A Solution Manual For

Collection of Eigenvalues and Eigenvectors problems

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Contents

1 From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346

1.1 problem problem 1

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Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 1.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det \begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 5\lambda + 6 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 3$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 1 & -1 & 0 \end{array} \right]
$$

$$
R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$
\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
2	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
3	1	2	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$$

1.2 problem problem 2

Internal problem ID [10263]
Internal file name [`OUTPUT/9210_Monday_June_06_2022_01_44_32_PM_79379111/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 2.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det \begin{bmatrix} 5-\lambda & -6 \\ 3 & -4-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - \lambda - 2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 6 & -6 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & -6 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{cc|c} 6 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 3 & -6 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -6 & 0 \\ 3 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-1	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2	1	2	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$$

1.3 problem problem 3

Internal problem ID [10264]
Internal file name [`OUTPUT/9211_Monday_June_06_2022_01_44_32_PM_35359149/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 3.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 8-\lambda & -6 \\ 3 & -1-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 7\lambda + 10 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 5$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
5	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 6 & -6 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & -6 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 6 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 5$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 3 & -6 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -6 & 0 \\ 3 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
2	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
5	1	2	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$$

1.4 problem problem 4

Internal problem ID [10265]
Internal file name [`OUTPUT/9212_Monday_June_06_2022_01_44_32_PM_7112981/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 4.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2	1	2	No	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}^{-1}$$

1.5 problem problem 5

Internal problem ID [10266]
Internal file name [`OUTPUT/9213_Monday_June_06_2022_01_44_33_PM_81964955/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 5.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 10 & -9 \\ 6 & -5 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 10 & -9 \\ 6 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 10-\lambda & -9 \\ 6 & -5-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 5\lambda + 4 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 4$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
4	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} 10 & -9 \\ 6 & -5 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 10 & -9 \\ 6 & -5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 9 & -9 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 9 & -9 & 0 \\ 6 & -6 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{2R_1}{3} \implies \left[\begin{array}{cc|c} 9 & -9 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 9 & -9 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$
\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 4$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 10 & -9 \\ 6 & -5 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 10 & -9 \\ 6 & -5 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 6 & -9 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & -9 & 0 \\ 6 & -9 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 6 & -9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -9 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
4	1	2	No	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 10 & -9 \\ 6 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}^{-1}$$

1.6 problem problem 6

Internal problem ID [10267]
Internal file name [`OUTPUT/9214_Monday_June_06_2022_01_44_34_PM_15847638/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 6.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 6-\lambda & -4 \\ 3 & -1-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 5\lambda + 6 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 3$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 4 & -4 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 4 & -4 & 0 \\ 3 & -3 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{3R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} 4 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$
\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 3 & -4 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -4 & 0 \\ 3 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{4t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
2	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
3	1	2	No	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}^{-1}$$

1.7 problem problem 7

Internal problem ID [10268]
Internal file name [`OUTPUT/9215_Monday_June_06_2022_01_44_34_PM_99424410/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 7.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 10-\lambda & -8 \\ 6 & -4-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 6\lambda + 8 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 4$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
4	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 8 & -8 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 8 & -8 & 0 \\ 6 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{4} \implies \left[\begin{array}{cc|c} 8 & -8 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 8 & -8 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 4$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 6 & -8 \\ 6 & -8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & -8 & 0 \\ 6 & -8 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 6 & -8 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{4t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
2	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
4	1	2	No	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}^{-1}$$

1.8 problem problem 8

Internal problem ID [10269]
Internal file name [`OUTPUT/9216_Monday_June_06_2022_01_44_35_PM_9147535/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 8.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 7 & -6 \\ 12 & -10 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 7 & -6 \\ 12 & -10 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 7-\lambda & -6 \\ 12 & -10-\lambda \end{bmatrix} = 0$$
$$\lambda^2 + 3\lambda + 2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = -1$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 7 & -6 \\ 12 & -10 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 7 & -6 \\ 12 & -10 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 8 & -6 \\ 12 & -9 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 8 & -6 & 0 \\ 12 & -9 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 8 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 8 & -6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \frac{3t}{4}\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{4} \\ t \end{bmatrix} = t\begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{3t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{3t}{4} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Considering $\lambda = -2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 7 & -6 \\ 12 & -10 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 7 & -6 \\ 12 & -10 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 9 & -6 \\ 12 & -8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 9 & -6 & 0 \\ 12 & -8 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{4R_1}{3} \implies \left[\begin{array}{cc|c} 9 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 9 & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-1	1	2	No	$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$
-2	1	2	No	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 7 & -6 \\ 12 & -10 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}^{-1}$$

1.9 problem problem 9

Internal problem ID [10270]
Internal file name [`OUTPUT/9217_Monday_June_06_2022_01_44_36_PM_96430611/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 9.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 8 & -10 \\ 2 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 8 & -10 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 8-\lambda & -10 \\ 2 & -1-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 7\lambda + 12 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 4$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue
4	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 8 & -10 \\ 2 & -1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 8 & -10 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 5 & -10 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 5 & -10 & 0 \\ 2 & -4 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{cc|c} 5 & -10 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 5 & -10 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$
\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering $\lambda = 4$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 8 & -10 \\ 2 & -1 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 8 & -10 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 4 & -10 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -10 & 0 \\ 2 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{cc|c} 4 & -10 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -10 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{5t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
3	1	2	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
4	1	2	No	$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 8 & -10 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}^{-1}$$

1.10 problem problem 10

Internal problem ID [10271]
Internal file name [`OUTPUT/9218_Monday_June_06_2022_01_44_36_PM_67236026/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 10.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 9 & -10 \\ 2 & 0 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 9 & -10 \\ 2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 9-\lambda & -10 \\ 2 & -\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 9\lambda + 20 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 5$$
$$\lambda_2 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
4	1	real eigenvalue
5	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 4$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 9 & -10 \\ 2 & 0 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 9 & -10 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 5 & -10 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 5 & -10 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{cc|c} 5 & -10 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & -10 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering $\lambda = 5$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 9 & -10 \\ 2 & 0 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 9 & -10 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 4 & -10 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -10 & 0 \\ 2 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 4 & -10 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -10 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{5t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
4	1	2	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
5	1	2	No	$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 9 & -10 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}^{-1}$$

1.11 problem problem 11

Internal problem ID [10272]
Internal file name [`OUTPUT/9219_Monday_June_06_2022_01_44_36_PM_98956009/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 11.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 19 & -10 \\ 21 & -10 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 19 & -10 \\ 21 & -10 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 19 - \lambda & -10 \\ 21 & -10 - \lambda \end{bmatrix} = 0$$
$$\lambda^2 - 9\lambda + 20 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 5$$
$$\lambda_2 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
4	1	real eigenvalue
5	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 4$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 19 & -10 \\ 21 & -10 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 19 & -10 \\ 21 & -10 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 15 & -10 \\ 21 & -14 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 15 & -10 & 0 \\ 21 & -14 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{7R_1}{5} \implies \left[\begin{array}{cc|c} 15 & -10 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 15 & -10 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \frac{2t}{3}\}$

Hence the solution is

$$
\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = t\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Considering $\lambda = 5$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 19 & -10 \\ 21 & -10 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 19 & -10 \\ 21 & -10 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 14 & -10 \\ 21 & -15 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 14 & -10 & 0 \\ 21 & -15 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 14 & -10 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 14 & -10 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{5t}{7}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{5t}{7} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5t}{7} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{5t}{7} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{7} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{5t}{7} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{7} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{5t}{7} \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
4	1	2	No	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
5	1	2	No	$\begin{bmatrix} 5 \\ 7 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 19 & -10 \\ 21 & -10 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}^{-1}$$

1.12 problem problem 12

Internal problem ID [10273]
Internal file name [OUTPUT/9220_Monday_June_06_2022_01_44_37_PM_29226975/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 12.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 13 & -15 \\ 6 & -6 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 13 & -15 \\ 6 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 13 - \lambda & -15 \\ 6 & -6 - \lambda \end{bmatrix} = 0$$
$$\lambda^2 - 7\lambda + 12 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 4$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue
4	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 13 & -15 \\ 6 & -6 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 13 & -15 \\ 6 & -6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 10 & -15 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 10 & -15 & 0 \\ 6 & -9 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{3R_1}{5} \Longrightarrow \left[\begin{array}{cc|c} 10 & -15 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 10 & -15 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \frac{3t}{2}\}$

Hence the solution is

$$
\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Considering $\lambda = 4$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 13 & -15 \\ 6 & -6 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 13 & -15 \\ 6 & -6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 9 & -15 \\ 6 & -10 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 9 & -15 & 0 \\ 6 & -10 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 9 & -15 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 9 & -15 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{5t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
3	1	2	No	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$
4	1	2	No	$\begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 13 & -15 \\ 6 & -6 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}^{-1}$$

1.13 problem problem 13

Internal problem ID [10274]
Internal file name [`OUTPUT/9221_Monday_June_06_2022_01_44_38_PM_65388942/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 13.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 2-\lambda & 0 & 0 \\ 2 & -2-\lambda & -1 \\ -2 & 6 & 3-\lambda \end{bmatrix} = 0$$

$$-(-2+\lambda)\,\lambda(\lambda-1) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 0$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 2 & -2 & -1 & 0 \\ -2 & 6 & 3 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ -2 & 6 & 3 & 0 \end{array}\right]
$$

$$R_3 = R_3 + R_1 \implies \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 6 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \implies \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -\frac{t}{2}\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} 0 \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \end{aligned}$$

$$\left(\begin{bmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & -1 \\ -2 & 6 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & -3 & -1 & 0 \\ -2 & 6 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ -2 & 6 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_1 \implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 6 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = 0, v_2 = -\frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & -1 \\ -2 & 6 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & -4 & -1 & 0 \\ -2 & 6 & 1 & 0 \end{array}\right]
$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$
\left[\begin{array}{ccc|c} 2 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 6 & 1 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]
$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a

row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c}2 & -4 & -1 & 0\\0 & 2 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}2 & -4 & -1\\0 & 2 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = 0\right\}$

Hence the solution is

$$\begin{bmatrix}\frac{t}{2}\\0\\t\end{bmatrix}=\begin{bmatrix}\frac{t}{2}\\0\\t\end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix}\frac{t}{2}\\0\\t\end{bmatrix}=t\begin{bmatrix}\frac{1}{2}\\0\\1\end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix}\frac{t}{2}\\0\\t\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\0\\1\end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix}\frac{t}{2}\\0\\t\end{bmatrix}=\begin{bmatrix}1\\0\\2\end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
0	1	3	No	$\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$
1	1	3	No	$\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ 2 & 3 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ 2 & 3 & 2 \end{bmatrix}^{-1}$$

