A Solution Manual For

An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906

AN ELEMENTARY TREATISE

ON

DIFFERENTIAL EQUATIONS

BY

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1.1 problem Ex 1

Internal problem ID [11122]
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Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 8. Exact differential equations. Page 11
Problem number: Ex 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{\frac{2yx+1}{y}+\frac{(y-x)\,y'}{y^2}=0}$$

1.1.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\frac{2u(x)\,x^2+1}{u\,(x)\,x}+\frac{(u(x)\,x-x)\,(u'(x)\,x+u(x))}{u\,(x)^2\,x^2}=0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2(2x^2+1)}{x\,(u-1)} \end{aligned}$$

Where $f(x) = -\frac{2x^2+1}{x}$ and $g(u) = \frac{u^2}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{u-1}} du = -\frac{2x^2+1}{x} dx$$

$$\int \frac{1}{\frac{u^2}{u-1}} du = \int -\frac{2x^2+1}{x} dx$$

$$\frac{1}{u} + \ln(u) = -x^2 - \ln(x) + c_2$$

The solution is

$$\frac{1}{u(x)} + \ln(u(x)) + x^2 + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{x}{y} + \ln\left(\frac{y}{x}\right) + x^2 + \ln(x) - c_2 = 0$$

$$\frac{x}{y} + \ln\left(\frac{y}{x}\right) + x^2 + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{x}{y} + \ln\left(\frac{y}{x}\right) + x^2 + \ln(x) - c_2 = 0 \tag{1}$$

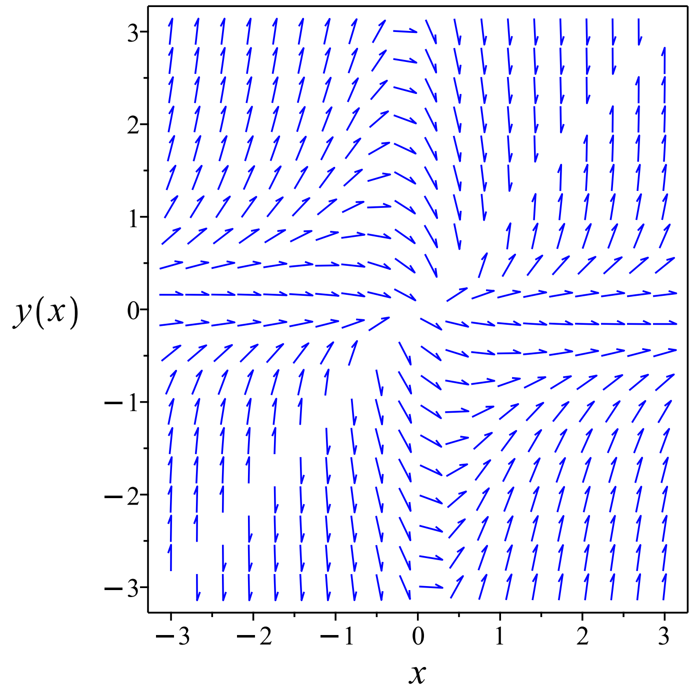


Figure 1: Slope field plot

<u>Verification of solutions</u>

$$\frac{x}{y}+\ln\left(\frac{y}{x}\right)+x^{2}+\ln\left(x\right)-c_{2}=0$$

Verified OK.

1.1.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{y-x}{y^2}\right)\mathrm{d}y = \left(-\frac{2xy+1}{y}\right)\mathrm{d}x$$
$$\left(\frac{2xy+1}{y}\right)\mathrm{d}x + \left(\frac{y-x}{y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{2xy+1}{y}$$
$$N(x,y) = \frac{y-x}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{2xy+1}{y}\right)$$
$$= -\frac{1}{y^2}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y-x}{y^2}\right)\\
&= -\frac{1}{y^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2xy+1}{y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x(xy+1)}{y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x^2}{y} - \frac{x(xy+1)}{y^2} + f'(y) \tag{4}
$$

$$
= -\frac{x}{y^2} + f'(y)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y-x}{y^2}$. Therefore equation (4) becomes

$$
\frac{y-x}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$

$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(xy+1)}{y} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(xy+1)}{y} + \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-x^2 + \mathrm{LambertW}\left(-x\,\mathrm{e}^{x^2-c_1}\right) + c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x^2 + \mathrm{LambertW}\left(-x\,\mathrm{e}^{x^2-c_1}\right) + c_1} \tag{1}$$

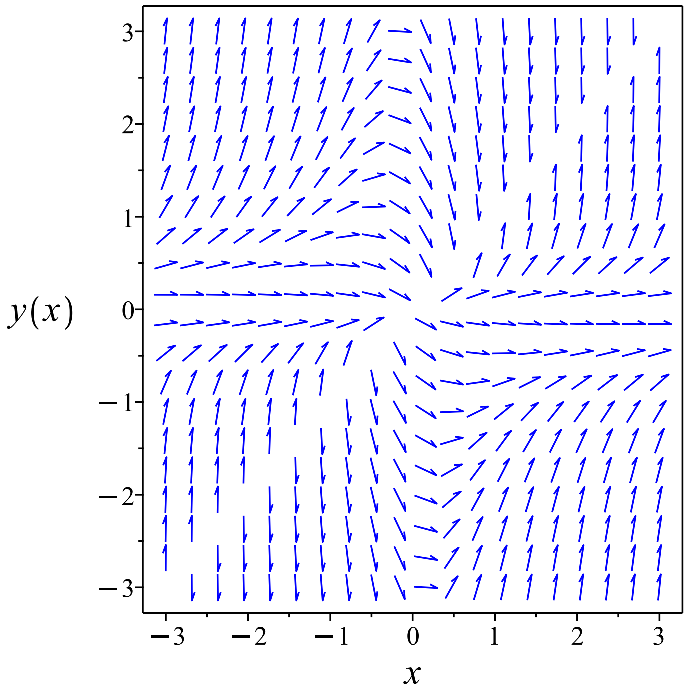


Figure 2: Slope field plot

Verification of solutions

$$y = e^{-x^2 + \text{LambertW}\left(-x\,e^{x^2 - c_1}\right) + c_1}$$

Verified OK.

1.1.3 Maple step by step solution

Let's solve

$\frac{2yx+1}{y} + \frac{(y-x)y'}{y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ∘ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

- ∘ Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $\frac{2x}{y} - \frac{2xy+1}{y^2} = -\frac{1}{y^2}$

- Simplify

  $-\frac{1}{y^2} = -\frac{1}{y^2}$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int \frac{2xy+1}{y}dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \frac{x^2y+x}{y} + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $\frac{y-x}{y^2} = -\frac{x^2y+x}{y^2} + \frac{x^2}{y} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = \frac{y-x}{y^2} + \frac{x^2y+x}{y^2} - \frac{x^2}{y}$

- Solve for $f_1(y)$

  $f_1(y) = \ln(y)$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = \frac{x^2y+x}{y} + \ln(y)$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{x^2y+x}{y} + \ln(y) = c_1$

- Solve for y

  $y = \mathrm{e}^{-x^2 + \mathit{LambertW}\left(-x\,\mathrm{e}^{x^2-c_1}\right)+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 18

```
dsolve((2*x*y(x)+1)/y(x)+ (y(x)-x)/y(x)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{\text{LambertW}\left(-\mathrm{e}^{x^2} c_1 x\right)}$$

✓ Solution by Mathematica

Time used: 7.151 (sec). Leaf size: 29

```
DSolve[(2*x*y[x]+1)/y[x]+ (y[x]-x)/y[x]^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{W\left(x\left(-e^{x^2-c_1}\right)\right)}$$
$$y(x) \to 0$$

1.2 problem Ex 2

Internal problem ID [11123]
Internal file name [`OUTPUT/10108_Wednesday_November_23_2022_11_50_42_AM_71634107/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 8. Exact differential equations. Page 11
Problem number: Ex 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{\frac{y^2-2x^2}{y^2x-x^3}+\frac{(2y^2-x^2)\,y'}{y^3-x^2y}=0}$$

1.2.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\frac{u(x)^2\,x^2-2x^2}{u\,(x)^2\,x^3-x^3}+\frac{\left(2u(x)^2\,x^2-x^2\right)\left(u'(x)\,x+u(x)\right)}{u\,(x)^3\,x^3-x^3u\,(x)}=0$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=-\frac{3(u^3-u)}{x\,(2u^2-1)}\end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = \frac{u^3-u}{2u^2-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^3-u}{2u^2-1}}\,du = -\frac{3}{x}\,dx$$

$$\int \frac{1}{\frac{u^3-u}{2u^2-1}}\,du = \int -\frac{3}{x}\,dx$$

$$\frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} + \ln(u) = -3\ln(x) + c_2$$

Raising both side to exponential gives

$$\mathrm{e}^{\frac{\ln(u-1)}{2}+\frac{\ln(u+1)}{2}+\ln(u)} = \mathrm{e}^{-3\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u-1}\,\sqrt{u+1}\,u = \frac{c_3}{x^3}$$

The solution is

$$\sqrt{u(x)-1}\,\sqrt{u(x)+1}\,u(x) = \frac{c_3}{x^3}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\sqrt{\frac{y}{x}-1}\,\sqrt{\frac{y}{x}+1}\,y}{x} = \frac{c_3}{x^3}$$

$$\frac{\sqrt{\frac{y-x}{x}}\,\sqrt{\frac{y+x}{x}}\,y}{x} = \frac{c_3}{x^3}$$

Which simplifies to

$$\sqrt{\frac{y-x}{x}}\,\sqrt{\frac{y+x}{x}}\,y = \frac{c_3}{x^2}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{y-x}{x}}\,\sqrt{\frac{y+x}{x}}\,y = \frac{c_3}{x^2} \tag{1}$$

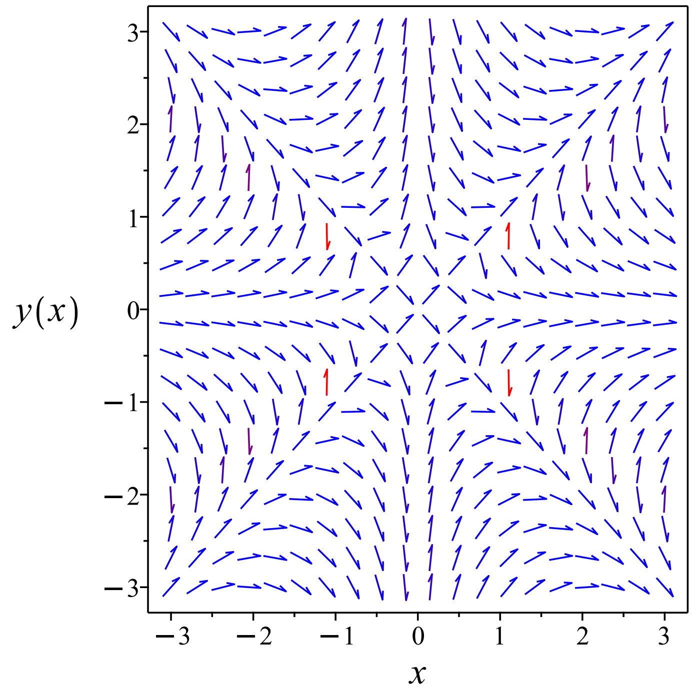


Figure 3: Slope field plot

Verification of solutions

$$\sqrt{\frac{y-x}{x}}\,\sqrt{\frac{y+x}{x}}\,y = \frac{c_3}{x^2}$$

Verified OK.

1.2.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(-2x^2+y^2)}{x\,(-x^2+2y^2)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{y(-2x^2+y^2)(b_3-a_2)}{x(-x^2+2y^2)} - \frac{y^2(-2x^2+y^2)^2 a_3}{x^2(-x^2+2y^2)^2} \\
&- \left(\frac{4y}{-x^2+2y^2} + \frac{y(-2x^2+y^2)}{x^2(-x^2+2y^2)} - \frac{2y(-2x^2+y^2)}{(-x^2+2y^2)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{-2x^2+y^2}{x(-x^2+2y^2)} - \frac{2y^2}{x(-x^2+2y^2)}\right. \\
&\left.+ \frac{4y^2(-2x^2+y^2)}{x(-x^2+2y^2)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^6b_2 - 6x^4y^2a_3 - 3x^4y^2b_2 - 6x^3y^3a_2 + 6x^3y^3b_3 + 3x^2y^4a_3 + 6x^2y^4b_2 - 3y^6a_3 + 2x^5b_1 - 2x^4ya_1 + x^3y^2b_1}{(x^2-2y^2)^2 x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&3x^6b_2 - 6x^4y^2a_3 - 3x^4y^2b_2 - 6x^3y^3a_2 + 6x^3y^3b_3 + 3x^2y^4a_3 + 6x^2y^4b_2 \\
&- 3y^6a_3 + 2x^5b_1 - 2x^4ya_1 + x^3y^2b_1 - x^2y^3a_1 + 2x\,y^4b_1 - 2y^5a_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-6a_2v_1^3v_2^3 - 6a_3v_1^4v_2^2 + 3a_3v_1^2v_2^4 - 3a_3v_2^6 + 3b_2v_1^6 - 3b_2v_1^4v_2^2 + 6b_2v_1^2v_2^4 \\
&+ 6b_3v_1^3v_2^3 - 2a_1v_1^4v_2 - a_1v_1^2v_2^3 - 2a_1v_2^5 + 2b_1v_1^5 + b_1v_1^3v_2^2 + 2b_1v_1v_2^4 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} & 3b_2v_1^6 + 2b_1v_1^5 + (-6a_3 - 3b_2)\, v_1^4v_2^2 - 2a_1v_1^4v_2 + (-6a_2 + 6b_3)\, v_1^3v_2^3 \\ & + b_1v_1^3v_2^2 + (3a_3 + 6b_2)\, v_1^2v_2^4 - a_1v_1^2v_2^3 + 2b_1v_1v_2^4 - 3a_3v_2^6 - 2a_1v_2^5 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -2a_1 &= 0 \\ -a_1 &= 0 \\ -3a_3 &= 0 \\ 2b_1 &= 0 \\ 3b_2 &= 0 \\ -6a_2 + 6b_3 &= 0 \\ -6a_3 - 3b_2 &= 0 \\ 3a_3 + 6b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{y(-2x^2+y^2)}{x\,(-x^2+2y^2)}\right)(x) \\ &= \frac{3x^2y-3y^3}{x^2-2y^2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{3x^2y-3y^3}{x^2-2y^2}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(y-x)}{6} + \frac{\ln(y)}{3} + \frac{\ln(y+x)}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(-2x^2+y^2)}{x\,(-x^2+2y^2)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x}{3x^2 - 3y^2} \\
S_y &= \frac{x^2 - 2y^2}{3x^2y - 3y^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{3x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln (R)}{3} + c_1 \tag{4}$$

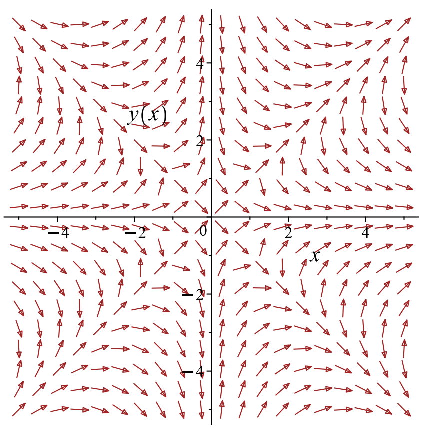
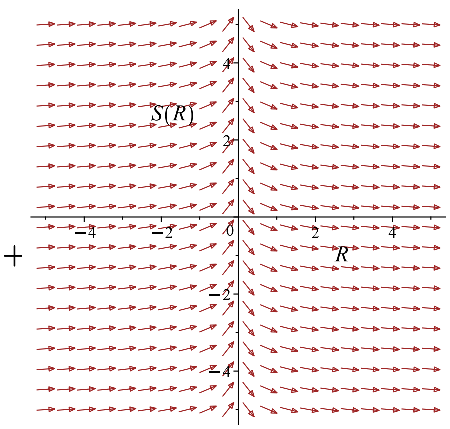
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln (y - x)}{6} + \frac{\ln (y)}{3} + \frac{\ln (y + x)}{6} = -\frac{\ln (x)}{3} + c_1$$

Which simplifies to

$$\frac{\ln (y - x)}{6} + \frac{\ln (y)}{3} + \frac{\ln (y + x)}{6} = -\frac{\ln (x)}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-2x^2+y^2)}{x(-x^2+2y^2)}$ 	$R = x$ $S = \frac{\ln(y-x)}{6} + \frac{\ln(y)}{3} +$	$\frac{dS}{dR} = -\frac{1}{3R}$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y-x)}{6} + \frac{\ln(y)}{3} + \frac{\ln(y+x)}{6} = -\frac{\ln(x)}{3} + c_1 \tag{1}$$

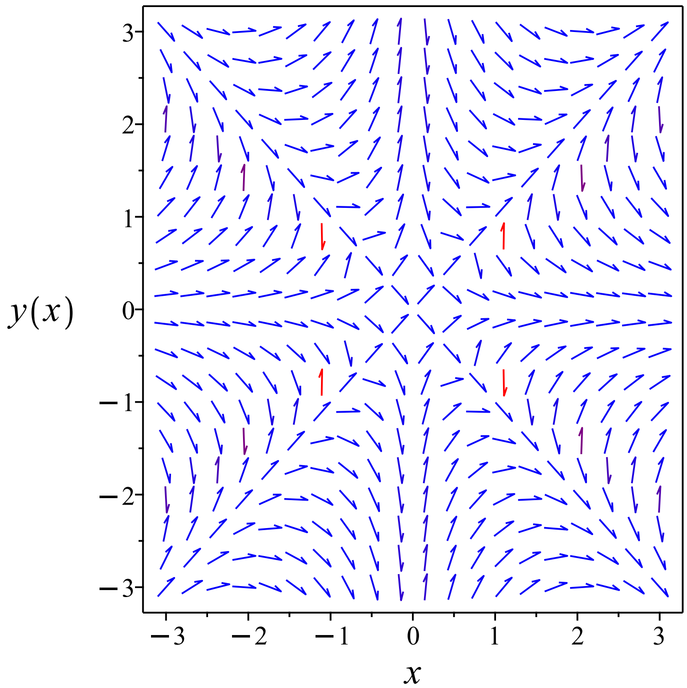


Figure 4: Slope field plot

Verification of solutions

$$\frac{\ln(y-x)}{6}+\frac{\ln(y)}{3}+\frac{\ln(y+x)}{6}=-\frac{\ln(x)}{3}+c_1$$

Verified OK.

1.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-x^2+2y^2}{-x^2 y+y^3}\right) \mathrm{d}y = \left(-\frac{-2x^2+y^2}{-x^3+x\,y^2}\right) \mathrm{d}x$$
$$\left(\frac{-2x^2+y^2}{-x^3+x\,y^2}\right) \mathrm{d}x + \left(\frac{-x^2+2y^2}{-x^2 y+y^3}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{-2x^2+y^2}{-x^3+x\,y^2}$$
$$N(x,y) = \frac{-x^2+2y^2}{-x^2 y+y^3}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-2x^2+y^2}{-x^3+x\,y^2}\right) \\ &= \frac{2yx}{(x^2-y^2)^2}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x^2+2y^2}{-x^2y+y^3}\right) \\
&= \frac{2yx}{\left(x^2-y^2\right)^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{-2x^2+y^2}{-x^3+x\,y^2} \mathrm{d}x \\
\phi &= \frac{\ln(y+x)}{2} + \ln(x) + \frac{\ln(-y+x)}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{1}{2y+2x} - \frac{1}{2(-y+x)} + f'(y) \qquad (4) \\
&= -\frac{y}{x^2-y^2} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x^2+2y^2}{-x^2y+y^3}$. Therefore equation (4) becomes

$$
\frac{-x^2+2y^2}{-x^2y+y^3} = -\frac{y}{x^2-y^2} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right)\mathrm{d}y$$

$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(y+x)}{2} + \ln(x) + \frac{\ln(-y+x)}{2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(y+x)}{2} + \ln(x) + \frac{\ln(-y+x)}{2} + \ln(y)$$

Summary
The solution(s) found are the following

$$\frac{\ln(y+x)}{2} + \ln(x) + \frac{\ln(-y+x)}{2} + \ln(y) = c_1 \tag{1}$$

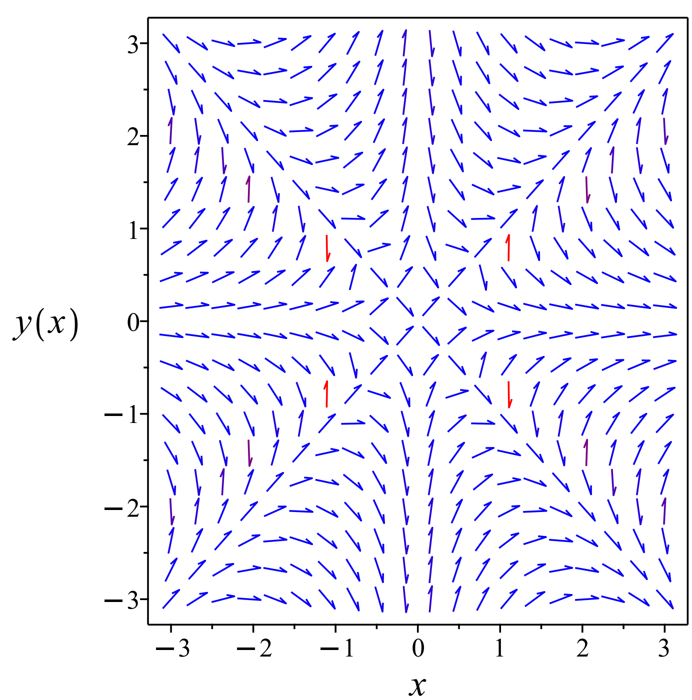


Figure 5: Slope field plot

Verification of solutions

$$\frac{\ln(y+x)}{2}+\ln(x)+\frac{\ln(-y+x)}{2}+\ln(y)=c_1$$

Verified OK.

1.2.4 Maple step by step solution

Let's solve

$\frac{y^2-2x^2}{y^2x-x^3}+\frac{(2y^2-x^2)y'}{y^3-x^2y}=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $\frac{2y}{-x^3+x\,y^2}-\frac{2(-2x^2+y^2)xy}{(-x^3+x\,y^2)^2}=-\frac{2x}{-x^2y+y^3}+\frac{2(-x^2+2y^2)xy}{(-x^2y+y^3)^2}$

  - Simplify

    $\frac{2yx}{(x^2-y^2)^2}=\frac{2yx}{(x^2-y^2)^2}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\frac{-2x^2+y^2}{-x^3+x\,y^2}dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{\ln(y+x)}{2}+\ln(x)+\frac{\ln(-y+x)}{2}+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

$$\frac{-x^2+2y^2}{-x^2y+y^3} = \frac{1}{2(y+x)} - \frac{1}{2(-y+x)} + \frac{d}{dy}f_1(y)$$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y) = \frac{-x^2+2y^2}{-x^2y+y^3} - \frac{1}{2(y+x)} + \frac{1}{2(-y+x)}$$

- Solve for $f_1(y)$

$$f_1(y) = -\frac{\ln(-y+x)}{2} + \frac{\ln(y-x)}{2} + \ln(y)$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = \frac{\ln(y+x)}{2} + \ln(x) + \frac{\ln(y-x)}{2} + \ln(y)$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\frac{\ln(y+x)}{2} + \ln(x) + \frac{\ln(y-x)}{2} + \ln(y) = c_1$$

- Solve for y

$$\left\{y = \frac{-2x^2+\sqrt{2x^4-2\sqrt{x^8+4(e^{c_1})^2x^2}}}{2x} + x, y = -\frac{2x^2+\sqrt{2x^4-2\sqrt{x^8+4(e^{c_1})^2x^2}}}{2x} + x, y = \frac{\sqrt{2}\sqrt{x^4+\sqrt{x^8+4(e^{c_1})^2x^2}-2}}{2x}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.266 (sec). Leaf size: 71

```
dsolve((y(x)^2-2*x^2)/(x*y(x)^2-x^3)+ (2*y(x)^2-x^2)/(y(x)^3-x^2*y(x))*diff(y(x),x)=0,y(x),
```

$$y(x) = \frac{\sqrt{\frac{2c_1x^3-2\sqrt{c_1^2x^6+4}}{c_1x^3}}\, x}{2}$$

$$y(x) = \frac{\sqrt{2}\,\sqrt{\frac{c_1x^3+\sqrt{c_1^2x^6+4}}{c_1x^3}}\, x}{2}$$

✓ Solution by Mathematica

Time used: 15.598 (sec). Leaf size: 277

```
DSolve[(y[x]^2-2*x^2)/(x*y[x]^2-x^3)+ (2*y[x]^2-x^2)/(y[x]^3-x^2*y[x])*y'[x]==0,y[x],x,Inclu
```

$$y(x) \to -\frac{\sqrt{x^2 - \frac{\sqrt{x^6-4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{x^2 - \frac{\sqrt{x^6-4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \to -\frac{\sqrt{\frac{x^3+\sqrt{x^6-4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{\frac{x^3+\sqrt{x^6-4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \to -\frac{\sqrt{x^2 - \frac{\sqrt{x^6}}{x}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{x^2 - \frac{\sqrt{x^6}}{x}}}{\sqrt{2}}$$

$$y(x) \to -\frac{\sqrt{\frac{\sqrt{x^6}+x^3}{x}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{\frac{\sqrt{x^6}+x^3}{x}}}{\sqrt{2}}$$

1.3 problem Ex 3

Internal problem ID [11124]
Internal file name [`OUTPUT/10109_Wednesday_November_23_2022_11_50_44_AM_4922352/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 8. Exact differential equations. Page 11
Problem number: Ex 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{\frac{1}{\sqrt{x^2+y^2}}+\left(\frac{1}{y}-\frac{x}{y\sqrt{x^2+y^2}}\right)y'=0}$$

1.3.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y}{\sqrt{x^2+y^2}-x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(b_3 - a_2)}{\sqrt{x^2+y^2} - x} - \frac{y^2 a_3}{\left(\sqrt{x^2+y^2} - x\right)^2} - \frac{y\left(\frac{x}{\sqrt{x^2+y^2}} - 1\right)(x a_2 + y a_3 + a_1)}{\left(\sqrt{x^2+y^2} - x\right)^2} - \left(-\frac{1}{\sqrt{x^2+y^2} - x} + \frac{y^2}{\left(\sqrt{x^2+y^2} - x\right)^2 \sqrt{x^2+y^2}}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{\left(x^2+y^2\right)^{\frac{3}{2}} b_2 - x^3 b_2 - x\,y^2 a_3 - 2x\,y^2 b_2 + y^3 a_2 - y^3 b_3 - \sqrt{x^2+y^2}\,x b_1 + \sqrt{x^2+y^2}\,y a_1 + x^2 b_1 - x y a_1}{\left(\sqrt{x^2+y^2} - x\right)^2 \sqrt{x^2+y^2}} = 0$$

Setting the numerator to zero gives

$$\left(x^2+y^2\right)^{\frac{3}{2}} b_2 - x^3 b_2 - x\,y^2 a_3 - 2x\,y^2 b_2 + y^3 a_2 - y^3 b_3 - \sqrt{x^2+y^2}\,x b_1 + \sqrt{x^2+y^2}\,y a_1 + x^2 b_1 - x y a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$\left(x^2+y^2\right)^{\frac{3}{2}} b_2 - \left(x^2+y^2\right) x b_2 + \left(x^2+y^2\right) y a_2 - x^2 y a_2 - x\,y^2 a_3 - x\,y^2 b_2 - y^3 b_3 + \left(x^2+y^2\right) b_1 - \sqrt{x^2+y^2}\,x b_1 + \sqrt{x^2+y^2}\,y a_1 - x y a_1 - y^2 b_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\sqrt{x^2+y^2}\,x^2 b_2 + \sqrt{x^2+y^2}\,y^2 b_2 - x^3 b_2 - x\,y^2 a_3 - 2x\,y^2 b_2 + y^3 a_2 - y^3 b_3 - \sqrt{x^2+y^2}\,x b_1 + \sqrt{x^2+y^2}\,y a_1 + x^2 b_1 - x y a_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2 + y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{gathered} v_2^3 a_2 - v_1 v_2^2 a_3 - v_1^3 b_2 + v_3 v_1^2 b_2 - 2 v_1 v_2^2 b_2 + v_3 v_2^2 b_2 \\ - v_2^3 b_3 - v_1 v_2 a_1 + v_3 v_2 a_1 + v_1^2 b_1 - v_3 v_1 b_1 = 0 \end{gathered} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{gathered} -v_1^3 b_2 + v_3 v_1^2 b_2 + v_1^2 b_1 + (-a_3 - 2b_2) v_1 v_2^2 - v_1 v_2 a_1 \\ - v_3 v_1 b_1 + (-b_3 + a_2) v_2^3 + v_3 v_2^2 b_2 + v_3 v_2 a_1 = 0 \end{gathered} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ -a_1 &= 0 \\ -b_1 &= 0 \\ -b_2 &= 0 \\ -a_3 - 2b_2 &= 0 \\ -b_3 + a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y}{\sqrt{x^2+y^2}-x}\right)(x) \\
&= \frac{y\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}-x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}-x}} dy
\end{aligned}$$

Which results in

$$S = \ln(y) + \frac{x\ln\left(\frac{2x^2+2\sqrt{x^2}\,\sqrt{x^2+y^2}}{y}\right)}{\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y}{\sqrt{x^2+y^2} - x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{\sqrt{x^2+y^2} + x}{\sqrt{x^2+y^2}\, x} \\
S_y &= \frac{y}{\sqrt{x^2+y^2}\, \left(\sqrt{x^2+y^2} + x\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

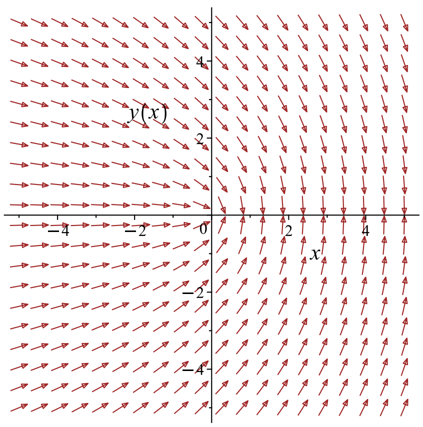
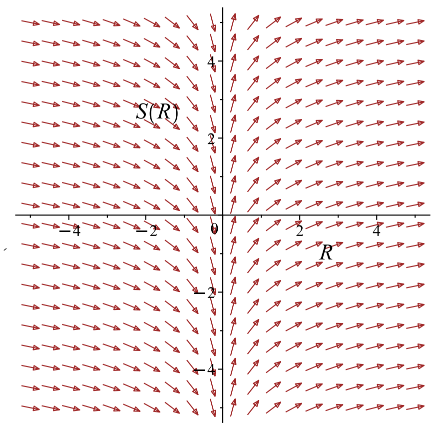
We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{\sqrt{x^2+y^2}-x}$ 	$R = x$ $S = \ln(2) + \ln(x) + \ln\big($	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$\ln(2) + \ln(x) + \ln\left(\sqrt{x^2 + y^2} + x\right) = \ln(x) + c_1 \tag{1}$$

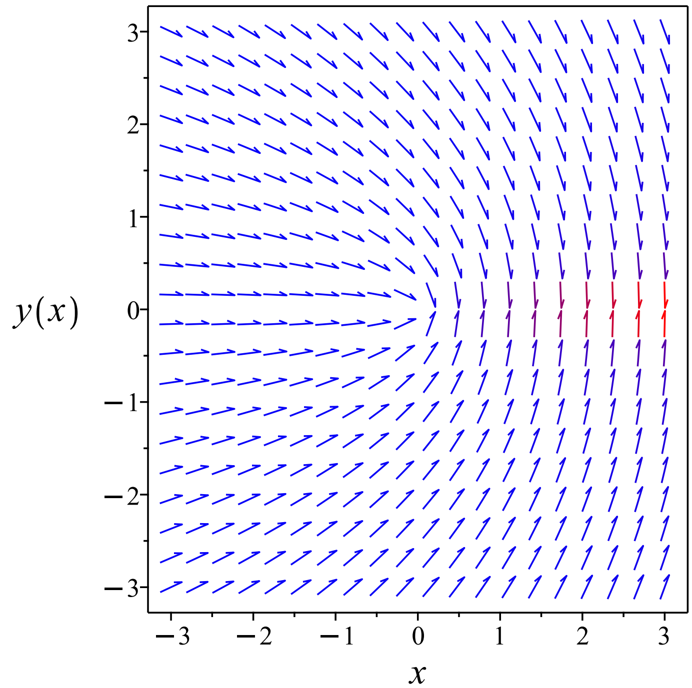


Figure 6: Slope field plot

Verification of solutions

$$\ln(2) + \ln(x) + \ln\left(\sqrt{x^2 + y^2} + x\right) = \ln(x) + c_1$$

Verified OK.

1.3.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y} - \frac{x}{y\sqrt{x^2+y^2}}\right)\mathrm{d}y = \left(-\frac{1}{\sqrt{x^2+y^2}}\right)\mathrm{d}x$$
$$\left(\frac{1}{\sqrt{x^2+y^2}}\right)\mathrm{d}x + \left(\frac{1}{y} - \frac{x}{y\sqrt{x^2+y^2}}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{1}{\sqrt{x^2+y^2}}$$
$$N(x,y) = \frac{1}{y} - \frac{x}{y\sqrt{x^2+y^2}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{x^2+y^2}}\right)\\ &= -\frac{y}{(x^2+y^2)^{\frac{3}{2}}}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y} - \frac{x}{y\sqrt{x^2+y^2}}\right) \\
&= -\frac{y}{(x^2+y^2)^{\frac{3}{2}}}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{1}{\sqrt{x^2+y^2}} \mathrm{d}x \\
\phi &= \ln\left(\sqrt{x^2+y^2} + x\right) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{y}{\sqrt{x^2+y^2}\,\left(\sqrt{x^2+y^2}+x\right)} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y} - \frac{x}{y\sqrt{x^2+y^2}}$. Therefore equation (4) becomes

$$
\frac{1}{y} - \frac{x}{y\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}\,\left(\sqrt{x^2+y^2}+x\right)} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln\left(\sqrt{x^2+y^2}+x\right)+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln\left(\sqrt{x^2+y^2}+x\right)$$

Summary

The solution(s) found are the following

$$\ln\left(\sqrt{x^2+y^2}+x\right) = c_1 \tag{1}$$

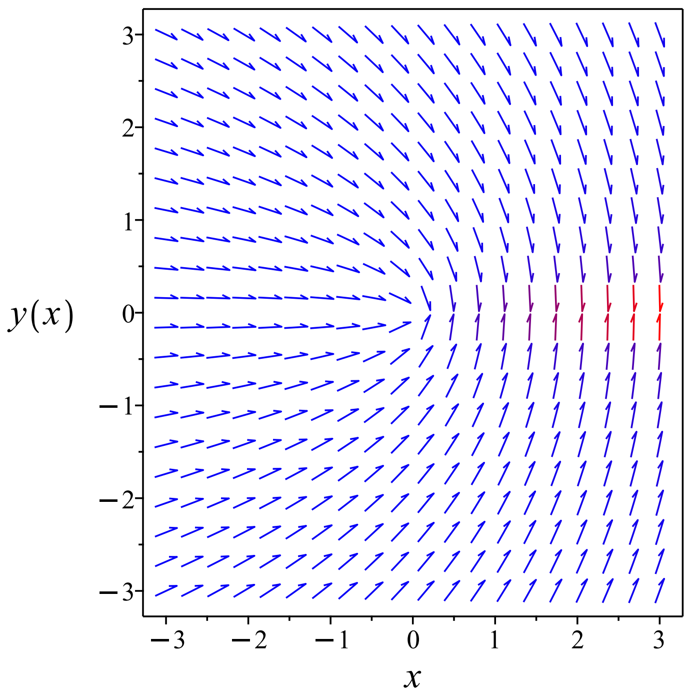


Figure 7: Slope field plot

Verification of solutions

$$\ln\left(\sqrt{x^2+y^2}+x\right) = c_1$$

Verified OK.

1.3.3 Maple step by step solution

Let's solve

$$\frac{1}{\sqrt{x^2+y^2}}+\left(\frac{1}{y}-\frac{x}{y\sqrt{x^2+y^2}}\right)y'=0$$

- Highest derivative means the order of the ODE is 1

y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$

  - Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

  $-\frac{y}{(x^2+y^2)^{\frac{3}{2}}}=-\frac{1}{y\sqrt{x^2+y^2}}+\frac{x^2}{y(x^2+y^2)^{\frac{3}{2}}}$

  - Simplify

  $-\frac{y}{(x^2+y^2)^{\frac{3}{2}}}=-\frac{y}{(x^2+y^2)^{\frac{3}{2}}}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

$\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

$F(x,y)=\int\frac{1}{\sqrt{x^2+y^2}}dx+f_1(y)$

- Evaluate integral

$F(x,y)=\ln\left(\sqrt{x^2+y^2}+x\right)+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

$N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

$\frac{1}{y}-\frac{x}{y\sqrt{x^2+y^2}}=\frac{y}{\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}+x\right)}+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y) = \frac{1}{y} - \frac{x}{y\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}+x\right)}$$

- Solve for $f_1(y)$

  $f_1(y) = 0$
- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = \ln\left(\sqrt{x^2+y^2}+x\right)$
- Substitute $F(x, y)$ into the solution of the ODE

  $\ln\left(\sqrt{x^2+y^2}+x\right) = c_1$
- Solve for y

  $\left\{y = \sqrt{-2\,\mathrm{e}^{c_1}x + (\mathrm{e}^{c_1})^2}, y = -\sqrt{-2\,\mathrm{e}^{c_1}x + (\mathrm{e}^{c_1})^2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 18

```
dsolve(1/sqrt(x^2+y(x)^2)+ ( 1/y(x)-(x/(y(x)*sqrt(x^2+y(x)^2))))*diff(y(x),x)=0,y(x), singso
```

$$-c_1 + \sqrt{y(x)^2 + x^2} + x = 0$$

✓ Solution by Mathematica

Time used: 0.893 (sec). Leaf size: 62

```
DSolve[1/Sqrt[x^2+y[x]^2]+ ( 1/y[x]-(x/(y[x]*Sqrt[x^2+y[x]^2])))*y'[x]==0,y[x],x,IncludeSing
```

$$y(x) \to -e^{\frac{c_1}{2}}\sqrt{-2x+e^{c_1}}$$
$$y(x) \to e^{\frac{c_1}{2}}\sqrt{-2x+e^{c_1}}$$
$$y(x) \to 0$$
$$y(x) \to \text{ComplexInfinity}$$

1.4 problem Ex 4

Internal problem ID [11125]
Internal file name [`OUTPUT/10110_Wednesday_November_23_2022_11_50_45_AM_26166275/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 8. Exact differential equations. Page 11
Problem number: Ex 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y + y'x = -x}$$

1.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = -1$$

Hence the ode is

$$y' + \frac{y}{x} = -1$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{x}dx} \\ = x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(-1)$$
$$\frac{d}{dx}(xy) = (x)(-1)$$
$$d(xy) = (-x)\, dx$$

Integrating gives

$$xy = \int -x\, dx$$
$$xy = -\frac{x^2}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = -\frac{x}{2} + \frac{c_1}{x}$$

Summary
The solution(s) found are the following

$$y = -\frac{x}{2} + \frac{c_1}{x} \tag{1}$$

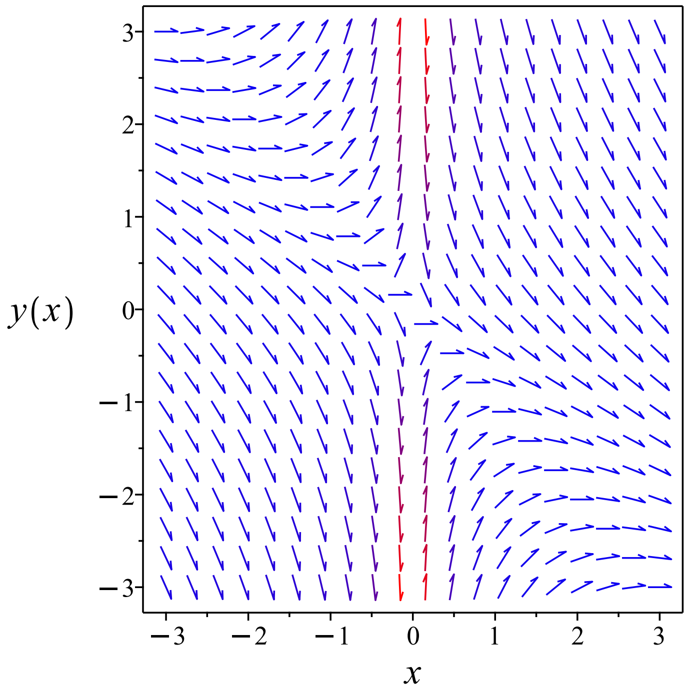


Figure 8: Slope field plot

Verification of solutions

$$y = -\frac{x}{2} + \frac{c_1}{x}$$

Verified OK.

1.4.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x + (u'(x)\,x + u(x))\,x = -x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-2u-1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -2u - 1$. Integrating both sides gives

$$\frac{1}{-2u-1}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{-2u-1}\,du = \int \frac{1}{x}\,dx$$
$$-\frac{\ln(-2u-1)}{2} = \ln(x) + c_2$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-2u-1}} = e^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{\sqrt{-2u-1}} = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= -\frac{\left(c_3^2 e^{2c_2} x^2 + 1\right) e^{-2c_2}}{2x\, c_3^2} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(c_3^2 e^{2c_2} x^2 + 1\right) e^{-2c_2}}{2x\, c_3^2} \tag{1}$$

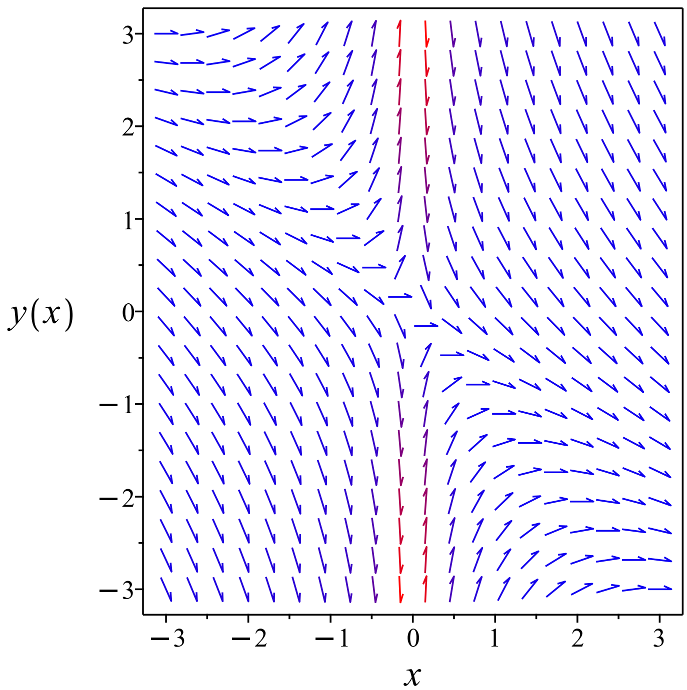


Figure 9: Slope field plot

Verification of solutions

$$y = -\frac{(c_3^2 e^{2c_2} x^2 + 1)\, e^{-2c_2}}{2x\, c_3^2}$$

Verified OK.

1.4.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-y - x}{x} \tag{1}$$

Which becomes

$$0 = (-x)\, dy + (-x - y)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + (-x - y)\, dx = d\left(-\frac{1}{2}x^2 - xy\right)$$

Hence (2) becomes

$$0 = d\left(-\frac{1}{2}x^2 - xy\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{-x^2 + 2c_1}{2x} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{-x^2 + 2c_1}{2x} + c_1 \tag{1}$$

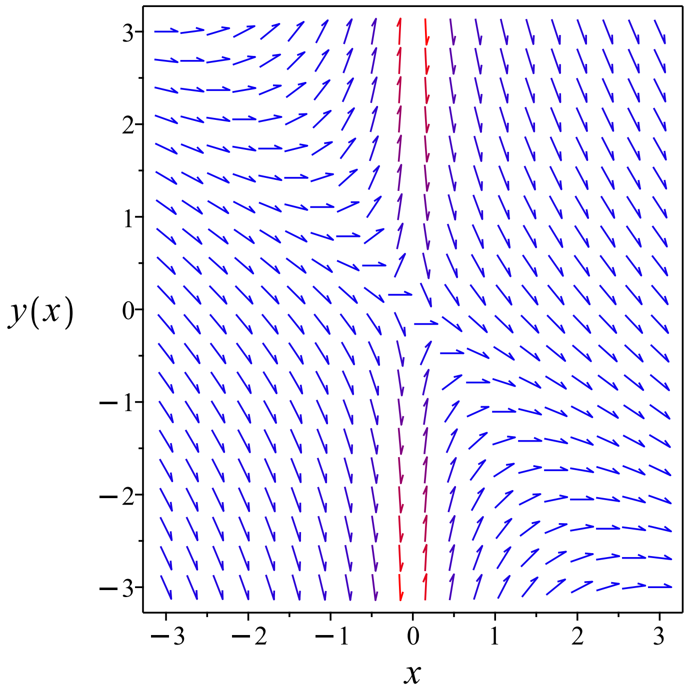


Figure 10: Slope field plot

Verification of solutions

$$y = \frac{-x^2 + 2c_1}{2x} + c_1$$

Verified OK.

1.4.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y + x}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = xy$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y + x}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = -\frac{x^2}{2} + c_1$$

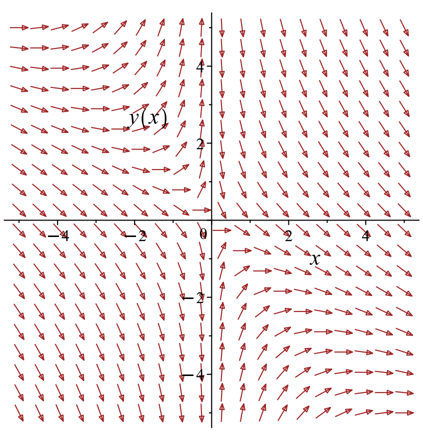
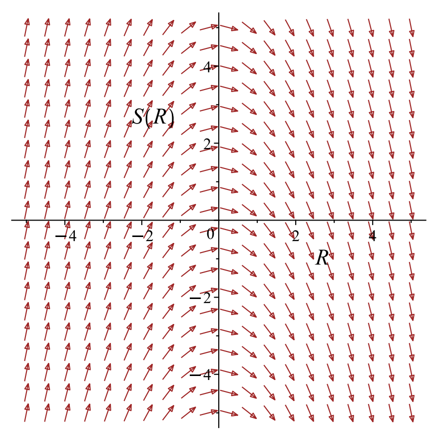
Which simplifies to

$$yx = -\frac{x^2}{2} + c_1$$

Which gives

$$y = \frac{-x^2 + 2c_1}{2x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y+x}{x}$ 	$R = x$ $S = xy$	$\frac{dS}{dR} = -R$ 

Summary

The solution(s) found are the following

$$y = \frac{-x^2 + 2c_1}{2x} \tag{1}$$

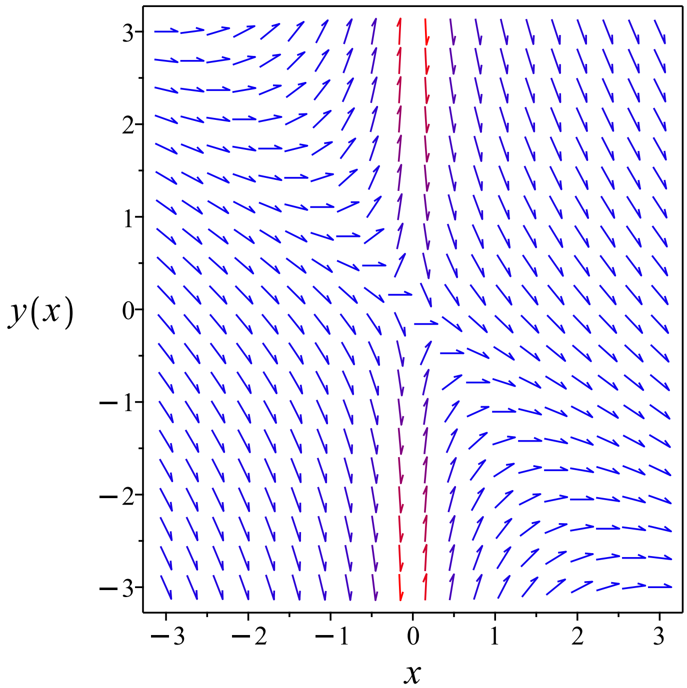


Figure 11: Slope field plot

Verification of solutions

$$y = \frac{-x^2 + 2c_1}{2x}$$

Verified OK.

1.4.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x)\,\mathrm{d}y &= (-x - y)\,\mathrm{d}x \\
(y + x)\,\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y + x \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y + x) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int y+x\,\mathrm{d}x$$
$$\phi=\frac{x(2y+x)}{2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x$. Therefore equation (4) becomes

$$x=x+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=\frac{x(2y+x)}{2}+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=\frac{x(2y+x)}{2}$$

The solution becomes

$$y = \frac{-x^2 + 2c_1}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{-x^2 + 2c_1}{2x} \tag{1}$$

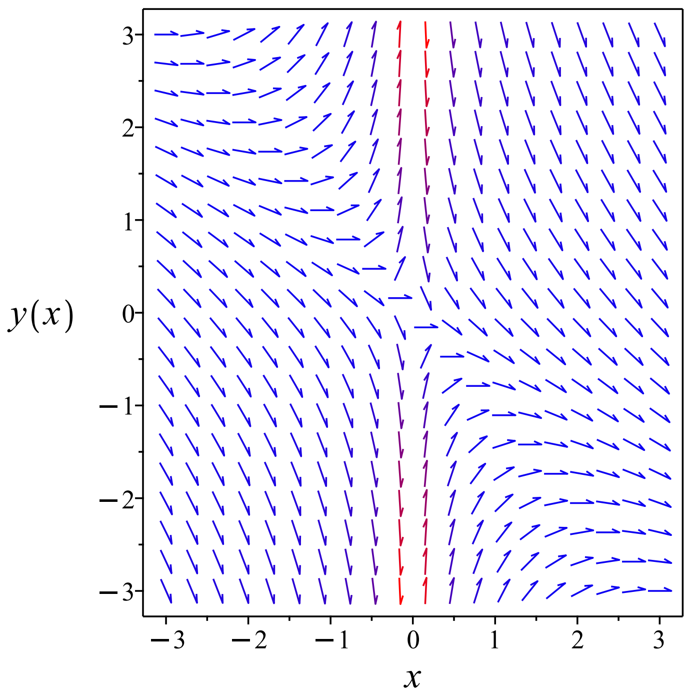


Figure 12: Slope field plot

Verification of solutions

$$y = \frac{-x^2 + 2c_1}{2x}$$

Verified OK.

1.4.6 Maple step by step solution

Let's solve

$y + y'x = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -1 - \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = -1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = -\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\mu(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\mu(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int -\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int -x dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{x^2}{2} + c_1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve((y(x)+x)+ x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{2} + \frac{c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 17

```
DSolve[(y[x]+x)+ x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{2} + \frac{c_1}{x}$$

1.5 problem Ex 5

Internal problem ID [11126]
Internal file name [`OUTPUT/10111_Wednesday_November_23_2022_11_50_46_AM_71070688/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 8. Exact differential equations. Page 11
Problem number: Ex 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{-2y + (2y - 2x - 3)\, y' = -6x - 1}$$

1.5.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-6x + 2y - 1}{2y - 2x - 3} \tag{1}$$

Which becomes

$$(-2y + 3)\, dy = (-2x)\, dy + (6x - 2y + 1)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-2x)\, dy + (6x - 2y + 1)\, dx = d\big(3x^2 - 2xy + x\big)$$

Hence (2) becomes

$$(-2y+3)\,dy = d\left(3x^2 - 2xy + x\right)$$

Integrating both sides gives gives these solutions

$$y = x + \frac{3}{2} + \frac{\sqrt{-8x^2 - 4c_1 + 8x + 9}}{2} + c_1$$
$$y = x + \frac{3}{2} - \frac{\sqrt{-8x^2 - 4c_1 + 8x + 9}}{2} + c_1$$

Summary

The solution(s) found are the following

$$y = x + \frac{3}{2} + \frac{\sqrt{-8x^2 - 4c_1 + 8x + 9}}{2} + c_1 \tag{1}$$
$$y = x + \frac{3}{2} - \frac{\sqrt{-8x^2 - 4c_1 + 8x + 9}}{2} + c_1 \tag{2}$$

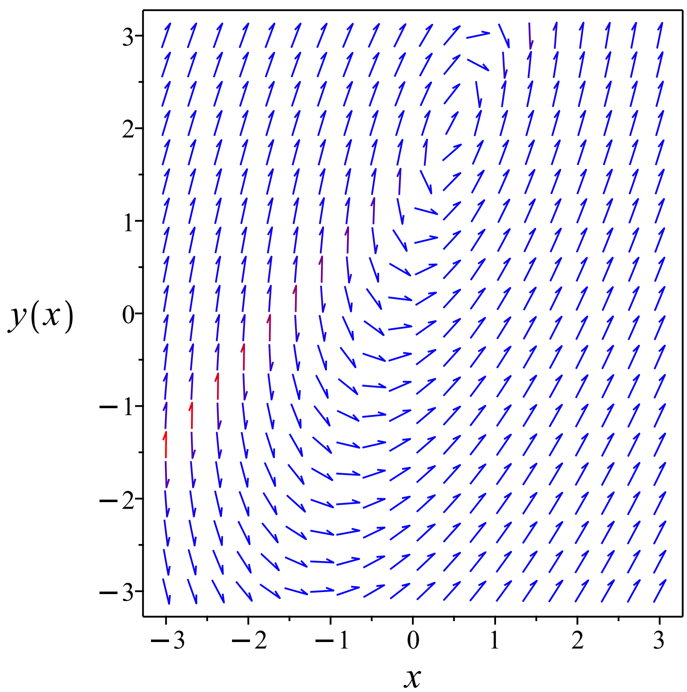


Figure 13: Slope field plot

Verification of solutions

$$y = x + \frac{3}{2} + \frac{\sqrt{-8x^2 - 4c_1 + 8x + 9}}{2} + c_1$$

Verified OK.

$$y = x + \frac{3}{2} - \frac{\sqrt{-8x^2 - 4c_1 + 8x + 9}}{2} + c_1$$

Verified OK.

1.5.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{-6X - 6x_0 + 2Y(X) + 2y_0 - 1}{2Y(X) + 2y_0 - 2X - 2x_0 - 3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = \frac{1}{2}$$
$$y_0 = 2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-6X + 2Y(X)}{2Y(X) - 2X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{-3X + Y}{Y - X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = 3X - Y$ and $N = -Y + X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{du}{dX}X + u = \frac{u-3}{u-1}$$
$$\frac{du}{dX} = \frac{\frac{u(X)-3}{u(X)-1} - u(X)}{X}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{u(X)-3}{u(X)-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) Xu(X) - \left(\frac{d}{dX}u(X)\right) X + u(X)^2 - 2u(X) + 3 = 0$$

Or

$$X(u(X) - 1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 - 2u(X) + 3 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X)g(u) \\ &= -\frac{u^2 - 2u + 3}{X(u-1)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-2u+3}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-2u+3}{u-1}}\,du = -\frac{1}{X}\,dX$$
$$\int \frac{1}{\frac{u^2-2u+3}{u-1}}\,du = \int -\frac{1}{X}\,dX$$
$$\frac{\ln(u^2 - 2u + 3)}{2} = -\ln(X) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2 - 2u + 3} = e^{-\ln(X)+c_2}$$

Which simplifies to

$$\sqrt{u^2 - 2u + 3} = \frac{c_3}{X}$$

Which simplifies to

$$\sqrt{u(X)^2 - 2u(X) + 3} = \frac{c_3 e^{c_2}}{X}$$

The solution is

$$\sqrt{u(X)^2 - 2u(X) + 3} = \frac{c_3 e^{c_2}}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\sqrt{\frac{Y(X)^2}{X^2} - \frac{2Y(X)}{X} + 3} = \frac{c_3 e^{c_2}}{X}$$

Using the solution for $Y(X)$

$$\sqrt{\frac{Y(X)^2 - 2Y(X)X + 3X^2}{X^2}} = \frac{c_3 e^{c_2}}{X}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 2$$
$$X = x + \frac{1}{2}$$

Then the solution in y becomes

$$\sqrt{\frac{(y-2)^2 - 2(y-2)\left(x - \frac{1}{2}\right) + 3\left(x - \frac{1}{2}\right)^2}{\left(x - \frac{1}{2}\right)^2}} = \frac{c_3 e^{c_2}}{x - \frac{1}{2}}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{(y-2)^2 - 2(y-2)\left(x - \frac{1}{2}\right) + 3\left(x - \frac{1}{2}\right)^2}{\left(x - \frac{1}{2}\right)^2}} = \frac{c_3 e^{c_2}}{x - \frac{1}{2}} \tag{1}$$

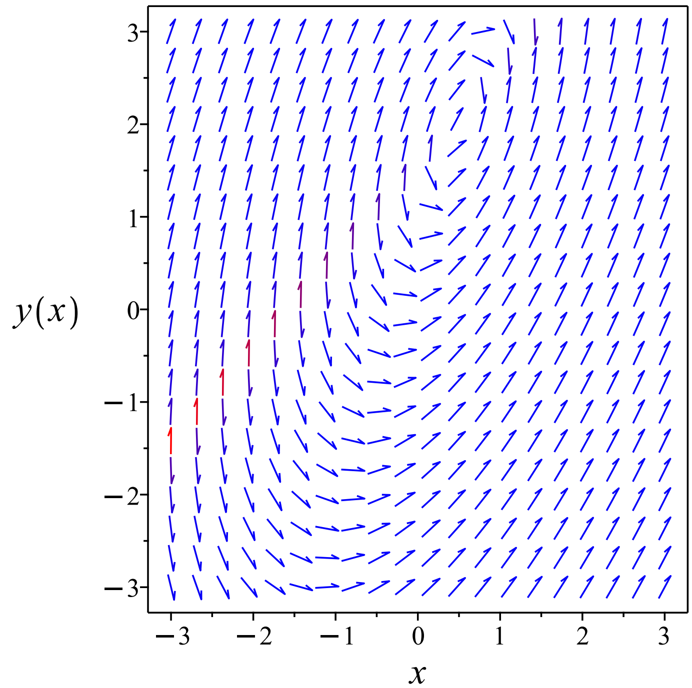


Figure 14: Slope field plot

Verification of solutions

$$\sqrt{\frac{(y-2)^2-2(y-2)\left(x-\frac{1}{2}\right)+3\left(x-\frac{1}{2}\right)^2}{\left(x-\frac{1}{2}\right)^2}}=\frac{c_3 \mathrm{e}^{c_2}}{x-\frac{1}{2}}$$

Verified OK.

1.5.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{-6x+2y-1}{2y-2x-3}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{(-6x+2y-1)(b_3-a_2)}{2y-2x-3} - \frac{(-6x+2y-1)^2 a_3}{(2y-2x-3)^2} \\ &- \left(-\frac{6}{2y-2x-3} + \frac{-12x+4y-2}{(2y-2x-3)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(\frac{2}{2y-2x-3} - \frac{2(-6x+2y-1)}{(2y-2x-3)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{12x^2a_2 + 36x^2a_3 + 4x^2b_2 - 12x^2b_3 - 24xya_2 - 24xya_3 + 8xyb_2 + 24xyb_3 + 4y^2a_2 - 4y^2a_3 - 4y^2b_2 - 4y}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-12x^2a_2 - 36x^2a_3 - 4x^2b_2 + 12x^2b_3 + 24xya_2 + 24xya_3 - 8xyb_2 - 24xyb_3 \\ &- 4y^2a_2 + 4y^2a_3 + 4y^2b_2 + 4y^2b_3 - 36xa_2 - 12xa_3 - 8xb_1 + 16xb_2 + 20xb_3 \\ &+ 8ya_1 + 8ya_2 - 12ya_3 - 12yb_2 - 4yb_3 - 16a_1 - 3a_2 - a_3 + 4b_1 + 9b_2 + 3b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-12a_2v_1^2 + 24a_2v_1v_2 - 4a_2v_2^2 - 36a_3v_1^2 + 24a_3v_1v_2 + 4a_3v_2^2 - 4b_2v_1^2 - 8b_2v_1v_2 \\ &+ 4b_2v_2^2 + 12b_3v_1^2 - 24b_3v_1v_2 + 4b_3v_2^2 + 8a_1v_2 - 36a_2v_1 + 8a_2v_2 - 12a_3v_1 - 12a_3v_2 \\ &- 8b_1v_1 + 16b_2v_1 - 12b_2v_2 + 20b_3v_1 - 4b_3v_2 - 16a_1 - 3a_2 - a_3 + 4b_1 + 9b_2 + 3b_3 \\ &= 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(-12a_2 - 36a_3 - 4b_2 + 12b_3) v_1^2 + (24a_2 + 24a_3 - 8b_2 - 24b_3) v_1 v_2 \\ &+ (-36a_2 - 12a_3 - 8b_1 + 16b_2 + 20b_3) v_1 + (-4a_2 + 4a_3 + 4b_2 + 4b_3) v_2^2 \\ &+ (8a_1 + 8a_2 - 12a_3 - 12b_2 - 4b_3) v_2 - 16a_1 - 3a_2 - a_3 + 4b_1 + 9b_2 + 3b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-12a_2 - 36a_3 - 4b_2 + 12b_3 &= 0 \\ -4a_2 + 4a_3 + 4b_2 + 4b_3 &= 0 \\ 24a_2 + 24a_3 - 8b_2 - 24b_3 &= 0 \\ 8a_1 + 8a_2 - 12a_3 - 12b_2 - 4b_3 &= 0 \\ -36a_2 - 12a_3 - 8b_1 + 16b_2 + 20b_3 &= 0 \\ -16a_1 - 3a_2 - a_3 + 4b_1 + 9b_2 + 3b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= a_1 \\ a_2 &= -4a_3 - 2a_1 \\ a_3 &= a_3 \\ b_1 &= 4a_1 + \frac{11a_3}{2} \\ b_2 &= -3a_3 \\ b_3 &= -2a_1 - 2a_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -4x + y \\ \eta &= -3x + \frac{11}{2} - 2y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -3x + \frac{11}{2} - 2y - \left(\frac{-6x + 2y - 1}{2y - 2x - 3}\right)(-4x + y) \\
&= \frac{36x^2 - 24xy + 12y^2 + 12x - 36y + 33}{-4y + 4x + 6} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{36x^2 - 24xy + 12y^2 + 12x - 36y + 33}{-4y + 4x + 6}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{\ln\left(12x^2 - 8xy + 4y^2 + 4x - 12y + 11\right)}{6}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{-6x + 2y - 1}{2y - 2x - 3}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-12x + 4y - 2}{36x^2 + (-24y + 12)\, x + 12y^2 - 36y + 33} \\
S_y &= \frac{-4y + 4x + 6}{36x^2 + (-24y + 12)\, x + 12y^2 - 36y + 33}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln \left(12x^2 + (-8y + 4)\, x + 4y^2 - 12y + 11\right)}{6} = c_1$$

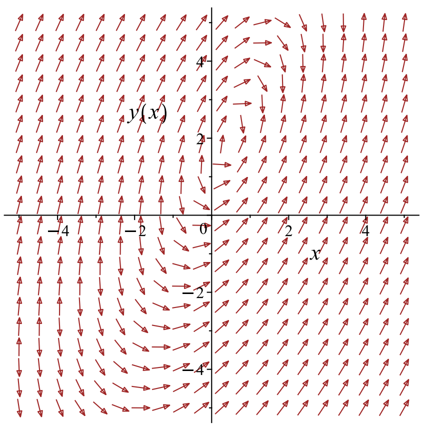
Which simplifies to

$$-\frac{\ln \left(12x^2 + (-8y + 4)\, x + 4y^2 - 12y + 11\right)}{6} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{-6x+2y-1}{2y-2x-3}$



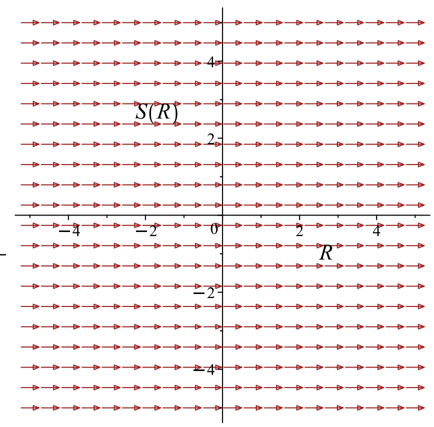
</td><td>

$R = x$

$S = -\frac{\ln(12x^2 + (-8y +}{}$

</td><td>

$\frac{dS}{dR} = 0$



</td></tr>
</table>

<u>Summary</u>

The solution(s) found are the following

$$-\frac{\ln(12x^2 + (-8y+4)x + 4y^2 - 12y + 11)}{6} = c_1 \tag{1}$$

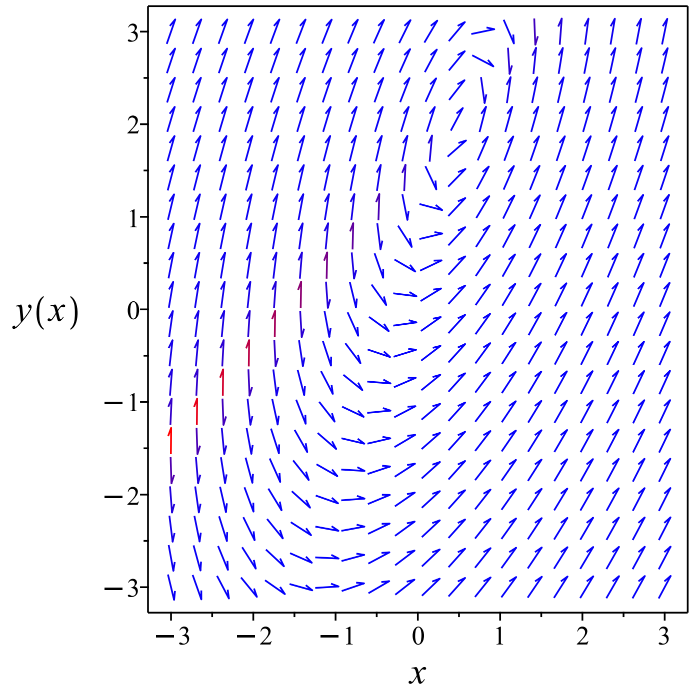


Figure 15: Slope field plot

Verification of solutions

$$-\frac{\ln\left(12x^2+(-8y+4)\,x+4y^2-12y+11\right)}{6}=c_1$$

Verified OK.

1.5.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2y - 2x - 3)\,\mathrm{d}y &= (-6x + 2y - 1)\,\mathrm{d}x \\ (6x - 2y + 1)\,\mathrm{d}x + (2y - 2x - 3)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 6x - 2y + 1 \\ N(x, y) &= 2y - 2x - 3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(6x - 2y + 1) \\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2y - 2x - 3) \\ &= -2\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 6x - 2y + 1 \, \mathrm{d}x$$
$$\phi = x(3x - 2y + 1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -2x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y - 2x - 3$. Therefore equation (4) becomes

$$2y - 2x - 3 = -2x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y - 3$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (2y - 3) \, \mathrm{d}y$$
$$f(y) = y^2 - 3y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(3x - 2y + 1) + y^2 - 3y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(3x - 2y + 1) + y^2 - 3y$$

Summary

The solution(s) found are the following

$$x(3x - 2y + 1) + y^2 - 3y = c_1 \tag{1}$$

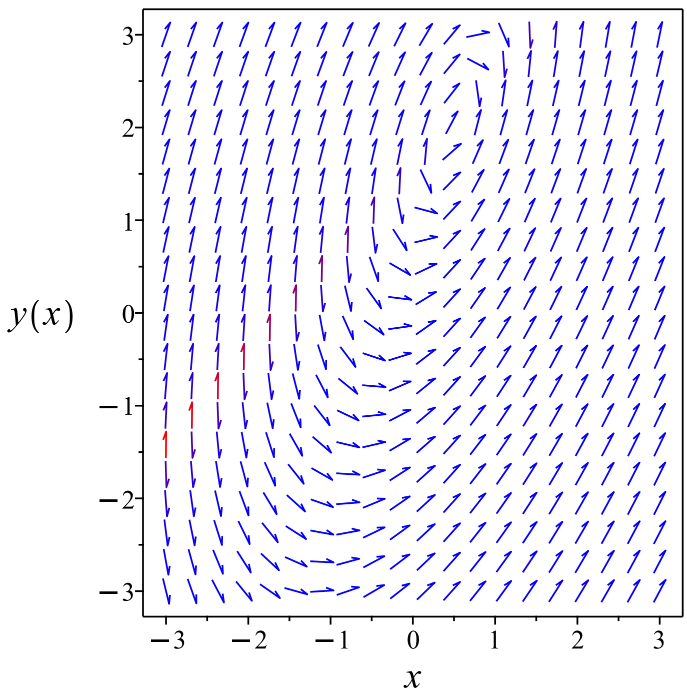


Figure 16: Slope field plot

Verification of solutions

$$x(3x - 2y + 1) + y^2 - 3y = c_1$$

Verified OK.

1.5.5 Maple step by step solution

Let's solve

$-2y + (2y - 2x - 3)\, y' = -6x - 1$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  ◦ Evaluate derivatives

    $-2 = -2$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (6x - 2y + 1)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = 3x^2 - 2xy + x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $2y - 2x - 3 = -2x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 2y - 3$

- Solve for $f_1(y)$

  $f_1(y) = y^2 - 3y$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = 3x^2 - 2xy + y^2 + x - 3y$$

- Substitute $F(x, y)$ into the solution of the ODE

$$3x^2 - 2xy + y^2 + x - 3y = c_1$$

- Solve for y

$$\left\{y = x + \frac{3}{2} - \frac{\sqrt{-8x^2 + 4c_1 + 8x + 9}}{2}, y = x + \frac{3}{2} + \frac{\sqrt{-8x^2 + 4c_1 + 8x + 9}}{2}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.469 (sec). Leaf size: 33

