x]+(a*n*x^(n-1)*b*m*x^(m-1))*y'[x]+d*y[x]==0,y[x],x,IncludeSing
```

Not solved

33.26 problem 264

Internal problem ID [11088]
Internal file name [`OUTPUT/10344_Wednesday_January_24_2024_10_18_10_PM_59869510/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.2-8. Other equations.
Problem number: 264.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{(a\,x^n+b)^{m+1}\,y''+(a\,x^n+b)\,y'-anm\,x^{n-1}y=0}$$

33.26.1 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$(a\,x^n+b)^{m+1}\,y''+(a\,x^n+b)\,y'-anm\,x^{n-1}y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=r(x) \tag{2}$$

Where

$$\begin{aligned}p(x)&=(a\,x^n+b)^{-m}\\ q(x)&=-anm\,x^{n-1}(a\,x^n+b)^{-m-1}\\ r(x)&=0\end{aligned}$$

The Lagrange adjoint ode is given by

$$\xi''-(\xi\,p)'+\xi q=0$$
$$\xi''-\left((a\,x^n+b)^{-m}\,\xi(x)\right)'+\left(-anm\,x^{n-1}(a\,x^n+b)^{-m-1}\,\xi(x)\right)=0$$
$$\xi''(x)-(a\,x^n+b)^{-m}\,\xi'(x)+\left(\frac{anm\,x^n(a\,x^n+b)^{-m}}{x\,(a\,x^n+b)}-anm\,x^{n-1}(a\,x^n+b)^{-m-1}\right)\xi(x)=0$$

Which is solved for $\xi(x)$. This is second order ode with missing dependent variable $\xi(x)$. Let

$$p(x) = \xi'(x)$$

Then

$$p'(x) = \xi''(x)$$

Hence the ode becomes

$$x(-a\,x^n - b)\,p'(x) + x(a\,x^n + b)^{-m+1}\,p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned}
p' &= F(x,p)\\
&= f(x)g(p)\\
&= \frac{(a\,x^n + b)^{-m+1}\,p}{a\,x^n + b}
\end{aligned}$$

Where $f(x) = \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= \frac{(a\,x^n + b)^{-m+1}}{a\,x^n + b}\,dx\\
\int \frac{1}{p}\,dp &= \int \frac{(a\,x^n + b)^{-m+1}}{a\,x^n + b}\,dx\\
\ln(p) &= \int \frac{(a\,x^n + b)^{-m+1}}{a\,x^n + b}dx + c_1\\
p &= e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx + c_1}\\
&= c_1 e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx}
\end{aligned}$$

Since $p = \xi'(x)$ then the new first order ode to solve is

$$\xi'(x) = c_1 e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx}$$

Writing the ode as

$$\begin{aligned}
\xi'(x) &= c_1 e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx}\\
\xi'(x) &= \omega(x,\xi)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_\xi - \xi_x) - \omega^2 \xi_\xi - \omega_x \xi - \omega_\xi \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + \xi a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + \xi b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + c_1 \mathrm{e}^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} (b_3 - a_2) - c_1^2 \mathrm{e}^{\int 2(a\,x^n+b)^{-m}dx} a_3 - \frac{c_1 (a\,x^n + b)^{-m+1} \mathrm{e}^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} (x a_2 + \xi a_3 + a_1)}{a\,x^n + b} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\mathrm{e}^{\int 2(a\,x^n+b)^{-m}dx} x^n c_1^2 a a_3 + \mathrm{e}^{\int 2(a\,x^n+b)^{-m}dx} c_1^2 b a_3 + \mathrm{e}^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} x^n c_1 a a_2 - \mathrm{e}^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} x^n c_1 a b_3 + \mathrm{e}^{\int \frac{(a}{}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-\mathrm{e}^{\int 2(a\,x^n+b)^{-m}dx} x^n c_1^2 a a_3 - \mathrm{e}^{\int 2(a\,x^n+b)^{-m}dx} c_1^2 b a_3 - \mathrm{e}^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} x^n c_1 a a_2 \\
&+ \mathrm{e}^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} x^n c_1 a b_3 - \mathrm{e}^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} (a\,x^n + b)^{-m+1} c_1 x a_2 \\
&- \mathrm{e}^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} (a\,x^n + b)^{-m+1} c_1 \xi a_3 - \mathrm{e}^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} (a\,x^n + b)^{-m+1} c_1 a_1 \\
&- \mathrm{e}^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} c_1 b a_2 + \mathrm{e}^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} c_1 b b_3 + x^n a b_2 + b b_2 = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$-\left(x^n e^{2\left(\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx\right)} c_1^2 a a_3 (a\,x^n+b)^m + x^n e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} c_1 a a_2 (a\,x^n+b)^m - x^n e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} c_1 a b_3 (a\,x^n+b)^m + x^n e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} c_1 a x a_2 + x^n e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} c_1 a \xi a_3 + e^{2\left(\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx\right)} c_1^2 b a_3 (a\,x^n+b)^m + x^n e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} c_1 a a_1 + e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} c_1 b a_2 (a\,x^n+b)^m - e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} c_1 b b_3 (a\,x^n+b)^m + e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} c_1 b x a_2 + e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} c_1 b \xi a_3 - x^n a b_2 (a\,x^n+b)^m + e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} c_1 b a_1 - b b_2 (a\,x^n+b)^m\right) (a\,x^n+b)^{-m} = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, \xi\}$ in them.

$$\left\{x, \xi, x^n, (a\,x^n+b)^m, (a\,x^n+b)^{-m+1}, \int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx, e^{2\left(\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx\right)}, e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, \xi\}$ in them

$$\left\{x = v_1, \xi = v_2, x^n = v_3, (a\,x^n+b)^m = v_4, (a\,x^n+b)^{-m+1} = v_5, \int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx = v_6, e^{2\left(\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx\right)} = v_7, e^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx} = v_8\right\}$$

The above PDE (6E) now becomes

$$-\frac{a\,c_1^2 a_3 v_3 v_4 v_7 + a c_1 a_2 v_1 v_3 v_8 + a c_1 a_2 v_3 v_4 v_8 + a c_1 a_3 v_2 v_3 v_8 - a c_1 b_3 v_3 v_4 v_8 + b\,c_1^2 a_3 v_4 v_7 + a c_1 a_1 v_3 v_8 + b c_1 a_2 \cdots}{v_4} = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$$

Equation (7E) now becomes

$$\begin{aligned} &-\frac{c_1 a a_2 v_8 v_1 v_3}{v_4} - \frac{c_1 b a_2 v_8 v_1}{v_4} - \frac{c_1 a a_3 v_8 v_2 v_3}{v_4} - \frac{c_1 b a_3 v_8 v_2}{v_4} \\ &- c_1^2 a a_3 v_7 v_3 + (-a c_1 a_2 + a c_1 b_3) v_8 v_3 + a b_2 v_3 - \frac{c_1 a a_1 v_8 v_3}{v_4} \\ &- c_1^2 b a_3 v_7 + (-b c_1 a_2 + b c_1 b_3) v_8 + b b_2 - \frac{v_8 c_1 b a_1}{v_4} = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a b_2 &= 0 \\ b b_2 &= 0 \\ -c_1 a a_1 &= 0 \\ -a c_1 a_2 &= 0 \\ -c_1 a a_3 &= 0 \\ -c_1 b a_1 &= 0 \\ -b c_1 a_2 &= 0 \\ -c_1 b a_3 &= 0 \\ -c_1^2 a a_3 &= 0 \\ -c_1^2 b a_3 &= 0 \\ -a c_1 a_2 + a c_1 b_3 &= 0 \\ -b c_1 a_2 + b c_1 b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 0$$
$$\eta = 1$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,\xi) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{d\xi}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial \xi}\right) S(x,\xi) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{1} dy \end{aligned}$$

Which results in

$$S = \xi$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,\xi)S_\xi}{R_x + \omega(x,\xi)R_\xi} \tag{2}$$

Where in the above R_x, R_ξ, S_x, S_ξ are all partial derivatives and $\omega(x,\xi)$ is the right hand side of the original ode given by

$$\omega(x,\xi) = c_1 \mathrm{e}^{\int \frac{(a\,x^n+b)^{-m+1}}{a\,x^n+b}dx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_\xi &= 0 \\ S_x &= 0 \\ S_\xi &= 1 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = c_1 \mathrm{e}^{\int (a\,x^n+b)^{-m}dx} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, ξ in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = c_1 \mathrm{e}^{\int (a\,R^n+b)^{-m}dR}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int c_1 \mathrm{e}^{\int (a\,R^n+b)^{-m}dR} dR + c_2 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, ξ coordinates. This results in

$$\xi(x) = \int c_1 \mathrm{e}^{\int (a\,x^n+b)^{-m}dx} dx + c_2$$

Which simplifies to

$$\xi(x) = \int c_1 \mathrm{e}^{\int (a\,x^n+b)^{-m}dx} dx + c_2$$

Which gives

$$\xi(x) = \int c_1 \mathrm{e}^{\int (a\,x^n+b)^{-m}dx} dx + c_2$$

The original ode (2) now reduces to first order ode

$$\begin{aligned} \xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)} \end{aligned}$$

$$y' + y\left((a\,x^n+b)^{-m} - \frac{c_3 \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}}{\int c_3 \mathrm{e}^{\int (a\,x^n+b)^{-m}dx} dx + c_2}\right) = 0$$

Which is now a first order ode. This is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-(a\,x^n+b)^{-m}\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)c_3 - (a\,x^n+b)^{-m}c_2 + c_3\mathrm{e}^{\int (a\,x^n+b)^{-m}dx}}{c_3\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)+c_2}$$

$$q(x) = 0$$

Hence the ode is

$$y' - \frac{\left(-(a\,x^n+b)^{-m}\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)c_3 - (a\,x^n+b)^{-m}c_2 + c_3\mathrm{e}^{\int (a\,x^n+b)^{-m}dx}\right)y}{c_3\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)+c_2} = 0$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{-(a\,x^n+b)^{-m}\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)c_3-(a\,x^n+b)^{-m}c_2+c_3\mathrm{e}^{\int (a\,x^n+b)^{-m}dx}}{c_3\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)+c_2}dx}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\int -\frac{-(a\,x^n+b)^{-m}\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)c_3-(a\,x^n+b)^{-m}c_2+c_3\mathrm{e}^{\int (a\,x^n+b)^{-m}dx}}{c_3\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)+c_2}dx}y\right) = 0$$

Integrating gives

$$\mathrm{e}^{\int -\frac{-(a\,x^n+b)^{-m}\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)c_3-(a\,x^n+b)^{-m}c_2+c_3\mathrm{e}^{\int (a\,x^n+b)^{-m}dx}}{c_3\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)+c_2}dx}y = c_3$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{-(a\,x^n+b)^{-m}\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)c_3-(a\,x^n+b)^{-m}c_2+c_3\mathrm{e}^{\int (a\,x^n+b)^{-m}dx}}{c_3\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)+c_2}dx}$ results in

$$y = c_3\mathrm{e}^{-\left(\int \frac{(a\,x^n+b)^{-m}\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)c_3+(a\,x^n+b)^{-m}c_2-c_3\mathrm{e}^{\int (a\,x^n+b)^{-m}dx}}{c_3\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)+c_2}dx\right)}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_3 \mathrm{e}^{-\left(\int \frac{(a x^n+b)^{-m}\left(\int \mathrm{e}^{\int (a x^n+b)^{-m}dx}dx\right)c_3+(a x^n+b)^{-m}c_2-c_3\mathrm{e}^{\int (a x^n+b)^{-m}dx}}{c_3\left(\int \mathrm{e}^{\int (a x^n+b)^{-m}dx}dx\right)+c_2}dx\right)}$$

Summary

The solution(s) found are the following

$$y = c_3 \mathrm{e}^{-\left(\int \frac{(a x^n+b)^{-m}\left(\int \mathrm{e}^{\int (a x^n+b)^{-m}dx}dx\right)c_3+(a x^n+b)^{-m}c_2-c_3\mathrm{e}^{\int (a x^n+b)^{-m}dx}}{c_3\left(\int \mathrm{e}^{\int (a x^n+b)^{-m}dx}dx\right)+c_2}dx\right)} \tag{1}$$

Verification of solutions

$$y = c_3 \mathrm{e}^{-\left(\int \frac{(a x^n+b)^{-m}\left(\int \mathrm{e}^{\int (a x^n+b)^{-m}dx}dx\right)c_3+(a x^n+b)^{-m}c_2-c_3\mathrm{e}^{\int (a x^n+b)^{-m}dx}}{c_3\left(\int \mathrm{e}^{\int (a x^n+b)^{-m}dx}dx\right)+c_2}dx\right)}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 41

```
dsolve((a*x^n+b)^(m+1)*diff(y(x),x$2)+(a*x^n+b)*diff(y(x),x)-a*n*m*x^(n-1)*y(x)=0,y(x), sing
```

$$y(x) = \mathrm{e}^{-\left(\int (a\,x^n+b)^{-m}dx\right)}\left(c_1+\left(\int \mathrm{e}^{\int (a\,x^n+b)^{-m}dx}dx\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.249 (sec). Leaf size: 116

```
DSolve[(a*x^n+b)^(m+1)*y''[x]+(a*x^n+b)*y'[x]-a*n*m*x^(n-1)*y[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to \exp\left(-x(ax^n+b)^{-m}\left(\frac{ax^n}{b}+1\right)^m \text{Hypergeometric2F1}\left(m,\frac{1}{n},1+\frac{1}{n},-\frac{ax^n}{b}\right)\right)\left(\int_1^x \exp\left(\text{Hypergeometric2F1}\left(m,\frac{1}{n},1+\frac{1}{n},-\frac{aK[1]^n}{b}\right)K[1]\left(aK[1]^n+b\right)^{-m}\left(\frac{aK[1]^n}{b}+1\right)^{m}\right. \right. \left. + c_2\right)$$

34 Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions

34.1 problem 1

Internal problem ID [11089]
Internal file name [`OUTPUT/10345_Wednesday_January_24_2024_10_18_11_PM_93345610/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode_form_A"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \mathrm{e}^{\lambda x} y a = 0}$$

34.1.1 Solving as second order bessel ode form A ode

Writing the ode as

$$y'' + \mathrm{e}^{\lambda x} y a = 0 \tag{1}$$

An ode of the form

$$a y'' + b y' + (c e^{rx} + m) y = 0 \tag{1}$$

can be transformed to Bessel ode using the transformation

$$\begin{aligned} x &= \ln(t) \\ e^x &= t \end{aligned}$$

Where a, b, c, m are not functions of x and where b and m are allowed to be be zero. Using this transformation gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} \\ &= \frac{dy}{dt} e^x \\ &= t\frac{dy}{dt} \end{aligned} \tag{2}$$

And

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right)\\
&= \frac{d}{dx}\left(t\frac{dy}{dt}\right)\\
&= \frac{d}{dt}\frac{dt}{dx}\left(t\frac{dy}{dt}\right)\\
&= \frac{dt}{dx}\frac{d}{dt}\left(t\frac{dy}{dt}\right)\\
&= t\frac{d}{dt}\left(t\frac{dy}{dt}\right)\\
&= t\left(\frac{dy}{dt}+t\frac{d^2y}{dt^2}\right) \qquad (3)
\end{aligned}$$

Substituting (2,3) into (1) gives

$$\begin{aligned}
at\left(\frac{dy}{dt}+t\frac{d^2y}{dt^2}\right)+bt\frac{dy}{dt}+(ce^{rx}+m)y &= 0\\
(aty'+at^2y'')+bty'+(ct^r+m)y &= 0\\
at^2y''+(b+a)\,ty'+(ct^r+m)y &= 0\\
t^2y''+\frac{b+a}{a}ty'+\left(\frac{c}{a}t^r+\frac{m}{a}\right)y &= 0 \qquad (4)
\end{aligned}$$

Which is Bessel ODE. Comparing the above to the general known Bowman form of Bessel ode which is

$$t^2y''+(1-2\alpha)\,ty'+\left(\beta^2\gamma^2t^{2\gamma}-\left(n^2\gamma^2-\alpha^2\right)\right)y=0 \qquad \text{(C)}$$

And now comparing (4) and (C) shows that

$$\begin{aligned}
(1-2\alpha) &= \frac{b+a}{a} \qquad (5)\\
\beta^2\gamma^2 &= \frac{c}{a} \qquad (6)\\
2\gamma &= r \qquad (7)\\
\left(n^2\gamma^2-\alpha^2\right) &= -\frac{m}{a} \qquad (8)
\end{aligned}$$

(5) gives $\alpha=\frac{1}{2}-\frac{b+a}{2a}$. (7) gives $\gamma=\frac{r}{2}$. (8) now becomes $\left(n^2\left(\frac{r}{2}\right)^2-\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2\right)=-\frac{m}{a}$ or $n^2=\frac{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}{\left(\frac{r}{2}\right)^2}$. Hence $n=\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}$ by taking the positive root.