1.14 problem problem 14

Internal problem ID [10275]
Internal file name [OUTPUT/9222_Monday_June_06_2022_01_44_39_PM_18921205/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 14.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 5 & 0 & 0 \\ 4 & -4 & -2 \\ -2 & 12 & 6 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 5 & 0 & 0 \\ 4 & -4 & -2 \\ -2 & 12 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 5-\lambda & 0 & 0 \\ 4 & -4-\lambda & -2 \\ -2 & 12 & 6-\lambda \end{bmatrix} = 0$$

$$-(-5+\lambda)\,\lambda(\lambda-2) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
2	1	real eigenvalue
5	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 0$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 5 & 0 & 0 \\ 4 & -4 & -2 \\ -2 & 12 & 6 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 5 & 0 & 0 \\ 4 & -4 & -2 \\ -2 & 12 & 6 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 5 & 0 & 0 \\ 4 & -4 & -2 \\ -2 & 12 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 5 & 0 & 0 & 0 \\ 4 & -4 & -2 & 0 \\ -2 & 12 & 6 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{4R_1}{5} \implies \left[\begin{array}{ccc|c} 5 & 0 & 0 & 0 \\ 0 & -4 & -2 & 0 \\ -2 & 12 & 6 & 0 \end{array}\right]
$$

$$R_3 = R_3 + \frac{2R_1}{5} \implies \left[\begin{array}{ccc|c} 5 & 0 & 0 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 12 & 6 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \implies \left[\begin{array}{ccc|c} 5 & 0 & 0 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -\frac{t}{2}\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} 0 \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \end{aligned}$$

$$\left(\begin{bmatrix} 5 & 0 & 0 \\ 4 & -4 & -2 \\ -2 & 12 & 6 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 5 & 0 & 0 \\ 4 & -4 & -2 \\ -2 & 12 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 4 & -6 & -2 \\ -2 & 12 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 4 & -6 & -2 & 0 \\ -2 & 12 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{4R_1}{3} \implies \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & -6 & -2 & 0 \\ -2 & 12 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_1}{3} \implies \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & -6 & -2 & 0 \\ 0 & 12 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & -6 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -6 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = 0, v_2 = -\frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

Considering $\lambda = 5$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$A\boldsymbol{v} = \lambda \boldsymbol{v}$$
$$A\boldsymbol{v} - \lambda \boldsymbol{v} = \boldsymbol{0}$$
$$(A - \lambda I)\boldsymbol{v} = \boldsymbol{0}$$

$$\left(\begin{bmatrix} 5 & 0 & 0 \\ 4 & -4 & -2 \\ -2 & 12 & 6 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 5 & 0 & 0 \\ 4 & -4 & -2 \\ -2 & 12 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 4 & -9 & -2 \\ -2 & 12 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 4 & -9 & -2 & 0 \\ -2 & 12 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 4 & -9 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 12 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & -9 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{15}{2} & 0 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a

row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 4 & -9 & -2 & 0 \\ 0 & \frac{15}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -9 & -2 \\ 0 & \frac{15}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
0	1	3	No	$\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$
5	1	3	No	$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ 2 & 3 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 5 & 0 & 0 \\ 4 & -4 & -2 \\ -2 & 12 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ 2 & 3 & 2 \end{bmatrix}^{-1}$$

1.15 problem problem 15

Internal problem ID [10276]
Internal file name [OUTPUT/9223_Monday_June_06_2022_01_44_40_PM_54044065/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 15.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & -1 \\ -2 & 2 & 3 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & -1 \\ -2 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 2-\lambda & -2 & 0 \\ 2 & -2-\lambda & -1 \\ -2 & 2 & 3-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 3\lambda^2 - 2\lambda = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 0$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & -1 \\ -2 & 2 & 3 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & -1 \\ -2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & -1 \\ -2 & 2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 2 & -2 & -1 & 0 \\ -2 & 2 & 3 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -2 & 2 & 3 & 0 \end{array}\right]
$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & -1 \\ -2 & 2 & 3 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & -1 \\ -2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & -3 & -1 \\ -2 & 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 2 & -3 & -1 & 0 \\ -2 & 2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -2 & 2 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & -1 \\ -2 & 2 & 3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & -1 \\ -2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & 0 \\ 2 & -4 & -1 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -2 & 0 & 0 \\ 2 & -4 & -1 & 0 \\ -2 & 2 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 2 & -4 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ -2 & 2 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -4 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -4 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
0	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
1	1	3	No	$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & -1 \\ -2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}^{-1}$$

1.16 problem problem 16

Internal problem ID [10277]
Internal file name [OUTPUT/9224_Monday_June_06_2022_01_44_41_PM_73663570/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 16.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & 0 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 0 & -1 \\ -2 & 3-\lambda & -1 \\ -6 & 6 & -\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 4\lambda^2 - 3\lambda = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 0$$
$$\lambda_2 = 3$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
1	1	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 0$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & 0 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -2 & 3 & -1 & 0 \\ -6 & 6 & 0 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ -6 & 6 & 0 & 0 \end{array}\right]
$$

$$R_3 = R_3 + 6R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 6 & -6 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & 0 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 & -1 \\ -2 & 2 & -1 \\ -6 & 6 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ -2 & 2 & -1 & 0 \\ -6 & 6 & -1 & 0 \end{array}\right]
$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$
\left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ -6 & 6 & -1 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - 3R_1 \implies \left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + 2R_2 \implies \left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & 0 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -2 & 0 & -1 \\ -2 & 0 & -1 \\ -6 & 6 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 0 & -1 & 0 \\ -2 & 0 & -1 & 0 \\ -6 & 6 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \implies \left[\begin{array}{ccc|c} -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & 6 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_1 \implies \left[\begin{array}{ccc|c} -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -2 & 0 & -1 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}, v_2 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
0	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
1	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
3	1	3	No	$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}^{-1}$$

1.17 problem problem 17

Internal problem ID [10278]
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Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 17.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det \begin{bmatrix} 3-\lambda & 5 & -2 \\ 0 & 2-\lambda & 0 \\ 0 & 2 & 1-\lambda \end{bmatrix} = 0$$

$$-(-3+\lambda)(-2+\lambda)(-1+\lambda) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 5 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 5 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 5 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 5 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$A\boldsymbol{v} = \lambda\boldsymbol{v}$$
$$A\boldsymbol{v} - \lambda\boldsymbol{v} = \boldsymbol{0}$$
$$(A - \lambda I)\boldsymbol{v} = \boldsymbol{0}$$
$$\left(\begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\left(\begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 5 & -2 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 1 & 5 & -2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 5 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \end{aligned}$$

$$\left(\begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 5 & -2 \\ 0 & -1 & 0 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 5 & -2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{5} \implies \left[\begin{array}{ccc|c} 0 & 5 & -2 & 0 \\ 0 & 0 & -\frac{2}{5} & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{2R_1}{5} \implies \left[\begin{array}{ccc|c} 0 & 5 & -2 & 0 \\ 0 & 0 & -\frac{2}{5} & 0 \\ 0 & 0 & -\frac{6}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_2 \implies \left[\begin{array}{ccc|c} 0 & 5 & -2 & 0 \\ 0 & 0 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 5 & -2 \\ 0 & 0 & -\frac{2}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	3	No	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$
3	1	3	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}^{-1}$$

1.18 problem problem 18

Internal problem ID [10279]
Internal file name [`OUTPUT/9226_Monday_June_06_2022_01_44_43_PM_70215540/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 18.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -6 & 8 & 2 \\ 12 & -15 & -3 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ -6 & 8 & 2 \\ 12 & -15 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 0 & 0 \\ -6 & 8-\lambda & 2 \\ 12 & -15 & -3-\lambda \end{bmatrix} = 0$$

$$-(-1+\lambda)\left(\lambda^2 - 5\lambda + 6\right) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ -6 & 8 & 2 \\ 12 & -15 & -3 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ -6 & 8 & 2 \\ 12 & -15 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -6 & 7 & 2 \\ 12 & -15 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -6 & 7 & 2 & 0 \\ 12 & -15 & -4 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} -6 & 7 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & -15 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -6 & 7 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -6 & 7 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6 & 7 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}, v_2 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{t}{3} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{t}{3} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ -6 & 8 & 2 \\ 12 & -15 & -3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ -6 & 8 & 2 \\ 12 & -15 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ -6 & 6 & 2 \\ 12 & -15 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ -6 & 6 & 2 & 0 \\ 12 & -15 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 - 6R_1 \implies \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 6 & 2 & 0 \\ 12 & -15 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 + 12R_1 \implies \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & -15 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{5R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -\frac{t}{3}\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 1 & 0 & 0 \\ -6 & 8 & 2 \\ 12 & -15 & -3 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 1 & 0 & 0 \\ -6 & 8 & 2 \\ 12 & -15 & -3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -2 & 0 & 0 \\ -6 & 5 & 2 \\ 12 & -15 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ -6 & 5 & 2 & 0 \\ 12 & -15 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \implies \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 12 & -15 & -6 & 0 \end{array}\right]$$

$$R_3 = R_3 + 6R_1 \implies \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & -15 & -6 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \implies \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = 0, v_2 = -\frac{2t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -\frac{2}{5} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{5} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	3	No	$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$
3	1	3	No	$\begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 3 & 3 & 5 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 0 & 0 \\ -6 & 8 & 2 \\ 12 & -15 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 3 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 3 & 3 & 5 \end{bmatrix}^{-1}$$

1.19 problem problem 19

Internal problem ID [10280]
Internal file name [`OUTPUT/9227_Monday_June_06_2022_01_44_44_PM_24262799/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 19.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 3 & 6 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 3 & 6 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 3-\lambda & 6 & -2 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$$

$$-(-3+\lambda)(-1+\lambda)^2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	2	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 3 & 6 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 3 & 6 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 6 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 6 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 6 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -3t + s\}$

Hence the solution is

$$\begin{bmatrix} -3t + s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -3t + s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -3t+s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -3t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \\ s \end{bmatrix}$$
$$= t\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -3t+s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} 3 & 6 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 3 & 6 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 6 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 6 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 0 & 6 & -2 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} 0 & 6 & -2 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 6 & -2 \\ 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	2	3	No	$\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$
3	1	3	No	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} -3 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 3 & 6 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$$

1.20 problem problem 20

Internal problem ID [10281]
Internal file name [`OUTPUT/9228_Monday_June_06_2022_01_44_45_PM_52282663/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 20.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 7 & 2 \\ 10 & -15 & -4 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ -4 & 7 & 2 \\ 10 & -15 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 0 & 0 \\ -4 & 7-\lambda & 2 \\ 10 & -15 & -4-\lambda \end{bmatrix} = 0$$

$$-(-1+\lambda)\left(\lambda^2 - 3\lambda + 2\right) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	2	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 0 & 0 \\ -4 & 7 & 2 \\ 10 & -15 & -4 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 0 & 0 \\ -4 & 7 & 2 \\ 10 & -15 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 & 0 \\ -4 & 6 & 2 \\ 10 & -15 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -4 & 6 & 2 & 0 \\ 10 & -15 & -5 & 0 \end{array}\right]
$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$
\left[\begin{array}{ccc|c} -4 & 6 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 10 & -15 & -5 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + \frac{5R_1}{2} \implies \left[\begin{array}{ccc|c} -4 & 6 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 6 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2} + \frac{s}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} + \frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} + \frac{s}{2} \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} \frac{3t}{2} + \frac{s}{2} \\ t \\ s \end{bmatrix} &= \begin{bmatrix} \frac{3t}{2} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{s}{2} \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} \frac{3t}{2} + \frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right)$$