```
dsolve((6*x-2*y(x)+1)+(2*y(x)-2*x-3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-\sqrt{1 - 8\left(x - \frac{1}{2}\right)^2 c_1^2} + (2x + 3)\, c_1}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.208 (sec). Leaf size: 67

```
DSolve[(6*x-2*y[x]+1)+(2*y[x]-2*x-3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}i\sqrt{8x^2 - 8x - 9 - 4c_1} + x + \frac{3}{2}$$
$$y(x) \to \frac{1}{2}i\sqrt{8x^2 - 8x - 9 - 4c_1} + x + \frac{3}{2}$$

2 Chapter 2, differential equations of the first order and the first degree. Article 9. Variables searated or separable. Page 13

2.1 problem Ex 1

Internal problem ID [11127]
Internal file name [`OUTPUT/10112_Wednesday_November_23_2022_11_50_47_AM_67654846/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 9. Variables searated or separable. Page 13
Problem number: Ex 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\sec(x)\cos(y)^2-\cos(x)\sin(y)\,y'=0}$$

2.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{\sec(x)\cos(y)\cot(y)}{\cos(x)}\end{aligned}$$

Where $f(x)=\frac{\sec(x)}{\cos(x)}$ and $g(y)=\cos(y)\cot(y)$. Integrating both sides gives

$$\frac{1}{\cos(y)\cot(y)}\,dy=\frac{\sec(x)}{\cos(x)}\,dx$$

$$\int \frac{1}{\cos(y)\cot(y)}dy = \int \frac{\sec(x)}{\cos(x)}dx$$
$$\frac{1}{\cos(y)} = \tan(x) + c_1$$

Which results in

$$y = \arccos\left(\frac{1}{\tan(x) + c_1}\right)$$

Summary
The solution(s) found are the following

$$y = \arccos\left(\frac{1}{\tan(x) + c_1}\right) \tag{1}$$

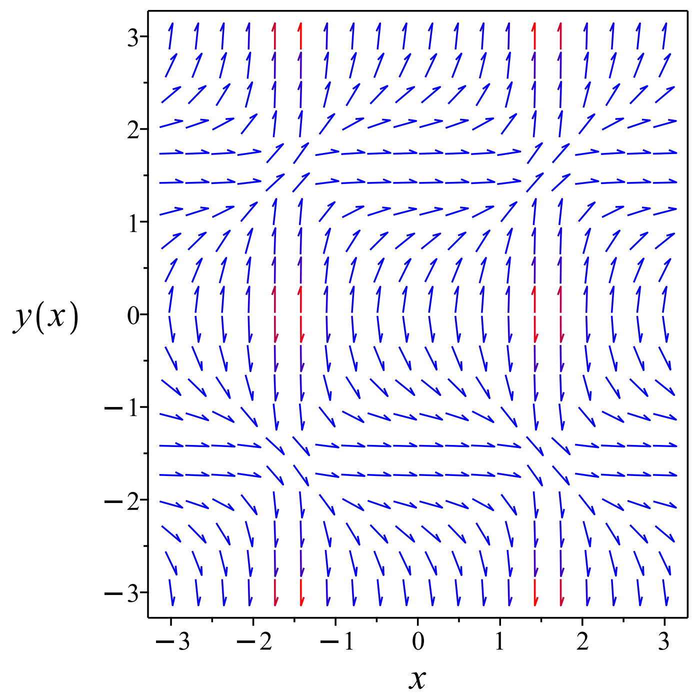


Figure 17: Slope field plot

Verification of solutions

$$y = \arccos\left(\frac{1}{\tan(x) + c_1}\right)$$

Verified OK.

2.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\sec(x)\cos(y)^2}{\cos(x)\sin(y)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 8: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{\cos(x)}{\sec(x)}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{\cos(x)}{\sec(x)}} dx \end{aligned}$$

Which results in

$$S = \tan(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\sec(x)\cos(y)^2}{\cos(x)\sin(y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \sec(x)^2 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \tan(y)\sec(y) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \tan(R)\sec(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \sec(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\tan(x) = \sec(y) + c_1
$$

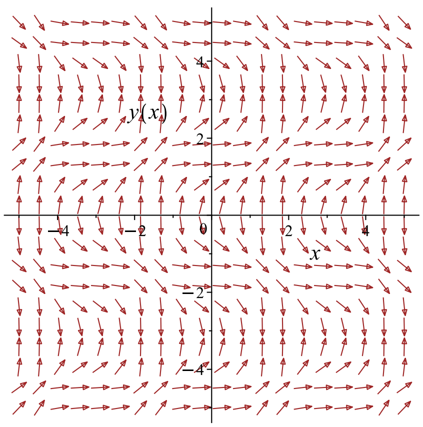
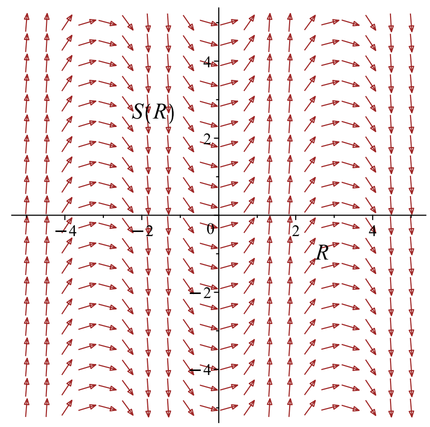
Which simplifies to

$$
\tan(x) = \sec(y) + c_1
$$

Which gives

$$
y = \pi - \operatorname{arcsec}(-\tan(x) + c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sec(x)\cos(y)^2}{\cos(x)\sin(y)}$ 	$R = y$ $S = \tan(x)$	$\frac{dS}{dR} = \tan(R)\sec(R)$ 

Summary
The solution(s) found are the following

$$y = \pi - \text{arcsec}\left(-\tan(x) + c_1\right) \tag{1}$$

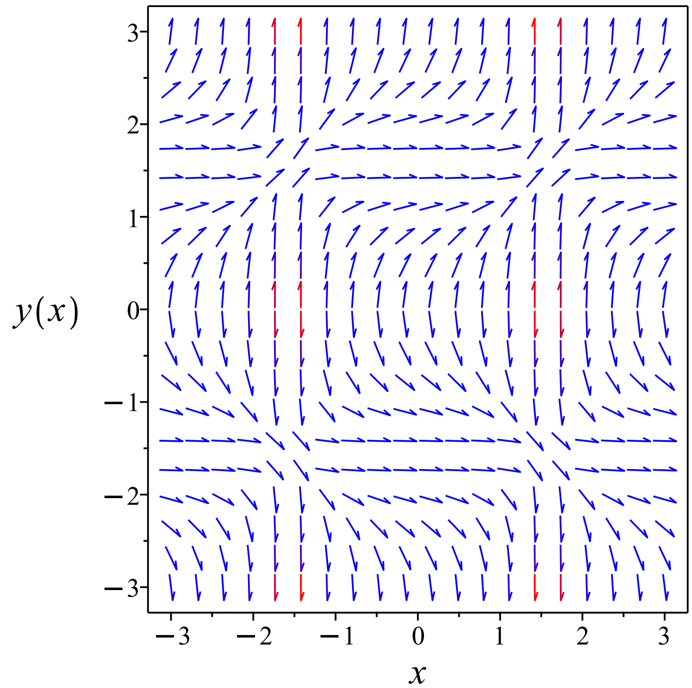


Figure 18: Slope field plot

Verification of solutions

$$y = \pi - \operatorname{arcsec}\left(-\tan\left(x\right) + c_1\right)$$

Verified OK.

2.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{\sin(y)}{\cos(y)^2}\right) \mathrm{d}y &= \left(\frac{\sec(x)}{\cos(x)}\right) \mathrm{d}x \\
\left(-\frac{\sec(x)}{\cos(x)}\right) \mathrm{d}x + \left(\frac{\sin(y)}{\cos(y)^2}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{\sec(x)}{\cos(x)} \\
N(x, y) &= \frac{\sin(y)}{\cos(y)^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sec(x)}{\cos(x)}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\sin(y)}{\cos(y)^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{\sec(x)}{\cos(x)} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\tan(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\sin(y)}{\cos(y)^2}$. Therefore equation (4) becomes

$$
\frac{\sin(y)}{\cos(y)^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= \frac{\sin(y)}{\cos(y)^2} \\
&= \tan(y)\sec(y)
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int (\tan(y)\sec(y))\,\mathrm{d}y$$
$$f(y) = \sec(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\tan(x) + \sec(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\tan(x) + \sec(y)$$

The solution becomes

$$y = \operatorname{arcsec}(\tan(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \operatorname{arcsec}(\tan(x) + c_1) \tag{1}$$

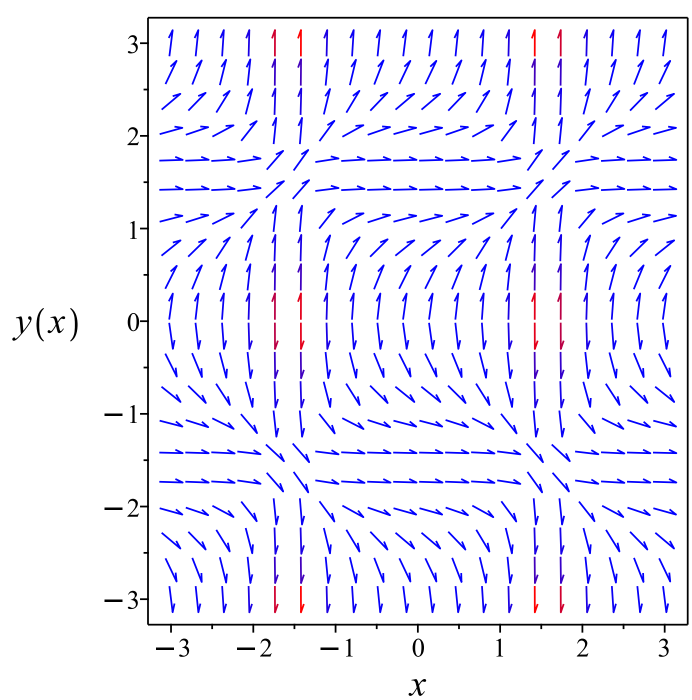


Figure 19: Slope field plot

Verification of solutions

$$y = \operatorname{arcsec}(\tan(x) + c_1)$$

Verified OK.

2.1.4 Maple step by step solution

Let's solve

$\sec(x)\cos(y)^2 - \cos(x)\sin(y)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' \sin(y)}{\cos(y)^2} = \frac{\sec(x)}{\cos(x)}$

- Integrate both sides with respect to x

  $\int \frac{y' \sin(y)}{\cos(y)^2} dx = \int \frac{\sec(x)}{\cos(x)} dx + c_1$

- Evaluate integral

  $\frac{1}{\cos(y)} = \tan(x) + c_1$

- Solve for y

  $y = \arccos\left(\frac{1}{\tan(x)+c_1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve((sec(x)*cos(y(x))^2)-(cos(x)*sin(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arccos\left(\frac{1}{\tan(x) + c_1}\right)$$

✓ Solution by Mathematica

Time used: 1.366 (sec). Leaf size: 45

```
DSolve[(Sec[x]*Cos[y[x]]^2)-(Cos[x]*Sin[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$\begin{aligned}
y(x) &\to -\sec^{-1}(\tan(x) + 2c_1) \\
y(x) &\to \sec^{-1}(\tan(x) + 2c_1) \\
y(x) &\to -\frac{\pi}{2} \\
y(x) &\to \frac{\pi}{2}
\end{aligned}$$

2.2 problem Ex 2

Internal problem ID [11128]
Internal file name [`OUTPUT/10113_Wednesday_November_23_2022_11_50_49_AM_47658040/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 9. Variables searated or separable. Page 13
Problem number: Ex 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(1+x)\,y^2 - y'x^3 = 0}$$

2.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{(1+x)\,y^2}{x^3} \end{aligned}$$

Where $f(x) = \frac{1+x}{x^3}$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = \frac{1+x}{x^3}\,dx$$

$$\int \frac{1}{y^2}\, dy = \int \frac{1+x}{x^3}\, dx$$
$$-\frac{1}{y} = -\frac{1}{2x^2} - \frac{1}{x} + c_1$$

Which results in

$$y = -\frac{2x^2}{2c_1x^2 - 2x - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{2x^2}{2c_1x^2 - 2x - 1} \tag{1}$$

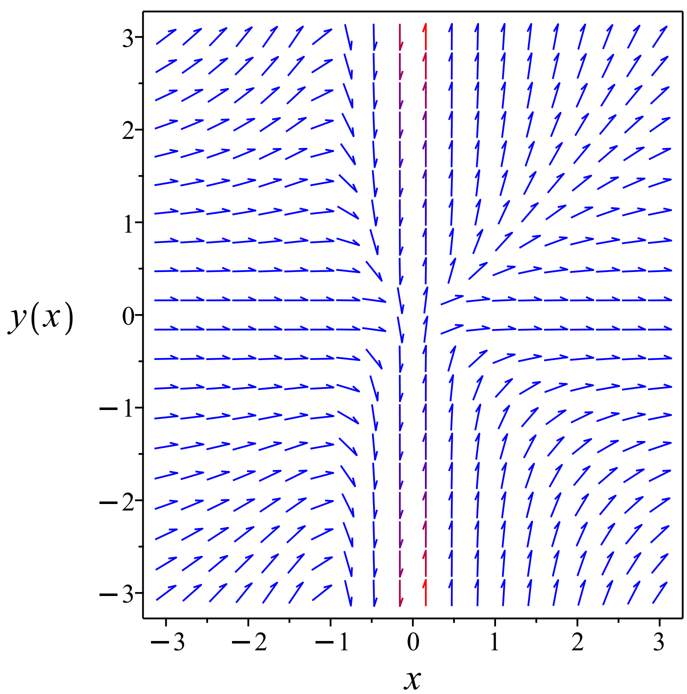


Figure 20: Slope field plot

Verification of solutions

$$y = -\frac{2x^2}{2c_1x^2 - 2x - 1}$$

Verified OK.

2.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{(1+x)\, y^2}{x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 11: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= \frac{x^3}{1+x} \\ \eta(x, y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x^3}{1+x}} dx\end{aligned}$$

Which results in

$$S = -\frac{1}{2x^2} - \frac{1}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{(1+x)\, y^2}{x^3}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1+x}{x^3} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{1}{R} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{-2x-1}{2x^2} = -\frac{1}{y} + c_1
$$

Which simplifies to

$$
\frac{-2x-1}{2x^2} = -\frac{1}{y} + c_1
$$

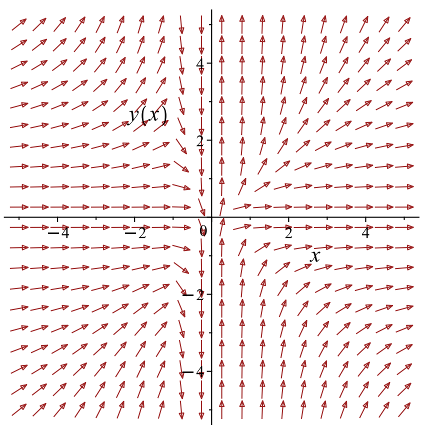
Which gives

$$
y = \frac{2x^2}{2c_1x^2 + 2x + 1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{(1+x)y^2}{x^3}$



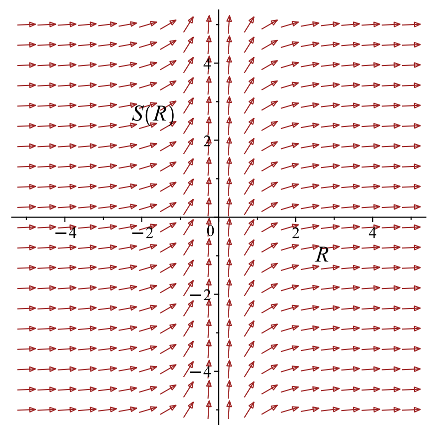
</td><td>

$R = y$

$S = \frac{-2x-1}{2x^2}$

</td><td>

$\frac{dS}{dR} = \frac{1}{R^2}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{2x^2}{2c_1x^2 + 2x + 1} \tag{1}$$

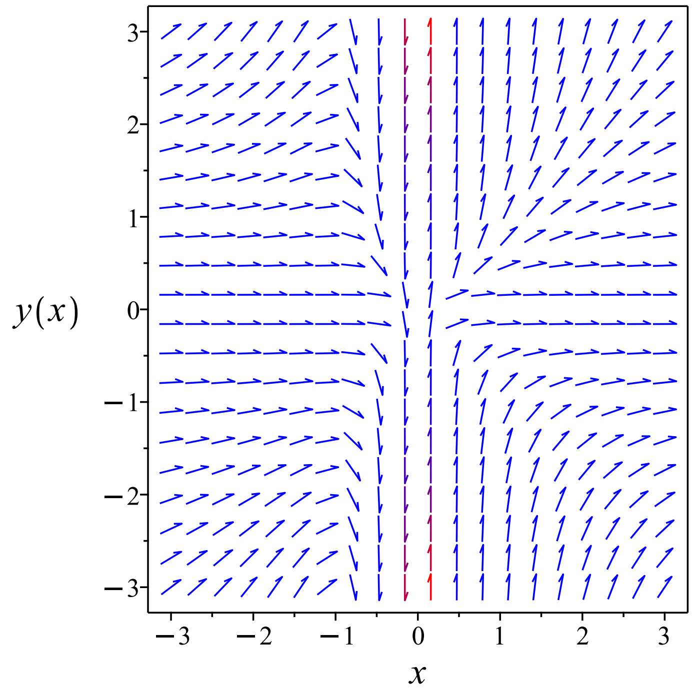


Figure 21: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{2x^2}{2c_1x^2 + 2x + 1}$$

Verified OK.

2.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^2}\right) \mathrm{d}y = \left(\frac{1 + x}{x^3}\right) \mathrm{d}x$$
$$\left(-\frac{1 + x}{x^3}\right) \mathrm{d}x + \left(\frac{1}{y^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1 + x}{x^3}$$
$$N(x, y) = \frac{1}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1 + x}{x^3}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1+x}{x^3}\,\mathrm{d}x\\ \phi &= \frac{2x+1}{2x^2} + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2}\right)\mathrm{d}y$$
$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{2x+1}{2x^2} - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{2x+1}{2x^2} - \frac{1}{y}$$

The solution becomes

$$y = -\frac{2x^2}{2c_1x^2 - 2x - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{2x^2}{2c_1x^2 - 2x - 1} \tag{1}$$

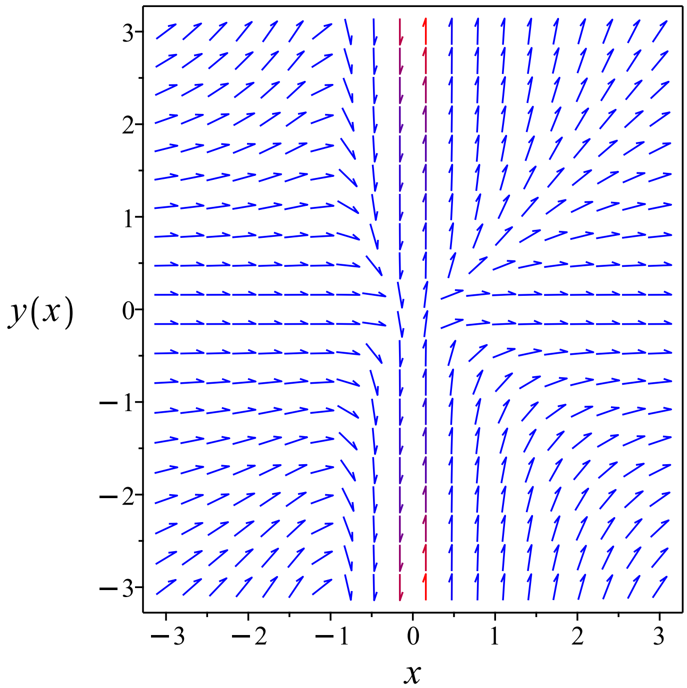


Figure 22: Slope field plot

Verification of solutions

$$y = -\frac{2x^2}{2c_1x^2 - 2x - 1}$$

Verified OK.

2.2.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{(1+x)\, y^2}{x^3} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x^3} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = \frac{1+x}{x^3}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{(1+x)u}{x^3}}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - \left(f_2' + f_1 f_2\right) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= \frac{1}{x^3} - \frac{3(1+x)}{x^4} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{(1+x)\, u''(x)}{x^3} - \left(\frac{1}{x^3} - \frac{3(1+x)}{x^4}\right) u'(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 + \frac{(2x+1)\, c_2}{x^2}
$$

The above shows that

$$
u'(x) = -\frac{2c_2(1+x)}{x^3}
$$

Using the above in (1) gives the solution

$$
y = \frac{2c_2}{c_1 + \frac{(2x+1)c_2}{x^2}}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{2x^2}{c_3 x^2 + 2x + 1}
$$

Summary
The solution(s) found are the following

$$y = \frac{2x^2}{c_3x^2 + 2x + 1} \tag{1}$$

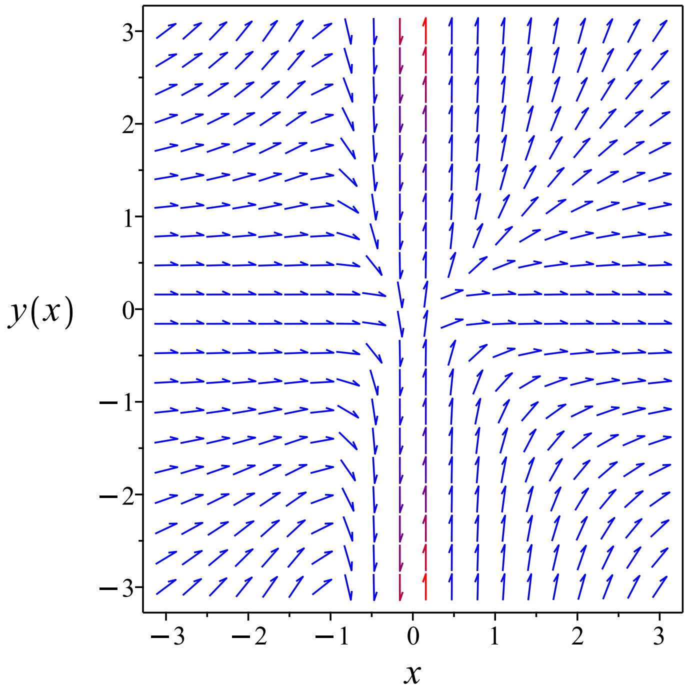


Figure 23: Slope field plot

Verification of solutions

$$y = \frac{2x^2}{c_3x^2 + 2x + 1}$$

Verified OK.

2.2.5 Maple step by step solution

Let's solve

$(1 + x)\, y^2 - y' x^3 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = \frac{1+x}{x^3}$

- Integrate both sides with respect to x

$\int \frac{y'}{y^2} dx = \int \frac{1+x}{x^3} dx + c_1$

- Evaluate integral

$-\frac{1}{y} = -\frac{1}{2x^2} - \frac{1}{x} + c_1$

- Solve for y

$y = -\frac{2x^2}{2c_1x^2 - 2x - 1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve((1+x)*y(x)^2-x^3*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{2x^2}{2c_1x^2 + 2x + 1}$$

✓ Solution by Mathematica

Time used: 0.231 (sec). Leaf size: 29

```
DSolve[(1+x)*y[x]^2-x^3*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2x^2}{-2c_1x^2 + 2x + 1}$$
$$y(x) \to 0$$

2.3 problem Ex 3

Internal problem ID [11129]
Internal file name [`OUTPUT/10114_Wednesday_November_23_2022_11_50_50_AM_16874930/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 9. Variables searated or separable. Page 13
Problem number: Ex 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{2(1-y^2)\,xy+(x^2+1)\,(1+y^2)\,y'=0}$$

2.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(2y^3-2y)}{(x^2+1)\,(y^2+1)} \end{aligned}$$

Where $f(x)=\frac{x}{x^2+1}$ and $g(y)=\frac{2y^3-2y}{y^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{2y^3-2y}{y^2+1}}\,dy=\frac{x}{x^2+1}\,dx$$

$$\int \frac{1}{\frac{2y^3-2y}{y^2+1}}\,dy = \int \frac{x}{x^2+1}\,dx$$

$$\frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \frac{\ln(y)}{2} = \frac{\ln(x^2+1)}{2} + c_1$$

The above can be written as

$$\begin{aligned}\left(\frac{1}{2}\right)(\ln(y-1)+\ln(y+1)-\ln(y)) &= \frac{\ln(x^2+1)}{2} + 2c_1\\ \ln(y-1)+\ln(y+1)-\ln(y) &= (2)\left(\frac{\ln(x^2+1)}{2} + 2c_1\right)\\ &= \ln\left(x^2+1\right) + 4c_1\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(y-1)+\ln(y+1)-\ln(y)} = e^{\ln\left(x^2+1\right)+2c_1}$$

Which simplifies to

$$\begin{aligned}\frac{y^2-1}{y} &= 2c_1\left(x^2+1\right)\\ &= c_2\left(x^2+1\right)\end{aligned}$$

The solution is

$$\frac{y^2-1}{y} = c_2\left(x^2+1\right)$$

Summary
The solution(s) found are the following

$$\frac{y^2-1}{y} = c_2\left(x^2+1\right) \tag{1}$$

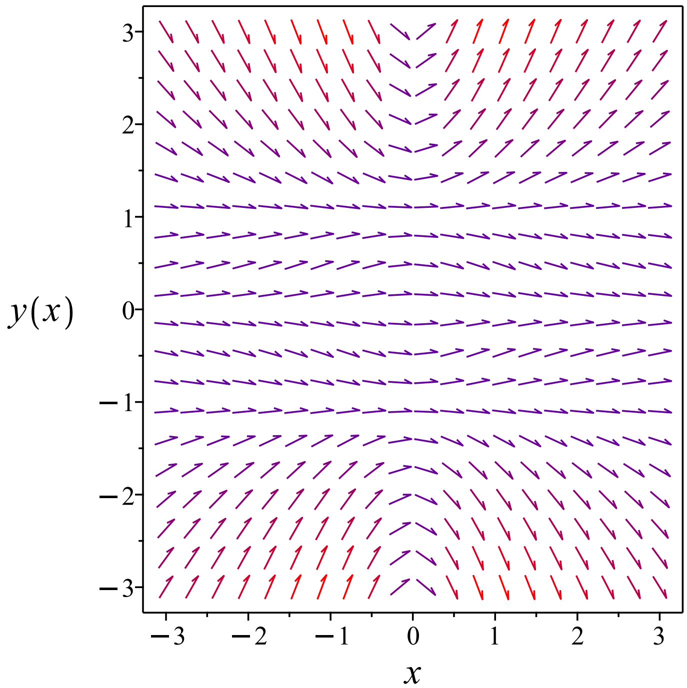


Figure 24: Slope field plot

Verification of solutions

$$\frac{y^2-1}{y} = c_2\left(x^2+1\right)$$

Verified OK.

2.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2xy(y^2-1)}{x^2y^2+x^2+y^2+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 14: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{x^2 + 1}{x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x^2+1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{\ln(x^2+1)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2xy(y^2-1)}{x^2y^2 + x^2 + y^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{x}{x^2+1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y^2+1}{2y^3 - 2y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^2+1}{2R^3 - 2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R-1)}{2} + \frac{\ln(R+1)}{2} - \frac{\ln(R)}{2} + c_1 \tag{4}$$

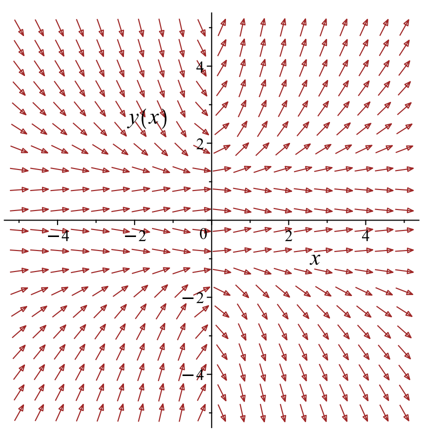
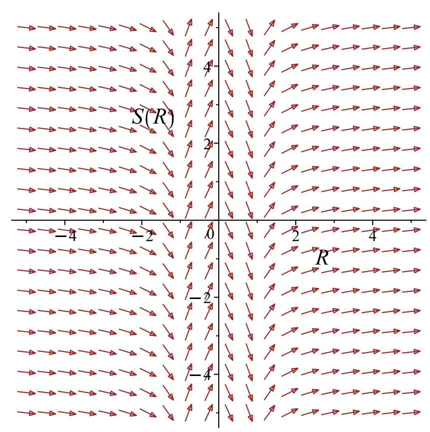
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2+1)}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \frac{\ln(y)}{2} + c_1$$

Which simplifies to

$$\frac{\ln(x^2+1)}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \frac{\ln(y)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2xy(y^2-1)}{x^2y^2+x^2+y^2+1}$ 	$R = y$ $S = \frac{\ln(x^2+1)}{2}$	$\frac{dS}{dR} = \frac{R^2+1}{2R^3-2R}$ 

Summary
The solution(s) found are the following

$$\frac{\ln(x^2+1)}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \frac{\ln(y)}{2} + c_1 \tag{1}$$

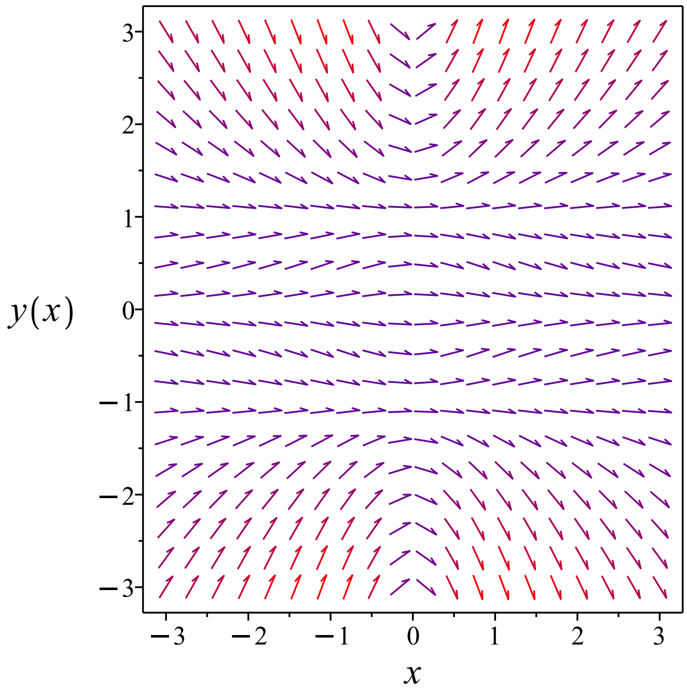


Figure 25: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+1)}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \frac{\ln(y)}{2} + c_1$$

Verified OK.

2.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y^2+1}{2y^3-2y}\right)\mathrm{d}y &= \left(\frac{x}{x^2+1}\right)\mathrm{d}x \\ \left(-\frac{x}{x^2+1}\right)\mathrm{d}x + \left(\frac{y^2+1}{2y^3-2y}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{x}{x^2+1} \\ N(x, y) &= \frac{y^2+1}{2y^3-2y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y^2+1}{2y^3-2y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{x^2+1}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{\ln(x^2+1)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2+1}{2y^3-2y}$. Therefore equation (4) becomes

$$\frac{y^2+1}{2y^3-2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y^2+1}{2y\,(y^2-1)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y^2+1}{2y^3-2y}\right)\mathrm{d}y$$
$$f(y) = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \frac{\ln(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} + \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} + \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \frac{\ln(y)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{\ln(x^2+1)}{2} + \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \frac{\ln(y)}{2} = c_1 \tag{1}$$

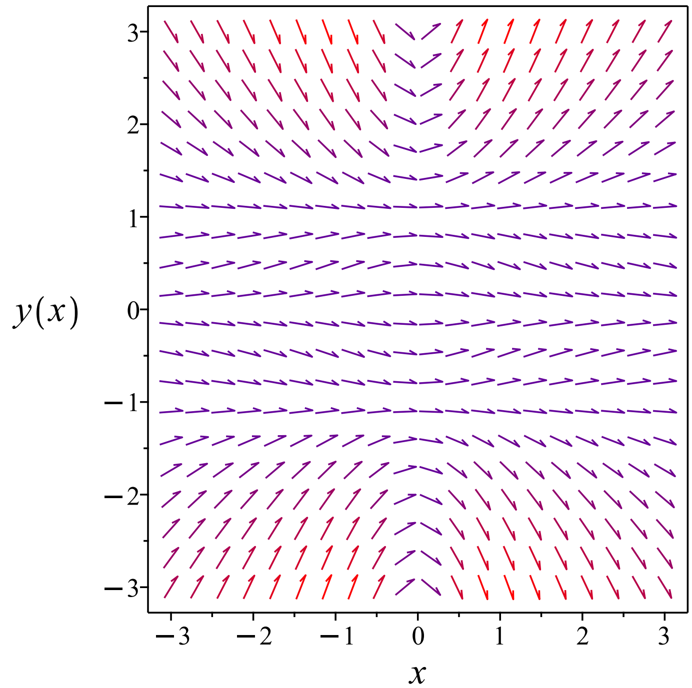


Figure 26: Slope field plot

Verification of solutions

$$-\frac{\ln\left(x^2+1\right)}{2}+\frac{\ln\left(y-1\right)}{2}+\frac{\ln\left(y+1\right)}{2}-\frac{\ln\left(y\right)}{2}=c_1$$

Verified OK.

2.3.4 Maple step by step solution

Let's solve

$2(1-y^2)\,xy+(x^2+1)\,(1+y^2)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(1+y^2)}{(1-y^2)y}=-\frac{2x}{x^2+1}$

- Integrate both sides with respect to x

  $\int\frac{y'(1+y^2)}{(1-y^2)y}dx=\int-\frac{2x}{x^2+1}dx+c_1$

- Evaluate integral

$$-\ln(y-1)-\ln(y+1)+\ln(y)=-\ln(x^2+1)+c_1$$

- Solve for y

$$\left\{y=-\frac{-x^2+\sqrt{x^4+4(e^{c_1})^2+2x^2+1}-1}{2\,e^{c_1}}, y=\frac{x^2+1+\sqrt{x^4+4(e^{c_1})^2+2x^2+1}}{2\,e^{c_1}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 61

```
dsolve(2*(1-y(x)^2)*x*y(x)+(1+x^2)*(1+y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1x^2}{2}+\frac{c_1}{2}-\frac{\sqrt{4+(x^2+1)^2\,c_1^2}}{2}$$

$$y(x)=\frac{c_1x^2}{2}+\frac{c_1}{2}+\frac{\sqrt{4+(x^2+1)^2\,c_1^2}}{2}$$

✓ Solution by Mathematica

Time used: 8.437 (sec). Leaf size: 98

```
DSolve[2*(1-y[x]^2)*x*y[x]+(1+x^2)*(1+y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x)\to\frac{1}{2}\left(-e^{c_1}(x^2+1)-\sqrt{4+e^{2c_1}(x^2+1)^2}\right)$$

$$y(x)\to\frac{1}{2}\left(\sqrt{4+e^{2c_1}(x^2+1)^2}-e^{c_1}(x^2+1)\right)$$

$$y(x)\to-1$$

$$y(x)\to 0$$

$$y(x)\to 1$$

2.4 problem Ex 4

Internal problem ID [11130]
Internal file name [`OUTPUT/10115_Wednesday_November_23_2022_11_50_51_AM_23588908/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 9. Variables searated or separable. Page 13
Problem number: Ex 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\sin(x)\cos(y)^2 + \cos(x)^2 y' = 0}$$

2.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{\sin(x)\cos(y)^2}{\cos(x)^2} \end{aligned}$$

Where $f(x) = -\frac{\sin(x)}{\cos(x)^2}$ and $g(y) = \cos(y)^2$. Integrating both sides gives

$$\frac{1}{\cos(y)^2}\,dy = -\frac{\sin(x)}{\cos(x)^2}\,dx$$

$$\int \frac{1}{\cos \left(y\right)^{2}}d y = \int -\frac{\sin \left(x\right)}{\cos \left(x\right)^{2}}d x$$

$$\tan \left(y\right) = -\frac{1}{\cos \left(x\right)}+c_{1}$$

Which results in

$$y = \arctan \left(\frac{c_{1} \cos \left(x\right)-1}{\cos \left(x\right)}\right)$$

Summary

The solution(s) found are the following

$$y = \arctan \left(\frac{c_{1} \cos \left(x\right)-1}{\cos \left(x\right)}\right) \tag{1}$$

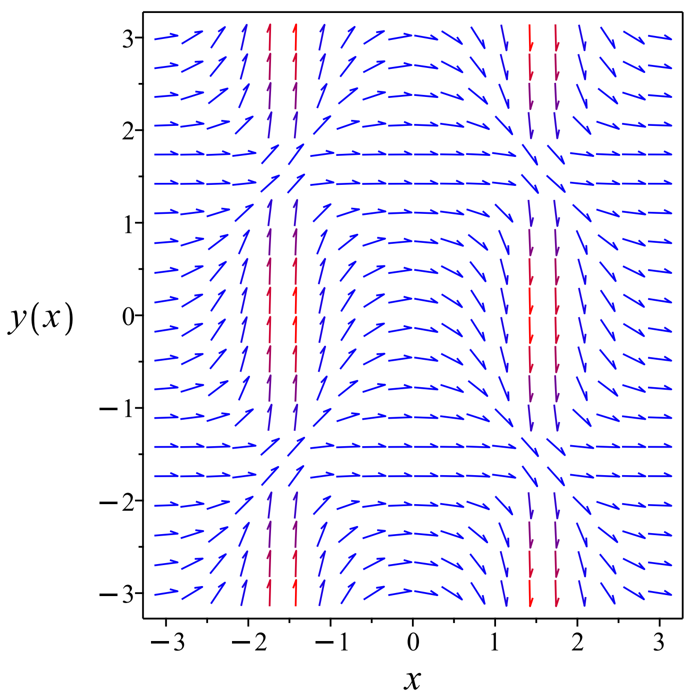


Figure 27: Slope field plot

Verification of solutions

$$y = \arctan \left(\frac{c_{1} \cos \left(x\right)-1}{\cos \left(x\right)}\right)$$

Verified OK.

2.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\sin(x)\cos(y)^2}{\cos(x)^2} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 17: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = -\frac{\cos(x)^2}{\sin(x)}$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{\cos(x)^2}{\sin(x)}} dx \end{aligned}$$

Which results in

$$S = -\frac{1}{\cos(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\sin(x)\cos(y)^2}{\cos(x)^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\sec(x)\tan(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sec(y)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sec(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \tan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\sec(x) = \tan(y) + c_1$$

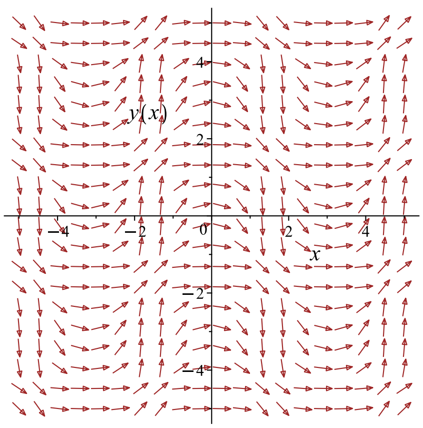
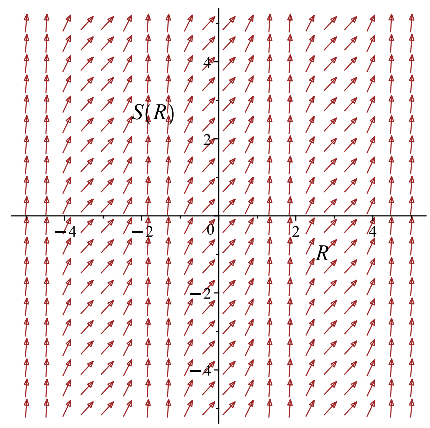
Which simplifies to

$$-\sec(x) = \tan(y) + c_1$$

Which gives

$$y = -\arctan(\sec(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sin(x)\cos(y)^2}{\cos(x)^2}$ 	$R = y$ $S = -\sec(x)$	$\frac{dS}{dR} = \sec(R)^2$ 

Summary

The solution(s) found are the following

$$y = -\arctan(\sec(x) + c_1) \tag{1}$$

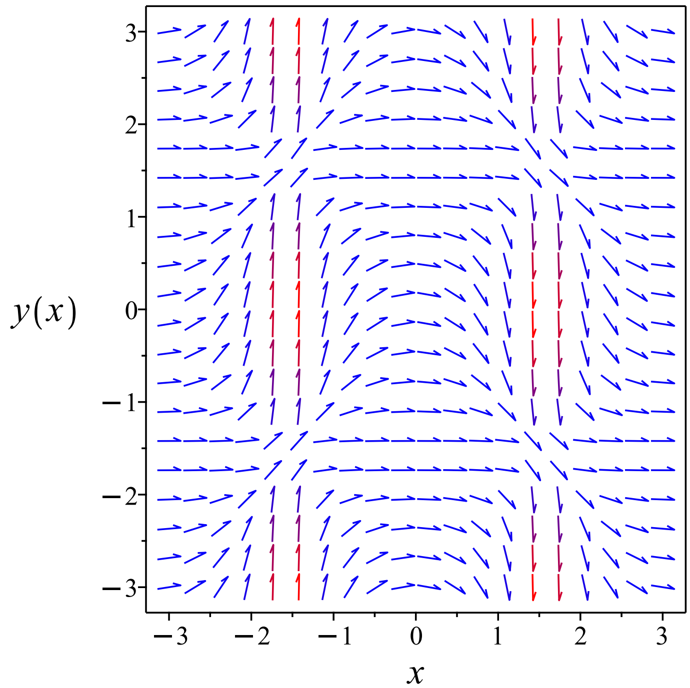


Figure 28: Slope field plot

Verification of solutions

$$y = -\arctan(\sec(x) + c_1)$$

Verified OK.

2.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{\cos(y)^2}\right) \mathrm{d}y = \left(\frac{\sin(x)}{\cos(x)^2}\right) \mathrm{d}x$$
$$\left(-\frac{\sin(x)}{\cos(x)^2}\right) \mathrm{d}x + \left(-\frac{1}{\cos(y)^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{\sin(x)}{\cos(x)^2}$$
$$N(x, y) = -\frac{1}{\cos(y)^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sin(x)}{\cos(x)^2}\right) \\ &= 0 \end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{\cos(y)^2}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{\sin(x)}{\cos(x)^2}\,\mathrm{d}x$$
$$\phi = -\sec(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{\cos(y)^2}$. Therefore equation (4) becomes

$$-\frac{1}{\cos(y)^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{\cos(y)^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\sec(y)^2\right)\mathrm{d}y$$
$$f(y) = -\tan(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sec(x) - \tan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sec(x) - \tan(y)$$

Summary

The solution(s) found are the following

$$-\sec(x) - \tan(y) = c_1 \tag{1}$$

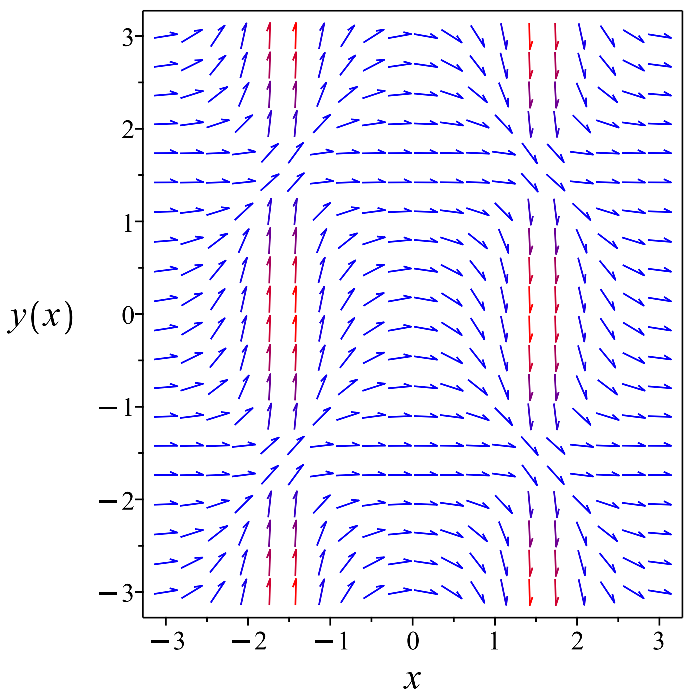


Figure 29: Slope field plot

Verification of solutions

$$-\sec(x) - \tan(y) = c_1$$

Verified OK.

2.4.4 Maple step by step solution

Let's solve

$\sin(x)\cos(y)^2+\cos(x)^2 y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\cos(y)^2}=-\frac{\sin(x)}{\cos(x)^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\cos(y)^2}dx=\int -\frac{\sin(x)}{\cos(x)^2}dx+c_1$

- Evaluate integral

  $\tan(y)=-\frac{1}{\cos(x)}+c_1$

- Solve for y

  $y=\arctan\left(\frac{c_1\cos(x)-1}{\cos(x)}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(sin(x)*cos(y(x))^2+cos(x)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=-\arctan(\sec(x)+c_1)$$

✓ Solution by Mathematica

Time used: 2.833 (sec). Leaf size: 31

```
DSolve[Sin[x]*Cos[y[x]]^2+Cos[x]^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arctan(-\sec(x) + c_1)$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

3 Chapter 2, differential equations of the first order and the first degree. Article 10. Homogeneous equations. Page 15

3.1 problem Ex 1

Internal problem ID [11131]
Internal file name [`OUTPUT/10116_Wednesday_November_23_2022_11_50_52_AM_1551047/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 10. Homogeneous equations. Page 15
Problem number: Ex 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{x\,\mathrm{e}^{\frac{y}{x}}+y-y'x=0}$$

3.1.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y'=\mathrm{e}^{\frac{y}{x}}+\frac{y}{x} \tag{A}$$

The given ode has the form

$$y'=\frac{y}{x}+g(x)\,f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x)=u(x)\,x$ then

$$\frac{dy}{dx}=\frac{du}{dx}x+u$$

Hence the given ode becomes

$$
\begin{aligned}
\frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\
u' &= \frac{1}{x} g(x)\, f(bu)^{\frac{n}{m}} \qquad (2)
\end{aligned}
$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$
\begin{aligned}
g(x) &= 1 \\
b &= 1 \\
f\left(\frac{bx}{y}\right) &= e^{\frac{y}{x}}
\end{aligned}
$$

Substituting the above in (2) results in the $u(x)$ ode as

$$
u'(x) = \frac{e^{u(x)}}{x}
$$

Which is now solved as separable In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= \frac{e^{u}}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = e^{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{e^{u}}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{e^{u}}\, du &= \int \frac{1}{x}\, dx \\
-e^{-u} &= \ln(x) + c_1
\end{aligned}
$$

The solution is

$$
-e^{-u(x)} - \ln(x) - c_1 = 0
$$

Therefore the solution is found using $y = ux$. Hence

$$
-e^{-\frac{y}{x}} - \ln(x) - c_1 = 0
$$

Summary

The solution(s) found are the following

$$-e^{-\frac{y}{x}}-\ln \left(x \right)-c_{1} = 0 \tag{1}$$

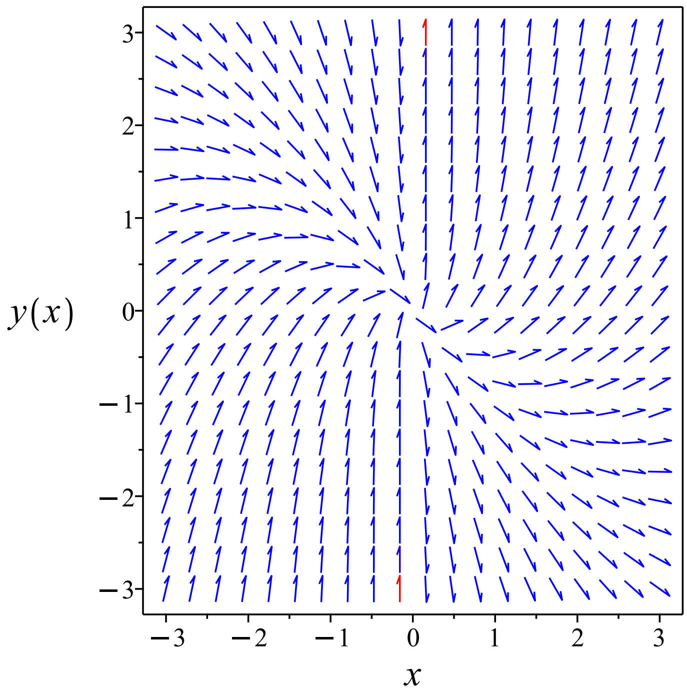


Figure 30: Slope field plot

Verification of solutions

$$-e^{-\frac{y}{x}}-\ln \left(x \right)-c_{1} = 0$$

Verified OK.

3.1.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x\,e^{u(x)}+u(x)\,x-\left(u'(x)\,x+u(x)\right)x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{e^{u}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = e^u$. Integrating both sides gives

$$\frac{1}{e^u} du = \frac{1}{x} dx$$
$$\int \frac{1}{e^u} du = \int \frac{1}{x} dx$$
$$-e^{-u} = \ln(x) + c_2$$

The solution is

$$-e^{-u(x)} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-e^{-\frac{y}{x}} - \ln(x) - c_2 = 0$$
$$-e^{-\frac{y}{x}} - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$-e^{-\frac{y}{x}} - \ln(x) - c_2 = 0 \tag{1}$$

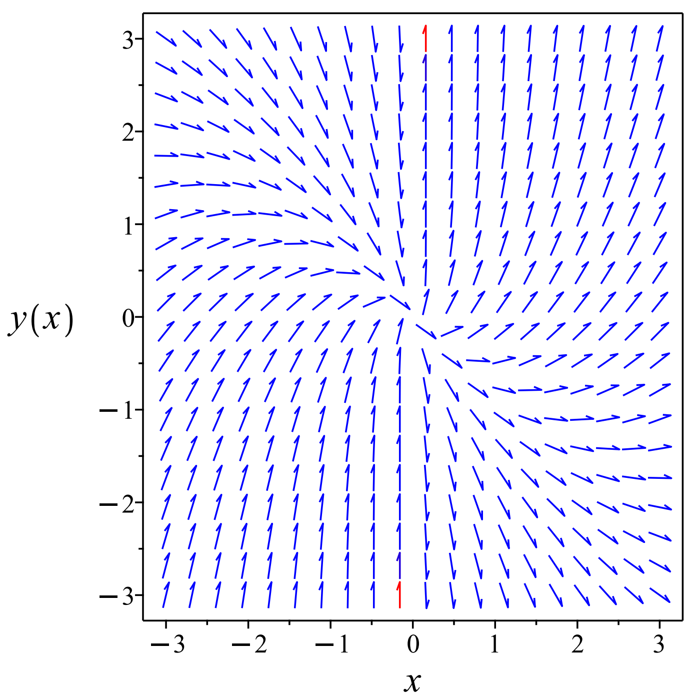


Figure 31: Slope field plot

Verification of solutions

$$-\mathrm{e}^{-\frac{y}{x}} - \ln(x) - c_2 = 0$$

Verified OK.

3.1.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x\,\mathrm{e}^{\frac{y}{x}} + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 20: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x^2 \\
\eta(x, y) &= xy
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{xy}{x^2} \\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x^2}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= -\frac{1}{x}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x\,\mathrm{e}^{\frac{y}{x}} + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{y}{x^2} \\ R_y &= \frac{1}{x} \\ S_x &= \frac{1}{x^2} \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{-\frac{y}{x}}}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -S(R)\, e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 e^{e^{-R}} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 e^{e^{-\frac{y}{x}}}$$

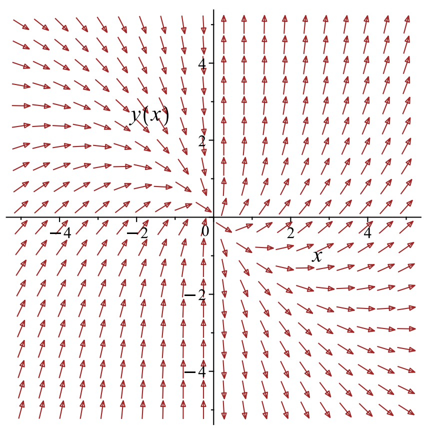
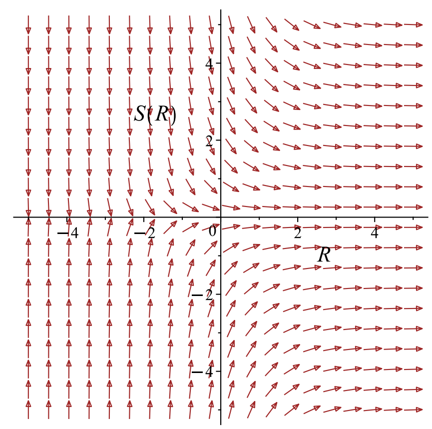
Which simplifies to

$$-\frac{1}{x} = c_1 e^{e^{-\frac{y}{x}}}$$

Which gives

$$y = -\ln\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x\,\mathrm{e}^{\frac{y}{x}}+y}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -S(R)\,\mathrm{e}^{-R}$ 

Summary

The solution(s) found are the following

$$y = -\ln\left(\ln\left(-\frac{1}{c_1 x}\right)\right)x \tag{1}$$

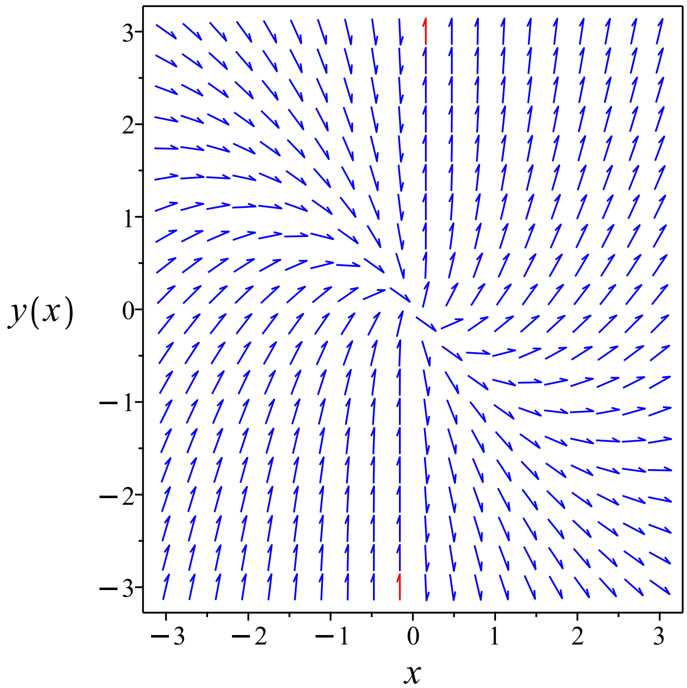


Figure 32: Slope field plot

Verification of solutions

$$y = -\ln\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x*exp(y(x)/x)+y(x))-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \ln\left(-\frac{1}{\ln(x) + c_1}\right) x$$

✓ Solution by Mathematica

Time used: 0.527 (sec). Leaf size: 18

```
DSolve[(x*Exp[y[x]/x]+y[x])-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x \log(-\log(x) - c_1)$$

3.2 problem Ex 2

Internal problem ID [11132]
Internal file name [`OUTPUT/10117_Wednesday_November_23_2022_11_50_53_AM_96544115/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 10. Homogeneous equations. Page 15
Problem number: Ex 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{2x^2y+3y^3-\left(x^3+2y^2x\right)y'=0}$$

3.2.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2x^3u(x)+3u(x)^3\,x^3-\left(x^3+2u(x)^2\,x^3\right)\left(u'(x)\,x+u(x)\right)=0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= \frac{u(u^2+1)}{(2u^2+1)\,x}
\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u(u^2+1)}{2u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u^2+1)}{2u^2+1}}\, du = \frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u(u^2+1)}{2u^2+1}}\, du = \int \frac{1}{x}\, dx$$

$$\ln(u) + \frac{\ln(u^2+1)}{2} = \ln(x) + c_2$$

Raising both side to exponential gives

$$e^{\ln(u)+\frac{\ln\left(u^2+1\right)}{2}} = e^{\ln(x)+c_2}$$

Which simplifies to

$$u\sqrt{u^2+1} = c_3 x$$

The solution is

$$u(x)\sqrt{u(x)^2+1} = c_3 x$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y\sqrt{\frac{y^2}{x^2}+1}}{x} = c_3 x$$

$$\frac{y\sqrt{\frac{x^2+y^2}{x^2}}}{x} = c_3 x$$

Summary

The solution(s) found are the following

$$\frac{y\sqrt{\frac{x^2+y^2}{x^2}}}{x} = c_3 x \tag{1}$$

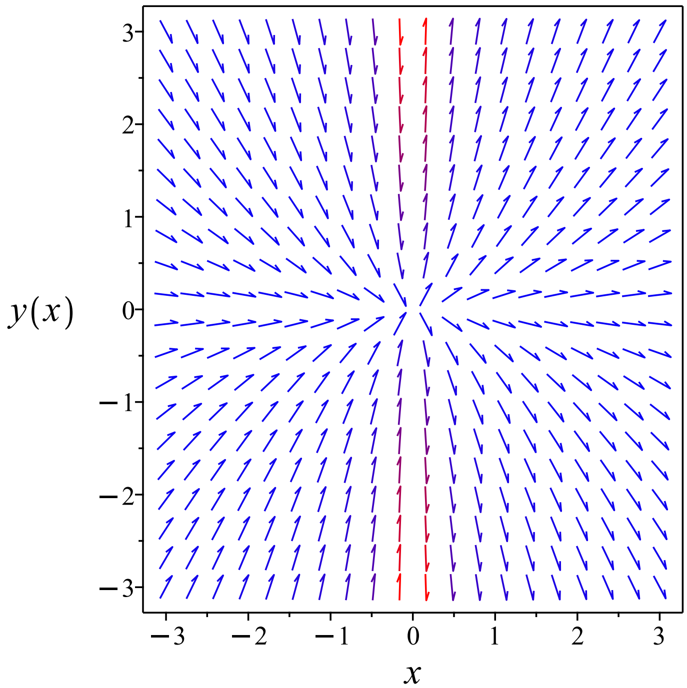


Figure 33: Slope field plot

<u>Verification of solutions</u>

$$\frac{y\sqrt{\frac{x^2+y^2}{x^2}}}{x} = c_3 x$$

Verified OK.

3.2.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y(2x^2+3y^2)}{x\,(x^2+2y^2)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{y(2x^2+3y^2)\,(b_3-a_2)}{x\,(x^2+2y^2)} - \frac{y^2(2x^2+3y^2)^2\,a_3}{x^2\,(x^2+2y^2)^2} \\
&- \left(\frac{4y}{x^2+2y^2} - \frac{y(2x^2+3y^2)}{x^2\,(x^2+2y^2)} - \frac{2y(2x^2+3y^2)}{(x^2+2y^2)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(\frac{2x^2+3y^2}{x\,(x^2+2y^2)} + \frac{6y^2}{x\,(x^2+2y^2)} - \frac{4y^2(2x^2+3y^2)}{x\,(x^2+2y^2)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^6b_2 + 2x^4y^2a_3 + x^4y^2b_2 + 2x^3y^3a_2 - 2x^3y^3b_3 + 7x^2y^4a_3 + 2x^2y^4b_2 + 3y^6a_3 + 2x^5b_1 - 2x^4ya_1 + 5x^3y^2b_1}{(x^2+2y^2)^2\,x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-x^6b_2 - 2x^4y^2a_3 - x^4y^2b_2 - 2x^3y^3a_2 + 2x^3y^3b_3 - 7x^2y^4a_3 - 2x^2y^4b_2 \\
&- 3y^6a_3 - 2x^5b_1 + 2x^4ya_1 - 5x^3y^2b_1 + 5x^2y^3a_1 - 6x\,y^4b_1 + 6y^5a_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-2a_2v_1^3v_2^3 - 2a_3v_1^4v_2^2 - 7a_3v_1^2v_2^4 - 3a_3v_2^6 - b_2v_1^6 - b_2v_1^4v_2^2 - 2b_2v_1^2v_2^4 \\
&+ 2b_3v_1^3v_2^3 + 2a_1v_1^4v_2 + 5a_1v_1^2v_2^3 + 6a_1v_2^5 - 2b_1v_1^5 - 5b_1v_1^3v_2^2 - 6b_1v_1v_2^4 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_2 v_1^6 - 2b_1 v_1^5 + (-2a_3 - b_2) v_1^4 v_2^2 + 2a_1 v_1^4 v_2 + (-2a_2 + 2b_3) v_1^3 v_2^3 \\ - 5b_1 v_1^3 v_2^2 + (-7a_3 - 2b_2) v_1^2 v_2^4 + 5a_1 v_1^2 v_2^3 - 6b_1 v_1 v_2^4 - 3a_3 v_2^6 + 6a_1 v_2^5 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 &= 0 \\ 5a_1 &= 0 \\ 6a_1 &= 0 \\ -3a_3 &= 0 \\ -6b_1 &= 0 \\ -5b_1 &= 0 \\ -2b_1 &= 0 \\ -b_2 &= 0 \\ -2a_2 + 2b_3 &= 0 \\ -7a_3 - 2b_2 &= 0 \\ -2a_3 - b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y(2x^2+3y^2)}{x\,(x^2+2y^2)}\right)(x) \\
&= \frac{-x^2y - y^3}{x^2+2y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta}dy \\
&= \int \frac{1}{\frac{-x^2y-y^3}{x^2+2y^2}}dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln\left(x^2+y^2\right)}{2} - \ln\left(y\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(2x^2+3y^2)}{x\,(x^2+2y^2)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{x}{x^2+y^2} \\
S_y &= -\frac{y}{x^2+y^2} - \frac{1}{y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{3}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{3}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -3\ln(R) + c_1 \tag{4}
$$

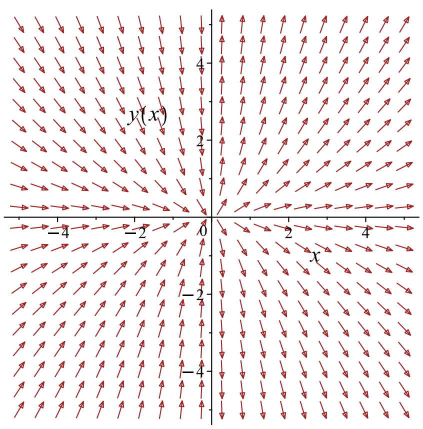
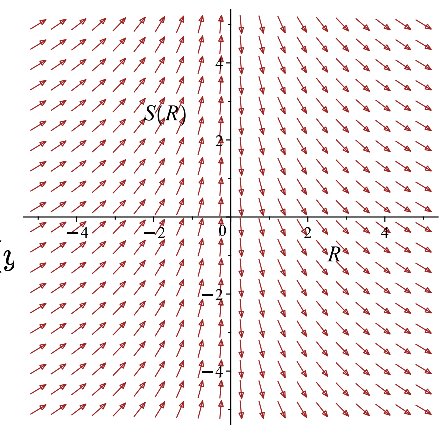
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{\ln(x^2+y^2)}{2} - \ln(y) = -3\ln(x) + c_1
$$

Which simplifies to

$$
-\frac{\ln(x^2+y^2)}{2} - \ln(y) = -3\ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(2x^2+3y^2)}{x(x^2+2y^2)}$  	$R = x$ $S = -\frac{\ln(x^2+y^2)}{2} - \ln(y$	$\frac{dS}{dR} = -\frac{3}{R}$ 

Summary
The solution(s) found are the following

$$-\frac{\ln(x^2+y^2)}{2} - \ln(y) = -3\ln(x) + c_1 \tag{1}$$

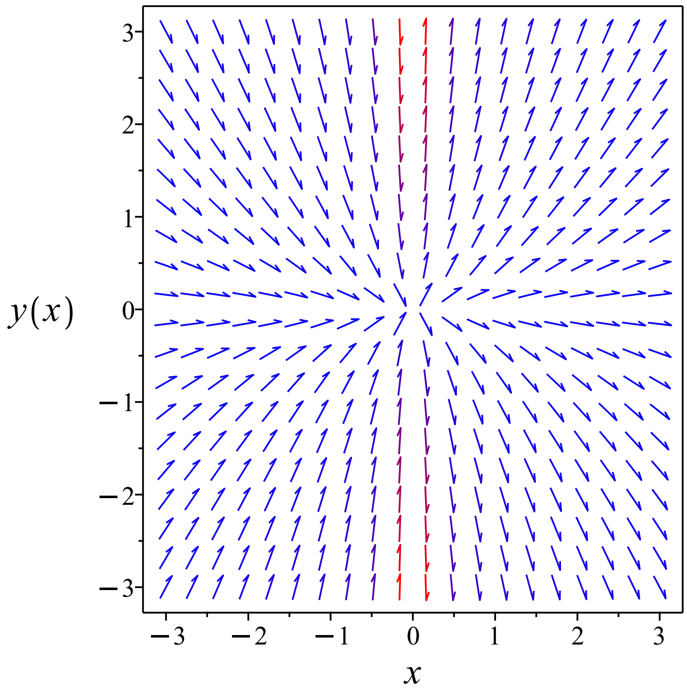


Figure 34: Slope field plot

Verification of solutions

$$-\frac{\ln(x^2+y^2)}{2} - \ln(y) = -3\ln(x) + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.641 (sec). Leaf size: 89

```
dsolve((2*x^2*y(x)+3*y(x)^3)-(x^3+2*x*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2-2\sqrt{4c_1x^2+1}}\,x}{2}$$
$$y(x) = \frac{\sqrt{-2-2\sqrt{4c_1x^2+1}}\,x}{2}$$
$$y(x) = -\frac{\sqrt{-2+2\sqrt{4c_1x^2+1}}\,x}{2}$$
$$y(x) = \frac{\sqrt{-2+2\sqrt{4c_1x^2+1}}\,x}{2}$$