And finally (6) gives $\beta^2 = \frac{c}{a\gamma^2}$ or $\beta = \sqrt{\frac{c}{a}\frac{1}{\gamma}} = \sqrt{\frac{c}{a}\frac{2}{r}}$ (also taking the positive root). Hence

$$
\begin{aligned}
\alpha &= \frac{1}{2} - \frac{b+a}{2a} \\
n &= \frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2} \\
\beta &= \sqrt{\frac{c}{a}\frac{2}{r}} \\
\gamma &= \frac{r}{2}
\end{aligned}
$$

But the solution to (C) which is general form of Bessel ode is known and given by

$$
y(t) = t^\alpha (c_1 J_n(\beta t^\gamma) + c_2 Y_n(\beta t^\gamma))
$$

Substituting the above values found into this solution gives

$$
y(t) = t^{\frac{1}{2} - \frac{b+a}{2a}} \left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} t^{\frac{r}{2}} \right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} t^{\frac{r}{2}} \right) \right)
$$

Since $e^x = t$ then the above becomes

$$
\begin{aligned}
y(x) &= e^{x\left(\frac{1}{2} - \frac{b+a}{2a}\right)} \left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}} \right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}} \right) \right) \\
&= e^{x\left(\frac{-b}{2a}\right)} \left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{-b}{2a}\right)^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}} \right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{-b}{2a}\right)^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}} \right) \right) \\
&= e^{x\left(\frac{-b}{2a}\right)} \left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a} + \frac{b^2}{4a^2}}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}} \right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a} + \frac{b^2}{4a^2}}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}} \right) \right) \\
&= e^{x\left(\frac{-b}{2a}\right)} \left(c_1 J_{\frac{2}{r}\sqrt{-\frac{4ma+b^2}{4a^2}}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}} \right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{4ma+b^2}{4a^2}}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}} \right) \right) \\
&= e^{x\left(\frac{-b}{2a}\right)} \left(c_1 J_{\frac{1}{ra}\sqrt{-4ma+b^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}} \right) + c_2 Y_{\frac{1}{ra}\sqrt{-4ma+b^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}} \right) \right) \qquad (9)
\end{aligned}
$$

Equation (9) above is the solution to $ay'' + by' + (ce^{rx} + m)y = 0$. Therefore we just need now to compare this form to the ode given and use (9) to obtain the final solution.

Comparing form (1) to the ode we are solving shows that

$$
\begin{aligned}
a &= 1 \\
b &= 0 \\
c &= a \\
r &= \lambda \\
m &= 0
\end{aligned}
$$

Substituting these in (9) gives the solution as

$$y = c_1 \operatorname{BesselJ}\left(0, \frac{2\sqrt{a}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) + c_2 \operatorname{BesselY}\left(0, \frac{2\sqrt{a}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}\left(0, \frac{2\sqrt{a}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) + c_2 \operatorname{BesselY}\left(0, \frac{2\sqrt{a}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}\left(0, \frac{2\sqrt{a}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) + c_2 \operatorname{BesselY}\left(0, \frac{2\sqrt{a}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      a*u(t)+lambda^2*diff(u(t),t)+lambda^2*t*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$2)+a*exp(lambda*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}\left(0, \frac{2\sqrt{a}\,\mathrm{e}^{\frac{x\lambda}{2}}}{\lambda}\right) + c_2 \operatorname{BesselY}\left(0, \frac{2\sqrt{a}\,\mathrm{e}^{\frac{x\lambda}{2}}}{\lambda}\right)$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 55

```
DSolve[y''[x]+a*Exp[\[Lambda]*x]*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}\left(0, \frac{2\sqrt{a}\sqrt{e^{x\lambda}}}{\lambda}\right) + 2c_2 \operatorname{BesselY}\left(0, \frac{2\sqrt{a}\sqrt{e^{x\lambda}}}{\lambda}\right)$$

34.2 problem 2

Internal problem ID [11090]
Internal file name [`OUTPUT/10346_Wednesday_January_24_2024_10_18_11_PM_62097031/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode_form_A"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + (a\,\mathrm{e}^{x} - b)\,y = 0}$$

34.2.1 Solving as second order bessel ode form A ode

Writing the ode as

$$y'' + (a\,\mathrm{e}^{x} - b)\,y = 0 \tag{1}$$

An ode of the form

$$ay'' + by' + (ce^{rx} + m)y = 0 \tag{1}$$

can be transformed to Bessel ode using the transformation

$$\begin{aligned} x &= \ln(t) \\ e^{x} &= t \end{aligned}$$

Where a, b, c, m are not functions of x and where b and m are allowed to be be zero. Using this transformation gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} \\ &= \frac{dy}{dt}e^{x} \\ &= t\frac{dy}{dt} \end{aligned} \tag{2}$$

And

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \frac{d}{dx}\left(t\frac{dy}{dt}\right) \\
&= \frac{d}{dt}\frac{dt}{dx}\left(t\frac{dy}{dt}\right) \\
&= \frac{dt}{dx}\frac{d}{dt}\left(t\frac{dy}{dt}\right) \\
&= t\frac{d}{dt}\left(t\frac{dy}{dt}\right) \\
&= t\left(\frac{dy}{dt}+t\frac{d^2y}{dt^2}\right) \qquad (3)
\end{aligned}$$

Substituting (2,3) into (1) gives

$$\begin{aligned}
at\left(\frac{dy}{dt}+t\frac{d^2y}{dt^2}\right)+bt\frac{dy}{dt}+(ce^{rx}+m)y &= 0 \\
\left(aty'+at^2y''\right)+bty'+(ct^r+m)y &= 0 \\
at^2y''+(b+a)\,ty'+(ct^r+m)y &= 0 \\
t^2y''+\frac{b+a}{a}ty'+\left(\frac{c}{a}t^r+\frac{m}{a}\right)y &= 0 \qquad (4)
\end{aligned}$$

Which is Bessel ODE. Comparing the above to the general known Bowman form of Bessel ode which is

$$t^2y''+(1-2\alpha)\,ty'+\left(\beta^2\gamma^2t^{2\gamma}-\left(n^2\gamma^2-\alpha^2\right)\right)y=0 \qquad (C)$$

And now comparing (4) and (C) shows that

$$\begin{aligned}
(1-2\alpha) &= \frac{b+a}{a} \qquad (5) \\
\beta^2\gamma^2 &= \frac{c}{a} \qquad (6) \\
2\gamma &= r \qquad (7) \\
\left(n^2\gamma^2-\alpha^2\right) &= -\frac{m}{a} \qquad (8)
\end{aligned}$$

(5) gives $\alpha=\frac{1}{2}-\frac{b+a}{2a}$. (7) gives $\gamma=\frac{r}{2}$. (8) now becomes $\left(n^2\left(\frac{r}{2}\right)^2-\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2\right)=-\frac{m}{a}$ or $n^2=\frac{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}{\left(\frac{r}{2}\right)^2}$. Hence $n=\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}$ by taking the positive root.

And finally (6) gives $\beta^2 = \frac{c}{a\gamma^2}$ or $\beta = \sqrt{\frac{c}{a}\frac{1}{\gamma}} = \sqrt{\frac{c}{a}\frac{2}{r}}$ (also taking the positive root). Hence

$$\begin{aligned}
\alpha &= \frac{1}{2} - \frac{b+a}{2a} \\
n &= \frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2} \\
\beta &= \sqrt{\frac{c}{a}\frac{2}{r}} \\
\gamma &= \frac{r}{2}
\end{aligned}$$

But the solution to (C) which is general form of Bessel ode is known and given by

$$y(t) = t^\alpha(c_1 J_n(\beta t^\gamma) + c_2 Y_n(\beta t^\gamma))$$

Substituting the above values found into this solution gives

$$y(t) = t^{\frac{1}{2}-\frac{b+a}{2a}}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}t^{\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}t^{\frac{r}{2}}\right)\right)$$

Since $e^x = t$ then the above becomes

$$\begin{aligned}
y(x) &= e^{x\left(\frac{1}{2}-\frac{b+a}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{-b}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{-b}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\frac{b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\frac{b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{4ma+b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{4ma+b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{1}{ra}\sqrt{-4ma+b^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{1}{ra}\sqrt{-4ma+b^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \quad (9)
\end{aligned}$$

Equation (9) above is the solution to $ay'' + by' + (ce^{rx} + m)y = 0$. Therefore we just need now to compare this form to the ode given and use (9) to obtain the final solution.

Comparing form (1) to the ode we are solving shows that

$$\begin{aligned}
a &= 1 \\
b &= 0 \\
c &= a \\
r &= 1 \\
m &= -b
\end{aligned}$$

Substituting these in (9) gives the solution as

$$y = c_1 \operatorname{BesselJ}\left(2\sqrt{b}, 2\sqrt{a}\, e^{\frac{x}{2}}\right) + c_2 \operatorname{BesselY}\left(2\sqrt{b}, 2\sqrt{a}\, e^{\frac{x}{2}}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}\left(2\sqrt{b}, 2\sqrt{a}\, e^{\frac{x}{2}}\right) + c_2 \operatorname{BesselY}\left(2\sqrt{b}, 2\sqrt{a}\, e^{\frac{x}{2}}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}\left(2\sqrt{b}, 2\sqrt{a}\, e^{\frac{x}{2}}\right) + c_2 \operatorname{BesselY}\left(2\sqrt{b}, 2\sqrt{a}\, e^{\frac{x}{2}}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
   Change of variables used:
      [x = ln(t)]
   Linear ODE actually solved:
      (a*t-b)*u(t)+t*diff(u(t),t)+t^2*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$2)+(a*exp(x)-b)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}\left(2\sqrt{b}, 2\sqrt{a}\,\mathrm{e}^{\frac{x}{2}}\right) + c_2 \operatorname{BesselY}\left(2\sqrt{b}, 2\sqrt{a}\,\mathrm{e}^{\frac{x}{2}}\right)$$

✓ Solution by Mathematica

Time used: 0.081 (sec). Leaf size: 76

```
DSolve[y''[x]+(a*Exp[x]-b)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) \to {} & c_1 \operatorname{Gamma}\left(1-2\sqrt{b}\right) \operatorname{BesselJ}\left(-2\sqrt{b}, 2\sqrt{a}\sqrt{e^x}\right) \\ & + c_2 \operatorname{Gamma}\left(2\sqrt{b}+1\right) \operatorname{BesselJ}\left(2\sqrt{b}, 2\sqrt{a}\sqrt{e^x}\right) \end{aligned}$$

34.3 problem 3

Internal problem ID [11091]
Internal file name [`OUTPUT/10347_Wednesday_January_24_2024_10_18_11_PM_29877762/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + a\left(\lambda\,\mathrm{e}^{\lambda x} - a\,\mathrm{e}^{2\lambda x}\right) y = 0}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      (-a^2*t+a*lambda)*u(t)+lambda^2*diff(u(t),t)+lambda^2*t*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)+a*(lambda*exp(lambda*x)-a*exp(2*lambda*x))*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{a\,e^{x\lambda}}{\lambda}} \left(c_1 + \text{expIntegral}_1\left(-\frac{2a\,e^{x\lambda}}{\lambda}\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 1.32 (sec). Leaf size: 37

```
DSolve[y''[x]+a*(\[Lambda]*Exp[\[Lambda]*x]-a*Exp[2*\[Lambda]*x])*y[x]==0,y[x],x,IncludeSing
```

$$y(x) \to e^{-\frac{ae^{\lambda x}}{\lambda}} \left(c_2 \,\text{ExpIntegralEi}\left(\frac{2ae^{x\lambda}}{\lambda}\right) + c_1\right)$$

34.4 problem 4

Internal problem ID [11092]
Internal file name [`OUTPUT/10348_Wednesday_January_24_2024_10_18_11_PM_34915829/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' - \left(a^2 \mathrm{e}^{2x} + a(1+2b)\,\mathrm{e}^{x} + b^2\right) y = 0}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
   Change of variables used:
      [x = ln(t)]
   Linear ODE actually solved:
      (-a^2*t^2-2*a*b*t-a*t-b^2)*u(t)+t*diff(u(t),t)+t^2*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.297 (sec). Leaf size: 76

```
dsolve(diff(y(x),x$2)-(a^2*exp(2*x)+a*(2*b+1)*exp(x)+b^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = -c_2 a^{-2b} \operatorname{WhittakerM}\left(-b, \frac{1}{2}-b, 2a\,e^{x}\right) + c_1 e^{bx+a\,e^{x}} + \left(a\,e^{x}\right)^{-b} e^{-a\,e^{x}} c_2 a^{-2b}\left(b2^{-b+1}-2^{-b}\right)$$

✓ Solution by Mathematica

Time used: 1.8 (sec). Leaf size: 57

```
DSolve[y''[x]-(a^2*Exp[2*x]+a*(2*b+1)*Exp[x]+b^2)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to e^{ae^x}(e^x)^{-b}\left(c_1(e^x)^{2b} - 4^b c_2(ae^x)^{2b}\,\Gamma(-2b, 2ae^x)\right)$$

34.5 problem 5

Internal problem ID [11093]
Internal file name [`OUTPUT/10349_Wednesday_January_24_2024_10_18_11_PM_29171069/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' - \left(a\,\mathrm{e}^{2\lambda x} + b\,\mathrm{e}^{\lambda x} + c\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      (-a*t^2-b*t-c)*u(t)+lambda^2*t*diff(u(t),t)+lambda^2*t^2*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.343 (sec). Leaf size: 73

```
dsolve(diff(y(x),x$2)-(a*exp(2*lambda*x)+b*exp(lambda*x)+c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x\lambda}{2}} \left(\mathrm{WhittakerM}\left(-\frac{b}{2\lambda\sqrt{a}}, \frac{\sqrt{c}}{\lambda}, \frac{2\sqrt{a}\, e^{x\lambda}}{\lambda}\right) c_1 + \mathrm{WhittakerW}\left(-\frac{b}{2\lambda\sqrt{a}}, \frac{\sqrt{c}}{\lambda}, \frac{2\sqrt{a}\, e^{x\lambda}}{\lambda}\right) c_2 \right)$$

✓ Solution by Mathematica

Time used: 1.158 (sec). Leaf size: 145

```
DSolve[y''[x]-(a*Exp[2*\[Lambda]*x]+b*Exp[\[Lambda]*x]+c)*y[x]==0,y[x],x,IncludeSingularSolu
```

$$y(x) \to e^{-\frac{\sqrt{a}e^{\lambda x}}{\lambda}}\left(e^{\lambda x}\right)^{\frac{\sqrt{c}}{\lambda}}\left(c_1 \operatorname{HypergeometricU}\left(\frac{\frac{b}{\sqrt{a}}+\lambda+2\sqrt{c}}{2\lambda}, \frac{2\sqrt{c}}{\lambda}+1, \frac{2\sqrt{a}e^{x\lambda}}{\lambda}\right) + c_2 L^{\frac{2\sqrt{c}}{\lambda}}_{-\frac{\frac{b}{\sqrt{a}}+\lambda+2\sqrt{c}}{2\lambda}}\left(\frac{2\sqrt{a}e^{x\lambda}}{\lambda}\right)\right)$$

34.6 problem 6

Internal problem ID [11094]
Internal file name [OUTPUT/10350_Wednesday_January_24_2024_10_18_11_PM_91888170/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$y'' + \left(a\,\mathrm{e}^{4\lambda x} + b\,\mathrm{e}^{3\lambda x} + \mathrm{e}^{2\lambda x} c - \frac{\lambda^2}{4}\right) y = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: indirect Equivalence to 0F1 under \`\`^ @ Moebius\`\` is reso
      <- hypergeometric successful
   <- special function solution successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      (4*a*t^4+4*b*t^3+4*c*t^2-lambda^2)*u(t)+4*lambda^2*t*diff(u(t),t)+4*lambda^2*t^2*diff(
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.36 (sec). Leaf size: 221