Which can be normalized to

$$\left(\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right)$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ -4 & 7 & 2 \\ 10 & -15 & -4 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ -4 & 7 & 2 \\ 10 & -15 & -4 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ -4 & 5 & 2 \\ 10 & -15 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ -4 & 5 & 2 & 0 \\ 10 & -15 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 - 4R_1 \implies \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 10 & -15 & -6 & 0 \end{array}\right]$$

$$R_3 = R_3 + 10R_1 \implies \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & -15 & -6 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = 0, v_2 = -\frac{2t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -\frac{2}{5} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{5} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	2	3	No	$\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & -2 \\ 0 & 2 & 5 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 7 & 2 \\ 10 & -15 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & -2 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & -2 \\ 0 & 2 & 5 \end{bmatrix}^{-1}$$

1.21 problem problem 21

Internal problem ID [10282]
Internal file name [`OUTPUT/9229_Monday_June_06_2022_01_44_46_PM_44517223/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 21.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 4-\lambda & -3 & 1 \\ 2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & -3 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \implies \left[\begin{array}{ccc|c} 3 & -3 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$R_3 = R_3 - 3R_2 \implies \left[\begin{array}{ccc|c} 3 & -3 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -3 & 1 \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2} - \frac{s}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} - \frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} - \frac{s}{2} \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} \frac{3t}{2} - \frac{s}{2} \\ t \\ s \end{bmatrix} &= \begin{bmatrix} \frac{3t}{2} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{s}{2} \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} \frac{3t}{2} - \frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right)$$

Which can be normalized to

$$\left(\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right)$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
2	2	3	No	$\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{-1}$$

1.22 problem problem 22

Internal problem ID [10283]
Internal file name [`OUTPUT/9230_Monday_June_06_2022_01_44_46_PM_55902958/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 22.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 5 & -6 & 3 \\ 6 & -7 & 3 \\ 6 & -6 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 5 & -6 & 3 \\ 6 & -7 & 3 \\ 6 & -6 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 5-\lambda & -6 & 3 \\ 6 & -7-\lambda & 3 \\ 6 & -6 & 2-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 3\lambda + 2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	2	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 5 & -6 & 3 \\ 6 & -7 & 3 \\ 6 & -6 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 5 & -6 & 3 \\ 6 & -7 & 3 \\ 6 & -6 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 6 & -6 & 3 \\ 6 & -6 & 3 \\ 6 & -6 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 6 & -6 & 3 & 0 \\ 6 & -6 & 3 & 0 \\ 6 & -6 & 3 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 6 & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & -6 & 3 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 6 & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -6 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t - \frac{s}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} t - \frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} t - \frac{s}{2} \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} t - \frac{s}{2} \\ t \\ s \end{bmatrix} &= \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{s}{2} \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t - \frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right)$$

Which can be normalized to

$$\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\2\end{bmatrix}\right)$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}A\boldsymbol{v} &= \lambda\boldsymbol{v}\\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0}\\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0}\\ \left(\begin{bmatrix}5 & -6 & 3\\6 & -7 & 3\\6 & -6 & 2\end{bmatrix} - (2)\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} &= \begin{bmatrix}0\\0\\0\end{bmatrix}\\ \left(\begin{bmatrix}5 & -6 & 3\\6 & -7 & 3\\6 & -6 & 2\end{bmatrix} - \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} &= \begin{bmatrix}0\\0\\0\end{bmatrix}\\ \begin{bmatrix}3 & -6 & 3\\6 & -9 & 3\\6 & -6 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} &= \begin{bmatrix}0\\0\\0\end{bmatrix}\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c}3 & -6 & 3 & 0\\6 & -9 & 3 & 0\\6 & -6 & 0 & 0\end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c}3 & -6 & 3 & 0\\0 & 3 & -3 & 0\\6 & -6 & 0 & 0\end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c}3 & -6 & 3 & 0\\0 & 3 & -3 & 0\\0 & 6 & -6 & 0\end{array}\right]$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 3 & -6 & 3 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -6 & 3 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-1	2	3	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 5 & -6 & 3 \\ 6 & -7 & 3 \\ 6 & -6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}^{-1}$$

1.23 problem problem 23

Internal problem ID [10284]
Internal file name [`OUTPUT/9231_Monday_June_06_2022_01_44_47_PM_10020561/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 23.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 2 & 2 & 2 \\ 0 & 2-\lambda & 2 & 2 \\ 0 & 0 & 3-\lambda & 2 \\ 0 & 0 & 0 & 4-\lambda \end{bmatrix} = 0$$

$$-(1-\lambda)(-2+\lambda)(-3+\lambda)(-4+\lambda) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$
$$\lambda_4 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue
4	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 2 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cccc|c} 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(3,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3, v_4\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & 2 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{2} \Longrightarrow \left[\begin{array}{cccc|c} -1 & 2 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$R_4 = R_4 - 2R_3 \Longrightarrow \left[\begin{array}{cccc|c} -1 & 2 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 2 & 2 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} -2 & 2 & 2 & 2 & 0 \\ 0 & -1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{R_3}{2} \Longrightarrow \left[\begin{array}{cccc|c} -2 & 2 & 2 & 2 & 0 \\ 0 & -1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 & 2 & 2 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2, v_4\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t, v_2 = 2t, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ 2t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 3t \\ 2t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ 2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 3t \\ 2t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = 4$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 & 2 & 2 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} -3 & 2 & 2 & 2 & 0 \\ 0 & -2 & 2 & 2 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 2 & 2 & 2 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 4t, v_2 = 3t, v_3 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 4t \\ 3t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 4t \\ 3t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 4t \\ 3t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 4t \\ 3t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	4	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
2	1	4	No	$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
3	1	4	No	$\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$
4	1	4	No	$\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

1.24 problem problem 24

Internal problem ID [10285]
Internal file name [OUTPUT/9232_Monday_June_06_2022_01_44_48_PM_25813954/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 24.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 0 & 4 & 0 \\ 0 & 1-\lambda & 4 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix} = 0$$

$$-(1-\lambda)(-1+\lambda)(-3+\lambda)^2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 3$$
$$\lambda_4 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	2	real eigenvalue
3	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

Since the current pivot $A(2,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 4 gives

$$\left[\begin{array}{cccc|c} 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1, v_2\}$ and the leading variables are $\{v_3, v_4\}$. Let $v_1 = t$. Let $v_2 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ s \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ s \\ 0 \\ 0 \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ s \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ 0 \\ 0 \end{bmatrix}$$

$$= t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t \\ s \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0}
\end{aligned}
$$

$$
\left(\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\left(\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} -2 & 0 & 4 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cccc|c} -2 & 0 & 4 & 0 & 0 \\ 0 & -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -2 & 0 & 4 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3, v_4\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Let $v_4 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ 2t \\ t \\ s \end{bmatrix} = \begin{bmatrix} 2t \\ 2t \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned}\begin{bmatrix} 2t \\ 2t \\ t \\ s \end{bmatrix} &= \begin{bmatrix} 2t \\ 2t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} 2t \\ 2t \\ t \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	2	4	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
3	2	4	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

1.25 problem problem 25

Internal problem ID [10286]
Internal file name [`OUTPUT/9233_Monday_June_06_2022_01_44_50_PM_49169769/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 25.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$-(1-\lambda)(-1+\lambda)(-2+\lambda)^2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 2$$
$$\lambda_4 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	2	real eigenvalue
2	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]
$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(2,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 4 gives

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1, v_2\}$ and the leading variables are $\{v_3, v_4\}$. Let $v_1 = t$. Let $v_2 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ s \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ s \\ 0 \\ 0 \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ s \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ 0 \\ 0 \end{bmatrix}$$

$$= t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t \\ s \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3, v_4\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Let $v_4 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \\ s \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned}\begin{bmatrix} t \\ t \\ t \\ s \end{bmatrix} &= \begin{bmatrix} t \\ t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ s \end{bmatrix} \\ &= t\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t \\ t \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	2	4	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
2	2	4	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

1.26 problem problem 26

Internal problem ID [10287]
Internal file name [OUTPUT/9234_Monday_June_06_2022_01_44_50_PM_96183690/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 26.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 4 & 0 & 0 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 0 & 0 & -5 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 4 & 0 & 0 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 0 & 0 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 4-\lambda & 0 & 0 & -3 \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & -1-\lambda & 0 \\ 6 & 0 & 0 & -5-\lambda \end{bmatrix} = 0$$

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$
$$\lambda_3 = 1$$
$$\lambda_4 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 4 & 0 & 0 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 0 & 0 & -5 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 4 & 0 & 0 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 0 & 0 & -5 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 0 & -3 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} 5 & 0 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & -4 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{6R_1}{5} \Longrightarrow \left[\begin{array}{cccc|c} 5 & 0 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & 0 \end{array}\right]$$

Since the current pivot $A(3,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} 5 & 0 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 0 & 0 & -3 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2, v_4\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = -2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \end{aligned}$$

$$\left(\begin{bmatrix} 4 & 0 & 0 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 0 & 0 & -5 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 4 & 0 & 0 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 0 & 0 & -5 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 0 & -3 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} 6 & 0 & 0 & -3 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & 0 & -3 & 0 \end{array}\right]$$

$$R_4 = R_4 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 6 & 0 & 0 & -3 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & 0 & 0 & -3 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = 0, v_3 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$A\boldsymbol{v} = \lambda\boldsymbol{v}$$
$$A\boldsymbol{v} - \lambda\boldsymbol{v} = \mathbf{0}$$
$$(A - \lambda I)\boldsymbol{v} = \mathbf{0}$$

$$\left(\begin{bmatrix} 4 & 0 & 0 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 0 & 0 & -5 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 4 & 0 & 0 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 0 & 0 & -5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 6 & 0 & 0 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 6 & 0 & 0 & -6 & 0 \end{array}\right]$$

$$R_4 = R_4 - 2R_1 \Longrightarrow \left[\begin{array}{cccc|c} 3 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 4 & 0 & 0 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 0 & 0 & -5 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 4 & 0 & 0 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 0 & 0 & -5 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 6 & 0 & 0 & -7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 6 & 0 & 0 & -7 & 0 \end{array}\right]$$

$$R_4 = R_4 - 3R_1 \Longrightarrow \left[\begin{array}{cccc|c} 2 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a

row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & -3 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

Since the current pivot $A(3,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & -3 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-1	1	4	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$
-2	1	4	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$
1	1	4	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
2	1	4	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 4 & 0 & 0 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}^{-1}$$

1.27 problem problem 27

Internal problem ID [10288]
Internal file name [OUTPUT/9235_Monday_June_06_2022_01_44_52_PM_14054475/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 27.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = 0$$
$$\lambda^2 + 1 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Considering $\lambda = i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
$-i$	1	2	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
i	1	2	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}$$

1.28 problem problem 28

Internal problem ID [10289]
Internal file name [OUTPUT/9236_Monday_June_06_2022_01_44_53_PM_82695140/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 28.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} -\lambda & -6 \\ 6 & -\lambda \end{bmatrix} = 0$$
$$\lambda^2 + 36 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 6i$$
$$\lambda_2 = -6i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$6i$	1	complex eigenvalue
$-6i$	1	complex eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 6i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix} - (6i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix} - \begin{bmatrix} 6i & 0 \\ 0 & 6i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -6i & -6 \\ 6 & -6i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} -6i & -6 & 0 \\ 6 & -6i & 0 \end{array}\right]
$$

$$
R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -6i & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -6i & -6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$
\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Considering $\lambda = -6i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix} - (-6i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix} - \begin{bmatrix} -6i & 0 \\ 0 & -6i \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 6i & -6 \\ 6 & 6i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 6i & -6 & 0 \\ 6 & 6i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 6i & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6i & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
$6i$	1	2	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
$-6i$	1	2	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 6i & 0 \\ 0 & -6i \end{bmatrix}$$

$$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6i & 0 \\ 0 & -6i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}$$