✓ Solution by Mathematica

Time used: 47.499 (sec). Leaf size: 277

```
DSolve[(2*x^2*y[x]+3*y[x]^3)-(x^3+2*x*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to -\frac{\sqrt{-x^2-\sqrt{x^4+4e^{2c_1}x^6}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-x^2-\sqrt{x^4+4e^{2c_1}x^6}}}{\sqrt{2}}$$
$$y(x) \to -\frac{\sqrt{-x^2+\sqrt{x^4+4e^{2c_1}x^6}}}{\sqrt{2}}$$
$$y(x) \to \sqrt{-\frac{x^2}{2}+\frac{1}{2}\sqrt{x^4+4e^{2c_1}x^6}}$$
$$y(x) \to -\frac{\sqrt{-\sqrt{x^4}-x^2}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-\sqrt{x^4}-x^2}}{\sqrt{2}}$$
$$y(x) \to -\frac{\sqrt{\sqrt{x^4}-x^2}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{\sqrt{x^4}-x^2}}{\sqrt{2}}$$

3.3 problem Ex 3

Internal problem ID [11133]
Internal file name [`OUTPUT/10118_Wednesday_November_23_2022_11_50_55_AM_49001000/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 10. Homogeneous equations. Page 15
Problem number: Ex 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 - yx + y'x^2 = 0}$$

3.3.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 - u(x)\,x^2 + (u'(x)\,x + u(x))\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u^2$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u^2}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u^2}\,du &= \int -\frac{1}{x}\,dx \\
-\frac{1}{u} &= -\ln(x) + c_2
\end{aligned}
$$

The solution is

$$
-\frac{1}{u(x)} + \ln(x) - c_2 = 0
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
-\frac{x}{y} + \ln(x) - c_2 = 0 \\
-\frac{x}{y} + \ln(x) - c_2 = 0
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
-\frac{x}{y} + \ln(x) - c_2 = 0 \tag{1}
$$

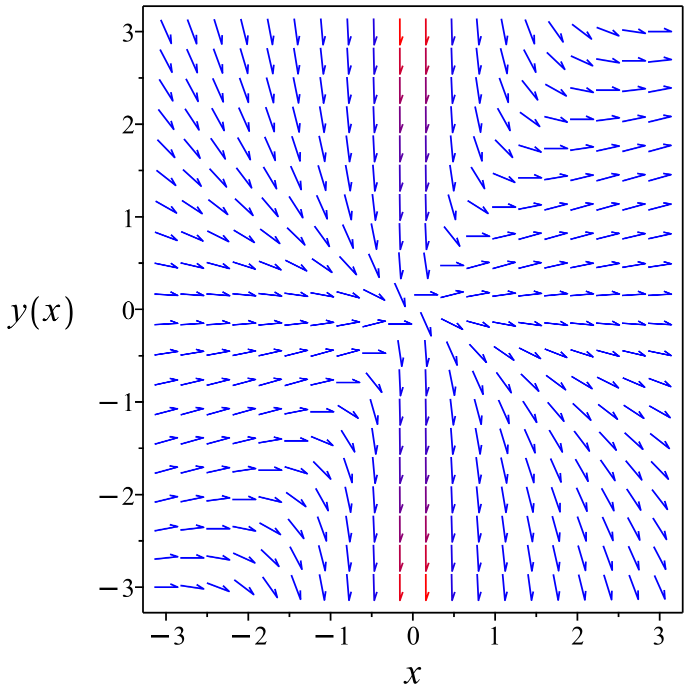


Figure 35: Slope field plot

Verification of solutions

$$-\frac{x}{y}+\ln(x)-c_2=0$$

Verified OK.

3.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(y-x)}{x^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 22: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{y^2}{x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y^2}{x}} dy
\end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(y - x)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{y} \\
S_y &= \frac{x}{y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = -\ln(x) + c_1$$

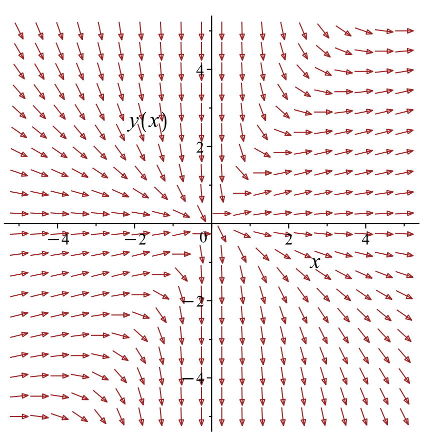
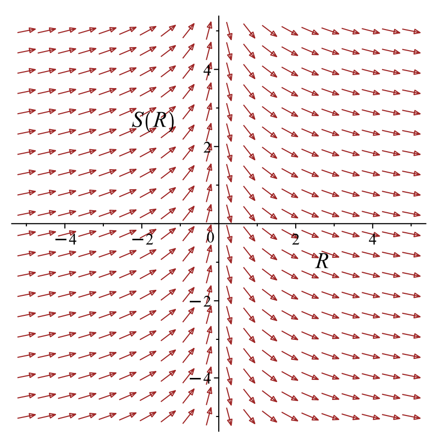
Which simplifies to

$$-\frac{x}{y} = -\ln(x) + c_1$$

Which gives

$$y = \frac{x}{\ln(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(y-x)}{x^2}$ 	$R = x$ $S = -\frac{x}{y}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary
The solution(s) found are the following

$$y = \frac{x}{\ln(x) - c_1} \tag{1}$$

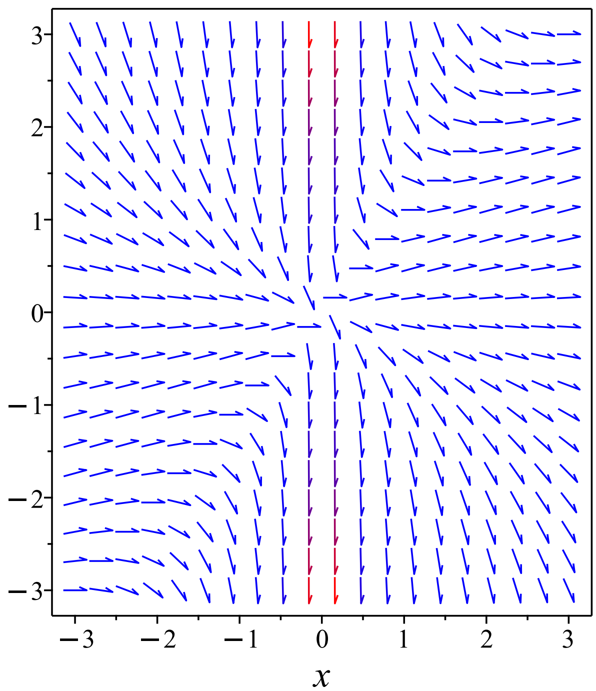


Figure 36: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) - c_1}$$

Verified OK.

3.3.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(y-x)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y - \frac{1}{x^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= -\frac{1}{x^2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{xy} - \frac{1}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} - \frac{1}{x^2} \\ w' &= -\frac{w}{x} + \frac{1}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = \frac{1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x}dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\left(\frac{1}{x^2}\right) \\ \frac{d}{dx}(wx) &= (x)\left(\frac{1}{x^2}\right) \\ d(wx) &= \frac{1}{x}\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} wx &= \int \frac{1}{x}\, dx \\ wx &= \ln(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = \frac{\ln(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$w(x) = \frac{\ln(x) + c_1}{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{\ln(x) + c_1}{x}$$

Or

$$y = \frac{x}{\ln(x) + c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{\ln(x) + c_1} \tag{1}$$

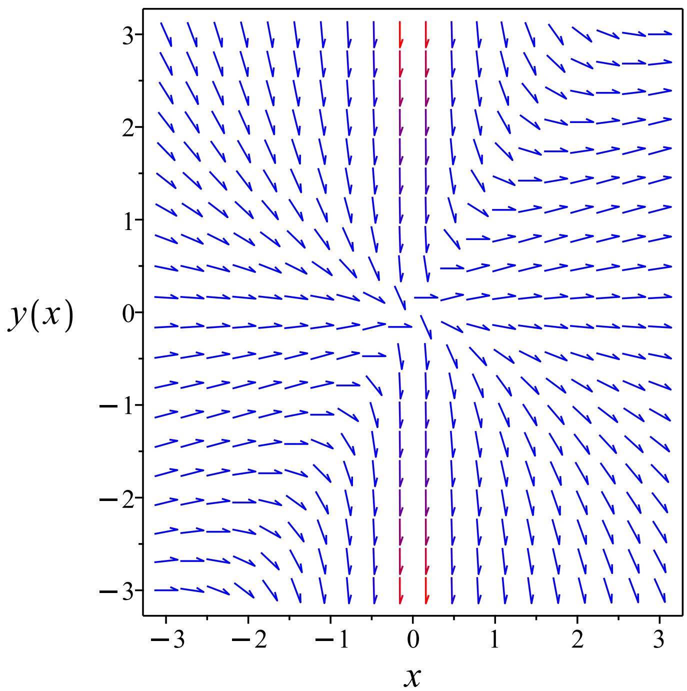


Figure 37: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) + c_1}$$

Verified OK.

3.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2)\,\mathrm{d}y &= (xy - y^2)\,\mathrm{d}x\\ (-xy + y^2)\,\mathrm{d}x + (x^2)\,\mathrm{d}y &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -xy + y^2\\ N(x,y) &= x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-xy+y^2\right)\\&=-x+2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(x^2\right)\\&=2x\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M=y^2-yx$ and $N=x^2$ by this integrating factor the ode becomes exact. The new M,N are

$$\begin{aligned}M&=\frac{y^2-yx}{xy^2}\\N&=\frac{x}{y^2}\end{aligned}$$

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\\frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{x}{y^2}\right)\mathrm{d}y&=\left(-\frac{-xy+y^2}{x\,y^2}\right)\mathrm{d}x\\ \left(\frac{-xy+y^2}{x\,y^2}\right)\mathrm{d}x+\left(\frac{x}{y^2}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\frac{-xy+y^2}{x\,y^2}\\ N(x,y)&=\frac{x}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(\frac{-xy+y^2}{x\,y^2}\right)\\ &=\frac{1}{y^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(\frac{x}{y^2}\right)\\ &=\frac{1}{y^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-xy + y^2}{x\,y^2} \mathrm{d}x$$

$$\phi = \ln(x) - \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{y^2}$. Therefore equation (4) becomes

$$\frac{x}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) - \frac{x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) - \frac{x}{y}$$

The solution becomes

$$y = \frac{x}{\ln(x) - c_1}$$

Summary
The solution(s) found are the following

$$y = \frac{x}{\ln(x) - c_1} \tag{1}$$

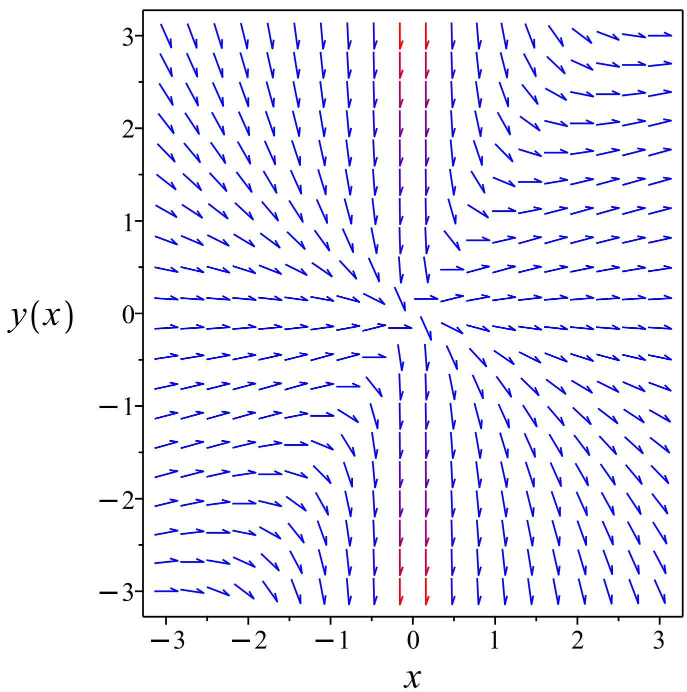


Figure 38: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) - c_1}$$

Verified OK.

3.3.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(y-x)}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x} - \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = -\frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2}{x^3} \\ f_1 f_2 &= -\frac{1}{x^3} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x^2} - \frac{u'(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 \ln(x) + c_1$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$y = \frac{c_2 x}{c_2 \ln(x) + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x}{\ln(x) + c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{\ln(x) + c_3} \tag{1}$$

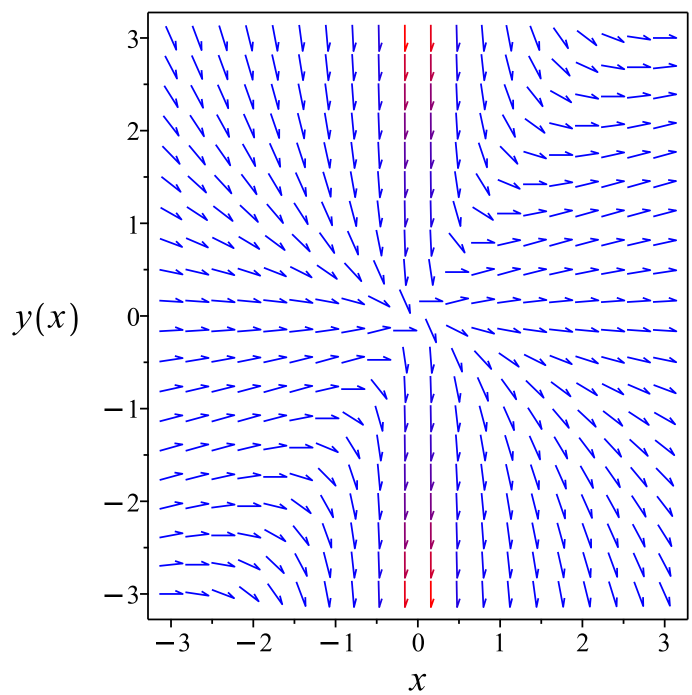


Figure 39: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln (x) + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve((y(x)^2-x*y(x))+x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{\ln (x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 19

```
DSolve[(y[x]^2-x*y[x])+x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{\log(x) + c_1}$$
$$y(x) \to 0$$

3.4 problem Ex 4

Internal problem ID [11134]
Internal file name [`OUTPUT/10119_Wednesday_November_23_2022_11_50_56_AM_16570338/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 10. Homogeneous equations. Page 15
Problem number: Ex 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{2x^2y + y^3 - y'x^3 = 0}$$

3.4.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2x^3u(x) + u(x)^3\,x^3 - (u'(x)\,x + u(x))\,x^3 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(u^2+1)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u(u^2+1)$. Integrating both sides gives

$$\frac{1}{u\left(u^2+1\right)}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{u\left(u^2+1\right)}\,du = \int \frac{1}{x}\,dx$$
$$\ln\left(u\right) - \frac{\ln\left(u^2+1\right)}{2} = \ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(u)-\frac{\ln\left(u^2+1\right)}{2}} = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{u}{\sqrt{u^2+1}} = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x^2 c_3 \sqrt{-\frac{1}{c_3^2 x^2 - 1}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^2 c_3 \sqrt{-\frac{1}{c_3^2 x^2 - 1}} \tag{1}$$

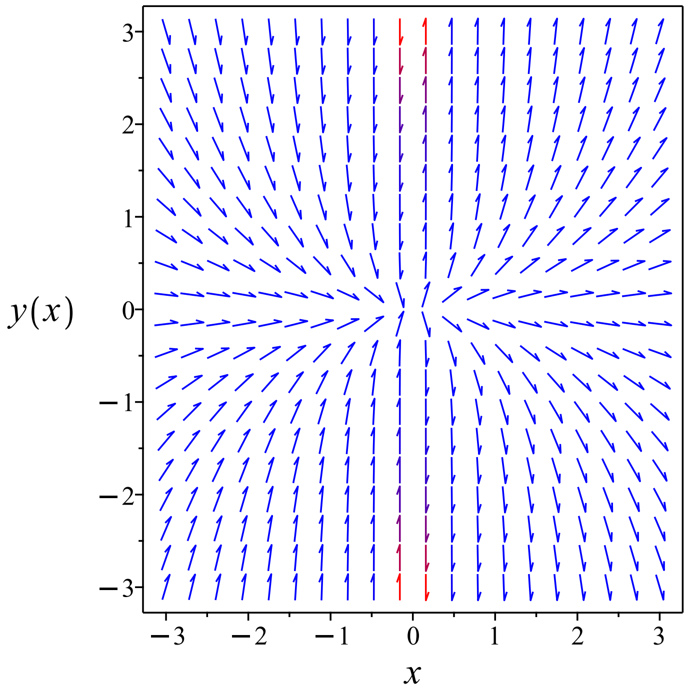


Figure 40: Slope field plot

<u>Verification of solutions</u>

$$y = x^2 c_3 \sqrt{-\frac{1}{c_3^2 x^2 - 1}}$$

Verified OK.

3.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(2x^2 + y^2)}{x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 24: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{y^3}{x^4}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^3}{x^4}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^4}{2y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(2x^2 + y^2)}{x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2x^3}{y^2} \\ S_y &= \frac{x^4}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

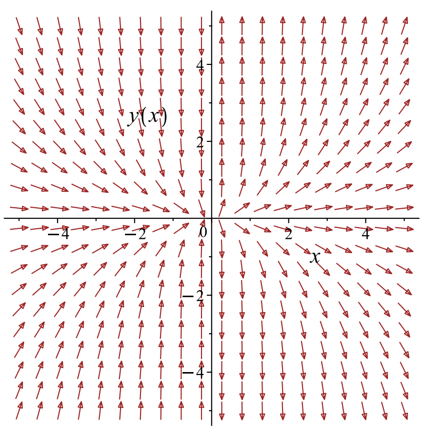
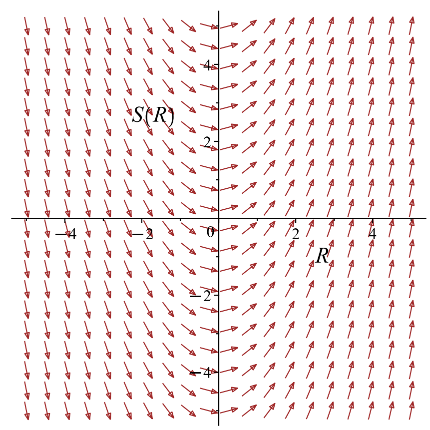
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^4}{2y^2} = \frac{x^2}{2} + c_1$$

Which simplifies to

$$-\frac{x^4}{2y^2} = \frac{x^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(2x^2+y^2)}{x^3}$ 	$R = x$ $S = -\frac{x^4}{2y^2}$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$-\frac{x^4}{2y^2} = \frac{x^2}{2} + c_1 \tag{1}$$

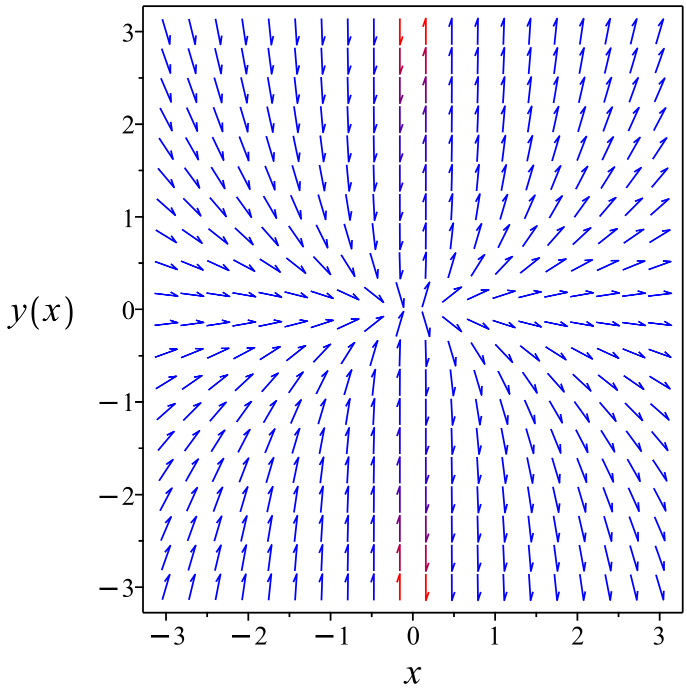


Figure 41: Slope field plot

Verification of solutions

$$-\frac{x^4}{2y^2} = \frac{x^2}{2} + c_1$$

Verified OK.

3.4.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(2x^2 + y^2)}{x^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{2}{x}y + \frac{1}{x^3}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{2}{x} \\ f_1(x) &= \frac{1}{x^3} \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y' \frac{1}{y^3} = \frac{2}{x\, y^2} + \frac{1}{x^3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{2} &= \frac{2w(x)}{x} + \frac{1}{x^3} \\ w' &= -\frac{4w}{x} - \frac{2}{x^3} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= -\frac{2}{x^3} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{4w(x)}{x} = -\frac{2}{x^3}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{4}{x}dx} \\ &= x^4\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{2}{x^3}\right) \\ \frac{d}{dx}\left(x^4 w\right) &= \left(x^4\right)\left(-\frac{2}{x^3}\right) \\ d\left(x^4 w\right) &= (-2x)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^4 w &= \int -2x\, dx \\ x^4 w &= -x^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^4$ results in

$$w(x) = -\frac{1}{x^2} + \frac{c_1}{x^4}$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = -\frac{1}{x^2} + \frac{c_1}{x^4}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{x^2}{\sqrt{-x^2 + c_1}} \\ y(x) &= -\frac{x^2}{\sqrt{-x^2 + c_1}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{\sqrt{-x^2 + c_1}} \tag{1}$$

$$y = -\frac{x^2}{\sqrt{-x^2 + c_1}} \tag{2}$$

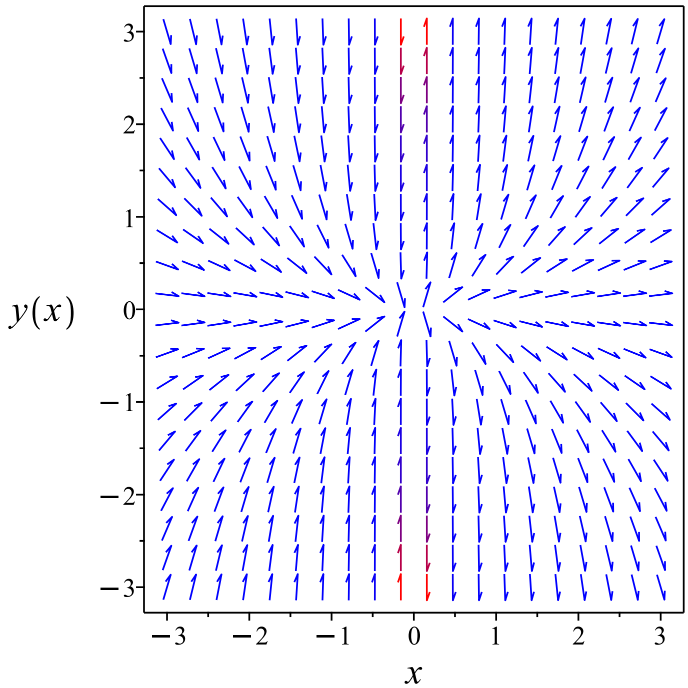


Figure 42: Slope field plot

Verification of solutions

$$y = \frac{x^2}{\sqrt{-x^2 + c_1}}$$

Verified OK.

$$y = -\frac{x^2}{\sqrt{-x^2 + c_1}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(2*x^2*y(x)+y(x)^3-x^3*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{\sqrt{-x^2+c_1}}$$
$$y(x) = -\frac{x^2}{\sqrt{-x^2+c_1}}$$

✓ Solution by Mathematica

Time used: 0.181 (sec). Leaf size: 47

```
DSolve[2*x^2*y[x]+y[x]^3-x^3*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2}{\sqrt{-x^2+c_1}}$$
$$y(x) \to \frac{x^2}{\sqrt{-x^2+c_1}}$$
$$y(x) \to 0$$

3.5 problem Ex 5

Internal problem ID [11135]
Internal file name [`OUTPUT/10120_Wednesday_November_23_2022_11_50_57_AM_90587072/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 10. Homogeneous equations. Page 15
Problem number: Ex 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y^3 + y'x^3 = 0}$$

3.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y^3}{x^3} \end{aligned}$$

Where $f(x) = -\frac{1}{x^3}$ and $g(y) = y^3$. Integrating both sides gives

$$\frac{1}{y^3}\,dy = -\frac{1}{x^3}\,dx$$

$$\int \frac{1}{y^3}\, dy = \int -\frac{1}{x^3}\, dx$$
$$-\frac{1}{2y^2} = \frac{1}{2x^2} + c_1$$

Which results in

$$y = -\frac{x}{\sqrt{-2c_1x^2 - 1}}$$
$$y = \frac{x}{\sqrt{-2c_1x^2 - 1}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{x}{\sqrt{-2c_1x^2 - 1}} \tag{1}$$
$$y = \frac{x}{\sqrt{-2c_1x^2 - 1}} \tag{2}$$

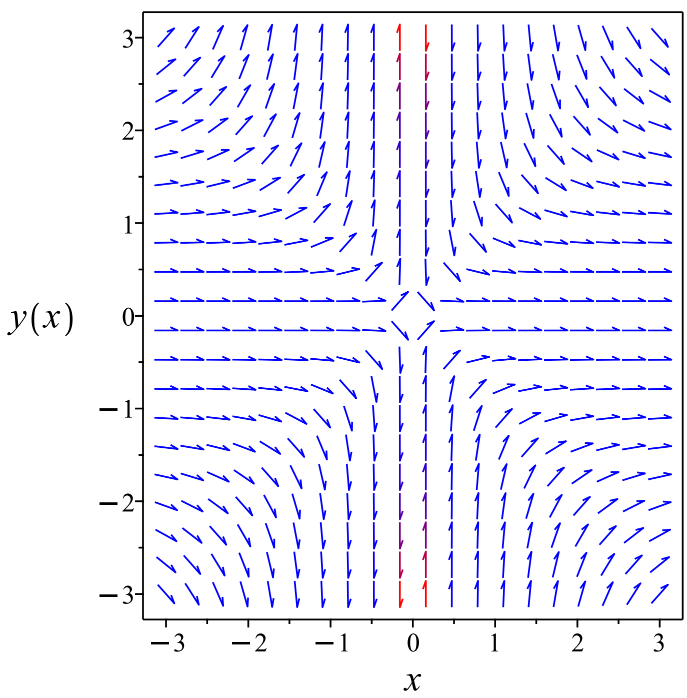


Figure 43: Slope field plot

Verification of solutions

$$y = -\frac{x}{\sqrt{-2c_1x^2 - 1}}$$

Verified OK.

$$y = \frac{x}{\sqrt{-2c_1x^2 - 1}}$$

Verified OK.

3.5.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^3\,x^3 + (u'(x)\,x + u(x))\,x^3 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(u^2+1)}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u(u^2+1)$. Integrating both sides gives

$$\frac{1}{u\,(u^2+1)}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{u\,(u^2+1)}\,du = \int -\frac{1}{x}\,dx$$

$$\ln(u) - \frac{\ln(u^2+1)}{2} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{\ln(u) - \frac{\ln(u^2+1)}{2}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{u}{\sqrt{u^2+1}} = \frac{c_3}{x}$$

The solution is

$$\frac{u(x)}{\sqrt{u(x)^2+1}} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y}{x\sqrt{\frac{y^2}{x^2}+1}} = \frac{c_3}{x}$$

$$\frac{y}{x\sqrt{\frac{x^2+y^2}{x^2}}} = \frac{c_3}{x}$$

Which simplifies to

$$\frac{y}{\sqrt{\frac{x^2+y^2}{x^2}}} = c_3$$

Summary

The solution(s) found are the following

$$\frac{y}{\sqrt{\frac{x^2+y^2}{x^2}}} = c_3 \tag{1}$$

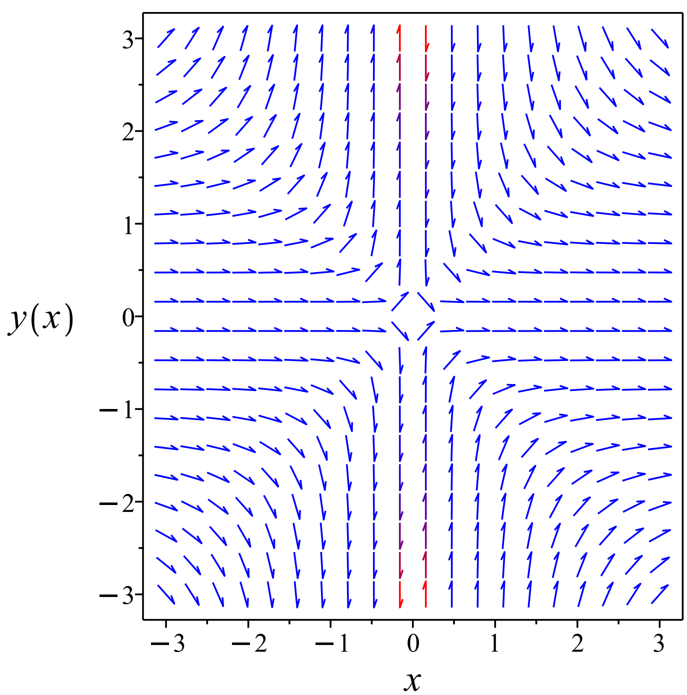


Figure 44: Slope field plot

Verification of solutions

$$\frac{y}{\sqrt{\frac{x^2+y^2}{x^2}}} = c_3$$

Verified OK.

3.5.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y^3}{x^3} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 26: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = -x^3$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$S = \int \frac{1}{\xi} dx$$
$$= \int \frac{1}{-x^3} dx$$

Which results in

$$S = \frac{1}{2x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^3}{x^3}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{1}{x^3} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

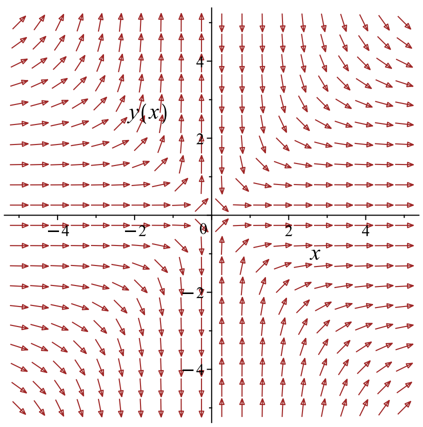
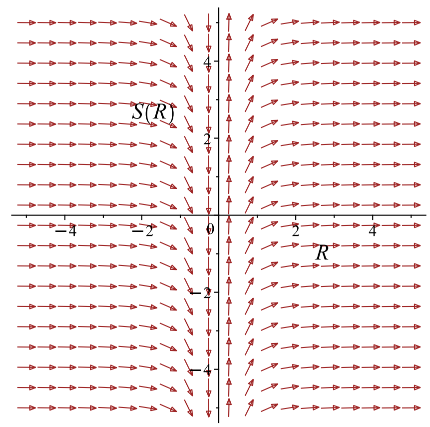
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{2x^2} = -\frac{1}{2y^2} + c_1$$

Which simplifies to

$$\frac{1}{2x^2} = -\frac{1}{2y^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^3}{x^3}$ 	$R = y$ $S = \frac{1}{2x^2}$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Summary

The solution(s) found are the following

$$\frac{1}{2x^2} = -\frac{1}{2y^2} + c_1 \tag{1}$$

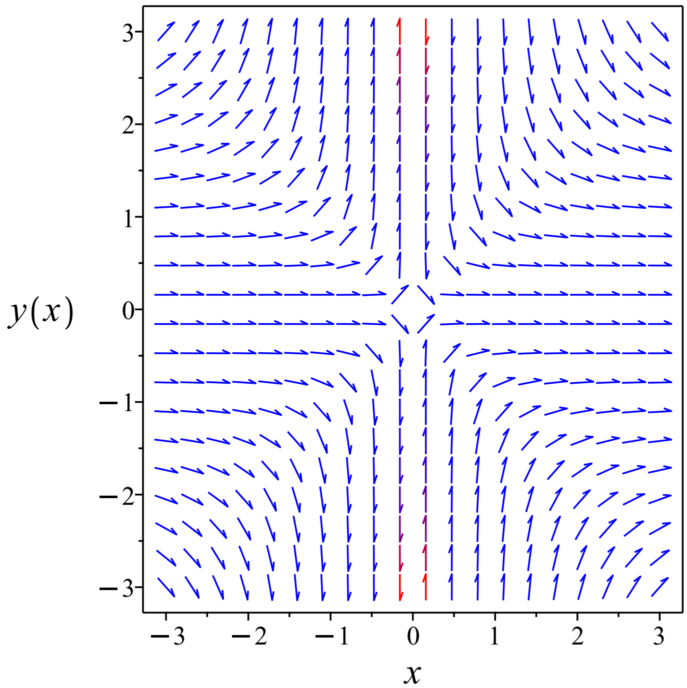


Figure 45: Slope field plot

Verification of solutions

$$\frac{1}{2x^2} = -\frac{1}{2y^2} + c_1$$

Verified OK.

3.5.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{y^3}\right) \mathrm{d}y &= \left(\frac{1}{x^3}\right) \mathrm{d}x \\
\left(-\frac{1}{x^3}\right) \mathrm{d}x + \left(-\frac{1}{y^3}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x^3} \\
N(x, y) &= -\frac{1}{y^3}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^3}\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y^3}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x^3}\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{1}{2x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y^3}$. Therefore equation (4) becomes

$$-\frac{1}{y^3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y^3}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y^3}\right) \mathrm{d}y$$
$$f(y) = \frac{1}{2y^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{2x^2} + \frac{1}{2y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{2x^2} + \frac{1}{2y^2}$$

Summary

The solution(s) found are the following

$$\frac{1}{2x^2} + \frac{1}{2y^2} = c_1 \tag{1}$$

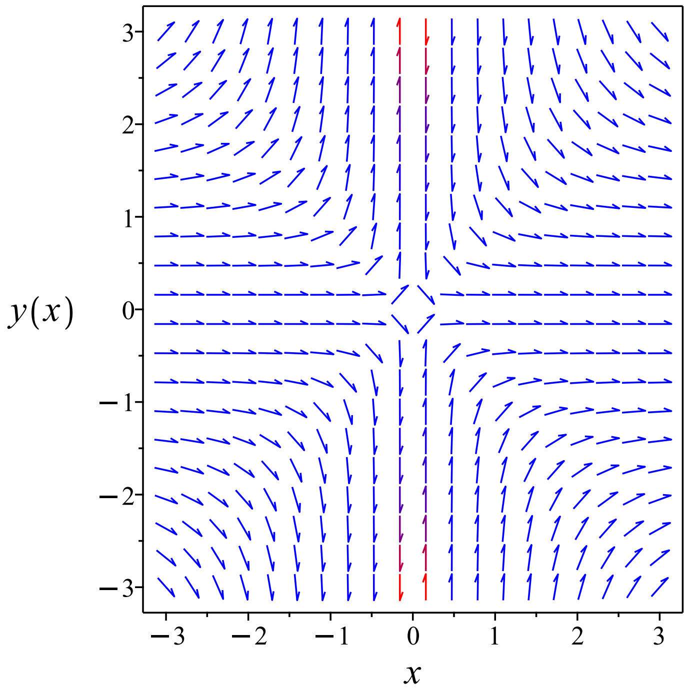


Figure 46: Slope field plot

Verification of solutions

$$\frac{1}{2x^2} + \frac{1}{2y^2} = c_1$$

Verified OK.

3.5.5 Maple step by step solution

Let's solve

$y^3 + y'x^3 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = -\frac{1}{x^3}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3} dx = \int -\frac{1}{x^3} dx + c_1$

- Evaluate integral

$$-\frac{1}{2y^2} = \frac{1}{2x^2} + c_1$$

- Solve for y

$$\left\{y = \frac{x}{\sqrt{-2c_1x^2-1}}, y = -\frac{x}{\sqrt{-2c_1x^2-1}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(y(x)^3+x^3*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{\sqrt{c_1x^2 - 1}}$$
$$y(x) = -\frac{x}{\sqrt{c_1x^2 - 1}}$$

✓ Solution by Mathematica

Time used: 0.356 (sec). Leaf size: 45

```
DSolve[y[x]^3+x^3*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{\sqrt{-1 - 2c_1x^2}}$$
$$y(x) \to \frac{x}{\sqrt{-1 - 2c_1x^2}}$$
$$y(x) \to 0$$

3.6 problem Ex 6

Internal problem ID [11136]
Internal file name [`OUTPUT/10121_Wednesday_November_23_2022_11_50_57_AM_10851037/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 10. Homogeneous equations. Page 15
Problem number: Ex 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y\cos\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)y'=-x}$$

3.6.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y'=\frac{y}{x}+\frac{1}{\cos\left(\frac{y}{x}\right)} \tag{A}$$

The given ode has the form

$$y'=\frac{y}{x}+g(x)\,f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n,m are integers. The solution is given in Kamke page 20. Using the substitution $y(x)=u(x)\,x$ then

$$\frac{dy}{dx}=\frac{du}{dx}x+u$$

Hence the given ode becomes

$$\frac{du}{dx}x + u = u + g(x)\, f(bu)^{\frac{n}{m}}$$
$$u' = \frac{1}{x}g(x)\, f(bu)^{\frac{n}{m}} \tag{2}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$g(x) = 1$$
$$b = 1$$
$$f\left(\frac{bx}{y}\right) = \cos\left(\frac{y}{x}\right)$$

Substituting the above in (2) results in the $u(x)$ ode as

$$u'(x) = \frac{1}{x\cos(u(x))}$$

Which is now solved as separable In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{1}{x\cos(u)} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{1}{\cos(u)}$. Integrating both sides gives

$$\frac{1}{\frac{1}{\cos(u)}}\, du = \frac{1}{x}\, dx$$
$$\int \frac{1}{\frac{1}{\cos(u)}}\, du = \int \frac{1}{x}\, dx$$
$$\sin(u) = \ln(x) + c_1$$

The solution is

$$\sin(u(x)) - \ln(x) - c_1 = 0$$

Therefore the solution is found using $y = ux$. Hence

$$\sin\left(\frac{y}{x}\right) - \ln(x) - c_1 = 0$$

Summary
The solution(s) found are the following

$$\sin\left(\frac{y}{x}\right)-\ln(x)-c_1=0 \tag{1}$$

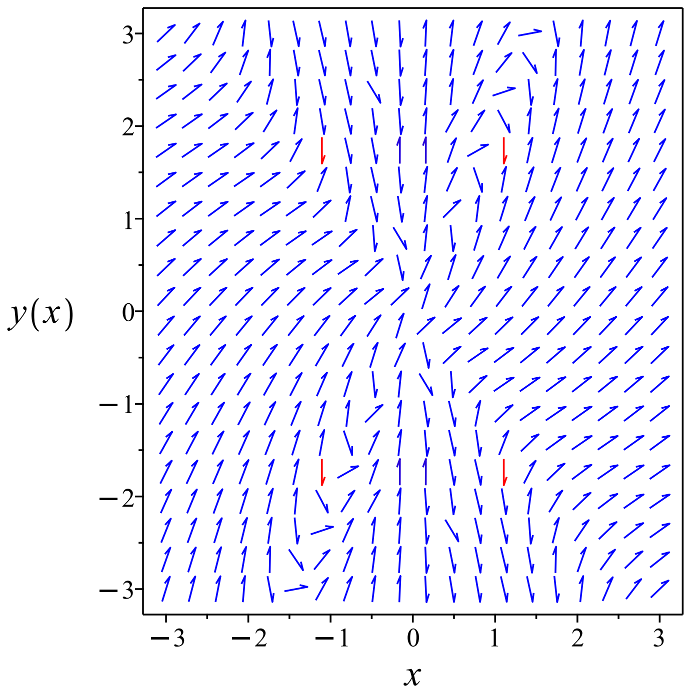


Figure 47: Slope field plot

Verification of solutions

$$\sin\left(\frac{y}{x}\right)-\ln(x)-c_1=0$$

Verified OK.

3.6.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x\cos(u(x))-x\cos(u(x))\,(u'(x)\,x+u(x))=-x$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=\frac{1}{\cos(u)\,x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{1}{\cos(u)}$. Integrating both sides gives

$$\frac{1}{\frac{1}{\cos(u)}} du = \frac{1}{x} dx$$

$$\int \frac{1}{\frac{1}{\cos(u)}} du = \int \frac{1}{x} dx$$

$$\sin(u) = \ln(x) + c_2$$

The solution is

$$\sin(u(x)) - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sin\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

$$\sin\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

<u>Summary</u>
The solution(s) found are the following

$$\sin\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

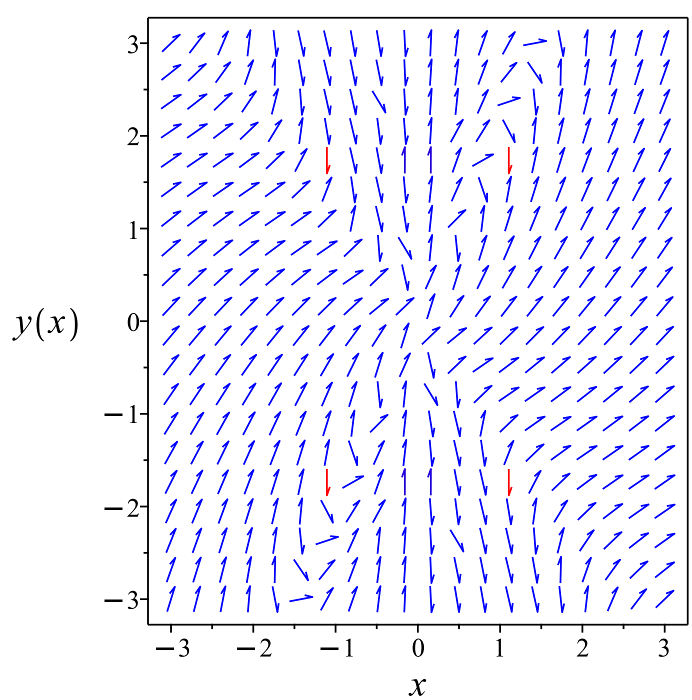


Figure 48: Slope field plot

Verification of solutions

$$\sin\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Verified OK.

3.6.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{x + y\cos\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 29: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= x^2 \\
\eta(x,y) &= xy
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{xy}{x^2} \\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x^2}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= -\frac{1}{x}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x + y\cos\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{y}{x^2} \\ R_y &= \frac{1}{x} \\ S_x &= \frac{1}{x^2} \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\cos\left(\frac{y}{x}\right)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\cos(R)\, S(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \mathrm{e}^{-\sin(R)} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 \mathrm{e}^{-\sin\left(\frac{y}{x}\right)}$$

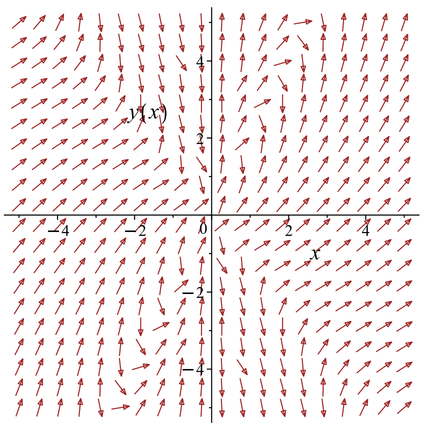
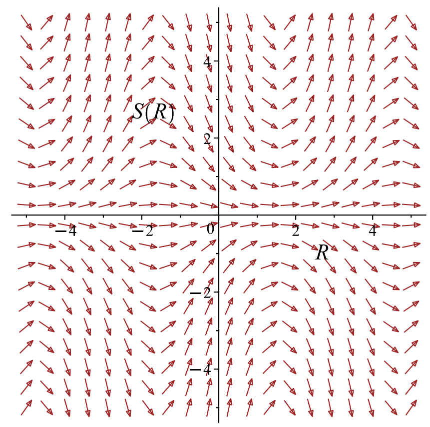
Which simplifies to

$$-\frac{1}{x} = c_1 \mathrm{e}^{-\sin\left(\frac{y}{x}\right)}$$

Which gives

$$y = -\arcsin\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x+y\cos\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -\cos(R)\, S(R)$ 

Summary

The solution(s) found are the following

$$y = -\arcsin\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x \tag{1}$$

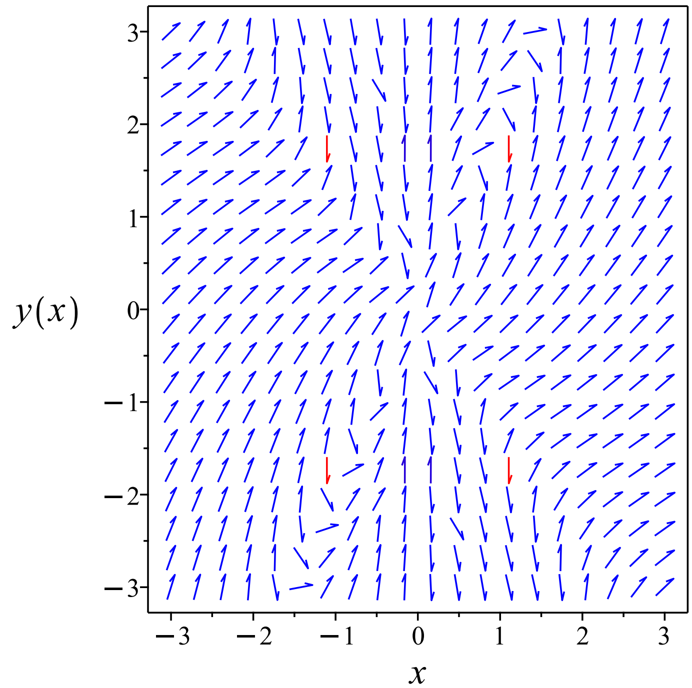


Figure 49: Slope field plot

Verification of solutions

$$y = -\arcsin\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

Verified OK.

3.6.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-x \cos\left(\frac{y}{x}\right)\right) \mathrm{d}y &= \left(-x - y \cos\left(\frac{y}{x}\right)\right) \mathrm{d}x \\
\left(x + y \cos\left(\frac{y}{x}\right)\right) \mathrm{d}x + \left(-x \cos\left(\frac{y}{x}\right)\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= x + y \cos\left(\frac{y}{x}\right) \\
N(x, y) &= -x \cos\left(\frac{y}{x}\right)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x + y \cos\left(\frac{y}{x}\right)\right) \\
&= \cos\left(\frac{y}{x}\right) - \frac{y \sin\left(\frac{y}{x}\right)}{x}
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x\cos\left(\frac{y}{x}\right)\right)\\ &= -\cos\left(\frac{y}{x}\right) - \frac{y\sin\left(\frac{y}{x}\right)}{x}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{\sec\left(\frac{y}{x}\right)}{x}\left(\left(\cos\left(\frac{y}{x}\right) - \frac{y\sin\left(\frac{y}{x}\right)}{x}\right) - \left(-\cos\left(\frac{y}{x}\right) - \frac{y\sin\left(\frac{y}{x}\right)}{x}\right)\right)\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}\left(x + y\cos\left(\frac{y}{x}\right)\right)\\ &= \frac{x + y\cos\left(\frac{y}{x}\right)}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^2}\left(-x\cos\left(\frac{y}{x}\right)\right)\\ &= -\frac{\cos\left(\frac{y}{x}\right)}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{x + y\cos\left(\frac{y}{x}\right)}{x^2}\right) + \left(-\frac{\cos\left(\frac{y}{x}\right)}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{x + y\cos\left(\frac{y}{x}\right)}{x^2}\,\mathrm{d}x$$
$$\phi = -\ln\left(\frac{1}{x}\right) - \sin\left(\frac{y}{x}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{\cos\left(\frac{y}{x}\right)}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\cos\left(\frac{y}{x}\right)}{x}$. Therefore equation (4) becomes

$$-\frac{\cos\left(\frac{y}{x}\right)}{x} = -\frac{\cos\left(\frac{y}{x}\right)}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln\left(\frac{1}{x}\right) - \sin\left(\frac{y}{x}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln\left(\frac{1}{x}\right) - \sin\left(\frac{y}{x}\right)$$

Summary

The solution(s) found are the following

$$-\ln\left(\frac{1}{x}\right) - \sin\left(\frac{y}{x}\right) = c_1 \tag{1}$$

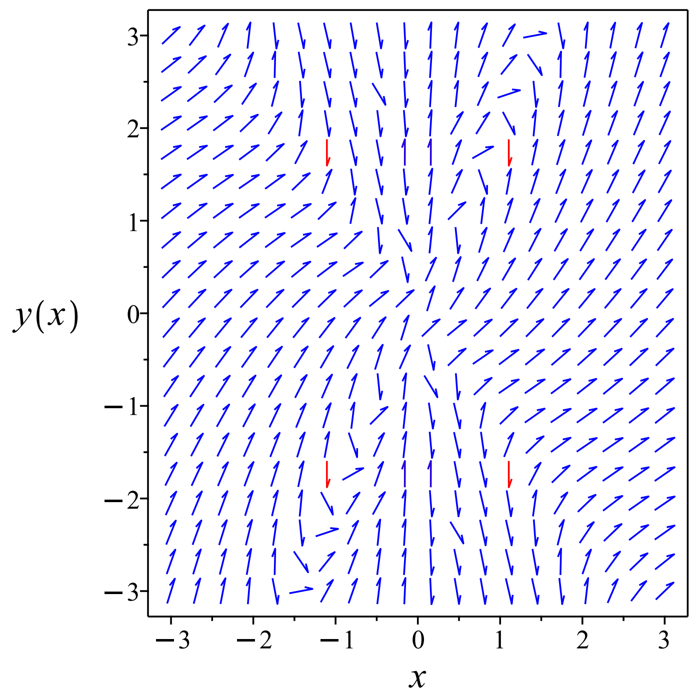


Figure 50: Slope field plot

Verification of solutions

$$-\ln\left(\frac{1}{x}\right) - \sin\left(\frac{y}{x}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve((x+y(x)*cos(y(x)/x))-x*cos(y(x)/x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arcsin(\ln(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.395 (sec). Leaf size: 13

```
DSolve[(x+y[x]*Cos[y[x]/x])-x*Cos[y[x]/x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \arcsin(\log(x) + c_1)$$

4 Chapter 2, differential equations of the first order and the first degree. Article 11. Equations in which M and N are linear but not homogeneous. Page 16

4.1 problem Ex 1

Internal problem ID [11137]
Internal file name [`OUTPUT/10122_Wednesday_November_23_2022_11_50_59_AM_15601945/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 11. Equations in which M and N are linear but not homogeneous. Page 16
Problem number: Ex 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{3y + (x + y + 1)\, y' = -4x - 1}$$

4.1.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{4X + 4x_0 + 3Y(X) + 3y_0 + 1}{X + x_0 + Y(X) + y_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 2$$
$$y_0 = -3$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{4X + 3Y(X)}{X + Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{4X + 3Y}{X + Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -4X - 3Y$ and $N = X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= \frac{-3u - 4}{u + 1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-3u(X)-4}{u(X)+1} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{\frac{-3u(X)-4}{u(X)+1} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX} u(X)\right) X u(X) + \left(\frac{d}{dX} u(X)\right) X + u(X)^2 + 4u(X) + 4 = 0
$$

Or

$$
X(u(X) + 1) \left(\frac{d}{dX} u(X)\right) + (u(X) + 2)^2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= -\frac{(u + 2)^2}{X (u + 1)}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{(u+2)^2}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{(u+2)^2}{u+1}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{(u+2)^2}{u+1}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{1}{u+2} + \ln(u+2) = -\ln(X) + c_2$$

The solution is

$$\frac{1}{u(X)+2} + \ln(u(X)+2) + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{1}{\frac{Y(X)}{X}+2} + \ln\left(\frac{Y(X)}{X}+2\right) + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{X}{Y(X)+2X} + \ln\left(\frac{Y(X)+2X}{X}\right) + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 3$$
$$X = x + 2$$

Then the solution in y becomes

$$\frac{x-2}{y-1+2x} + \ln\left(\frac{y-1+2x}{x-2}\right) + \ln(x-2) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{x-2}{y-1+2x} + \ln\left(\frac{y-1+2x}{x-2}\right) + \ln(x-2) - c_2 = 0 \tag{1}$$

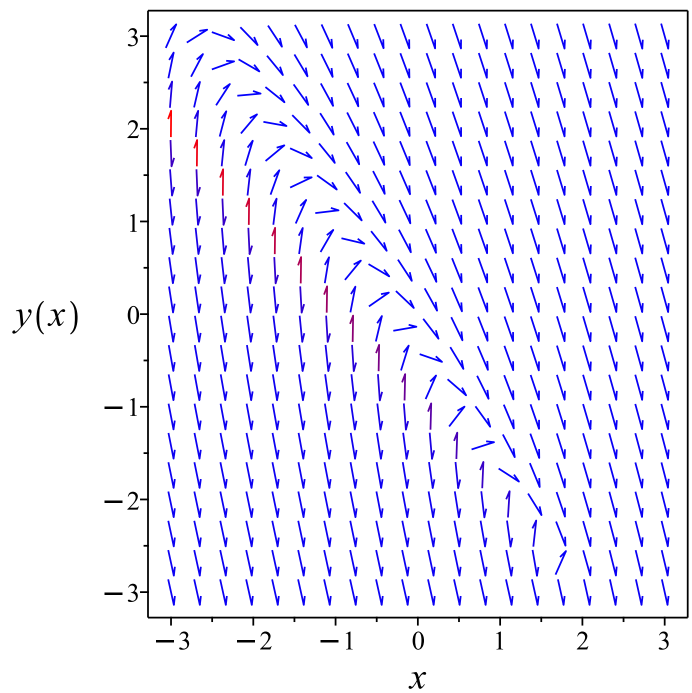


Figure 51: Slope field plot

Verification of solutions

$$\frac{x-2}{y-1+2x}+\ln\left(\frac{y-1+2x}{x-2}\right)+\ln(x-2)-c_2=0$$

Verified OK.

4.1.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{4x+3y+1}{x+y+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(4x+3y+1)(b_3-a_2)}{x+y+1} - \frac{(4x+3y+1)^2 a_3}{(x+y+1)^2} - \left(-\frac{4}{x+y+1} + \frac{4x+3y+1}{(x+y+1)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{3}{x+y+1} + \frac{4x+3y+1}{(x+y+1)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{4x^2a_2 - 16x^2a_3 - 4x^2b_3 + 8xya_2 - 24xya_3 + 2xyb_2 - 8xyb_3 + 3y^2a_2 - 8y^2a_3 + y^2b_2 - 3y^2b_3 + 8xa_2 - 8x}{(x+y+1)^2} = 0$$

Setting the numerator to zero gives

$$4x^2a_2 - 16x^2a_3 - 4x^2b_3 + 8xya_2 - 24xya_3 + 2xyb_2 - 8xyb_3 + 3y^2a_2 - 8y^2a_3 + y^2b_2 - 3y^2b_3 + 8xa_2 - 8xa_3 - xb_1 + 4xb_2 - 5xb_3 + ya_1 + 4ya_2 - 3ya_3 + 2yb_2 - 2yb_3 + 3a_1 + a_2 - a_3 + 2b_1 + b_2 - b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$4a_2v_1^2 + 8a_2v_1v_2 + 3a_2v_2^2 - 16a_3v_1^2 - 24a_3v_1v_2 - 8a_3v_2^2 + 2b_2v_1v_2 + b_2v_2^2 - 4b_3v_1^2 - 8b_3v_1v_2 - 3b_3v_2^2 + a_1v_2 + 8a_2v_1 + 4a_2v_2 - 8a_3v_1 - 3a_3v_2 - b_1v_1 + 4b_2v_1 + 2b_2v_2 - 5b_3v_1 - 2b_3v_2 + 3a_1 + a_2 - a_3 + 2b_1 + b_2 - b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(4a_2 - 16a_3 - 4b_3)\, v_1^2 + (8a_2 - 24a_3 + 2b_2 - 8b_3)\, v_1 v_2 \\ + (8a_2 - 8a_3 - b_1 + 4b_2 - 5b_3)\, v_1 + (3a_2 - 8a_3 + b_2 - 3b_3)\, v_2^2 \\ + (a_1 + 4a_2 - 3a_3 + 2b_2 - 2b_3)\, v_2 + 3a_1 + a_2 - a_3 + 2b_1 + b_2 - b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}4a_2 - 16a_3 - 4b_3 &= 0 \\ 3a_2 - 8a_3 + b_2 - 3b_3 &= 0 \\ 8a_2 - 24a_3 + 2b_2 - 8b_3 &= 0 \\ a_1 + 4a_2 - 3a_3 + 2b_2 - 2b_3 &= 0 \\ 8a_2 - 8a_3 - b_1 + 4b_2 - 5b_3 &= 0 \\ 3a_1 + a_2 - a_3 + 2b_1 + b_2 - b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -5a_3 - 2b_3 \\ a_2 &= 4a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 8a_3 + 3b_3 \\ b_2 &= -4a_3 \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x - 2 \\ \eta &= y + 3\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi \\ &= y + 3 - \left(-\frac{4x + 3y + 1}{x + y + 1}\right)(x - 2) \\ &= \frac{4x^2 + 4xy + y^2 - 4x - 2y + 1}{x + y + 1} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4x^2+4xy+y^2-4x-2y+1}{x+y+1}} dy \end{aligned}$$

Which results in

$$S = \ln(2x + y - 1) - \frac{-x + 2}{2x + y - 1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{4x + 3y + 1}{x + y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{4x + 3y + 1}{(2x + y - 1)^2} \\ S_y &= \frac{x + y + 1}{(2x + y - 1)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(y-1+2x)\ln(y-1+2x)+x-2}{y-1+2x} = c_1$$

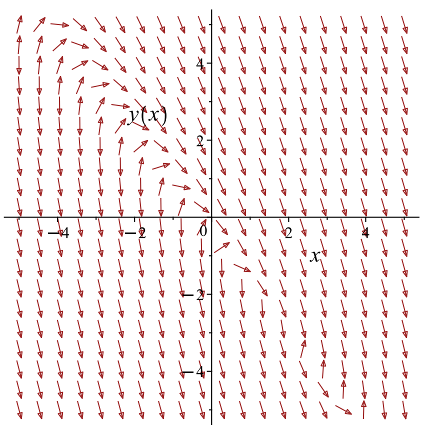
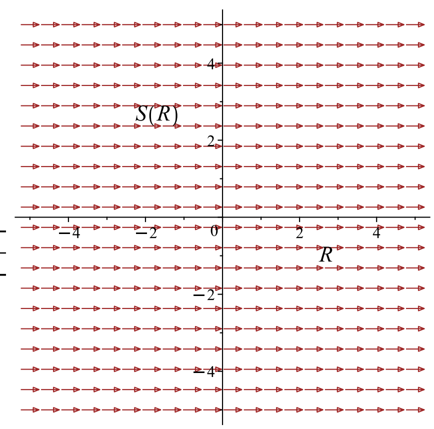
Which simplifies to

$$\frac{(y-1+2x)\ln(y-1+2x)+x-2}{y-1+2x} = c_1$$

Which gives

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-(x-2)\mathrm{e}^{-c_1}\right)+c_1} - 2x + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4x+3y+1}{x+y+1}$ 	$R = x$ $S = \frac{(2x+y-1)\ln(2x+}{2x+y-}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = e^{\operatorname{LambertW}\left(-(x-2)e^{-c_1}\right)+c_1} - 2x + 1 \tag{1}$$

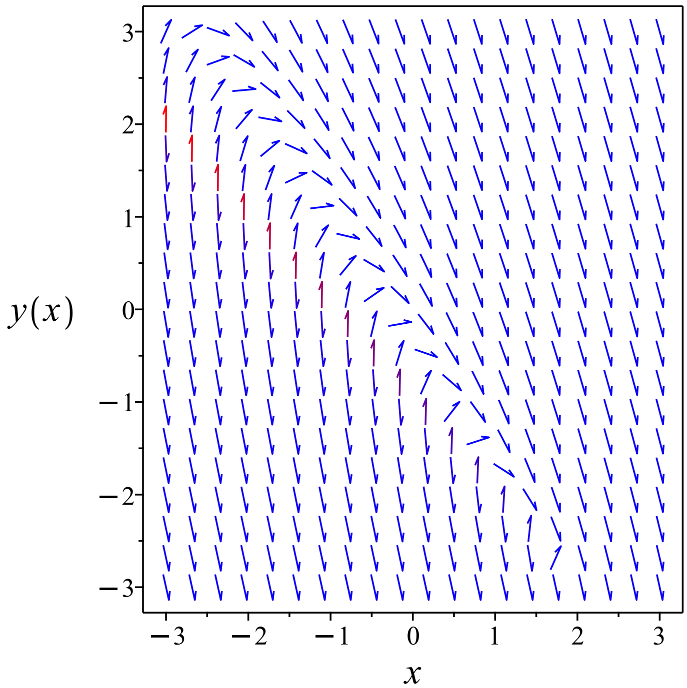


Figure 52: Slope field plot

Verification of solutions

$$y = e^{\mathrm{LambertW}\left(-(x-2)e^{-c_1}\right)+c_1} - 2x + 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 29

```
dsolve((4*x+3*y(x)+1)+(x+y(x)+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -3 - \frac{(x-2)\left(2\,\text{LambertW}\left(c_1(x-2)\right)+1\right)}{\text{LambertW}\left(c_1\left(x-2\right)\right)}$$

✓ Solution by Mathematica

Time used: 1.385 (sec). Leaf size: 159

```
DSolve[(4*x+3*y[x]+1)+(x+y[x]+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{(-2)^{2/3}\left(-2x\log\left(\frac{3(-2)^{2/3}(y(x)+2x-1)}{y(x)+x+1}\right)+(2x-1)\log\left(-\frac{3(-2)^{2/3}(x-2)}{y(x)+x+1}\right)+\log\left(\frac{3(-2)^{2/3}(y(x)+2x-1)}{y(x)+x+1}\right)+ \cdots}{9(y(x)+2x-1)}\right.$$

4.2 problem Ex 2

Internal problem ID [11138]
Internal file name [`OUTPUT/10123_Wednesday_November_23_2022_11_51_00_AM_73242900/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 11. Equations in which M and N are linear but not homogeneous. Page 16
Problem number: Ex 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y+(x+y+3)\,y'=-4x-2}$$

4.2.1 Solving as homogeneousTypeMapleC ode

Let $Y=y+y_0$ and $X=x+x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X)=\frac{-4X-4x_0+Y(X)+y_0-2}{X+x_0+Y(X)+y_0+3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0=-1$$
$$y_0=-2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X)=\frac{-4X+Y(X)}{X+Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{-4X + Y}{X + Y}
\end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -4X + Y$ and $N = X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= \frac{u - 4}{u + 1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)-4}{u(X)+1} - u(X)}{X}
\end{aligned}$$

Or

$$\frac{d}{dX} u(X) - \frac{\frac{u(X)-4}{u(X)+1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX} u(X)\right) X u(X) + \left(\frac{d}{dX} u(X)\right) X + u(X)^2 + 4 = 0$$

Or

$$X(u(X) + 1) \left(\frac{d}{dX} u(X)\right) + u(X)^2 + 4 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= -\frac{u^2 + 4}{X (u + 1)}
\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+4}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+4}{u+1}}\, du = -\frac{1}{X}\, dX$$

$$\int \frac{1}{\frac{u^2+4}{u+1}}\, du = \int -\frac{1}{X}\, dX$$

$$\frac{\ln (u^2+4)}{2} + \frac{\arctan\left(\frac{u}{2}\right)}{2} = -\ln (X) + c_2$$

The solution is

$$\frac{\ln \left(u(X)^2+4\right)}{2} + \frac{\arctan\left(\frac{u(X)}{2}\right)}{2} + \ln (X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln \left(\frac{Y(X)^2}{X^2}+4\right)}{2} + \frac{\arctan\left(\frac{Y(X)}{2X}\right)}{2} + \ln (X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln \left(\frac{Y(X)^2}{X^2}+4\right)}{2} + \frac{\arctan\left(\frac{Y(X)}{2X}\right)}{2} + \ln (X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 2$$
$$X = x - 1$$

Then the solution in y becomes

$$\frac{\ln \left(\frac{(y+2)^2}{(1+x)^2}+4\right)}{2} + \frac{\arctan\left(\frac{y+2}{2+2x}\right)}{2} + \ln (1+x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{(y+2)^2}{(1+x)^2}+4\right)}{2}+\frac{\arctan\left(\frac{y+2}{2+2x}\right)}{2}+\ln(1+x)-c_2=0 \tag{1}$$

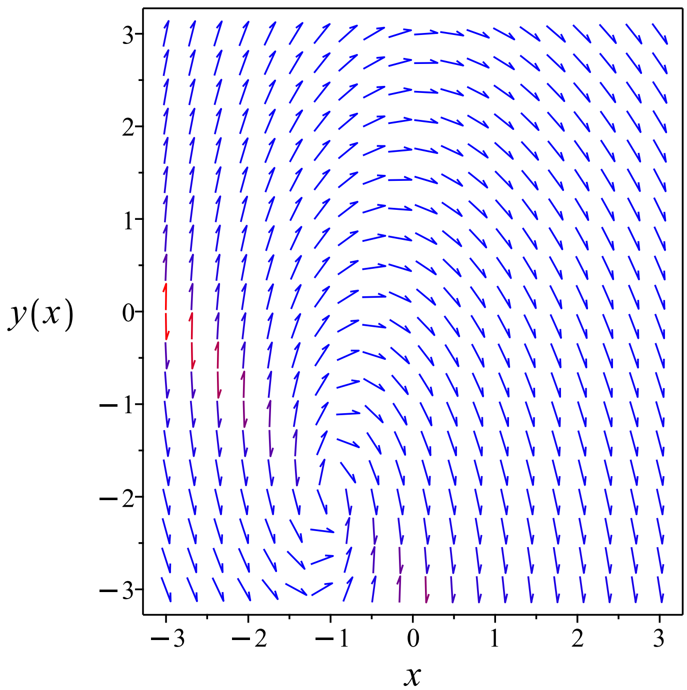


Figure 53: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{(y+2)^2}{(1+x)^2}+4\right)}{2}+\frac{\arctan\left(\frac{y+2}{2+2x}\right)}{2}+\ln(1+x)-c_2=0$$

Verified OK.

4.2.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-4x+y-2}{x+y+3}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 + \frac{(-4x+y-2)(b_3-a_2)}{x+y+3} - \frac{(-4x+y-2)^2 a_3}{(x+y+3)^2} \\ - \left(-\frac{4}{x+y+3} - \frac{-4x+y-2}{(x+y+3)^2}\right)(xa_2+ya_3+a_1) \\ - \left(\frac{1}{x+y+3} - \frac{-4x+y-2}{(x+y+3)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{4x^2a_2 - 16x^2a_3 - 4x^2b_2 - 4x^2b_3 + 8xya_2 + 8xya_3 + 2xyb_2 - 8xyb_3 - y^2a_2 + 4y^2a_3 + y^2b_2 + y^2b_3 + 24xa_2}{(x+}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} 4x^2a_2 - 16x^2a_3 - 4x^2b_2 - 4x^2b_3 + 8xya_2 + 8xya_3 + 2xyb_2 - 8xyb_3 - y^2a_2 \\ + 4y^2a_3 + y^2b_2 + y^2b_3 + 24xa_2 - 16xa_3 - 5xb_1 + xb_2 - 14xb_3 + 5ya_1 \\ - ya_2 + 14ya_3 + 6yb_2 - 4yb_3 + 10a_1 + 6a_2 - 4a_3 - 5b_1 + 9b_2 - 6b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&4a_2v_1^2+8a_2v_1v_2-a_2v_2^2-16a_3v_1^2+8a_3v_1v_2+4a_3v_2^2-4b_2v_1^2+2b_2v_1v_2+b_2v_2^2 \\
&-4b_3v_1^2-8b_3v_1v_2+b_3v_2^2+5a_1v_2+24a_2v_1-a_2v_2-16a_3v_1+14a_3v_2-5b_1v_1 \\
&+b_2v_1+6b_2v_2-14b_3v_1-4b_3v_2+10a_1+6a_2-4a_3-5b_1+9b_2-6b_3=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(4a_2-16a_3-4b_2-4b_3)\,v_1^2+(8a_2+8a_3+2b_2-8b_3)\,v_1v_2 \\
&+(24a_2-16a_3-5b_1+b_2-14b_3)\,v_1+(-a_2+4a_3+b_2+b_3)\,v_2^2 \\
&+(5a_1-a_2+14a_3+6b_2-4b_3)\,v_2+10a_1+6a_2-4a_3-5b_1+9b_2-6b_3=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-a_2+4a_3+b_2+b_3&=0\\
4a_2-16a_3-4b_2-4b_3&=0\\
8a_2+8a_3+2b_2-8b_3&=0\\
5a_1-a_2+14a_3+6b_2-4b_3&=0\\
24a_2-16a_3-5b_1+b_2-14b_3&=0\\
10a_1+6a_2-4a_3-5b_1+9b_2-6b_3&=0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1&=2a_3+b_3\\
a_2&=b_3\\
a_3&=a_3\\
b_1&=-4a_3+2b_3\\
b_2&=-4a_3\\
b_3&=b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi&=1+x\\
\eta&=y+2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y+2-\left(\frac{-4x+y-2}{x+y+3}\right)(1+x) \\
&= \frac{4x^2+y^2+8x+4y+8}{x+y+3} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{4x^2+y^2+8x+4y+8}{x+y+3}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(4x^2+y^2+8x+4y+8\right)}{2} + \frac{2(1+x)\arctan\left(\frac{2y+4}{4+4x}\right)}{4+4x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-4x+y-2}{x+y+3}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{4x - y + 2}{4x^2 + y^2 + 8x + 4y + 8} \\
S_y &= \frac{x + y + 3}{4x^2 + y^2 + 8x + 4y + 8}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

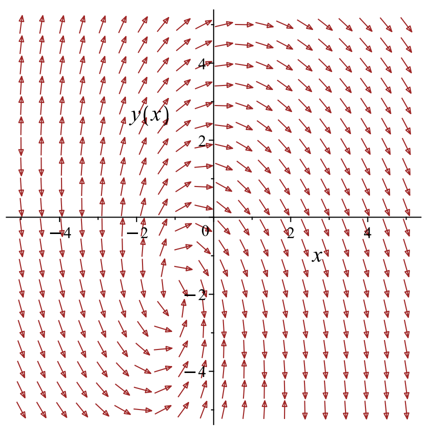
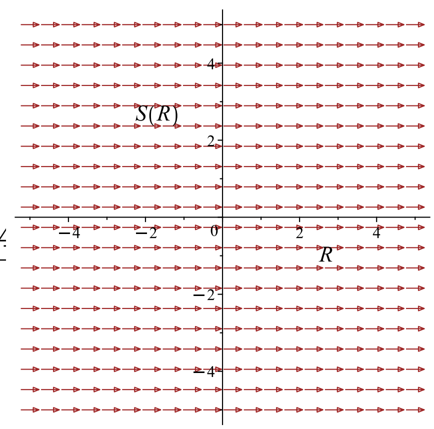
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln\left(y^2 + 4x^2 + 4y + 8x + 8\right)}{2} + \frac{\arctan\left(\frac{y+2}{2+2x}\right)}{2} = c_1
$$

Which simplifies to

$$
\frac{\ln\left(y^2 + 4x^2 + 4y + 8x + 8\right)}{2} + \frac{\arctan\left(\frac{y+2}{2+2x}\right)}{2} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-4x+y-2}{x+y+3}$ 	$R = x$ $S = \frac{\ln(4x^2 + y^2 + 8x + 4}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y^2 + 4x^2 + 4y + 8x + 8)}{2} + \frac{\arctan\left(\frac{y+2}{2+2x}\right)}{2} = c_1 \tag{1}$$

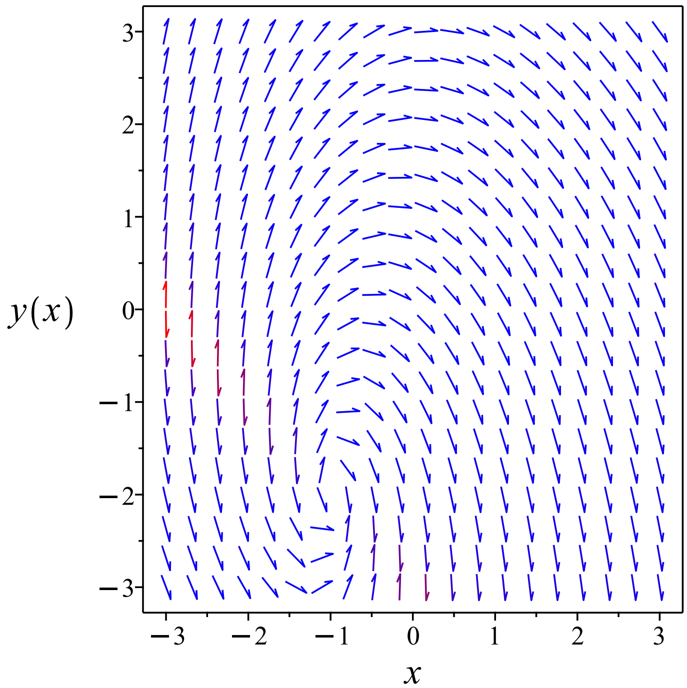


Figure 54: Slope field plot

Verification of solutions

$$\frac{\ln(y^2+4x^2+4y+8x+8)}{2}+\frac{\arctan\left(\frac{y+2}{2+2x}\right)}{2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 36