```
dsolve(diff(y(x),x$2)+(a*exp(4*lambda*x)+b*exp(3*lambda*x)+c*exp(2*lambda*x)-1/4*lambda^2)*y
```

$$y(x) = c_1 \operatorname{hypergeom}\left(\left[\frac{4\lambda a^{\frac{3}{2}} + 4iac - ib^2}{16\lambda a^{\frac{3}{2}}}\right], \left[\frac{1}{2}\right], \frac{i(2\,e^{x\lambda}a + b)^2}{4\lambda a^{\frac{3}{2}}}\right) e^{-\frac{ie^{2x\lambda}a+\lambda^2 x\sqrt{a}+ib\,e^{x\lambda}}{2\lambda\sqrt{a}}} + c_2 \operatorname{hypergeom}\left(\left[\frac{12\lambda a^{\frac{3}{2}} + 4iac - ib^2}{16\lambda a^{\frac{3}{2}}}\right], \left[\frac{3}{2}\right], \frac{i(2\,e^{x\lambda}a + b)^2}{4\lambda a^{\frac{3}{2}}}\right) \left(2a\,e^{-\frac{ie^{2x\lambda}a-\lambda^2 x\sqrt{a}+ib\,e^{x\lambda}}{2\lambda\sqrt{a}}} + b\,e^{-\frac{ie^{2x\lambda}a+\lambda^2 x\sqrt{a}+ib\,e^{x\lambda}}{2\lambda\sqrt{a}}}\right)$$

✓ Solution by Mathematica

Time used: 1.895 (sec). Leaf size: 178

```
DSolve[y''[x]+(a*Exp[4*\[Lambda]*x]+b*Exp[3*\[Lambda]*x]+c*Exp[2*\[Lambda]*x]-1/4*\[Lambda]^
```

$$y(x) \to \frac{e^{-\frac{ie^{\lambda x}(ae^{\lambda x}+b)}{2\sqrt{a}\lambda}}\left(c_1 \operatorname{HermiteH}\left(\frac{i(b^2-4ac+4ia^{3/2}\lambda)}{8a^{3/2}\lambda}, \frac{\sqrt[4]{-1}(2e^{x\lambda}a+b)}{2a^{3/4}\sqrt{\lambda}}\right) + c_2 \operatorname{Hypergeometric1F1}\left(\frac{-ib^2+4iac+4a^{3/2}\lambda}{16a^{3/2}\lambda}\right.\right.}{\sqrt{e^{\lambda x}}}$$

34.7 problem 7

Internal problem ID [11095]
Internal file name [`OUTPUT/10351_Wednesday_January_24_2024_10_18_11_PM_18234304/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + \left(a\,\mathrm{e}^{2\lambda x}\left(b\,\mathrm{e}^{\lambda x}+c\right)^{n} - \frac{\lambda^{2}}{4}\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 0F1 ODE
      <- Whittaker successful
   <- special function solution successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      (4*a*t^2*(b*t+c)^n-lambda^2)*u(t)+4*lambda^2*t*diff(u(t),t)+4*lambda^2*t^2*diff(diff(u
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.844 (sec). Leaf size: 218

```
dsolve(diff(y(x),x$2)+(a*exp(2*lambda*x)*(b*exp(lambda*x)+c)^n-1/4*lambda^2)*y(x)=0,y(x), si
```

$$y(x) = \frac{e^{-\frac{x\lambda}{2}}\Gamma\left(\frac{n+1}{n+2}\right)^2\left(-\frac{a(b\,e^{x\lambda}+c)^{n+2}}{\lambda^2 b^2(n+2)^2}\right)^{\frac{1}{2n+4}} c_1(n+2)\,\mathrm{BesselI}\left(-\frac{1}{n+2}, 2\sqrt{-\frac{a(b\,e^{x\lambda}+c)^{n+2}}{\lambda^2 b^2(n+2)^2}}\right) + \csc\left(\frac{\pi(n+1)}{n+2}\right)\left(-\frac{a(b\,e^{x\lambda}\cdots}{\lambda^2 b^2(n\cdots}\right.}{(n+2)\,\Gamma\left(\frac{n+1}{n+2}\right)}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+(a*Exp[2*\[Lambda]*x]*(b*Exp[\[Lambda]*x]+c)^n-1/4*\[Lambda]^2)*y[x]==0,y[x],x
```

Not solved

34.8 problem 8

Internal problem ID [11096]
Internal file name [OUTPUT/10352_Wednesday_January_24_2024_10_18_11_PM_64980096/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode_form_A"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + ay' + b\,\mathrm{e}^{2ax} y = 0}$$

34.8.1 Solving as second order bessel ode form A ode

Writing the ode as

$$y'' + ay' + b\,\mathrm{e}^{2ax} y = 0 \tag{1}$$

An ode of the form

$$ay'' + by' + (ce^{rx} + m)y = 0 \tag{1}$$

can be transformed to Bessel ode using the transformation

$$\begin{aligned} x &= \ln(t) \\ e^x &= t \end{aligned}$$

Where a, b, c, m are not functions of x and where b and m are allowed to be be zero. Using this transformation gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} \\ &= \frac{dy}{dt}e^x \\ &= t\frac{dy}{dt} \end{aligned} \tag{2}$$

And

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \frac{d}{dx}\left(t\frac{dy}{dt}\right) \\
&= \frac{d}{dt}\frac{dt}{dx}\left(t\frac{dy}{dt}\right) \\
&= \frac{dt}{dx}\frac{d}{dt}\left(t\frac{dy}{dt}\right) \\
&= t\frac{d}{dt}\left(t\frac{dy}{dt}\right) \\
&= t\left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right) \qquad (3)
\end{aligned}
$$

Substituting (2,3) into (1) gives

$$
\begin{aligned}
at\left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right) + bt\frac{dy}{dt} + (ce^{rx} + m)y &= 0 \\
(aty' + at^2y'') + bty' + (ct^r + m)y &= 0 \\
at^2y'' + (b+a)\,ty' + (ct^r + m)y &= 0 \\
t^2y'' + \frac{b+a}{a}ty' + \left(\frac{c}{a}t^r + \frac{m}{a}\right)y &= 0 \qquad (4)
\end{aligned}
$$

Which is Bessel ODE. Comparing the above to the general known Bowman form of Bessel ode which is

$$
t^2y'' + (1-2\alpha)\,ty' + \left(\beta^2\gamma^2t^{2\gamma} - \left(n^2\gamma^2 - \alpha^2\right)\right)y = 0 \qquad (C)
$$

And now comparing (4) and (C) shows that

$$
\begin{aligned}
(1-2\alpha) &= \frac{b+a}{a} \qquad (5) \\
\beta^2\gamma^2 &= \frac{c}{a} \qquad (6) \\
2\gamma &= r \qquad (7) \\
\left(n^2\gamma^2 - \alpha^2\right) &= -\frac{m}{a} \qquad (8)
\end{aligned}
$$

(5) gives $\alpha = \frac{1}{2} - \frac{b+a}{2a}$. (7) gives $\gamma = \frac{r}{2}$. (8) now becomes $\left(n^2\left(\frac{r}{2}\right)^2 - \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2\right) = -\frac{m}{a}$ or $n^2 = \frac{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}{\left(\frac{r}{2}\right)^2}$. Hence $n = \frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}$ by taking the positive root.

And finally (6) gives $\beta^2 = \frac{c}{a\gamma^2}$ or $\beta = \sqrt{\frac{c}{a}\frac{1}{\gamma}} = \sqrt{\frac{c}{a}\frac{2}{r}}$ (also taking the positive root). Hence

$$\begin{aligned}
\alpha &= \frac{1}{2} - \frac{b+a}{2a} \\
n &= \frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2} \\
\beta &= \sqrt{\frac{c}{a}\frac{2}{r}} \\
\gamma &= \frac{r}{2}
\end{aligned}$$

But the solution to (C) which is general form of Bessel ode is known and given by

$$y(t) = t^{\alpha}(c_1 J_n(\beta t^{\gamma}) + c_2 Y_n(\beta t^{\gamma}))$$

Substituting the above values found into this solution gives

$$y(t) = t^{\frac{1}{2}-\frac{b+a}{2a}}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}t^{\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}t^{\frac{r}{2}}\right)\right)$$

Since $e^x = t$ then the above becomes

$$\begin{aligned}
y(x) &= e^{x\left(\frac{1}{2}-\frac{b+a}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{-b}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{-b}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\frac{b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\frac{b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{4ma+b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{4ma+b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{1}{ra}\sqrt{-4ma+b^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{1}{ra}\sqrt{-4ma+b^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}e^{x\frac{r}{2}}\right)\right) \qquad (9)
\end{aligned}$$

Equation (9) above is the solution to $ay'' + by' + (ce^{rx} + m)y = 0$. Therefore we just need now to compare this form to the ode given and use (9) to obtain the final solution.

Comparing form (1) to the ode we are solving shows that

$$\begin{aligned}
a &= 1 \\
b &= a \\
c &= b \\
r &= 2a \\
m &= 0
\end{aligned}$$

Substituting these in (9) gives the solution as

$$y = \frac{c_1 \mathrm{e}^{-\frac{ax}{2}} \sqrt{2}\, \sin\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}}} - \frac{c_2 \mathrm{e}^{-\frac{ax}{2}} \sqrt{2}\, \cos\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-\frac{ax}{2}} \sqrt{2}\, \sin\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}}} - \frac{c_2 \mathrm{e}^{-\frac{ax}{2}} \sqrt{2}\, \cos\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-\frac{ax}{2}} \sqrt{2}\, \sin\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}}} - \frac{c_2 \mathrm{e}^{-\frac{ax}{2}} \sqrt{2}\, \cos\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)}{\sqrt{\pi}\, \sqrt{\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
   Change of variables used:
      [x = ln(t)/a]
   Linear ODE actually solved:
      b*t*u(t)+2*a^2*diff(u(t),t)+a^2*t*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$2)+a*diff(y(x),x)+b*exp(2*a*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-ax}\left(c_1 \sin\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right) + c_2 \cos\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.133 (sec). Leaf size: 78

```
DSolve[y''[x]+a*y'[x]+b*Exp[2*a*x]*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{a}e^{-\frac{ax}{2}}\left(2c_1 \cos\left(\frac{\sqrt{be^{2ax}}}{a}\right) + c_2 \sin\left(\frac{\sqrt{be^{2ax}}}{a}\right)\right)}{\sqrt{2}\sqrt[4]{be^{2ax}}}$$

34.9 problem 9

Internal problem ID [11097]
Internal file name [`OUTPUT/10353_Wednesday_January_24_2024_10_18_12_PM_31832114/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode_form_A", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{y'' - ay' + b\,\mathrm{e}^{2ax}y = 0}$$

34.9.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$y'' - ay' + b\,\mathrm{e}^{2ax}y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -a \\ q(x) &= b\,\mathrm{e}^{2ax} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\
&= \int e^{-\left(\int -adx\right)}dx \\
&= \int e^{ax}\,dx \\
&= \int e^{ax}dx \\
&= \frac{e^{ax}}{a}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{b\,e^{2ax}}{e^{2ax}} \\
&= b
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + by(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = b$. Let the solution be $y(\tau) = e^{\lambda \tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda \tau} + b\, e^{\lambda \tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda \tau}$ gives

$$\lambda^2 + b = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = b$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(b)}\\ &= \pm\sqrt{-b}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-b}\\ \lambda_2 &= -\sqrt{-b}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-b}\\ \lambda_2 &= -\sqrt{-b}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}\\ y(\tau) &= c_1 e^{(\sqrt{-b})\tau} + c_2 e^{(-\sqrt{-b})\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{\sqrt{-b}\,\tau} + c_2 e^{-\sqrt{-b}\,\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \mathrm{e}^{\frac{\sqrt{-b}\,\mathrm{e}^{ax}}{a}} + c_2 \mathrm{e}^{-\frac{\sqrt{-b}\,\mathrm{e}^{ax}}{a}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{\sqrt{-b}\,\mathrm{e}^{ax}}{a}} + c_2 \mathrm{e}^{-\frac{\sqrt{-b}\,\mathrm{e}^{ax}}{a}} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{\sqrt{-b}\,\mathrm{e}^{ax}}{a}} + c_2 \mathrm{e}^{-\frac{\sqrt{-b}\,\mathrm{e}^{ax}}{a}}$$

Verified OK.

34.9.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$y'' - ay' + b\,\mathrm{e}^{2ax} y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -a \\ q(x) &= b\,\mathrm{e}^{2ax} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{b\,\mathrm{e}^{2ax}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = \frac{b\,\mathrm{e}^{2ax} a}{c\sqrt{b\,\mathrm{e}^{2ax}}}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{b\,\mathrm{e}^{2ax} a}{c\sqrt{b\,\mathrm{e}^{2ax}}} - a\frac{\sqrt{b\,\mathrm{e}^{2ax}}}{c}}{\left(\frac{\sqrt{b\,\mathrm{e}^{2ax}}}{c}\right)^2} \\
&= 0
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{b\,\mathrm{e}^{2ax}}dx}{c} \\
&= \frac{\sqrt{b\,\mathrm{e}^{2ax}}}{ca}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right) + c_2 \sin\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right) + c_2 \sin\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right) \qquad (1)$$

Verification of solutions

$$y = c_1 \cos\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right) + c_2 \sin\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)$$

Verified OK.

34.9.3 Solving as second order bessel ode form A ode

Writing the ode as

$$y'' - ay' + b\,e^{2ax}y = 0 \tag{1}$$

An ode of the form

$$ay'' + by' + (ce^{rx} + m)y = 0 \tag{1}$$

can be transformed to Bessel ode using the transformation

$$\begin{aligned} x &= \ln(t) \\ e^x &= t \end{aligned}$$

Where a, b, c, m are not functions of x and where b and m are allowed to be be zero. Using this transformation gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} \\ &= \frac{dy}{dt}e^x \\ &= t\frac{dy}{dt} \end{aligned} \tag{2}$$

And

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{dx}\left(t\frac{dy}{dt}\right) \\ &= \frac{d}{dt}\frac{dt}{dx}\left(t\frac{dy}{dt}\right) \\ &= \frac{dt}{dx}\frac{d}{dt}\left(t\frac{dy}{dt}\right) \\ &= t\frac{d}{dt}\left(t\frac{dy}{dt}\right) \\ &= t\left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right) \end{aligned} \tag{3}$$

Substituting (2,3) into (1) gives

$$\begin{aligned} at\left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right) + bt\frac{dy}{dt} + (ce^{rx} + m)y &= 0 \\ (aty' + at^2y'') + bty' + (ct^r + m)y &= 0 \\ at^2y'' + (b+a)\,ty' + (ct^r + m)y &= 0 \\ t^2y'' + \frac{b+a}{a}ty' + \left(\frac{c}{a}t^r + \frac{m}{a}\right)y &= 0 \end{aligned} \tag{4}$$

Which is Bessel ODE. Comparing the above to the general known Bowman form of Bessel ode which is

$$t^2y'' + (1-2\alpha)\,ty' + \left(\beta^2\gamma^2 t^{2\gamma} - \left(n^2\gamma^2 - \alpha^2\right)\right)y = 0 \tag{C}$$

And now comparing (4) and (C) shows that

$$(1-2\alpha) = \frac{b+a}{a} \tag{5}$$
$$\beta^2\gamma^2 = \frac{c}{a} \tag{6}$$
$$2\gamma = r \tag{7}$$
$$\left(n^2\gamma^2 - \alpha^2\right) = -\frac{m}{a} \tag{8}$$

(5) gives $\alpha = \frac{1}{2} - \frac{b+a}{2a}$. (7) gives $\gamma = \frac{r}{2}$. (8) now becomes $\left(n^2\left(\frac{r}{2}\right)^2 - \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2\right) = -\frac{m}{a}$ or $n^2 = \frac{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}{\left(\frac{r}{2}\right)^2}$. Hence $n = \frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}$ by taking the positive root. And finally (6) gives $\beta^2 = \frac{c}{a\gamma^2}$ or $\beta = \sqrt{\frac{c}{a}\frac{1}{\gamma}} = \sqrt{\frac{c}{a}\frac{2}{r}}$ (also taking the positive root). Hence

$$\alpha = \frac{1}{2} - \frac{b+a}{2a}$$
$$n = \frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}$$
$$\beta = \sqrt{\frac{c}{a}\frac{2}{r}}$$
$$\gamma = \frac{r}{2}$$

But the solution to (C) which is general form of Bessel ode is known and given by

$$y(t) = t^{\alpha}(c_1 J_n(\beta t^{\gamma}) + c_2 Y_n(\beta t^{\gamma}))$$

Substituting the above values found into this solution gives

$$y(t) = t^{\frac{1}{2}-\frac{b+a}{2a}}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}t^{\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}\frac{2}{r}}t^{\frac{r}{2}}\right)\right)$$

Since $e^x = t$ then the above becomes

$$
\begin{aligned}
y(x) &= e^{x\left(\frac{1}{2}-\frac{b+a}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)+c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{1}{2}-\frac{b+a}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{-b}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)+c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\left(\frac{-b}{2a}\right)^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a}+\frac{b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)+c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a}+\frac{b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{2}{r}\sqrt{-\frac{4ma+b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)+c_2 Y_{\frac{2}{r}\sqrt{-\frac{4ma+b^2}{4a^2}}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)}\left(c_1 J_{\frac{1}{ra}\sqrt{-4ma+b^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)+c_2 Y_{\frac{1}{ra}\sqrt{-4ma+b^2}}\left(\sqrt{\frac{c}{a}}\frac{2}{r}e^{x\frac{r}{2}}\right)\right) \qquad (9)
\end{aligned}
$$

Equation (9) above is the solution to $ay'' + by' + (ce^{rx} + m)y = 0$. Therefore we just need now to compare this form to the ode given and use (9) to obtain the final solution.

Comparing form (1) to the ode we are solving shows that

$$
\begin{aligned}
a &= 1 \\
b &= -a \\
c &= b \\
r &= 2a \\
m &= 0
\end{aligned}
$$

Substituting these in (9) gives the solution as

$$
y = \frac{c_1 \mathrm{e}^{\frac{ax}{2}}\sqrt{2}\,\sin\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}}} - \frac{c_2 \mathrm{e}^{\frac{ax}{2}}\sqrt{2}\,\cos\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \mathrm{e}^{\frac{ax}{2}}\sqrt{2}\,\sin\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}}} - \frac{c_2 \mathrm{e}^{\frac{ax}{2}}\sqrt{2}\,\cos\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}}} \qquad (1)
$$

Verification of solutions

$$
y = \frac{c_1 \mathrm{e}^{\frac{ax}{2}}\sqrt{2}\,\sin\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}}} - \frac{c_2 \mathrm{e}^{\frac{ax}{2}}\sqrt{2}\,\cos\left(\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}\right)}{\sqrt{\pi}\,\sqrt{\frac{\sqrt{b}\,\mathrm{e}^{ax}}{a}}}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)-a*diff(y(x),x)+b*exp(2*a*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin\left(\frac{\sqrt{b}\,e^{ax}}{a}\right) + c_2 \cos\left(\frac{\sqrt{b}\,e^{ax}}{a}\right)$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 42