1.29 problem problem 29

Internal problem ID [10290]
Internal file name [`OUTPUT/9237_Monday_June_06_2022_01_44_53_PM_40858430/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 29.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} -\lambda & -3 \\ 12 & -\lambda \end{bmatrix} = 0$$
$$\lambda^2 + 36 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 6i$$
$$\lambda_2 = -6i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$6i$	1	complex eigenvalue
$-6i$	1	complex eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 6i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix} - (6i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix} - \begin{bmatrix} 6i & 0 \\ 0 & 6i \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -6i & -3 \\ 12 & -6i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -6i & -3 & 0 \\ 12 & -6i & 0 \end{array}\right]$$

$$R_2 = -2iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -6i & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6i & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{it}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{\mathrm{I}}{2}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{it}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{\mathrm{I}}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{i}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{I}{2}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{i}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{I}{2}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 2 \end{bmatrix}$$

Considering $\lambda = -6i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix} - (-6i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix} - \begin{bmatrix} -6i & 0 \\ 0 & -6i \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 6i & -3 \\ 12 & 6i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 6i & -3 & 0 \\ 12 & 6i & 0 \end{array}\right]$$

$$R_2 = 2iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 6i & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6i & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{it}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{I}{2}t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{it}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{I}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{i}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{I}{2}t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{i}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{I}{2}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
$6i$	1	2	No	$\begin{bmatrix} i \\ 2 \end{bmatrix}$
$-6i$	1	2	No	$\begin{bmatrix} -i \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 6i & 0 \\ 0 & -6i \end{bmatrix}$$

$$P = \begin{bmatrix} i & -i \\ 2 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix} = \begin{bmatrix} i & -i \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 6i & 0 \\ 0 & -6i \end{bmatrix} \begin{bmatrix} i & -i \\ 2 & 2 \end{bmatrix}^{-1}$$

1.30 problem problem 30

Internal problem ID [10291]
Internal file name [`OUTPUT/9238_Monday_June_06_2022_01_44_54_PM_24323647/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 30.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 0 & -12 \\ 12 & 0 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 0 & -12 \\ 12 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} -\lambda & -12 \\ 12 & -\lambda \end{bmatrix} = 0$$
$$\lambda^2 + 144 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 12i$$
$$\lambda_2 = -12i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-12i$	1	complex eigenvalue
$12i$	1	complex eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -12i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 0 & -12 \\ 12 & 0 \end{bmatrix} - (-12i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 0 & -12 \\ 12 & 0 \end{bmatrix} - \begin{bmatrix} -12i & 0 \\ 0 & -12i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 12i & -12 \\ 12 & 12i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 12i & -12 & 0 \\ 12 & 12i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 12i & -12 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 12i & -12 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = t\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Considering $\lambda = 12i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 0 & -12 \\ 12 & 0 \end{bmatrix} - (12i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 0 & -12 \\ 12 & 0 \end{bmatrix} - \begin{bmatrix} 12i & 0 \\ 0 & 12i \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -12i & -12 \\ 12 & -12i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -12i & -12 & 0 \\ 12 & -12i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -12i & -12 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -12i & -12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
$-12i$	1	2	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$12i$	1	2	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -12i & 0 \\ 0 & 12i \end{bmatrix}$$

$$P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 0 & -12 \\ 12 & 0 \end{bmatrix} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -12i & 0 \\ 0 & 12i \end{bmatrix} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}^{-1}$$

1.31 problem problem 31

Internal problem ID [10292]
Internal file name [OUTPUT/9239_Monday_June_06_2022_01_44_55_PM_96866979/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 31.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 0 & 24 \\ -6 & 0 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 0 & 24 \\ -6 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} -\lambda & 24 \\ -6 & -\lambda \end{bmatrix} = 0$$
$$\lambda^2 + 144 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 12i$$
$$\lambda_2 = -12i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-12i$	1	complex eigenvalue
$12i$	1	complex eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -12i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 0 & 24 \\ -6 & 0 \end{bmatrix} - (-12i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 0 & 24 \\ -6 & 0 \end{bmatrix} - \begin{bmatrix} -12i & 0 \\ 0 & -12i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 12i & 24 \\ -6 & 12i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 12i & 24 & 0 \\ -6 & 12i & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{iR_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 12i & 24 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 12i & 24 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2it\}$

Hence the solution is

$$\begin{bmatrix} 2\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} 2it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2\,\mathrm{I}\,t \\ t \end{bmatrix} = t\begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

Considering $\lambda = 12i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 0 & 24 \\ -6 & 0 \end{bmatrix} - (12i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 0 & 24 \\ -6 & 0 \end{bmatrix} - \begin{bmatrix} 12i & 0 \\ 0 & 12i \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -12i & 24 \\ -6 & -12i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -12i & 24 & 0 \\ -6 & -12i & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{iR_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -12i & 24 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -12i & 24 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2it\}$

Hence the solution is

$$\begin{bmatrix} -2\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -2it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2\,\mathrm{I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -2\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
$-12i$	1	2	No	$\begin{bmatrix} 2i \\ 1 \end{bmatrix}$
$12i$	1	2	No	$\begin{bmatrix} -2i \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -12i & 0 \\ 0 & 12i \end{bmatrix}$$

$$P = \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 0 & 24 \\ -6 & 0 \end{bmatrix} = \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -12i & 0 \\ 0 & 12i \end{bmatrix} \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix}^{-1}$$

1.32 problem problem 32

Internal problem ID [10293]
Internal file name [OUTPUT/9240_Monday_June_06_2022_01_44_56_PM_75504834/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 32.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 0 & -4 \\ 36 & 0 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 0 & -4 \\ 36 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det \begin{bmatrix} -\lambda & -4 \\ 36 & -\lambda \end{bmatrix} = 0$$
$$\lambda^2 + 144 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 12i$$
$$\lambda_2 = -12i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-12i$	1	complex eigenvalue
$12i$	1	complex eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -12i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 0 & -4 \\ 36 & 0 \end{bmatrix} - (-12i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 0 & -4 \\ 36 & 0 \end{bmatrix} - \begin{bmatrix} -12i & 0 \\ 0 & -12i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 12i & -4 \\ 36 & 12i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 12i & -4 & 0 \\ 36 & 12i & 0 \end{array}\right]$$

$$R_2 = 3iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 12i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 12i & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{it}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{\mathrm{I}}{3}t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{it}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{\mathrm{I}}{3}t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{i}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{\text{I}}{3}t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{i}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{\text{I}}{3}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 3 \end{bmatrix}$$

Considering $\lambda = 12i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 0 & -4 \\ 36 & 0 \end{bmatrix} - (12i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 0 & -4 \\ 36 & 0 \end{bmatrix} - \begin{bmatrix} 12i & 0 \\ 0 & 12i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -12i & -4 \\ 36 & -12i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -12i & -4 & 0 \\ 36 & -12i & 0 \end{array}\right]$$

$$R_2 = -3iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -12i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -12i & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{it}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{\mathrm{I}}{3}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{it}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{\mathrm{I}}{3}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{i}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{\mathrm{I}}{3}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{i}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{\mathrm{I}}{3}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 3 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
$-12i$	1	2	No	$\begin{bmatrix} -i \\ 3 \end{bmatrix}$
$12i$	1	2	No	$\begin{bmatrix} i \\ 3 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -12i & 0 \\ 0 & 12i \end{bmatrix}$$

$$P = \begin{bmatrix} -i & i \\ 3 & 3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 0 & -4 \\ 36 & 0 \end{bmatrix} = \begin{bmatrix} -i & i \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -12i & 0 \\ 0 & 12i \end{bmatrix} \begin{bmatrix} -i & i \\ 3 & 3 \end{bmatrix}^{-1}$$

1.33 problem problem 40

Internal problem ID [10294]
Internal file name [`OUTPUT/9241_Monday_June_06_2022_01_44_56_PM_60806489/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 40.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 32 & -67 & 47 \\ 7 & -14 & 13 \\ -7 & 15 & -6 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 32 & -67 & 47 \\ 7 & -14 & 13 \\ -7 & 15 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 32-\lambda & -67 & 47 \\ 7 & -14-\lambda & 13 \\ -7 & 15 & -6-\lambda \end{bmatrix} = 0$$
$$-\lambda^3 + 12\lambda^2 - 47\lambda + 60 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 5$$
$$\lambda_2 = 3$$
$$\lambda_3 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue
4	1	real eigenvalue
5	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 32 & -67 & 47 \\ 7 & -14 & 13 \\ -7 & 15 & -6 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 32 & -67 & 47 \\ 7 & -14 & 13 \\ -7 & 15 & -6 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 29 & -67 & 47 \\ 7 & -17 & 13 \\ -7 & 15 & -9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 29 & -67 & 47 & 0 \\ 7 & -17 & 13 & 0 \\ -7 & 15 & -9 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{7R_1}{29} \Longrightarrow \left[\begin{array}{ccc|c} 29 & -67 & 47 & 0 \\ 0 & -\frac{24}{29} & \frac{48}{29} & 0 \\ -7 & 15 & -9 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{7R_1}{29} \Longrightarrow \left[\begin{array}{ccc|c} 29 & -67 & 47 & 0 \\ 0 & -\frac{24}{29} & \frac{48}{29} & 0 \\ 0 & -\frac{34}{29} & \frac{68}{29} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{17R_2}{12} \Longrightarrow \left[\begin{array}{ccc|c} 29 & -67 & 47 & 0 \\ 0 & -\frac{24}{29} & \frac{48}{29} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 29 & -67 & 47 \\ 0 & -\frac{24}{29} & \frac{48}{29} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 3t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Considering $\lambda = 4$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 32 & -67 & 47 \\ 7 & -14 & 13 \\ -7 & 15 & -6 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 32 & -67 & 47 \\ 7 & -14 & 13 \\ -7 & 15 & -6 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 28 & -67 & 47 \\ 7 & -18 & 13 \\ -7 & 15 & -10 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 28 & -67 & 47 & 0 \\ 7 & -18 & 13 & 0 \\ -7 & 15 & -10 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} 28 & -67 & 47 & 0 \\ 0 & -\frac{5}{4} & \frac{5}{4} & 0 \\ -7 & 15 & -10 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} 28 & -67 & 47 & 0 \\ 0 & -\frac{5}{4} & \frac{5}{4} & 0 \\ 0 & -\frac{7}{4} & \frac{7}{4} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{7R_2}{5} \Longrightarrow \left[\begin{array}{ccc|c} 28 & -67 & 47 & 0 \\ 0 & -\frac{5}{4} & \frac{5}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 28 & -67 & 47 \\ 0 & -\frac{5}{4} & \frac{5}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{5t}{7}, v_2 = t\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{5t}{7} \\ t \\ t \end{bmatrix} = \begin{bmatrix} \frac{5t}{7} \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{5t}{7} \\ t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{7} \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{5t}{7} \\ t \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{7} \\ 1 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{5t}{7} \\ t \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 7 \end{bmatrix}$$