```
dsolve((4*x-y(x)+2)+(x+y(x)+3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -2 + \tan\left(\text{RootOf}\left(2\ln(2) + \ln\left(\sec(\_Z)^2\right) - \_Z + 2\ln(1+x) + 2c_1\right)\right)(-2x-2)$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 67

```
DSolve[(4*x-y[x]+2)+(x+y[x]+3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2\arctan\left(\frac{1}{2} - \frac{5(x+1)}{2(y(x)+x+3)}\right) + 2\log\left(\frac{4x^2+y(x)^2+4y(x)+8x+8}{5(x+1)^2}\right) + 4\log(x+1) + 5c_1 = 0, y(x)\right]$$

4.3 problem Ex 3

Internal problem ID [11139]
Internal file name [`OUTPUT/10124_Wednesday_November_23_2022_11_51_01_AM_89746317/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 11. Equations in which M and N are linear but not homogeneous. Page 16
Problem number: Ex 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y-(4x+2y-1)\,y'=-2x}$$

4.3.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{2x+y}{4x+2y-1}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{(2x+y)(b_3 - a_2)}{4x+2y-1} - \frac{(2x+y)^2 a_3}{(4x+2y-1)^2} \\ &- \left(\frac{2}{4x+2y-1} - \frac{4(2x+y)}{(4x+2y-1)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(\frac{1}{4x+2y-1} - \frac{2(2x+y)}{(4x+2y-1)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{8x^2a_2 + 4x^2a_3 - 16x^2b_2 - 8x^2b_3 + 8xya_2 + 4xya_3 - 16xyb_2 - 8xyb_3 + 2y^2a_2 + y^2a_3 - 4y^2b_2 - 2y^2b_3 -}{(4x+2y-1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-8x^2a_2 - 4x^2a_3 + 16x^2b_2 + 8x^2b_3 - 8xya_2 - 4xya_3 + 16xyb_2 + 8xyb_3 - 2y^2a_2 \\ &- y^2a_3 + 4y^2b_2 + 2y^2b_3 + 4xa_2 - 7xb_2 - 2xb_3 + ya_2 + 2ya_3 - 4yb_2 + 2a_1 + b_1 + b_2 \\ &= 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-8a_2v_1^2 - 8a_2v_1v_2 - 2a_2v_2^2 - 4a_3v_1^2 - 4a_3v_1v_2 - a_3v_2^2 + 16b_2v_1^2 \\ &+ 16b_2v_1v_2 + 4b_2v_2^2 + 8b_3v_1^2 + 8b_3v_1v_2 + 2b_3v_2^2 + 4a_2v_1 \\ &+ a_2v_2 + 2a_3v_2 - 7b_2v_1 - 4b_2v_2 - 2b_3v_1 + 2a_1 + b_1 + b_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-8a_2 - 4a_3 + 16b_2 + 8b_3)\, v_1^2 + (-8a_2 - 4a_3 + 16b_2 + 8b_3)\, v_1 v_2 + (4a_2 - 7b_2 - 2b_3)\, v_1 + (-2a_2 - a_3 + 4b_2 + 2b_3)\, v_2^2 + (a_2 + 2a_3 - 4b_2)\, v_2 + 2a_1 + b_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
2a_1 + b_1 + b_2 &= 0\\
a_2 + 2a_3 - 4b_2 &= 0\\
4a_2 - 7b_2 - 2b_3 &= 0\\
-8a_2 - 4a_3 + 16b_2 + 8b_3 &= 0\\
-2a_2 - a_3 + 4b_2 + 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= a_1\\
a_2 &= 2b_2\\
a_3 &= b_2\\
b_1 &= -2a_1 - b_2\\
b_2 &= b_2\\
b_3 &= \frac{b_2}{2}
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 1\\
\eta &= -2
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi\\
&= -2 - \left(\frac{2x + y}{4x + 2y - 1}\right)(1)\\
&= \frac{-10x - 5y + 2}{4x + 2y - 1}\\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-10x-5y+2}{4x+2y-1}} dy \end{aligned}$$

Which results in

$$S = -\frac{2y}{5} + \frac{\ln(10x + 5y - 2)}{25}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2x + y}{4x + 2y - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{50x + 25y - 10} \\ S_y &= \frac{-4x - 2y + 1}{10x + 5y - 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{5} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{5}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R}{5} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2y}{5} + \frac{\ln(10x + 5y - 2)}{25} = -\frac{x}{5} + c_1$$

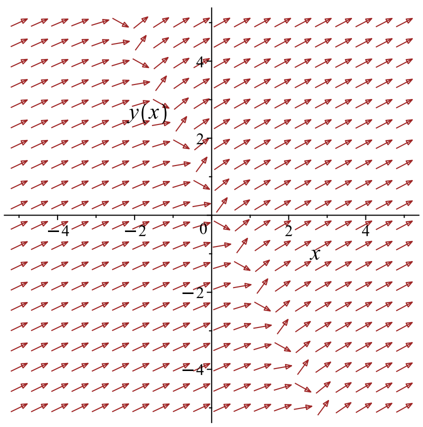
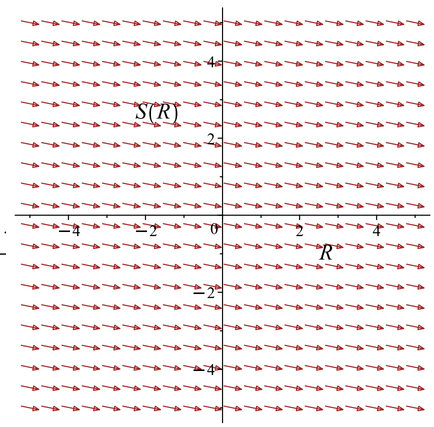
Which simplifies to

$$-\frac{2y}{5} + \frac{\ln(10x + 5y - 2)}{25} = -\frac{x}{5} + c_1$$

Which gives

$$y = \frac{\mathrm{e}^{-\operatorname{LambertW}\left(-2\,\mathrm{e}^{-25x+4+25c_1}\right)-25x+4+25c_1}}{5} - 2x + \frac{2}{5}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x+y}{4x+2y-1}$ 	$R = x$ $S = -\frac{2y}{5} + \frac{\ln(10x + 5y}{25}$	$\frac{dS}{dR} = -\frac{1}{5}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{-\operatorname{LambertW}\left(-2\,e^{-25x+4+25c_1}\right)-25x+4+25c_1}}{5} - 2x + \frac{2}{5} \tag{1}$$

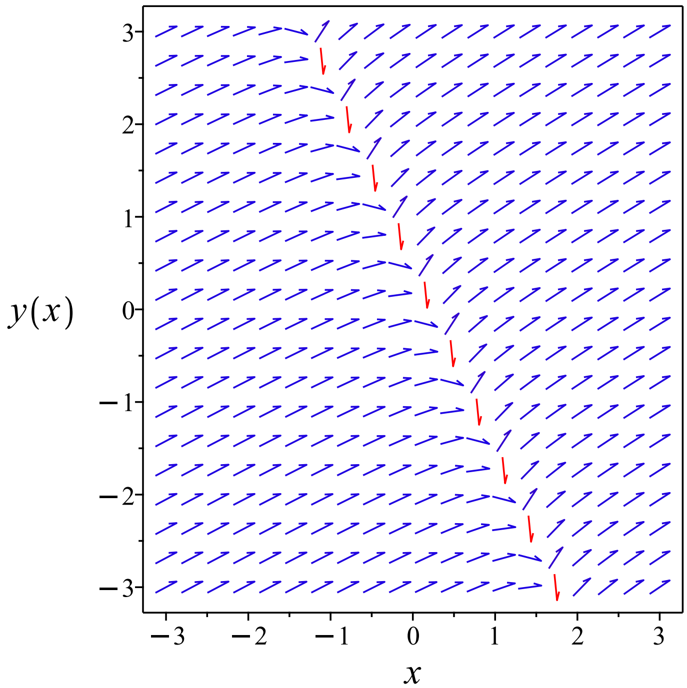


Figure 55: Slope field plot

Verification of solutions

$$y = \frac{e^{-\operatorname{LambertW}\left(-2\,e^{-25x+4+25c_1}\right)-25x+4+25c_1}}{5} - 2x + \frac{2}{5}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -2, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 23

```
dsolve((2*x+y(x))-(4*x+2*y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\operatorname{LambertW}\left(-2\,e^{4-25x+25c_1}\right)}{10} + \frac{2}{5} - 2x$$

✓ Solution by Mathematica

Time used: 4.725 (sec). Leaf size: 39

```
DSolve[(2*x+y[x])-(4*x+2*y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{10}W\left(-e^{-25x-1+c_1}\right) - 2x + \frac{2}{5}$$
$$y(x) \to \frac{2}{5} - 2x$$

5 Chapter 2, differential equations of the first order and the first degree. Article 12. Equations of form $yf_1(xy) + xf_2(xy)y' = 0$. Page 18

5.1 problem Ex 1

Internal problem ID [11140]
Internal file name [`OUTPUT/10125_Wednesday_November_23_2022_11_51_02_AM_64659519/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 12. Equations of form $yf_1(xy) + xf_2(xy)y' = 0$. Page 18
Problem number: Ex 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Riccati]
```

$$\boxed{y + 2y^2x - y^3x^2 + 2x^2yy' = 0}$$

5.1.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x^2y^2 - 2xy - 1}{2x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(x^2y^2 - 2xy - 1)(b_3 - a_2)}{2x^2} - \frac{(x^2y^2 - 2xy - 1)^2 a_3}{4x^4} \\ & - \left(\frac{2x\,y^2 - 2y}{2x^2} - \frac{x^2y^2 - 2xy - 1}{x^3}\right)(xa_2 + ya_3 + a_1) \\ & - \frac{(2x^2y - 2x)(xb_2 + yb_3 + b_1)}{2x^2} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4y^4a_3 + 4x^5yb_2 + 2x^4y^2a_2 + 2x^4y^2b_3 - 4x^3y^3a_3 + 4x^4yb_1 - 8b_2x^4 + 6x^2y^2a_3 - 4x^3b_1 + 4x^2ya_1 + 2x^2a_2}{4x^4}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -x^4y^4a_3 - 4x^5yb_2 - 2x^4y^2a_2 - 2x^4y^2b_3 + 4x^3y^3a_3 - 4x^4yb_1 + 8b_2x^4 \\ & - 6x^2y^2a_3 + 4x^3b_1 - 4x^2ya_1 - 2x^2a_2 - 2x^2b_3 - 8xya_3 - 4xa_1 - a_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -a_3v_1^4v_2^4 - 2a_2v_1^4v_2^2 + 4a_3v_1^3v_2^3 - 4b_2v_1^5v_2 - 2b_3v_1^4v_2^2 - 4b_1v_1^4v_2 - 6a_3v_1^2v_2^2 \\ & + 8b_2v_1^4 - 4a_1v_1^2v_2 + 4b_1v_1^3 - 2a_2v_1^2 - 8a_3v_1v_2 - 2b_3v_1^2 - 4a_1v_1 - a_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-4b_2v_1^5v_2 - a_3v_1^4v_2^4 + (-2a_2 - 2b_3)\,v_1^4v_2^2 - 4b_1v_1^4v_2 + 8b_2v_1^4 + 4a_3v_1^3v_2^3 + 4b_1v_1^3 \\ - 6a_3v_1^2v_2^2 - 4a_1v_1^2v_2 + (-2a_2 - 2b_3)\,v_1^2 - 8a_3v_1v_2 - 4a_1v_1 - a_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -4a_1 &= 0 \\ -8a_3 &= 0 \\ -6a_3 &= 0 \\ -a_3 &= 0 \\ 4a_3 &= 0 \\ -4b_1 &= 0 \\ 4b_1 &= 0 \\ -4b_2 &= 0 \\ 8b_2 &= 0 \\ -2a_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(\frac{x^2y^2 - 2xy - 1}{2x^2}\right)(-x) \\ &= \frac{x^2y^2 - 1}{2x} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2y^2-1}{2x}} dy\end{aligned}$$

Which results in

$$S = -\ln(xy + 1) + \ln(xy - 1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2y^2 - 2xy - 1}{2x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2y}{x^2y^2 - 1} \\
S_y &= \frac{2x}{x^2y^2 - 1}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(yx + 1) + \ln(yx - 1) = \ln(x) + c_1$$

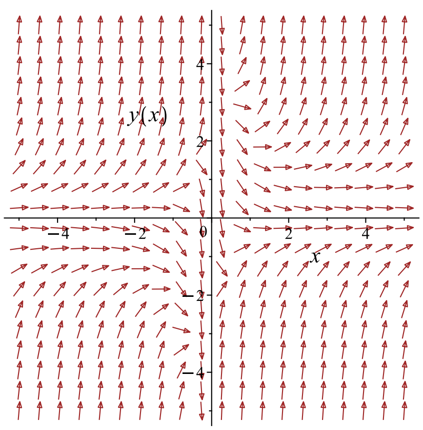
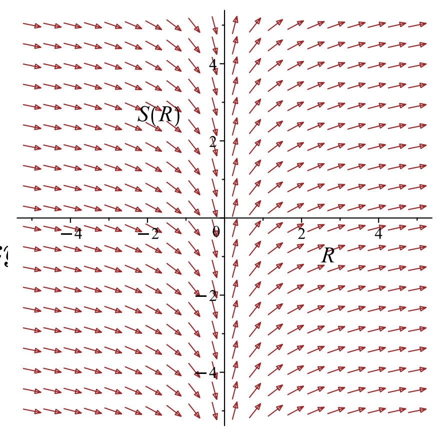
Which simplifies to

$$-\ln(yx + 1) + \ln(yx - 1) = \ln(x) + c_1$$

Which gives

$$y = -\frac{e^{c_1}x + 1}{x(e^{c_1}x - 1)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2y^2-2xy-1}{2x^2}$ 	$R = x$ $S = -\ln(xy + 1) + \ln(x$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{e^{c_1}x + 1}{x\left(e^{c_1}x - 1\right)} \tag{1}$$

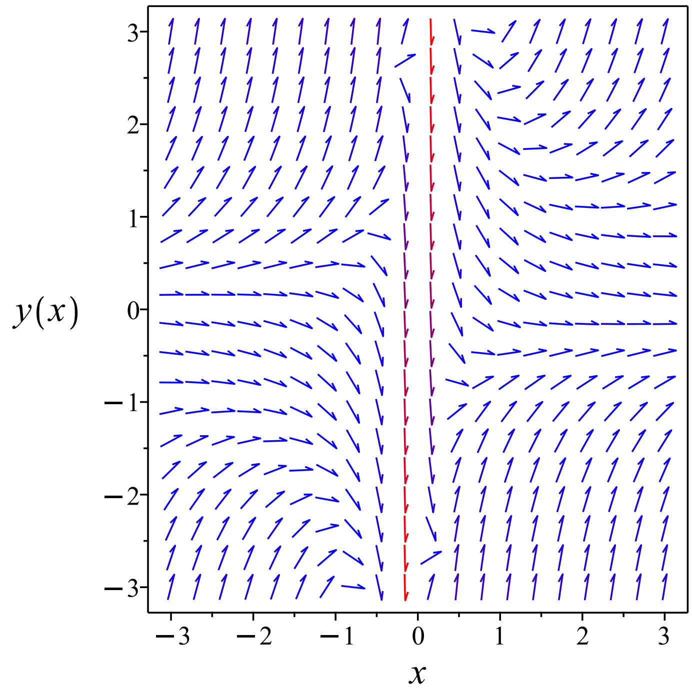


Figure 56: Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{e^{c_1}x + 1}{x\left(e^{c_1}x - 1\right)}$$

Verified OK.

5.1.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x^2y)\,\mathrm{d}y &= (y^3x^2 - 2x\,y^2 - y)\,\mathrm{d}x\\ (-y^3x^2 + 2x\,y^2 + y)\,\mathrm{d}x + (2x^2y)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -y^3x^2 + 2x\,y^2 + y\\ N(x,y) &= 2x^2y \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y^3x^2 + 2x\,y^2 + y)\\ &= -3x^2y^2 + 4xy + 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x^2y)\\ &= 4xy \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{2x^2 y}\left(\left(-3x^2y^2 + 4xy + 1\right) - (4xy)\right) \\
&= \frac{-3x^2y^2 + 1}{2x^2 y}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{-y^3x^2 + 2x\,y^2 + y}\left((4xy) - \left(-3x^2y^2 + 4xy + 1\right)\right) \\
&= \frac{-3x^2y^2 + 1}{y\,(x^2y^2 - 2xy - 1)}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(4xy) - \left(-3x^2y^2 + 4xy + 1\right)}{x\left(-y^3x^2 + 2x\,y^2 + y\right) - y\left(2x^2y\right)} \\
&= \frac{-3x^2y^2 + 1}{xy\,(x^2y^2 - 1)}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = \frac{-3t^2 + 1}{t\,(t^2 - 1)}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(\frac{-3t^2+1}{t\left(t^2-1\right)}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln\left(t\left(t^2-1\right)\right)}\\ &= \frac{1}{t^3-t}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^3y^3-xy}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^3y^3-xy}\left(-y^3x^2+2x\,y^2+y\right)\\ &= \frac{-x^2y^2+2xy+1}{y^2x^3-x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^3y^3-xy}\left(2x^2y\right)\\ &= \frac{2x}{x^2y^2-1}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{-x^2y^2+2xy+1}{y^2x^3-x}\right)+\left(\frac{2x}{x^2y^2-1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-x^2y^2 + 2xy + 1}{y^2x^3 - x} \, \mathrm{d}x$$

$$\phi = \ln(xy - 1) - \ln(xy + 1) - \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{xy + 1} + \frac{x}{xy - 1} + f'(y) \tag{4}$$

$$= \frac{2x}{x^2y^2 - 1} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2x}{x^2y^2-1}$. Therefore equation (4) becomes

$$\frac{2x}{x^2y^2 - 1} = \frac{2x}{x^2y^2 - 1} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(xy - 1) - \ln(xy + 1) - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(xy - 1) - \ln(xy + 1) - \ln(x)$$

The solution becomes

$$y = -\frac{\mathrm{e}^{c_1}x + 1}{x\left(\mathrm{e}^{c_1}x - 1\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{c_1}x + 1}{x\left(e^{c_1}x - 1\right)} \tag{1}$$

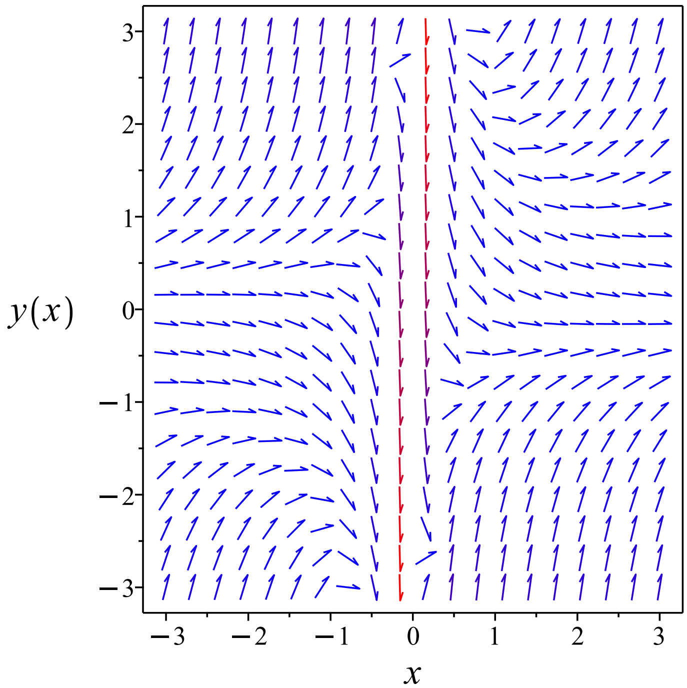


Figure 57: Slope field plot

Verification of solutions

$$y = -\frac{e^{c_1}x + 1}{x\left(e^{c_1}x - 1\right)}$$

Verified OK.

5.1.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2y^2 - 2xy - 1}{2x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{2} - \frac{y}{x} - \frac{1}{2x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{1}{2x^2}$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = \frac{1}{2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= -\frac{1}{2x} \\ f_2^2 f_0 &= -\frac{1}{8x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{2} + \frac{u'(x)}{2x} - \frac{u(x)}{8x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_1 x + c_2}{\sqrt{x}}$$

The above shows that

$$u'(x) = \frac{c_1 x - c_2}{2x^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 x - c_2}{x (c_1 x + c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 x + 1}{x (c_3 x + 1)}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_3 x + 1}{x (c_3 x + 1)} \tag{1}$$

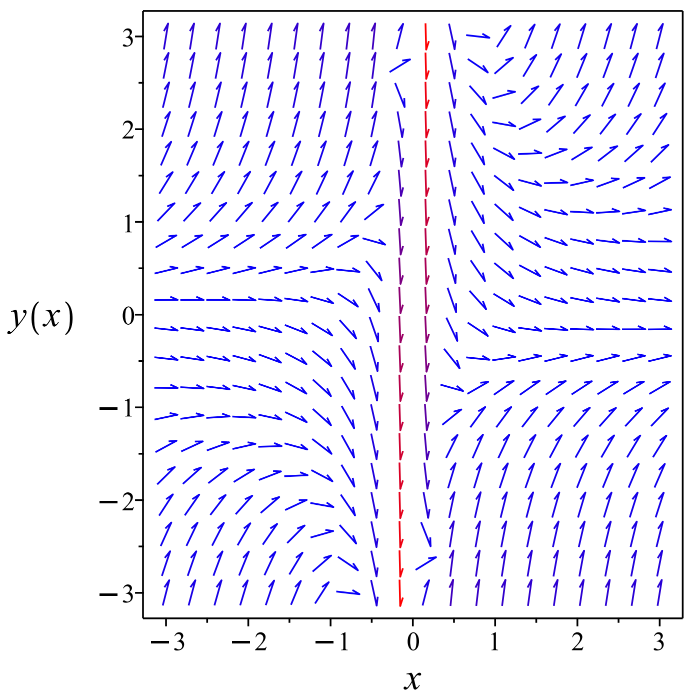


Figure 58: Slope field plot

Verification of solutions

$$y = \frac{-c_3 x + 1}{x (c_3 x + 1)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve((y(x)+2*x*y(x)^2-x^2*y(x)^3)+(2*x^2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = \frac{\tanh\left(-\frac{\ln(x)}{2} + \frac{c_1}{2}\right)}{x}$$

✓ Solution by Mathematica

Time used: 1.44 (sec). Leaf size: 71

```
DSolve[(y[x]+2*x*y[x]^2-x^2*y[x]^3)+(2*x^2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to 0$$

$$y(x) \to \frac{i\tan\left(\frac{1}{2}i\log(x) + c_1\right)}{x}$$

$$y(x) \to 0$$

$$y(x) \to \frac{-x + e^{2i\text{Interval}[\{0,\pi\}]}}{x^2 + xe^{2i\text{Interval}[\{0,\pi\}]}}$$

5.2 problem Ex 2

Internal problem ID [11141]
Internal file name [`OUTPUT/10126_Wednesday_November_23_2022_11_51_04_AM_56554848/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 12. Equations of form $yf_1(xy) + xf_2(xy)y' = 0$. Page 18
Problem number: Ex 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{2y + 3y^2x + (x + 2x^2y)\, y' = 0}$$

5.2.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(3xy+2)}{x\,(2xy+1)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(3xy+2)(b_3-a_2)}{x(2xy+1)} - \frac{y^2(3xy+2)^2 a_3}{x^2(2xy+1)^2} - \left(-\frac{3y^2}{x(2xy+1)} + \frac{y(3xy+2)}{x^2(2xy+1)} + \frac{2y^2(3xy+2)}{x(2xy+1)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{3xy+2}{x(2xy+1)} - \frac{3y}{2xy+1} + \frac{2y(3xy+2)}{(2xy+1)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{10x^4y^2b_2 - 15x^2y^4a_3 + 6x^3y^2b_1 - 6x^2y^3a_1 + 10x^3yb_2 - x^2y^2a_2 - x^2y^2b_3 - 20x\,y^3a_3 + 6x^2yb_1 - 8x\,y^2a_1 +}{x^2(2xy+1)^2}$$

$= 0$

Setting the numerator to zero gives

$$10x^4y^2b_2 - 15x^2y^4a_3 + 6x^3y^2b_1 - 6x^2y^3a_1 + 10x^3yb_2 - x^2y^2a_2 - x^2y^2b_3 - 20x\,y^3a_3 + 6x^2yb_1 - 8x\,y^2a_1 + 3b_2x^2 - 6y^2a_3 + 2xb_1 - 2ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-15a_3v_1^2v_2^4 + 10b_2v_1^4v_2^2 - 6a_1v_1^2v_2^3 + 6b_1v_1^3v_2^2 - a_2v_1^2v_2^2 - 20a_3v_1v_2^3 + 10b_2v_1^3v_2 - b_3v_1^2v_2^2 - 8a_1v_1v_2^2 + 6b_1v_1^2v_2 - 6a_3v_2^2 + 3b_2v_1^2 - 2a_1v_2 + 2b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &10b_2v_1^4v_2^2 + 6b_1v_1^3v_2^2 + 10b_2v_1^3v_2 - 15a_3v_1^2v_2^4 - 6a_1v_1^2v_2^3 + (-a_2 - b_3)\, v_1^2v_2^2 \\ &+ 6b_1v_1^2v_2 + 3b_2v_1^2 - 20a_3v_1v_2^3 - 8a_1v_1v_2^2 + 2b_1v_1 - 6a_3v_2^2 - 2a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -8a_1 &= 0 \\ -6a_1 &= 0 \\ -2a_1 &= 0 \\ -20a_3 &= 0 \\ -15a_3 &= 0 \\ -6a_3 &= 0 \\ 2b_1 &= 0 \\ 6b_1 &= 0 \\ 3b_2 &= 0 \\ 10b_2 &= 0 \\ -a_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y(3xy+2)}{x\,(2xy+1)}\right)(-x) \\
&= \frac{-x\,y^2 - y}{2xy+1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x\,y^2 - y}{2xy+1}} dy
\end{aligned}
$$

Which results in

$$
S = -\ln\left(y(xy+1)\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y(3xy+2)}{x\,(2xy+1)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y}{xy+1} \\
S_y &= \frac{-2xy-1}{y\,(xy+1)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{2}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{2}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = 2\ln(R) + c_1 \tag{4}
$$

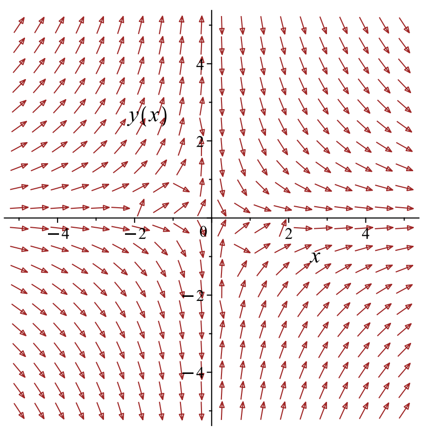
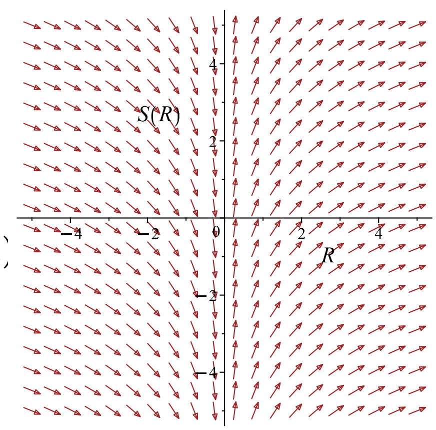
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln(y) - \ln(yx+1) = 2\ln(x) + c_1
$$

Which simplifies to

$$
-\ln(y) - \ln(yx+1) = 2\ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(3xy+2)}{x(2xy+1)}$  	$R = x$ $S = -\ln(y) - \ln(xy + 1)$	$\frac{dS}{dR} = \frac{2}{R}$ 

Summary

The solution(s) found are the following

$$-\ln(y) - \ln(yx + 1) = 2\ln(x) + c_1 \tag{1}$$

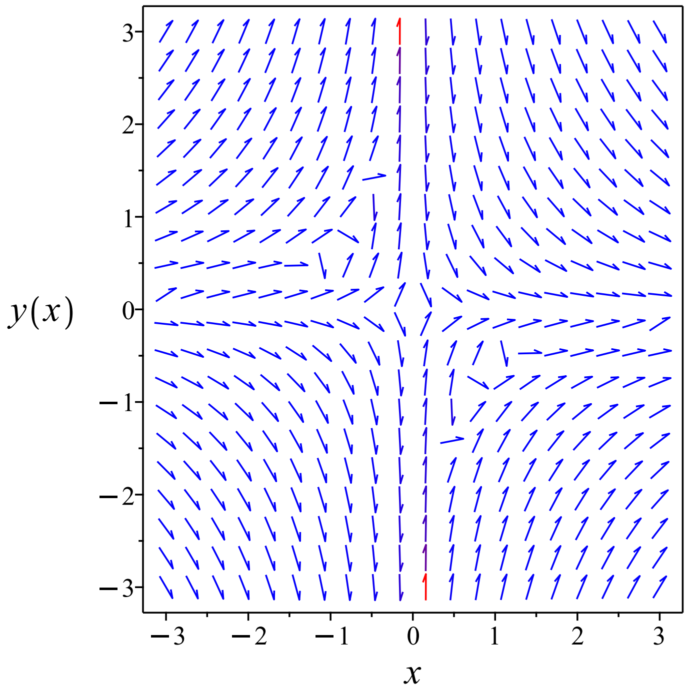


Figure 59: Slope field plot

Verification of solutions

$$-\ln(y) - \ln(yx+1) = 2\ln(x) + c_1$$

Verified OK.

5.2.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x^2y + x)\,\mathrm{d}y &= (-3x\,y^2 - 2y)\,\mathrm{d}x\\ (3x\,y^2 + 2y)\,\mathrm{d}x + (2x^2y + x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 3x\,y^2 + 2y\\ N(x, y) &= 2x^2y + x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x\,y^2 + 2y)\\ &= 6xy + 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x^2y + x)\\ &= 4xy + 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{2x^2y + x}((6xy + 2) - (4xy + 1)) \\ &= \frac{1}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int \frac{1}{x} \,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(x)} \\ &= x \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= x\left(3x\,y^2 + 2y\right) \\ &= 3x^2y^2 + 2xy \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= x\left(2x^2y + x\right) \\ &= 2y\,x^3 + x^2 \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(3x^2y^2 + 2xy\right) + \left(2y\,x^3 + x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int 3x^2y^2 + 2xy \mathrm{d}x$$

$$\phi = y x^2(xy+1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2(xy+1) + y x^3 + f'(y) \tag{4}$$

$$= 2y x^3 + x^2 + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y x^3 + x^2$. Therefore equation (4) becomes

$$2y x^3 + x^2 = 2y x^3 + x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y x^2(xy+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y x^2(xy+1)$$

Summary
The solution(s) found are the following

$$yx^2(yx+1) = c_1 \tag{1}$$

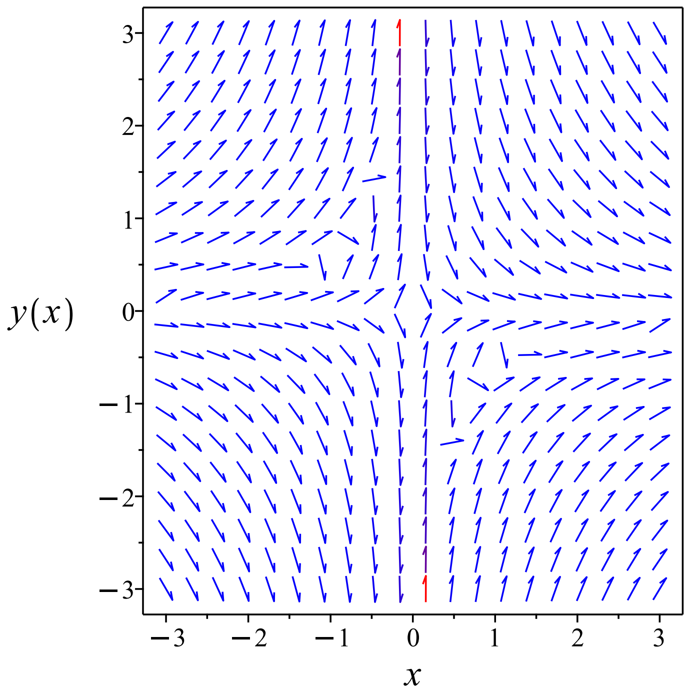


Figure 60: Slope field plot

Verification of solutions

$$yx^2(yx+1) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
dsolve((2*y(x)+3*x*y(x)^2)+(x+2*x^2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-x + \sqrt{x\,(4c_1 + x)}}{2x^2}$$
$$y(x) = \frac{-x - \sqrt{x\,(4c_1 + x)}}{2x^2}$$

✓ Solution by Mathematica

Time used: 0.888 (sec). Leaf size: 69

```
DSolve[(2*y[x]+3*x*y[x]^2)+(x+2*x^2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^{3/2} + \sqrt{x^2(x + 4c_1)}}{2x^{5/2}}$$
$$y(x) \to \frac{-x^{3/2} + \sqrt{x^2(x + 4c_1)}}{2x^{5/2}}$$

5.3 problem Ex 3

Internal problem ID [11142]
Internal file name [`OUTPUT/10127_Wednesday_November_23_2022_11_51_05_AM_56605166/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 12. Equations of form $yf_1(xy) + xf_2(xy)y' = 0$. Page 18
Problem number: Ex 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y + y^2x + (x - x^2y)\, y' = 0}$$

5.3.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y(xy+1)}{x\,(xy-1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{y(xy+1)(b_3 - a_2)}{x(xy-1)} - \frac{y^2(xy+1)^2 a_3}{x^2(xy-1)^2} \\
&- \left(\frac{y^2}{x(xy-1)} - \frac{y(xy+1)}{x^2(xy-1)} - \frac{y^2(xy+1)}{x(xy-1)^2}\right)(xa_2 + ya_3 + a_1) \\
&- \left(\frac{xy+1}{x(xy-1)} + \frac{y}{xy-1} - \frac{y(xy+1)}{(xy-1)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{x^3y^2b_1 - x^2y^3a_1 - 2x^2y^2a_2 - 2x^2y^2b_3 - 2x^2yb_1 - 2x\,y^2a_1 - 2b_2x^2 + 2y^2a_3 - xb_1 + ya_1}{x^2(xy-1)^2} = 0
$$

Setting the numerator to zero gives

$$
-x^3y^2b_1 + x^2y^3a_1 + 2x^2y^2a_2 + 2x^2y^2b_3 + 2x^2yb_1 + 2x\,y^2a_1 + 2b_2x^2 - 2y^2a_3 + xb_1 - ya_1 = 0 \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
a_1v_1^2v_2^3 - b_1v_1^3v_2^2 + 2a_2v_1^2v_2^2 + 2b_3v_1^2v_2^2 + 2a_1v_1v_2^2 + 2b_1v_1^2v_2 - 2a_3v_2^2 + 2b_2v_1^2 - a_1v_2 + b_1v_1 = 0 \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_1 v_1^3 v_2^2 + a_1 v_1^2 v_2^3 + (2a_2 + 2b_3)\, v_1^2 v_2^2 + 2b_1 v_1^2 v_2 + 2b_2 v_1^2 + 2a_1 v_1 v_2^2 + b_1 v_1 - 2a_3 v_2^2 - a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ -a_1 &= 0 \\ 2a_1 &= 0 \\ -2a_3 &= 0 \\ -b_1 &= 0 \\ 2b_1 &= 0 \\ 2b_2 &= 0 \\ 2a_2 + 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(\frac{y(xy+1)}{x\,(xy-1)}\right)(-x) \\ &= \frac{2y^2 x}{xy-1} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2y^2 x}{xy-1}} dy\end{aligned}$$

Which results in

$$S = \frac{1}{2xy} + \frac{\ln(y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(xy + 1)}{x(xy - 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{2x^2 y} \\ S_y &= \frac{xy - 1}{2y^2 x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)\, xy + 1}{2xy} = \frac{\ln(x)}{2} + c_1$$

Which simplifies to

$$\frac{\ln(y)\, xy + 1}{2xy} = \frac{\ln(x)}{2} + c_1$$

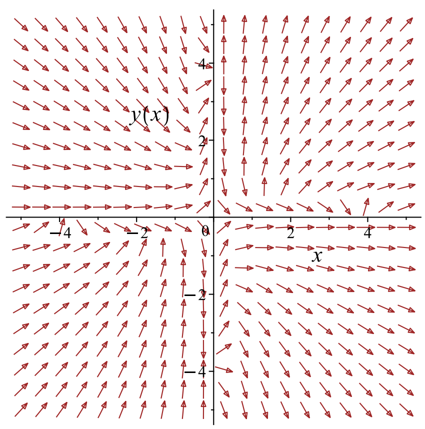
Which gives

$$y = -\frac{1}{x \operatorname{LambertW}\left(-\frac{e^{-2c_1}}{x^2}\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{y(xy+1)}{x(xy-1)}$



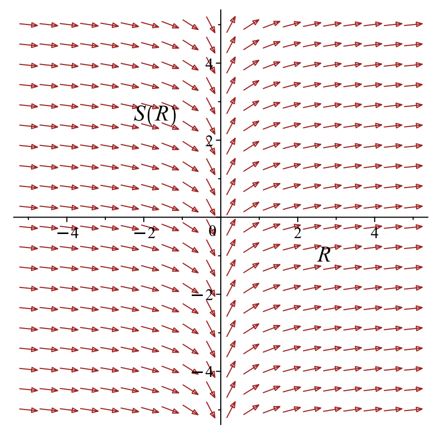
</td><td>

$R = x$

$S = \frac{\ln(y)\, xy + 1}{2xy}$

</td><td>

$\frac{dS}{dR} = \frac{1}{2R}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\frac{1}{x \,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-2c_1}}{x^2}\right)} \tag{1}$$

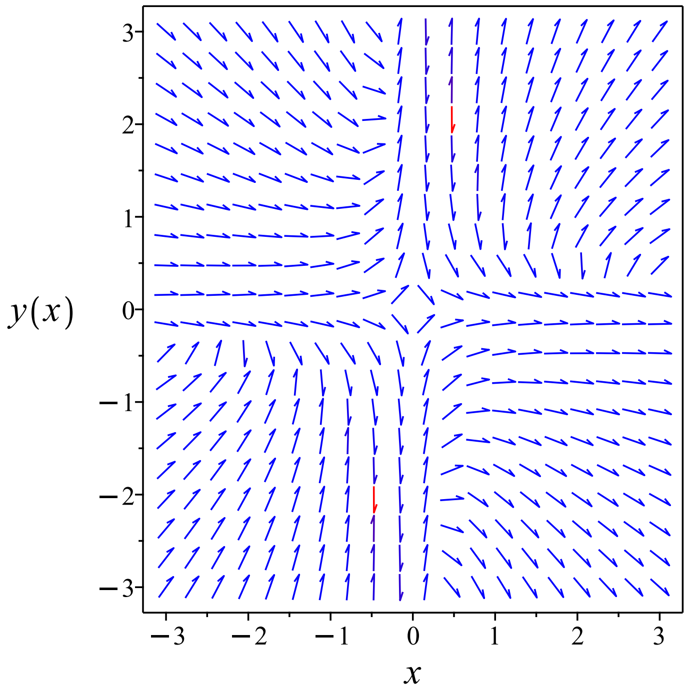


Figure 61: Slope field plot

Verification of solutions

$$y = -\frac{1}{x \operatorname{LambertW}\left(-\frac{e^{-2c_1}}{x^2}\right)}$$

Verified OK.

5.3.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-x^2 y + x\right) \mathrm{d}y = \left(-x\,y^2 - y\right) \mathrm{d}x$$
$$\left(x\,y^2 + y\right) \mathrm{d}x + \left(-x^2 y + x\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = x\,y^2 + y$$
$$N(x, y) = -x^2 y + x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x\,y^2 + y\right)\\ &= 2xy + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^2 y + x\right)\\ &= -2xy + 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{-x^2y + x}((2xy + 1) - (-2xy + 1)) \\
&= -\frac{4y}{xy - 1}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{x\,y^2 + y}((-2xy + 1) - (2xy + 1)) \\
&= -\frac{4x}{xy + 1}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(-2xy + 1) - (2xy + 1)}{x\,(x\,y^2 + y) - y\,(-x^2y + x)} \\
&= -\frac{2}{xy}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{2}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \, \mathrm{d}t} \\
&= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{x^2 y^2}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2 y^2}\left(x\,y^2 + y\right) \\
&= \frac{xy+1}{y\,x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2 y^2}\left(-x^2 y + x\right) \\
&= \frac{-xy+1}{x\,y^2}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{xy+1}{y\,x^2}\right) + \left(\frac{-xy+1}{x\,y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{xy+1}{y\,x^2} \, \mathrm{d}x$$

$$\phi = -\frac{1}{xy} + \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x\,y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-xy+1}{x\,y^2}$. Therefore equation (4) becomes

$$\frac{-xy+1}{x\,y^2} = \frac{1}{x\,y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{1}{y}\right) \mathrm{d}y$$

$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{xy} + \ln(x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{xy} + \ln(x) - \ln(y)$$

The solution becomes

$$y = -\frac{1}{x \operatorname{LambertW}\left(-\frac{e^{c_1}}{x^2}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x \operatorname{LambertW}\left(-\frac{e^{c_1}}{x^2}\right)} \tag{1}$$

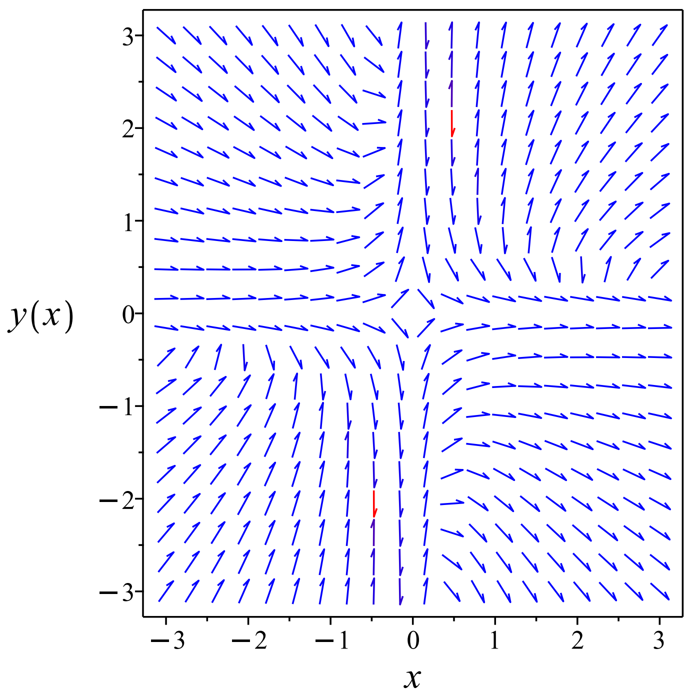


Figure 62: Slope field plot

Verification of solutions

$$y = -\frac{1}{x \operatorname{LambertW}\left(-\frac{e^{c_1}}{x^2}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve((y(x)+x*y(x)^2)+(x-x^2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{\operatorname{LambertW}\left(-\frac{c_1}{x^2}\right) x}$$

✓ Solution by Mathematica

Time used: 8.358 (sec). Leaf size: 35

```
DSolve[(y[x]+x*y[x]^2)+(x-x^2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{xW\left(\frac{e^{-1+\frac{9c_1}{2^{2/3}}}}{x^2}\right)}$$

$$y(x) \to 0$$

6 Chapter 2, differential equations of the first order and the first degree. Article 13. Linear equations of first order. Page 19

6.1 problem Ex 1

Internal problem ID [11143]
Internal file name [`OUTPUT/10128_Wednesday_November_23_2022_11_51_06_AM_91300626/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 13. Linear equations of first order. Page 19
Problem number: Ex 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cot(x) = \sec(x)}$$

6.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cot(x)$$
$$q(x) = \sec(x)$$

Hence the ode is

$$y' + y\cot(x) = \sec(x)$$

The integrating factor μ is

$$\mu = e^{\int \cot(x)dx}$$
$$= \sin(x)$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(\sec(x))$$
$$\frac{d}{dx}(\sin(x) y) = (\sin(x))(\sec(x))$$
$$d(\sin(x) y) = \tan(x)\, dx$$

Integrating gives

$$\sin(x) y = \int \tan(x)\, dx$$
$$\sin(x) y = -\ln(\cos(x)) + c_1$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$y = -\csc(x)\ln(\cos(x)) + c_1 \csc(x)$$

which simplifies to

$$y = \csc(x)(-\ln(\cos(x)) + c_1)$$

Summary
The solution(s) found are the following

$$y = \csc(x)(-\ln(\cos(x)) + c_1) \tag{1}$$

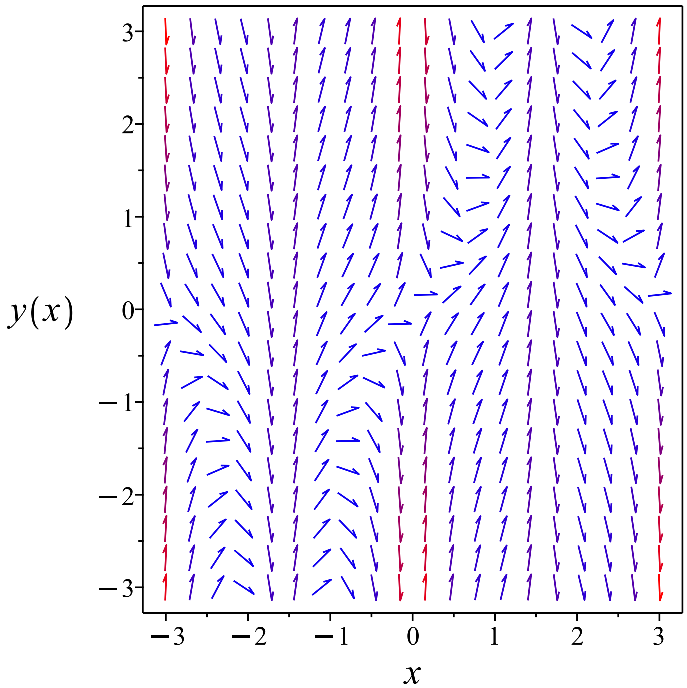


Figure 63: Slope field plot

Verification of solutions

$$y = \csc(x)\left(-\ln(\cos(x)) + c_1\right)$$

Verified OK.

6.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y\cot(x) + \sec(x)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 31: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = 0$$
$$\eta(x, y) = \frac{1}{\sin(x)} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sin(x)}} dy \end{aligned}$$

Which results in

$$S = \sin(x) y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y \cot(x) + \sec(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x) y \\ S_y &= \sin(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \tan(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \tan(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(\cos(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sin(x)\, y = -\ln(\cos(x)) + c_1$$

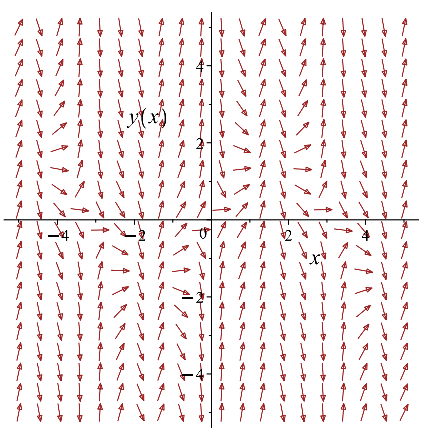
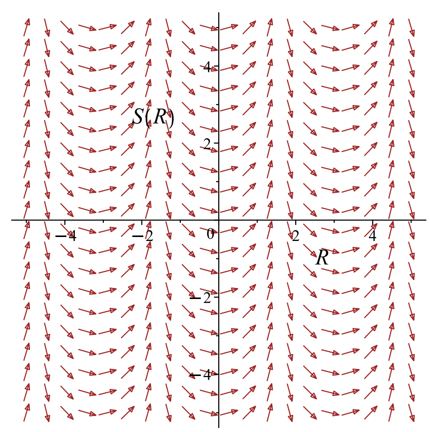
Which simplifies to

$$\sin(x)\, y = -\ln(\cos(x)) + c_1$$

Which gives

$$y = -\frac{\ln(\cos(x)) - c_1}{\sin(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dy}{dx} = -y\cot(x) + \sec(x)$

</td><td>$R = x$
$S = \sin(x)\, y$</td><td>$\frac{dS}{dR} = \tan(R)$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\frac{\ln(\cos(x)) - c_1}{\sin(x)} \tag{1}$$

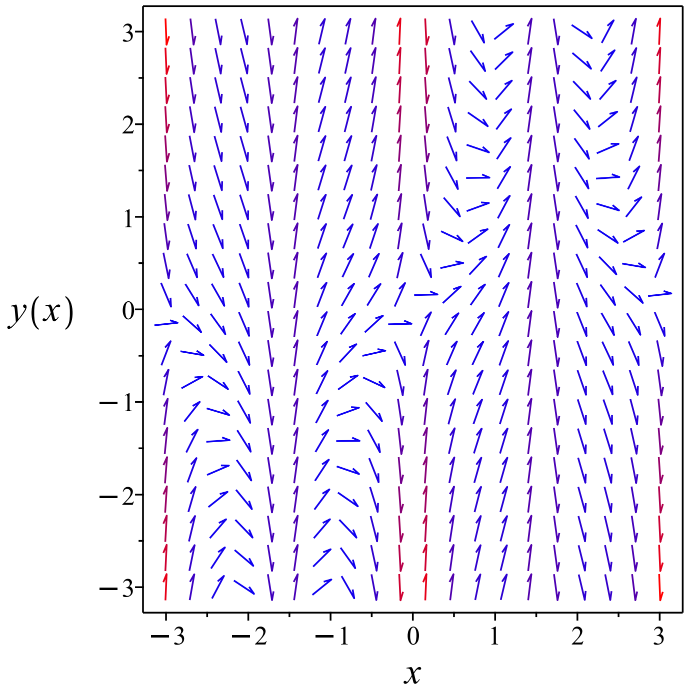


Figure 64: Slope field plot

Verification of solutions

$$y = -\frac{\ln(\cos(x)) - c_1}{\sin(x)}$$

Verified OK.

6.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= (-y \cot(x) + \sec(x))\,\mathrm{d}x \\
(y \cot(x) - \sec(x))\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= y \cot(x) - \sec(x) \\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y \cot(x) - \sec(x)) \\
&= \cot(x)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((\cot(x)) - (0)) \\
&= \cot(x)
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \cot(x) \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{\ln(\sin(x))} \\
&= \sin(x)
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \sin(x)\,(y \cot(x) - \sec(x)) \\
&= \cos(x)\, y - \tan(x)
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \sin(x)\,(1) \\
&= \sin(x)
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(\cos(x)\, y - \tan(x)) + (\sin(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \cos(x) y - \tan(x) \mathrm{d}x \\
\phi &= \sin(x) y + \ln(\cos(x)) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)$. Therefore equation (4) becomes

$$
\sin(x) = \sin(x) + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \sin(x) y + \ln(\cos(x)) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \sin(x) y + \ln(\cos(x))
$$

The solution becomes

$$
y = -\frac{\ln(\cos(x)) - c_1}{\sin(x)}
$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln(\cos(x)) - c_1}{\sin(x)} \tag{1}$$

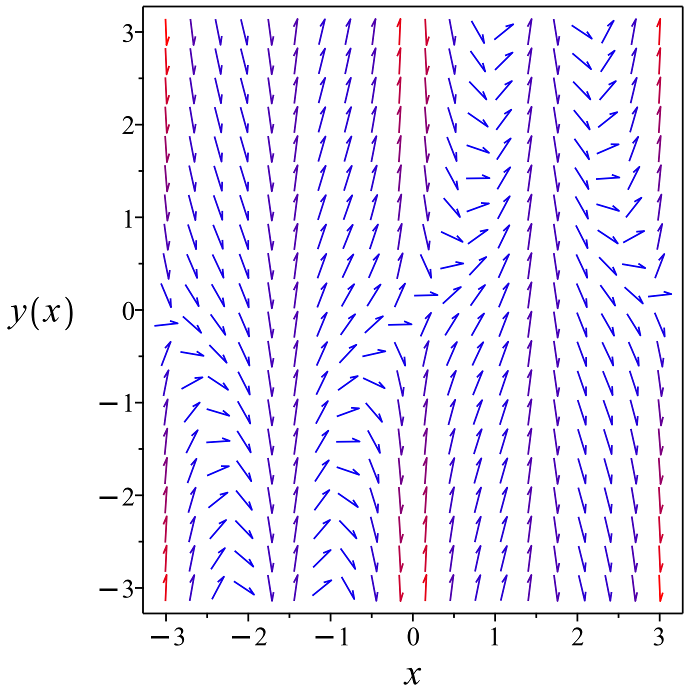


Figure 65: Slope field plot

Verification of solutions

$$y = -\frac{\ln(\cos(x)) - c_1}{\sin(x)}$$

Verified OK.

6.1.4 Maple step by step solution

Let's solve

$y' + y\cot(x) = \sec(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -y\cot(x) + \sec(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' + y\cot(x) = \sec(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\,(y' + y\cot(x)) = \mu(x)\sec(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$\mu(x)\,(y' + y\cot(x)) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

$\mu'(x) = \mu(x)\cot(x)$

- Solve to find the integrating factor

$\mu(x) = \sin(x)$

- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \mu(x)\sec(x)\,dx + c_1$

- Evaluate the integral on the lhs

$\mu(x)\,y = \int \mu(x)\sec(x)\,dx + c_1$

- Solve for y

$y = \frac{\int \mu(x)\sec(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \sin(x)$

$y = \frac{\int \sec(x)\sin(x)dx + c_1}{\sin(x)}$

- Evaluate the integrals on the rhs

$y = \frac{\ln(\sec(x)) + c_1}{\sin(x)}$

- Simplify

$y = \csc(x)\,(\ln(\sec(x)) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+y(x)*cot(x)=sec(x),y(x), singsol=all)
```

$$y(x) = \csc(x)(-\ln(\cos(x)) + c_1)$$

✓ Solution by Mathematica

Time used: 0.081 (sec). Leaf size: 16

```
DSolve[y'[x]+y[x]*Cot[x]==Sec[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \csc(x)(-\log(\cos(x)) + c_1)$$

6.2 problem Ex 2

Internal problem ID [11144]
Internal file name [`OUTPUT/10129_Wednesday_November_23_2022_11_51_06_AM_3118788/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 13. Linear equations of first order. Page 19
Problem number: Ex 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x + y(1+x) = e^x}$$

6.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-x-1}{x}$$
$$q(x) = \frac{e^x}{x}$$

Hence the ode is

$$y' - \frac{(-x-1)\,y}{x} = \frac{e^x}{x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{-x-1}{x}dx} \\
&= e^{x+\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = x\,e^{x}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{e^{x}}{x}\right) \\
\frac{d}{dx}(x\,e^{x} y) &= (x\,e^{x})\left(\frac{e^{x}}{x}\right) \\
d(x\,e^{x} y) &= e^{2x}\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x\,e^{x} y &= \int e^{2x}\,dx \\
x\,e^{x} y &= \frac{e^{2x}}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x\,e^{x}$ results in

$$
y = \frac{e^{-x} e^{2x}}{2x} + \frac{c_1 e^{-x}}{x}
$$

which simplifies to

$$
y = \frac{2c_1 e^{-x} + e^{x}}{2x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2c_1 e^{-x} + e^{x}}{2x} \tag{1}
$$

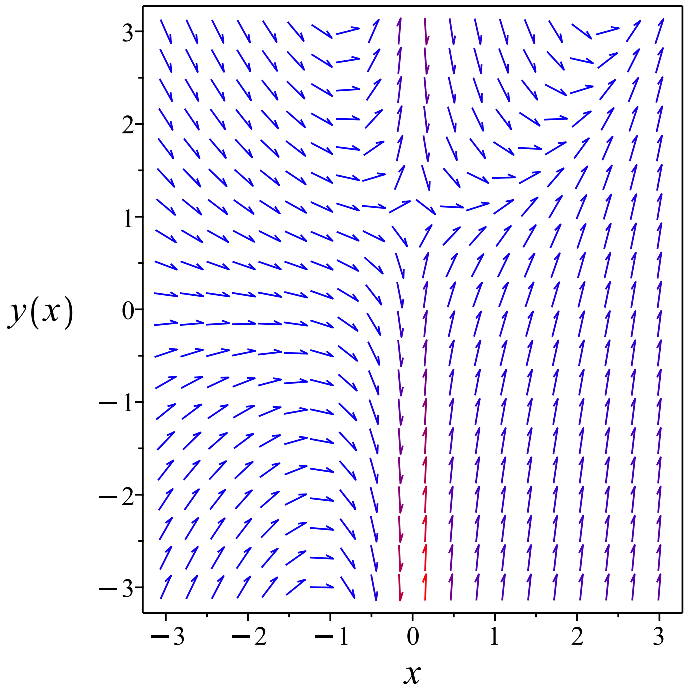


Figure 66: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{2c_1 e^{-x} + e^x}{2x}$$

Verified OK.

6.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-xy + e^x - y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 34: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x - \ln(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x-\ln(x)}} dy \end{aligned}$$

Which results in

$$S = x\,e^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-xy + e^{x} - y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{x} y(1+x) \\ S_y &= x\,e^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x\,e^{x} y = \frac{e^{2x}}{2} + c_1$$

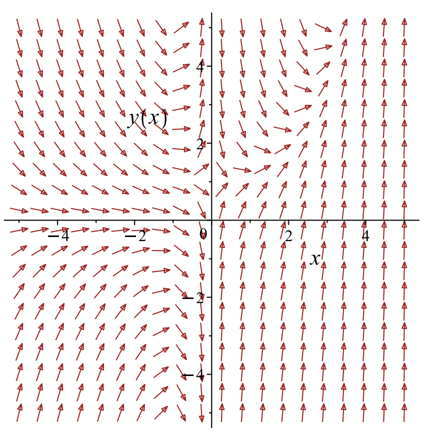
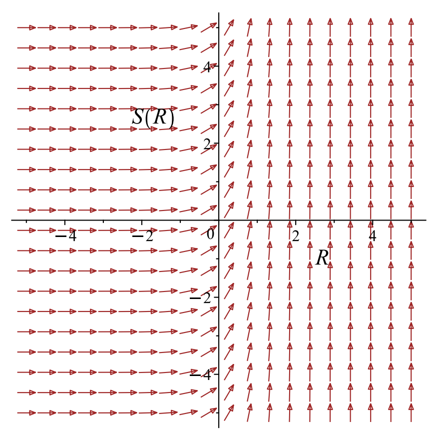
Which simplifies to

$$x\,e^{x} y = \frac{e^{2x}}{2} + c_1$$

Which gives

$$y = \frac{(e^{2x} + 2c_1)\,e^{-x}}{2x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-xy + e^{x} - y}{x}$ 	$R = x$ $S = x\,e^{x} y$	$\frac{dS}{dR} = e^{2R}$ 

Summary

The solution(s) found are the following

$$y = \frac{(e^{2x} + 2c_1)\,e^{-x}}{2x} \tag{1}$$

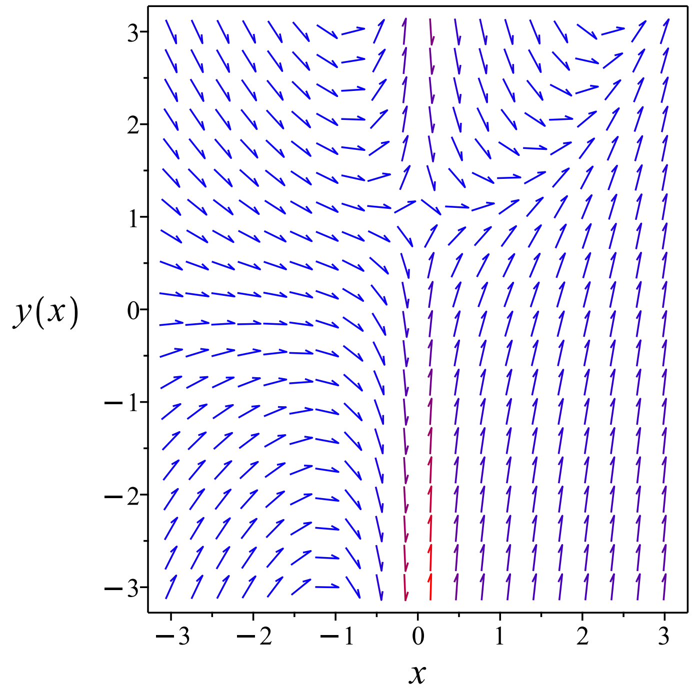


Figure 67: Slope field plot

Verification of solutions

$$y = \frac{(e^{2x} + 2c_1) e^{-x}}{2x}$$

Verified OK.

6.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\, \mathrm{d}y &= (-y(1 + x) + \mathrm{e}^x)\, \mathrm{d}x\\ (y(1 + x) - \mathrm{e}^x)\, \mathrm{d}x + (x)\, \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y(1 + x) - \mathrm{e}^x\\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(1 + x) - \mathrm{e}^x)\\ &= 1 + x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((1 + x) - (1)) \\
&= 1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x}(y(1 + x) - \mathrm{e}^{x}) \\
&= (y(1 + x) - \mathrm{e}^{x})\,\mathrm{e}^{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(x) \\
&= x\,\mathrm{e}^{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
((y(1 + x) - \mathrm{e}^{x})\,\mathrm{e}^{x}) + (x\,\mathrm{e}^{x})\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int (y(1+x) - \mathrm{e}^{x})\,\mathrm{e}^{x}\,\mathrm{d}x$$
$$\phi = x\,\mathrm{e}^{x} y - \frac{\mathrm{e}^{2x}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^{x}$. Therefore equation (4) becomes

$$x\,\mathrm{e}^{x} = x\,\mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x\,\mathrm{e}^{x} y - \frac{\mathrm{e}^{2x}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,\mathrm{e}^{x} y - \frac{\mathrm{e}^{2x}}{2}$$

The solution becomes

$$y = \frac{(\mathrm{e}^{2x} + 2c_1)\,\mathrm{e}^{-x}}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{(e^{2x} + 2c_1) e^{-x}}{2x} \tag{1}$$

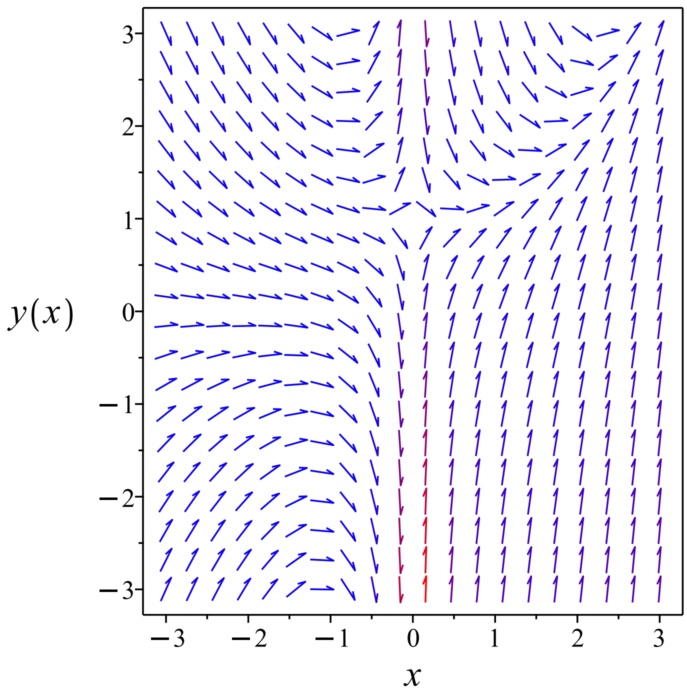


Figure 68: Slope field plot

Verification of solutions

$$y = \frac{(e^{2x} + 2c_1) e^{-x}}{2x}$$

Verified OK.

6.2.4 Maple step by step solution

Let's solve

$y'x + y(1 + x) = e^x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = -\frac{(1+x)y}{x} + \frac{e^x}{x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + \frac{(1+x)y}{x} = \frac{e^x}{x}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + \frac{(1+x)y}{x}\right) = \frac{\mu(x)e^x}{x}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x)\left(y' + \frac{(1+x)y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{\mu(x)(1+x)}{x}$$

- Solve to find the integrating factor

$$\mu(x) = x\, e^x$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)e^x}{x} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \frac{\mu(x)e^x}{x} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x)e^x}{x} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = xe^x$

$$y = \frac{\int (e^x)^2 dx + c_1}{x\, e^x}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{(e^x)^2}{2} + c_1}{x\, e^x}$$

- Simplify

$$y = \frac{2c_1 e^{-x} + e^x}{2x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x)+(1+x)*y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{x} + 2c_1\mathrm{e}^{-x}}{2x}$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 25

```
DSolve[x*y'[x]+(1+x)*y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{x} + 2c_1e^{-x}}{2x}$$

6.3 problem Ex 3

Internal problem ID [11145]
Internal file name [`OUTPUT/10130_Wednesday_November_23_2022_11_51_07_AM_33825994/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 13. Linear equations of first order. Page 19
Problem number: Ex 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{2y}{1+x} = (1+x)^3}$$

6.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{1+x}$$
$$q(x) = (1+x)^3$$

Hence the ode is

$$y' - \frac{2y}{1+x} = (1+x)^3$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{1+x}dx}$$
$$= \frac{1}{(1+x)^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left((1+x)^3\right)$$
$$\frac{d}{dx}\left(\frac{y}{(1+x)^2}\right) = \left(\frac{1}{(1+x)^2}\right)\left((1+x)^3\right)$$
$$d\left(\frac{y}{(1+x)^2}\right) = (1+x)\,dx$$

Integrating gives

$$\frac{y}{(1+x)^2} = \int 1+x\,dx$$
$$\frac{y}{(1+x)^2} = x + \frac{1}{2}x^2 + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(1+x)^2}$ results in

$$y = (1+x)^2\left(x + \frac{1}{2}x^2\right) + c_1(1+x)^2$$

which simplifies to

$$y = \frac{(1+x)^2 (x^2 + 2c_1 + 2x)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{(1+x)^2 (x^2 + 2c_1 + 2x)}{2} \tag{1}$$

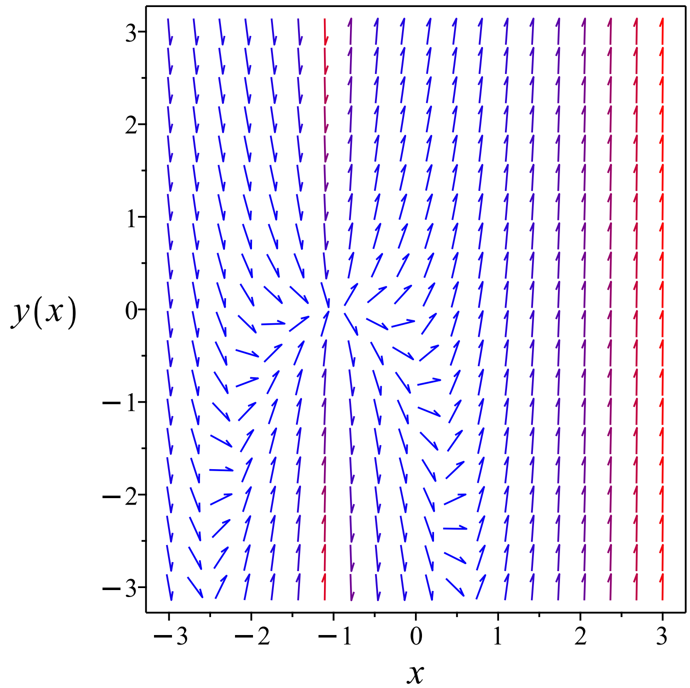


Figure 69: Slope field plot

Verification of solutions

$$y = \frac{(1+x)^2 (x^2 + 2c_1 + 2x)}{2}$$

Verified OK.