```
DSolve[y''[x]-a*y'[x]+b*Exp[2*a*x]*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(\frac{\sqrt{b}e^{ax}}{a}\right) + c_2 \sin\left(\frac{\sqrt{b}e^{ax}}{a}\right)$$

34.10 problem 10

Internal problem ID [11098]
Internal file name [`OUTPUT/10354_Wednesday_January_24_2024_10_18_13_PM_80628882/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__bessel__ode__form__A"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + ay' + \left(b\,\mathrm{e}^{\lambda x} + c\right) y = 0}$$

34.10.1 Solving as second order bessel ode form A ode

Writing the ode as

$$y'' + ay' + \left(b\,\mathrm{e}^{\lambda x} + c\right) y = 0 \tag{1}$$

An ode of the form

$$ay'' + by' + (ce^{rx} + m)y = 0 \tag{1}$$

can be transformed to Bessel ode using the transformation

$$\begin{aligned} x &= \ln(t) \\ e^x &= t \end{aligned}$$

Where a, b, c, m are not functions of x and where b and m are allowed to be be zero. Using this transformation gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} \\ &= \frac{dy}{dt}e^x \\ &= t\frac{dy}{dt} \end{aligned} \tag{2}$$

And

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \frac{d}{dx}\left(t\frac{dy}{dt}\right) \\
&= \frac{d}{dt}\frac{dt}{dx}\left(t\frac{dy}{dt}\right) \\
&= \frac{dt}{dx}\frac{d}{dt}\left(t\frac{dy}{dt}\right) \\
&= t\frac{d}{dt}\left(t\frac{dy}{dt}\right) \\
&= t\left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right) \qquad (3)
\end{aligned}
$$

Substituting (2,3) into (1) gives

$$
\begin{aligned}
at\left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right) + bt\frac{dy}{dt} + (ce^{rx} + m)y &= 0 \\
(aty' + at^2y'') + bty' + (ct^r + m)y &= 0 \\
at^2y'' + (b+a)\,ty' + (ct^r + m)y &= 0 \\
t^2y'' + \frac{b+a}{a}ty' + \left(\frac{c}{a}t^r + \frac{m}{a}\right)y &= 0 \qquad (4)
\end{aligned}
$$

Which is Bessel ODE. Comparing the above to the general known Bowman form of Bessel ode which is

$$
t^2y'' + (1-2\alpha)\,ty' + \left(\beta^2\gamma^2t^{2\gamma} - \left(n^2\gamma^2 - \alpha^2\right)\right)y = 0 \qquad \text{(C)}
$$

And now comparing (4) and (C) shows that

$$
\begin{aligned}
(1-2\alpha) &= \frac{b+a}{a} \qquad (5) \\
\beta^2\gamma^2 &= \frac{c}{a} \qquad (6) \\
2\gamma &= r \qquad (7) \\
\left(n^2\gamma^2 - \alpha^2\right) &= -\frac{m}{a} \qquad (8)
\end{aligned}
$$

(5) gives $\alpha = \frac{1}{2} - \frac{b+a}{2a}$. (7) gives $\gamma = \frac{r}{2}$. (8) now becomes $\left(n^2\left(\frac{r}{2}\right)^2 - \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2\right) = -\frac{m}{a}$ or $n^2 = \frac{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}{\left(\frac{r}{2}\right)^2}$. Hence $n = \frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}$ by taking the positive root.

And finally (6) gives $\beta^2 = \frac{c}{a\gamma^2}$ or $\beta = \sqrt{\frac{c}{a}\frac{1}{\gamma}} = \sqrt{\frac{c}{a}\frac{2}{r}}$ (also taking the positive root). Hence

$$
\begin{aligned}
\alpha &= \frac{1}{2} - \frac{b+a}{2a} \\
n &= \frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2} \\
\beta &= \sqrt{\frac{c}{a}\frac{2}{r}} \\
\gamma &= \frac{r}{2}
\end{aligned}
$$

But the solution to (C) which is general form of Bessel ode is known and given by

$$
y(t) = t^{\alpha}(c_1 J_n(\beta t^{\gamma}) + c_2 Y_n(\beta t^{\gamma}))
$$

Substituting the above values found into this solution gives

$$
y(t) = t^{\frac{1}{2} - \frac{b+a}{2a}} \left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} t^{\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} t^{\frac{r}{2}}\right)\right)
$$

Since $e^x = t$ then the above becomes

$$
\begin{aligned}
y(x) &= e^{x\left(\frac{1}{2} - \frac{b+a}{2a}\right)} \left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{1}{2} - \frac{b+a}{2a}\right)^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)} \left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{-b}{2a}\right)^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a} + \left(\frac{-b}{2a}\right)^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)} \left(c_1 J_{\frac{2}{r}\sqrt{-\frac{m}{a} + \frac{b^2}{4a^2}}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{m}{a} + \frac{b^2}{4a^2}}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)} \left(c_1 J_{\frac{2}{r}\sqrt{-\frac{4ma+b^2}{4a^2}}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{2}{r}\sqrt{-\frac{4ma+b^2}{4a^2}}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}}\right)\right) \\
&= e^{x\left(\frac{-b}{2a}\right)} \left(c_1 J_{\frac{1}{ra}\sqrt{-4ma+b^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}}\right) + c_2 Y_{\frac{1}{ra}\sqrt{-4ma+b^2}} \left(\sqrt{\frac{c}{a}\frac{2}{r}} e^{x\frac{r}{2}}\right)\right) \qquad (9)
\end{aligned}
$$

Equation (9) above is the solution to $ay'' + by' + (ce^{rx} + m)y = 0$. Therefore we just need now to compare this form to the ode given and use (9) to obtain the final solution.

Comparing form (1) to the ode we are solving shows that

$$
\begin{aligned}
a &= 1 \\
b &= a \\
c &= b \\
r &= \lambda \\
m &= c
\end{aligned}
$$

Substituting these in (9) gives the solution as

$$y = c_1 \mathrm{e}^{-\frac{ax}{2}} \operatorname{BesselJ}\left(\frac{\sqrt{a^2-4c}}{\lambda}, \frac{2\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) + c_2 \mathrm{e}^{-\frac{ax}{2}} \operatorname{BesselY}\left(\frac{\sqrt{a^2-4c}}{\lambda}, \frac{2\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{ax}{2}} \operatorname{BesselJ}\left(\frac{\sqrt{a^2-4c}}{\lambda}, \frac{2\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) + c_2 \mathrm{e}^{-\frac{ax}{2}} \operatorname{BesselY}\left(\frac{\sqrt{a^2-4c}}{\lambda}, \frac{2\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{ax}{2}} \operatorname{BesselJ}\left(\frac{\sqrt{a^2-4c}}{\lambda}, \frac{2\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right) + c_2 \mathrm{e}^{-\frac{ax}{2}} \operatorname{BesselY}\left(\frac{\sqrt{a^2-4c}}{\lambda}, \frac{2\sqrt{b}\,\mathrm{e}^{\frac{\lambda x}{2}}}{\lambda}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      (b*t+c)*u(t)+(a*lambda*t+lambda^2*t)*diff(u(t),t)+lambda^2*t^2*diff(diff(u(t),t),t) =
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 69

```
dsolve(diff(y(x),x$2)+a*diff(y(x),x)+(b*exp(lambda*x)+c)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{ax}{2}} \left(\mathrm{BesselJ}\left(\frac{\sqrt{a^2-4c}}{\lambda}, \frac{2\sqrt{b}\,\mathrm{e}^{\frac{x\lambda}{2}}}{\lambda}\right) c_1 + \mathrm{BesselY}\left(\frac{\sqrt{a^2-4c}}{\lambda}, \frac{2\sqrt{b}\,\mathrm{e}^{\frac{x\lambda}{2}}}{\lambda}\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.185 (sec). Leaf size: 123

```
DSolve[y''[x]+a*y'[x]+(b*Exp[\[Lambda]*x]+c)*y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{-\frac{ax}{2}} \left(c_1 \operatorname{Gamma}\left(1 - \frac{\sqrt{a^2-4c}}{\lambda}\right) \operatorname{BesselJ}\left(-\frac{\sqrt{a^2-4c}}{\lambda}, \frac{2\sqrt{be^{x\lambda}}}{\lambda}\right) + c_2 \operatorname{Gamma}\left(\frac{\lambda+\sqrt{a^2-4c}}{\lambda}\right) \operatorname{BesselJ}\left(\frac{\sqrt{a^2-4c}}{\lambda}, \frac{2\sqrt{be^{x\lambda}}}{\lambda}\right)\right)$$

34.11 problem 11

Internal problem ID [11099]
Internal file name [`OUTPUT/10355_Wednesday_January_24_2024_10_18_13_PM_46931920/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$y'' - y' + \left(e^{3\lambda x} a + b\, e^{2\lambda x} + \frac{1}{4} - \frac{\lambda^2}{4}\right) y = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      (4*a*t^3+4*b*t^2-lambda^2+1)*u(t)+(4*lambda^2*t-4*lambda*t)*diff(u(t),t)+4*lambda^2*t^
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.234 (sec). Leaf size: 51

```
dsolve(diff(y(x),x$2)-diff(y(x),x)+(a*exp(3*lambda*x)+b*exp(2*lambda*x)+1/4-1/4*lambda^2  )*
```

$$y(x) = e^{-\frac{x(\lambda-1)}{2}} \left(\text{AiryAi}\left(-\frac{e^{x\lambda}a+b}{\lambda^{\frac{2}{3}}a^{\frac{2}{3}}}\right)c_1 + \text{AiryBi}\left(-\frac{e^{x\lambda}a+b}{\lambda^{\frac{2}{3}}a^{\frac{2}{3}}}\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 1.332 (sec). Leaf size: 77

```
DSolve[y''[x]-y'[x]+(a*Exp[3*\[Lambda]*x]+b*Exp[2*\[Lambda]*x]+1/4-1/4*\[Lambda]^2  )*y[x]==
```

$$y(x) \to \frac{e^{x/2}\left(c_1 \operatorname{AiryAi}\left(\frac{(e^{x\lambda}a+b)\sqrt[3]{-\frac{a}{\lambda^2}}}{a}\right) + c_2 \operatorname{AiryBi}\left(\frac{(e^{x\lambda}a+b)\sqrt[3]{-\frac{a}{\lambda^2}}}{a}\right)\right)}{\sqrt{e^{\lambda x}}}$$

34.12 problem 12

Internal problem ID [11100]
Internal file name [`OUTPUT/10356_Wednesday_January_24_2024_10_18_13_PM_71465684/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$y'' - y' + \left(a\,\mathrm{e}^{2\lambda x}\left(b\,\mathrm{e}^{\lambda x} + c\right)^{n} + \frac{1}{4} - \frac{\lambda^{2}}{4}\right) y = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 0F1 ODE
      <- Whittaker successful
   <- special function solution successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      (4*a*t^2*(b*t+c)^n-lambda^2+1)*u(t)+(4*lambda^2*t-4*lambda*t)*diff(u(t),t)+4*lambda^2*
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.796 (sec). Leaf size: 224

```
dsolve(diff(y(x),x$2)-diff(y(x),x)+(a*exp(2*lambda*x)*(b*exp(lambda*x)+c)^n+1/4-1/4*lambda^2
```

$$y(x) = \frac{\Gamma\left(\frac{n+1}{n+2}\right)^2 \mathrm{e}^{-\frac{x(\lambda-1)}{2}} \left(-\frac{a\left(b\,\mathrm{e}^{x\lambda}+c\right)^{n+2}}{\lambda^2 b^2 (n+2)^2}\right)^{\frac{1}{2n+4}} c_1 (n+2)\,\mathrm{BesselI}\left(-\frac{1}{n+2}, 2\sqrt{-\frac{a\left(b\,\mathrm{e}^{x\lambda}+c\right)^{n+2}}{\lambda^2 b^2 (n+2)^2}}\right) + \csc\left(\frac{\pi(n+1)}{n+2}\right)\left(-\frac{a(b}{\lambda^2}\right.}{(n+2)\,\Gamma\left(\frac{n+1}{n+2}\right)}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]-y'[x]+(a*Exp[2*\[Lambda]*x]*(b*Exp[\[Lambda]*x]+c)^n+1/4-1/4*\[Lambda]^2 )*y[
```

Not solved

34.13 problem 13

Internal problem ID [11101]
Internal file name [`OUTPUT/10357_Wednesday_January_24_2024_10_18_13_PM_34806798/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2a\,\mathrm{e}^{\lambda x} y' + a\,\mathrm{e}^{\lambda x}\left(a\,\mathrm{e}^{\lambda x} + \lambda\right) y = 0}$$

34.13.1 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2a\,\mathrm{e}^{\lambda x}$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int 2a\,\mathrm{e}^{\lambda x}\,dx} \\
&= \mathrm{e}^{\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 0$$
$$\left(\mathrm{e}^{\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}} y\right)'' = 0$$

Integrating once gives

$$\left(\mathrm{e}^{\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}}$$

Or

$$y = c_1 x\,\mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}} + c_2 \mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}} + c_2 \mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 x\,\mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}} + c_2 \mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}$$

Verified OK.

34.13.2 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = 2a\,\mathrm{e}^{\lambda x}$$
$$q(x) = a^2 \mathrm{e}^{2\lambda x} + a\lambda\,\mathrm{e}^{\lambda x}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= a^2 e^{2\lambda x} + a\lambda\,e^{\lambda x} - \frac{\left(2a\,e^{\lambda x}\right)'}{2} - \frac{\left(2a\,e^{\lambda x}\right)^2}{4} \\
&= a^2 e^{2\lambda x} + a\lambda\,e^{\lambda x} - \frac{\left(2a\lambda\,e^{\lambda x}\right)}{2} - \frac{\left(4a^2 e^{2\lambda x}\right)}{4} \\
&= a^2 e^{2\lambda x} + a\lambda\,e^{\lambda x} - \left(a\lambda\,e^{\lambda x}\right) - a^2 e^{2\lambda x} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\,z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= e^{-\left(\int \frac{p(x)}{2}dx\right)} \\
&= e^{-\int \frac{2a\,e^{\lambda x}}{2}} \\
&= e^{-\frac{a\,e^{\lambda x}}{\lambda}}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\,e^{-\frac{a\,e^{\lambda x}}{\lambda}} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(x)\,e^{-\frac{a\,e^{\lambda x}}{\lambda}} = 0
$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$
v(x) = c_1 x + c_2
$$

Now that $v(x)$ is known, then

$$
\begin{aligned}
y &= v(x)\,z(x) \\
&= (c_1 x + c_2)\,(z(x))
\end{aligned} \tag{7}
$$

But from (5)

$$
z(x) = e^{-\frac{a\,e^{\lambda x}}{\lambda}}
$$

Hence (7) becomes

$$y = \mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}(c_1 x + c_2)$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}(c_1 x + c_2) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}(c_1 x + c_2)$$

Verified OK.

34.13.3 Solving using Kovacic algorithm

Writing the ode as

$$\begin{aligned} y'' + 2a\,\mathrm{e}^{\lambda x} y' + a\,\mathrm{e}^{\lambda x}\left(a\,\mathrm{e}^{\lambda x} + \lambda\right) y &= 0 & (1)\\ Ay'' + By' + Cy &= 0 & (2) \end{aligned}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2a\,\mathrm{e}^{\lambda x} \\ C &= \mathrm{e}^{\lambda x} a\left(a\,\mathrm{e}^{\lambda x} + \lambda\right) \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 225: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2a\,\mathrm{e}^{\lambda x}}{1}\,dx} \\
&= z_1 e^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}} \\
&= z_1 \left(\mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2a\,\mathrm{e}^{\lambda x}}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{2a\,\mathrm{e}^{\lambda x}}{\lambda}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\ln\left(\mathrm{e}^{\lambda x}\right)}{\lambda}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}\right) + c_2 \left(\mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}} \left(\frac{\ln\left(\mathrm{e}^{\lambda x}\right)}{\lambda}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-\frac{a\,e^{\lambda x}}{\lambda}} c_1 + \frac{c_2 e^{-\frac{a\,e^{\lambda x}}{\lambda}} \ln\left(e^{\lambda x}\right)}{\lambda} \tag{1}$$

Verification of solutions

$$y = e^{-\frac{a\,e^{\lambda x}}{\lambda}} c_1 + \frac{c_2 e^{-\frac{a\,e^{\lambda x}}{\lambda}} \ln\left(e^{\lambda x}\right)}{\lambda}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+2*a*exp(lambda*x)*diff(y(x),x)+a*exp(lambda*x)*(a*exp(lambda*x)+lambda
```