Considering $\lambda = 5$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 32 & -67 & 47 \\ 7 & -14 & 13 \\ -7 & 15 & -6 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 32 & -67 & 47 \\ 7 & -14 & 13 \\ -7 & 15 & -6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 27 & -67 & 47 \\ 7 & -19 & 13 \\ -7 & 15 & -11 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 27 & -67 & 47 & 0 \\ 7 & -19 & 13 & 0 \\ -7 & 15 & -11 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{7R_1}{27} \Longrightarrow \left[\begin{array}{ccc|c} 27 & -67 & 47 & 0 \\ 0 & -\frac{44}{27} & \frac{22}{27} & 0 \\ -7 & 15 & -11 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + \frac{7R_1}{27} \Longrightarrow \left[\begin{array}{ccc|c} 27 & -67 & 47 & 0 \\ 0 & -\frac{44}{27} & \frac{22}{27} & 0 \\ 0 & -\frac{64}{27} & \frac{32}{27} & 0 \end{array}\right]
$$

$$
R_3 = R_3 - \frac{16R_2}{11} \Longrightarrow \left[\begin{array}{ccc|c} 27 & -67 & 47 & 0 \\ 0 & -\frac{44}{27} & \frac{22}{27} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 27 & -67 & 47 \\ 0 & -\frac{44}{27} & \frac{22}{27} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
3	1	3	No	$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$
4	1	3	No	$\begin{bmatrix} 5 \\ 7 \\ 7 \end{bmatrix}$
5	1	3	No	$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$P = \begin{bmatrix} 3 & 5 & -1 \\ 2 & 7 & 1 \\ 1 & 7 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 32 & -67 & 47 \\ 7 & -14 & 13 \\ -7 & 15 & -6 \end{bmatrix} = \begin{bmatrix} 3 & 5 & -1 \\ 2 & 7 & 1 \\ 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 5 & -1 \\ 2 & 7 & 1 \\ 1 & 7 & 2 \end{bmatrix}^{-1}$$

1.34 problem problem 41

Internal problem ID [10295]
Internal file name [OUTPUT/9242_Monday_June_06_2022_01_44_57_PM_12178831/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.1, Introduction to Eigenvalues, Eigenvalues and Eigenvectors. Page 346
Problem number: problem 41.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 22 & -9 & -8 & -8 \\ 10 & -7 & -14 & 2 \\ 10 & 0 & 8 & -10 \\ 29 & -9 & -3 & -15 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 22 & -9 & -8 & -8 \\ 10 & -7 & -14 & 2 \\ 10 & 0 & 8 & -10 \\ 29 & -9 & -3 & -15 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 22-\lambda & -9 & -8 & -8 \\ 10 & -7-\lambda & -14 & 2 \\ 10 & 0 & 8-\lambda & -10 \\ 29 & -9 & -3 & -15-\lambda \end{bmatrix} = 0$$

$$\lambda^4 - 8\lambda^3 + 11\lambda^2 + 32\lambda - 60 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = 5$$
$$\lambda_4 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue
5	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 22 & -9 & -8 & -8 \\ 10 & -7 & -14 & 2 \\ 10 & 0 & 8 & -10 \\ 29 & -9 & -3 & -15 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 22 & -9 & -8 & -8 \\ 10 & -7 & -14 & 2 \\ 10 & 0 & 8 & -10 \\ 29 & -9 & -3 & -15 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 24 & -9 & -8 & -8 \\ 10 & -5 & -14 & 2 \\ 10 & 0 & 10 & -10 \\ 29 & -9 & -3 & -13 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} 24 & -9 & -8 & -8 & 0 \\ 10 & -5 & -14 & 2 & 0 \\ 10 & 0 & 10 & -10 & 0 \\ 29 & -9 & -3 & -13 & 0 \end{array} \right]$$

$$R_2 = R_2 - \frac{5R_1}{12} \Longrightarrow \left[\begin{array}{cccc|c} 24 & -9 & -8 & -8 & 0 \\ 0 & -\frac{5}{4} & -\frac{32}{3} & \frac{16}{3} & 0 \\ 10 & 0 & 10 & -10 & 0 \\ 29 & -9 & -3 & -13 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{5R_1}{12} \Longrightarrow \left[\begin{array}{cccc|c} 24 & -9 & -8 & -8 & 0 \\ 0 & -\frac{5}{4} & -\frac{32}{3} & \frac{16}{3} & 0 \\ 0 & \frac{15}{4} & \frac{40}{3} & -\frac{20}{3} & 0 \\ 29 & -9 & -3 & -13 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{29R_1}{24} \Longrightarrow \left[\begin{array}{cccc|c} 24 & -9 & -8 & -8 & 0 \\ 0 & -\frac{5}{4} & -\frac{32}{3} & \frac{16}{3} & 0 \\ 0 & \frac{15}{4} & \frac{40}{3} & -\frac{20}{3} & 0 \\ 0 & \frac{15}{8} & \frac{20}{3} & -\frac{10}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \Longrightarrow \left[\begin{array}{cccc|c} 24 & -9 & -8 & -8 & 0 \\ 0 & -\frac{5}{4} & -\frac{32}{3} & \frac{16}{3} & 0 \\ 0 & 0 & -\frac{56}{3} & \frac{28}{3} & 0 \\ 0 & \frac{15}{8} & \frac{20}{3} & -\frac{10}{3} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{3R_2}{2} \Longrightarrow \left[\begin{array}{cccc|c} 24 & -9 & -8 & -8 & 0 \\ 0 & -\frac{5}{4} & -\frac{32}{3} & \frac{16}{3} & 0 \\ 0 & 0 & -\frac{56}{3} & \frac{28}{3} & 0 \\ 0 & 0 & -\frac{28}{3} & \frac{14}{3} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{R_3}{2} \Longrightarrow \left[\begin{array}{cccc|c} 24 & -9 & -8 & -8 & 0 \\ 0 & -\frac{5}{4} & -\frac{32}{3} & \frac{16}{3} & 0 \\ 0 & 0 & -\frac{56}{3} & \frac{28}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 24 & -9 & -8 & -8 \\ 0 & -\frac{5}{4} & -\frac{32}{3} & \frac{16}{3} \\ 0 & 0 & -\frac{56}{3} & \frac{28}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = 0, v_3 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ 0 \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{t}{2} \\ 0 \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 22 & -9 & -8 & -8 \\ 10 & -7 & -14 & 2 \\ 10 & 0 & 8 & -10 \\ 29 & -9 & -3 & -15 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 22 & -9 & -8 & -8 \\ 10 & -7 & -14 & 2 \\ 10 & 0 & 8 & -10 \\ 29 & -9 & -3 & -15 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 20 & -9 & -8 & -8 \\ 10 & -9 & -14 & 2 \\ 10 & 0 & 6 & -10 \\ 29 & -9 & -3 & -17 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} 20 & -9 & -8 & -8 & 0 \\ 10 & -9 & -14 & 2 & 0 \\ 10 & 0 & 6 & -10 & 0 \\ 29 & -9 & -3 & -17 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{cccc|c} 20 & -9 & -8 & -8 & 0 \\ 0 & -\frac{9}{2} & -10 & 6 & 0 \\ 10 & 0 & 6 & -10 & 0 \\ 29 & -9 & -3 & -17 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cccc|c} 20 & -9 & -8 & -8 & 0 \\ 0 & -\frac{9}{2} & -10 & 6 & 0 \\ 0 & \frac{9}{2} & 10 & -6 & 0 \\ 29 & -9 & -3 & -17 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{29R_1}{20} \Longrightarrow \left[\begin{array}{cccc|c} 20 & -9 & -8 & -8 & 0 \\ 0 & -\frac{9}{2} & -10 & 6 & 0 \\ 0 & \frac{9}{2} & 10 & -6 & 0 \\ 0 & \frac{81}{20} & \frac{43}{5} & -\frac{27}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} 20 & -9 & -8 & -8 & 0 \\ 0 & -\frac{9}{2} & -10 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{81}{20} & \frac{43}{5} & -\frac{27}{5} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{9R_2}{10} \Longrightarrow \left[\begin{array}{cccc|c} 20 & -9 & -8 & -8 & 0 \\ 0 & -\frac{9}{2} & -10 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{5} & 0 & 0 \end{array}\right]$$

Since the current pivot $A(3,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} 20 & -9 & -8 & -8 & 0 \\ 0 & -\frac{9}{2} & -10 & 6 & 0 \\ 0 & 0 & -\frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 20 & -9 & -8 & -8 \\ 0 & -\frac{9}{2} & -10 & 6 \\ 0 & 0 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t, v_2 = \frac{4t}{3}, v_3 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} t \\ \frac{4t}{3} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} t \\ \frac{4t}{3} \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ \frac{4t}{3} \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ \frac{4t}{3} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} t \\ \frac{4t}{3} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 3 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \end{aligned}$$

$$\left(\begin{bmatrix} 22 & -9 & -8 & -8 \\ 10 & -7 & -14 & 2 \\ 10 & 0 & 8 & -10 \\ 29 & -9 & -3 & -15 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 22 & -9 & -8 & -8 \\ 10 & -7 & -14 & 2 \\ 10 & 0 & 8 & -10 \\ 29 & -9 & -3 & -15 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 19 & -9 & -8 & -8 \\ 10 & -10 & -14 & 2 \\ 10 & 0 & 5 & -10 \\ 29 & -9 & -3 & -18 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} 19 & -9 & -8 & -8 & 0 \\ 10 & -10 & -14 & 2 & 0 \\ 10 & 0 & 5 & -10 & 0 \\ 29 & -9 & -3 & -18 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{10R_1}{19} \Longrightarrow \left[\begin{array}{cccc|c} 19 & -9 & -8 & -8 & 0 \\ 0 & -\frac{100}{19} & -\frac{186}{19} & \frac{118}{19} & 0 \\ 10 & 0 & 5 & -10 & 0 \\ 29 & -9 & -3 & -18 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{10R_1}{19} \implies \left[\begin{array}{cccc|c} 19 & -9 & -8 & -8 & 0 \\ 0 & -\frac{100}{19} & -\frac{186}{19} & \frac{118}{19} & 0 \\ 0 & \frac{90}{19} & \frac{175}{19} & -\frac{110}{19} & 0 \\ 29 & -9 & -3 & -18 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{29R_1}{19} \implies \left[\begin{array}{cccc|c} 19 & -9 & -8 & -8 & 0 \\ 0 & -\frac{100}{19} & -\frac{186}{19} & \frac{118}{19} & 0 \\ 0 & \frac{90}{19} & \frac{175}{19} & -\frac{110}{19} & 0 \\ 0 & \frac{90}{19} & \frac{175}{19} & -\frac{110}{19} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{9R_2}{10} \implies \left[\begin{array}{cccc|c} 19 & -9 & -8 & -8 & 0 \\ 0 & -\frac{100}{19} & -\frac{186}{19} & \frac{118}{19} & 0 \\ 0 & 0 & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & \frac{90}{19} & \frac{175}{19} & -\frac{110}{19} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{9R_2}{10} \implies \left[\begin{array}{cccc|c} 19 & -9 & -8 & -8 & 0 \\ 0 & -\frac{100}{19} & -\frac{186}{19} & \frac{118}{19} & 0 \\ 0 & 0 & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{2}{5} & -\frac{1}{5} & 0 \end{array}\right]$$

$$R_4 = R_4 - R_3 \implies \left[\begin{array}{cccc|c} 19 & -9 & -8 & -8 & 0 \\ 0 & -\frac{100}{19} & -\frac{186}{19} & \frac{118}{19} & 0 \\ 0 & 0 & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 19 & -9 & -8 & -8 \\ 0 & -\frac{100}{19} & -\frac{186}{19} & \frac{118}{19} \\ 0 & 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{4}, v_2 = \frac{t}{4}, v_3 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{4} \\ \frac{t}{4} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{4} \\ \frac{t}{4} \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{4} \\ \frac{t}{4} \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{3t}{4} \\ \frac{t}{4} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{3t}{4} \\ \frac{t}{4} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