6.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^4 + 4x^3 + 6x^2 + 4x + 2y + 1}{1+x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 37: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= (1+x)^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\left(1+x\right)^{2}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\left(1+x\right)^{2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^4 + 4x^3 + 6x^2 + 4x + 2y + 1}{1+x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{\left(1+x\right)^{3}} \\ S_y &= \frac{1}{\left(1+x\right)^{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 + x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1 + R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{2}R^2 + R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{(1+x)^2} = x + \frac{1}{2}x^2 + c_1$$

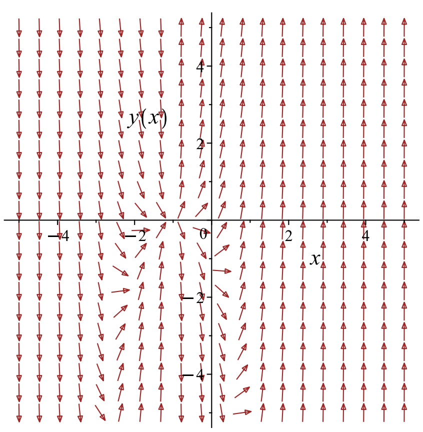
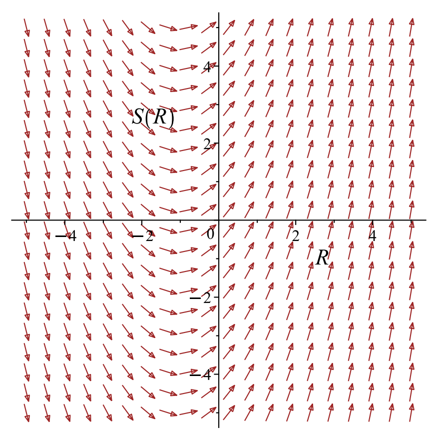
Which simplifies to

$$\frac{y}{(1+x)^2} = x + \frac{1}{2}x^2 + c_1$$

Which gives

$$y = \frac{(1+x)^2 (x^2 + 2c_1 + 2x)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^4+4x^3+6x^2+4x+2y+1}{1+x}$ 	$R = x$ $S = \frac{y}{(1+x)^2}$	$\frac{dS}{dR} = 1 + R$ 

Summary

The solution(s) found are the following

$$y = \frac{(1+x)^2 (x^2 + 2c_1 + 2x)}{2} \tag{1}$$

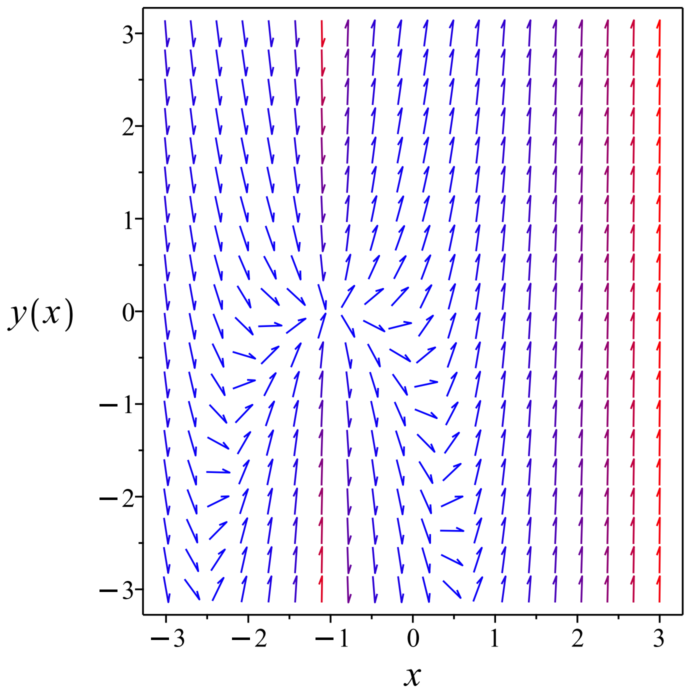


Figure 70: Slope field plot

Verification of solutions

$$y = \frac{(1+x)^2 (x^2 + 2c_1 + 2x)}{2}$$

Verified OK.

6.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\mathrm{d} y=\left(\frac{2 y}{1+x}+(1+x)^3\right) \mathrm{d} x$$

$$\left(-\frac{2 y}{1+x}-(1+x)^3\right) \mathrm{d} x+\mathrm{d} y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -\frac{2 y}{1+x}-(1+x)^3 \\
N(x, y) &= 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2 y}{1+x}-(1+x)^3\right) \\
&= -\frac{2}{1+x}
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{2}{1+x}\right) - (0)\right) \\
&= -\frac{2}{1+x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int -\frac{2}{1+x} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(1+x)} \\
&= \frac{1}{(1+x)^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{(1+x)^2}\left(-\frac{2y}{1+x} - (1+x)^3\right) \\
&= \frac{-x^4 - 4x^3 - 6x^2 - 4x - 2y - 1}{(1+x)^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(1+x)^2}(1) \\
&= \frac{1}{(1+x)^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-x^4 - 4x^3 - 6x^2 - 4x - 2y - 1}{(1+x)^3}\right) + \left(\frac{1}{(1+x)^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-x^4 - 4x^3 - 6x^2 - 4x - 2y - 1}{(1+x)^3} \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} - x + \frac{y}{(1+x)^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{(1+x)^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{(1+x)^2}$. Therefore equation (4) becomes

$$\frac{1}{(1+x)^2} = \frac{1}{(1+x)^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - x + \frac{y}{(1+x)^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - x + \frac{y}{(1+x)^2}$$

The solution becomes

$$y = \frac{(1+x)^2 (x^2 + 2c_1 + 2x)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{(1+x)^2 (x^2 + 2c_1 + 2x)}{2} \tag{1}$$

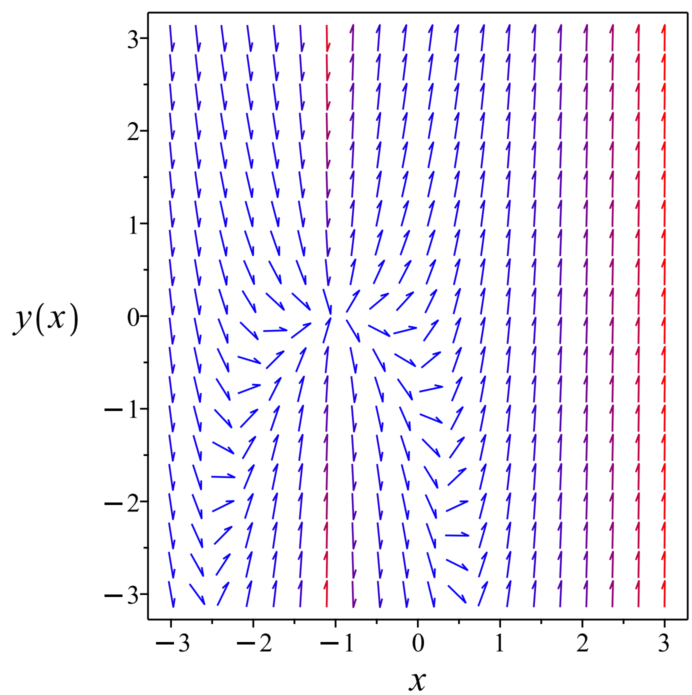


Figure 71: Slope field plot

Verification of solutions

$$y = \frac{(1+x)^2 (x^2+2c_1+2x)}{2}$$

Verified OK.

6.3.4 Maple step by step solution

Let's solve

$y' - \frac{2y}{1+x} = (1+x)^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{1+x} + (1+x)^3$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{1+x} = (1+x)^3$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{2y}{1+x}\right) = \mu(x)(1+x)^3$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{2y}{1+x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{2\mu(x)}{1+x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{(1+x)^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)(1+x)^3 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)(1+x)^3 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)(1+x)^3 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{(1+x)^2}$

$$y = (1+x)^2 \left(\int (1+x)\, dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = (1+x)^2 \left(x + \tfrac{1}{2}x^2 + c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x)-2*y(x)/(1+x)=(x+1)^3,y(x), singsol=all)
```

$$y(x) = \left(x + \frac{1}{2}x^2 + c_1\right)(1+x)^2$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 22

```
DSolve[y'[x]-2*y[x]/(1+x)==(x+1)^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x+1)^2 \left(\frac{x^2}{2} + x + c_1\right)$$

6.4 problem Ex 4

Internal problem ID [11146]
Internal file name [`OUTPUT/10131_Wednesday_November_23_2022_11_51_08_AM_5086682/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 13. Linear equations of first order. Page 19
Problem number: Ex 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x^3+x)\,y'+4x^2y=2}$$

6.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{4x}{x^2+1}$$
$$q(x)=\frac{2}{x\,(x^2+1)}$$

Hence the ode is

$$y'+\frac{4xy}{x^2+1}=\frac{2}{x\,(x^2+1)}$$

The integrating factor μ is

$$\mu = e^{\int \frac{4x}{x^2+1}dx}$$
$$= (x^2+1)^2$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{2}{x(x^2+1)}\right)$$
$$\frac{d}{dx}\left((x^2+1)^2 y\right) = \left((x^2+1)^2\right)\left(\frac{2}{x(x^2+1)}\right)$$
$$d\left((x^2+1)^2 y\right) = \left(\frac{2x^2+2}{x}\right) dx$$

Integrating gives

$$(x^2+1)^2 y = \int \frac{2x^2+2}{x}\, dx$$
$$(x^2+1)^2 y = x^2 + 2\ln(x) + c_1$$

Dividing both sides by the integrating factor $\mu = (x^2+1)^2$ results in

$$y = \frac{x^2 + 2\ln(x)}{(x^2+1)^2} + \frac{c_1}{(x^2+1)^2}$$

which simplifies to

$$y = \frac{x^2 + 2\ln(x) + c_1}{(x^2+1)^2}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2 + 2\ln(x) + c_1}{(x^2+1)^2} \tag{1}$$

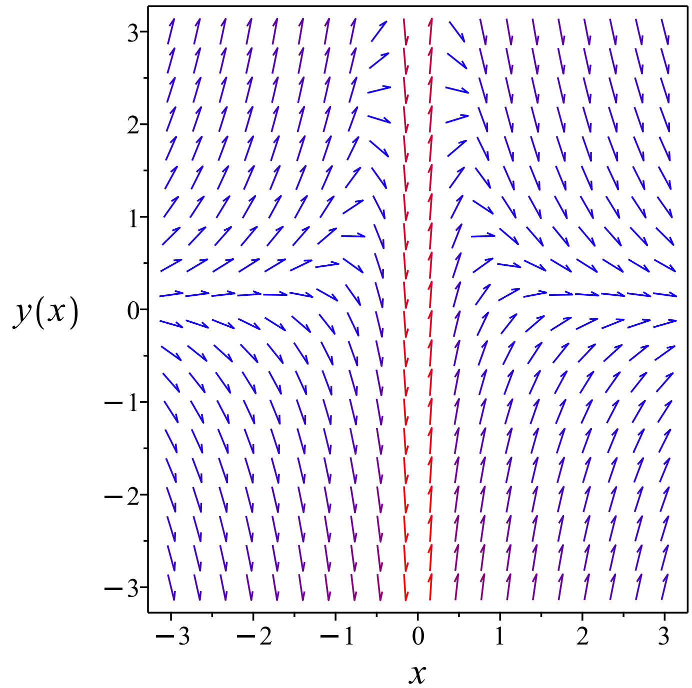


Figure 72: Slope field plot

Verification of solutions

$$y = \frac{x^2 + 2\ln(x) + c_1}{(x^2+1)^2}$$

Verified OK.

6.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2(2x^2y - 1)}{x(x^2+1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 40: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\left(x^2 + 1\right)^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(x^2+1)^2}} dy \end{aligned}$$

Which results in

$$S = (x^2 + 1)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2(2x^2 y - 1)}{x(x^2 + 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 4(x^2 + 1) yx \\ S_y &= (x^2 + 1)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2x^2 + 2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2R^2 + 2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R^2 + 2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x^2+1)^2 y = x^2 + 2\ln(x) + c_1$$

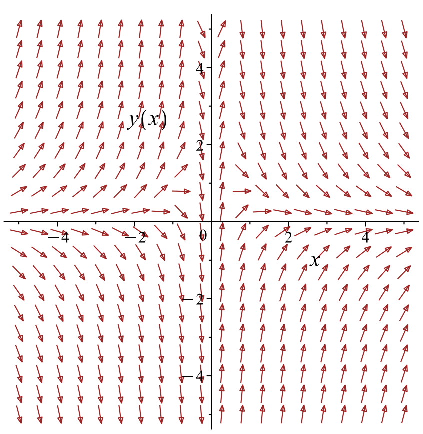
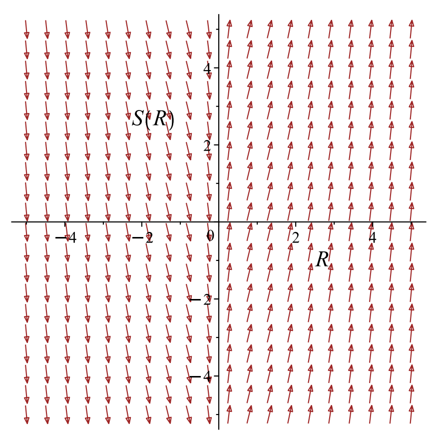
Which simplifies to

$$(x^2+1)^2 y = x^2 + 2\ln(x) + c_1$$

Which gives

$$y = \frac{x^2 + 2\ln(x) + c_1}{(x^2+1)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2(2x^2y-1)}{x(x^2+1)}$ 	$R = x$ $S = (x^2+1)^2 y$	$\frac{dS}{dR} = \frac{2R^2+2}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{x^2 + 2\ln(x) + c_1}{(x^2 + 1)^2} \tag{1}$$

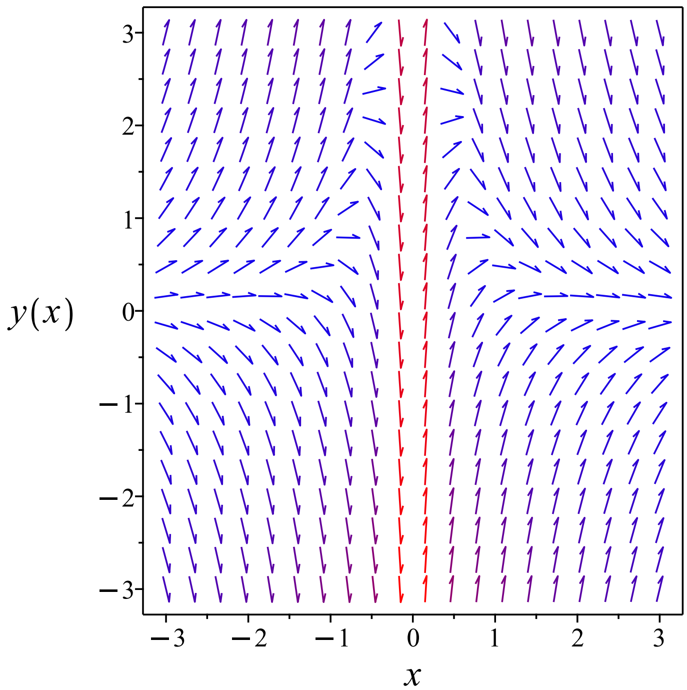


Figure 73: Slope field plot

Verification of solutions

$$y = \frac{x^2 + 2\ln(x) + c_1}{(x^2 + 1)^2}$$

Verified OK.

6.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^3+x\right)\mathrm{d}y &= \left(-4x^2y+2\right)\mathrm{d}x\\ \left(4x^2y-2\right)\mathrm{d}x + \left(x^3+x\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 4x^2y-2\\ N(x,y) &= x^3+x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(4x^2y-2\right)\\ &= 4x^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^3+x\right)\\ &= 3x^2+1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^3+x}\left(\left(4x^2\right)-\left(3x^2+1\right)\right)\\ &= \frac{x^2-1}{x^3+x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{x^2-1}{x^3+x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(x)+\ln(x^2+1)}\\ &= \frac{x^2+1}{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{x^2+1}{x}\left(4x^2y-2\right)\\ &= \frac{\left(4x^2y-2\right)\left(x^2+1\right)}{x}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{x^2+1}{x}(x^3+x) \\
&= (x^2+1)^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{(4x^2y-2)(x^2+1)}{x}\right) + \left((x^2+1)^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{(4x^2y-2)(x^2+1)}{x}\,\mathrm{d}x \\
\phi &= -2\ln(x) + x^4y + (2y-1)x^2 + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x^4 + 2x^2 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x^2+1)^2$. Therefore equation (4) becomes

$$
(x^2+1)^2 = x^4 + 2x^2 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (1) \, \mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\ln(x) + x^4 y + (2y - 1)\, x^2 + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2\ln(x) + x^4 y + (2y - 1)\, x^2 + y$$

The solution becomes

$$y = \frac{x^2 + 2\ln(x) + c_1}{x^4 + 2x^2 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + 2\ln(x) + c_1}{x^4 + 2x^2 + 1} \tag{1}$$

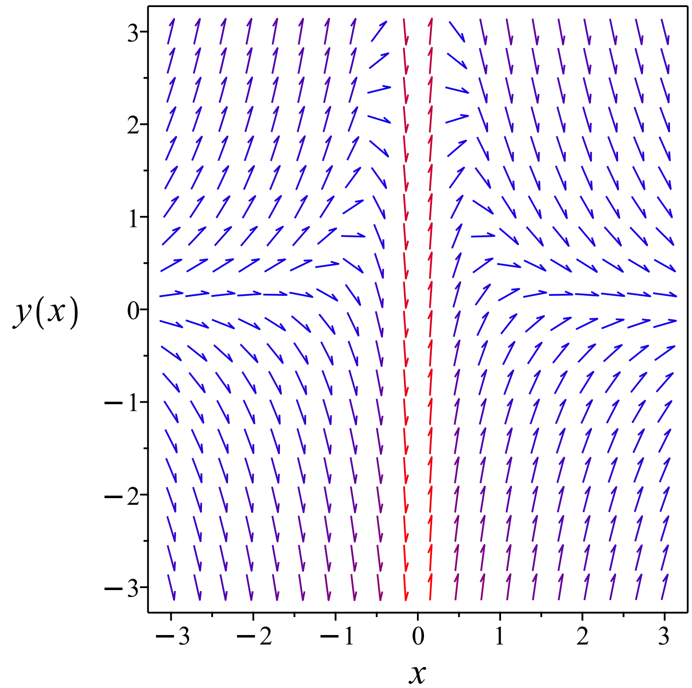


Figure 74: Slope field plot

Verification of solutions

$$y = \frac{x^2 + 2\ln(x) + c_1}{x^4 + 2x^2 + 1}$$

Verified OK.

6.4.4 Maple step by step solution

Let's solve

$(x^3 + x) y' + 4x^2 y = 2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{4xy}{x^2+1} + \frac{2}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{4xy}{x^2+1} = \frac{2}{x(x^2+1)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + \frac{4xy}{x^2+1}\right) = \frac{2\mu(x)}{x(x^2+1)}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y' + \frac{4xy}{x^2+1}\right) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{4\mu(x)x}{x^2+1}$$

- Solve to find the integrating factor

$$\mu(x) = (x^2+1)^2$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \frac{2\mu(x)}{x(x^2+1)}dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \frac{2\mu(x)}{x(x^2+1)}dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{2\mu(x)}{x\left(x^2+1\right)}dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = (x^2+1)^2$

$$y = \frac{\int \frac{2\left(x^2+1\right)}{x}dx + c_1}{(x^2+1)^2}$$

- Evaluate the integrals on the rhs

$$y = \frac{x^2 + 2\ln(x) + c_1}{(x^2+1)^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve((x+x^3)*diff(y(x),x)+4*x^2*y(x)=2,y(x), singsol=all)
```

$$y(x) = \frac{x^2 + 2\ln(x) + c_1}{(x^2 + 1)^2}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 23

```
DSolve[(x+x^3)*y'[x]+4*x^2*y[x]==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2 + 2\log(x) + c_1}{(x^2 + 1)^2}$$

6.5 problem Ex 5

Internal problem ID [11147]
Internal file name [`OUTPUT/10132_Wednesday_November_23_2022_11_51_09_AM_65944221/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 13. Linear equations of first order. Page 19
Problem number: Ex 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x^2 + (-2x + 1)\, y = x^2}$$

6.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2x - 1}{x^2}$$
$$q(x) = 1$$

Hence the ode is

$$y' - \frac{(2x - 1)\, y}{x^2} = 1$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2x-1}{x^2}dx}$$
$$= e^{-\frac{1}{x}-2\ln(x)}$$

Which simplifies to

$$\mu = \frac{e^{-\frac{1}{x}}}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu$$
$$\frac{d}{dx}\left(\frac{e^{-\frac{1}{x}}y}{x^2}\right) = \frac{e^{-\frac{1}{x}}}{x^2}$$
$$d\left(\frac{e^{-\frac{1}{x}}y}{x^2}\right) = \frac{e^{-\frac{1}{x}}}{x^2}dx$$

Integrating gives

$$\frac{e^{-\frac{1}{x}}y}{x^2} = \int \frac{e^{-\frac{1}{x}}}{x^2}\,dx$$
$$\frac{e^{-\frac{1}{x}}y}{x^2} = e^{-\frac{1}{x}} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{e^{-\frac{1}{x}}}{x^2}$ results in

$$y = x^2 e^{\frac{1}{x}} e^{-\frac{1}{x}} + c_1 x^2 e^{\frac{1}{x}}$$

which simplifies to

$$y = x^2\left(1 + c_1 e^{\frac{1}{x}}\right)$$

Summary
The solution(s) found are the following

$$y = x^2\left(1 + c_1 e^{\frac{1}{x}}\right) \tag{1}$$

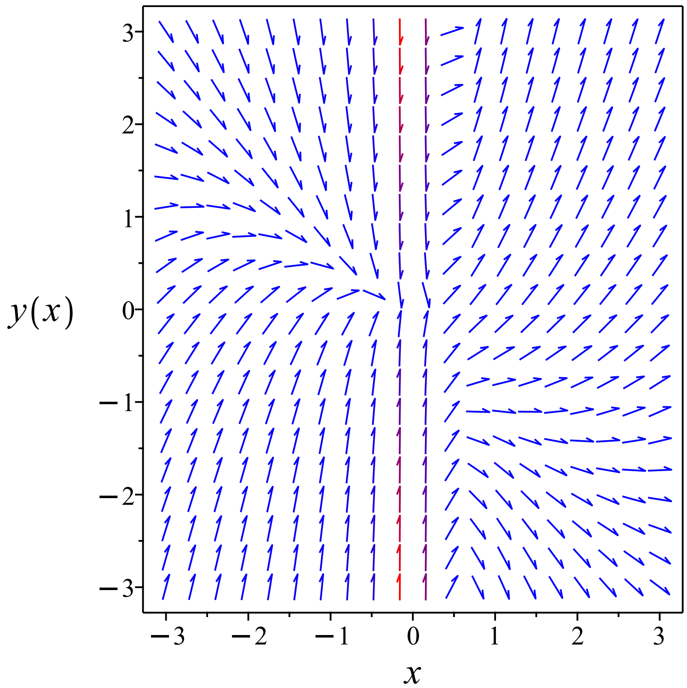


Figure 75: Slope field plot

Verification of solutions

$$y = x^2\left(1 + c_1 e^{\frac{1}{x}}\right)$$

Verified OK.

6.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + 2xy - y}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 43: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{\frac{1}{x} + 2\ln(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{1}{x} + 2\ln(x)}} dy \end{aligned}$$

Which results in

$$S = \frac{e^{-\frac{1}{x}} y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 + 2xy - y}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y(-2x + 1) e^{-\frac{1}{x}}}{x^4} \\ S_y &= \frac{e^{-\frac{1}{x}}}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{-\frac{1}{x}}}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{-\frac{1}{R}}}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = e^{-\frac{1}{R}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{e^{-\frac{1}{x}} y}{x^2} = e^{-\frac{1}{x}} + c_1$$

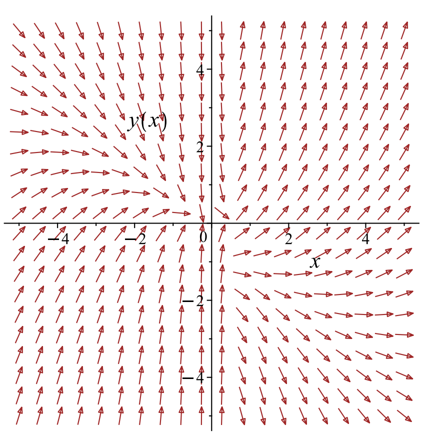
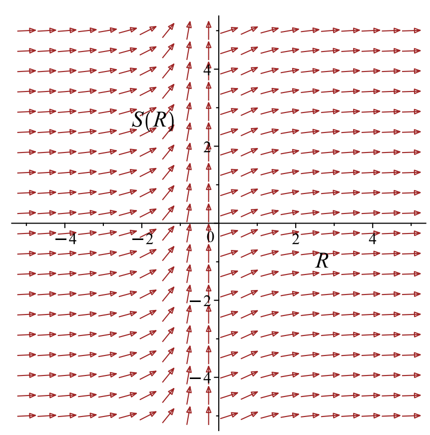
Which simplifies to

$$\frac{e^{-\frac{1}{x}} y}{x^2} = e^{-\frac{1}{x}} + c_1$$

Which gives

$$y = x^2 \left(e^{-\frac{1}{x}} + c_1\right) e^{\frac{1}{x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+2xy-y}{x^2}$ 	$R = x$ $S = \frac{e^{-\frac{1}{x}} y}{x^2}$	$\frac{dS}{dR} = \frac{e^{-\frac{1}{R}}}{R^2}$ 

Summary

The solution(s) found are the following

$$y = x^2 \left(e^{-\frac{1}{x}} + c_1\right) e^{\frac{1}{x}} \tag{1}$$

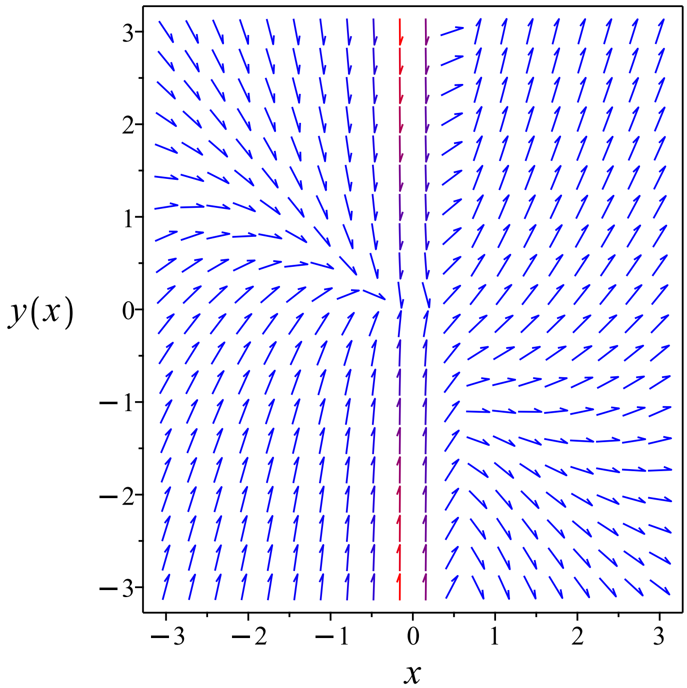


Figure 76: Slope field plot

Verification of solutions

$$y = x^2 \left(e^{-\frac{1}{x}} + c_1\right) e^{\frac{1}{x}}$$

Verified OK.

6.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2)\,\mathrm{d}y &= \left(-(-2x+1)\,y + x^2\right)\mathrm{d}x \\ \left((-2x+1)\,y - x^2\right)\mathrm{d}x + (x^2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= (-2x+1)\,y - x^2 \\ N(x, y) &= x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left((-2x+1)\,y - x^2\right) \\ &= -2x + 1\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2\right) \\
&= 2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^2}((-2x+1) - (2x)) \\
&= \frac{-4x+1}{x^2}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{-4x+1}{x^2} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{1}{x} - 4\ln(x)} \\
&= \frac{\mathrm{e}^{-\frac{1}{x}}}{x^4}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\mathrm{e}^{-\frac{1}{x}}}{x^4}\left((-2x+1)\,y - x^2\right) \\
&= -\frac{\mathrm{e}^{-\frac{1}{x}}\left(x^2 + 2xy - y\right)}{x^4}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\mathrm{e}^{-\frac{1}{x}}}{x^4}\left(x^2\right) \\
&= \frac{\mathrm{e}^{-\frac{1}{x}}}{x^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x}&=0\\
\left(-\frac{\mathrm{e}^{-\frac{1}{x}}(x^2+2xy-y)}{x^4}\right)+\left(\frac{\mathrm{e}^{-\frac{1}{x}}}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x}&=0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int\overline{M}\,\mathrm{d}x\\
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int-\frac{\mathrm{e}^{-\frac{1}{x}}(x^2+2xy-y)}{x^4}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi=-\frac{(x^2-y)\,\mathrm{e}^{-\frac{1}{x}}}{x^2}+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y}=\frac{\mathrm{e}^{-\frac{1}{x}}}{x^2}+f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\frac{\mathrm{e}^{-\frac{1}{x}}}{x^2}$. Therefore equation (4) becomes

$$
\frac{\mathrm{e}^{-\frac{1}{x}}}{x^2}=\frac{\mathrm{e}^{-\frac{1}{x}}}{x^2}+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y)=0
$$

Therefore

$$
f(y)=c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(x^2 - y)\,\mathrm{e}^{-\frac{1}{x}}}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(x^2 - y)\,\mathrm{e}^{-\frac{1}{x}}}{x^2}$$

The solution becomes

$$y = x^2\left(\mathrm{e}^{-\frac{1}{x}} + c_1\right)\mathrm{e}^{\frac{1}{x}}$$

Summary

The solution(s) found are the following

$$y = x^2\left(\mathrm{e}^{-\frac{1}{x}} + c_1\right)\mathrm{e}^{\frac{1}{x}} \tag{1}$$

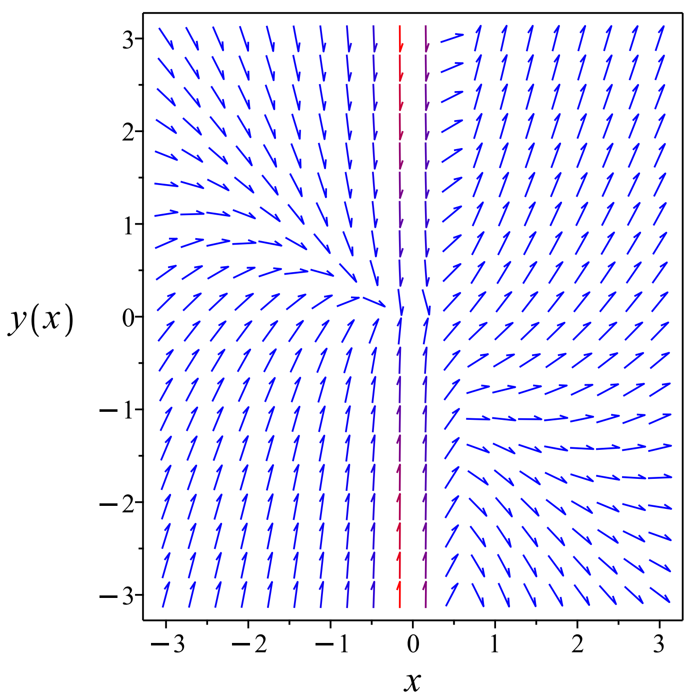


Figure 77: Slope field plot

Verification of solutions

$$y = x^2 \left(\mathrm{e}^{-\frac{1}{x}} + c_1\right) \mathrm{e}^{\frac{1}{x}}$$

Verified OK.

6.5.4 Maple step by step solution

Let's solve

$y'x^2 + (-2x+1)\, y = x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 1 + \frac{(2x-1)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{(2x-1)y}{x^2} = 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' - \frac{(2x-1)y}{x^2}\right) = \mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' - \frac{(2x-1)y}{x^2}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)(2x-1)}{x^2}$

- Solve to find the integrating factor

  $\mu(x) = \frac{\mathrm{e}^{-\frac{1}{x}}}{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{\mathrm{e}^{-\frac{1}{x}}}{x^2}$

$$y = \frac{x^2 \left(\int \frac{e^{-\frac{1}{x}}}{x^2} dx + c_1\right)}{e^{-\frac{1}{x}}}$$

- Evaluate the integrals on the rhs

$$y = \frac{x^2 \left(e^{-\frac{1}{x}} + c_1\right)}{e^{-\frac{1}{x}}}$$

- Simplify

$$y = x^2 \left(1 + c_1 e^{\frac{1}{x}}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x^2*diff(y(x),x)+(1-2*x)*y(x)=x^2,y(x), singsol=all)
```

$$y(x) = x^2 \left(1 + e^{\frac{1}{x}} c_1\right)$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 19

```
DSolve[x^2*y'[x]+(1-2*x)*y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 \left(1 + c_1 e^{\frac{1}{x}}\right)$$

7 Chapter 2, differential equations of the first order and the first degree. Article 14. Equations reducible to linear equations (Bernoulli). Page 21

7.1 problem Ex 1

Internal problem ID [11148]
Internal file name [OUTPUT/10133_Wednesday_November_23_2022_11_51_10_AM_42627324/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 14. Equations reducible to linear equations (Bernoulli). Page 21
Problem number: Ex 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{\left(-x^2+1\right)y'-2y(1+x)-y^{\frac{5}{2}}=0}$$

7.1.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=-\frac{y^{\frac{5}{2}}+2xy+2y}{x^2-1}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 46: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^{\frac{5}{2}} (x-1)^3
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^{\frac{5}{2}} (x-1)^3} dy \end{aligned}$$

Which results in

$$S = -\frac{2}{3y^{\frac{3}{2}} (x-1)^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y^{\frac{5}{2}} + 2xy + 2y}{x^2 - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{y^{\frac{3}{2}} (x-1)^4} \\ S_y &= \frac{1}{y^{\frac{5}{2}} (x-1)^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{(x-1)^4 (1+x)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{(R-1)^4 (1+R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{1}{6\left(R-1\right)^{3}}-\frac{1}{8\left(R-1\right)^{2}}+\frac{1}{8R-8}+\frac{\ln\left(R-1\right)}{16}-\frac{\ln\left(1+R\right)}{16}+c_1 \tag{4}$$

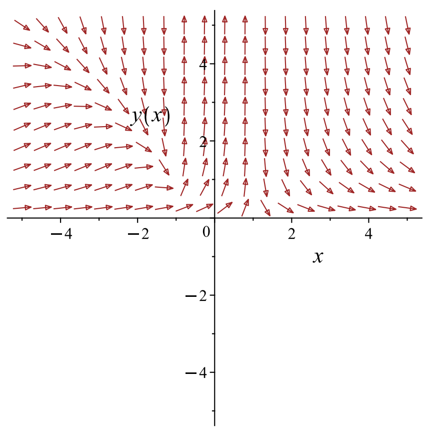
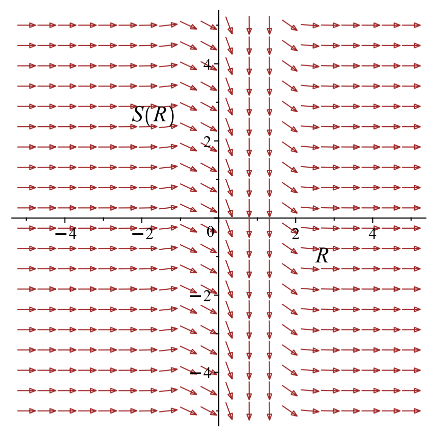
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2}{3y^{\frac{3}{2}}\left(x-1\right)^{3}} = \frac{1}{6\left(x-1\right)^{3}}-\frac{1}{8\left(x-1\right)^{2}}+\frac{1}{8x-8}+\frac{\ln\left(x-1\right)}{16}-\frac{\ln\left(1+x\right)}{16}+c_1$$

Which simplifies to

$$-\frac{2}{3y^{\frac{3}{2}}\left(x-1\right)^{3}} = \frac{1}{6\left(x-1\right)^{3}}-\frac{1}{8\left(x-1\right)^{2}}+\frac{1}{8x-8}+\frac{\ln\left(x-1\right)}{16}-\frac{\ln\left(1+x\right)}{16}+c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^{\frac{5}{2}}+2xy+2y}{x^{2}-1}$ 	$R = x$ $S = -\frac{2}{3y^{\frac{3}{2}}\left(x-1\right)^{3}}$	$\frac{dS}{dR} = -\frac{1}{\left(R-1\right)^{4}\left(1+R\right)}$ 

Summary

The solution(s) found are the following

$$-\frac{2}{3y^{\frac{3}{2}}\left(x-1\right)^{3}} = \frac{1}{6\left(x-1\right)^{3}}-\frac{1}{8\left(x-1\right)^{2}}+\frac{1}{8x-8}+\frac{\ln\left(x-1\right)}{16}-\frac{\ln\left(1+x\right)}{16}+c_1 \tag{1}$$

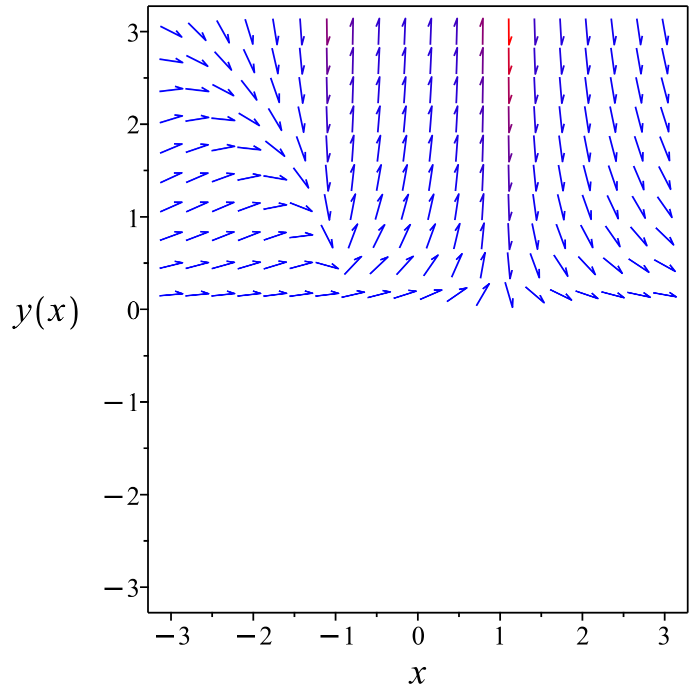


Figure 78: Slope field plot

Verification of solutions

$$-\frac{2}{3y^{\frac{3}{2}}\left(x-1\right)^{3}}=\frac{1}{6\left(x-1\right)^{3}}-\frac{1}{8\left(x-1\right)^{2}}+\frac{1}{8x-8}+\frac{\ln\left(x-1\right)}{16}-\frac{\ln\left(1+x\right)}{16}+c_1$$

Verified OK.

7.1.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= -\frac{y^{\frac{5}{2}}+2xy+2y}{x^2-1}\end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2+2x}{x^2-1}y-\frac{1}{x^2-1}y^{\frac{5}{2}} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y+f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n}+f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{2+2x}{x^2-1} \\
f_1(x) &= -\frac{1}{x^2-1} \\
n &= \frac{5}{2}
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^{\frac{5}{2}}$ gives

$$
y'\frac{1}{y^{\frac{5}{2}}} = -\frac{2+2x}{(x^2-1)\,y^{\frac{3}{2}}} - \frac{1}{x^2-1} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^{\frac{3}{2}}}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{3}{2y^{\frac{5}{2}}}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{2w'(x)}{3} &= -\frac{(2+2x)\,w(x)}{x^2-1} - \frac{1}{x^2-1} \\
w' &= \frac{3(2+2x)\,w}{2\,(x^2-1)} + \frac{3}{2\,(x^2-1)}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3}{x-1} \\
q(x) &= \frac{3}{2x^2-2}
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{3w(x)}{x-1} = \frac{3}{2x^2-2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{3}{x-1}dx} \\ &= \frac{1}{(x-1)^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{3}{2x^2-2}\right) \\ \frac{d}{dx}\left(\frac{w}{(x-1)^3}\right) &= \left(\frac{1}{(x-1)^3}\right)\left(\frac{3}{2x^2-2}\right) \\ d\left(\frac{w}{(x-1)^3}\right) &= \left(\frac{3}{2(x-1)^4(1+x)}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{(x-1)^3} &= \int \frac{3}{2(x-1)^4(1+x)} dx \\ \frac{w}{(x-1)^3} &= -\frac{1}{4(x-1)^3} + \frac{3}{16(x-1)^2} - \frac{3}{16(x-1)} - \frac{3\ln(x-1)}{32} + \frac{3\ln(1+x)}{32} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(x-1)^3}$ results in

$$w(x) = (x-1)^3\left(-\frac{1}{4(x-1)^3} + \frac{3}{16(x-1)^2} - \frac{3}{16(x-1)} - \frac{3\ln(x-1)}{32} + \frac{3\ln(1+x)}{32}\right) + c_1(x-1)^3$$

Replacing w in the above by $\frac{1}{y^{\frac{3}{2}}}$ using equation (5) gives the final solution.

$$\frac{1}{y^{\frac{3}{2}}} = (x-1)^3\left(-\frac{1}{4(x-1)^3} + \frac{3}{16(x-1)^2} - \frac{3}{16(x-1)} - \frac{3\ln(x-1)}{32} + \frac{3\ln(1+x)}{32}\right) + c_1(x-1)^3$$

Solving for y gives

$$y(x) = \frac{2^{\frac{1}{3}} \left(-1024 \left(\frac{3(x-1)^3 \ln(1+x)}{32} - \frac{3(x-1)^3 \ln(x-1)}{32} + c_1 x^3 + \left(-3c_1 - \frac{3}{16}\right) x^2 + \left(3c_1 + \frac{9}{16}\right) x - c_1 - \frac{5}{8}\right)^2\right)^{\frac{2}{3}}}{128 \left(\frac{3(x-1)^3 \ln(1+x)}{32} - \frac{3(x-1)^3 \ln(x-1)}{32} + c_1 x^3 + \left(-3c_1 - \frac{3}{16}\right) x^2 + \left(3c_1 + \frac{9}{16}\right) x - c_1 - \frac{5}{8}\right)^2}$$

$$y(x) = -\frac{2^{\frac{1}{3}} \left(-1024 \left(\frac{3(x-1)^3 \ln(1+x)}{32} - \frac{3(x-1)^3 \ln(x-1)}{32} + c_1 x^3 + \left(-3c_1 - \frac{3}{16}\right) x^2 + \left(3c_1 + \frac{9}{16}\right) x - c_1 - \frac{5}{8}\right)^2\right)^{\frac{2}{3}} (}{512 \left(\frac{3(x-1)^3 \ln(1+x)}{32} - \frac{3(x-1)^3 \ln(x-1)}{32} + c_1 x^3 + \left(-3c_1 - \frac{3}{16}\right) x^2 + \left(3c_1 + \frac{9}{16}\right) x - c_1 - \frac{5}{8}\right)^2}$$

$$y(x) = -\frac{2^{\frac{1}{3}} \left(-1024 \left(\frac{3(x-1)^3 \ln(1+x)}{32} - \frac{3(x-1)^3 \ln(x-1)}{32} + c_1 x^3 + \left(-3c_1 - \frac{3}{16}\right) x^2 + \left(3c_1 + \frac{9}{16}\right) x - c_1 - \frac{5}{8}\right)^2\right)^{\frac{2}{3}} (}{512 \left(\frac{3(x-1)^3 \ln(1+x)}{32} - \frac{3(x-1)^3 \ln(x-1)}{32} + c_1 x^3 + \left(-3c_1 - \frac{3}{16}\right) x^2 + \left(3c_1 + \frac{9}{16}\right) x - c_1 - \frac{5}{8}\right)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{2^{\frac{1}{3}} \left(-1024 \left(\frac{3(x-1)^3 \ln(1+x)}{32} - \frac{3(x-1)^3 \ln(x-1)}{32} + c_1 x^3 + \left(-3c_1 - \frac{3}{16}\right) x^2 + \left(3c_1 + \frac{9}{16}\right) x - c_1 - \frac{5}{8}\right)^2\right)^{\frac{2}{3}}}{128 \left(\frac{3(x-1)^3 \ln(1+x)}{32} - \frac{3(x-1)^3 \ln(x-1)}{32} + c_1 x^3 + \left(-3c_1 - \frac{3}{16}\right) x^2 + \left(3c_1 + \frac{9}{16}\right) x - c_1 - \frac{5}{8}\right)^2} \tag{1}$$

$$y = -\frac{2^{\frac{1}{3}} \left(-1024 \left(\frac{3(x-1)^3 \ln(1+x)}{32} - \frac{3(x-1)^3 \ln(x-1)}{32} + c_1 x^3 + \left(-3c_1 - \frac{3}{16}\right) x^2 + \left(3c_1 + \frac{9}{16}\right) x - c_1 - \frac{5}{8}\right)^2\right)^{\frac{2}{3}} (i -}{512 \left(\frac{3(x-1)^3 \ln(1+x)}{32} - \frac{3(x-1)^3 \ln(x-1)}{32} + c_1 x^3 + \left(-3c_1 - \frac{3}{16}\right) x^2 + \left(3c_1 + \frac{9}{16}\right) x - c_1 - \frac{5}{8}\right)^2} \tag{2}$$

$$y = -\frac{2^{\frac{1}{3}} \left(-1024 \left(\frac{3(x-1)^3 \ln(1+x)}{32} - \frac{3(x-1)^3 \ln(x-1)}{32} + c_1 x^3 + \left(-3c_1 - \frac{3}{16}\right) x^2 + \left(3c_1 + \frac{9}{16}\right) x - c_1 - \frac{5}{8}\right)^2\right)^{\frac{2}{3}} (\sqrt{3}}{512 \left(\frac{3(x-1)^3 \ln(1+x)}{32} - \frac{3(x-1)^3 \ln(x-1)}{32} + c_1 x^3 + \left(-3c_1 - \frac{3}{16}\right) x^2 + \left(3c_1 + \frac{9}{16}\right) x - c_1 - \frac{5}{8}\right)^2} \tag{3}$$

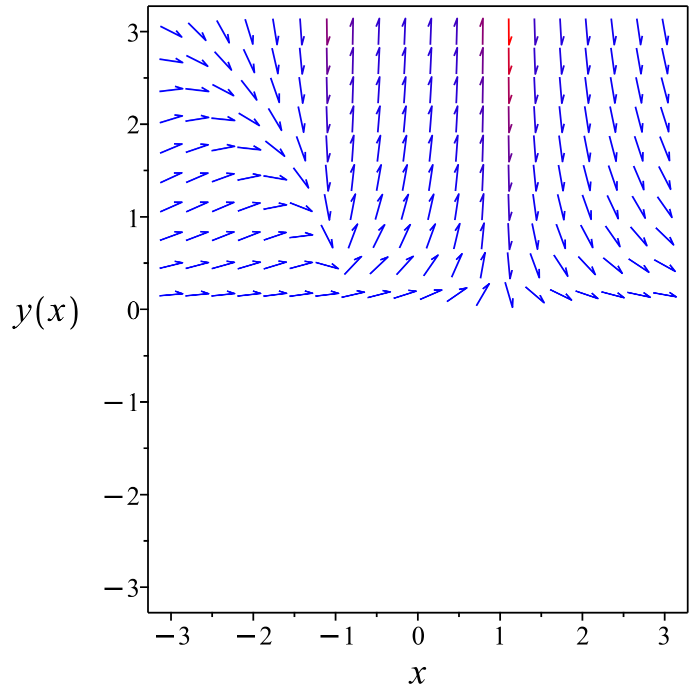


Figure 79: Slope field plot

Verification of solutions

$$y = \frac{2^{\frac{1}{3}}\left(-1024\left(\frac{3(x-1)^3\ln(1+x)}{32} - \frac{3(x-1)^3\ln(x-1)}{32} + c_1x^3 + \left(-3c_1 - \frac{3}{16}\right)x^2 + \left(3c_1 + \frac{9}{16}\right)x - c_1 - \frac{5}{8}\right)^2\right)^{\frac{2}{3}}}{128\left(\frac{3(x-1)^3\ln(1+x)}{32} - \frac{3(x-1)^3\ln(x-1)}{32} + c_1x^3 + \left(-3c_1 - \frac{3}{16}\right)x^2 + \left(3c_1 + \frac{9}{16}\right)x - c_1 - \frac{5}{8}\right)^2}$$

Verified OK.

$$y = -\frac{2^{\frac{1}{3}}\left(-1024\left(\frac{3(x-1)^3\ln(1+x)}{32} - \frac{3(x-1)^3\ln(x-1)}{32} + c_1x^3 + \left(-3c_1 - \frac{3}{16}\right)x^2 + \left(3c_1 + \frac{9}{16}\right)x - c_1 - \frac{5}{8}\right)^2\right)^{\frac{2}{3}}(i -}{512\left(\frac{3(x-1)^3\ln(1+x)}{32} - \frac{3(x-1)^3\ln(x-1)}{32} + c_1x^3 + \left(-3c_1 - \frac{3}{16}\right)x^2 + \left(3c_1 + \frac{9}{16}\right)x - c_1 - \frac{5}{8}\right)^2}$$

Verified OK.

$$y = -\frac{2^{\frac{1}{3}}\left(-1024\left(\frac{3(x-1)^3\ln(1+x)}{32} - \frac{3(x-1)^3\ln(x-1)}{32} + c_1x^3 + \left(-3c_1 - \frac{3}{16}\right)x^2 + \left(3c_1 + \frac{9}{16}\right)x - c_1 - \frac{5}{8}\right)^2\right)^{\frac{2}{3}}(\sqrt{3}}{512\left(\frac{3(x-1)^3\ln(1+x)}{32} - \frac{3(x-1)^3\ln(x-1)}{32} + c_1x^3 + \left(-3c_1 - \frac{3}{16}\right)x^2 + \left(3c_1 + \frac{9}{16}\right)x - c_1 - \frac{5}{8}\right)^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 64

```
dsolve((1-x^2)*diff(y(x),x)-2*(1+x)*y(x)=y(x)^(5/2),y(x), singsol=all)
```

$$-\frac{-1 + \left(-\frac{3(-1+x)^3\ln(-1+x)}{32} + \frac{3(-1+x)^3\ln(1+x)}{32} + c_1x^3 + \left(-3c_1 - \frac{3}{16}\right)x^2 + \left(3c_1 + \frac{9}{16}\right)x - c_1 - \frac{5}{8}\right)y(x)^{\frac{3}{2}}}{y(x)^{\frac{3}{2}}} = 0$$

✓ Solution by Mathematica

Time used: 1.042 (sec). Leaf size: 76

```
DSolve[(1-x^2)*y'[x]-2*(1+x)*y[x]==y[x]^(5/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{8\sqrt[3]{2}}{(32c_1x^3 - 6x^2 - 96c_1x^2 + 18x - 3(x-1)^3\log(x-1) + 3(x-1)^3\log(x+1) + 96c_1x - 20 - 32c_1)^{2/3}}$$
$$y(x) \to 0$$

7.2 problem Ex 2

Internal problem ID [11149]
Internal file name [`OUTPUT/10134_Wednesday_November_23_2022_11_51_33_AM_58770289/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 14. Equations reducible to linear equations (Bernoulli). Page 21
Problem number: Ex 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{yy' + y^2x = x}$$

7.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x(y^2-1)}{y} \end{aligned}$$

Where $f(x) = -x$ and $g(y) = \frac{y^2-1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2-1}{y}}\,dy = -x\,dx$$

$$\int \frac{1}{\frac{y^2-1}{y}}\, dy = \int -x\, dx$$

$$\frac{\ln (y-1)}{2}+\frac{\ln (y+1)}{2} = -\frac{x^2}{2}+c_1$$

The above can be written as

$$\left(\frac{1}{2}\right)(\ln (y-1)+\ln (y+1)) = -\frac{x^2}{2}+2c_1$$

$$\begin{aligned}\ln (y-1)+\ln (y+1) &= (2)\left(-\frac{x^2}{2}+2c_1\right)\\ &= -x^2+4c_1\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln (y-1)+\ln (y+1)} = \mathrm{e}^{-x^2+2c_1}$$

Which simplifies to

$$\begin{aligned}y^2-1 &= 2c_1 \mathrm{e}^{-x^2}\\ &= c_2 \mathrm{e}^{-x^2}\end{aligned}$$

The solution is

$$y^2-1 = c_2 \mathrm{e}^{-x^2}$$

Summary

The solution(s) found are the following

$$y^2-1 = c_2 \mathrm{e}^{-x^2} \tag{1}$$

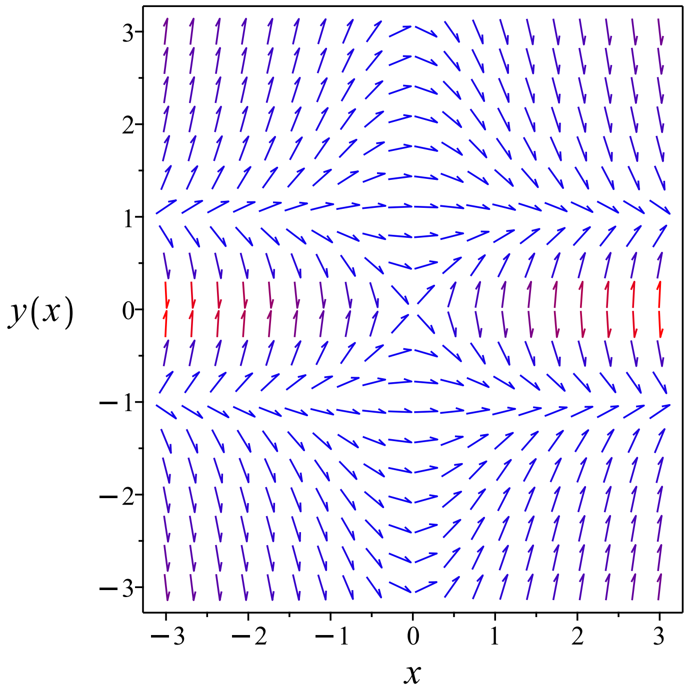


Figure 80: Slope field plot

Verification of solutions

$$y^2 - 1 = c_2 e^{-x^2}$$

Verified OK.

7.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x(y^2 - 1)}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 48: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{1}{x} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x(y^2-1)}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2-1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2-1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R-1)}{2} + \frac{\ln(R+1)}{2} + c_1 \tag{4}$$

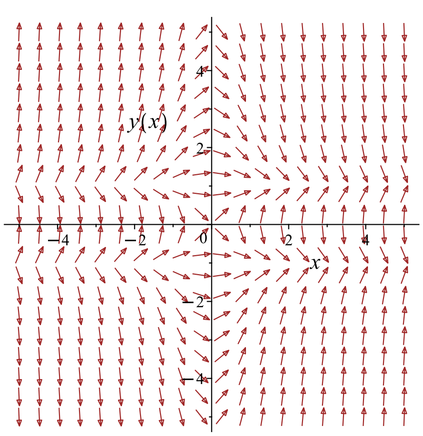
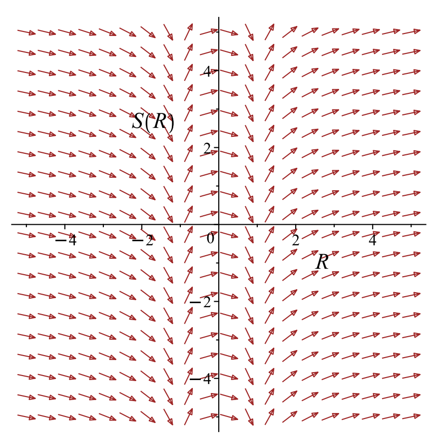
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} + c_1$$

Which simplifies to

$$-\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x(y^2-1)}{y}$ 	$R = y$ $S = -\frac{x^2}{2}$	$\frac{dS}{dR} = \frac{R}{R^2-1}$ 

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} + c_1 \tag{1}$$

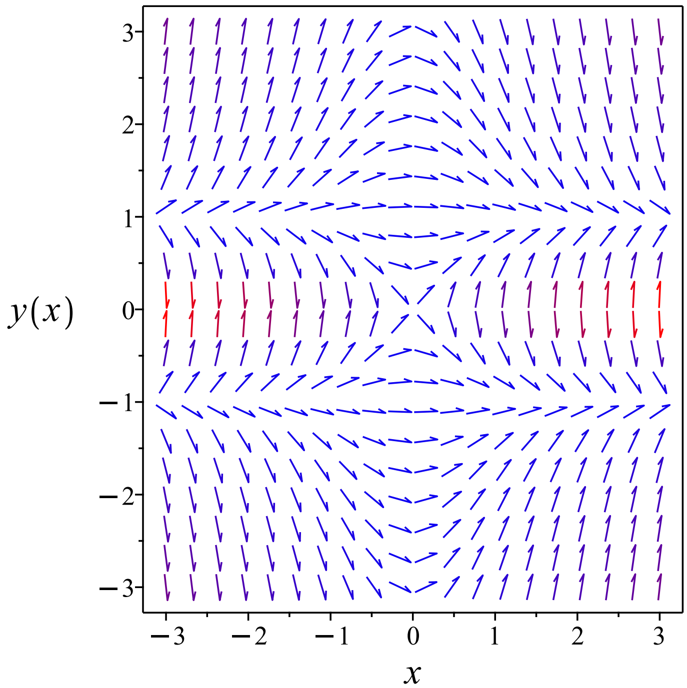


Figure 81: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} + c_1$$

Verified OK.

7.2.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x(y^2-1)}{y} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -xy + x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -x \\ f_1(x) &= x \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -x\,y^2 + x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -w(x)\,x + x \\ w' &= -2wx + 2x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2x \\ q(x) &= 2x \end{aligned}$$

Hence the ode is

$$w'(x) + 2w(x)\,x = 2x$$

The integrating factor μ is

$$\mu = e^{\int 2x dx}$$
$$= e^{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)(2x)$$
$$\frac{d}{dx}\left(e^{x^2} w\right) = \left(e^{x^2}\right)(2x)$$
$$d\left(e^{x^2} w\right) = \left(2x\, e^{x^2}\right) dx$$

Integrating gives

$$e^{x^2} w = \int 2x\, e^{x^2}\, dx$$
$$e^{x^2} w = e^{x^2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{x^2}$ results in

$$w(x) = e^{-x^2} e^{x^2} + c_1 e^{-x^2}$$

which simplifies to

$$w(x) = 1 + c_1 e^{-x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = 1 + c_1 e^{-x^2}$$

Solving for y gives

$$y(x) = \sqrt{1 + c_1 e^{-x^2}}$$
$$y(x) = -\sqrt{1 + c_1 e^{-x^2}}$$

Summary

The solution(s) found are the following

$$y = \sqrt{1 + c_1 e^{-x^2}} \tag{1}$$
$$y = -\sqrt{1 + c_1 e^{-x^2}} \tag{2}$$

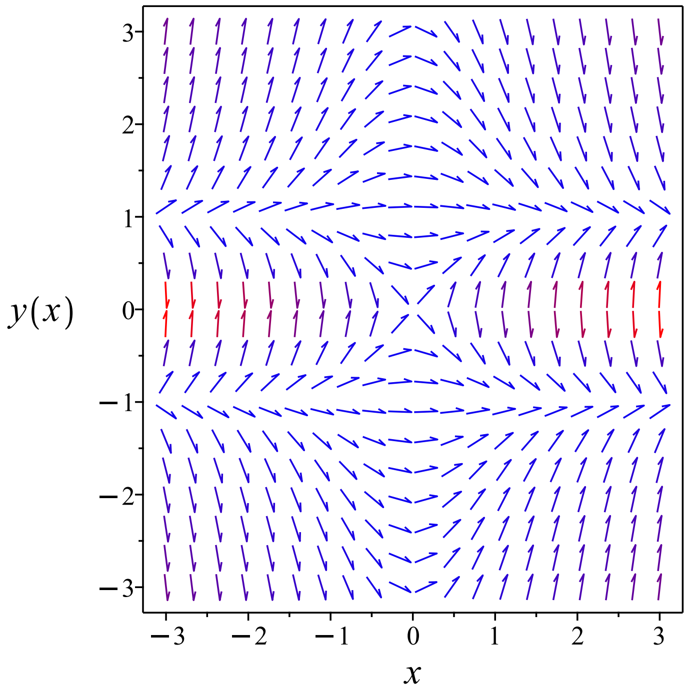


Figure 82: Slope field plot

Verification of solutions

$$y = \sqrt{1 + c_1 e^{-x^2}}$$

Verified OK.

$$y = -\sqrt{1 + c_1 e^{-x^2}}$$

Verified OK.

7.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{y}{y^2-1}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+\left(-\frac{y}{y^2-1}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\\ N(x,y)&=-\frac{y}{y^2-1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-x)\\ &=0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{y}{y^2-1}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y}{y^2-1}$. Therefore equation (4) becomes

$$-\frac{y}{y^2-1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{y}{y^2-1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{y}{y^2-1}\right) \mathrm{d}y$$
$$f(y) = -\frac{\ln(y-1)}{2} - \frac{\ln(y+1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{\ln(y-1)}{2} - \frac{\ln(y+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{\ln(y-1)}{2} - \frac{\ln(y+1)}{2}$$

Summary
The solution(s) found are the following

$$-\frac{x^2}{2} - \frac{\ln(y-1)}{2} - \frac{\ln(y+1)}{2} = c_1 \tag{1}$$

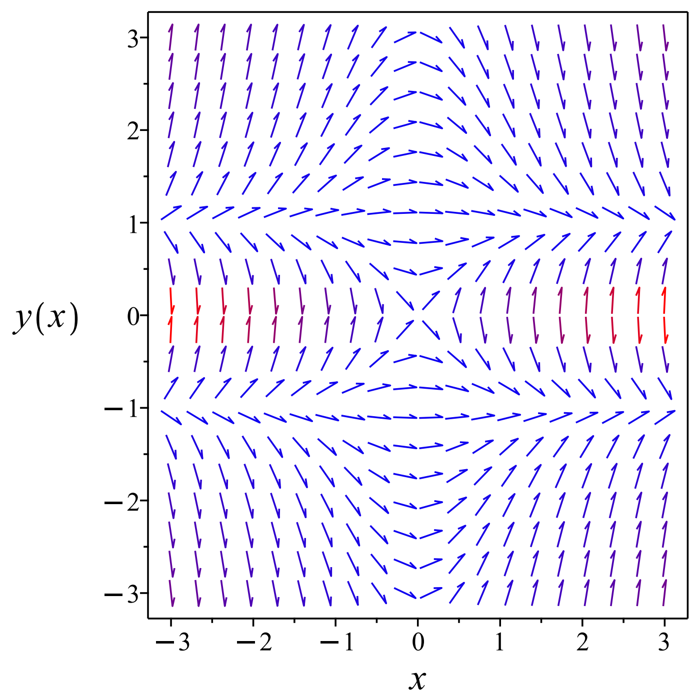


Figure 83: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} - \frac{\ln(y-1)}{2} - \frac{\ln(y+1)}{2} = c_1$$

Verified OK.

7.2.5 Maple step by step solution

Let's solve

$yy' + y^2x = x$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$\frac{y'y}{(y-1)(y+1)} = -x$

- Integrate both sides with respect to x

$\int \frac{y'y}{(y-1)(y+1)}dx = \int -xdx + c_1$

- Evaluate integral

$\frac{\ln((y-1)(y+1))}{2} = -\frac{x^2}{2} + c_1$

- Solve for y

$\left\{y = \sqrt{1 + \mathrm{e}^{-x^2+2c_1}}, y = -\sqrt{1 + \mathrm{e}^{-x^2+2c_1}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(y(x)*diff(y(x),x)+x*y(x)^2=x,y(x), singsol=all)
```

$$y(x) = \sqrt{\mathrm{e}^{-x^2}c_1 + 1}$$
$$y(x) = -\sqrt{\mathrm{e}^{-x^2}c_1 + 1}$$

✓ Solution by Mathematica

Time used: 2.1 (sec). Leaf size: 57

```
DSolve[y[x]*y'[x]+x*y[x]^2==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{1 + e^{-x^2+2c_1}}$$
$$y(x) \to \sqrt{1 + e^{-x^2+2c_1}}$$
$$y(x) \to -1$$
$$y(x) \to 1$$

7.3 problem Ex 3

Internal problem ID [11150]
Internal file name [`OUTPUT/10135_Wednesday_November_23_2022_11_51_34_AM_79946896/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 14. Equations reducible to linear equations (Bernoulli). Page 21
Problem number: Ex 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' \sin(y) + \sin(x) \cos(y) = \sin(x)}$$

7.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\sin(x)\,(-\csc(y) + \cot(y)) \end{aligned}$$

Where $f(x) = -\sin(x)$ and $g(y) = -\csc(y) + \cot(y)$. Integrating both sides gives

$$\frac{1}{-\csc(y) + \cot(y)}\, dy = -\sin(x)\, dx$$

$$\int \frac{1}{-\csc(y) + \cot(y)}\, dy = \int -\sin(x)\, dx$$

$$-\ln(-1 + \cos(y)) = \cos(x) + c_1$$

Raising both side to exponential gives

$$\frac{1}{-1+\cos(y)} = e^{\cos(x)+c_1}$$

Which simplifies to

$$\frac{1}{-1+\cos(y)} = c_2 e^{\cos(x)}$$

Summary

The solution(s) found are the following

$$y = \arccos\left(\frac{\left(c_2 e^{\cos(x)+c_1}+1\right) e^{-\cos(x)-c_1}}{c_2}\right) \tag{1}$$

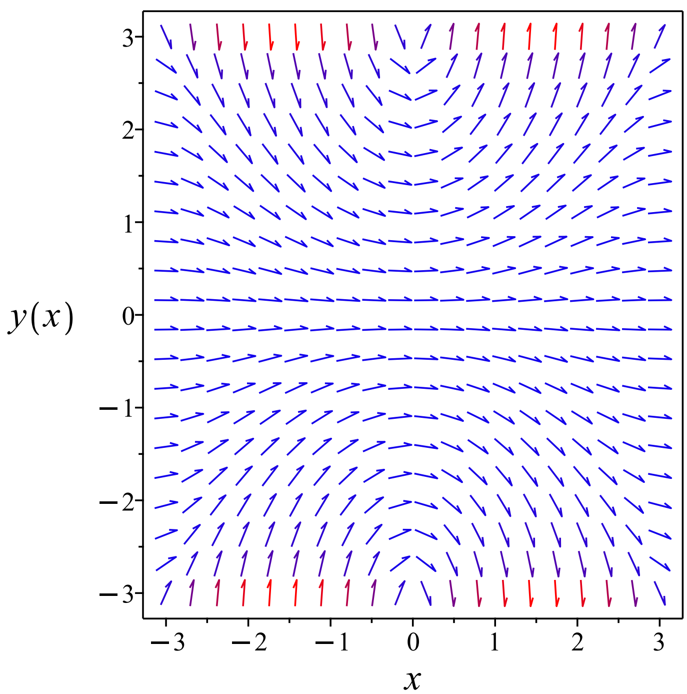


Figure 84: Slope field plot

Verification of solutions

$$y = \arccos\left(\frac{\left(c_2 e^{\cos(x)+c_1}+1\right) e^{-\cos(x)-c_1}}{c_2}\right)$$

Verified OK.

7.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= -\frac{\sin(x)(-1+\cos(y))}{\sin(y)} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 51: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = -\frac{1}{\sin(x)}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{\sin(x)}} dx \end{aligned}$$

Which results in

$$S = \cos(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sin(x)(-1+\cos(y))}{\sin(y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\sin(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\sin(y)}{-1+\cos(y)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{\sin(R)}{-1+\cos(R)}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(-1+\cos(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\cos(x) = -\ln(-1+\cos(y)) + c_1
$$

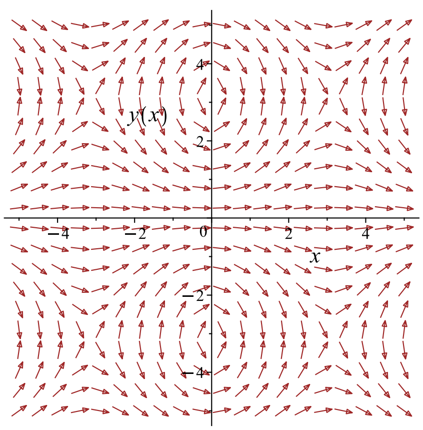
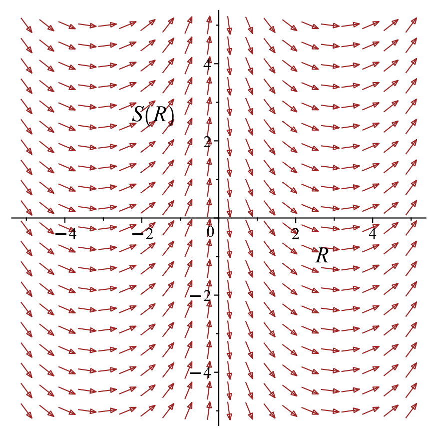
Which simplifies to

$$
\cos(x) = -\ln(-1+\cos(y)) + c_1
$$

Which gives

$$
y = \arccos\left(e^{c_1-\cos(x)} + 1\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sin(x)(-1+\cos(y))}{\sin(y)}$  	$R = y$ $S = \cos(x)$	$\frac{dS}{dR} = \frac{\sin(R)}{-1+\cos(R)}$ 

Summary

The solution(s) found are the following

$$y = \arccos\left(e^{c_1 - \cos(x)} + 1\right) \tag{1}$$

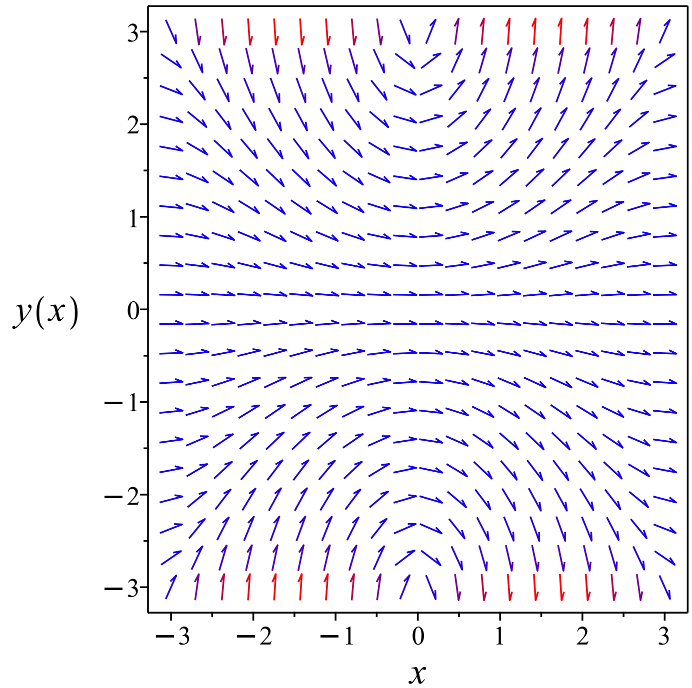


Figure 85: Slope field plot

Verification of solutions

$$y = \arccos\left(\mathrm{e}^{c_1 - \cos(x)} + 1\right)$$

Verified OK.

7.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1A}$$

Therefore

$$\left(-\frac{\sin(y)}{-1+\cos(y)}\right) dy = (\sin(x))\, dx$$
$$(-\sin(x))\, dx + \left(-\frac{\sin(y)}{-1+\cos(y)}\right) dy = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\sin(x)$$
$$N(x, y) = -\frac{\sin(y)}{-1+\cos(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-\sin(x))$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{\sin(y)}{-1+\cos(y)}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\sin(x) \, \mathrm{d}x$$
$$\phi = \cos(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\sin(y)}{-1+\cos(y)}$. Therefore equation (4) becomes

$$-\frac{\sin(y)}{-1+\cos(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{\sin(y)}{-1+\cos(y)}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{\sin(y)}{-1+\cos(y)}\right) \mathrm{d}y$$
$$f(y) = \ln(-1+\cos(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(x) + \ln(-1 + \cos(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(x) + \ln(-1 + \cos(y))$$

Summary

The solution(s) found are the following

$$\cos(x) + \ln(-1 + \cos(y)) = c_1 \tag{1}$$

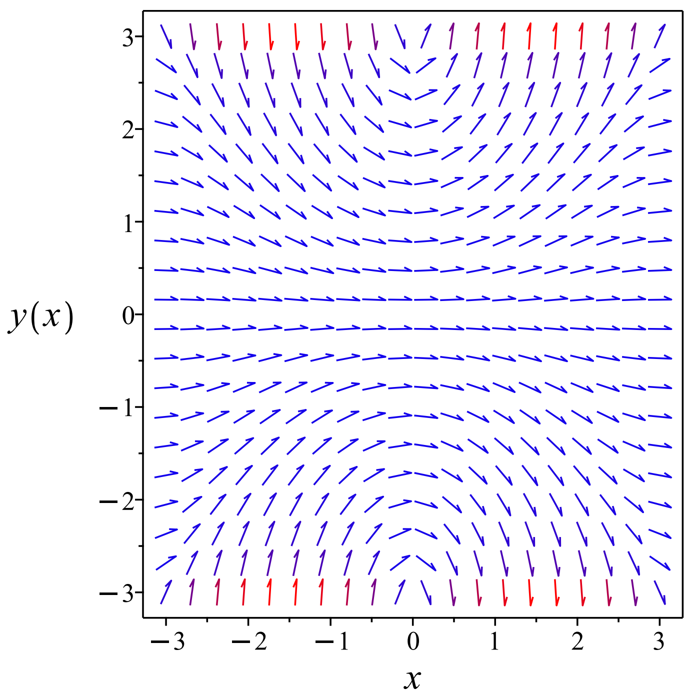


Figure 86: Slope field plot

Verification of solutions

$$\cos(x) + \ln(-1 + \cos(y)) = c_1$$

Verified OK.

7.3.4 Maple step by step solution

Let's solve

$y' \sin(y) + \sin(x)\cos(y) = \sin(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' \sin(y)}{-1+\cos(y)} = -\sin(x)$

- Integrate both sides with respect to x

  $\int \frac{y' \sin(y)}{-1+\cos(y)} dx = \int -\sin(x)\, dx + c_1$

- Evaluate integral

  $-\ln(-1+\cos(y)) = \cos(x) + c_1$

- Solve for y

  $y = \arccos\left(e^{-\cos(x)-c_1} + 1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 14

```
dsolve(sin(y(x))*diff(y(x),x)+sin(x)*cos(y(x))=sin(x),y(x), singsol=all)
```

$$y(x) = \arccos\left(e^{-\cos(x)} c_1 + 1\right)$$

✓ Solution by Mathematica

Time used: 1.53 (sec). Leaf size: 81

```
DSolve[Sin[y[x]]*y'[x]+Sin[x]*Cos[y[x]]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

$$\text{Solve}\left[2\cos(x)\tan\left(\frac{y(x)}{2}\right)e^{\operatorname{arctanh}(\cos(y(x)))} - \sqrt{\sin^2(y(x))}\csc\left(\frac{y(x)}{2}\right)\sec\left(\frac{y(x)}{2}\right)\left(\log\left(\sec^2\left(\frac{y(x)}{2}\right)\right) - 2\log\left(\tan\left(\frac{y(x)}{2}\right)\right)\right) = c_1, y(x)\right]$$

$$y(x) \to 0$$

7.4 problem Ex 4

Internal problem ID [11151]
Internal file name [`OUTPUT/10136_Wednesday_November_23_2022_11_51_35_AM_46179607/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 14. Equations reducible to linear equations (Bernoulli). Page 21
Problem number: Ex 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{4y'x + 3y + e^x x^4 y^5 = 0}$$

7.4.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(e^x x^4 y^4 + 3)}{4x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 54: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^5 x^3
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^5 x^3} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{4x^3 y^4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(e^x x^4 y^4 + 3)}{4x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3}{4x^4 y^4} \\ S_y &= \frac{1}{y^5 x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{e^x}{4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{e^R}{4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{e^{R}}{4} + c_1 \tag{4}$$

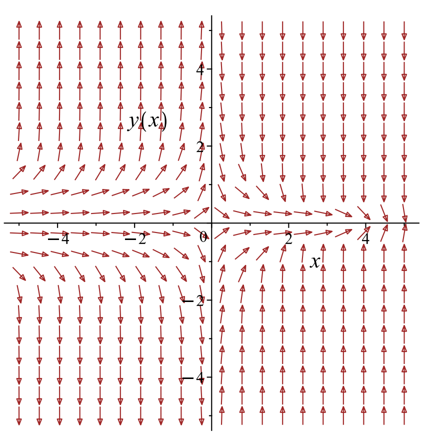
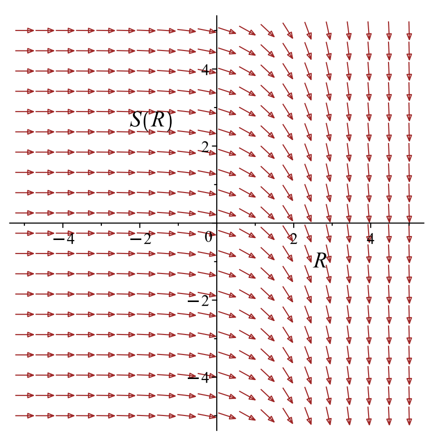
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{4x^3y^4} = -\frac{e^{x}}{4} + c_1$$

Which simplifies to

$$-\frac{1}{4x^3y^4} = -\frac{e^{x}}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(e^{x}x^4y^4+3)}{4x}$ 	$R = x$ $S = -\frac{1}{4x^3y^4}$	$\frac{dS}{dR} = -\frac{e^{R}}{4}$ 

Summary
The solution(s) found are the following

$$-\frac{1}{4x^3y^4} = -\frac{e^{x}}{4} + c_1 \tag{1}$$

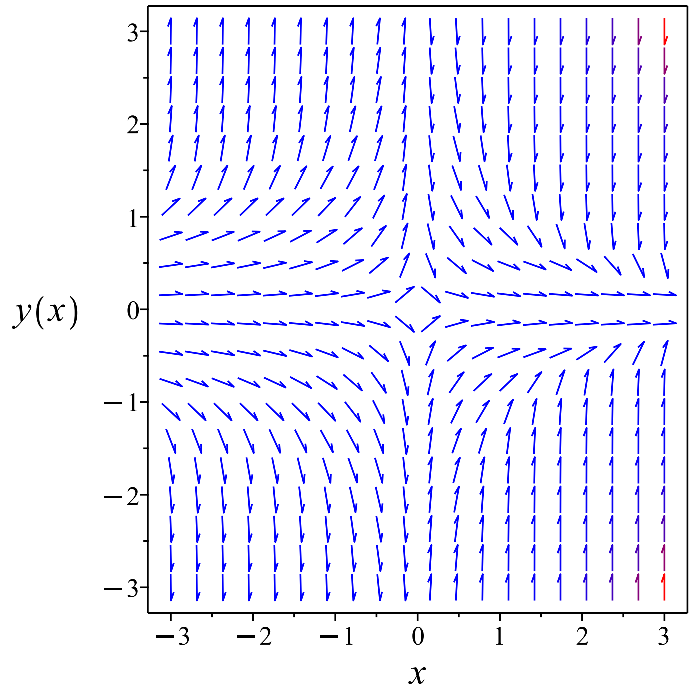


Figure 87: Slope field plot

Verification of solutions

$$-\frac{1}{4x^3y^4} = -\frac{e^x}{4} + c_1$$

Verified OK.