$$y(x) = e^{-\frac{a\,e^{x\lambda}}{\lambda}} (c_2 x + c_1)$$

✓ Solution by Mathematica
Time used: 0.109 (sec). Leaf size: 26

```
DSolve[y''[x]+2*a*Exp[\[Lambda]*x]*y'[x]+a*Exp[\[Lambda]*x]*(a*Exp[\[Lambda]*x]+\[Lambda])*y
```

$$y(x) \to (c_2 x + c_1) e^{-\frac{a e^{\lambda x}}{\lambda}}$$

34.14 problem 14

Internal problem ID [11102]
Internal file name [OUTPUT/10358_Wednesday_January_24_2024_10_18_14_PM_71707269/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$y'' + (a + b)\,\mathrm{e}^{\lambda x} y' + a\,\mathrm{e}^{\lambda x}\left(b\,\mathrm{e}^{\lambda x} + \lambda\right) y = 0$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      (a*b*t+a*lambda)*u(t)+(a*lambda*t+b*lambda*t+lambda^2)*diff(u(t),t)+lambda^2*t*diff(di
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+(a+b)*exp(lambda*x)*diff(y(x),x)+a*exp(lambda*x)*(b*exp(lambda*x)+lamb
```

$$y(x) = \mathrm{e}^{-\frac{a\,\mathrm{e}^{x\lambda}}{\lambda}}\left(c_1 + \mathrm{expIntegral}_1\left(-\frac{\mathrm{e}^{x\lambda}(a-b)}{\lambda}\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 2.377 (sec). Leaf size: 40

```
DSolve[y''[x]+(a+b)*Exp[\[Lambda]*x]*y'[x]+a*Exp[\[Lambda]*x]*(b*Exp[\[Lambda]*x]+\[Lambda])
```

$$y(x) \to e^{-\frac{ae^{\lambda x}}{\lambda}}\left(c_2\,\mathrm{ExpIntegralEi}\left(\frac{(a-b)e^{x\lambda}}{\lambda}\right)+c_1\right)$$

34.15 problem 15

Internal problem ID [11103]
Internal file name [`OUTPUT/10359_Wednesday_January_24_2024_10_18_14_PM_13126995/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + a\,e^{\lambda x} y' - b\,e^{\mu x}\left(a\,e^{\lambda x} + e^{\mu x} b + \mu\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 46

```
dsolve(diff(y(x),x$2)+a*exp(lambda*x)*diff(y(x),x)-b*exp(mu*x)*(a*exp(lambda*x)+b*exp(mu*x)+
```

$$y(x) = \left(\left(\int e^{\frac{-2b\,e^{x\mu}\lambda - e^{x\lambda} a\mu}{\mu\lambda}} dx\right) c_1 + c_2\right) e^{\frac{b\,e^{x\mu}}{\mu}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+a*Exp[\[Lambda]*x]*y'[x]-b*Exp[\[Mu]*x]*(a*Exp[\[Lambda]*x]+b*Exp[\[Mu]*x]+\[M
```

Not solved

34.16 problem 16

Internal problem ID [11104]
Internal file name [`OUTPUT/10360_Wednesday_January_24_2024_10_18_15_PM_73998394/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + 2k\,e^{\mu x}y' + \left(a\,e^{2\lambda x} + b\,e^{\lambda x} + k^2 e^{2\mu x} + k\mu\,e^{\mu x} + c\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+2*k*exp(mu*x)*diff(y(x),x)+(a*exp(2*lambda*x)+b*exp(lambda*x)+k^2*exp(
```

No solution found

✓ Solution by Mathematica

Time used: 2.531 (sec). Leaf size: 232

```
DSolve[y''[x]+2*k*Exp[\[Mu]*x]*y'[x]+(a*Exp[2*\[Lambda]*x]+b*Exp[\[Lambda]*x]+k^2*Exp[2*\[Mu
```

$$y(x) \rightarrow 2^{\frac{1}{2}-\frac{i\sqrt{c}}{\lambda}}(e^x)^{\frac{1}{2}-\frac{\lambda}{2}}\left((e^x)^\mu\right)^{-\frac{1}{2}/\mu}\left((e^x)^\lambda\right)^{\frac{1}{2}-\frac{i\sqrt{c}}{\lambda}} e^{-\frac{k(e^x)^\mu}{\mu}+\frac{i\sqrt{a}(e^x)^\lambda}{\lambda}}\left(c_1 \operatorname{HypergeometricU}\left(-\frac{\frac{ib}{\sqrt{a}}-\lambda+2i\sqrt{c}}{2\lambda}, 1-\frac{2i\sqrt{c}}{\lambda}, -\frac{2i\sqrt{a}(e^x)^\lambda}{\lambda}\right)+c_2 L_{\frac{\frac{ib}{\sqrt{a}}-\lambda+2i\sqrt{c}}{2\lambda}}^{-\frac{2i\sqrt{c}}{\lambda}}\left(-\frac{2i\sqrt{a}(e^x)^\lambda}{\lambda}\right)\right)$$

34.17 problem 17

Internal problem ID [11105]
Internal file name [OUTPUT/10361_Wednesday_January_24_2024_10_18_15_PM_64620218/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - (a + 2\,\mathrm{e}^{ax} b)\, y' + b^2 \mathrm{e}^{2ax} y = 0}$$

34.17.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$y'' + (-2\,\mathrm{e}^{ax} b - a)\, y' + b^2 \mathrm{e}^{2ax} y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -2\,\mathrm{e}^{ax} b - a \\ q(x) &= b^2 \mathrm{e}^{2ax} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int (-2\,e^{ax} b - a)dx\right)} dx \\
&= \int e^{\frac{a^2 x + 2\,e^{ax} b}{a}} dx \\
&= \int e^{\frac{a^2 x + 2\,e^{ax} b}{a}} dx \\
&= \frac{e^{\frac{2b\,e^{ax}}{a}}}{2b}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{b^2 e^{2ax}}{e^{\frac{2a^2 x + 4\,e^{ax} b}{a}}} \\
&= b^2 e^{-\frac{4b\,e^{ax}}{a}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2} y(\tau) + b^2 e^{-\frac{4b\,e^{ax}}{a}} y(\tau) = 0$$

But in terms of τ

$$b^2 e^{-\frac{4b\,e^{ax}}{a}} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,\sqrt{\frac{e^{\frac{2b\,e^{ax}}{a}}}{b}}\left(c_1 + c_2\ln\left(\frac{e^{\frac{2b\,e^{ax}}{a}}}{b}\right) - c_2\ln(2)\right)}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{2}\, \sqrt{\frac{e^{\frac{2b\, e^{ax}}{a}}}{b}}\, \left(c_1 + c_2 \ln\left(\frac{e^{\frac{2b\, e^{ax}}{a}}}{b}\right) - c_2 \ln(2)\right)}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\, \sqrt{\frac{e^{\frac{2b\, e^{ax}}{a}}}{b}}\, \left(c_1 + c_2 \ln\left(\frac{e^{\frac{2b\, e^{ax}}{a}}}{b}\right) - c_2 \ln(2)\right)}{2}$$

Verified OK.

34.17.2 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -2\, e^{ax} b - a \\ q(x) &= b^2 e^{2ax} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= b^2 e^{2ax} - \frac{(-2\, e^{ax} b - a)'}{2} - \frac{(-2\, e^{ax} b - a)^2}{4} \\ &= b^2 e^{2ax} - \frac{(-2a\, e^{ax} b)}{2} - \frac{((-2\, e^{ax} b - a)^2)}{4} \\ &= b^2 e^{2ax} - (-a\, e^{ax} b) - \frac{(-2\, e^{ax} b - a)^2}{4} \\ &= -\frac{a^2}{4} \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{-2\,\mathrm{e}^{ax} b - a}{2}} \\ &= \mathrm{e}^{\frac{a^2 x + 2\,\mathrm{e}^{ax} b}{2a}} \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\,\mathrm{e}^{\frac{a^2 x + 2\,\mathrm{e}^{ax} b}{2a}} \tag{4}$$

Applying this change of variable to the original ode results in

$$\mathrm{e}^{\frac{a^2 x + 2\,\mathrm{e}^{ax} b}{2a}} \left(-a^2 v(x) + 4v''(x)\right) = 0$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 4, B = 0, C = -a^2$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 \mathrm{e}^{\lambda x} - a^2 \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$-a^2 + 4\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 0, C = -a^2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(4)} \pm \frac{1}{(2)\,(4)}\sqrt{0^2 - (4)\,(4)\,(-a^2)} \\ &= \pm\frac{\sqrt{a^2}}{2} \end{aligned}$$

Hence

$$\lambda_1 = +\frac{\sqrt{a^2}}{2}$$

$$\lambda_2 = -\frac{\sqrt{a^2}}{2}$$

Which simplifies to

$$\lambda_1 = \frac{\sqrt{a^2}}{2}$$
$$\lambda_2 = -\frac{\sqrt{a^2}}{2}$$

Since roots are real and distinct, then the solution is

$$v(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$v(x) = c_1 e^{\left(\frac{\sqrt{a^2}}{2}\right)x} + c_2 e^{\left(-\frac{\sqrt{a^2}}{2}\right)x}$$

Or

$$v(x) = c_1 \mathrm{e}^{\frac{\sqrt{a^2}\,x}{2}} + c_2 \mathrm{e}^{-\frac{\sqrt{a^2}\,x}{2}}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 \mathrm{e}^{\frac{\sqrt{a^2}\,x}{2}} + c_2 \mathrm{e}^{-\frac{\sqrt{a^2}\,x}{2}}\right)(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \mathrm{e}^{\frac{a^2 x + 2\,\mathrm{e}^{a x} b}{2a}}$$

Hence (7) becomes

$$y = \left(c_1 \mathrm{e}^{\frac{\sqrt{a^2}\,x}{2}} + c_2 \mathrm{e}^{-\frac{\sqrt{a^2}\,x}{2}}\right) \mathrm{e}^{\frac{a^2 x + 2\,\mathrm{e}^{a x} b}{2a}}$$

Summary

The solution(s) found are the following

$$y = \left(c_1 \mathrm{e}^{\frac{\sqrt{a^2}\,x}{2}} + c_2 \mathrm{e}^{-\frac{\sqrt{a^2}\,x}{2}}\right) \mathrm{e}^{\frac{a^2 x + 2\,\mathrm{e}^{a x} b}{2a}} \tag{1}$$

Verification of solutions

$$y = \left(c_1 \mathrm{e}^{\frac{\sqrt{a^2}\,x}{2}} + c_2 \mathrm{e}^{-\frac{\sqrt{a^2}\,x}{2}}\right) \mathrm{e}^{\frac{a^2 x + 2\,\mathrm{e}^{a x} b}{2a}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$2)-(a+2*b*exp(a*x))*diff(y(x),x)+b^2*exp(2*a*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\frac{a^2 x+2\,\mathrm{e}^{ax} b}{2a}} \left(c_1 \sinh\left(\frac{ax}{2}\right) + c_2 \cosh\left(\frac{ax}{2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.079 (sec). Leaf size: 35

```
DSolve[y''[x]-(a+2*b*Exp[a*x])*y'[x]+b^2*Exp[2*a*x]*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^{\frac{be^{ax}}{a}}(bc_2e^{ax} + ac_1)}{a}$$

34.18 problem 18

Internal problem ID [11106]
Internal file name [`OUTPUT/10362_Wednesday_January_24_2024_10_18_16_PM_88334329/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + \left(a\,\mathrm{e}^{2\lambda x} + \lambda\right) y' - a\lambda\,\mathrm{e}^{2\lambda x} y = 0}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
      Solution has integrals. Trying a special function solution free of integrals...
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         <- Kummer successful
      <- special function solution successful
         -> Trying to convert hypergeometric functions to elementary form...
         <- elementary form is not straightforward to achieve - returning special function s
      <- Kovacics algorithm successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      -t*a*u(t)+(a*t^2+2*lambda)*diff(u(t),t)+t*lambda*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.593 (sec). Leaf size: 79

```
dsolve(diff(y(x),x$2)+(a*exp(2*lambda*x)+lambda)*diff(y(x),x)-a*lambda*exp(2*lambda*x)*y(x)=
```

$$y(x) = c_2\sqrt{\pi}\left(\mathrm{e}^{x\lambda}a + \mathrm{e}^{-x\lambda}\lambda\right)\mathrm{erf}\left(\frac{\sqrt{2}\,\mathrm{e}^{x\lambda}\sqrt{a}}{2\sqrt{\lambda}}\right) + \sqrt{a}\,\sqrt{\lambda}\,\mathrm{e}^{-\frac{a\,\mathrm{e}^{2x\lambda}}{2\lambda}}\sqrt{2}\,c_2 + c_1\left(\mathrm{e}^{x\lambda}a + \mathrm{e}^{-x\lambda}\lambda\right)$$

✓ Solution by Mathematica

Time used: 0.283 (sec). Leaf size: 129

```
DSolve[y''[x]+(a*Exp[2*\[Lambda]*x]+\[Lambda])*y'[x]-a*\[Lambda]*Exp[2*\[Lambda]*x]*y[x]==0,
```

$$y(x) \to \frac{\sqrt{2\pi}c_2\left(ae^{2\lambda x} + \lambda\right)\mathrm{erf}\left(\frac{\sqrt{a\lambda e^{2\lambda x}}}{\sqrt{2}\lambda}\right) - 4i\sqrt{2}ac_1e^{2\lambda x} + 2c_2e^{-\frac{ae^{2\lambda x}}{2\lambda}}\sqrt{a\lambda e^{2\lambda x}} - 4i\sqrt{2}c_1\lambda}{4\sqrt{a\lambda e^{2\lambda x}}}$$