Considering $\lambda = 5$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 22 & -9 & -8 & -8 \\ 10 & -7 & -14 & 2 \\ 10 & 0 & 8 & -10 \\ 29 & -9 & -3 & -15 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 22 & -9 & -8 & -8 \\ 10 & -7 & -14 & 2 \\ 10 & 0 & 8 & -10 \\ 29 & -9 & -3 & -15 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 17 & -9 & -8 & -8 \\ 10 & -12 & -14 & 2 \\ 10 & 0 & 3 & -10 \\ 29 & -9 & -3 & -20 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cccc|c} 17 & -9 & -8 & -8 & 0 \\ 10 & -12 & -14 & 2 & 0 \\ 10 & 0 & 3 & -10 & 0 \\ 29 & -9 & -3 & -20 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{10R_1}{17} \Longrightarrow \left[\begin{array}{cccc|c} 17 & -9 & -8 & -8 & 0 \\ 0 & -\frac{114}{17} & -\frac{158}{17} & \frac{114}{17} & 0 \\ 10 & 0 & 3 & -10 & 0 \\ 29 & -9 & -3 & -20 & 0 \end{array}\right]
$$

$$R_3 = R_3 - \frac{10R_1}{17} \Longrightarrow \left[\begin{array}{cccc|c} 17 & -9 & -8 & -8 & 0 \\ 0 & -\frac{114}{17} & -\frac{158}{17} & \frac{114}{17} & 0 \\ 0 & \frac{90}{17} & \frac{131}{17} & -\frac{90}{17} & 0 \\ 29 & -9 & -3 & -20 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{29R_1}{17} \Longrightarrow \left[\begin{array}{cccc|c} 17 & -9 & -8 & -8 & 0 \\ 0 & -\frac{114}{17} & -\frac{158}{17} & \frac{114}{17} & 0 \\ 0 & \frac{90}{17} & \frac{131}{17} & -\frac{90}{17} & 0 \\ 0 & \frac{108}{17} & \frac{181}{17} & -\frac{108}{17} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{15R_2}{19} \Longrightarrow \left[\begin{array}{cccc|c} 17 & -9 & -8 & -8 & 0 \\ 0 & -\frac{114}{17} & -\frac{158}{17} & \frac{114}{17} & 0 \\ 0 & 0 & \frac{7}{19} & 0 & 0 \\ 0 & \frac{108}{17} & \frac{181}{17} & -\frac{108}{17} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{18R_2}{19} \Longrightarrow \left[\begin{array}{cccc|c} 17 & -9 & -8 & -8 & 0 \\ 0 & -\frac{114}{17} & -\frac{158}{17} & \frac{114}{17} & 0 \\ 0 & 0 & \frac{7}{19} & 0 & 0 \\ 0 & 0 & \frac{35}{19} & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 - 5R_3 \Longrightarrow \left[\begin{array}{cccc|c} 17 & -9 & -8 & -8 & 0 \\ 0 & -\frac{114}{17} & -\frac{158}{17} & \frac{114}{17} & 0 \\ 0 & 0 & \frac{7}{19} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 17 & -9 & -8 & -8 \\ 0 & -\frac{114}{17} & -\frac{158}{17} & \frac{114}{17} \\ 0 & 0 & \frac{7}{19} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-2	1	4	No	$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$
2	1	4	No	$\begin{bmatrix} 3 \\ 4 \\ 0 \\ 3 \end{bmatrix}$
3	1	4	No	$\begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}$
5	1	4	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 4 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 22 & -9 & -8 & -8 \\ 10 & -7 & -14 & 2 \\ 10 & 0 & 8 & -10 \\ 29 & -9 & -3 & -15 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 4 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 4 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 3 & 4 & 1 \end{bmatrix}^{-1}$$

2 From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354

2.1 problem problem 1

Internal problem ID [10296]
Internal file name [`OUTPUT/9243_Monday_June_06_2022_01_44_59_PM_66444117/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 1.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 4\lambda + 3 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 4 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or, by letting $t=1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \implies \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
3	1	2	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$$

2.2 problem problem 2

Internal problem ID [10297]
Internal file name [`OUTPUT/9244_Monday_June_06_2022_01_45_00_PM_98617278/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 2.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 6 & -6 \\ 4 & -4 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 6 & -6 \\ 4 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 6-\lambda & -6 \\ 4 & -4-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 2\lambda = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 0$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 6 & -6 \\ 4 & -4 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 6 & -6 \\ 4 & -4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 6 & -6 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & -6 & 0 \\ 4 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 6 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 6 & -6 \\ 4 & -4 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 6 & -6 \\ 4 & -4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 4 & -6 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -6 & 0 \\ 4 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 4 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
0	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2	1	2	No	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 6 & -6 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}^{-1}$$

2.3 problem problem 3

Internal problem ID [10298]
Internal file name [`OUTPUT/9245_Monday_June_06_2022_01_45_00_PM_77891827/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 3.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 5-\lambda & -3 \\ 2 & -\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 5\lambda + 6 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 3$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
2	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
3	1	2	No	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}^{-1}$$

2.4 problem problem 4

Internal problem ID [10299]
Internal file name [OUTPUT/9246_Monday_June_06_2022_01_45_01_PM_52924244/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 4.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 5-\lambda & -4 \\ 3 & -2-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 4 & -4 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 4 & -4 & 0 \\ 3 & -3 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{3R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} 4 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$
\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 3 & -4 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -4 & 0 \\ 3 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{4t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2	1	2	No	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}^{-1}$$

2.5 problem problem 5

Internal problem ID [10300]
Internal file name [`OUTPUT/9247_Monday_June_06_2022_01_45_02_PM_55297811/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 5.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 9 & -8 \\ 6 & -5 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 9 & -8 \\ 6 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 9-\lambda & -8 \\ 6 & -5-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 4\lambda + 3 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 9 & -8 \\ 6 & -5 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 9 & -8 \\ 6 & -5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 8 & -8 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 8 & -8 & 0 \\ 6 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} 8 & -8 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 8 & -8 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 9 & -8 \\ 6 & -5 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 9 & -8 \\ 6 & -5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 6 & -8 \\ 6 & -8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & -8 & 0 \\ 6 & -8 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \implies \left[\begin{array}{cc|c} 6 & -8 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{4t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
3	1	2	No	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 9 & -8 \\ 6 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}^{-1}$$

2.6 problem problem 6

Internal problem ID [10301]
Internal file name [`OUTPUT/9248_Monday_June_06_2022_01_45_02_PM_17597627/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 6.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 10 & -6 \\ 12 & -7 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 10 & -6 \\ 12 & -7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det \begin{bmatrix} 10-\lambda & -6 \\ 12 & -7-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 10 & -6 \\ 12 & -7 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 10 & -6 \\ 12 & -7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 9 & -6 \\ 12 & -8 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 9 & -6 & 0 \\ 12 & -8 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{4R_1}{3} \implies \left[\begin{array}{cc|c} 9 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 9 & -6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \frac{2t}{3}\}$

Hence the solution is

$$
\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = t\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 10 & -6 \\ 12 & -7 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 10 & -6 \\ 12 & -7 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 8 & -6 \\ 12 & -9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 8 & -6 & 0 \\ 12 & -9 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{2} \implies \left[\begin{array}{cc|c} 8 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 8 & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{3t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{3t}{4} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	2	No	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
2	1	2	No	$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 10 & -6 \\ 12 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}^{-1}$$

2.7 problem problem 7

Internal problem ID [10302]
Internal file name [`OUTPUT/9249_Monday_June_06_2022_01_45_03_PM_99935789/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 7.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 6-\lambda & -10 \\ 2 & -3-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 5 & -10 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 5 & -10 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{cc|c} 5 & -10 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & -10 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 4 & -10 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -10 & 0 \\ 2 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{cc|c} 4 & -10 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -10 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{5t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	2	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
2	1	2	No	$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}^{-1}$$

2.8 problem problem 8

Internal problem ID [10303]
Internal file name [`OUTPUT/9250_Monday_June_06_2022_01_45_03_PM_78447792/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 8.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 11 & -15 \\ 6 & -8 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 11 & -15 \\ 6 & -8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 11-\lambda & -15 \\ 6 & -8-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 11 & -15 \\ 6 & -8 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 11 & -15 \\ 6 & -8 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 10 & -15 \\ 6 & -9 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 10 & -15 & 0 \\ 6 & -9 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{5} \Longrightarrow \left[\begin{array}{cc|c} 10 & -15 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 10 & -15 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = t\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 11 & -15 \\ 6 & -8 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 11 & -15 \\ 6 & -8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 9 & -15 \\ 6 & -10 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 9 & -15 & 0 \\ 6 & -10 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 9 & -15 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 9 & -15 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{5t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{5t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	2	No	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$
2	1	2	No	$\begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 11 & -15 \\ 6 & -8 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}^{-1}$$

2.9 problem problem 9

Internal problem ID [10304]
Internal file name [OUTPUT/9251_Monday_June_06_2022_01_45_04_PM_73590790/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 9.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} -1-\lambda & 4 \\ -1 & 3-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 2\lambda + 1 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	2	2	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}^{-1}$$

2.10 problem problem 10

Internal problem ID [10305]
Internal file name [OUTPUT/9252_Monday_June_06_2022_01_45_04_PM_54819035/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 10.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 4\lambda + 4 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$
\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
2	2	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{-1}$$

2.11 problem problem 11

Internal problem ID [10306]
Internal file name [`OUTPUT/9253_Monday_June_06_2022_01_45_04_PM_89645115/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 11.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 5 & 1 \\ -9 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 5 & 1 \\ -9 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 5-\lambda & 1 \\ -9 & -1-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 4\lambda + 4 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 5 & 1 \\ -9 & -1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 5 & 1 \\ -9 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 3 & 1 \\ -9 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 3 & 1 & 0 \\ -9 & -3 & 0 \end{array} \right]
$$

$$
R_2 = R_2 + 3R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{3}\right\}$

Hence the solution is

$$
\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
2	2	2	No	$\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 5 & 1 \\ -9 & -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}^{-1}$$