7.4.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(e^x x^4 y^4 + 3)}{4x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{3}{4x}y - \frac{x^3 e^x}{4}y^5 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{3}{4x} \\
f_1(x) &= -\frac{x^3 e^x}{4} \\
n &= 5
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^5$ gives

$$
y' \frac{1}{y^5} = -\frac{3}{4x\, y^4} - \frac{x^3 e^x}{4} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^4}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{4}{y^5} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{4} &= -\frac{3w(x)}{4x} - \frac{x^3 e^x}{4} \\
w' &= \frac{3w}{x} + x^3 e^x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x) w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= x^3 e^x
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{3w(x)}{x} = x^3 e^x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{3}{x}dx} \\ &= \frac{1}{x^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(x^3 e^x\right) \\ \frac{d}{dx}\left(\frac{w}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(x^3 e^x\right) \\ d\left(\frac{w}{x^3}\right) &= e^x \, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^3} &= \int e^x \, dx \\ \frac{w}{x^3} &= e^x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$w(x) = x^3 e^x + c_1 x^3$$

which simplifies to

$$w(x) = x^3(e^x + c_1)$$

Replacing w in the above by $\frac{1}{y^4}$ using equation (5) gives the final solution.

$$\frac{1}{y^4} = x^3(e^x + c_1)$$

Solving for y gives

$$y(x) = \frac{1}{\sqrt{\sqrt{(e^x + c_1)\, x}\, x}}$$

$$y(x) = -\frac{1}{\sqrt{\sqrt{(e^x + c_1)\, x}\, x}}$$

$$y(x) = -\frac{1}{\sqrt{-\sqrt{(e^x + c_1)\, x}\, x}}$$

$$y(x) = \frac{1}{\sqrt{-\sqrt{(e^x + c_1)\, x}\, x}}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{\sqrt{(e^x + c_1)\, x}\, x}} \tag{1}$$

$$y = -\frac{1}{\sqrt{\sqrt{(e^x + c_1)\, x}\, x}} \tag{2}$$

$$y = -\frac{1}{\sqrt{-\sqrt{(e^x + c_1)\, x}\, x}} \tag{3}$$

$$y = \frac{1}{\sqrt{-\sqrt{(e^x + c_1)\, x}\, x}} \tag{4}$$

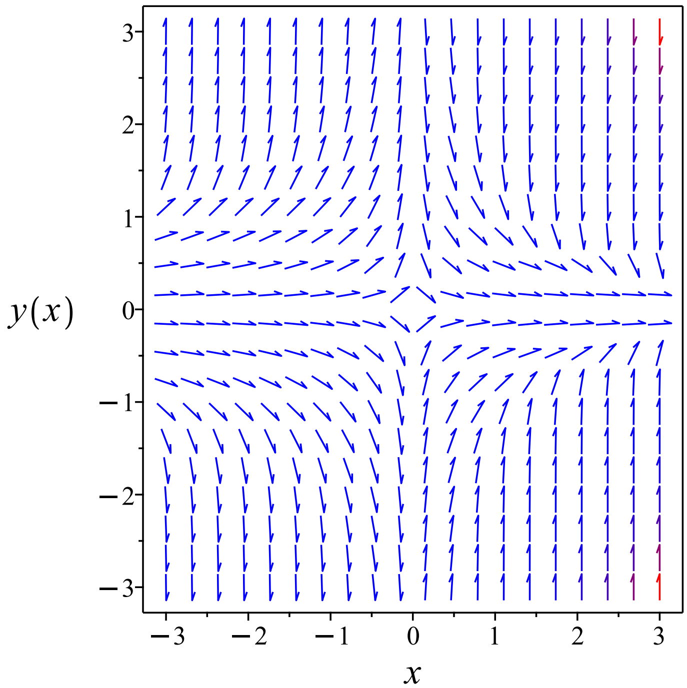


Figure 88: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{\sqrt{(e^x + c_1)\, x}\, x}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{\sqrt{(e^x + c_1)\, x}\, x}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{-\sqrt{(e^x + c_1)\, x}\, x}}$$

Verified OK.

$$y = \frac{1}{\sqrt{-\sqrt{(e^x + c_1)\, x}\, x}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 67

```
dsolve(4*x*diff(y(x),x)+3*y(x)+exp(x)*x^4*y(x)^5=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{\sqrt{(e^x + c_1)\,x}\,x}}$$
$$y(x) = \frac{1}{\sqrt{-\sqrt{(e^x + c_1)\,x}\,x}}$$
$$y(x) = -\frac{1}{\sqrt{\sqrt{(e^x + c_1)\,x}\,x}}$$
$$y(x) = -\frac{1}{\sqrt{-\sqrt{(e^x + c_1)\,x}\,x}}$$

✓ Solution by Mathematica

Time used: 14.931 (sec). Leaf size: 88

```
DSolve[4*x*y'[x]+3*y[x]+Exp[x]*x^4*y[x]^5==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt[4]{x^3 (e^x + c_1)}}$$
$$y(x) \to -\frac{i}{\sqrt[4]{x^3 (e^x + c_1)}}$$
$$y(x) \to \frac{i}{\sqrt[4]{x^3 (e^x + c_1)}}$$
$$y(x) \to \frac{1}{\sqrt[4]{x^3 (e^x + c_1)}}$$
$$y(x) \to 0$$

7.5 problem Ex 5

Internal problem ID [11152]
Internal file name [`OUTPUT/10137_Wednesday_November_23_2022_11_51_36_AM_47359439/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 14. Equations reducible to linear equations (Bernoulli). Page 21
Problem number: Ex 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{y' - \frac{y+1}{1+x} - \sqrt{y+1} = 0}$$

7.5.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{\sqrt{y+1}\,x + \sqrt{y+1} + y + 1}{1+x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2+\frac{\left(\sqrt{y+1}\,x+\sqrt{y+1}+y+1\right)(b_3-a_2)}{1+x}\\
&-\frac{\left(\sqrt{y+1}\,x+\sqrt{y+1}+y+1\right)^2 a_3}{(1+x)^2}\\
&-\left(\frac{\sqrt{y+1}}{1+x}-\frac{\sqrt{y+1}\,x+\sqrt{y+1}+y+1}{(1+x)^2}\right)(xa_2+ya_3+a_1)\\
&-\frac{\left(\frac{x}{2\sqrt{y+1}}+\frac{1}{2\sqrt{y+1}}+1\right)(xb_2+yb_3+b_1)}{1+x}=0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$-\frac{8xya_3+2a_2+4a_3+b_1-2b_3+x^3b_2+x^2b_1+2x^2b_2+2xb_1+2(y+1)^{\frac{3}{2}}a_3-2\sqrt{y+1}\,a_1+2\sqrt{y+1}\,a_2 \cdots}{\cdots}$$

$$=0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-8xya_3-2a_2-4a_3-b_1+2b_3-x^3b_2-x^2b_1-2x^2b_2-2xb_1-2(y+1)^{\frac{3}{2}}a_3\\
&+2\sqrt{y+1}\,a_1-2\sqrt{y+1}\,a_2-2a_3\sqrt{y+1}-2\sqrt{y+1}\,b_1+2b_2\sqrt{y+1}\\
&+2\sqrt{y+1}\,b_3-2a_2y-4y^2a_3+x^2yb_3+2xyb_3-4xa_3-2(y+1)^{\frac{3}{2}}x^2a_3\\
&-4(y+1)^{\frac{3}{2}}xa_3-2\sqrt{y+1}\,xb_1+2\sqrt{y+1}\,xb_2+2\sqrt{y+1}\,xb_3\\
&+2\sqrt{y+1}\,ya_1-2\sqrt{y+1}\,ya_2-2\sqrt{y+1}\,ya_3-4x\,y^2a_3-2x^2a_2\\
&-2x^2a_2y+2x^2b_3-4xa_2y+4xb_3-4xa_2-8ya_3-xb_2+yb_3=0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-4(y+1)\,xya_3 - b_1 - x^3 b_2 - x^2 b_1 - 2x^2 b_2 - 2x b_1 - 2(y+1)^{\frac{3}{2}}\,a_3 \\
&- 2(y+1)\,a_2 - 4(y+1)\,a_3 + 2(y+1)\,b_3 + 2\sqrt{y+1}\,a_1 - 2\sqrt{y+1}\,a_2 \\
&- 2a_3\sqrt{y+1} - 2\sqrt{y+1}\,b_1 + 2b_2\sqrt{y+1} + 2\sqrt{y+1}\,b_3 - x^2 y b_3 \\
&- 2xyb_3 - 2(y+1)^{\frac{3}{2}}\,x^2 a_3 - 4(y+1)^{\frac{3}{2}}\,x a_3 - 2(y+1)\,x^2 a_2 \\
&+ 2(y+1)\,x^2 b_3 - 4(y+1)\,x a_2 - 4(y+1)\,x a_3 + 4(y+1)\,x b_3 \\
&- 4(y+1)\,y a_3 - 2\sqrt{y+1}\,x b_1 + 2\sqrt{y+1}\,x b_2 + 2\sqrt{y+1}\,x b_3 \\
&+ 2\sqrt{y+1}\,y a_1 - 2\sqrt{y+1}\,y a_2 - 2\sqrt{y+1}\,y a_3 - x b_2 - y b_3 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-8xya_3 - 2\sqrt{y+1}\,x^2 a_3 - 4\sqrt{y+1}\,x a_3 - 2\sqrt{y+1}\,x^2 a_3 y - 4\sqrt{y+1}\,x a_3 y \\
&- 2a_2 - 4a_3 - b_1 + 2b_3 - x^3 b_2 - x^2 b_1 - 2x^2 b_2 - 2x b_1 + 2\sqrt{y+1}\,a_1 \\
&- 2\sqrt{y+1}\,a_2 - 4a_3\sqrt{y+1} - 2\sqrt{y+1}\,b_1 + 2b_2\sqrt{y+1} + 2\sqrt{y+1}\,b_3 \\
&- 2a_2 y - 4y^2 a_3 + x^2 y b_3 + 2xyb_3 - 4x a_3 - 2\sqrt{y+1}\,x b_1 + 2\sqrt{y+1}\,x b_2 \\
&+ 2\sqrt{y+1}\,x b_3 + 2\sqrt{y+1}\,y a_1 - 2\sqrt{y+1}\,y a_2 - 4\sqrt{y+1}\,y a_3 - 4x\,y^2 a_3 \\
&- 2x^2 a_2 - 2x^2 a_2 y + 2x^2 b_3 - 4x a_2 y + 4x b_3 - 4x a_2 - 8y a_3 - x b_2 + y b_3 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{y+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{y+1} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2v_3 v_1^2 a_3 v_2 - 2v_1^2 a_2 v_2 - 2v_3 v_1^2 a_3 - 4v_1 v_2^2 a_3 - 4v_3 v_1 a_3 v_2 - v_1^3 b_2 + v_1^2 v_2 b_3 \\
&+ 2v_3 v_2 a_1 - 2v_1^2 a_2 - 4v_1 a_2 v_2 - 2v_3 v_2 a_2 - 8v_1 v_2 a_3 - 4v_3 v_1 a_3 - 4v_2^2 a_3 \\
&- 4v_3 v_2 a_3 - v_1^2 b_1 - 2v_3 v_1 b_1 - 2v_1^2 b_2 + 2v_3 v_1 b_2 + 2v_1^2 b_3 + 2v_1 v_2 b_3 + 2v_3 v_1 b_3 \\
&+ 2v_3 a_1 - 4v_1 a_2 - 2a_2 v_2 - 2v_3 a_2 - 4v_1 a_3 - 8v_2 a_3 - 4a_3 v_3 - 2v_1 b_1 - 2v_3 b_1 \\
&- v_1 b_2 + 2b_2 v_3 + 4v_1 b_3 + v_2 b_3 + 2v_3 b_3 - 2a_2 - 4a_3 - b_1 + 2b_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-v_1^3 b_2 - 2v_3 v_1^2 a_3 v_2 + (-2a_2 + b_3) v_1^2 v_2 - 2v_3 v_1^2 a_3 + (-2a_2 - b_1 - 2b_2 + 2b_3) v_1^2 \\
&- 4v_1 v_2^2 a_3 - 4v_3 v_1 a_3 v_2 + (-4a_2 - 8a_3 + 2b_3) v_1 v_2 \\
&+ (-4a_3 - 2b_1 + 2b_2 + 2b_3) v_1 v_3 + (-4a_2 - 4a_3 - 2b_1 - b_2 + 4b_3) v_1 \\
&- 4v_2^2 a_3 + (2a_1 - 2a_2 - 4a_3) v_2 v_3 + (-2a_2 - 8a_3 + b_3) v_2 \\
&+ (2a_1 - 2a_2 - 4a_3 - 2b_1 + 2b_2 + 2b_3) v_3 - 2a_2 - 4a_3 - b_1 + 2b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-4a_3 &= 0 \\
-2a_3 &= 0 \\
-b_2 &= 0 \\
-2a_2 + b_3 &= 0 \\
2a_1 - 2a_2 - 4a_3 &= 0 \\
-4a_2 - 8a_3 + 2b_3 &= 0 \\
-2a_2 - 8a_3 + b_3 &= 0 \\
-2a_2 - 4a_3 - b_1 + 2b_3 &= 0 \\
-2a_2 - b_1 - 2b_2 + 2b_3 &= 0 \\
-4a_3 - 2b_1 + 2b_2 + 2b_3 &= 0 \\
-4a_2 - 4a_3 - 2b_1 - b_2 + 4b_3 &= 0 \\
2a_1 - 2a_2 - 4a_3 - 2b_1 + 2b_2 + 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= a_2 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 2a_2 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 1 + x$$
$$\eta = 2y + 2$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 2y + 2 - \left(\frac{\sqrt{y+1}\,x + \sqrt{y+1} + y + 1}{1+x}\right)(1 + x) \\
&= -\sqrt{y+1}\,x - \sqrt{y+1} + y + 1 \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{y+1}\,x - \sqrt{y+1} + y + 1} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(-x^2 - 2x + y)}{x^2 + 2x + 1} + \frac{x(x+2)\ln(-x^2 - 2x + y)}{x^2 + 2x + 1} + \frac{2\ln\left(\sqrt{y+1} - 1 - x\right)}{2 + 2x} - \frac{2\ln\left(\sqrt{y+1} + x + 1\right)}{2 + 2x} +$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\sqrt{y+1}\,x + \sqrt{y+1} + y + 1}{1+x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2}{\sqrt{y+1} - 1 - x} \\
S_y &= \frac{1}{\sqrt{y+1}\left(\sqrt{y+1} - 1 - x\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sqrt{y+1}\,(1+x) - y - 1}{\sqrt{y+1}\left(-\sqrt{y+1} + 1 + x\right)(1+x)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R+1) + c_1 \tag{4}$$

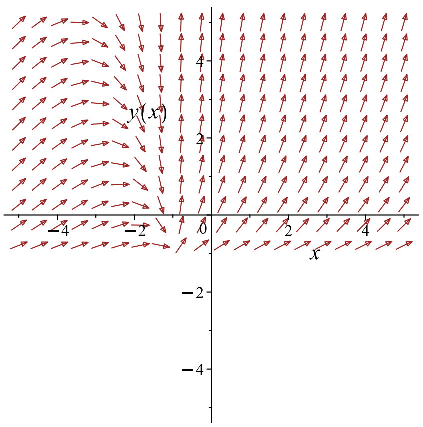
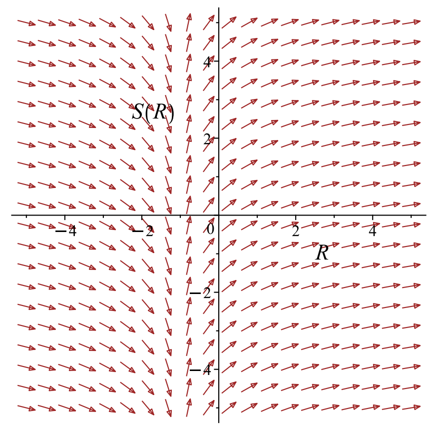
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(-x^2 - 2x + y\right) + \ln\left(\sqrt{y+1} - 1 - x\right) - \ln\left(\sqrt{y+1} + x + 1\right) = \ln(1+x) + c_1$$

Which simplifies to

$$\ln\left(-x^2 - 2x + y\right) + \ln\left(\sqrt{y+1} - 1 - x\right) - \ln\left(\sqrt{y+1} + x + 1\right) = \ln(1+x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{y+1}\,x+\sqrt{y+1}+y+1}{1+x}$ 	$R = x$ $S = \ln(-x^2 - 2x + y) + $ [illegible] $+1+x$	$\frac{dS}{dR} = \frac{1}{R+1}$ 

Summary

The solution(s) found are the following

$$\ln(-x^2 - 2x + y) + \ln\left(\sqrt{y+1} - 1 - x\right) - \ln\left(\sqrt{y+1} + x + 1\right) = \ln(1 + x) + c_1 \tag{1}$$

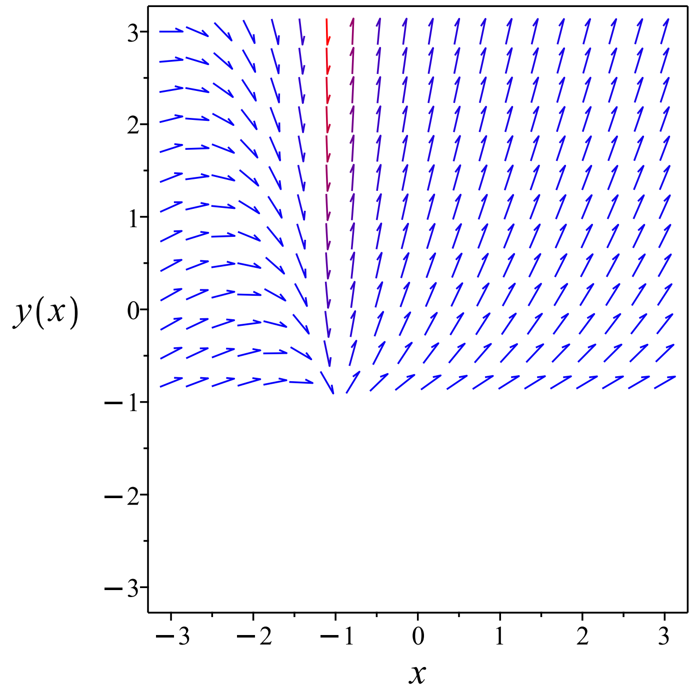


Figure 89: Slope field plot

Verification of solutions

$$\ln\left(-x^2-2x+y\right)+\ln\left(\sqrt{y+1}-1-x\right)-\ln\left(\sqrt{y+1}+x+1\right)=\ln\left(1+x\right)+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = (2*y(x)+2)/(x+1), y(x)`      *** Sublev
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 81

```
dsolve(diff(y(x),x)- (y(x)+1)/(x+1)=sqrt(1+y(x)),y(x), singsol=all)
```

$$\frac{(-c_1y(x)+1+c_1x^2+(2c_1+1)\,x)\,\sqrt{y\,(x)+1}-(1+x)\,(-c_1y(x)-1+c_1x^2+(2c_1-1)\,x)}{(x^2+2x-y\,(x))\left(-\sqrt{y\,(x)+1}+1+x\right)} = 0$$

✓ Solution by Mathematica

Time used: 0.418 (sec). Leaf size: 60

```
DSolve[y'[x]- (y[x]+1)/(x+1)==Sqrt[1+y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{2\sqrt{y(x)+1}\arctan\left(\frac{x+1}{\sqrt{-y(x)-1}}\right)}{\sqrt{-y(x)-1}}+\log\left(y(x)-(x+1)^2+1\right)-\log(x+1)=c_1,y(x)\right]$$

8 Chapter 2, differential equations of the first order and the first degree. Article 15. Page 22

8.1 problem Ex 1

Internal problem ID [11153]
Internal file name [`OUTPUT/10138_Sunday_November_27_2022_04_33_47_PM_9550685/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 15. Page 22
Problem number: Ex 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{x^4y(3y + 2y'x) + x^2(4y + 3y'x) = 0}$$

8.1.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(3x^2y + 4)}{x(2x^2y + 3)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(3x^2y+4)(b_3-a_2)}{x(2x^2y+3)} - \frac{y^2(3x^2y+4)^2 a_3}{x^2(2x^2y+3)^2} - \left(\frac{y(3x^2y+4)}{x^2(2x^2y+3)} - \frac{6y^2}{2x^2y+3} + \frac{4y^2(3x^2y+4)}{(2x^2y+3)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{3x^2y+4}{x(2x^2y+3)} - \frac{3xy}{2x^2y+3} + \frac{2xy(3x^2y+4)}{(2x^2y+3)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{10x^6y^2b_2 - 15x^4y^4a_3 + 6x^5y^2b_1 - 6x^4y^3a_1 + 30x^4yb_2 + 2x^3y^2a_2 + x^3y^2b_3 - 39x^2y^3a_3 + 18x^3yb_1 - 15x^2y^2}{x^2(2x^2y+3)^2} = 0$$

Setting the numerator to zero gives

$$10x^6y^2b_2 - 15x^4y^4a_3 + 6x^5y^2b_1 - 6x^4y^3a_1 + 30x^4yb_2 + 2x^3y^2a_2 + x^3y^2b_3 - 39x^2y^3a_3 + 18x^3yb_1 - 15x^2y^2a_1 + 21b_2x^2 - 28y^2a_3 + 12xb_1 - 12ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-15a_3v_1^4v_2^4 + 10b_2v_1^6v_2^2 - 6a_1v_1^4v_2^3 + 6b_1v_1^5v_2^2 + 2a_2v_1^3v_2^2 - 39a_3v_1^2v_2^3 + 30b_2v_1^4v_2 + b_3v_1^3v_2^2 - 15a_1v_1^2v_2^2 + 18b_1v_1^3v_2 - 28a_3v_2^2 + 21b_2v_1^2 - 12a_1v_2 + 12b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} 10b_2v_1^6v_2^2 + 6b_1v_1^5v_2^2 - 15a_3v_1^4v_2^4 - 6a_1v_1^4v_2^3 + 30b_2v_1^4v_2 + (2a_2 + b_3)\, v_1^3v_2^2 \\ + 18b_1v_1^3v_2 - 39a_3v_1^2v_2^3 - 15a_1v_1^2v_2^2 + 21b_2v_1^2 + 12b_1v_1 - 28a_3v_2^2 - 12a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -15a_1 &= 0 \\ -12a_1 &= 0 \\ -6a_1 &= 0 \\ -39a_3 &= 0 \\ -28a_3 &= 0 \\ -15a_3 &= 0 \\ 6b_1 &= 0 \\ 12b_1 &= 0 \\ 18b_1 &= 0 \\ 10b_2 &= 0 \\ 21b_2 &= 0 \\ 30b_2 &= 0 \\ 2a_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= -2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= -2y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -2y - \left(-\frac{y(3x^2y+4)}{x\,(2x^2y+3)}\right)(x) \\
&= \frac{-x^2y^2 - 2y}{2x^2y+3} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2y^2-2y}{2x^2y+3}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{\ln(x^2y+2)}{2} - \frac{3\ln(y)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y(3x^2y+4)}{x\,(2x^2y+3)}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{xy}{x^2y+2} \\
S_y &= \frac{-2x^2y-3}{y\left(x^2y+2\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\ln(R) + c_1 \tag{4}$$

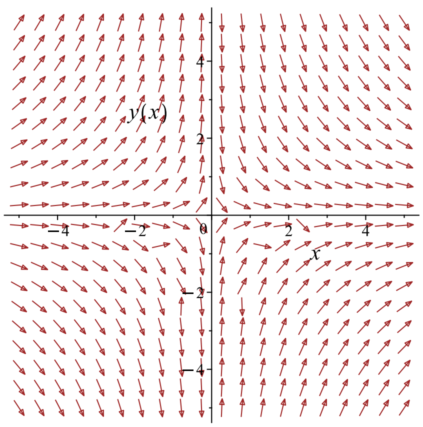
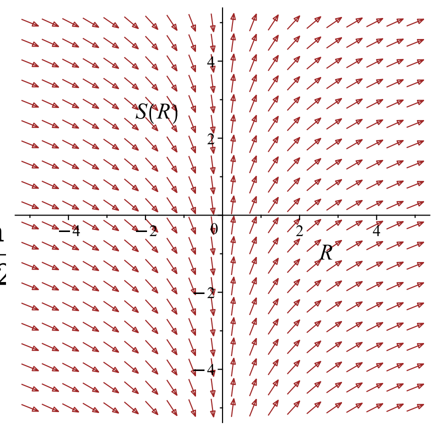
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln\left(x^2y+2\right)}{2} - \frac{3\ln(y)}{2} = 2\ln(x) + c_1$$

Which simplifies to

$$-\frac{\ln\left(x^2y+2\right)}{2} - \frac{3\ln(y)}{2} = 2\ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(3x^2y+4)}{x(2x^2y+3)}$ 	$R = x$ $S = -\frac{\ln(x^2y+2)}{2} - \frac{3\ln}{2}$	$\frac{dS}{dR} = \frac{2}{R}$ 

Summary

The solution(s) found are the following

$$-\frac{\ln(x^2y+2)}{2} - \frac{3\ln(y)}{2} = 2\ln(x) + c_1 \tag{1}$$

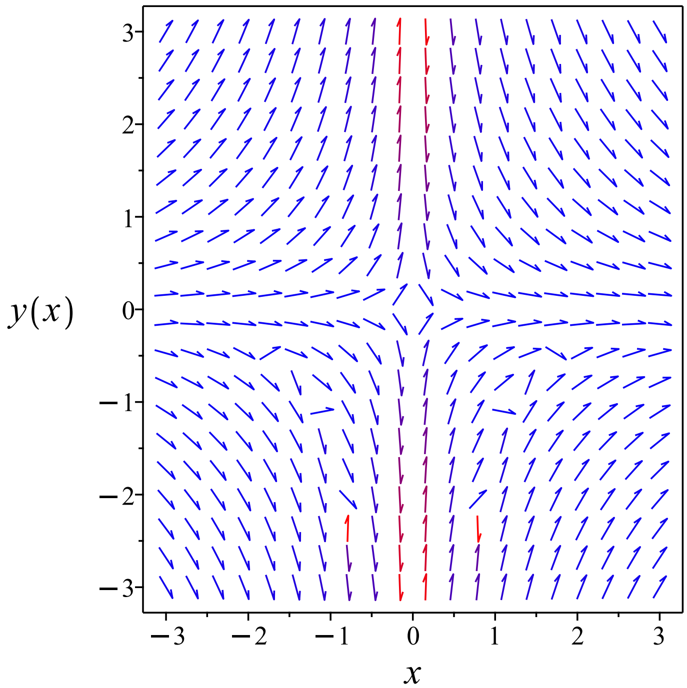


Figure 90: Slope field plot

Verification of solutions

$$-\frac{\ln(x^2y+2)}{2}-\frac{3\ln(y)}{2}=2\ln(x)+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.547 (sec). Leaf size: 39

```
dsolve(x^4*y(x)*(3*y(x)+2*x*diff(y(x),x))+ x^2*(4*y(x)+3*x*diff(y(x),x))=0,y(x), singsol=all
```

$$y(x) = \frac{\operatorname{RootOf}\left(x^2\_Z^8 - 2\_Z^2 c_1 - c_1\right)^6 x^2 - 2c_1}{x^2 c_1}$$

✓ Solution by Mathematica

Time used: 60.464 (sec). Leaf size: 1769

```
DSolve[x^4*y[x]*(3*y[x]+2*x*y'[x])+ x^2*(4*y[x]+3*x*y'[x])==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to -\frac{1}{2x^2} + \frac{\sqrt{\frac{3}{x^4} - \frac{2\ 6^{2/3}e^{-2c_1}}{\sqrt[3]{e^{-6c_1}\left(\sqrt{48e^{6c_1}x^{18}+81e^{8c_1}x^{16}}-9e^{4c_1}x^8\right)}} + \frac{\sqrt[3]{6}\sqrt[3]{e^{-6c_1}\left(\sqrt{48e^{6c_1}x^{18}+81e^{8c_1}x^{16}}-9e^{4c_1}x^8\right)}}{x^6}}}{2\sqrt{3}}$$

$$-\frac{1}{2}\sqrt{\frac{2}{x^4} + \frac{2\ 2^{2/3}e^{-2c_1}}{\sqrt[3]{3}\sqrt[3]{e^{-6c_1}\left(\sqrt{48e^{6c_1}x^{18}+81e^{8c_1}x^{16}}-9e^{4c_1}x^8\right)}} - \frac{\sqrt[3]{2}\sqrt[3]{e^{-6c_1}\left(\sqrt{48e^{6c_1}x^{18}+81e^{8c_1}x^{16}}-9e\right)}}{3^{2/3}x^6}}$$

$$y(x) \to -\frac{1}{2x^2} + \frac{\sqrt{\frac{3}{x^4} - \frac{2\ 6^{2/3}e^{-2c_1}}{\sqrt[3]{e^{-6c_1}\left(\sqrt{48e^{6c_1}x^{18}+81e^{8c_1}x^{16}}-9e^{4c_1}x^8\right)}} + \frac{\sqrt[3]{6}\sqrt[3]{e^{-6c_1}\left(\sqrt{48e^{6c_1}x^{18}+81e^{8c_1}x^{16}}-9e^{4c_1}x^8\right)}}{x^6}}}{2\sqrt{3}}$$

$$+\frac{1}{2}\sqrt{\frac{2}{x^4} + \frac{2\ 2^{2/3}e^{-2c_1}}{\sqrt[3]{3}\sqrt[3]{e^{-6c_1}\left(\sqrt{48e^{6c_1}x^{18}+81e^{8c_1}x^{16}}-9e^{4c_1}x^8\right)}} - \frac{\sqrt[3]{2}\sqrt[3]{e^{-6c_1}\left(\sqrt{48e^{6c_1}x^{18}+81e^{8c_1}x^{16}}-9e\right)}}{3^{2/3}x^6}}$$

$$y(x) \to -\frac{1}{2x^2} - \frac{\sqrt{\frac{3}{x^4} - \frac{2\ 6^{2/3}e^{-2c_1}}{\sqrt[3]{e^{-6c_1}\left(\sqrt{48e^{6c_1}x^{18}+81e^{8c_1}x^{16}}-9e^{4c_1}x^8\right)}} + \frac{\sqrt[3]{6}\sqrt[3]{e^{-6c_1}\left(\sqrt{48e^{6c_1}x^{18}+81e^{8c_1}x^{16}}-9e^{4c_1}x^8\right)}}{x^6}}}{2\sqrt{3}}$$

$$-\frac{1}{2}\sqrt{\frac{2}{x^4} + \frac{2\ 2^{2/3}e^{-2c_1}}{\sqrt[3]{3}\sqrt[3]{e^{-6c_1}\left(\sqrt{48e^{6c_1}x^{18}+81e^{8c_1}x^{16}}-9e^{4c_1}x^8\right)}} - \frac{\sqrt[3]{2}\sqrt[3]{e^{-6c_1}\left(\sqrt{48e^{6c_1}x^{18}+81e^{8c_1}x^{16}}-9e\right)}}{3^{2/3}x^6}}$$

8.2 problem Ex 2

Internal problem ID [11154]
Internal file name [`OUTPUT/10139_Sunday_November_27_2022_04_33_49_PM_62361066/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 15. Page 22
Problem number: Ex 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y^2(3y - 6y'x) - x(y - 2y'x) = 0}$$

8.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{2x} \end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= \frac{1}{2x}\, dx \\
\int \frac{1}{y}\, dy &= \int \frac{1}{2x}\, dx \\
\ln(y) &= \frac{\ln(x)}{2} + c_1 \\
y &= e^{\frac{\ln(x)}{2} + c_1} \\
&= c_1\sqrt{x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x} \tag{1}$$

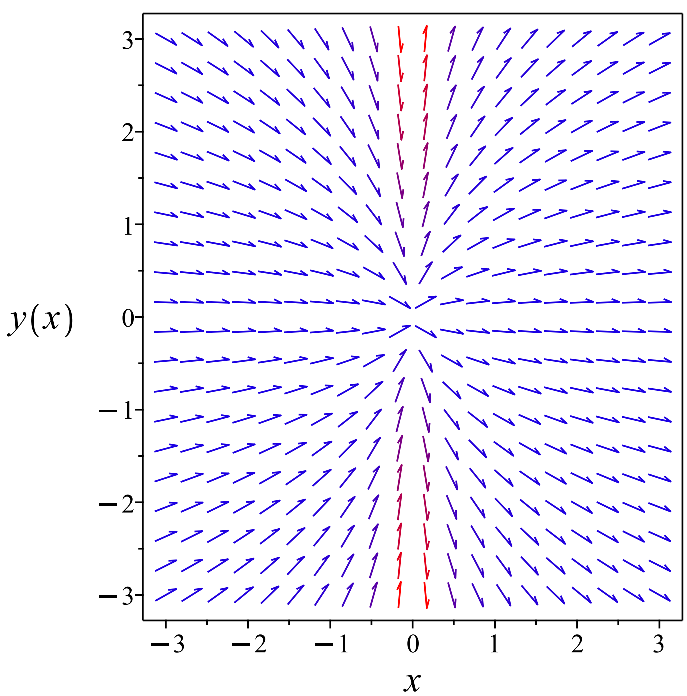


Figure 91: Slope field plot

Verification of solutions

$$y = c_1\sqrt{x}$$

Verified OK.

8.2.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{1}{2x} \\
q(x) &= 0
\end{aligned}$$

Hence the ode is

$$y' - \frac{y}{2x} = 0$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{2x}dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{x}}\right) &= 0
\end{aligned}$$

Integrating gives

$$\frac{y}{\sqrt{x}} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x}}$ results in

$$y = c_1\sqrt{x}$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x} \tag{1}$$

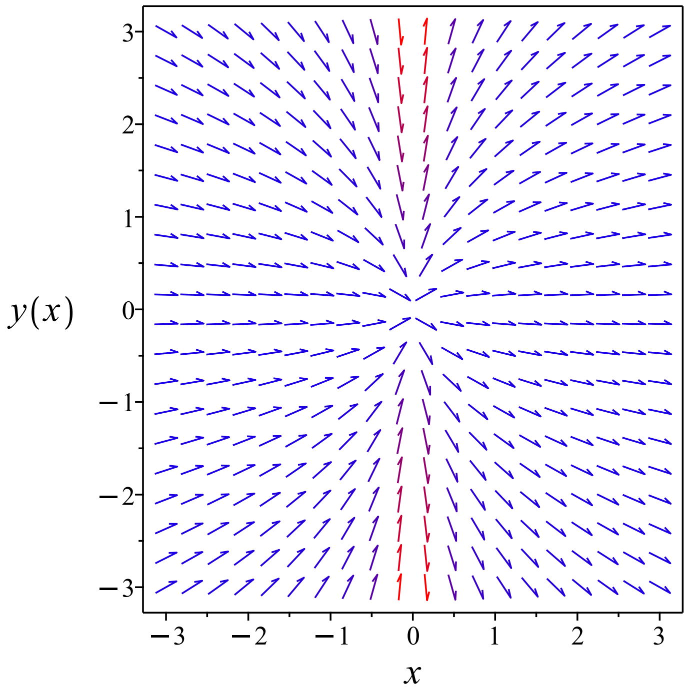


Figure 92: Slope field plot

<u>Verification of solutions</u>

$$y = c_1\sqrt{x}$$

Verified OK.

8.2.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2(3u(x)\,x - 6(u'(x)\,x + u(x))\,x) - x(u(x)\,x - 2(u'(x)\,x + u(x))\,x) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{2x} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{1}{2x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{1}{2x}\, dx \\
\ln(u) &= -\frac{\ln(x)}{2} + c_2 \\
u &= \mathrm{e}^{-\frac{\ln(x)}{2}+c_2} \\
&= \frac{c_2}{\sqrt{x}}
\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= \sqrt{x}\, c_2
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \sqrt{x}\, c_2 \tag{1}$$

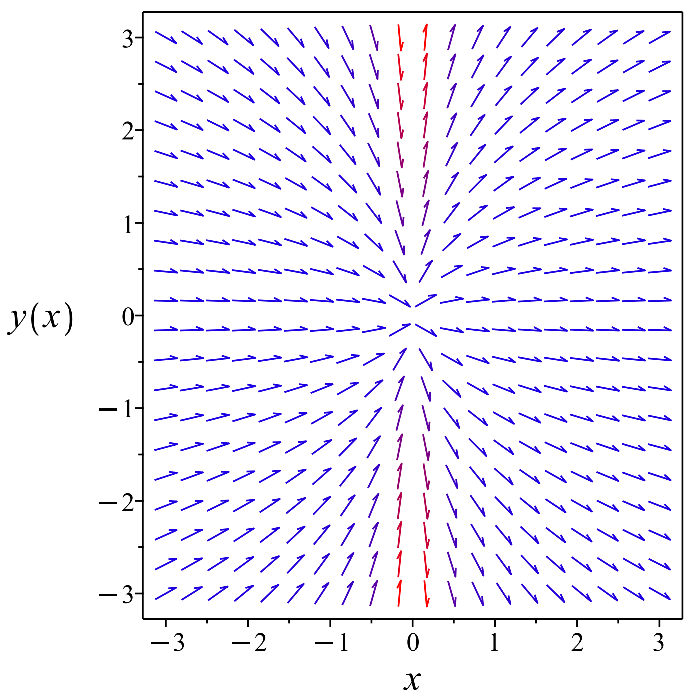


Figure 93: Slope field plot

Verification of solutions

$$y = \sqrt{x}\, c_2$$

Verified OK.

8.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{y}{2x} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 56: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \sqrt{x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{x}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sqrt{x}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{2x^{\frac{3}{2}}} \\ S_y &= \frac{1}{\sqrt{x}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\sqrt{x}} = c_1$$

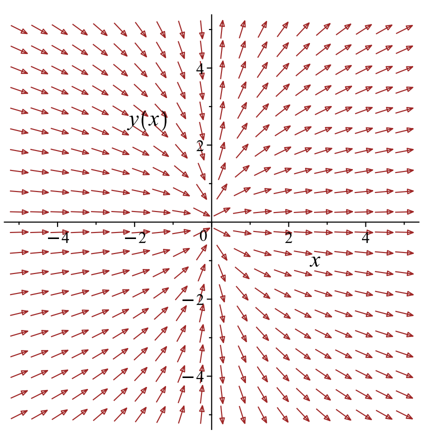
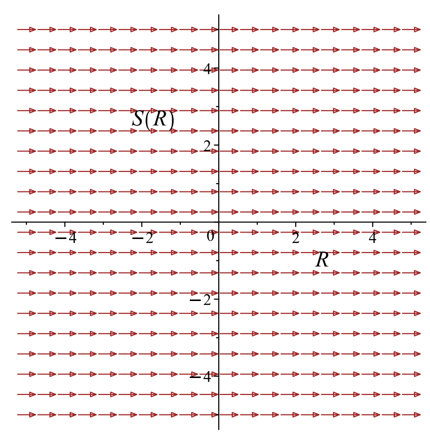
Which simplifies to

$$\frac{y}{\sqrt{x}} = c_1$$

Which gives

$$y = c_1 \sqrt{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{2x}$ 	$R = x$ $S = \frac{y}{\sqrt{x}}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x} \tag{1}$$

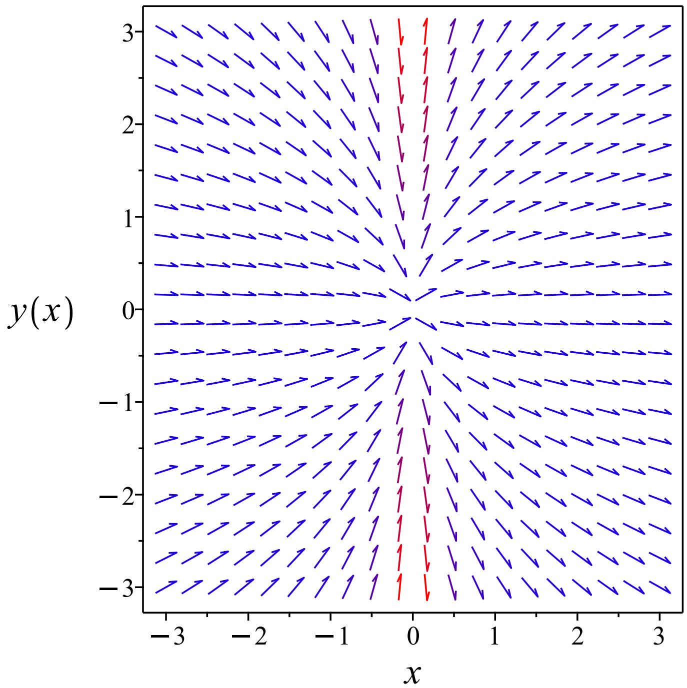


Figure 94: Slope field plot

Verification of solutions

$$y = c_1\sqrt{x}$$

Verified OK.

8.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{2}{y}\right)\mathrm{d}y = \left(\frac{1}{x}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{2}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = \frac{2}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2}{y}$. Therefore equation (4) becomes

$$
\frac{2}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{2}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{2}{y}\right) \mathrm{d}y \\
f(y) &= 2\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + 2\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + 2\ln(y)$$

The solution becomes

$$y = e^{\frac{\ln(x)}{2} + \frac{c_1}{2}}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{\ln(x)}{2} + \frac{c_1}{2}} \tag{1}$$

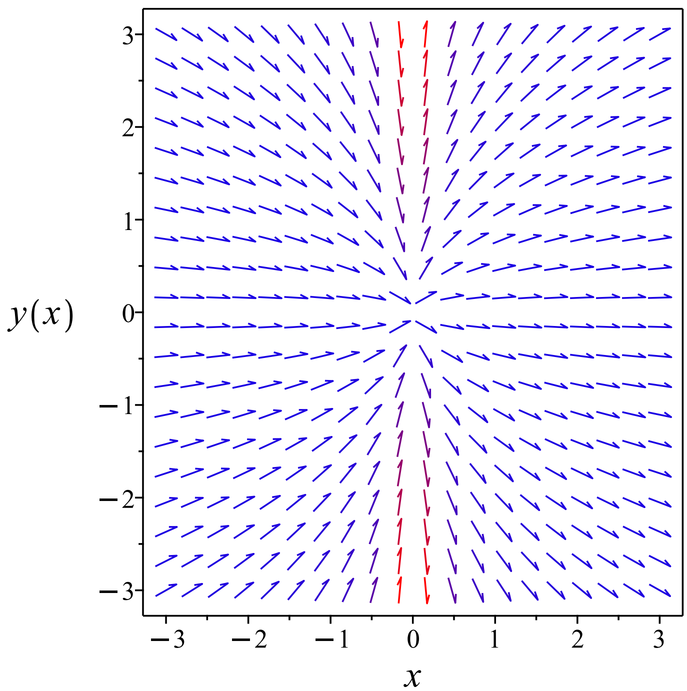


Figure 95: Slope field plot

Verification of solutions

$$y = e^{\frac{\ln(x)}{2} + \frac{c_1}{2}}$$

Verified OK.

8.2.6 Maple step by step solution

Let's solve

$y^2(3y - 6y'x) - x(y - 2y'x) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{2x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{2x}dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{\ln(x)}{2} + c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{e^{-2c_1}x}}{e^{-2c_1}}, y = -\frac{\sqrt{e^{-2c_1}x}}{e^{-2c_1}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(y(x)^2*(3*y(x)-6*x*diff(y(x),x))-x*(y(x)-2*x*diff(y(x),x))=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{3}\sqrt{x}}{3}$$
$$y(x) = \frac{\sqrt{3}\sqrt{x}}{3}$$
$$y(x) = c_1\sqrt{x}$$

✓ Solution by Mathematica

Time used: 6.194 (sec). Leaf size: 74

```
DSolve[y[x]^2*(3*y[x]-6*x*y'[x])- x*(4*y[x]-2*x*y'[x])==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\frac{i\sqrt{x}\sqrt{W\left(-3e^{-3c_1}x^3\right)}}{\sqrt{3}}$$
$$y(x) \to \frac{i\sqrt{x}\sqrt{W\left(-3e^{-3c_1}x^3\right)}}{\sqrt{3}}$$
$$y(x) \to 0$$

8.3 problem Ex 3

Internal problem ID [11155]
Internal file name [`OUTPUT/10140_Sunday_November_27_2022_04_33_51_PM_2025625/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 15. Page 22
Problem number: Ex 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{2yx^3 - y^2 - (2x^4 + yx)\, y' = 0}$$

8.3.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(-2x^3 + y)}{x\,(2x^3 + y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(-2x^3+y)(b_3-a_2)}{x(2x^3+y)} - \frac{y^2(-2x^3+y)^2 a_3}{x^2(2x^3+y)^2} - \left(\frac{6yx}{2x^3+y} + \frac{y(-2x^3+y)}{x^2(2x^3+y)} + \frac{6y(-2x^3+y)x}{(2x^3+y)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{-2x^3+y}{x(2x^3+y)} - \frac{y}{x(2x^3+y)} + \frac{y(-2x^3+y)}{x(2x^3+y)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4x^7b_1 - 4x^6ya_1 - 8x^5yb_2 + 12x^4y^2a_2 - 4x^4y^2b_3 + 8x^3y^3a_3 - 4x^4yb_1 + 12x^3y^2a_1 - 2x^2y^2b_2 + 2y^4a_3 - x}{(2x^3+y)^2 x^2} = 0$$

Setting the numerator to zero gives

$$-4x^7b_1 + 4x^6ya_1 + 8x^5yb_2 - 12x^4y^2a_2 + 4x^4y^2b_3 - 8x^3y^3a_3 + 4x^4yb_1 - 12x^3y^2a_1 + 2x^2y^2b_2 - 2y^4a_3 + x\,y^2b_1 - y^3a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$4a_1v_1^6v_2 - 4b_1v_1^7 - 12a_2v_1^4v_2^2 - 8a_3v_1^3v_2^3 + 8b_2v_1^5v_2 + 4b_3v_1^4v_2^2 - 12a_1v_1^3v_2^2 + 4b_1v_1^4v_2 - 2a_3v_2^4 + 2b_2v_1^2v_2^2 - a_1v_2^3 + b_1v_1v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &-4b_1v_1^7 + 4a_1v_1^6v_2 + 8b_2v_1^5v_2 + (-12a_2 + 4b_3)\,v_1^4v_2^2 + 4b_1v_1^4v_2 \\ &- 8a_3v_1^3v_2^3 - 12a_1v_1^3v_2^2 + 2b_2v_1^2v_2^2 + b_1v_1v_2^2 - 2a_3v_2^4 - a_1v_2^3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -12a_1 &= 0 \\ -a_1 &= 0 \\ 4a_1 &= 0 \\ -8a_3 &= 0 \\ -2a_3 &= 0 \\ -4b_1 &= 0 \\ 4b_1 &= 0 \\ 2b_2 &= 0 \\ 8b_2 &= 0 \\ -12a_2 + 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 3a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= 3y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi \\ &= 3y - \left(-\frac{y(-2x^3 + y)}{x\,(2x^3 + y)}\right)(x) \\ &= \frac{4y\,x^3 + 4y^2}{2x^3 + y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4y\,x^3+4y^2}{2x^3+y}} dy\end{aligned}$$

Which results in

$$S = -\frac{\ln (x^3 + y)}{4} + \frac{\ln (y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(-2x^3 + y)}{x\,(2x^3 + y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{3x^2}{4x^3 + 4y} \\
S_y &= -\frac{1}{4x^3 + 4y} + \frac{1}{2y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{4x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{4R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln(R)}{4} + c_1 \tag{4}
$$

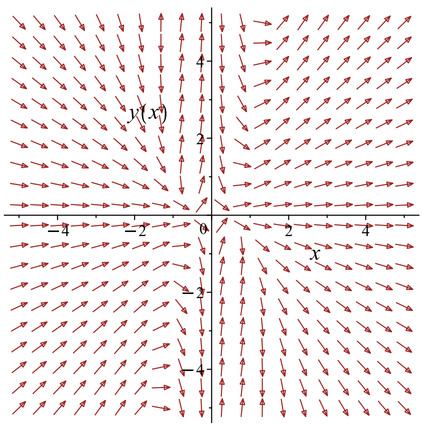
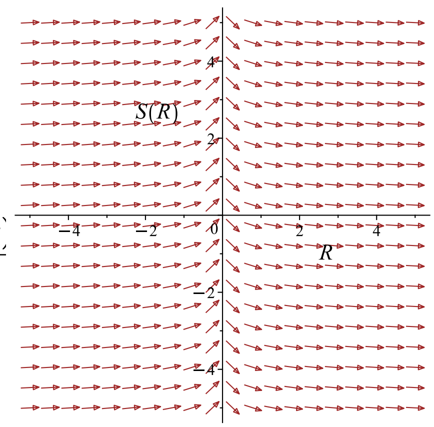
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{\ln(x^3 + y)}{4} + \frac{\ln(y)}{2} = -\frac{\ln(x)}{4} + c_1
$$

Which simplifies to

$$
-\frac{\ln(x^3 + y)}{4} + \frac{\ln(y)}{2} = -\frac{\ln(x)}{4} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-2x^3+y)}{x(2x^3+y)}$ 	$R = x$ $S = -\frac{\ln(x^3+y)}{4} + \frac{\ln(y)}{2}$	$\frac{dS}{dR} = -\frac{1}{4R}$ 

Summary

The solution(s) found are the following

$$-\frac{\ln(x^3+y)}{4} + \frac{\ln(y)}{2} = -\frac{\ln(x)}{4} + c_1 \tag{1}$$

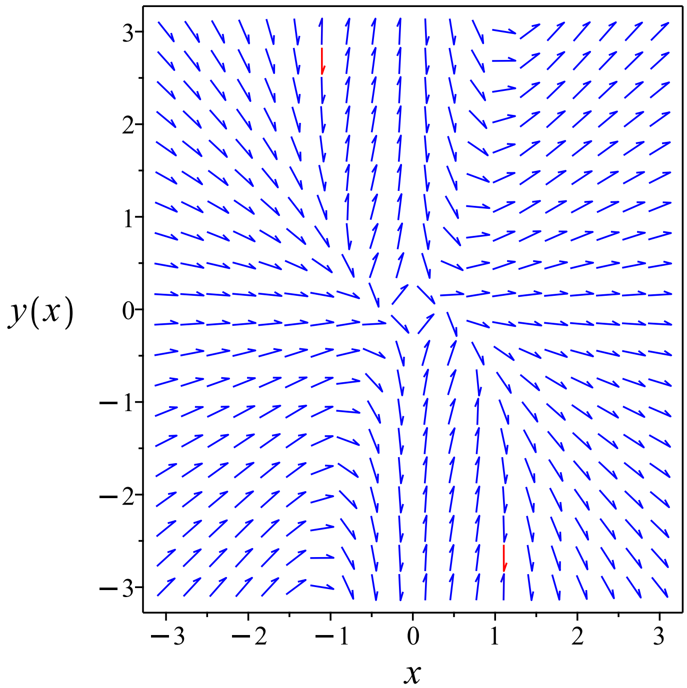


Figure 96: Slope field plot

Verification of solutions

$$-\frac{\ln(x^3+y)}{4}+\frac{\ln(y)}{2}=-\frac{\ln(x)}{4}+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.594 (sec). Leaf size: 47

```
dsolve((2*x^3*y(x)-y(x)^2)-(2*x^4+x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1\left(\sqrt{4x^4 + c_1^2} + c_1\right)}{2x}$$
$$y(x) = -\frac{c_1\left(-c_1 + \sqrt{4x^4 + c_1^2}\right)}{2x}$$

✓ Solution by Mathematica

Time used: 1.279 (sec). Leaf size: 76

```
DSolve[(2*x^3*y[x]-y[x]^2)-(2*x^4+x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2x^4}{-x + \frac{\sqrt{1+4c_1x^4}}{\sqrt{\frac{1}{x^2}}}}$$
$$y(x) \to -\frac{2x^4}{x + \frac{\sqrt{1+4c_1x^4}}{\sqrt{\frac{1}{x^2}}}}$$
$$y(x) \to 0$$

9 Chapter 2, differential equations of the first order and the first degree. Article 16. Integrating factors by inspection. Page 23

9.1 problem Ex 1

Internal problem ID [11156]
Internal file name [`OUTPUT/10141_Sunday_November_27_2022_04_33_52_PM_70744530/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 16. Integrating factors by inspection. Page 23
Problem number: Ex 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 - yx + y'x^2 = 0}$$

9.1.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 - u(x)\,x^2 + (u'(x)\,x + u(x))\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u^2$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u^2}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u^2}\,du &= \int -\frac{1}{x}\,dx \\
-\frac{1}{u} &= -\ln\left(x\right) + c_2
\end{aligned}
$$

The solution is

$$
-\frac{1}{u\left(x\right)} + \ln\left(x\right) - c_2 = 0
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
-\frac{x}{y} + \ln\left(x\right) - c_2 &= 0 \\
-\frac{x}{y} + \ln\left(x\right) - c_2 &= 0
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
-\frac{x}{y} + \ln\left(x\right) - c_2 = 0 \tag{1}
$$

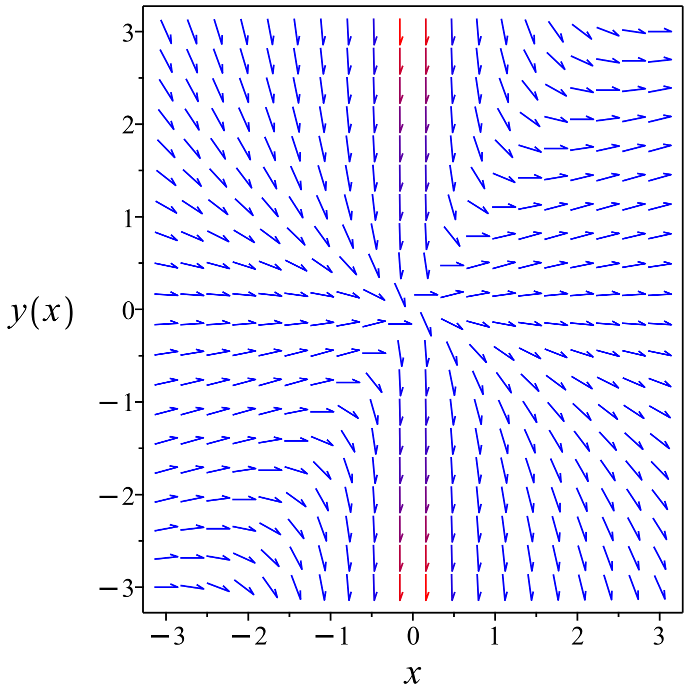


Figure 97: Slope field plot

Verification of solutions

$$-\frac{x}{y}+\ln(x)-c_2=0$$

Verified OK.

9.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=-\frac{y(y-x)}{x^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 59: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{y^2}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(y-x)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{y} \\ S_y &= \frac{x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = -\ln(x) + c_1$$

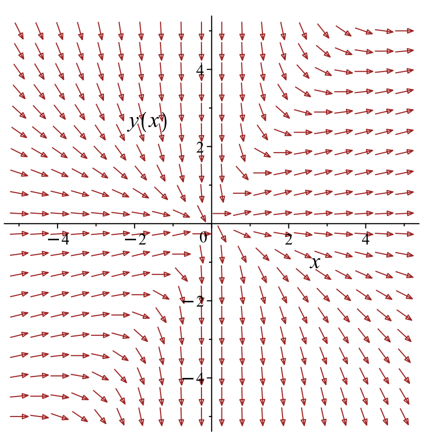
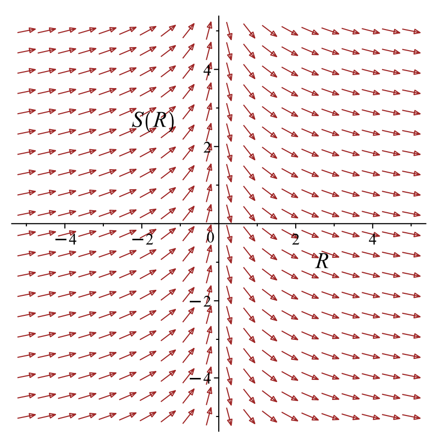
Which simplifies to

$$-\frac{x}{y} = -\ln(x) + c_1$$

Which gives

$$y = \frac{x}{\ln(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(y-x)}{x^2}$ 	$R = x$ $S = -\frac{x}{y}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{x}{\ln(x) - c_1} \tag{1}$$

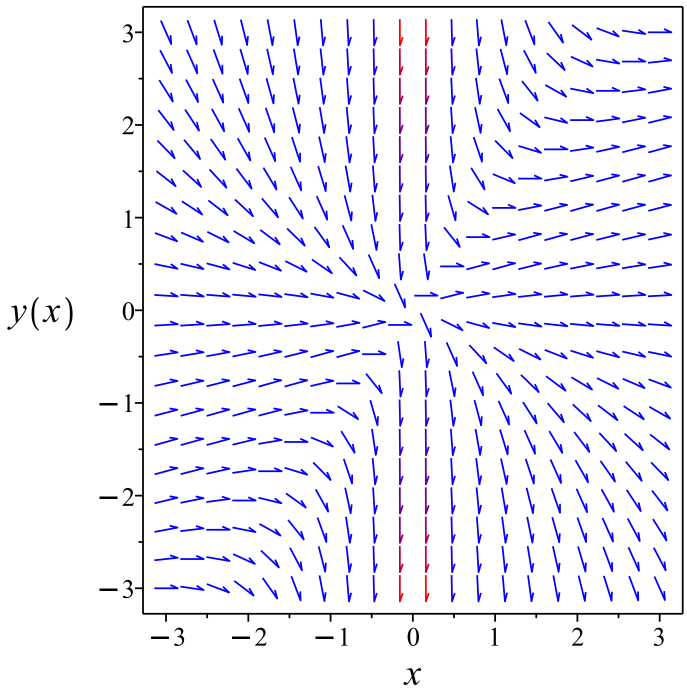


Figure 98: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) - c_1}$$

Verified OK.

9.1.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(y-x)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y - \frac{1}{x^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= -\frac{1}{x^2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{xy} - \frac{1}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} - \frac{1}{x^2} \\ w' &= -\frac{w}{x} + \frac{1}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{1}{x}\\ q(x) &= \frac{1}{x^2}\end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = \frac{1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx}\\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{1}{x^2}\right)\\ \frac{d}{dx}(wx) &= (x)\left(\frac{1}{x^2}\right)\\ d(wx) &= \frac{1}{x}\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}wx &= \int \frac{1}{x}\,dx\\ wx &= \ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = \frac{\ln(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$w(x) = \frac{\ln(x) + c_1}{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{\ln(x) + c_1}{x}$$

Or

$$y = \frac{x}{\ln(x) + c_1}$$

Summary
The solution(s) found are the following

$$y = \frac{x}{\ln(x) + c_1} \tag{1}$$

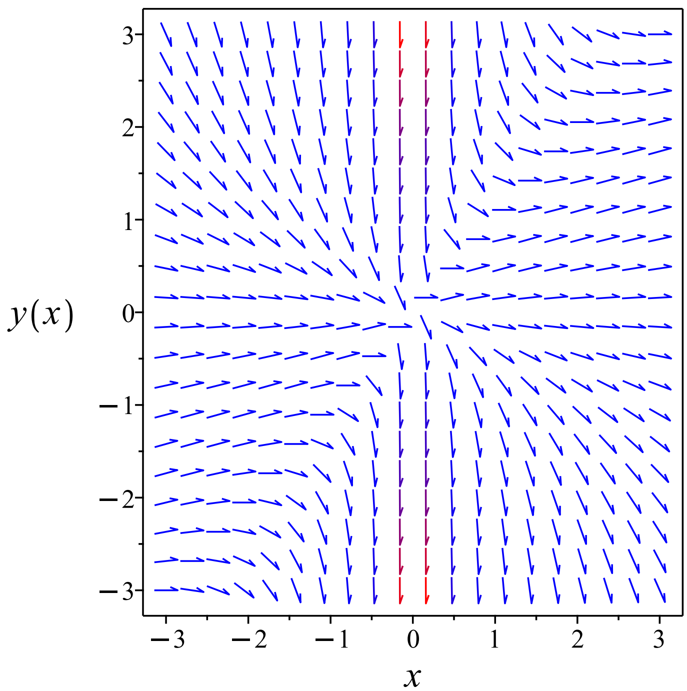


Figure 99: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) + c_1}$$

Verified OK.

9.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x^2)\,\mathrm{d}y &= (xy - y^2)\,\mathrm{d}x \\
(-xy + y^2)\,\mathrm{d}x + (x^2)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= -xy + y^2 \\
N(x,y) &= x^2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-xy + y^2\right)\\ &= -x + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2\right)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = y^2 - yx$ and $N = x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{y^2 - yx}{xy^2}\\ N &= \frac{x}{y^2}\end{aligned}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{x}{y^2}\right)\mathrm{d}y &= \left(-\frac{-xy + y^2}{x\,y^2}\right)\mathrm{d}x\\ \left(\frac{-xy + y^2}{x\,y^2}\right)\mathrm{d}x + \left(\frac{x}{y^2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \frac{-xy + y^2}{x\,y^2}\\ N(x, y) &= \frac{x}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-xy + y^2}{x\,y^2}\right)\\ &= \frac{1}{y^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right)\\ &= \frac{1}{y^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-xy + y^2}{x\, y^2} \, \mathrm{d}x$$

$$\phi = \ln(x) - \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{y^2}$. Therefore equation (4) becomes

$$\frac{x}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) - \frac{x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) - \frac{x}{y}$$

The solution becomes

$$y = \frac{x}{\ln(x) - c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{\ln(x) - c_1} \tag{1}$$

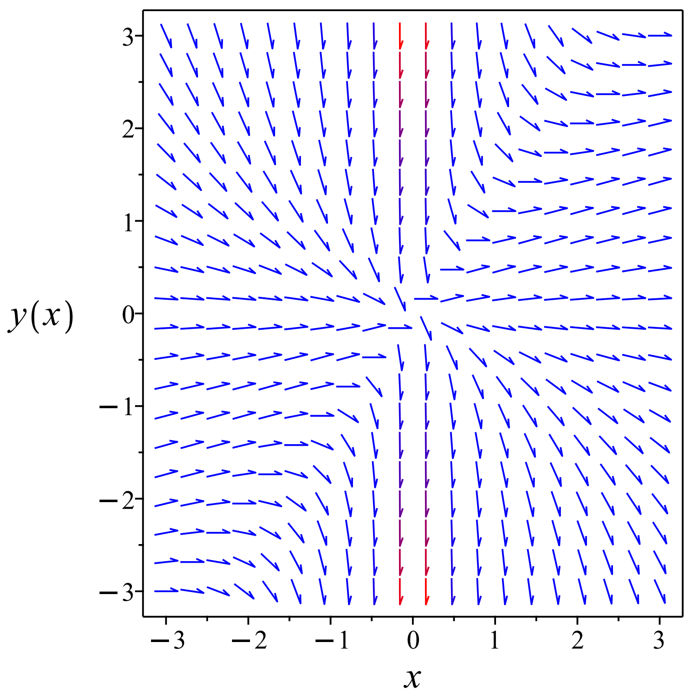


Figure 100: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) - c_1}$$

Verified OK.

9.1.5 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= -\frac{y(y-x)}{x^2}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = \frac{y}{x} - \frac{y^2}{x^2}
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = -\frac{1}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{u}{x^2}} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= \frac{2}{x^3} \\
f_1 f_2 &= -\frac{1}{x^3} \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-\frac{u''(x)}{x^2} - \frac{u'(x)}{x^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_2 \ln(x) + c_1
$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$y = \frac{c_2 x}{c_2 \ln(x) + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x}{\ln(x) + c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{\ln(x) + c_3} \tag{1}$$

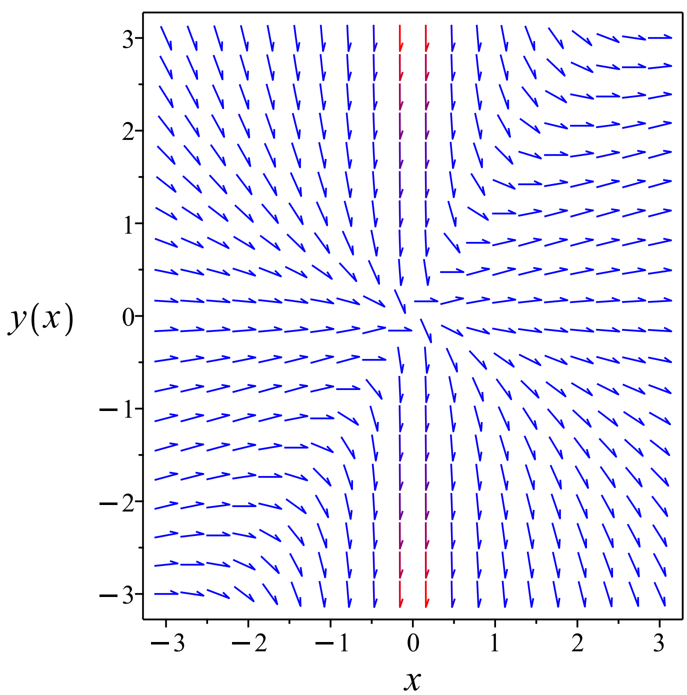


Figure 101: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve((y(x)^2-x*y(x))+x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.227 (sec). Leaf size: 19