34.19 problem 19

Internal problem ID [11107]
Internal file name [`OUTPUT/10363_Wednesday_January_24_2024_10_18_16_PM_44683069/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \left(a\,\mathrm{e}^{\lambda x} - \lambda\right) y' + yb\,\mathrm{e}^{2\lambda x} = 0}$$

34.19.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$y'' + \left(a\,\mathrm{e}^{\lambda x} - \lambda\right) y' + yb\,\mathrm{e}^{2\lambda x} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= a\,\mathrm{e}^{\lambda x} - \lambda \\ q(x) &= b\,\mathrm{e}^{2\lambda x} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int (a\,e^{\lambda x}-\lambda)dx\right)} dx \\
&= \int e^{\frac{x\,\lambda^2 - a\,e^{\lambda x}}{\lambda}} dx \\
&= \int e^{\frac{x\,\lambda^2 - a\,e^{\lambda x}}{\lambda}} dx \\
&= -\frac{e^{-\frac{a\,e^{\lambda x}}{\lambda}}}{a}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{b\,e^{2\lambda x}}{e^{\frac{2x\,\lambda^2 - 2a\,e^{\lambda x}}{\lambda}}} \\
&= b\,e^{\frac{2a\,e^{\lambda x}}{\lambda}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) + b\,e^{\frac{2a\,e^{\lambda x}}{\lambda}} y(\tau) = 0$$

But in terms of τ

$$b\,e^{\frac{2a\,e^{\lambda x}}{\lambda}} = \frac{b}{a^2\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{by(\tau)}{a^2\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)a^2\tau^2 + by(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$a^2\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + b\,\tau^r = 0$$

Simplifying gives

$$a^2r(r-1)\,\tau^r + 0\,\tau^r + b\,\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$a^2r(r-1) + 0 + b = 0$$

Or

$$a^2r^2 - a^2r + b = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{-a + \sqrt{a^2-4b}}{2a}$$
$$r_2 = \frac{a + \sqrt{a^2-4b}}{2a}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{-\frac{-a+\sqrt{a^2-4b}}{2a}} + c_2\tau^{\frac{a+\sqrt{a^2-4b}}{2a}}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1\left(-\frac{\mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}}{a}\right)^{-\frac{-a+\sqrt{a^2-4b}}{2a}} + c_2\left(-\frac{\mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}}{a}\right)^{\frac{a+\sqrt{a^2-4b}}{2a}}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(-\frac{\mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}}{a}\right)^{-\frac{-a+\sqrt{a^2-4b}}{2a}} + c_2 \left(-\frac{\mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}}{a}\right)^{\frac{a+\sqrt{a^2-4b}}{2a}} \tag{1}$$

Verification of solutions

$$y = c_1 \left(-\frac{\mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}}{a}\right)^{-\frac{-a+\sqrt{a^2-4b}}{2a}} + c_2 \left(-\frac{\mathrm{e}^{-\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}}}{a}\right)^{\frac{a+\sqrt{a^2-4b}}{2a}}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   <- constant coefficients successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      b*u(t)+a*lambda*diff(u(t),t)+lambda^2*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 53

```
dsolve(diff(y(x),x$2)+(a*exp(lambda*x)-lambda)*diff(y(x),x)+b*exp(2*lambda*x)*y(x)=0,y(x), s
```

$$y(x) = c_1 \mathrm{e}^{\frac{\left(-a+\sqrt{a^2-4b}\right)\mathrm{e}^{x\lambda}}{2\lambda}} + c_2 \mathrm{e}^{-\frac{\left(a+\sqrt{a^2-4b}\right)\mathrm{e}^{x\lambda}}{2\lambda}}$$

✓ Solution by Mathematica

Time used: 0.133 (sec). Leaf size: 61

```
DSolve[y''[x]+(a*Exp[\[Lambda]*x]-\[Lambda])*y'[x]+b*Exp[2*\[Lambda]*x]*y[x]==0,y[x],x,Inclu
```

$$y(x) \to e^{-\frac{\left(\sqrt{a^2-4b}+a\right)e^{\lambda x}}{2\lambda}} \left(c_2 e^{\frac{\sqrt{a^2-4b}e^{\lambda x}}{\lambda}} + c_1\right)$$

34.20 problem 20

Internal problem ID [11108]
Internal file name [OUTPUT/10364_Wednesday_January_24_2024_10_18_17_PM_66837224/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + \left(a\,\mathrm{e}^{\lambda x} + b\right) y' + c\left(a\,\mathrm{e}^{\lambda x} + b - c\right) y = 0}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      (a*c*t+b*c-c^2)*u(t)+(a*lambda*t^2+b*lambda*t+lambda^2*t)*diff(u(t),t)+lambda^2*t^2*di
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.297 (sec). Leaf size: 176

```
dsolve(diff(y(x),x$2)+(a*exp(lambda*x)+b)*diff(y(x),x)+c*(a*exp(lambda*x)+b-c)*y(x)=0,y(x),
```

$$
\begin{aligned}
y(x) = {} & \mathrm{e}^{\frac{-\mathrm{e}^{x\lambda}a-(b+3\lambda)x\lambda}{2\lambda}} c_2(-\lambda-2c+b)^2 \operatorname{WhittakerM}\left(-\frac{-\lambda-2c+b}{2\lambda}, -\frac{-2\lambda-2c+b}{2\lambda}, \frac{a\,\mathrm{e}^{x\lambda}}{\lambda}\right) \\
& + \left((\lambda+2c-b)\,\mathrm{e}^{\frac{-\mathrm{e}^{x\lambda}a-(b+3\lambda)x\lambda}{2\lambda}} + a\,\mathrm{e}^{\frac{-\mathrm{e}^{x\lambda}a-x\lambda(b+\lambda)}{2\lambda}}\right) c_2\lambda \operatorname{WhittakerM}\left(-\frac{b-2c+\lambda}{2\lambda}, -\frac{-2\lambda-2c+b}{2\lambda}, \frac{a\,\mathrm{e}^{x\lambda}}{\lambda}\right) + c_1\mathrm{e}^{-cx}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.244 (sec). Leaf size: 96

```
DSolve[y''[x]+(a*Exp[\[Lambda]*x]+b)*y'[x]+c*(a*Exp[\[Lambda]*x]+b-c)*y[x]==0,y[x],x,Include
```

$$y(x) \to (-1)^{-\frac{c}{\lambda}} c^{c/\lambda} \lambda^{\frac{c}{\lambda}-1} a^{-\frac{c}{\lambda}} \left(ce^{\lambda x}\right)^{-\frac{c}{\lambda}} \left(c_2(2c-b)(-1)^{c/\lambda} \Gamma\left(-\frac{b-2c}{\lambda}, 0, \frac{ae^{x\lambda}}{\lambda}\right) + c_1\lambda\right)$$

34.21 problem 21

Internal problem ID [11109]
Internal file name [`OUTPUT/10365_Wednesday_January_24_2024_10_18_17_PM_27204558/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$y'' + \left(a + b\,\mathrm{e}^{2\lambda x}\right) y' + \lambda\left(a - \lambda - b\,\mathrm{e}^{2\lambda x}\right) y = 0$$

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+(a+b*exp(2*lambda*x))*diff(y(x),x)+lambda*(a-lambda-b*exp(2*lambda*x))
```

No solution found

✓ Solution by Mathematica

Time used: 0.513 (sec). Leaf size: 248

```
DSolve[y''[x]+(a+b*Exp[2*\[Lambda]*x])*y'[x]+\[Lambda]*(a-\[Lambda]-b*Exp[2*\[Lambda]*x])*y[
```

$$y(x) \to \frac{-\frac{1}{2}c_2(a-2\lambda)e^{-\frac{be^{2\lambda x}}{2\lambda}}\left(b\lambda e^{2\lambda x}\right)^{-\frac{a}{2\lambda}}\left(b2^{\frac{a}{2\lambda}}\lambda^{a/\lambda}e^{2\lambda x} + \text{Gamma}\left(1-\frac{a}{2\lambda}\right)e^{\frac{be^{2\lambda x}}{2\lambda}}\left(a+be^{2\lambda x}\right)\left(b\lambda e^{2\lambda x}\right)^{\frac{a}{2\lambda}} - e^{\frac{be^{2\lambda x}}{2\lambda}}\right.}{\sqrt{2}\lambda\sqrt{b\lambda e^{2\lambda x}}}$$

34.22 problem 22

Internal problem ID [11110]
Internal file name [OUTPUT/10366_Wednesday_January_24_2024_10_18_17_PM_14375704/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + \left(a + b\,\mathrm{e}^{\lambda x} + b - 3\lambda\right) y' + a^2\lambda(-\lambda + b)\,\mathrm{e}^{2\lambda x} y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      (a^2*b*t-a^2*lambda*t)*u(t)+(b*t+a+b-2*lambda)*diff(u(t),t)+t*lambda*diff(diff(u(t),t)
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.406 (sec). Leaf size: 205

```
dsolve(diff(y(x),x$2)+(a+b*exp(lambda*x)+b-3*lambda)*diff(y(x),x)+a^2*lambda*(b-lambda)*exp(
```

$$y(x) = \mathrm{e}^{-\frac{\mathrm{e}^{x\lambda}\left(b+\sqrt{-4\lambda(b-\lambda)a^2+b^2}\right)}{2\lambda}}\left(\mathrm{KummerU}\left(\frac{\left(b+\sqrt{-4\lambda\,(b-\lambda)\,a^2+b^2}\right)(-2\lambda+b+a)}{2\sqrt{-4\lambda\,(b-\lambda)\,a^2+b^2}\,\lambda},\frac{-2\lambda+b+a}{\lambda},\frac{\sqrt{-4\lambda\,(\right.\right.$$

$$+\mathrm{KummerM}\left(\frac{\left(b+\sqrt{-4\lambda\,(b-\lambda)\,a^2+b^2}\right)(-2\lambda+b+a)}{2\sqrt{-4\lambda\,(b-\lambda)\,a^2+b^2}\,\lambda},\frac{-2\lambda+b+a}{\lambda},\frac{\sqrt{-4\lambda\,(b-\lambda)\,a^2+b^2}\,\mathrm{e}^{x\lambda}}{\lambda}\right)$$

✓ Solution by Mathematica

Time used: 3.799 (sec). Leaf size: 260

```
DSolve[y''[x]+(a+b*Exp[\[Lambda]*x]+b-3*\[Lambda])*y'[x]+a^2*\[Lambda]*(b-\[Lambda])*Exp[2*\
```

$$y(x) \to \exp\left(-\frac{e^{\lambda x}\left(\sqrt{-4a^2b\lambda+4a^2\lambda^2+b^2}+b\right)}{2\lambda}\right)\left(c_1\,\mathrm{HypergeometricU}\left(\frac{(a+b-2\lambda)\left(b+\sqrt{4\lambda^2a^2-4b\lambda a^2\right.}}{2\lambda\sqrt{4\lambda^2a^2-4b\lambda a^2+b^2}}\right.\right.$$

$$+c_2L^{\frac{a+b-3\lambda}{\lambda}}_{-\frac{(a+b-2\lambda)\left(b+\sqrt{4\lambda^2a^2-4b\lambda a^2+b^2}\right)}{2\lambda\sqrt{4\lambda^2a^2-4b\lambda a^2+b^2}}}\left(\frac{e^{x\lambda}\sqrt{4\lambda^2a^2-4b\lambda a^2+b^2}}{\lambda}\right)\Bigg)\Bigg)$$

34.23 problem 23

Internal problem ID [11111]
Internal file name [OUTPUT/10367_Wednesday_January_24_2024_10_18_18_PM_71646621/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$
\boxed{y'' + \left(2a\, \mathrm{e}^{\lambda x} - \lambda\right) y' + \left(a^2 \mathrm{e}^{2\lambda x} + \mathrm{e}^{\mu x} c\right) y = 0}
$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+(2*a*exp(lambda*x)-lambda)*diff(y(x),x)+(a^2*exp(2*lambda*x)+c*exp(mu*
```

No solution found

✓ Solution by Mathematica

Time used: 1.858 (sec). Leaf size: 164

```
DSolve[y''[x]+(2*a*Exp[\[Lambda]*x]-\[Lambda])*y'[x]+(a^2*Exp[2*\[Lambda]*x]+c*Exp[\[Mu]*x])
```

$$y(x) \rightarrow (-1)^{-\frac{\lambda}{\mu}} 2^{\frac{\lambda+\mu}{2\mu}} \left((e^x)^\lambda\right)^{\frac{\lambda-1}{2\lambda}} (e^x)^{\frac{1}{2}-\frac{\mu}{2}} e^{-\frac{a(e^x)^\lambda}{\lambda}} ((e^x)^\mu)^{\frac{\lambda+\mu}{2\mu}} \left(-\frac{c(e^x)^\mu}{\mu^2}\right)^{-\frac{\lambda}{2\mu}} \left(c_1(-1)^{\lambda/\mu} \operatorname{BesselI}\left(\frac{\lambda}{\mu}, 2\sqrt{-\frac{c\,(e^x)}{\mu^2}}\right.\right.$$
$$\left.\left. + c_2 K_{\frac{\lambda}{\mu}}\left(2\sqrt{-\frac{c\,(e^x)^\mu}{\mu^2}}\right)\right)$$

34.24 problem 24

Internal problem ID [11112]
Internal file name [`OUTPUT/10368_Wednesday_January_24_2024_10_18_18_PM_3875151/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \left(2a\,e^{\lambda x} + b\right)y' + \left(a^2 e^{2\lambda x} + a(b+\lambda)\,e^{\lambda x} + c\right)y = 0}$$

34.24.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= 2a\,e^{\lambda x} + b \\ q(x) &= ab\,e^{\lambda x} + a\lambda\,e^{\lambda x} + a^2 e^{2\lambda x} + c \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= ab\,\mathrm{e}^{\lambda x} + a\lambda\,\mathrm{e}^{\lambda x} + a^2\mathrm{e}^{2\lambda x} + c - \frac{\left(2a\,\mathrm{e}^{\lambda x} + b\right)'}{2} - \frac{\left(2a\,\mathrm{e}^{\lambda x} + b\right)^2}{4} \\
&= ab\,\mathrm{e}^{\lambda x} + a\lambda\,\mathrm{e}^{\lambda x} + a^2\mathrm{e}^{2\lambda x} + c - \frac{\left(2a\lambda\,\mathrm{e}^{\lambda x}\right)}{2} - \frac{\left(\left(2a\,\mathrm{e}^{\lambda x} + b\right)^2\right)}{4} \\
&= ab\,\mathrm{e}^{\lambda x} + a\lambda\,\mathrm{e}^{\lambda x} + a^2\mathrm{e}^{2\lambda x} + c - \left(a\lambda\,\mathrm{e}^{\lambda x}\right) - \frac{\left(2a\,\mathrm{e}^{\lambda x} + b\right)^2}{4} \\
&= c - \frac{b^2}{4}
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\,z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{2a\,\mathrm{e}^{\lambda x}+b}{2}} \\
&= \mathrm{e}^{-\frac{\lambda b x + 2a\,\mathrm{e}^{\lambda x}}{2\lambda}}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\,\mathrm{e}^{-\frac{\lambda b x + 2a\,\mathrm{e}^{\lambda x}}{2\lambda}} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
\mathrm{e}^{-\frac{\lambda b x + 2a\,\mathrm{e}^{\lambda x}}{2\lambda}}\left(-b^2 v(x) + 4cv(x) + 4v''(x)\right) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 4, B = 0, C = -b^2 + 4c$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
4\lambda^2\mathrm{e}^{\lambda x} + \left(-b^2 + 4c\right)\mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$-b^2 + 4\lambda^2 + 4c = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 0, C = -b^2 + 4c$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{0^2 - (4)(4)(-b^2 + 4c)}\\ &= \pm\frac{\sqrt{b^2 - 4c}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\frac{\sqrt{b^2 - 4c}}{2}\\ \lambda_2 &= -\frac{\sqrt{b^2 - 4c}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{\sqrt{b^2 - 4c}}{2}\\ \lambda_2 &= -\frac{\sqrt{b^2 - 4c}}{2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ v(x) &= c_1 e^{\left(\frac{\sqrt{b^2-4c}}{2}\right)x} + c_2 e^{\left(-\frac{\sqrt{b^2-4c}}{2}\right)x}\end{aligned}$$