2.12 problem problem 12

Internal problem ID [10307]
Internal file name [OUTPUT/9254_Monday_June_06_2022_01_45_05_PM_89895560/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 12.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 11 & 9 \\ -16 & -13 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 11 & 9 \\ -16 & -13 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 11-\lambda & 9 \\ -16 & -13-\lambda \end{bmatrix} = 0$$
$$\lambda^2 + 2\lambda + 1 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = -1$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 11 & 9 \\ -16 & -13 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 11 & 9 \\ -16 & -13 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 12 & 9 \\ -16 & -12 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 12 & 9 & 0 \\ -16 & -12 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + \frac{4R_1}{3} \implies \left[\begin{array}{cc|c} 12 & 9 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 12 & 9 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3t}{4}\right\}$

Hence the solution is

$$
\begin{bmatrix} -\frac{3t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3t}{4} \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} -\frac{3t}{4} \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{3t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{3t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-1	2	2	No	$\begin{bmatrix} -3 \\ 4 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 11 & 9 \\ -16 & -13 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix}^{-1}$$

2.13 problem problem 13

Internal problem ID [10308]
Internal file name [OUTPUT/9255_Monday_June_06_2022_01_45_05_PM_32218091/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 13.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 3 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$-(-1+\lambda)(-2+\lambda)^2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \implies \left[\begin{array}{ccc|c} 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} -1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$
\begin{bmatrix} 3t \\ t \\ s \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ s \end{bmatrix}
$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned}\begin{bmatrix}3t\\t\\s\end{bmatrix} &= \begin{bmatrix}3t\\t\\0\end{bmatrix}+\begin{bmatrix}0\\0\\s\end{bmatrix}\\ &= t\begin{bmatrix}3\\1\\0\end{bmatrix}+s\begin{bmatrix}0\\0\\1\end{bmatrix}\end{aligned}$$

By letting $t=1$ and $s=1$ then the above becomes

$$\begin{bmatrix}3t\\t\\s\end{bmatrix} = \begin{bmatrix}3\\1\\0\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix}3\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	3	No	$\begin{bmatrix}1\\0\\0\end{bmatrix}$
2	2	3	No	$\begin{bmatrix}3\\1\\0\end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

2.14 problem problem 14

Internal problem ID [10309]
Internal file name [`OUTPUT/9256_Monday_June_06_2022_01_45_06_PM_9268676/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 14.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 2-\lambda & -2 & 1 \\ 2 & -2-\lambda & 1 \\ 2 & -2 & 1-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + \lambda^2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	2	real eigenvalue
1	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 0$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t - \frac{s}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} t - \frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} t - \frac{s}{2} \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} t - \frac{s}{2} \\ t \\ s \end{bmatrix} &= \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{s}{2} \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t - \frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right)$$

Which can be normalized to

$$\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}\right)$$

Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -3 & 1 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
0	2	3	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
1	1	3	No	$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}^{-1}$$

2.15 problem problem 15

Internal problem ID [10310]
Internal file name [`OUTPUT/9257_Monday_June_06_2022_01_45_07_PM_16949892/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 15.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 3-\lambda & -3 & 1 \\ 2 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 2\lambda^2 - \lambda = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
1	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 0$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & -3 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \implies \left[\begin{array}{ccc|c} 3 & -3 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_2 \implies \left[\begin{array}{ccc|c} 3 & -3 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -3 & 1 \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2} - \frac{s}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2}-\frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{3t}{2}-\frac{s}{2} \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2}-\frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{s}{2} \\ 0 \\ s \end{bmatrix}$$
$$= t\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} \frac{3t}{2}-\frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right)$$

Which can be normalized to

$$\left(\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right)$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
0	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
1	2	3	No	$\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{-1}$$

2.16 problem problem 16

Internal problem ID [10311]
Internal file name [`OUTPUT/9258_Monday_June_06_2022_01_45_08_PM_32739529/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 16.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ -4 & 4 & 1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ -4 & 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 3-\lambda & -2 & 0 \\ 0 & 1-\lambda & 0 \\ -4 & 4 & 1-\lambda \end{bmatrix} = 0$$

$$-(-3+\lambda)(-1+\lambda)^2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	2	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ -4 & 4 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ -4 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 4 & 0 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ s \end{bmatrix} = \begin{bmatrix} t \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} t \\ t \\ s \end{bmatrix} &= \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ -4 & 4 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ -4 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -2 & 0 \\ 0 & -2 & 0 \\ -4 & 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -4 & 4 & -2 & 0 \end{array} \right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 3 gives

$$\left[\begin{array}{ccc|c} -4 & 4 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array} \right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 4 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 4 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}, v_2 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	2	3	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
3	1	3	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ -4 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}^{-1}$$

2.17 problem problem 17

Internal problem ID [10312]
Internal file name [OUTPUT/9259_Monday_June_06_2022_01_45_08_PM_38524352/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 17.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 7 & -8 & 3 \\ 6 & -7 & 3 \\ 2 & -2 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 7 & -8 & 3 \\ 6 & -7 & 3 \\ 2 & -2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 7-\lambda & -8 & 3 \\ 6 & -7-\lambda & 3 \\ 2 & -2 & 2-\lambda \end{bmatrix} = 0$$
$$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 7 & -8 & 3 \\ 6 & -7 & 3 \\ 2 & -2 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 7 & -8 & 3 \\ 6 & -7 & 3 \\ 2 & -2 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 8 & -8 & 3 \\ 6 & -6 & 3 \\ 2 & -2 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 8 & -8 & 3 & 0 \\ 6 & -6 & 3 & 0 \\ 2 & -2 & 3 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{3R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} 8 & -8 & 3 & 0 \\ 0 & 0 & \frac{3}{4} & 0 \\ 2 & -2 & 3 & 0 \end{array}\right]
$$

$$R_3 = R_3 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} 8 & -8 & 3 & 0 \\ 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{9}{4} & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} 8 & -8 & 3 & 0 \\ 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 8 & -8 & 3 \\ 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0}
\end{aligned}
$$

$$
\left(\begin{bmatrix} 7 & -8 & 3 \\ 6 & -7 & 3 \\ 2 & -2 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\left(\begin{bmatrix} 7 & -8 & 3 \\ 6 & -7 & 3 \\ 2 & -2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 6 & -8 & 3 \\ 6 & -8 & 3 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 6 & -8 & 3 & 0 \\ 6 & -8 & 3 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 6 & -8 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 6 & -8 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \end{array}\right]
$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$
\left[\begin{array}{ccc|c} 6 & -8 & 3 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -8 & 3 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}, v_2 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 7 & -8 & 3 \\ 6 & -7 & 3 \\ 2 & -2 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 7 & -8 & 3 \\ 6 & -7 & 3 \\ 2 & -2 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 5 & -8 & 3 \\ 6 & -9 & 3 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 5 & -8 & 3 & 0 \\ 6 & -9 & 3 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{6R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & -8 & 3 & 0 \\ 0 & \frac{3}{5} & -\frac{3}{5} & 0 \\ 2 & -2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & -8 & 3 & 0 \\ 0 & \frac{3}{5} & -\frac{3}{5} & 0 \\ 0 & \frac{6}{5} & -\frac{6}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 5 & -8 & 3 & 0 \\ 0 & \frac{3}{5} & -\frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & -8 & 3 \\ 0 & \frac{3}{5} & -\frac{3}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-1	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
1	1	3	No	$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 7 & -8 & 3 \\ 6 & -7 & 3 \\ 2 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}^{-1}$$

2.18 problem problem 18

Internal problem ID [10313]
Internal file name [OUTPUT/9260_Monday_June_06_2022_01_45_09_PM_79968995/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 18.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 6-\lambda & -5 & 2 \\ 4 & -3-\lambda & 2 \\ 2 & -2 & 3-\lambda \end{bmatrix} = 0$$
$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 5 & -5 & 2 \\ 4 & -4 & 2 \\ 2 & -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 5 & -5 & 2 & 0 \\ 4 & -4 & 2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{4R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & -5 & 2 & 0 \\ 0 & 0 & \frac{2}{5} & 0 \\ 2 & -2 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{2R_1}{5} \implies \left[\begin{array}{ccc|c} 5 & -5 & 2 & 0 \\ 0 & 0 & \frac{2}{5} & 0 \\ 0 & 0 & \frac{6}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_2 \implies \left[\begin{array}{ccc|c} 5 & -5 & 2 & 0 \\ 0 & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & -5 & 2 \\ 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 4 & -5 & 2 \\ 4 & -5 & 2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & -5 & 2 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4 & -5 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & -5 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 4 & -5 & 2 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -5 & 2 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}, v_2 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$A\boldsymbol{v} = \lambda \boldsymbol{v}$$
$$A\boldsymbol{v} - \lambda \boldsymbol{v} = \mathbf{0}$$
$$(A - \lambda I)\boldsymbol{v} = \mathbf{0}$$

$$\left(\begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -5 & 2 \\ 4 & -6 & 2 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & -5 & 2 & 0 \\ 4 & -6 & 2 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{4R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 3 & -5 & 2 & 0 \\ 0 & \frac{2}{3} & -\frac{2}{3} & 0 \\ 2 & -2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 3 & -5 & 2 & 0 \\ 0 & \frac{2}{3} & -\frac{2}{3} & 0 \\ 0 & \frac{4}{3} & -\frac{4}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 3 & -5 & 2 & 0 \\ 0 & \frac{2}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -5 & 2 \\ 0 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$
3	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}^{-1}$$

2.19 problem problem 19

Internal problem ID [10314]
Internal file name [OUTPUT/9261_Monday_June_06_2022_01_45_10_PM_43653147/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 19.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 1 & -1 \\ -2 & 4-\lambda & -1 \\ -4 & 4 & 1-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 1 & -1 \\ -2 & 3 & -1 \\ -4 & 4 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ -2 & 3 & -1 & 0 \\ -4 & 4 & 0 & 0 \end{array}\right]
$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$
\left[\begin{array}{ccc|c} -2 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -4 & 4 & 0 & 0 \end{array}\right]
$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 \\ -2 & 2 & -1 \\ -4 & 4 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ -2 & 2 & -1 & 0 \\ -4 & 4 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & 4 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -2 & 1 & -1 \\ -2 & 1 & -1 \\ -4 & 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ -2 & 1 & -1 & 0 \\ -4 & 4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 4 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}, v_2 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
3	1	3	No	$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}^{-1}$$

2.20 problem problem 20

Internal problem ID [10315]
Internal file name [OUTPUT/9262_Monday_June_06_2022_01_45_11_PM_53053529/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 20.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ -6 & 11 & 2 \\ 6 & -15 & 0 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 2 & 0 & 0 \\ -6 & 11 & 2 \\ 6 & -15 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 2-\lambda & 0 & 0 \\ -6 & 11-\lambda & 2 \\ 6 & -15 & -\lambda \end{bmatrix} = 0$$

$$-(-2+\lambda)\left(\lambda^2 - 11\lambda + 30\right) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 6$$
$$\lambda_3 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
5	1	real eigenvalue
6	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 2 & 0 & 0 \\ -6 & 11 & 2 \\ 6 & -15 & 0 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & 0 & 0 \\ -6 & 11 & 2 \\ 6 & -15 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -6 & 9 & 2 \\ 6 & -15 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -6 & 9 & 2 & 0 \\ 6 & -15 & -2 & 0 \end{array} \right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} -6 & 9 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & -15 & -2 & 0 \end{array} \right]$$

$$R_3 = R_3 + R_1 \implies \left[\begin{array}{ccc|c} -6 & 9 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -6 & 9 & 2 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6 & 9 & 2 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}, v_2 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{t}{3} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{t}{3} \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Considering $\lambda = 5$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 2 & 0 & 0 \\ -6 & 11 & 2 \\ 6 & -15 & 0 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & 0 & 0 \\ -6 & 11 & 2 \\ 6 & -15 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & 0 \\ -6 & 6 & 2 \\ 6 & -15 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ -6 & 6 & 2 & 0 \\ 6 & -15 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ 0 & 6 & 2 & 0 \\ 6 & -15 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & -15 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{5R_2}{2} \implies \left[\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = 0, v_2 = -\frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