```
DSolve[(y[x]^2-x*y[x])+x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{\log(x) + c_1}$$
$$y(x) \to 0$$

9.2 problem Ex 2

Internal problem ID [11157]
Internal file name [`OUTPUT/10142_Sunday_November_27_2022_04_33_53_PM_41838316/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 16. Integrating factors by inspection. Page 23
Problem number: Ex 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{\frac{y'x - y}{\sqrt{x^2 - y^2}} - y'x = 0}$$

9.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{x}{\sqrt{x^2 - y^2}} - x\right) \mathrm{d}y &= \left(\frac{y}{\sqrt{x^2 - y^2}}\right) \mathrm{d}x \\
\left(-\frac{y}{\sqrt{x^2 - y^2}}\right) \mathrm{d}x + \left(\frac{x}{\sqrt{x^2 - y^2}} - x\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{y}{\sqrt{x^2 - y^2}} \\
N(x, y) &= \frac{x}{\sqrt{x^2 - y^2}} - x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{\sqrt{x^2 - y^2}}\right) \\
&= -\frac{x^2}{(x^2 - y^2)^{\frac{3}{2}}}
\end{aligned}
$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2-y^2}} - x\right)$$
$$= -\frac{(x^2-y^2)^{\frac{3}{2}} + y^2}{(x^2-y^2)^{\frac{3}{2}}}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$A = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$
$$= -\frac{\sqrt{x^2-y^2}}{x\left(\sqrt{x^2-y^2}-1\right)}\left(\left(-\frac{1}{\sqrt{x^2-y^2}} - \frac{y^2}{(x^2-y^2)^{\frac{3}{2}}}\right) - \left(\frac{1}{\sqrt{x^2-y^2}} - \frac{x^2}{(x^2-y^2)^{\frac{3}{2}}} - 1\right)\right)$$
$$= -\frac{1}{x}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\mu = e^{\int A \, \mathrm{d}x}$$
$$= e^{\int -\frac{1}{x} \, \mathrm{d}x}$$

The result of integrating gives

$$\mu = e^{-\ln(x)}$$
$$= \frac{1}{x}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\overline{M} = \mu M$$
$$= \frac{1}{x}\left(-\frac{y}{\sqrt{x^2-y^2}}\right)$$
$$= -\frac{y}{\sqrt{x^2-y^2}\, x}$$

And

$$\overline{N} = \mu N$$
$$= \frac{1}{x}\left(\frac{x}{\sqrt{x^2-y^2}} - x\right)$$
$$= -1 + \frac{1}{\sqrt{x^2-y^2}}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$
$$\left(-\frac{y}{\sqrt{x^{2}-y^{2}}\,x}\right)+\left(-1+\frac{1}{\sqrt{x^{2}-y^{2}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int\overline{M}\,\mathrm{d}x$$
$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int-\frac{y}{\sqrt{x^{2}-y^{2}}\,x}\,\mathrm{d}x$$
$$\phi=\frac{y\left(\ln(2)+\ln\left(\frac{\sqrt{-y^{2}}\,\sqrt{x^{2}-y^{2}}-y^{2}}{x}\right)\right)}{\sqrt{-y^{2}}}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial\phi}{\partial y}&=\frac{\ln(2)+\ln\left(\frac{\sqrt{-y^{2}}\,\sqrt{x^{2}-y^{2}}-y^{2}}{x}\right)}{\sqrt{-y^{2}}}+\frac{y^{2}\left(\ln(2)+\ln\left(\frac{\sqrt{-y^{2}}\,\sqrt{x^{2}-y^{2}}-y^{2}}{x}\right)\right)}{(-y^{2})^{\frac{3}{2}}} \\ &\quad+\frac{y\left(-\frac{\sqrt{x^{2}-y^{2}}\,y}{\sqrt{-y^{2}}}-\frac{\sqrt{-y^{2}}\,y}{\sqrt{x^{2}-y^{2}}}-2y\right)}{\sqrt{-y^{2}}\,\left(\sqrt{-y^{2}}\,\sqrt{x^{2}-y^{2}}-y^{2}\right)}+f'(y) \\ &=-\frac{-2y^{2}+2\sqrt{-y^{2}}\,\sqrt{x^{2}-y^{2}}+x^{2}}{\sqrt{x^{2}-y^{2}}\,\left(-\sqrt{-y^{2}}\,\sqrt{x^{2}-y^{2}}+y^{2}\right)}+f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=-1+\frac{1}{\sqrt{x^{2}-y^{2}}}$. Therefore equation (4) becomes

$$-1+\frac{1}{\sqrt{x^{2}-y^{2}}}=-\frac{-2y^{2}+2\sqrt{-y^{2}}\,\sqrt{x^{2}-y^{2}}+x^{2}}{\sqrt{x^{2}-y^{2}}\,\left(-\sqrt{-y^{2}}\,\sqrt{x^{2}-y^{2}}+y^{2}\right)}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= -\frac{-\sqrt{x^2-y^2}\,y^2+\sqrt{-y^2}\,x^2-y^2\sqrt{-y^2}+\sqrt{-y^2}\,\sqrt{x^2-y^2}+x^2-y^2}{\sqrt{x^2-y^2}\,\left(\sqrt{-y^2}\,\sqrt{x^2-y^2}-y^2\right)} \\
&= \frac{\left(-y^2+\sqrt{-y^2}\right)\sqrt{x^2-y^2}+(-y+x)\,(y+x)\left(\sqrt{-y^2}+1\right)}{\sqrt{x^2-y^2}\,\left(-\sqrt{-y^2}\,\sqrt{x^2-y^2}+y^2\right)}
\end{aligned}
$$

Integrating the above w.r.t y results in

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{\left(-y^2+\sqrt{-y^2}\right)\sqrt{x^2-y^2}+(-y+x)\,(y+x)\left(\sqrt{-y^2}+1\right)}{\sqrt{x^2-y^2}\,\left(-\sqrt{-y^2}\,\sqrt{x^2-y^2}+y^2\right)}\right)\mathrm{d}y \\
f(y) &= -y+\frac{\sqrt{-y^2}\,\ln(y)}{y}+c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{y\left(\ln(2)+\ln\left(\frac{\sqrt{-y^2}\,\sqrt{x^2-y^2}-y^2}{x}\right)\right)}{\sqrt{-y^2}}-y+\frac{\sqrt{-y^2}\,\ln(y)}{y}+c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{y\left(\ln(2)+\ln\left(\frac{\sqrt{-y^2}\,\sqrt{x^2-y^2}-y^2}{x}\right)\right)}{\sqrt{-y^2}}-y+\frac{\sqrt{-y^2}\,\ln(y)}{y}
$$

Summary

The solution(s) found are the following

$$
\frac{y\left(\ln(2)+\ln\left(\frac{\sqrt{-y^2}\,\sqrt{x^2-y^2}-y^2}{x}\right)\right)}{\sqrt{-y^2}}-y+\frac{\sqrt{-y^2}\,\ln(y)}{y} = c_1 \tag{1}
$$

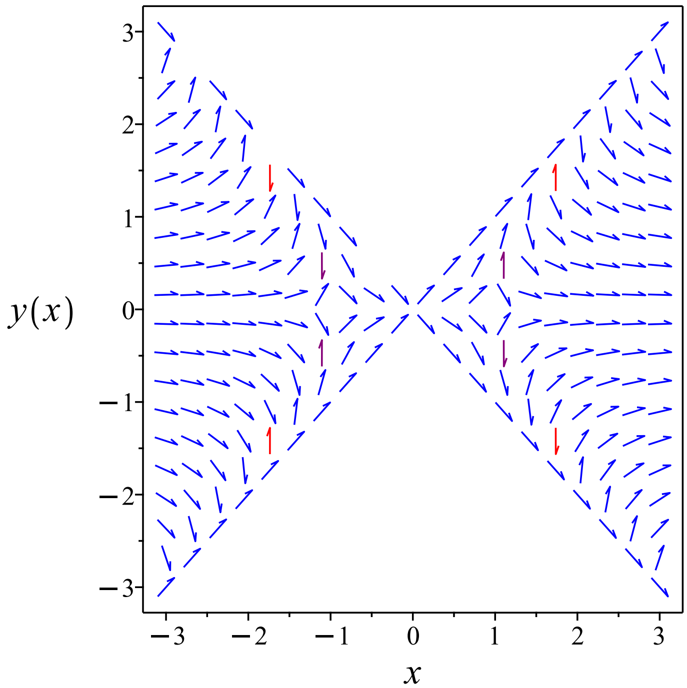


Figure 102: Slope field plot

Verification of solutions

$$\frac{y\left(\ln\left(2\right)+\ln\left(\frac{\sqrt{-y^2}\,\sqrt{x^2-y^2}-y^2}{x}\right)\right)}{\sqrt{-y^2}}-y+\frac{\sqrt{-y^2}\,\ln\left(y\right)}{y}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5`[0, (x^2-y^2)^(1/2)/((x^2-y^2)^(1/2)-1)]
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 27

```
dsolve((x*diff(y(x),x)-y(x))/sqrt(x^2-y(x)^2)=x*diff(y(x),x),y(x), singsol=all)
```

$$y(x) - \arctan\left(\frac{y(x)}{\sqrt{x^2 - y(x)^2}}\right) - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.896 (sec). Leaf size: 29

```
DSolve[(x*y'[x]-y[x])/Sqrt[x^2-y[x]^2]==x*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\arctan\left(\frac{\sqrt{x^2 - y(x)^2}}{y(x)}\right) + y(x) = c_1, y(x)\right]$$

9.3 problem Ex 3

Internal problem ID [11158]
Internal file name [`OUTPUT/10143_Sunday_November_27_2022_04_33_55_PM_16924072/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 16. Integrating factors by inspection. Page 23
Problem number: Ex 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y - (-y + x)\, y' = -x}$$

9.3.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x - (-u(x)\, x + x)\, (u'(x)\, x + u(x)) = -x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{u^2 + 1}{x\, (u - 1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u^2+1}{u-1}}\, du = \int -\frac{1}{x}\, dx$$

$$\frac{\ln(u^2+1)}{2} - \arctan(u) = -\ln(x) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2+1\right)}{2} - \arctan(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

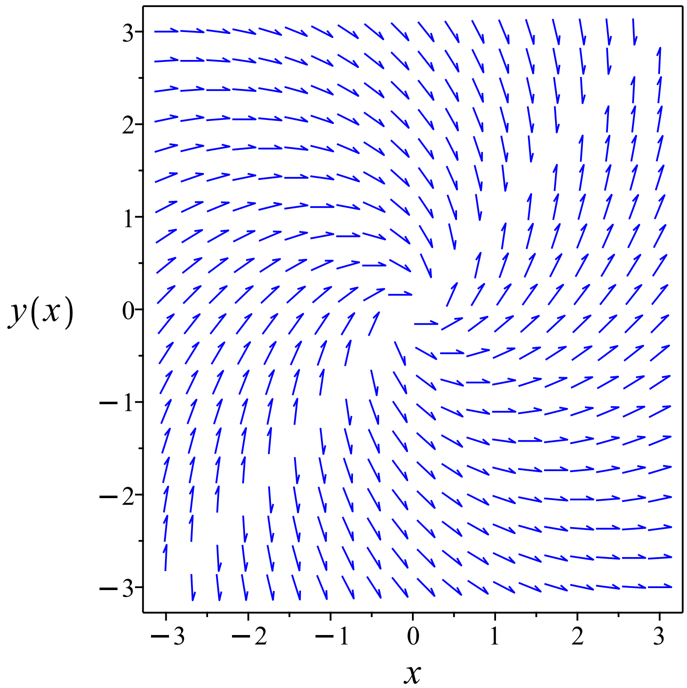


Figure 103: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Verified OK.

9.3.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y+x}{y-x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(y+x)(b_3-a_2)}{y-x} - \frac{(y+x)^2 a_3}{(y-x)^2} - \left(-\frac{1}{y-x} - \frac{y+x}{(y-x)^2}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{1}{y-x} + \frac{y+x}{(y-x)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 + 2xyb_2 + 2xyb_3 - y^2a_2 - y^2a_3 - y^2b_2 + y^2b_3 + 2xb_1 - 2ya}{(-y+x)^2} = 0$$

Setting the numerator to zero gives

$$-x^2a_2 - x^2a_3 - x^2b_2 + x^2b_3 + 2xya_2 - 2xya_3 - 2xyb_2 - 2xyb_3 + y^2a_2 + y^2a_3 + y^2b_2 - y^2b_3 - 2xb_1 + 2ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_1^2 + 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 - 2b_2v_1v_2 + b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 - b_3v_2^2 + 2a_1v_2 - 2b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-a_2 - a_3 - b_2 + b_3)\, v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3)\, v_1 v_2 \\
&- 2b_1 v_1 + (a_2 + a_3 + b_2 - b_3)\, v_2^2 + 2a_1 v_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
2a_1 &= 0 \\
-2b_1 &= 0 \\
-a_2 - a_3 - b_2 + b_3 &= 0 \\
a_2 + a_3 + b_2 - b_3 &= 0 \\
2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= -b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{y + x}{y - x}\right)(x) \\
&= \frac{-x^2 - y^2}{-y + x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2-y^2}{-y+x}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y + x}{y - x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y + x}{x^2 + y^2} \\ S_y &= \frac{y - x}{x^2 + y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

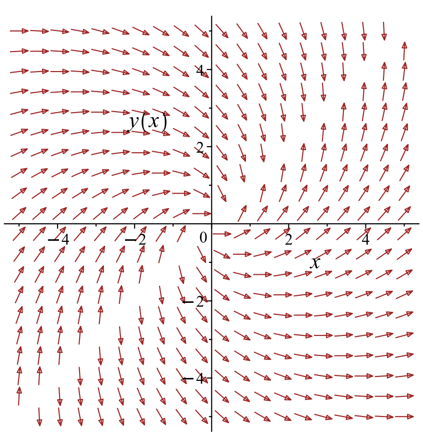
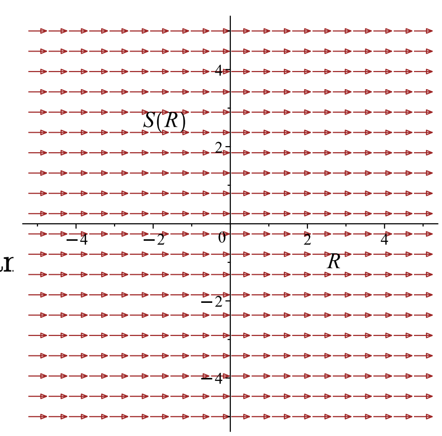
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Which simplifies to

$$\frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y+x}{y-x}$ 	$R = x$ $S = \frac{\ln(x^2 + y^2)}{2} - \arctan$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln (x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1 \tag{1}$$

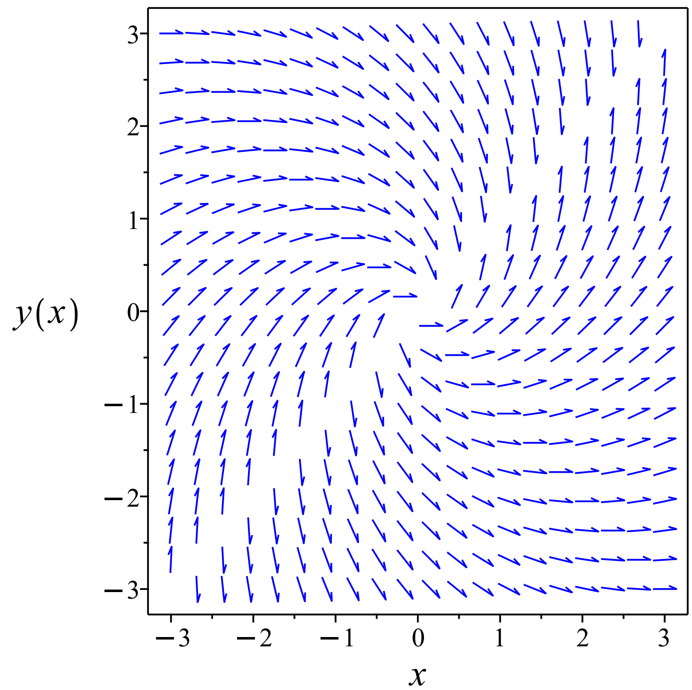


Figure 104: Slope field plot

Verification of solutions

$$\frac{\ln (x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Verified OK.

9.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(y-x)\,\mathrm{d}y&=(-y-x)\,\mathrm{d}x\\ (y+x)\,\mathrm{d}x+(y-x)\,\mathrm{d}y&=0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y+x\\ N(x,y)&=y-x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(y+x)\\ &=1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y-x)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = y + x$ and $N = y - x$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{y+x}{x^2+y^2}$$

$$N = \frac{y-x}{x^2+y^2}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{y-x}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{y+x}{x^2+y^2}\right)\mathrm{d}x$$

$$\left(\frac{y+x}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{y-x}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{y+x}{x^2+y^2}$$
$$N(x,y) = \frac{y-x}{x^2+y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y+x}{x^2+y^2}\right)\\ &= \frac{x^2-2xy-y^2}{\left(x^2+y^2\right)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y-x}{x^2+y^2}\right)\\ &= \frac{x^2-2xy-y^2}{\left(x^2+y^2\right)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{y+x}{x^2+y^2} \mathrm{d}x \\
\phi &= \frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{y}{x^2+y^2} - \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \qquad (4) \\
&= \frac{y-x}{x^2+y^2} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y-x}{x^2+y^2}$. Therefore equation (4) becomes

$$
\frac{y-x}{x^2+y^2} = \frac{y-x}{x^2+y^2} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right)
$$

Summary
The solution(s) found are the following

$$\frac{\ln (x^2+y^2)}{2}+\arctan \left(\frac{x}{y}\right)=c_1 \tag{1}$$

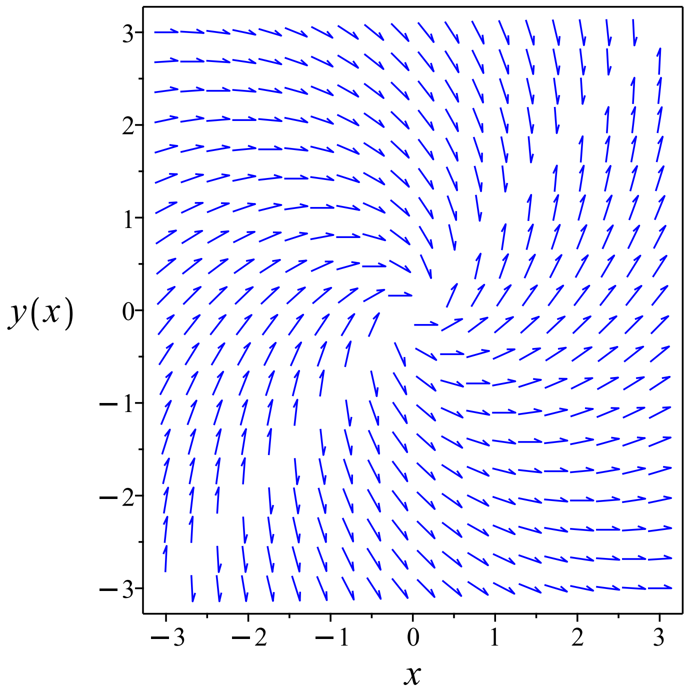


Figure 105: Slope field plot

Verification of solutions

$$\frac{\ln (x^2+y^2)}{2}+\arctan \left(\frac{x}{y}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve((x+y(x))-(x-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan\left(\mathrm{RootOf}\left(-2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x\right) + 2c_1\right)\right)x$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 36

```
DSolve[(x+y[x])-(x-y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right)-\arctan\left(\frac{y(x)}{x}\right) = -\log(x)+c_1, y(x)\right]$$

9.4 problem Ex 4

Internal problem ID [11159]
Internal file name [`OUTPUT/10144_Sunday_November_27_2022_04_33_56_PM_61308293/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 16. Integrating factors by inspection. Page 23
Problem number: Ex 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 - 2xyy' = -x^2}$$

9.4.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 - 2x^2u(x)\,(u'(x)\,x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2-1}{2ux} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{u}}\,du = -\frac{1}{2x}\,dx$$
$$\int \frac{1}{\frac{u^2-1}{u}}\,du = \int -\frac{1}{2x}\,dx$$
$$\frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} = -\frac{\ln(x)}{2} + c_2$$

The above can be written as

$$\left(\frac{1}{2}\right)(\ln(u-1) + \ln(u+1)) = -\frac{\ln(x)}{2} + 2c_2$$
$$\ln(u-1) + \ln(u+1) = (2)\left(-\frac{\ln(x)}{2} + 2c_2\right)$$
$$= -\ln(x) + 4c_2$$

Raising both side to exponential gives

$$e^{\ln(u-1)+\ln(u+1)} = e^{-\ln(x)+2c_2}$$

Which simplifies to

$$u^2 - 1 = \frac{2c_2}{x}$$
$$= \frac{c_3}{x}$$

The solution is

$$u(x)^2 - 1 = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x}$$
$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x}$$

Summary

The solution(s) found are the following

$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x} \tag{1}$$

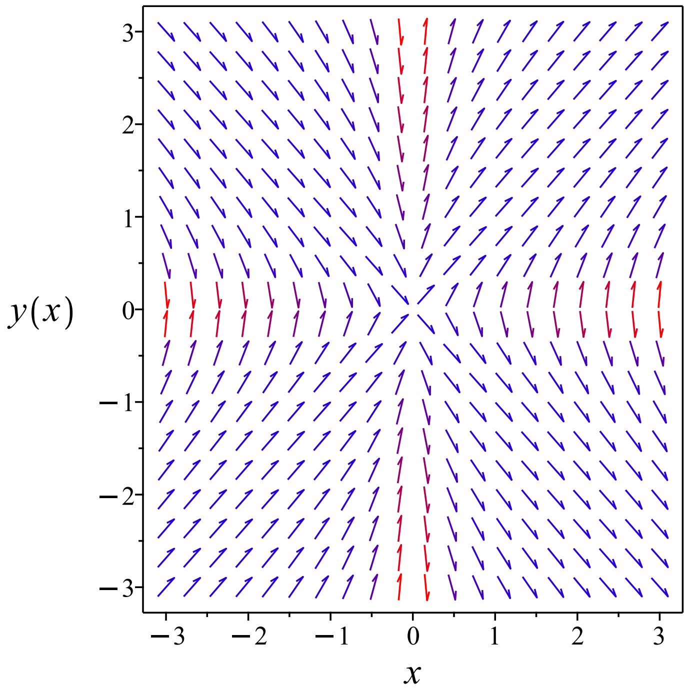


Figure 106: Slope field plot

Verification of solutions

$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x}$$

Verified OK.

9.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + y^2}{2yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 61: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x}{y}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2 + y^2}{2yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2}{2x^2} \\ S_y &= \frac{y}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \tag{4}$$

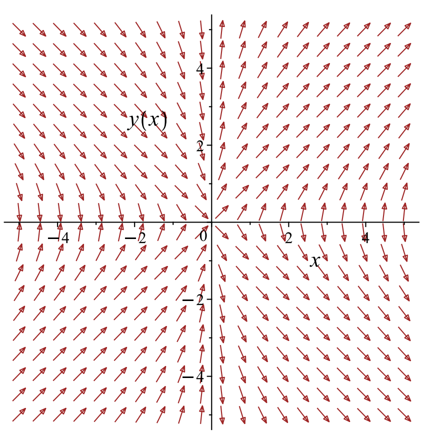
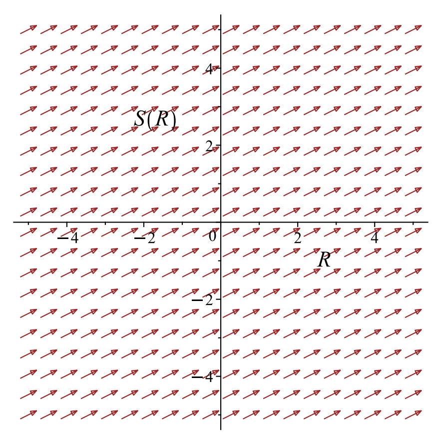
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x} = \frac{x}{2} + c_1$$

Which simplifies to

$$\frac{y^2}{2x} = \frac{x}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y^2}{2yx}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Summary

The solution(s) found are the following

$$\frac{y^2}{2x} = \frac{x}{2} + c_1 \tag{1}$$

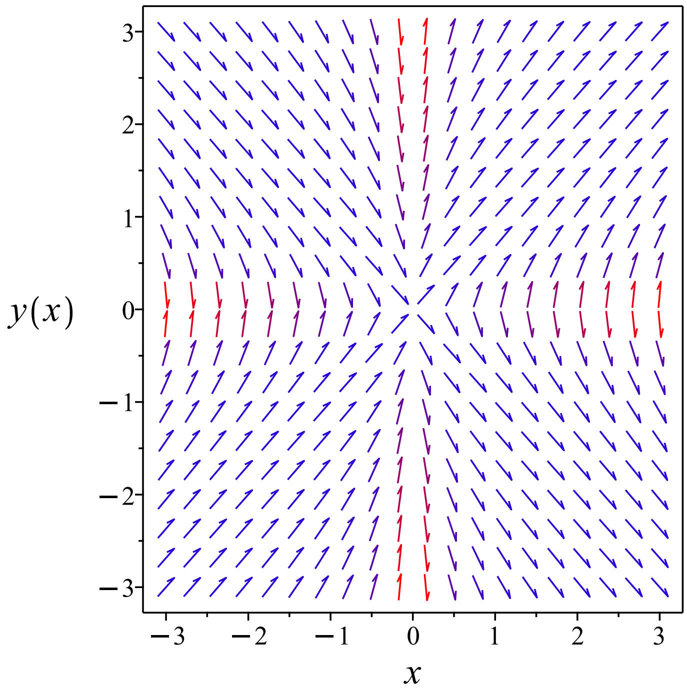


Figure 107: Slope field plot

Verification of solutions

$$\frac{y^2}{2x} = \frac{x}{2} + c_1$$

Verified OK.

9.4.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^2+y^2}{2yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y + \frac{x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{2x} \\ f_1(x) &= \frac{x}{2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{y^2}{2x} + \frac{x}{2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{w(x)}{2x} + \frac{x}{2} \\ w' &= \frac{w}{x} + x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= x \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(x) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(x) \\ d\left(\frac{w}{x}\right) &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int dx \\ \frac{w}{x} &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = c_1 x + x^2$$

which simplifies to

$$w(x) = x(x + c_1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x(x + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{x(x + c_1)} \\ y(x) &= -\sqrt{x(x + c_1)}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x(x + c_1)} \tag{1}$$

$$y = -\sqrt{x(x + c_1)} \tag{2}$$

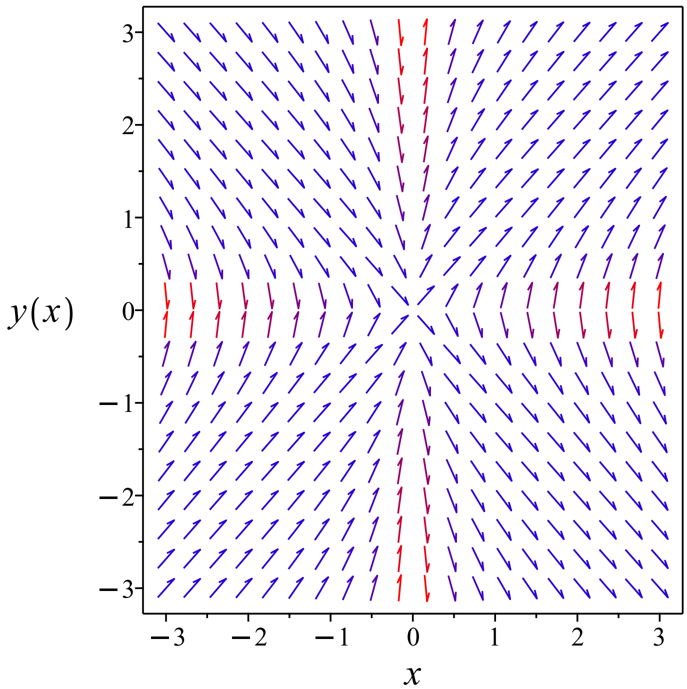


Figure 108: Slope field plot

Verification of solutions

$$y = \sqrt{x(x + c_1)}$$

Verified OK.

$$y = -\sqrt{x(x + c_1)}$$

Verified OK.

9.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2xy)\,\mathrm{d}y&=\left(-x^2-y^2\right)\mathrm{d}x\\ \left(x^2+y^2\right)\mathrm{d}x+(-2xy)\,\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x^2+y^2\\ N(x,y)&=-2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2 + y^2) \\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2xy) \\ &= -2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{2xy}((2y) - (-2y)) \\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{2}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^2}(x^2 + y^2) \\ &= \frac{x^2 + y^2}{x^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(-2xy) \\
&= -\frac{2y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^2+y^2}{x^2}\right) + \left(-\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{x^2+y^2}{x^2} \mathrm{d}x
\end{aligned}
$$

$$
\phi = x - \frac{y^2}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{2y}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{x}$. Therefore equation (4) becomes

$$
-\frac{2y}{x} = -\frac{2y}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x - \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x - \frac{y^2}{x}$$

Summary

The solution(s) found are the following

$$x - \frac{y^2}{x} = c_1 \tag{1}$$

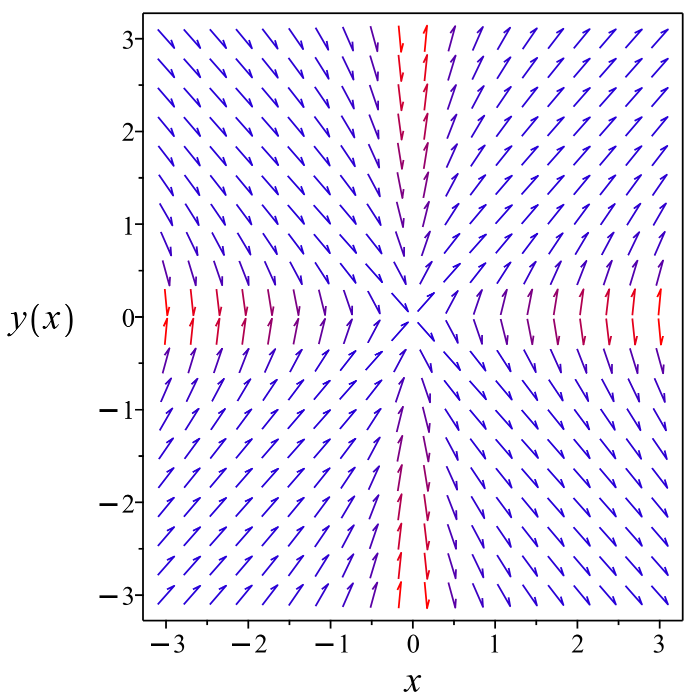


Figure 109: Slope field plot

Verification of solutions

$$x - \frac{y^2}{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve((x^2+y(x)^2)-2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{(c_1 + x)\,x}$$
$$y(x) = -\sqrt{(c_1 + x)\,x}$$

✓ Solution by Mathematica

Time used: 0.304 (sec). Leaf size: 38

```
DSolve[(x^2+y[x]^2)-2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x}\sqrt{x + c_1}$$
$$y(x) \to \sqrt{x}\sqrt{x + c_1}$$

9.5 problem Ex 5

Internal problem ID [11160]
Internal file name [`OUTPUT/10145_Sunday_November_27_2022_04_33_58_PM_25777141/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 16. Integrating factors by inspection. Page 23
Problem number: Ex 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{-y^2 + 2xyy' = -x}$$

9.5.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2 - x}{2yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 63: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x}{y}} dy
\end{aligned}$$

Which results in

$$S = \frac{y^2}{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^2 - x}{2yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y^2}{2x^2} \\
S_y &= \frac{y}{x}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

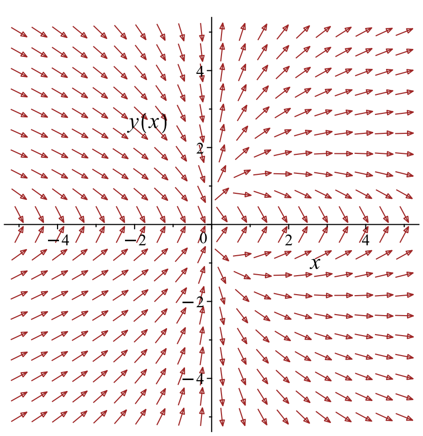
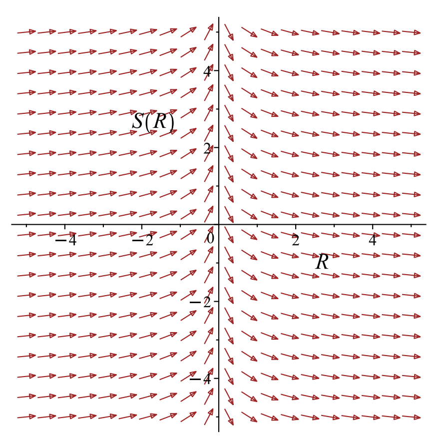
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x} = -\frac{\ln(x)}{2} + c_1$$

Which simplifies to

$$\frac{y^2}{2x} = -\frac{\ln(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2-x}{2yx}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$\frac{y^2}{2x} = -\frac{\ln(x)}{2} + c_1 \tag{1}$$

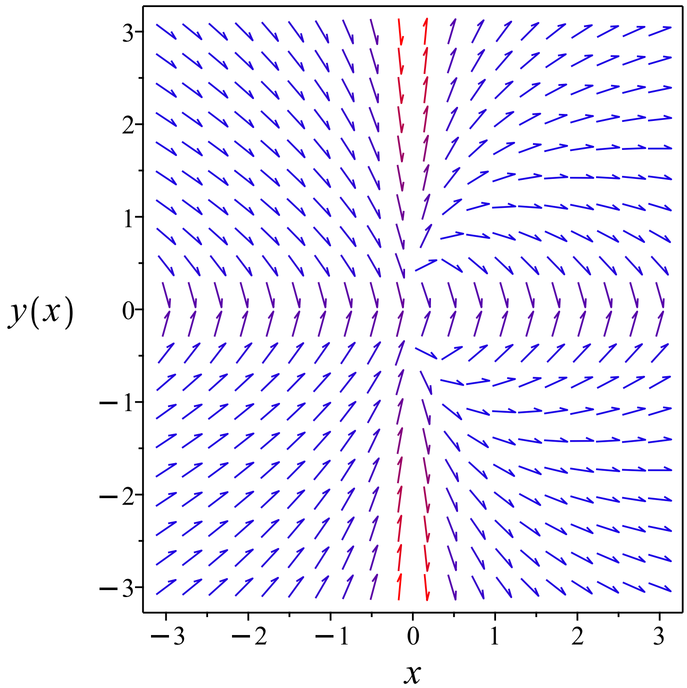


Figure 110: Slope field plot

Verification of solutions

$$\frac{y^2}{2x} = -\frac{\ln(x)}{2} + c_1$$

Verified OK.

9.5.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2 - x}{2yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y - \frac{1}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{2x} \\ f_1(x) &= -\frac{1}{2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{y^2}{2x} - \frac{1}{2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{w(x)}{2x} - \frac{1}{2} \\ w' &= \frac{w}{x} - 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -1 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = -1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-1) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(-1) \\ d\left(\frac{w}{x}\right) &= \left(-\frac{1}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int -\frac{1}{x}\, dx \\ \frac{w}{x} &= -\ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = -x\ln(x) + c_1 x$$

which simplifies to

$$w(x) = x(-\ln(x) + c_1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x(-\ln(x) + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{-x(\ln(x) - c_1)} \\ y(x) &= -\sqrt{x(-\ln(x) + c_1)}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-x\left(\ln\left(x\right)-c_1\right)} \tag{1}$$
$$y = -\sqrt{x\left(-\ln\left(x\right)+c_1\right)} \tag{2}$$

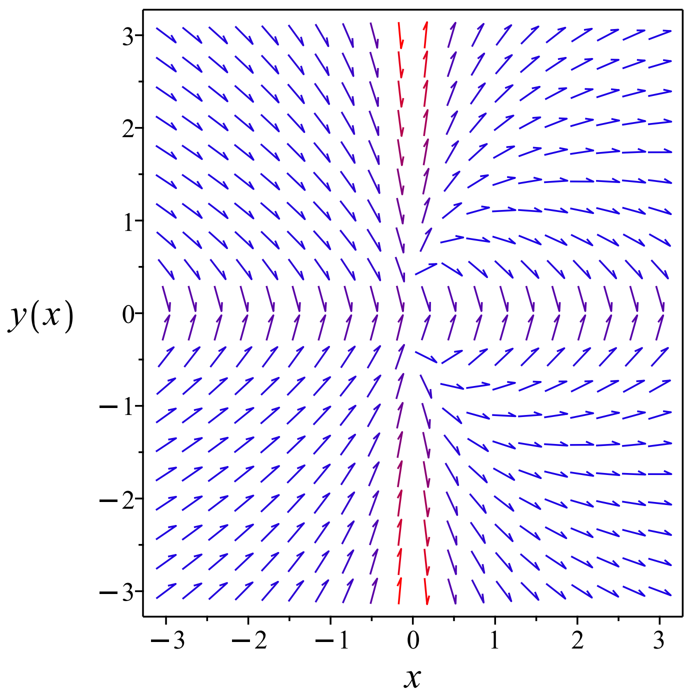


Figure 111: Slope field plot

Verification of solutions

$$y = \sqrt{-x\left(\ln\left(x\right)-c_1\right)}$$

Verified OK.

$$y = -\sqrt{x\left(-\ln\left(x\right)+c_1\right)}$$

Verified OK.

9.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy)\,\mathrm{d}y&=\left(y^2-x\right)\mathrm{d}x\\ \left(-y^2+x\right)\mathrm{d}x+(2xy)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y^2+x\\ N(x,y)&=2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y^2 + x) \\ &= -2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy) \\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{2xy}((-2y) - (2y)) \\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{2}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^2}(-y^2 + x) \\ &= \frac{-y^2 + x}{x^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(2xy) \\
&= \frac{2y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-y^2 + x}{x^2}\right) + \left(\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-y^2 + x}{x^2}\,\mathrm{d}x \\
\phi &= \frac{y^2}{x} + \ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{2y}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y}{x}$. Therefore equation (4) becomes

$$
\frac{2y}{x} = \frac{2y}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y^2}{x} + \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y^2}{x} + \ln(x)$$

Summary

The solution(s) found are the following

$$\frac{y^2}{x} + \ln(x) = c_1 \tag{1}$$

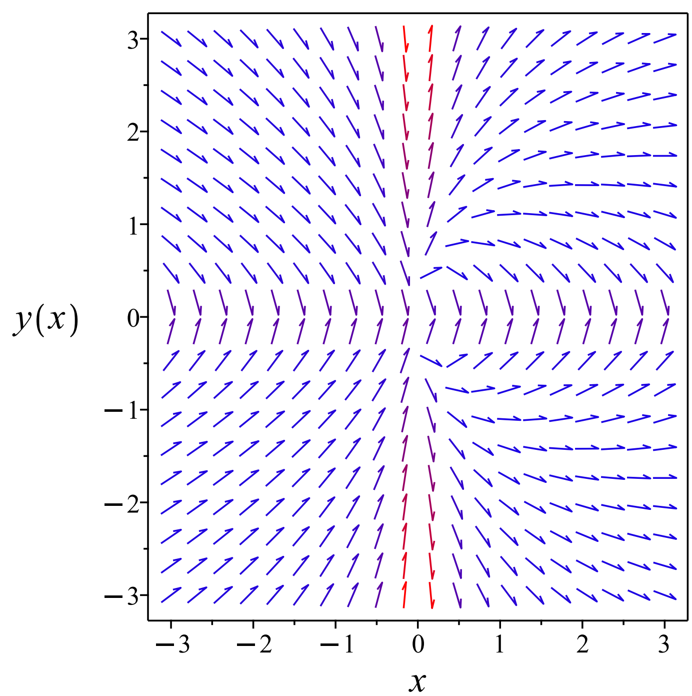


Figure 112: Slope field plot

Verification of solutions

$$\frac{y^2}{x}+\ln(x)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve((x-y(x)^2)+2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=\sqrt{-x(\ln(x)-c_1)}$$
$$y(x)=-\sqrt{(-\ln(x)+c_1)\,x}$$

✓ Solution by Mathematica

Time used: 0.298 (sec). Leaf size: 44

```
DSolve[(x-y[x]^2)+2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\sqrt{x}\sqrt{-\log(x)+c_1}$$
$$y(x)\to \sqrt{x}\sqrt{-\log(x)+c_1}$$

9.6 problem Ex 6

Internal problem ID [11161]
Internal file name [`OUTPUT/10146_Sunday_November_27_2022_04_33_59_PM_83276714/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 16. Integrating factors by inspection. Page 23
Problem number: Ex 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "exactByInspection", "homogeneousTypeD2"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Riccati]
```

$$\boxed{y'x - y - y^2 = x^2}$$

9.6.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x - u(x)\,x - u(x)^2\,x^2 = x^2$$

Integrating both sides gives

$$\int \frac{1}{u^2+1}du = x + c_2$$
$$\arctan(u) = x + c_2$$

Solving for u gives these solutions

$$u_1 = \tan(x + c_2)$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= x \tan (x + c_2)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = x \tan (x + c_2) \tag{1}
$$

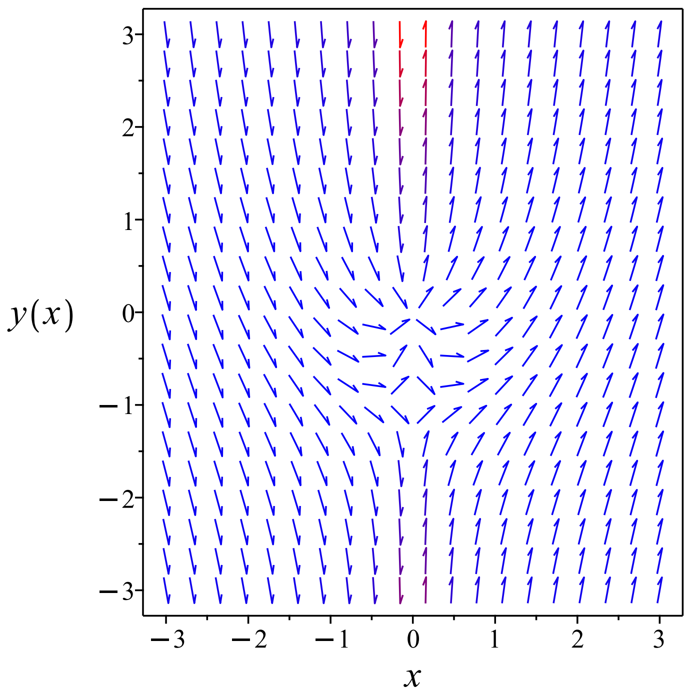


Figure 113: Slope field plot

Verification of solutions

$$
y = x \tan (x + c_2)
$$

Verified OK.

9.6.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(x^2+y^2+y\right)\mathrm{d}x\\ \left(-x^2-y^2-y\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^2-y^2-y\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^2 - y^2 - y)\\ &= -2y - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = -y^2 - x^2 - y$ and $N = x$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{-y^2 - x^2 - y}{x^2 + y^2}\\ N &= \frac{x}{x^2 + y^2}\end{aligned}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{x}{x^2+y^2}\right)\mathrm{d}y &= \left(-\frac{-x^2-y^2-y}{x^2+y^2}\right)\mathrm{d}x\\ \left(\frac{-x^2-y^2-y}{x^2+y^2}\right)\mathrm{d}x+\left(\frac{x}{x^2+y^2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \frac{-x^2-y^2-y}{x^2+y^2}\\ N(x,y) &= \frac{x}{x^2+y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-x^2-y^2-y}{x^2+y^2}\right)\\ &= \frac{-x^2+y^2}{(x^2+y^2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)\\ &= \frac{-x^2+y^2}{(x^2+y^2)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int \frac{-x^2-y^2-y}{x^2+y^2}\,\mathrm{d}x$$
$$\phi=-x-\arctan\left(\frac{x}{y}\right)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=\frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)}+f'(y) \tag{4}$$
$$=\frac{x}{x^2+y^2}+f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{x}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{x}{x^2+y^2}=\frac{x}{x^2+y^2}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=-x-\arctan\left(\frac{x}{y}\right)+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \arctan\left(\frac{x}{y}\right)$$

The solution becomes

$$y = -\frac{x}{\tan(x + c_1)}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{\tan(x + c_1)} \tag{1}$$

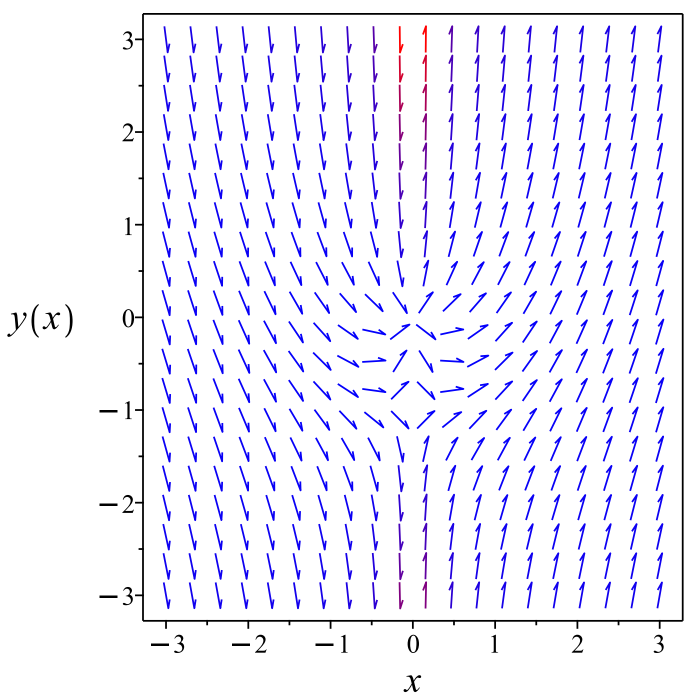


Figure 114: Slope field plot

Verification of solutions

$$y = -\frac{x}{\tan(x + c_1)}$$

Verified OK.

9.6.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^2+y^2+y}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x + \frac{y^2}{x} + \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{x^2} \\ f_1 f_2 &= \frac{1}{x^2} \\ f_2^2 f_0 &= \frac{1}{x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x} + \frac{u(x)}{x} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin(x) + c_2 \cos(x)$$

The above shows that

$$u'(x) = c_1 \cos(x) - c_2 \sin(x)$$

Using the above in (1) gives the solution

$$y = -\frac{(c_1 \cos(x) - c_2 \sin(x))\, x}{c_1 \sin(x) + c_2 \cos(x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-c_3 \cos(x) + \sin(x))\, x}{c_3 \sin(x) + \cos(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{(-c_3 \cos(x) + \sin(x))\, x}{c_3 \sin(x) + \cos(x)} \tag{1}$$

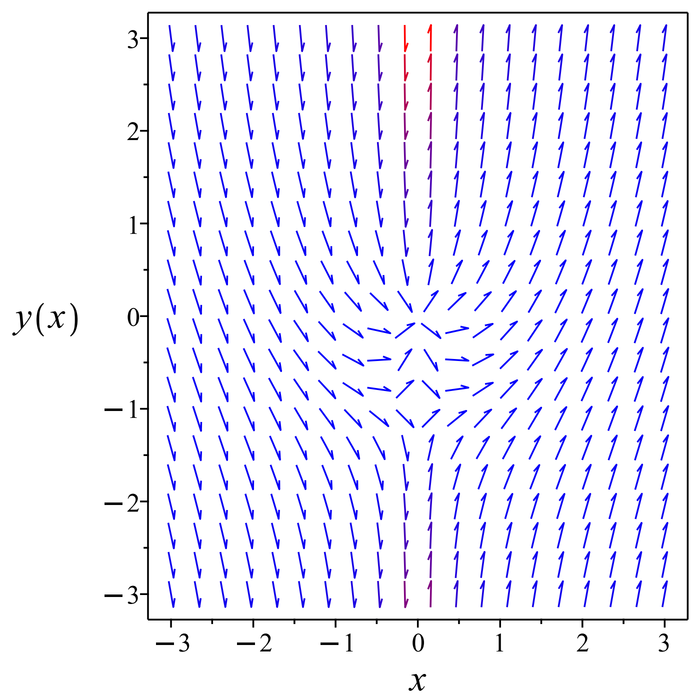


Figure 115: Slope field plot

Verification of solutions

$$y = \frac{(-c_3 \cos(x) + \sin(x))\, x}{c_3 \sin(x) + \cos(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 10

```
dsolve(x*diff(y(x),x)-y(x)=x^2+y(x)^2,y(x), singsol=all)
```

$$y(x) = \tan(c_1 + x)\, x$$

✓ Solution by Mathematica
Time used: 0.277 (sec). Leaf size: 12

```
DSolve[x*y'[x]-y[x]==x^2+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \tan(x + c_1)$$

10 Chapter 2, differential equations of the first order and the first degree. Article 17. Other forms which Integrating factors can be found. Page 25

10.1 problem Ex 1

Internal problem ID [11162]
Internal file name [`OUTPUT/10147_Sunday_November_27_2022_04_34_00_PM_42576888/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 17. Other forms which Integrating factors can be found. Page 25
Problem number: Ex 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{6yx + 3y^2 + (2x^2 + 3yx)\, y' = -3x^2}$$

10.1.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$6u(x)\, x^2 + 3u(x)^2\, x^2 + (2x^2 + 3u(x)\, x^2)\, (u'(x)\, x + u(x)) = -3x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{6u^2 + 8u + 3}{x\,(3u + 2)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{6u^2+8u+3}{3u+2}$. Integrating both sides gives

$$\frac{1}{\frac{6u^2+8u+3}{3u+2}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{6u^2+8u+3}{3u+2}}\, du = \int -\frac{1}{x}\, dx$$

$$\frac{\ln\left(6u^2+8u+3\right)}{4} = -\ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$\left(6u^2+8u+3\right)^{\frac{1}{4}} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\left(6u^2+8u+3\right)^{\frac{1}{4}} = \frac{c_3}{x}$$

Which simplifies to

$$\left(6u(x)^2+8u(x)+3\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\left(6u(x)^2+8u(x)+3\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{6y^2}{x^2}+\frac{8y}{x}+3\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

$$\left(\frac{6y^2+8yx+3x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\left(\frac{6y^2+8yx+3x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x} \tag{1}$$

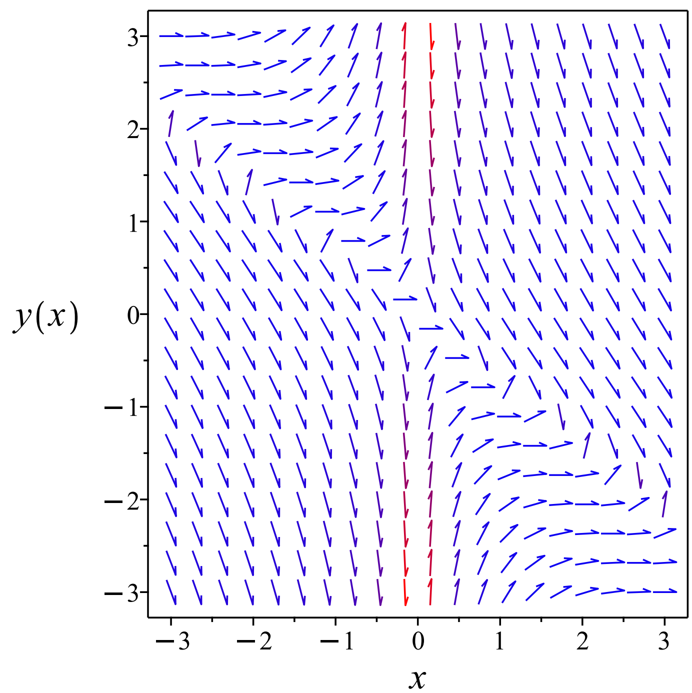


Figure 116: Slope field plot

Verification of solutions

$$\left(\frac{6y^2+8yx+3x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

10.1.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{3(x^2+2xy+y^2)}{x(2x+3y)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{3(x^2+2xy+y^2)(b_3-a_2)}{x(2x+3y)} - \frac{9(x^2+2xy+y^2)^2 a_3}{x^2(2x+3y)^2} \\ &- \left(-\frac{3(2x+2y)}{x(2x+3y)} + \frac{3x^2+6xy+3y^2}{x^2(2x+3y)} + \frac{6x^2+12xy+6y^2}{x(2x+3y)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{3(2x+2y)}{x(2x+3y)} + \frac{9x^2+18xy+9y^2}{x(2x+3y)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^4a_2 - 9x^4a_3 + 7x^4b_2 - 6x^4b_3 + 18x^3ya_2 - 36x^3ya_3 + 24x^3yb_2 - 18x^3yb_3 + 12x^2y^2a_2 - 57x^2y^2a_3 + 18x^2}{x^2(2x+3y}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &6x^4a_2 - 9x^4a_3 + 7x^4b_2 - 6x^4b_3 + 18x^3ya_2 - 36x^3ya_3 + 24x^3yb_2 - 18x^3yb_3 \\ &+ 12x^2y^2a_2 - 57x^2y^2a_3 + 18x^2y^2b_2 - 12x^2y^2b_3 - 48x\,y^3a_3 - 18y^4a_3 \\ &+ 3x^3b_1 - 3x^2ya_1 + 12x^2yb_1 - 12x\,y^2a_1 + 9x\,y^2b_1 - 9y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &6a_2v_1^4 + 18a_2v_1^3v_2 + 12a_2v_1^2v_2^2 - 9a_3v_1^4 - 36a_3v_1^3v_2 - 57a_3v_1^2v_2^2 - 48a_3v_1v_2^3 \\ &- 18a_3v_2^4 + 7b_2v_1^4 + 24b_2v_1^3v_2 + 18b_2v_1^2v_2^2 - 6b_3v_1^4 - 18b_3v_1^3v_2 - 12b_3v_1^2v_2^2 \\ &- 3a_1v_1^2v_2 - 12a_1v_1v_2^2 - 9a_1v_2^3 + 3b_1v_1^3 + 12b_1v_1^2v_2 + 9b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(6a_2 - 9a_3 + 7b_2 - 6b_3)\, v_1^4 + (18a_2 - 36a_3 + 24b_2 - 18b_3)\, v_1^3 v_2 \\ &+ 3b_1 v_1^3 + (12a_2 - 57a_3 + 18b_2 - 12b_3)\, v_1^2 v_2^2 + (-3a_1 + 12b_1)\, v_1^2 v_2 \\ &- 48a_3 v_1 v_2^3 + (-12a_1 + 9b_1)\, v_1 v_2^2 - 18a_3 v_2^4 - 9a_1 v_2^3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-9a_1 &= 0 \\ -48a_3 &= 0 \\ -18a_3 &= 0 \\ 3b_1 &= 0 \\ -12a_1 + 9b_1 &= 0 \\ -3a_1 + 12b_1 &= 0 \\ 6a_2 - 9a_3 + 7b_2 - 6b_3 &= 0 \\ 12a_2 - 57a_3 + 18b_2 - 12b_3 &= 0 \\ 18a_2 - 36a_3 + 24b_2 - 18b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x \\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{3(x^2+2xy+y^2)}{x\,(2x+3y)}\right)(x) \\
&= \frac{3x^2+8xy+6y^2}{2x+3y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3x^2+8xy+6y^2}{2x+3y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(3x^2+8xy+6y^2)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3(x^2+2xy+y^2)}{x\,(2x+3y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{3x + 4y}{6x^2 + 16xy + 12y^2} \\
S_y &= \frac{2x + 3y}{3x^2 + 8xy + 6y^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{2R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}
$$

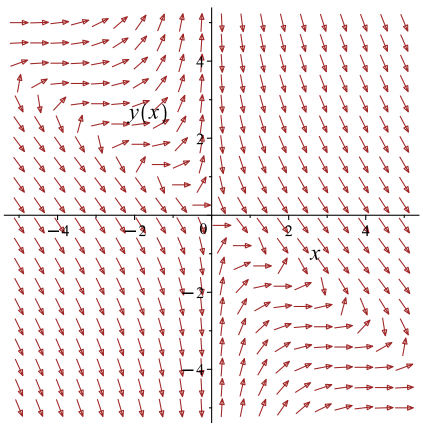
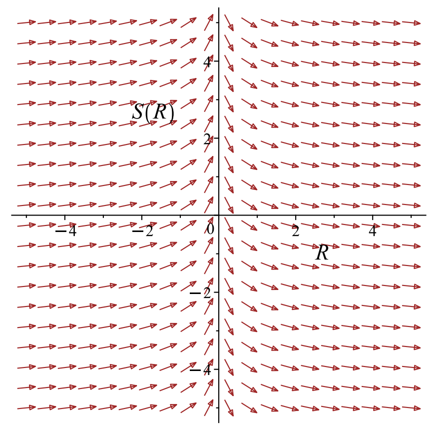
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(6y^2 + 8yx + 3x^2)}{4} = -\frac{\ln(x)}{2} + c_1
$$

Which simplifies to

$$
\frac{\ln(6y^2 + 8yx + 3x^2)}{4} = -\frac{\ln(x)}{2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3(x^2+2xy+y^2)}{x(2x+3y)}$ 	$R = x$ $S = \frac{\ln(3x^2+8xy+6y^2)}{4}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$\frac{\ln(6y^2+8yx+3x^2)}{4} = -\frac{\ln(x)}{2} + c_1 \tag{1}$$

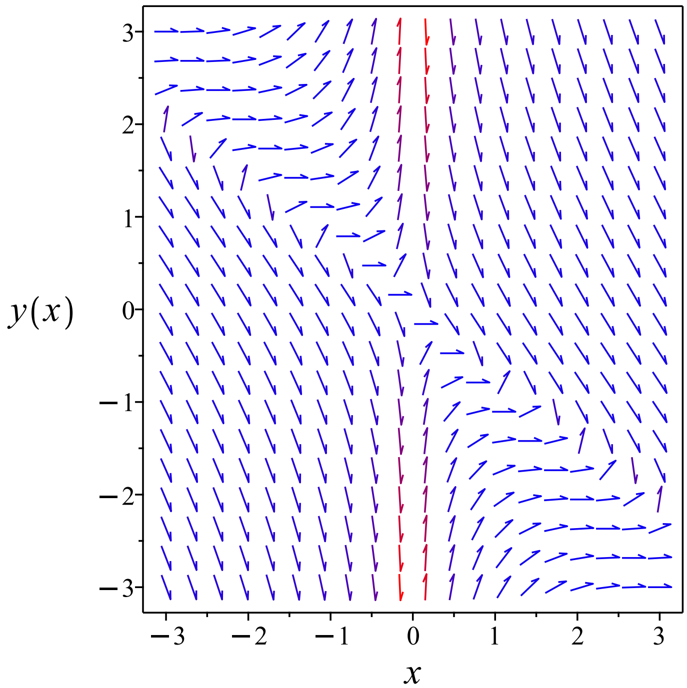


Figure 117: Slope field plot

Verification of solutions

$$\frac{\ln(6y^2+8yx+3x^2)}{4} = -\frac{\ln(x)}{2} + c_1$$

Verified OK.