Or

$$v(x) = c_1 \mathrm{e}^{\frac{\sqrt{b^2-4c}\,x}{2}} + c_2 \mathrm{e}^{-\frac{\sqrt{b^2-4c}\,x}{2}}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\,z(x)\\ &= \left(c_1 \mathrm{e}^{\frac{\sqrt{b^2-4c}\,x}{2}} + c_2 \mathrm{e}^{-\frac{\sqrt{b^2-4c}\,x}{2}}\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \mathrm{e}^{-\frac{\lambda b x + 2a\,\mathrm{e}^{\lambda x}}{2\lambda}}$$

Hence (7) becomes

$$y = \left(c_1 \mathrm{e}^{\frac{\sqrt{b^2-4c}\,x}{2}} + c_2 \mathrm{e}^{-\frac{\sqrt{b^2-4c}\,x}{2}}\right) \mathrm{e}^{-\frac{\lambda b x + 2a\,\mathrm{e}^{\lambda x}}{2\lambda}}$$

Summary

The solution(s) found are the following

$$y = \left(c_1 \mathrm{e}^{\frac{\sqrt{b^2-4c}\,x}{2}} + c_2 \mathrm{e}^{-\frac{\sqrt{b^2-4c}\,x}{2}}\right) \mathrm{e}^{-\frac{\lambda b x + 2a\,\mathrm{e}^{\lambda x}}{2\lambda}} \tag{1}$$

Verification of solutions

$$y = \left(c_1 \mathrm{e}^{\frac{\sqrt{b^2-4c}\,x}{2}} + c_2 \mathrm{e}^{-\frac{\sqrt{b^2-4c}\,x}{2}}\right) \mathrm{e}^{-\frac{\lambda b x + 2a\,\mathrm{e}^{\lambda x}}{2\lambda}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
dsolve(diff(y(x),x$2)+(2*a*exp(lambda*x)+b)*diff(y(x),x)+(a^2*exp(2*lambda*x)+a*(b+lambda)*e
```

$$y(x) = \mathrm{e}^{-\frac{b\lambda x + 2\,\mathrm{e}^{x\lambda} a}{2\lambda}} \left(c_1 \sinh\left(\frac{\sqrt{b^2-4c}\,x}{2}\right) + c_2 \cosh\left(\frac{\sqrt{b^2-4c}\,x}{2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.589 (sec). Leaf size: 82

```
DSolve[y''[x]+(2*a*Exp[\[Lambda]*x]+b)*y'[x]+(a^2*Exp[2*\[Lambda]*x]+a*(b+\[Lambda])*Exp[\[L
```

$$y(x) \to \frac{\left(c_2 e^{x\sqrt{b^2-4c}} + c_1\sqrt{b^2-4c}\right) e^{-\frac{ae^{\lambda x}}{\lambda}-\frac{1}{2}x\left(\sqrt{b^2-4c}+b\right)}}{\sqrt{b^2-4c}}$$

34.25 problem 25

Internal problem ID [11113]
Internal file name [`OUTPUT/10369_Wednesday_January_24_2024_10_18_18_PM_66780811/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + \left(a\,\mathrm{e}^{\lambda x} + 2b - \lambda\right) y' + \left(\mathrm{e}^{2\lambda x} c + ab\,\mathrm{e}^{\lambda x} + b^2 - b\lambda\right) y = 0}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      (a*b*t+c*t^2+b^2-b*lambda)*u(t)+(a*lambda*t^2+2*b*lambda*t)*diff(u(t),t)+lambda^2*t^2*
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.484 (sec). Leaf size: 74

```
dsolve(diff(y(x),x$2)+(a*exp(lambda*x)+2*b-lambda)*diff(y(x),x)+(c*exp(2*lambda*x)+a*b*exp(l
```

$$y(x) = c_1 \mathrm{e}^{\frac{-2b\lambda x + \mathrm{e}^{x\lambda}\sqrt{a^2-4c} - \mathrm{e}^{x\lambda}a}{2\lambda}} + c_2 \mathrm{e}^{-\frac{2b\lambda x + \mathrm{e}^{x\lambda}\sqrt{a^2-4c} + \mathrm{e}^{x\lambda}a}{2\lambda}}$$

✓ Solution by Mathematica

Time used: 2.119 (sec). Leaf size: 97

```
DSolve[y''[x]+(a*Exp[\[Lambda]*x]+2*b-\[Lambda])*y'[x]+(c*Exp[2*\[Lambda]*x]+a*b*Exp[\[Lambd
```

$$y(x) \to \frac{\left(e^{\lambda x}\right)^{-\frac{b}{\lambda}} e^{-\frac{\left(\sqrt{a^2-4c}+a\right)e^{\lambda x}}{2\lambda}} \left(c_2 \lambda e^{\frac{\sqrt{a^2-4c}e^{\lambda x}}{\lambda}} + c_1\sqrt{a^2-4c}\right)}{\sqrt{a^2-4c}}$$

34.26 problem 26

Internal problem ID [11114]
Internal file name [`OUTPUT/10370_Wednesday_January_24_2024_10_18_19_PM_28952538/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$y'' + (a\,e^{x} + b)\,y' + \left(c(a-c)\,e^{2x} + (ak + bc - 2ck + c)\,e^{x} + k(b-k)\right)y = 0$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
   Change of variables used:
      [x = ln(t)]
   Linear ODE actually solved:
      (a*c*t^2-c^2*t^2+a*k*t+b*c*t-2*c*k*t+b*k+c*t-k^2)*u(t)+(a*t^2+b*t+t)*diff(u(t),t)+t^2*
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.391 (sec). Leaf size: 114

```
dsolve(diff(y(x),x$2)+(a*exp(x)+b)*diff(y(x),x)+( c*(a-c)*exp(2*x)+ (a*k+b*c+c-2*c*k)*exp(x)
```

$$y(x) = -\operatorname{WhittakerM}\left(-\frac{b}{2}+k, -\frac{b}{2}+k+\frac{1}{2}, (-2c+a)\,\mathrm{e}^{x}\right)\mathrm{e}^{-\frac{a\,\mathrm{e}^{x}}{2}-\frac{bx}{2}}(-2c+a)^{-b+2k}\,c_2 + \left((-2c+a)\,\mathrm{e}^{x}\right)^{-\frac{b}{2}+k} c_2(-2c+a)^{-b+2k}\,(-1+b-2k)\,\mathrm{e}^{(-a+c)\mathrm{e}^{x}-\frac{bx}{2}} + c_1\mathrm{e}^{-kx-\mathrm{e}^{x}c}$$

✓ Solution by Mathematica

Time used: 3.806 (sec). Leaf size: 71

```
DSolve[y''[x]+(a*Exp[x]+b)*y'[x]+( c*(a-c)*Exp[2*x]+ (a*k+b*c+c-2*c*k)*Exp[x] + k*(b-k) )*y[
```

$$y(x) \to e^{-ce^x}(e^x)^{-k}\left(c_1 - c_2(e^x)^{2k-b}\left(e^x(a-2c)\right)^{b-2k}\Gamma(2k-b,(a-2c)e^x)\right)$$

34.27 problem 27

Internal problem ID [11115]
Internal file name [`OUTPUT/10371_Wednesday_January_24_2024_10_18_19_PM_37319352/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + \left(a\,e^{\lambda x} + b\right) y' + \left(\alpha\,e^{2\lambda x} + \beta\,e^{\lambda x} + \gamma\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
   Change of variables used:
      [x = ln(t)/lambda]
   Linear ODE actually solved:
      (alpha*t^2+beta*t+gamma)*u(t)+(a*lambda*t^2+b*lambda*t+lambda^2*t)*diff(u(t),t)+lambda
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.422 (sec). Leaf size: 141

```
dsolve(diff(y(x),x$2)+(a*exp(lambda*x)+b)*diff(y(x),x)+( alpha*exp(2*lambda*x)+ beta*exp(lam
```

$$y(x) = e^{\frac{-e^{x\lambda}a - x\lambda(b+\lambda)}{2\lambda}} \left(\text{WhittakerM}\left(-\frac{a(b+\lambda)-2\beta}{2\sqrt{a^2-4\alpha}\,\lambda}, \frac{\sqrt{b^2-4\gamma}}{2\lambda}, \frac{\sqrt{a^2-4\alpha}\,e^{x\lambda}}{\lambda}\right)c_1 + \text{WhittakerW}\left(-\frac{a(b+\lambda)-2\beta}{2\sqrt{a^2-4\alpha}\,\lambda}, \frac{\sqrt{b^2-4\gamma}}{2\lambda}, \frac{\sqrt{a^2-4\alpha}\,e^{x\lambda}}{\lambda}\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 2.375 (sec). Leaf size: 248

```
DSolve[y''[x]+(a*Exp[\[Lambda]*x]+b)*y'[x]+( alpha*Exp[2*\[Lambda]*x]+ \[Beta]*Exp[\[Lambda]
```

$$y(x) \to e^{-\frac{\left(\sqrt{a^2-4\alpha}+a\right)e^{\lambda x}}{2\lambda}} \left(e^{\lambda x}\right)^{\frac{\sqrt{b^2-4\gamma}-b}{2\lambda}} \left(c_1 \text{HypergeometricU}\left(\frac{-2\beta + a(b+\lambda) + \sqrt{a^2-4\alpha}\left(\lambda + \sqrt{b^2-4\gamma}\right)}{2\sqrt{a^2-4\alpha}\lambda}, \lambda +\right.\right.$$

$$\left.\left. + c_2 L_{\frac{2\beta - a(b+\lambda) - \sqrt{a^2-4\alpha}\left(\lambda + \sqrt{b^2-4\gamma}\right)}{2\sqrt{a^2-4\alpha}\lambda}}^{\frac{\sqrt{b^2-4\gamma}}{\lambda}}\left(\frac{\sqrt{a^2-4\alpha}e^{x\lambda}}{\lambda}\right)\right)\right)$$

34.28 problem 28

Internal problem ID [11116]
Internal file name [`OUTPUT/10372_Wednesday_January_24_2024_10_18_19_PM_40025974/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + \left(2a\,\mathrm{e}^{\lambda x} - \lambda\right) y' + \left(a^2 \mathrm{e}^{2\lambda x} + \mathrm{e}^{2\mu x} b + \mathrm{e}^{\mu x} c + k\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+(2*a*exp(lambda*x)-lambda)*diff(y(x),x)+( a^2*exp(2*lambda*x) + b*exp(
```

No solution found

✓ Solution by Mathematica

Time used: 2.29 (sec). Leaf size: 290

```
DSolve[y''[x]+(2*a*Exp[\[Lambda]*x]-\[Lambda])*y'[x]+( a^2*Exp[2*\[Lambda]*x] + b*Exp[2*\[Mu
```

$y(x)$

$$\to \left((e^x)^\lambda\right)^{\frac{\lambda-1}{2\lambda}} (e^x)^{\frac{1}{2}-\frac{\mu}{2}} 2^{\frac{\sqrt{\mu^2(\lambda^2-4k)}+\mu^2}{2\mu^2}} ((e^x)^\mu)^{\frac{\sqrt{\mu^2(\lambda^2-4k)}+\mu^2}{2\mu^2}} e^{-\frac{a(e^x)^\lambda}{\lambda}+\frac{i\sqrt{b}(e^x)^\mu}{\mu}} \left(c_1 \text{HypergeometricU}\left(\frac{\mu^2-\frac{ic\mu}{\sqrt{b}}}{\right.\right.$$

$$\left.-\frac{2i\sqrt{b}(e^x)^\mu}{\mu}\right) + c_2 L_{\frac{ic}{2\sqrt{b}\mu}-\frac{\mu^2+\sqrt{(\lambda^2-4k)\mu^2}}{2\mu^2}}^{\frac{\sqrt{(\lambda^2-4k)\mu^2}}{\mu^2}}\left(-\frac{2i\sqrt{b}(e^x)^\mu}{\mu}\right)\Bigg)$$

34.29 problem 29

Internal problem ID [11117]
Internal file name [OUTPUT/10373_Wednesday_January_24_2024_10_18_20_PM_10546755/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + \left(2a\,\mathrm{e}^{\lambda x} + b - \lambda\right) y' + \left(a^2 \mathrm{e}^{2\lambda x} + ab\,\mathrm{e}^{\lambda x} + \mathrm{e}^{2\mu x} c + d\,\mathrm{e}^{\mu x} + k\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+(2*a*exp(lambda*x)+b-lambda)*diff(y(x),x)+( a^2*exp(2*lambda*x) + a*b*
```

No solution found

✓ Solution by Mathematica

Time used: 2.625 (sec). Leaf size: 332

```
DSolve[y''[x]+(2*a*Exp[\[Lambda]*x]+b-\[Lambda])*y'[x]+( a^2*Exp[2*\[Lambda]*x] + a*b*Exp[\[
```

$y(x)$

$$\to (e^x)^{\frac{1}{2}-\frac{\mu}{2}}\left((e^x)^\lambda\right)^{-\frac{b-\lambda+1}{2\lambda}}2^{\frac{1}{2}\left(\frac{\sqrt{\mu^2(b^2-2b\lambda+\lambda^2-4k)}}{\mu^2}+1\right)}e^{-\frac{a(e^x)^\lambda}{\lambda}+\frac{i\sqrt{c}(e^x)^\mu}{\mu}}((e^x)^\mu)^{\frac{1}{2}\left(\frac{\sqrt{\mu^2(b^2-2b\lambda+\lambda^2-4k)}}{\mu^2}+1\right)}\left(c_1 \text{Hyper}\right.$$

$$\left.\left.-\frac{2i\sqrt{c}(e^x)^\mu}{\mu}\right)+c_2 L^{\frac{\sqrt{(b^2-2\lambda b+\lambda^2-4k)\mu^2}}{\mu^2}}_{-\frac{\mu^2-\frac{id\mu}{\sqrt{c}}+\sqrt{(b^2-2\lambda b+\lambda^2-4k)\mu^2}}{2\mu^2}}\left(-\frac{2i\sqrt{c}(e^x)^\mu}{\mu}\right)\right)$$

34.30 problem 30

Internal problem ID [11118]
Internal file name [OUTPUT/10374_Wednesday_January_24_2024_10_18_20_PM_27959611/index.tex]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + \left(a\,\mathrm{e}^{\lambda x} + \mathrm{e}^{\mu x} b\right) y' + a\,\mathrm{e}^{\lambda x} (\mathrm{e}^{\mu x} b + \lambda)\, y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+(a*exp(lambda*x)+b*exp(mu*x))*diff(y(x),x)+a*exp(lambda*x)*(b*exp(mu*x
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+(a*Exp[\[Lambda]*x]+b*Exp[\[Mu]*x])*y'[x]+a*Exp[\[Lambda]*x]*(b*Exp[\[Mu]*x]+\
```