Considering $\lambda = 6$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \end{aligned}$$

$$\left(\begin{bmatrix} 2 & 0 & 0 \\ -6 & 11 & 2 \\ 6 & -15 & 0 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & 0 & 0 \\ -6 & 11 & 2 \\ 6 & -15 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 0 & 0 \\ -6 & 5 & 2 \\ 6 & -15 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ -6 & 5 & 2 & 0 \\ 6 & -15 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 6 & -15 & -6 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & -15 & -6 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = 0, v_2 = -\frac{2t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -\frac{2}{5} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{5} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} 0 \\ -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
2	1	3	No	$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$
5	1	3	No	$\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$
6	1	3	No	$\begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 3 & 3 & 5 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 2 & 0 & 0 \\ -6 & 11 & 2 \\ 6 & -15 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 3 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 3 & 3 & 5 \end{bmatrix}^{-1}$$

2.21 problem problem 21

Internal problem ID [10316]
Internal file name [`OUTPUT/9263_Monday_June_06_2022_01_45_12_PM_89009704/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 21.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} -\lambda & 1 & 0 \\ -1 & 2-\lambda & 0 \\ -1 & 1 & 1-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	3	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ s \end{bmatrix} = \begin{bmatrix} t \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} t \\ t \\ s \end{bmatrix} &= \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	3	3	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

2.22 problem problem 22

Internal problem ID [10317]
Internal file name [`OUTPUT/9264_Monday_June_06_2022_01_45_13_PM_14505739/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 22.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 2 & -2 & 1 \\ -1 & 2 & 0 \\ -5 & 7 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 2 & -2 & 1 \\ -1 & 2 & 0 \\ -5 & 7 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det \begin{bmatrix} 2-\lambda & -2 & 1 \\ -1 & 2-\lambda & 0 \\ -5 & 7 & -1-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	3	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0}
\end{aligned}
$$

$$\left(\begin{bmatrix} 2 & -2 & 1 \\ -1 & 2 & 0 \\ -5 & 7 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & -2 & 1 \\ -1 & 2 & 0 \\ -5 & 7 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ -5 & 7 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -5 & 7 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -5 & 7 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 5R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	3	3	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 2 & -2 & 1 \\ -1 & 2 & 0 \\ -5 & 7 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^{-1}$$

2.23 problem problem 23

Internal problem ID [10318]
Internal file name [`OUTPUT/9265_Monday_June_06_2022_01_45_13_PM_1205736/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 23.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} -2 & 4 & -1 \\ -3 & 5 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -2 & 4 & -1 \\ -3 & 5 & -1 \\ -1 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} -2-\lambda & 4 & -1 \\ -3 & 5-\lambda & -1 \\ -1 & 1 & 1-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	2	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} -2 & 4 & -1 \\ -3 & 5 & -1 \\ -1 & 1 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} -2 & 4 & -1 \\ -3 & 5 & -1 \\ -1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -3 & 4 & -1 \\ -3 & 4 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} -3 & 4 & -1 & 0 \\ -3 & 4 & -1 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -3 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -3 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \end{array}\right]
$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -3 & 4 & -1 & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 4 & -1 \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} -2 & 4 & -1 \\ -3 & 5 & -1 \\ -1 & 1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} -2 & 4 & -1 \\ -3 & 5 & -1 \\ -1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -4 & 4 & -1 \\ -3 & 3 & -1 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} -4 & 4 & -1 & 0 \\ -3 & 3 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{3R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & 4 & -1 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ -1 & 1 & -1 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & 4 & -1 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{3}{4} & 0 \end{array}\right]
$$

$$
R_3 = R_3 - 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 4 & -1 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 4 & -1 \\ 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	2	3	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} -2 & 4 & -1 \\ -3 & 5 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$

2.24 problem problem 24

Internal problem ID [10319]
Internal file name [OUTPUT/9266_Monday_June_06_2022_01_45_14_PM_99820673/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 24.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 3-\lambda & -2 & 1 \\ 1 & -\lambda & 1 \\ -1 & 1 & 2-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -1 & 1 & 1 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{2} & 0 \end{array}\right]
$$

$$R_3 = R_3 - 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}A\boldsymbol{v} &= \lambda\boldsymbol{v}\\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0}\\ (A-\lambda I)\boldsymbol{v} &= \boldsymbol{0}\end{aligned}$$

$$\left(\begin{bmatrix}3 & -2 & 1\\ 1 & 0 & 1\\ -1 & 1 & 2\end{bmatrix} - (2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\left(\begin{bmatrix}3 & -2 & 1\\ 1 & 0 & 1\\ -1 & 1 & 2\end{bmatrix} - \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & 1\\ 1 & -2 & 1\\ -1 & 1 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c}1 & -2 & 1 & 0\\ 1 & -2 & 1 & 0\\ -1 & 1 & 0 & 0\end{array}\right]$$

$$R_2 = R_2 - R_1 \implies \left[\begin{array}{ccc|c}1 & -2 & 1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\end{array}\right]$$

$$R_3 = R_3 + R_1 \implies \left[\begin{array}{ccc|c}1 & -2 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0\end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c}1 & -2 & 1 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
2	2	3	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

2.25 problem problem 25

Internal problem ID [10320]
Internal file name [OUTPUT/9267_Monday_June_06_2022_01_45_15_PM_97416748/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 25.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 0 & -2 & 0 \\ 0 & 1-\lambda & -2 & 0 \\ 0 & 0 & -1-\lambda & 0 \\ 0 & 0 & 0 & -1-\lambda \end{bmatrix} = 0$$

$$-(1-\lambda)(-1+\lambda)(1+\lambda)^2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = -1$$
$$\lambda_2 = -1$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	2	real eigenvalue
1	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} 2 & 0 & -2 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3, v_4\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Let $v_4 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \\ s \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} t \\ t \\ t \\ s \end{bmatrix} &= \begin{bmatrix} t \\ t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t \\ t \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix}1\\1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)$$

Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v}\\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0}\\
(A-\lambda I)\boldsymbol{v} &= \mathbf{0}\\
\left(\begin{bmatrix}1&0&-2&0\\0&1&-2&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix} - (1)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} &= \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\\
\left(\begin{bmatrix}1&0&-2&0\\0&1&-2&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix} - \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} &= \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\\
\begin{bmatrix}0&0&-2&0\\0&0&-2&0\\0&0&-2&0\\0&0&0&-2\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} &= \begin{bmatrix}0\\0\\0\\0\end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c}0&0&-2&0&0\\0&0&-2&0&0\\0&0&-2&0&0\\0&0&0&-2&0\end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(2,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 4 gives

$$\left[\begin{array}{cccc|c} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1, v_2\}$ and the leading variables are $\{v_3, v_4\}$. Let $v_1 = t$. Let $v_2 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ s \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ s \\ 0 \\ 0 \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ s \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ 0 \\ 0 \end{bmatrix}$$

$$= t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t \\ s \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-1	2	4	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$
1	2	4	No	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1}$$

2.26 problem problem 26

Internal problem ID [10321]
Internal file name [`OUTPUT/9268_Monday_June_06_2022_01_45_16_PM_78720941/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 26.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 1 \\ 0 & 0 & 1-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$-(1-\lambda)(-1+\lambda)^2(-2+\lambda) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	3	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]
$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_4 = R_4 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1, v_2, v_3\}$ and the leading variables are $\{v_4\}$. Let $v_1 = t$. Let $v_2 = s$. Let $v_3 = r$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ s \\ r \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ s \\ r \\ 0 \end{bmatrix}$$

Since there are three free Variable, we have found three eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ s \\ r \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ 0 \\ 0 \end{bmatrix}$$

$$= t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ and $r = 1$ then the above becomes

$$\begin{bmatrix} t \\ s \\ r \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Hence the three eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0}
\end{aligned}
$$

$$
\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t, v_3 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	3	4	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
2	1	4	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

2.27 problem problem 27

Internal problem ID [10322]
Internal file name [OUTPUT/9269_Monday_June_06_2022_01_45_16_PM_44340434/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 27.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & 1-\lambda & 1 & 1 \\ 0 & 0 & 1-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$-(1-\lambda)(-1+\lambda)^2(-2+\lambda) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	3	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]
$$

$$R_4 = R_4 - R_3 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3, v_4\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = 2t, v_3 = t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ 2t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ 2t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ 2t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2t \\ 2t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	3	4	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
2	1	4	No	$\begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

2.28 problem problem 28

Internal problem ID [10323]
Internal file name [OUTPUT/9270_Monday_June_06_2022_01_45_17_PM_85864164/index.tex]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From Differential equations and linear algebra, 4th ed., Edwards and Penney. Section 6.2, Diagonalization of Matrices, Eigenvalues and Eigenvectors. Page 354
Problem number: problem 28.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 1 & 0 & 1 \\ 0 & 1-\lambda & 1 & 1 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$-(1-\lambda)(-1+\lambda)(-2+\lambda)^2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 2$$
$$\lambda_4 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	2	real eigenvalue
2	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(3,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3, v_4\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2, v_4\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	2	4	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
2	2	4	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^{-1}$$

3 From DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition. CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346

3.1 problem 31

Internal problem ID [10324]
Internal file name [`OUTPUT/9271_Monday_June_06_2022_01_45_18_PM_65125154/index.tex`]

Book: Collection of Eigenvalues and Eigenvectors problems
Section: From DIFFERENTIAL EQUATIONS with Boundary Value Problems. DENNIS G. ZILL, WARREN S. WRIGHT, MICHAEL R. CULLEN. Brooks/Cole. Boston, MA. 2013. 8th edition. CHAPTER 8 SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS. EXERCISES 8.2. Page 346
Problem number: 31.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 2-\lambda & 1 & 0 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$-(2-\lambda)^2 (-2+\lambda)^3 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$
$$\lambda_4 = 2$$
$$\lambda_5 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	5	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(2,5)$ is zero, then the current pivot row is replaced with a

row with a non-zero pivot. Swapping row 2 and row 4 gives

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1, v_3, v_4\}$ and the leading variables are $\{v_2, v_5\}$. Let $v_1 = t$. Let $v_3 = s$. Let $v_4 = r$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_5 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ s \\ r \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ s \\ r \\ 0 \end{bmatrix}$$

Since there are three free Variable, we have found three eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ s \\ r \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \\ 0 \\ 0 \end{bmatrix}$$

$$= t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ and $r = 1$ then the above becomes

$$\begin{bmatrix} t \\ 0 \\ s \\ r \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Hence the three eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
2	5	5	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^{-1}$$