Not solved

34.31 problem 31

Internal problem ID [11119]
Internal file name [`OUTPUT/10375_Wednesday_January_24_2024_10_18_20_PM_62185204/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + \mathrm{e}^{\lambda x}\left(a\,\mathrm{e}^{2\mu x} + b\right) y' + \mu\left(\mathrm{e}^{\lambda x}\left(b - a\,\mathrm{e}^{2\mu x}\right) - \mu\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 81

```
dsolve(diff(y(x),x$2)+exp(lambda*x)*(a*exp(2*mu*x)+b)*diff(y(x),x)+mu*(exp(lambda*x)*(b-a*ex
```

$$y(x) = \left(\left(\int \frac{e^{\frac{-e^{x(\lambda+2\mu)} a\lambda-2\left(\frac{\lambda}{2}+\mu\right)\left(-2\lambda\mu x+b\,e^{x\lambda}\right)}{\lambda(\lambda+2\mu)}}}{\left(e^{2x\mu}a+b\right)^2}dx\right)c_2+c_1\right)\left(a\,e^{x\mu}+e^{-x\mu}b\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+Exp[\[Lambda]*x]*(a*Exp[2*\[Mu]*x]+b)*y'[x]+\[Mu]*(Exp[\[Lambda]*x]*(b-a*Exp[2
```

Not solved

34.32 problem 32

Internal problem ID [11120]
Internal file name [`OUTPUT/10376_Wednesday_January_31_2024_08_14_02_PM_16279652/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + \left(a\,\mathrm{e}^{\lambda x} + \mathrm{e}^{\mu x} b + c\right) y' + \left(a\lambda\,\mathrm{e}^{\lambda x} + \mu\,\mathrm{e}^{\mu x} b\right) y = 0}$$

34.32.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(y'' + \left(a\,\mathrm{e}^{\lambda x} + \mathrm{e}^{\mu x} b + c\right) y' + \left(a\lambda\,\mathrm{e}^{\lambda x} + \mu\,\mathrm{e}^{\mu x} b\right) y\right) dx = 0$$
$$\left(a\,\mathrm{e}^{\lambda x} + \mathrm{e}^{\mu x} b + c\right) y + y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= a\,\mathrm{e}^{\lambda x}+\mathrm{e}^{\mu x} b+c \\
q(x) &= c_1
\end{aligned}
$$

Hence the ode is

$$
\left(a\,\mathrm{e}^{\lambda x}+\mathrm{e}^{\mu x} b+c\right) y+y' = c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (a\,\mathrm{e}^{\lambda x}+\mathrm{e}^{\mu x} b+c)dx} \\
&= \mathrm{e}^{c x+\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}+\frac{b\,\mathrm{e}^{\mu x}}{\mu}}
\end{aligned}
$$

Which simplifies to

$$
\mu = \mathrm{e}^{\frac{c x \lambda \mu+a\,\mathrm{e}^{\lambda x} \mu+\mathrm{e}^{\mu x} b \lambda}{\lambda \mu}}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(c_1) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{c x \lambda \mu+a\,\mathrm{e}^{\lambda x} \mu+\mathrm{e}^{\mu x} b \lambda}{\lambda \mu}} y\right) &= \left(\mathrm{e}^{\frac{c x \lambda \mu+a\,\mathrm{e}^{\lambda x} \mu+\mathrm{e}^{\mu x} b \lambda}{\lambda \mu}}\right)(c_1) \\
\mathrm{d}\left(\mathrm{e}^{\frac{c x \lambda \mu+a\,\mathrm{e}^{\lambda x} \mu+\mathrm{e}^{\mu x} b \lambda}{\lambda \mu}} y\right) &= \left(c_1 \mathrm{e}^{\frac{c x \lambda \mu+a\,\mathrm{e}^{\lambda x} \mu+\mathrm{e}^{\mu x} b \lambda}{\lambda \mu}}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\frac{c x \lambda \mu+a\,\mathrm{e}^{\lambda x} \mu+\mathrm{e}^{\mu x} b \lambda}{\lambda \mu}} y &= \int c_1 \mathrm{e}^{\frac{c x \lambda \mu+a\,\mathrm{e}^{\lambda x} \mu+\mathrm{e}^{\mu x} b \lambda}{\lambda \mu}}\,\mathrm{d}x \\
\mathrm{e}^{\frac{c x \lambda \mu+a\,\mathrm{e}^{\lambda x} \mu+\mathrm{e}^{\mu x} b \lambda}{\lambda \mu}} y &= \int c_1 \mathrm{e}^{\frac{c x \lambda \mu+a\,\mathrm{e}^{\lambda x} \mu+\mathrm{e}^{\mu x} b \lambda}{\lambda \mu}} dx + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{c x \lambda \mu+a\,\mathrm{e}^{\lambda x} \mu+\mathrm{e}^{\mu x} b \lambda}{\lambda \mu}}$ results in

$$
y = \mathrm{e}^{\frac{-a\,\mathrm{e}^{\lambda x} \mu-\lambda (c x \mu+\mathrm{e}^{\mu x} b)}{\mu \lambda}} \left(\int c_1 \mathrm{e}^{\frac{c x \lambda \mu+a\,\mathrm{e}^{\lambda x} \mu+\mathrm{e}^{\mu x} b \lambda}{\lambda \mu}} dx\right) + c_2 \mathrm{e}^{\frac{-a\,\mathrm{e}^{\lambda x} \mu-\lambda (c x \mu+\mathrm{e}^{\mu x} b)}{\mu \lambda}}
$$

which simplifies to

$$
y = \mathrm{e}^{\frac{-a\,\mathrm{e}^{\lambda x} \mu-\lambda (c x \mu+\mathrm{e}^{\mu x} b)}{\mu \lambda}} \left(c_1 \left(\int \mathrm{e}^{\frac{c x \lambda \mu+a\,\mathrm{e}^{\lambda x} \mu+\mathrm{e}^{\mu x} b \lambda}{\lambda \mu}} dx\right) + c_2\right)
$$

Summary
The solution(s) found are the following

$$y = e^{\frac{-a\,e^{\lambda x}\mu - \lambda\left(cx\mu + e^{\mu x}b\right)}{\mu\lambda}}\left(c_1\left(\int e^{\frac{cx\lambda\mu + a\,e^{\lambda x}\mu + e^{\mu x}b\lambda}{\lambda\mu}}dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = e^{\frac{-a\,e^{\lambda x}\mu - \lambda\left(cx\mu + e^{\mu x}b\right)}{\mu\lambda}}\left(c_1\left(\int e^{\frac{cx\lambda\mu + a\,e^{\lambda x}\mu + e^{\mu x}b\lambda}{\lambda\mu}}dx\right) + c_2\right)$$

Verified OK.

34.32.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + \left(a\,e^{\lambda x} + e^{\mu x}b + c\right)y' + \left(a\lambda\,e^{\lambda x} + \mu\,e^{\mu x}b\right)y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int \left(y'' + \left(a\,e^{\lambda x} + e^{\mu x}b + c\right)y' + \left(a\lambda\,e^{\lambda x} + \mu\,e^{\mu x}b\right)y\right)dx &= 0\\ \left(a\,e^{\lambda x} + e^{\mu x}b + c\right)y + y' &= c_1\end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= a\,e^{\lambda x} + e^{\mu x}b + c\\ q(x) &= c_1\end{aligned}$$

Hence the ode is

$$\left(a\,e^{\lambda x} + e^{\mu x}b + c\right)y + y' = c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \left(a\,e^{\lambda x} + e^{\mu x}b + c\right)dx}\\ &= e^{cx + \frac{a\,e^{\lambda x}}{\lambda} + \frac{b\,e^{\mu x}}{\mu}}\end{aligned}$$

Which simplifies to

$$\mu = \mathrm{e}^{\frac{cx\lambda\mu + a\,\mathrm{e}^{\lambda x}\mu + \mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(c_1) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{cx\lambda\mu + a\,\mathrm{e}^{\lambda x}\mu + \mathrm{e}^{\mu x} b\lambda}{\lambda\mu}} y\right) &= \left(\mathrm{e}^{\frac{cx\lambda\mu + a\,\mathrm{e}^{\lambda x}\mu + \mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}\right)(c_1) \\
\mathrm{d}\left(\mathrm{e}^{\frac{cx\lambda\mu + a\,\mathrm{e}^{\lambda x}\mu + \mathrm{e}^{\mu x} b\lambda}{\lambda\mu}} y\right) &= \left(c_1 \mathrm{e}^{\frac{cx\lambda\mu + a\,\mathrm{e}^{\lambda x}\mu + \mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}\right)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\mathrm{e}^{\frac{cx\lambda\mu + a\,\mathrm{e}^{\lambda x}\mu + \mathrm{e}^{\mu x} b\lambda}{\lambda\mu}} y &= \int c_1 \mathrm{e}^{\frac{cx\lambda\mu + a\,\mathrm{e}^{\lambda x}\mu + \mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}\,\mathrm{d}x \\
\mathrm{e}^{\frac{cx\lambda\mu + a\,\mathrm{e}^{\lambda x}\mu + \mathrm{e}^{\mu x} b\lambda}{\lambda\mu}} y &= \int c_1 \mathrm{e}^{\frac{cx\lambda\mu + a\,\mathrm{e}^{\lambda x}\mu + \mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}\,dx + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{cx\lambda\mu + a\,\mathrm{e}^{\lambda x}\mu + \mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}$ results in

$$y = \mathrm{e}^{\frac{-a\,\mathrm{e}^{\lambda x}\mu - \lambda(cx\mu + \mathrm{e}^{\mu x} b)}{\mu\lambda}} \left(\int c_1 \mathrm{e}^{\frac{cx\lambda\mu + a\,\mathrm{e}^{\lambda x}\mu + \mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}\,dx\right) + c_2 \mathrm{e}^{\frac{-a\,\mathrm{e}^{\lambda x}\mu - \lambda(cx\mu + \mathrm{e}^{\mu x} b)}{\mu\lambda}}$$

which simplifies to

$$y = \mathrm{e}^{\frac{-a\,\mathrm{e}^{\lambda x}\mu - \lambda(cx\mu + \mathrm{e}^{\mu x} b)}{\mu\lambda}} \left(c_1 \left(\int \mathrm{e}^{\frac{cx\lambda\mu + a\,\mathrm{e}^{\lambda x}\mu + \mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}\,dx\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{-a\,\mathrm{e}^{\lambda x}\mu - \lambda(cx\mu + \mathrm{e}^{\mu x} b)}{\mu\lambda}} \left(c_1 \left(\int \mathrm{e}^{\frac{cx\lambda\mu + a\,\mathrm{e}^{\lambda x}\mu + \mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}\,dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{\frac{-a\,\mathrm{e}^{\lambda x}\mu - \lambda(cx\mu + \mathrm{e}^{\mu x} b)}{\mu\lambda}} \left(c_1 \left(\int \mathrm{e}^{\frac{cx\lambda\mu + a\,\mathrm{e}^{\lambda x}\mu + \mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}\,dx\right) + c_2\right)$$

Verified OK.

34.32.3 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= a\,\mathrm{e}^{\lambda x} + \mathrm{e}^{\mu x} b + c \\ r(x) &= a\lambda\,\mathrm{e}^{\lambda x} + \mu\,\mathrm{e}^{\mu x} b \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= a\lambda\,\mathrm{e}^{\lambda x} + \mu\,\mathrm{e}^{\mu x} b \end{aligned}$$

Therefore (1) becomes

$$0 - \left(a\lambda\,\mathrm{e}^{\lambda x} + \mu\,\mathrm{e}^{\mu x} b\right) + \left(a\lambda\,\mathrm{e}^{\lambda x} + \mu\,\mathrm{e}^{\mu x} b\right) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$\left(a\,\mathrm{e}^{\lambda x} + \mathrm{e}^{\mu x} b + c\right) y + y' = c_1$$

We now have a first order ode to solve which is

$$\left(a\,\mathrm{e}^{\lambda x} + \mathrm{e}^{\mu x} b + c\right) y + y' = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= a\,\mathrm{e}^{\lambda x}+\mathrm{e}^{\mu x} b+c \\
q(x) &= c_1
\end{aligned}
$$

Hence the ode is

$$
\left(a\,\mathrm{e}^{\lambda x}+\mathrm{e}^{\mu x} b+c\right) y+y' = c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (a\,\mathrm{e}^{\lambda x}+\mathrm{e}^{\mu x} b+c)dx} \\
&= \mathrm{e}^{cx+\frac{a\,\mathrm{e}^{\lambda x}}{\lambda}+\frac{b\,\mathrm{e}^{\mu x}}{\mu}}
\end{aligned}
$$

Which simplifies to

$$
\mu = \mathrm{e}^{\frac{cx\lambda\mu+a\,\mathrm{e}^{\lambda x}\mu+\mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(c_1) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{cx\lambda\mu+a\,\mathrm{e}^{\lambda x}\mu+\mathrm{e}^{\mu x} b\lambda}{\lambda\mu}} y\right) &= \left(\mathrm{e}^{\frac{cx\lambda\mu+a\,\mathrm{e}^{\lambda x}\mu+\mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}\right)(c_1) \\
\mathrm{d}\left(\mathrm{e}^{\frac{cx\lambda\mu+a\,\mathrm{e}^{\lambda x}\mu+\mathrm{e}^{\mu x} b\lambda}{\lambda\mu}} y\right) &= \left(c_1 \mathrm{e}^{\frac{cx\lambda\mu+a\,\mathrm{e}^{\lambda x}\mu+\mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\frac{cx\lambda\mu+a\,\mathrm{e}^{\lambda x}\mu+\mathrm{e}^{\mu x} b\lambda}{\lambda\mu}} y &= \int c_1 \mathrm{e}^{\frac{cx\lambda\mu+a\,\mathrm{e}^{\lambda x}\mu+\mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}\,\mathrm{d}x \\
\mathrm{e}^{\frac{cx\lambda\mu+a\,\mathrm{e}^{\lambda x}\mu+\mathrm{e}^{\mu x} b\lambda}{\lambda\mu}} y &= \int c_1 \mathrm{e}^{\frac{cx\lambda\mu+a\,\mathrm{e}^{\lambda x}\mu+\mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}\,dx+c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{cx\lambda\mu+a\,\mathrm{e}^{\lambda x}\mu+\mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}$ results in

$$
y = \mathrm{e}^{\frac{-a\,\mathrm{e}^{\lambda x}\mu-\lambda(cx\mu+\mathrm{e}^{\mu x} b)}{\mu\lambda}}\left(\int c_1 \mathrm{e}^{\frac{cx\lambda\mu+a\,\mathrm{e}^{\lambda x}\mu+\mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}\,dx\right)+c_2 \mathrm{e}^{\frac{-a\,\mathrm{e}^{\lambda x}\mu-\lambda(cx\mu+\mathrm{e}^{\mu x} b)}{\mu\lambda}}
$$

which simplifies to

$$
y = \mathrm{e}^{\frac{-a\,\mathrm{e}^{\lambda x}\mu-\lambda(cx\mu+\mathrm{e}^{\mu x} b)}{\mu\lambda}}\left(c_1\left(\int \mathrm{e}^{\frac{cx\lambda\mu+a\,\mathrm{e}^{\lambda x}\mu+\mathrm{e}^{\mu x} b\lambda}{\lambda\mu}}\,dx\right)+c_2\right)
$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{-a\,\mathrm{e}^{\lambda x}\mu-\lambda\left(cx\mu+\mathrm{e}^{\mu x}b\right)}{\mu\lambda}}\left(c_1\left(\int \mathrm{e}^{\frac{cx\lambda\mu+a\,\mathrm{e}^{\lambda x}\mu+\mathrm{e}^{\mu x}b\lambda}{\lambda\mu}}dx\right)+c_2\right) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{\frac{-a\,\mathrm{e}^{\lambda x}\mu-\lambda\left(cx\mu+\mathrm{e}^{\mu x}b\right)}{\mu\lambda}}\left(c_1\left(\int \mathrm{e}^{\frac{cx\lambda\mu+a\,\mathrm{e}^{\lambda x}\mu+\mathrm{e}^{\mu x}b\lambda}{\lambda\mu}}dx\right)+c_2\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
No hypergeometric solution was found.
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 70

```
dsolve(diff(y(x),x$2)+(a*exp(lambda*x)+b*exp(mu*x)+c)*diff(y(x),x)+(a*lambda*exp(lambda*x)+b
```

$$y(x) = \left(c_1\left(\int \mathrm{e}^{\frac{cx\mu\lambda+\mathrm{e}^{x\lambda}a\mu+b\,\mathrm{e}^{x\mu}\lambda}{\mu\lambda}}dx\right)+c_2\right)\mathrm{e}^{\frac{-\mathrm{e}^{x\lambda}a\mu-\lambda\left(cx\mu+b\,\mathrm{e}^{x\mu}\right)}{\mu\lambda}}$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 77

```
DSolve[y''[x]+(a*Exp[\[Lambda]*x]+b*Exp[\[Mu]*x]+c)*y'[x]+(a*\[Lambda]*Exp[\[Lambda]*x]+b*\[
```

$$y(x) \to e^{-\frac{ae^{\lambda x}}{\lambda}-\frac{be^{\mu x}}{\mu}-cx}\left(\int_1^x e^{\frac{e^{\lambda K[1]}a}{\lambda}+cK[1]+\frac{be^{\mu K[1]}}{\mu}}c_1dK[1]+c_2\right)$$

34.33 problem 33

Internal problem ID [11121]
Internal file name [`OUTPUT/10377_Wednesday_January_31_2024_08_14_04_PM_76338301/index.tex`]

Book: Handbook of exact solutions for ordinary differential equations. By Polyanin and Zaitsev. Second edition
Section: Chapter 2, Second-Order Differential Equations. section 2.1.3-1. Equations with exponential functions
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + \left(a\,\mathrm{e}^{\lambda x} + \mathrm{e}^{\mu x} b + c\right) y' + \left(ab\,\mathrm{e}^{x(\lambda+\mu)} + \mathrm{e}^{\lambda x} ac + \mu\,\mathrm{e}^{\mu x} b\right) y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+(a*exp(lambda*x)+b*exp(mu*x)+c)*diff(y(x),x)+(a*b*exp((lambda+mu)*x)+a
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+(a*Exp[\[Lambda]*x]+b*Exp[\[Mu]*x]+c)*y'[x]+(a*b*Exp[(\[Lambda]+\[Mu])*x]+a*c*
```

Not solved