10.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(2x^2 + 3xy\right) \mathrm{d}y &= \left(-3x^2 - 6xy - 3y^2\right) \mathrm{d}x \\
\left(3x^2 + 6xy + 3y^2\right) \mathrm{d}x + \left(2x^2 + 3xy\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 3x^2 + 6xy + 3y^2 \\
N(x, y) &= 2x^2 + 3xy
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3x^2 + 6xy + 3y^2\right) \\
&= 6x + 6y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2x^2 + 3xy\right) \\
&= 4x + 3y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\,(2x+3y)}((6x+6y)-(4x+3y)) \\
&= \frac{1}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A\,\mathrm{d}x} \\
&= e^{\int \frac{1}{x}\,\mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{\ln(x)} \\
&= x
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= x\left(3x^2+6xy+3y^2\right) \\
&= 3(y+x)^2\,x
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= x\left(2x^2+3xy\right) \\
&= x^2(2x+3y)
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(3(y+x)^2\,x\right) + \left(x^2(2x+3y)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int 3(y+x)^2 \, x \,\mathrm{d}x$$
$$\phi = \frac{3}{4}x^4 + 2y\,x^3 + \frac{3}{2}x^2y^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x^3 + 3x^2y + f'(y) \tag{4}$$
$$= x^2(2x+3y) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2(2x+3y)$. Therefore equation (4) becomes

$$x^2(2x+3y) = x^2(2x+3y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3}{4}x^4 + 2y\,x^3 + \frac{3}{2}x^2y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3}{4}x^4 + 2y\,x^3 + \frac{3}{2}x^2y^2$$

Summary
The solution(s) found are the following

$$\frac{3x^4}{4} + 2yx^3 + \frac{3x^2y^2}{2} = c_1 \tag{1}$$

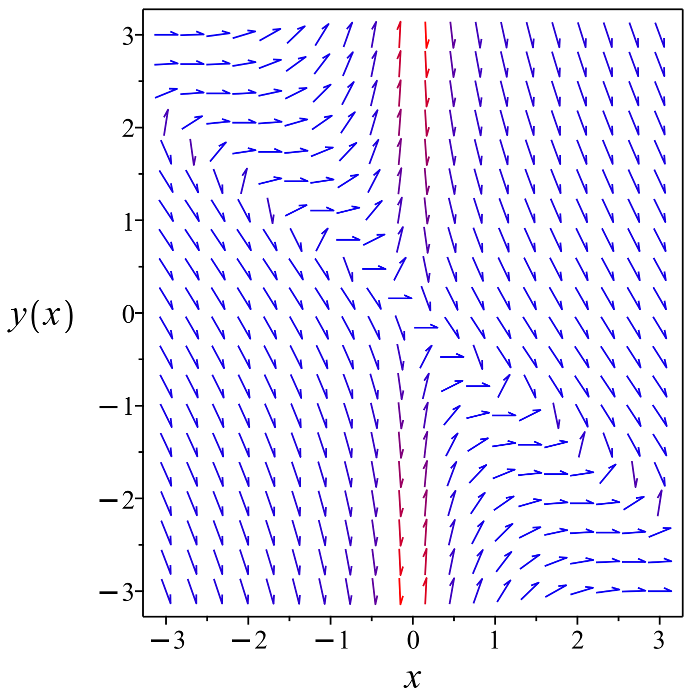


Figure 118: Slope field plot

Verification of solutions

$$\frac{3x^4}{4} + 2yx^3 + \frac{3x^2y^2}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 63

```
dsolve((3*x^2+6*x*y(x)+3*y(x)^2)+(2*x^2+3*x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-4c_1x^2 - \sqrt{-2c_1^2x^4 + 6}}{6c_1x}$$
$$y(x) = \frac{-4c_1x^2 + \sqrt{-2c_1^2x^4 + 6}}{6c_1x}$$

✓ Solution by Mathematica

Time used: 2.7 (sec). Leaf size: 135

```
DSolve[(3*x^2+6*x*y[x]+3*y[x]^2)+(2*x^2+3*x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to -\frac{4x^2 + \sqrt{-2x^4 + 6e^{4c_1}}}{6x}$$
$$y(x) \to \frac{-4x^2 + \sqrt{-2x^4 + 6e^{4c_1}}}{6x}$$
$$y(x) \to -\frac{\sqrt{2}\sqrt{-x^4} + 4x^2}{6x}$$
$$y(x) \to \frac{\sqrt{2}\sqrt{-x^4} - 4x^2}{6x}$$

10.2 problem Ex 2

Internal problem ID [11163]
Internal file name [OUTPUT/10148_Sunday_November_27_2022_04_34_02_PM_61047348/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 17. Other forms which Integrating factors can be found. Page 25
Problem number: Ex 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{(x^2 + y^2 + 2y)\, y' = -2x}$$

10.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(x^2+y^2+2y\right)\mathrm{d}y &= (-2x)\,\mathrm{d}x \\
(2x)\,\mathrm{d}x + \left(x^2+y^2+2y\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= 2x \\
N(x,y) &= x^2+y^2+2y
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2+y^2+2y\right) \\
&= 2x
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^2+y^2+2y}((0)-(2x)) \\
&= -\frac{2x}{x^2+y^2+2y}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{2x}((2x) - (0)) \\ &= 1\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int 1 \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{y} \\ &= \mathrm{e}^{y}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{y}(2x) \\ &= 2x\,\mathrm{e}^{y}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{y}\left(x^2 + y^2 + 2y\right) \\ &= \left(x^2 + y^2 + 2y\right)\mathrm{e}^{y}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (2x\,\mathrm{e}^{y}) + \left(\left(x^2 + y^2 + 2y\right)\mathrm{e}^{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 2x \, \mathrm{e}^{y} \, \mathrm{d}x \\
\phi &= x^{2}\mathrm{e}^{y} + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^{2}\mathrm{e}^{y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x^{2} + y^{2} + 2y) \, \mathrm{e}^{y}$. Therefore equation (4) becomes

$$\left(x^{2} + y^{2} + 2y\right) \mathrm{e}^{y} = x^{2}\mathrm{e}^{y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}
f'(y) &= \mathrm{e}^{y} y^{2} + 2 \, \mathrm{e}^{y} y \\
&= \mathrm{e}^{y} y(y + 2)
\end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int (\mathrm{e}^{y} y(y + 2)) \, \mathrm{d}y \\
f(y) &= \mathrm{e}^{y} y^{2} + c_{1}
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^{2}\mathrm{e}^{y} + \mathrm{e}^{y} y^{2} + c_{1}$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2 e^y + e^y y^2$$

Summary

The solution(s) found are the following

$$x^2 e^y + e^y y^2 = c_1 \tag{1}$$

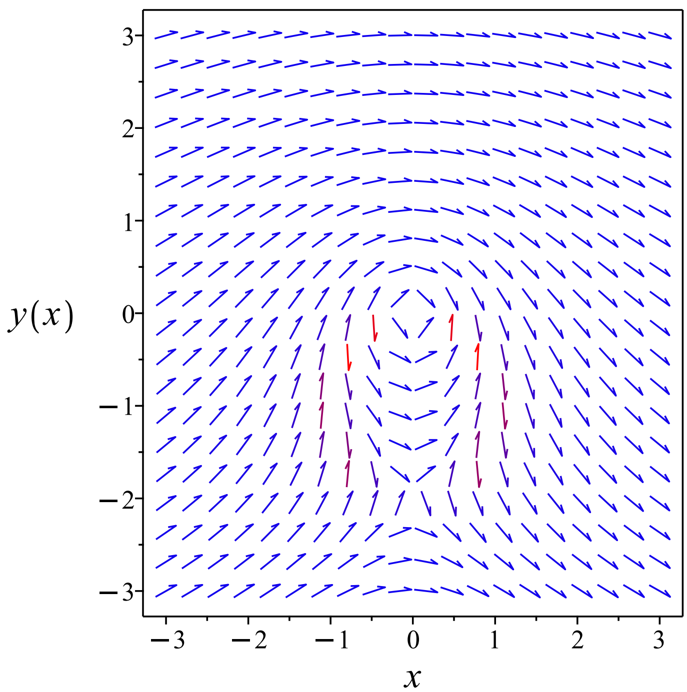


Figure 119: Slope field plot

Verification of solutions

$$x^2 e^y + e^y y^2 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve((2*x)+(x^2+y(x)^2+2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\left(y(x)^2+x^2\right) \mathrm{e}^{y(x)}+c_1=0$$

✓ Solution by Mathematica

Time used: 0.245 (sec). Leaf size: 24

```
DSolve[(2*x)+(x^2+y[x]^2+2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x^2e^{y(x)}+e^{y(x)}y(x)^2=c_1,y(x)\right]$$

10.3 problem Ex 3

Internal problem ID [11164]
Internal file name [OUTPUT/10149_Sunday_November_27_2022_04_34_03_PM_66933797/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 17. Other forms which Integrating factors can be found. Page 25
Problem number: Ex 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{y^4 + 2y + (y^3x + 2y^4 - 4x) y' = 0}$$

10.3.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x\,y^3 + 2y^4 - 4x\right)\mathrm{d}y &= \left(-y^4 - 2y\right)\mathrm{d}x\\ \left(y^4 + 2y\right)\mathrm{d}x + \left(x\,y^3 + 2y^4 - 4x\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y^4 + 2y\\ N(x,y) &= x\,y^3 + 2y^4 - 4x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^4 + 2y\right)\\ &= 4y^3 + 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\,y^3 + 2y^4 - 4x\right)\\ &= y^3 - 4\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x\,y^3 + 2y^4 - 4x}\left(\left(4y^3 + 2\right) - \left(y^3 - 4\right)\right)\\ &= \frac{3y^3 + 6}{x\,y^3 + 2y^4 - 4x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y^4+2y}\left(\left(y^3-4\right) - \left(4y^3+2\right)\right) \\
&= -\frac{3}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{3}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(y)} \\
&= \frac{1}{y^3}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^3}\left(y^4+2y\right) \\
&= \frac{y^3+2}{y^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^3}\left(x\,y^3+2y^4-4x\right) \\
&= \frac{x\,y^3+2y^4-4x}{y^3}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y^3+2}{y^2}\right) + \left(\frac{x\,y^3+2y^4-4x}{y^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{y^3+2}{y^2} \, \mathrm{d}x$$

$$\phi = \frac{(y^3+2)\, x}{y^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{2(y^3+2)\, x}{y^3} + 3x + f'(y) \tag{4}$$

$$= \frac{x(y^3-4)}{y^3} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x\, y^3 + 2y^4 - 4x}{y^3}$. Therefore equation (4) becomes

$$\frac{x\, y^3 + 2y^4 - 4x}{y^3} = \frac{x(y^3-4)}{y^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (2y) \, \mathrm{d}y$$

$$f(y) = y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(y^3+2)\,x}{y^2} + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(y^3+2)\,x}{y^2} + y^2$$

Summary

The solution(s) found are the following

$$\frac{(y^3+2)\,x}{y^2} + y^2 = c_1 \tag{1}$$

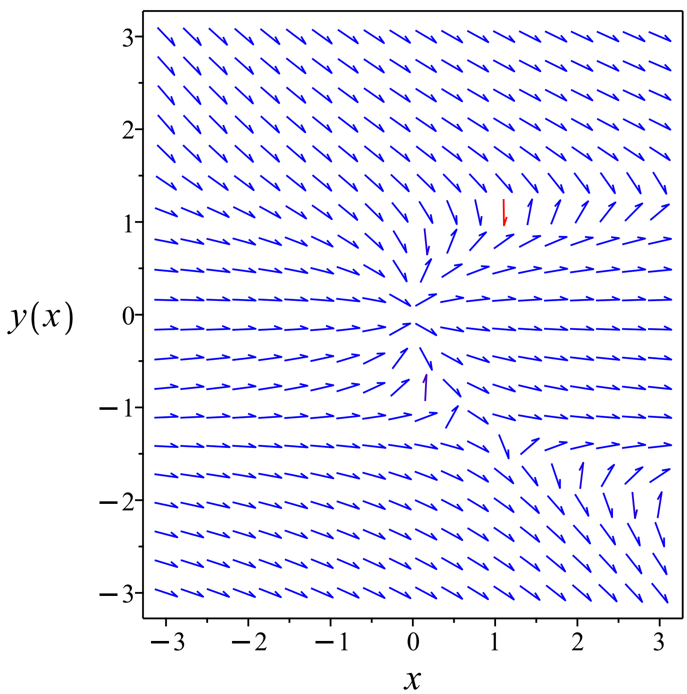


Figure 120: Slope field plot

Verification of solutions

$$\frac{(y^3+2)\,x}{y^2} + y^2 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve((y(x)^4+2*y(x))+(x*y(x)^3+2*y(x)^4-4*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$x-\frac{\left(-y(x)^{2}+c_{1}\right) y(x)^{2}}{y\left(x\right)^{3}+2}=0$$

✓ Solution by Mathematica

Time used: 60.318 (sec). Leaf size: 2021

```
DSolve[(y[x]^4+2*y[x])+(x*y[x]^3+2*y[x]^4-4*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to$$

$$-\frac{1}{2}\sqrt{\frac{\sqrt[3]{54x^3+\sqrt{(54x^3+144c_1x-2c_1^3)^2-4(24x+c_1^2)^3}+144c_1x-2c_1^3}}{3\sqrt[3]{2}}+\frac{}{3\sqrt[3]{54x^3+\sqrt{(54x^3+}}}}$$

$$-\frac{1}{2}\sqrt{-\frac{\sqrt[3]{54x^3+\sqrt{(54x^3+144c_1x-2c_1^3)^2-4(24x+c_1^2)^3}+144c_1x-2c_1^3}}{3\sqrt[3]{2}}-\frac{}{3\sqrt[3]{54x^3+\sqrt{(54x^3-}}}}$$

$$-\frac{x}{4}$$

$$y(x) \to$$

$$-\frac{1}{2}\sqrt{\frac{\sqrt[3]{54x^3+\sqrt{(54x^3+144c_1x-2c_1^3)^2-4(24x+c_1^2)^3}+144c_1x-2c_1^3}}{3\sqrt[3]{2}}+\frac{}{3\sqrt[3]{54x^3+\sqrt{(54x^3+}}}}$$

$$+\frac{1}{2}\sqrt{-\frac{\sqrt[3]{54x^3+\sqrt{(54x^3+144c_1x-2c_1^3)^2-4(24x+c_1^2)^3}+144c_1x-2c_1^3}}{3\sqrt[3]{2}}-\frac{}{3\sqrt[3]{54x^3+\sqrt{(54x^3-}}}}$$

$$-\frac{x}{4}$$

$$y(x)$$

$$\to \frac{1}{2}\sqrt{\frac{\sqrt[3]{54x^3+\sqrt{(54x^3+144c_1x-2c_1^3)^2-4(24x+c_1^2)^3}+144c_1x-2c_1^3}}{3\sqrt[3]{2}}+\frac{}{3\sqrt[3]{54x^3+\sqrt{(54x^3+14}}}}$$

$$-\frac{1}{2}\sqrt{-\frac{\sqrt[3]{54x^3+\sqrt{(54x^3+144c_1x-2c_1^3)^2-4(24x+c_1^2)^3}+144c_1x-2c_1^3}}{3\sqrt[3]{2}}-\frac{}{3\sqrt[3]{54x^3+\sqrt{(54x^3-}}}}$$

$$-\frac{x}{4}$$

$$y(x)$$

$$\to \frac{1}{2}\sqrt{\frac{\sqrt[3]{54x^3+\sqrt{(54x^3+144c_1x-2c_1^3)^2-4(24x+c_1^2)^3}+144c_1x-2c_1^3}}{3\sqrt[3]{2}}+\frac{}{3\sqrt[3]{54x^3+\sqrt{(54x^3+14}}}}$$

10.4 problem Ex 4

Internal problem ID [11165]
Internal file name [`OUTPUT/10150_Sunday_November_27_2022_04_34_04_PM_49776833/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 17. Other forms which Integrating factors can be found. Page 25
Problem number: Ex 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{yx^3 - y^4 + (y^3x - x^4)\, y' = 0}$$

10.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{x}\,dx \\
\ln(y) &= \ln(x) + c_1 \\
y &= e^{\ln(x)+c_1} \\
&= c_1 x
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x \tag{1}$$

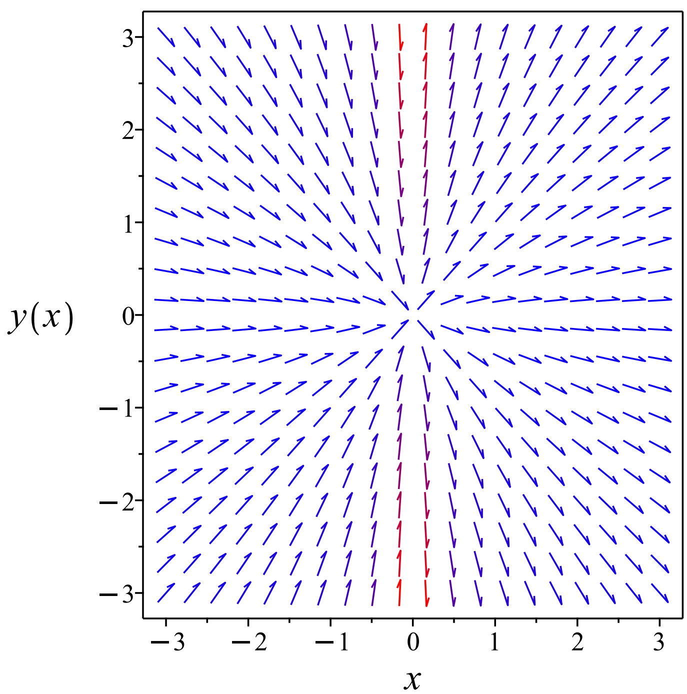


Figure 121: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

10.4.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x$$

Summary
The solution(s) found are the following

$$y = c_1 x \tag{1}$$

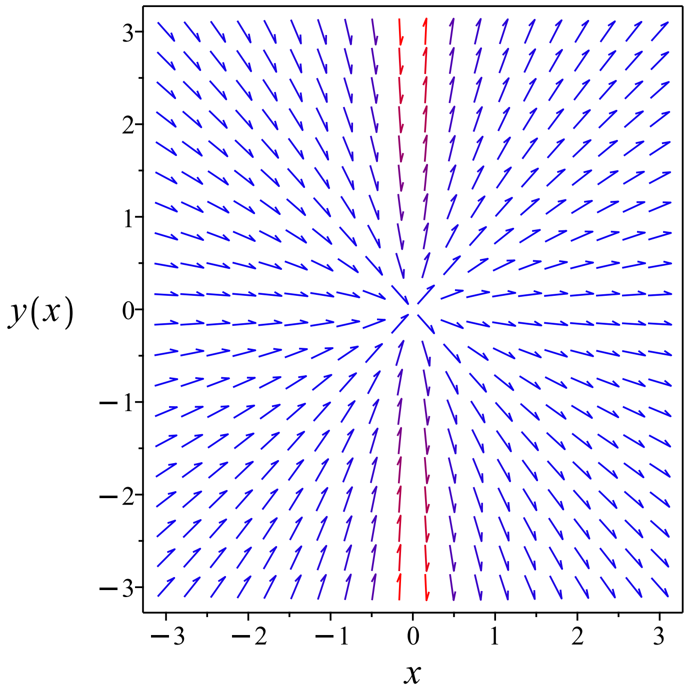


Figure 122: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

10.4.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x^4 - u(x)^4\,x^4 + \left(u(x)^3\,x^4 - x^4\right)\left(u'(x)\,x + u(x)\right) = 0$$

Integrating both sides gives

$$u(x) = \int 0\,\mathrm{d}x$$
$$= c_2$$

Therefore the solution y is

$$y = xu$$
$$= c_2 x$$

Summary
The solution(s) found are the following

$$y = c_2 x \tag{1}$$

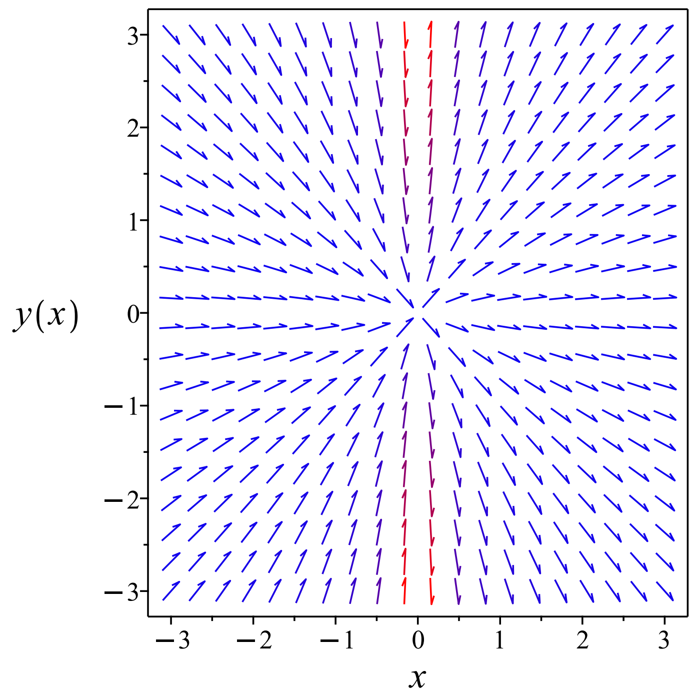


Figure 123: Slope field plot

Verification of solutions

$$y = c_2 x$$

Verified OK.

10.4.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 65: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy\end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = c_1$$

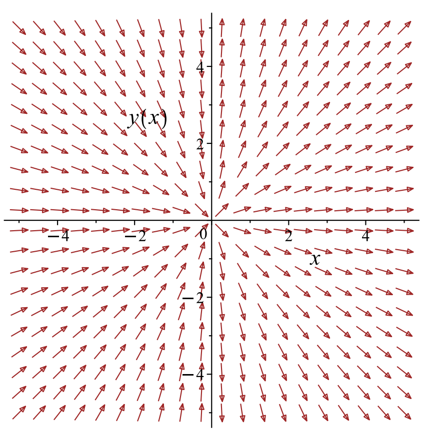
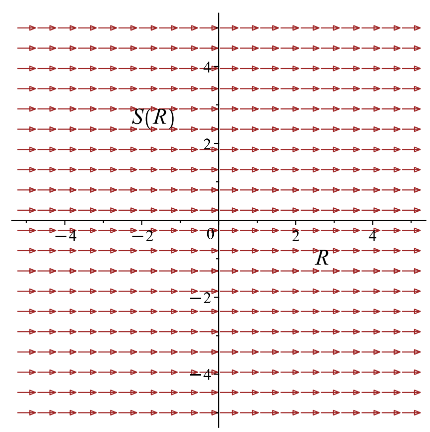
Which simplifies to

$$\frac{y}{x} = c_1$$

Which gives

$$y = c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 x \tag{1}$$

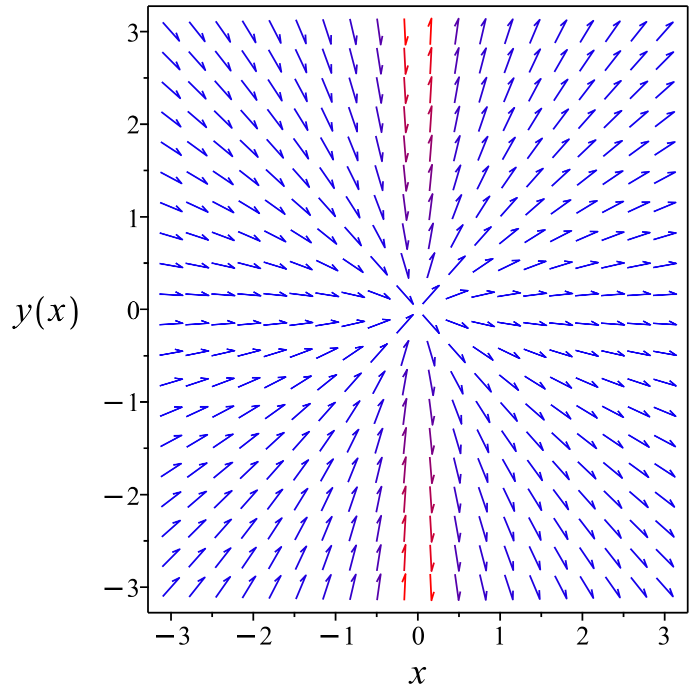


Figure 124: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 x$$

Verified OK.

10.4.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x \\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x} \\ N(x, y) &= \frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int M\, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int -\frac{1}{x}\, \mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y)$$

The solution becomes

$$y = e^{c_1} x$$

Summary

The solution(s) found are the following

$$y = e^{c_1} x \tag{1}$$

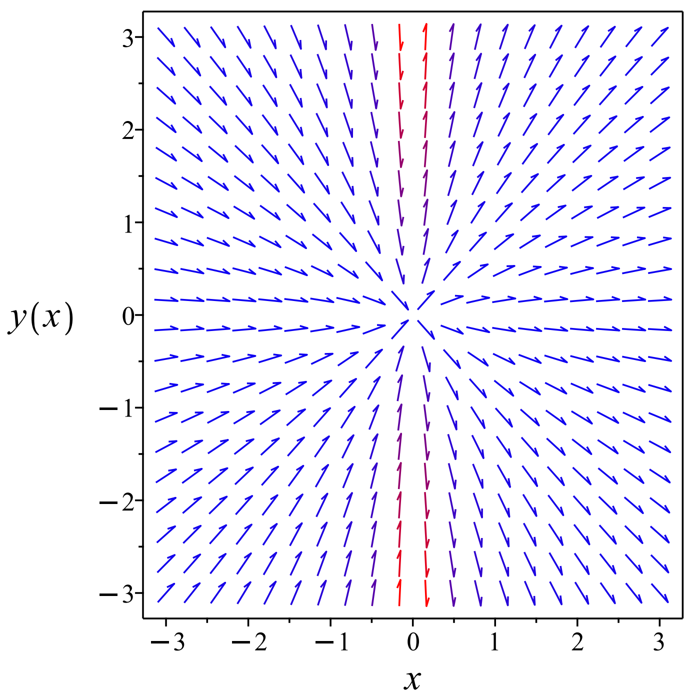


Figure 125: Slope field plot

Verification of solutions

$$y = e^{c_1} x$$

Verified OK.

10.4.6 Maple step by step solution

Let's solve

$yx^3 - y^4 + (y^3x - x^4)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = \mathrm{e}^{c_1} x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve((x^3*y(x)-y(x)^4)+(y(x)^3*x-x^4)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{(1 + i\sqrt{3})\, x}{2}$$
$$y(x) = \frac{(i\sqrt{3} - 1)\, x}{2}$$
$$y(x) = x$$
$$y(x) = c_1 x$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 99

```
DSolve[(x^3*y[x]-y[x]^4)+(y[x]^3*x-x^4)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x$$
$$y(x) \to -\frac{1}{2}i\left(\sqrt{3}-i\right)x$$
$$y(x) \to \frac{1}{2}i\left(\sqrt{3}+i\right)x$$
$$y(x) \to c_1 x$$
$$y(x) \to x$$
$$y(x) \to -\frac{1}{2}i\left(\sqrt{3}-i\right)x$$
$$y(x) \to \frac{1}{2}i\left(\sqrt{3}+i\right)x$$

10.5 problem Ex 6

Internal problem ID [11166]
Internal file name [`OUTPUT/10151_Sunday_November_27_2022_04_34_05_PM_58066198/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 17. Other forms which Integrating factors can be found. Page 25
Problem number: Ex 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y^2+2ymx+\left(y^2m-m\,x^2-2yx\right)y'=x^2}$$

10.5.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2+2u(x)\,x^2m+\left(u(x)^2\,x^2m-m\,x^2-2u(x)\,x^2\right)\left(u'(x)\,x+u(x)\right)=x^2$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=-\frac{(u^2+1)\,(um-1)}{x\,(m\,u^2-m-2u)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{(um-1)(u^2+1)}{m u^2-m-2u}$. Integrating both sides gives

$$\frac{1}{\frac{(um-1)(u^2+1)}{m u^2-m-2u}} du = -\frac{1}{x} dx$$

$$\int \frac{1}{\frac{(um-1)(u^2+1)}{m u^2-m-2u}} du = \int -\frac{1}{x} dx$$

$$-\ln(um-1) + \ln\left(u^2+1\right) = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{-\ln(um-1)+\ln(u^2+1)} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{u^2+1}{um-1} = \frac{c_3}{x}$$

The solution is

$$\frac{u(x)^2+1}{u(x)\,m-1} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\frac{y^2}{x^2}+1}{\frac{ym}{x}-1} = \frac{c_3}{x}$$

$$\frac{x^2+y^2}{x(ym-x)} = \frac{c_3}{x}$$

Which simplifies to

$$\frac{x^2+y^2}{ym-x} = c_3$$

Summary

The solution(s) found are the following

$$\frac{x^2+y^2}{ym-x} = c_3 \tag{1}$$

Verification of solutions

$$\frac{x^2+y^2}{ym-x} = c_3$$

Verified OK.

10.5.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2ymx - x^2 + y^2}{-m\,x^2 + y^2m - 2xy} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2 + ya_3 + a_1 \tag{1E}
$$

$$
\eta = xb_2 + yb_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(2ymx - x^2 + y^2)(b_3 - a_2)}{-m\,x^2 + y^2m - 2xy} - \frac{(2ymx - x^2 + y^2)^2 a_3}{(-m\,x^2 + y^2m - 2xy)^2} \\
&- \left(-\frac{2ym - 2x}{-m\,x^2 + y^2m - 2xy} + \frac{(2ymx - x^2 + y^2)(-2mx - 2y)}{(-m\,x^2 + y^2m - 2xy)^2}\right)(xa_2 \\
&+ ya_3 + a_1) - \left(-\frac{2mx + 2y}{-m\,x^2 + y^2m - 2xy}\right. \\
&\left.+ \frac{(2ymx - x^2 + y^2)(2ym - 2x)}{(-m\,x^2 + y^2m - 2xy)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{m^2x^4b_2 + 2m^2x^2y^2a_3 + 4m^2x^2y^2b_2 - 4m^2x\,y^3a_2 + 4m^2x\,y^3b_3 - 2m^2y^4a_3 - m^2y^4b_2 + 2m^2x^3b_1 - 2m^2x^2y}{}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-m^{2} x^{4} b_{2}-2 m^{2} x^{2} y^{2} a_{3}-4 m^{2} x^{2} y^{2} b_{2}+4 m^{2} x\, y^{3} a_{2}-4 m^{2} x\, y^{3} b_{3}+2 m^{2} y^{4} a_{3} \\
&+m^{2} y^{4} b_{2}-2 m^{2} x^{3} b_{1}+2 m^{2} x^{2} y a_{1}-2 m^{2} x\, y^{2} b_{1}+2 m^{2} y^{3} a_{1}+m\, x^{4} a_{2} \\
&-m\, x^{4} b_{3}+4 m\, x^{3} y a_{3}+4 m\, x^{3} y b_{2}-6 m\, x^{2} y^{2} a_{2}+6 m\, x^{2} y^{2} b_{3}-4 m x\, y^{3} a_{3} \\
&-4 m x\, y^{3} b_{2}+m\, y^{4} a_{2}-m\, y^{4} b_{3}-x^{4} a_{3}-2 x^{4} b_{2}+4 x^{3} y a_{2}-4 x^{3} y b_{3} \\
&+4 x^{2} y^{2} a_{3}+2 x^{2} y^{2} b_{2}+y^{4} a_{3}-2 x^{3} b_{1}+2 x^{2} y a_{1}-2 x\, y^{2} b_{1}+2 y^{3} a_{1}=0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x=v_{1}, y=v_{2}\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&4 m^{2} a_{2} v_{1} v_{2}^{3}-2 m^{2} a_{3} v_{1}^{2} v_{2}^{2}+2 m^{2} a_{3} v_{2}^{4}-m^{2} b_{2} v_{1}^{4}-4 m^{2} b_{2} v_{1}^{2} v_{2}^{2}+m^{2} b_{2} v_{2}^{4} \\
&-4 m^{2} b_{3} v_{1} v_{2}^{3}+2 m^{2} a_{1} v_{1}^{2} v_{2}+2 m^{2} a_{1} v_{2}^{3}-2 m^{2} b_{1} v_{1}^{3}-2 m^{2} b_{1} v_{1} v_{2}^{2}+m a_{2} v_{1}^{4} \\
&-6 m a_{2} v_{1}^{2} v_{2}^{2}+m a_{2} v_{2}^{4}+4 m a_{3} v_{1}^{3} v_{2}-4 m a_{3} v_{1} v_{2}^{3}+4 m b_{2} v_{1}^{3} v_{2}-4 m b_{2} v_{1} v_{2}^{3} \\
&-m b_{3} v_{1}^{4}+6 m b_{3} v_{1}^{2} v_{2}^{2}-m b_{3} v_{2}^{4}+4 a_{2} v_{1}^{3} v_{2}-a_{3} v_{1}^{4}+4 a_{3} v_{1}^{2} v_{2}^{2}+a_{3} v_{2}^{4} \\
&-2 b_{2} v_{1}^{4}+2 b_{2} v_{1}^{2} v_{2}^{2}-4 b_{3} v_{1}^{3} v_{2}+2 a_{1} v_{1}^{2} v_{2}+2 a_{1} v_{2}^{3}-2 b_{1} v_{1}^{3}-2 b_{1} v_{1} v_{2}^{2}=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_{1}, v_{2}\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(-m^{2} b_{2}+m a_{2}-m b_{3}-a_{3}-2 b_{2}\right) v_{1}^{4} \\
&+\left(4 m a_{3}+4 m b_{2}+4 a_{2}-4 b_{3}\right) v_{1}^{3} v_{2}+\left(-2 m^{2} b_{1}-2 b_{1}\right) v_{1}^{3} \\
&+\left(-2 m^{2} a_{3}-4 m^{2} b_{2}-6 m a_{2}+6 m b_{3}+4 a_{3}+2 b_{2}\right) v_{1}^{2} v_{2}^{2}+\left(2 m^{2} a_{1}+2 a_{1}\right) v_{1}^{2} v_{2} \\
&+\left(4 m^{2} a_{2}-4 m^{2} b_{3}-4 m a_{3}-4 m b_{2}\right) v_{1} v_{2}^{3}+\left(-2 m^{2} b_{1}-2 b_{1}\right) v_{1} v_{2}^{2} \\
&+\left(2 m^{2} a_{3}+m^{2} b_{2}+m a_{2}-m b_{3}+a_{3}\right) v_{2}^{4}+\left(2 m^{2} a_{1}+2 a_{1}\right) v_{2}^{3}=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
2m^2a_1 + 2a_1 &= 0\\
-2m^2b_1 - 2b_1 &= 0\\
4ma_3 + 4mb_2 + 4a_2 - 4b_3 &= 0\\
4m^2a_2 - 4m^2b_3 - 4ma_3 - 4mb_2 &= 0\\
-m^2b_2 + ma_2 - mb_3 - a_3 - 2b_2 &= 0\\
2m^2a_3 + m^2b_2 + ma_2 - mb_3 + a_3 &= 0\\
-2m^2a_3 - 4m^2b_2 - 6ma_2 + 6mb_3 + 4a_3 + 2b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x\\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi\\
&= y - \left(-\frac{2ymx - x^2 + y^2}{-m\,x^2 + y^2m - 2xy}\right)(x)\\
&= \frac{-m\,x^2y - m\,y^3 + x^3 + x\,y^2}{m\,x^2 - y^2m + 2xy}\\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-m\,x^2 y - m\,y^3 + x^3 + x\,y^2}{m\,x^2 - y^2 m + 2xy}} dy \end{aligned}$$

Which results in

$$S = \ln\left(x^2 + y^2\right) - \ln\left(ym - x\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2ymx - x^2 + y^2}{-m\,x^2 + y^2 m - 2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x}{x^2 + y^2} + \frac{1}{ym - x} \\ S_y &= \frac{2y}{x^2 + y^2} - \frac{m}{ym - x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln \left(x^2 + y^2\right) - \ln \left(ym - x\right) = c_1$$

Which simplifies to

$$\ln \left(x^2 + y^2\right) - \ln \left(ym - x\right) = c_1$$

Summary
The solution(s) found are the following

$$\ln \left(x^2 + y^2\right) - \ln \left(ym - x\right) = c_1 \tag{1}$$

Verification of solutions

$$\ln \left(x^2 + y^2\right) - \ln \left(ym - x\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 59

```
dsolve((y(x)^2-x^2+2*m*x*y(x))+(m*y(x)^2-m*x^2-2*x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{m - \sqrt{-4c_1^2x^2 - 4c_1x + m^2}}{2c_1}$$
$$y(x) = \frac{m + \sqrt{-4c_1^2x^2 - 4c_1x + m^2}}{2c_1}$$

✓ Solution by Mathematica

Time used: 3.604 (sec). Leaf size: 89

```
DSolve[(y[x]^2-x^2+2*m*x*y[x])+(m*y[x]^2-m*x^2-2*x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{1}{2}\left(-\sqrt{e^{2c_1}m^2 - 4x^2 + 4e^{c_1}x} - e^{c_1}m\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{e^{2c_1}m^2 - 4x^2 + 4e^{c_1}x} - e^{c_1}m\right)$$

11 Chapter 2, differential equations of the first order and the first degree. Article 18. Transformation of variables. Page 26

11.1 problem Ex 1

Internal problem ID [11167]
Internal file name [`OUTPUT/10152_Saturday_December_03_2022_08_02_57_AM_9550685/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 18. Transformation of variables. Page 26
Problem number: Ex 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x - y + 2x^2y = x^3}$$

11.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-2x^2 + 1}{x}$$
$$q(x) = x^2$$

Hence the ode is

$$y' - \frac{(-2x^2 + 1)\, y}{x} = x^2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{-2x^2+1}{x}dx} \\
&= e^{x^2-\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{e^{x^2}}{x}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(x^2\right) \\
\frac{d}{dx}\left(\frac{e^{x^2} y}{x}\right) &= \left(\frac{e^{x^2}}{x}\right)\left(x^2\right) \\
d\left(\frac{e^{x^2} y}{x}\right) &= \left(x\,e^{x^2}\right)\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{e^{x^2} y}{x} &= \int x\,e^{x^2}\,dx \\
\frac{e^{x^2} y}{x} &= \frac{e^{x^2}}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{e^{x^2}}{x}$ results in

$$
y = \frac{x\,e^{-x^2} e^{x^2}}{2} + c_1 x\,e^{-x^2}
$$

which simplifies to

$$
y = \frac{x}{2} + c_1 x\,e^{-x^2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x}{2} + c_1 x\,e^{-x^2} \tag{1}
$$

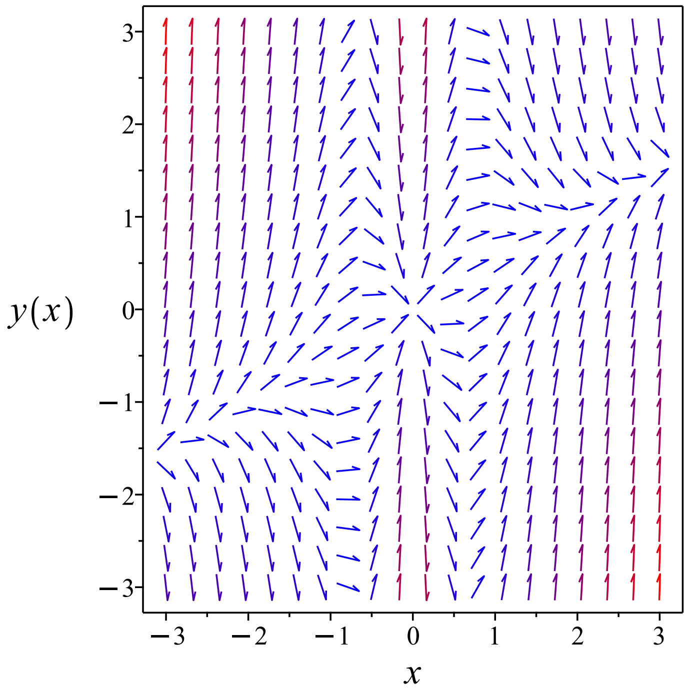


Figure 126: Slope field plot

Verification of solutions

$$y = \frac{x}{2} + c_1 x\,\mathrm{e}^{-x^2}$$

Verified OK.

11.1.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x - u(x)\,x + 2x^3 u(x) = x^3$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= x(-2u+1) \end{aligned}$$

Where $f(x) = x$ and $g(u) = -2u + 1$. Integrating both sides gives

$$\frac{1}{-2u+1}\,du = x\,dx$$

$$
\int \frac{1}{-2u+1}du = \int x\,dx
$$

$$
-\frac{\ln\left(-2u+1\right)}{2} = \frac{x^2}{2}+c_2
$$

Raising both side to exponential gives

$$
\frac{1}{\sqrt{-2u+1}} = \mathrm{e}^{\frac{x^2}{2}+c_2}
$$

Which simplifies to

$$
\frac{1}{\sqrt{-2u+1}} = c_3\mathrm{e}^{\frac{x^2}{2}}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= \frac{x\left(c_3^2\mathrm{e}^{x^2+2c_2}-1\right)\mathrm{e}^{-x^2-2c_2}}{2c_3^2}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{x\left(c_3^2\mathrm{e}^{x^2+2c_2}-1\right)\mathrm{e}^{-x^2-2c_2}}{2c_3^2} \tag{1}
$$

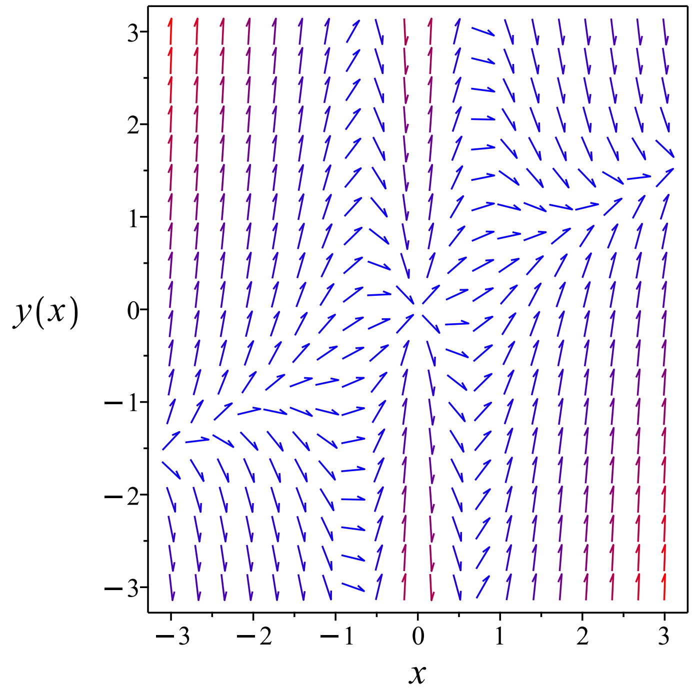


Figure 127: Slope field plot

Verification of solutions

$$y = \frac{x\left(c_3^2 \mathrm{e}^{x^2+2c_2} - 1\right) \mathrm{e}^{-x^2-2c_2}}{2c_3^2}$$

Verified OK.

11.1.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x^3 + 2x^2y - y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 68: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{-x^2+\ln(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x^2+\ln(x)}} dy \end{aligned}$$

Which results in

$$S = \frac{e^{x^2} y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-x^3 + 2x^2 y - y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{e^{x^2} y (2x^2 - 1)}{x^2} \\ S_y &= \frac{e^{x^2}}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x\, e^{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\, e^{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{e^{R^2}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{e^{x^2} y}{x} = \frac{e^{x^2}}{2} + c_1$$

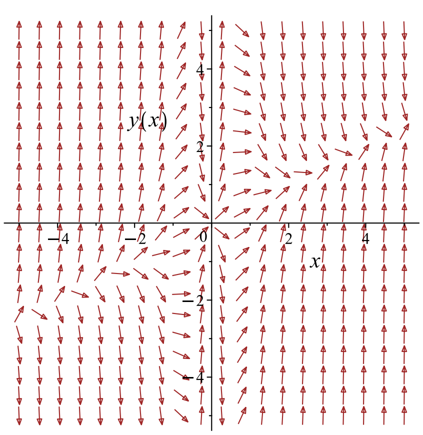
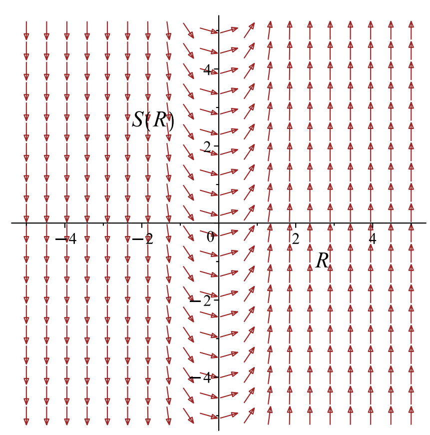
Which simplifies to

$$\frac{e^{x^2} y}{x} = \frac{e^{x^2}}{2} + c_1$$

Which gives

$$y = \frac{x\left(e^{x^2} + 2c_1\right) e^{-x^2}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^3+2x^2y-y}{x}$ 	$R = x$ $S = \frac{e^{x^2} y}{x}$	$\frac{dS}{dR} = R\,e^{R^2}$ 

Summary
The solution(s) found are the following

$$y = \frac{x\left(e^{x^2}+2c_1\right)e^{-x^2}}{2} \tag{1}$$

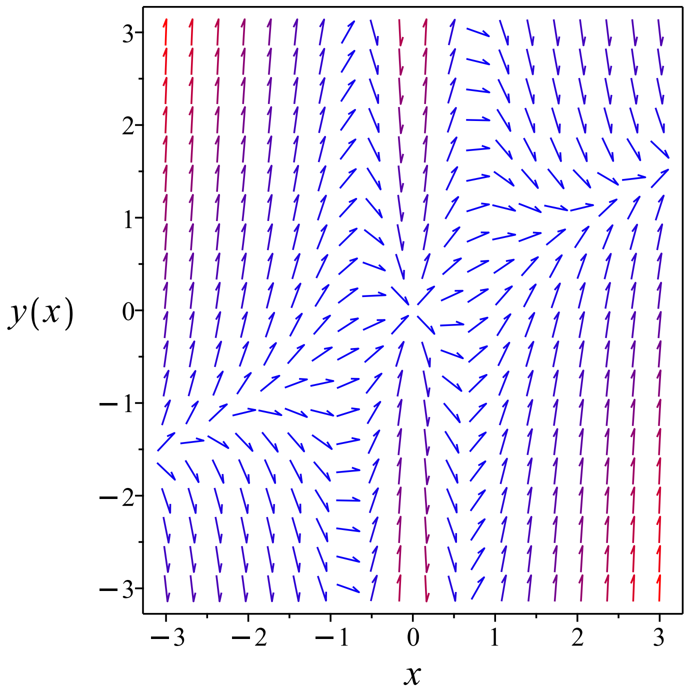


Figure 128: Slope field plot

Verification of solutions

$$y = \frac{x\left(e^{x^2}+2c_1\right)e^{-x^2}}{2}$$

Verified OK.

11.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(x^3-2x^2y+y\right)\mathrm{d}x\\ \left(-x^3+2x^2y-y\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^3+2x^2y-y\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3+2x^2y-y\right) \\
&= 2x^2-1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}\left(\left(2x^2-1\right)-(1)\right) \\
&= \frac{2x^2-2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int \frac{2x^2-2}{x} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x^2-2\ln(x)} \\
&= \frac{\mathrm{e}^{x^2}}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\mathrm{e}^{x^2}}{x^2}\left(-x^3+2x^2y-y\right) \\
&= -\frac{\mathrm{e}^{x^2}\left(x^3-2x^2y+y\right)}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\mathrm{e}^{x^2}}{x^2}(x) \\
&= \frac{\mathrm{e}^{x^2}}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{\mathrm{e}^{x^2}(x^3 - 2x^2 y + y)}{x^2}\right) + \left(\frac{\mathrm{e}^{x^2}}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{\mathrm{e}^{x^2}(x^3 - 2x^2 y + y)}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{(x - 2y)\,\mathrm{e}^{x^2}}{2x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{x^2}}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{x^2}}{x}$. Therefore equation (4) becomes

$$
\frac{\mathrm{e}^{x^2}}{x} = \frac{\mathrm{e}^{x^2}}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(x - 2y)\,\mathrm{e}^{x^2}}{2x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(x - 2y)\,\mathrm{e}^{x^2}}{2x}$$

The solution becomes

$$y = \frac{x\left(\mathrm{e}^{x^2} + 2c_1\right)\mathrm{e}^{-x^2}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{x\left(\mathrm{e}^{x^2} + 2c_1\right)\mathrm{e}^{-x^2}}{2} \tag{1}$$

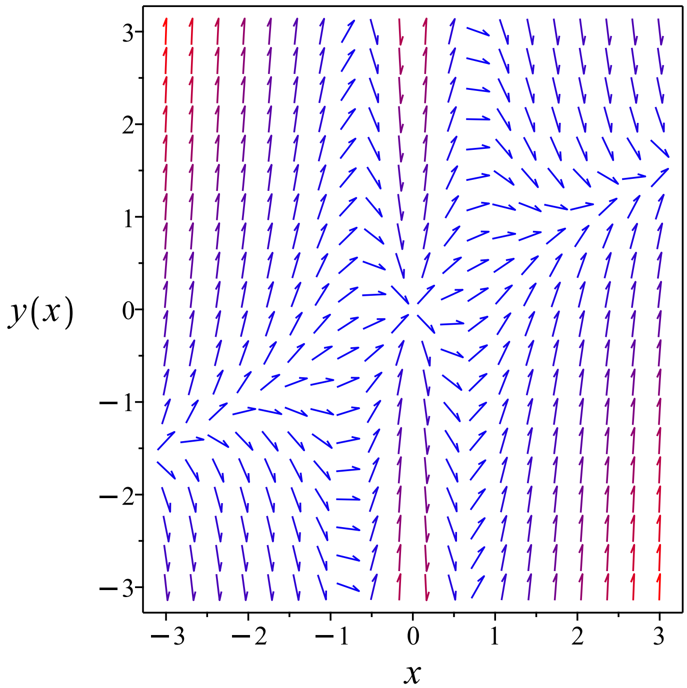


Figure 129: Slope field plot

Verification of solutions

$$y = \frac{x\left(e^{x^2} + 2c_1\right) e^{-x^2}}{2}$$

Verified OK.

11.1.5 Maple step by step solution

Let's solve

$y'x - y + 2x^2y = x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{(2x^2-1)y}{x} + x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{(2x^2-1)y}{x} = x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{(2x^2-1)y}{x}\right)=\mu(x)\,x^2$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{(2x^2-1)y}{x}\right)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)(2x^2-1)}{x}$
- Solve to find the integrating factor

  $\mu(x)=\frac{\mathrm{e}^{x^2}}{x}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\,x^2dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\mu(x)\,x^2dx+c_1$
- Solve for y

  $y=\frac{\int\mu(x)x^2dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\frac{\mathrm{e}^{x^2}}{x}$

  $y=\frac{x\left(\int x\,\mathrm{e}^{x^2}dx+c_1\right)}{\mathrm{e}^{x^2}}$
- Evaluate the integrals on the rhs

  $y=\frac{x\left(\frac{\mathrm{e}^{x^2}}{2}+c_1\right)}{\mathrm{e}^{x^2}}$
- Simplify

  $y=\frac{x}{2}+c_1x\,\mathrm{e}^{-x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x)-y(x)+2*x^2*y(x)-x^3=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{2} + x\,\mathrm{e}^{-x^2} c_1$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 21

```
DSolve[x*y'[x]-y[x]+2*x^2*y[x]-x^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\left(\frac{1}{2} + c_1 e^{-x^2}\right)$$

11.2 problem Ex 2

Internal problem ID [11168]
Internal file name [`OUTPUT/10153_Saturday_December_03_2022_08_02_58_AM_3788059/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 18. Transformation of variables. Page 26
Problem number: Ex 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeC", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], [_Abel, `2nd type`, `class C`],
  _dAlembert]
```

$$\boxed{(y+x)\,y'=1}$$

11.2.1 Solving as homogeneousTypeC ode

Let

$$z=y+x \tag{1}$$

Then

$$z'(x)=y'+1$$

Therefore

$$y'=z'(x)-1$$

Hence the given ode can now be written as

$$z'(x)-1=\frac{1}{z}$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{\frac{1}{z}+1} dz$$

$$x + c_1 = z - \ln(1+z)$$

Replacing z back by its value from (1) then the above gives the solution as

$$y = -\text{LambertW}\left(-e^{-x-c_1-1}\right) - x - 1$$

$$y = -\text{LambertW}\left(-e^{-x-c_1-1}\right) - x - 1$$

Summary

The solution(s) found are the following

$$y = -\text{LambertW}\left(-e^{-x-c_1-1}\right) - x - 1 \tag{1}$$

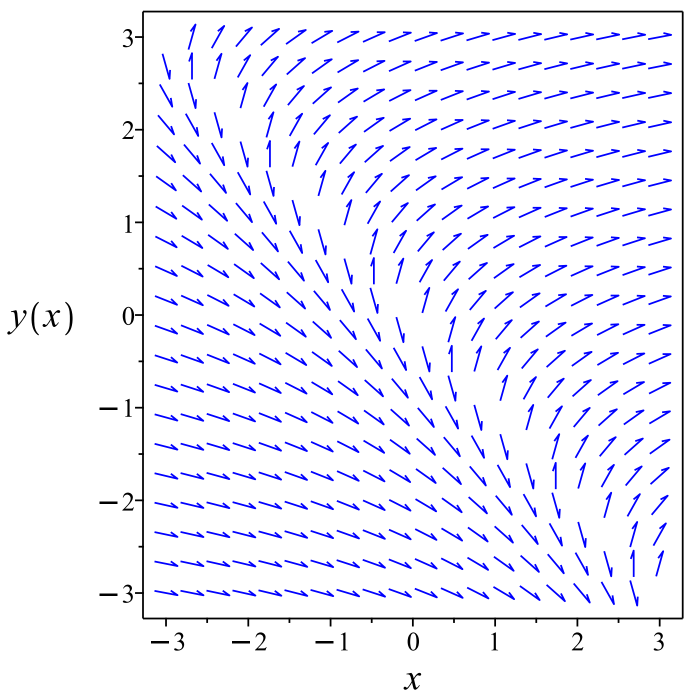


Figure 130: Slope field plot

Verification of solutions

$$y = -\text{LambertW}\left(-e^{-x-c_1-1}\right) - x - 1$$

Verified OK.

11.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{1}{y+x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 71: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 1 \\
\eta(x,y) &= -1
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{-1}{1} \\
&= -1
\end{aligned}
$$

This is easily solved to give

$$
y = -x + c_1
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = y + x
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{1}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= x
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{1}{y+x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 1 \\ S_x &= 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{1 + \frac{1}{y+x}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{1 + \frac{1}{R}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R - \ln(R+1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = y + x - \ln(x+y+1) + c_1$$

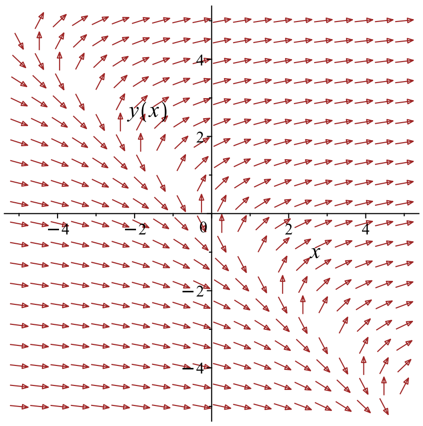
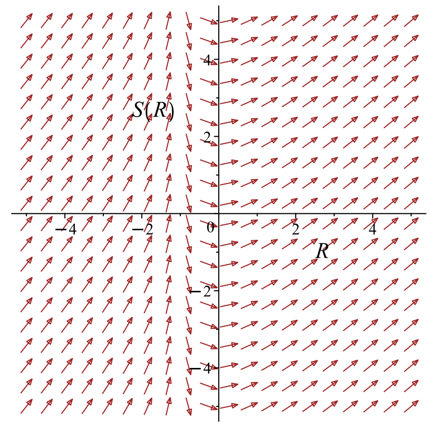
Which simplifies to

$$x = y + x - \ln(x+y+1) + c_1$$

Which gives

$$y = -\operatorname{LambertW}\left(-\mathrm{e}^{-x+c_1-1}\right) - x - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1}{y+x}$ 	$R = y + x$ $S = x$	$\frac{dS}{dR} = \frac{1}{1+\frac{1}{R}}$ 

Summary
The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\mathrm{e}^{-x+c_1-1}\right) - x - 1 \tag{1}$$

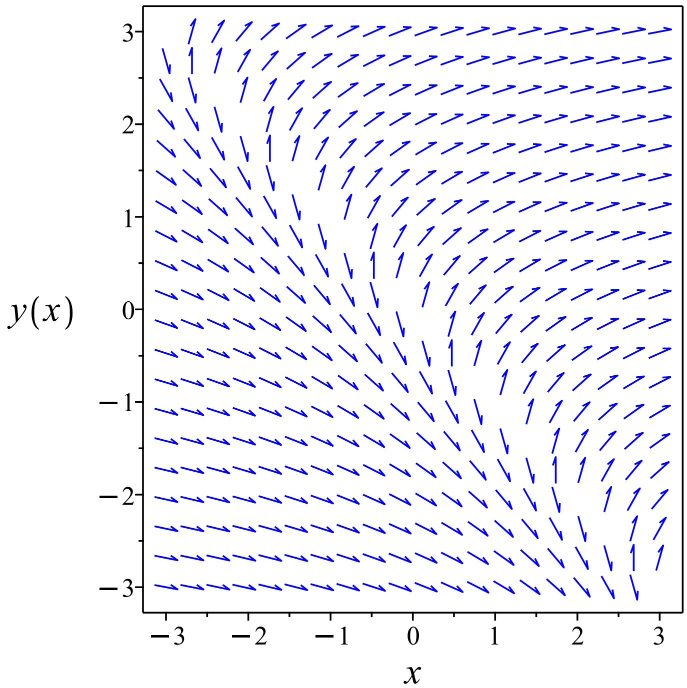


Figure 131: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-e^{-x+c_1-1}\right) - x - 1$$

Verified OK.

11.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(y + x)\, \mathrm{d}y &= \mathrm{d}x \\
-\mathrm{d}x + (y + x)\, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -1 \\
N(x, y) &= y + x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y + x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{y+x}((0)-(1))\\ &= -\frac{1}{y+x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -1((1)-(0))\\ &= -1\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B\,\mathrm{d}y}\\ &= e^{\int -1\,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-y}\\ &= \mathrm{e}^{-y}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-y}(-1)\\ &= -\mathrm{e}^{-y}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-y}(y+x)\\ &= (y+x)\,\mathrm{e}^{-y}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-y}\right) + \left((y+x)\,\mathrm{e}^{-y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^{-y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\mathrm{e}^{-y}x + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{-y}x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (y+x)\,\mathrm{e}^{-y}$. Therefore equation (4) becomes

$$
(y+x)\,\mathrm{e}^{-y} = \mathrm{e}^{-y}x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \mathrm{e}^{-y}y
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\mathrm{e}^{-y}y\right)\mathrm{d}y \\
f(y) &= -(y+1)\,\mathrm{e}^{-y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\mathrm{e}^{-y}x - (y+1)\,\mathrm{e}^{-y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\mathrm{e}^{-y}x - (y+1)\,\mathrm{e}^{-y}$$

The solution becomes

$$y = -\operatorname{LambertW}\left(c_1\mathrm{e}^{-x-1}\right) - x - 1$$

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(c_1\mathrm{e}^{-x-1}\right) - x - 1 \tag{1}$$

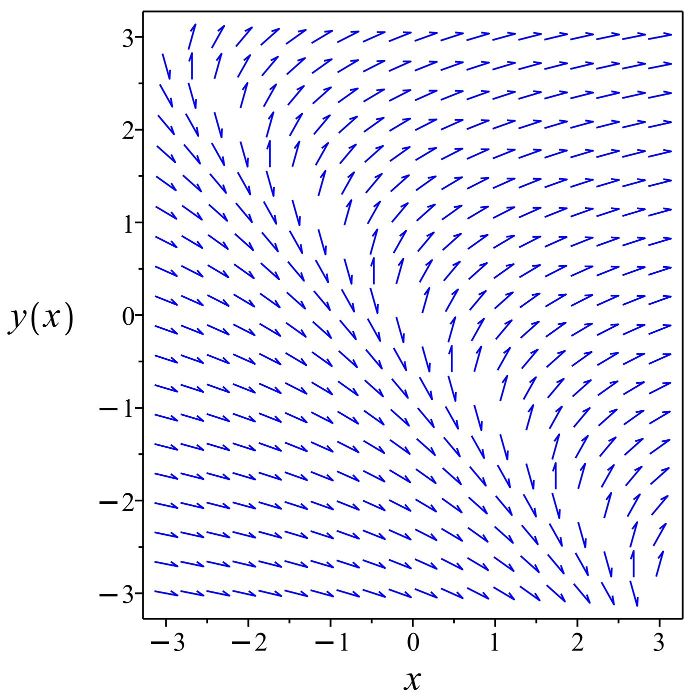


Figure 132: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(c_1\mathrm{e}^{-x-1}\right) - x - 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve((x+y(x))*diff(y(x),x)-1=0,y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}\left(-c_1 e^{-x-1}\right) - 1 - x$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 24

```
DSolve[(x+y[x])*y'[x]-1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -W\left(c_1\left(-e^{-x-1}\right)\right) - x - 1$$

11.3 problem Ex 3

Internal problem ID [11169]
Internal file name [`OUTPUT/10154_Saturday_December_03_2022_08_02_59_AM_16016325/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 18. Transformation of variables. Page 26
Problem number: Ex 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{yy' - y'x + y = -x}$$

11.3.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x(u'(x)\,x + u(x)) - (u'(x)\,x + u(x))\,x + u(x)\,x = -x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2+1}{x\,(u-1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2+1}{u-1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u^2+1)}{2} - \arctan(u) = -\ln(x) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2+1\right)}{2} - \arctan(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

<u>Summary</u>
The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

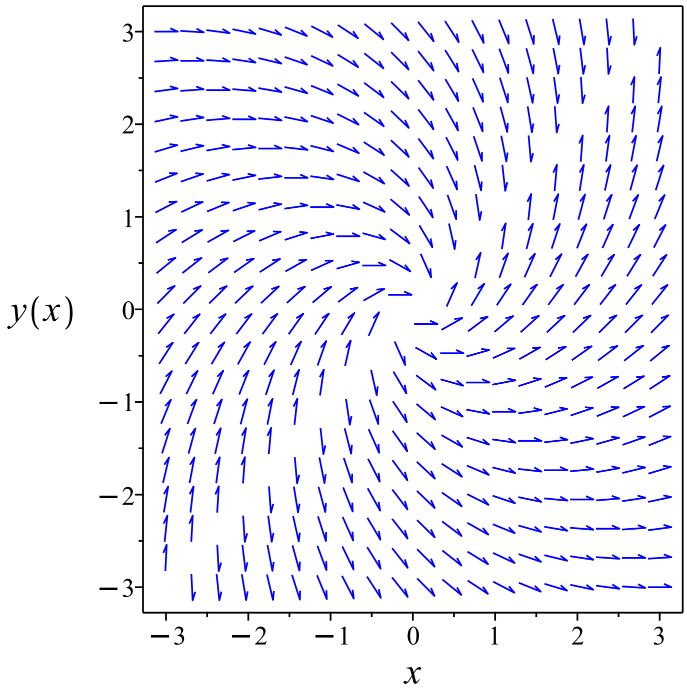


Figure 133: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2}-\arctan\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Verified OK.

11.3.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{y+x}{y-x}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$

$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(y+x)(b_3 - a_2)}{y-x} - \frac{(y+x)^2 a_3}{(y-x)^2} - \left(-\frac{1}{y-x} - \frac{y+x}{(y-x)^2}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{1}{y-x} + \frac{y+x}{(y-x)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 + 2xyb_2 + 2xyb_3 - y^2a_2 - y^2a_3 - y^2b_2 + y^2b_3 + 2xb_1 - 2ya}{(-y+x)^2} = 0$$

Setting the numerator to zero gives

$$-x^2a_2 - x^2a_3 - x^2b_2 + x^2b_3 + 2xya_2 - 2xya_3 - 2xyb_2 - 2xyb_3 + y^2a_2 + y^2a_3 + y^2b_2 - y^2b_3 - 2xb_1 + 2ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_1^2 + 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 - 2b_2v_1v_2 + b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 - b_3v_2^2 + 2a_1v_2 - 2b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-a_2 - a_3 - b_2 + b_3)\, v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3)\, v_1 v_2 \\ - 2b_1 v_1 + (a_2 + a_3 + b_2 - b_3)\, v_2^2 + 2a_1 v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}2a_1 &= 0\\ -2b_1 &= 0\\ -a_2 - a_3 - b_2 + b_3 &= 0\\ a_2 + a_3 + b_2 - b_3 &= 0\\ 2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= b_3\\ a_3 &= -b_2\\ b_1 &= 0\\ b_2 &= b_2\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi\\ &= y - \left(-\frac{y + x}{y - x}\right)(x)\\ &= \frac{-x^2 - y^2}{-y + x}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2-y^2}{-y+x}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y + x}{y - x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y + x}{x^2 + y^2} \\ S_y &= \frac{y - x}{x^2 + y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

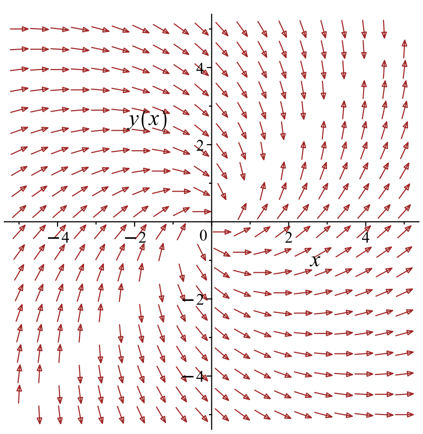
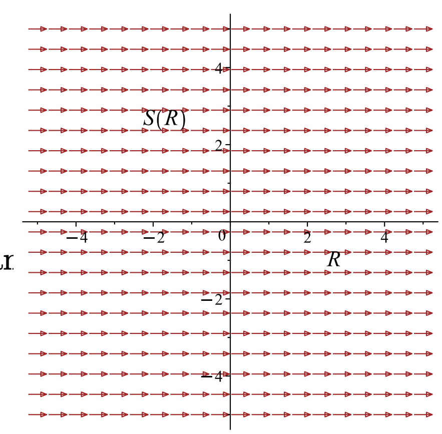
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Which simplifies to

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y+x}{y-x}$ 	$R = x$ $S = \frac{\ln(x^2+y^2)}{2} - \text{arctar}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln(x^2+y^2)}{2}-\arctan\left(\frac{y}{x}\right)=c_1 \tag{1}$$

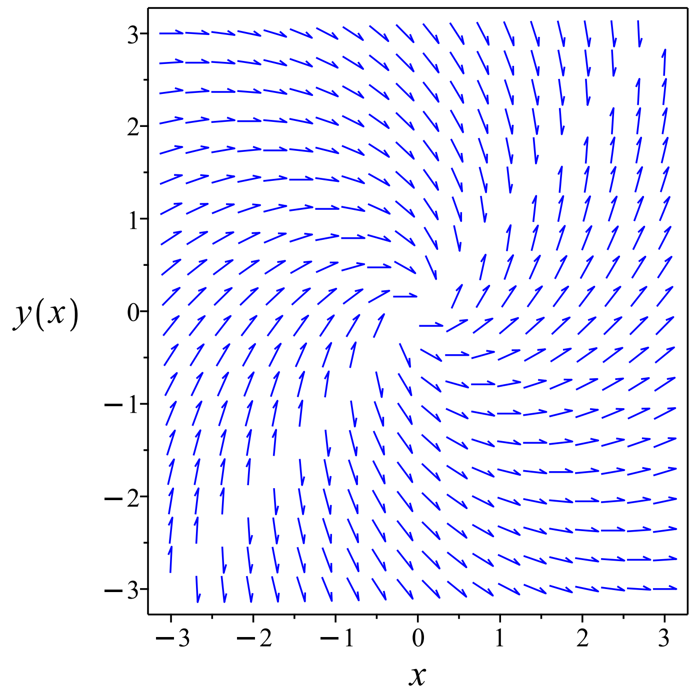


Figure 134: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+y^2)}{2}-\arctan\left(\frac{y}{x}\right)=c_1$$

Verified OK.

11.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y - x) \, \mathrm{d}y &= (-y - x) \, \mathrm{d}x \\ (y + x) \, \mathrm{d}x + (y - x) \, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y + x \\ N(x, y) &= y - x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y + x) \\ &= 1\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y-x) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = y + x$ and $N = y - x$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= \frac{y+x}{x^2+y^2} \\
N &= \frac{y-x}{x^2+y^2}
\end{aligned}
$$

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y-x}{x^2+y^2}\right)\mathrm{d}y &= \left(-\frac{y+x}{x^2+y^2}\right)\mathrm{d}x\\ \left(\frac{y+x}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{y-x}{x^2+y^2}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \frac{y+x}{x^2+y^2}\\ N(x,y) &= \frac{y-x}{x^2+y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y+x}{x^2+y^2}\right)\\ &= \frac{x^2-2xy-y^2}{(x^2+y^2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y-x}{x^2+y^2}\right)\\ &= \frac{x^2-2xy-y^2}{(x^2+y^2)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{y+x}{x^2+y^2} \,\mathrm{d}x$$

$$\phi = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{y}{x^2+y^2} - \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \tag{4}$$

$$= \frac{y-x}{x^2+y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y-x}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{y-x}{x^2+y^2} = \frac{y-x}{x^2+y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right)$$

Summary
The solution(s) found are the following

$$\frac{\ln(x^2+y^2)}{2}+\arctan\left(\frac{x}{y}\right)=c_1 \tag{1}$$

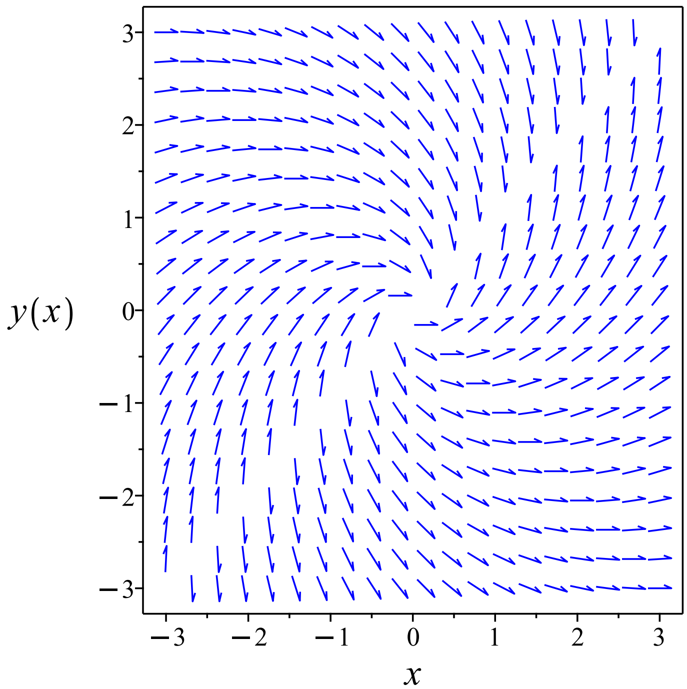


Figure 135: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+y^2)}{2}+\arctan\left(\frac{x}{y}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(x+y(x)*diff(y(x),x)+y(x)-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan\left(\mathrm{RootOf}\left(-2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x\right) + 2c_1\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 36

```
DSolve[x+y[x]*y'[x]+y[x]-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right)-\arctan\left(\frac{y(x)}{x}\right) = -\log(x)+c_1, y(x)\right]$$

11.4 problem Ex 4

Internal problem ID [11170]
Internal file name [`OUTPUT/10155_Saturday_December_03_2022_08_02_59_AM_16214539/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 18. Transformation of variables. Page 26
Problem number: Ex 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_rational, _Riccati]`

$$\boxed{y'x - ay + y^2 b = c\,x^{2a}}$$

11.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{ay - b\,y^2 + c\,x^{2a}}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{b\,y^2}{x} + \frac{c\,x^{2a}}{x} + \frac{ay}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{c\,x^{2a}}{x}$, $f_1(x) = \frac{a}{x}$ and $f_2(x) = -\frac{b}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{bu}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$f_2' = \frac{b}{x^2}$$
$$f_1 f_2 = -\frac{ba}{x^2}$$
$$f_2^2 f_0 = \frac{b^2 c\, x^{2a}}{x^3}$$

Substituting the above terms back in equation (2) gives

$$-\frac{b u''(x)}{x} - \left(\frac{b}{x^2} - \frac{ba}{x^2}\right) u'(x) + \frac{b^2 c\, x^{2a} u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(\frac{x^a \sqrt{-bc}}{a}\right) + c_2 \cos\left(\frac{x^a \sqrt{-bc}}{a}\right)$$

The above shows that

$$u'(x) = \frac{x^a \sqrt{-bc}\left(c_1 \cos\left(\frac{x^a \sqrt{-bc}}{a}\right) - c_2 \sin\left(\frac{x^a \sqrt{-bc}}{a}\right)\right)}{x}$$

Using the above in (1) gives the solution

$$y = \frac{x^a \sqrt{-bc}\left(c_1 \cos\left(\frac{x^a \sqrt{-bc}}{a}\right) - c_2 \sin\left(\frac{x^a \sqrt{-bc}}{a}\right)\right)}{b\left(c_1 \sin\left(\frac{x^a \sqrt{-bc}}{a}\right) + c_2 \cos\left(\frac{x^a \sqrt{-bc}}{a}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x^a \sqrt{-bc}\left(c_3 \cos\left(\frac{x^a \sqrt{-bc}}{a}\right) - \sin\left(\frac{x^a \sqrt{-bc}}{a}\right)\right)}{b\left(c_3 \sin\left(\frac{x^a \sqrt{-bc}}{a}\right) + \cos\left(\frac{x^a \sqrt{-bc}}{a}\right)\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{x^a \sqrt{-bc}\left(c_3 \cos\left(\frac{x^a \sqrt{-bc}}{a}\right) - \sin\left(\frac{x^a \sqrt{-bc}}{a}\right)\right)}{b\left(c_3 \sin\left(\frac{x^a \sqrt{-bc}}{a}\right) + \cos\left(\frac{x^a \sqrt{-bc}}{a}\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{x^a \sqrt{-bc}\left(c_3 \cos\left(\frac{x^a \sqrt{-bc}}{a}\right) - \sin\left(\frac{x^a \sqrt{-bc}}{a}\right)\right)}{b\left(c_3 \sin\left(\frac{x^a \sqrt{-bc}}{a}\right) + \cos\left(\frac{x^a \sqrt{-bc}}{a}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 35

```
dsolve(x*diff(y(x),x)-a*y(x)+b*y(x)^2=c*x^(2*a),y(x), singsol=all)
```

$$y(x) = \frac{\tanh\left(\frac{x^a \sqrt{b}\,\sqrt{c}+i c_1 a}{a}\right)\sqrt{c}\,x^a}{\sqrt{b}}$$

✓ Solution by Mathematica

Time used: 0.533 (sec). Leaf size: 153

```
DSolve[x*y'[x]-a*y[x]+b*y[x]^2==c*x^(2*a),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{c}x^a \left(-\cos\left(\frac{\sqrt{-b}\sqrt{c}x^a}{a}\right) + c_1 \sin\left(\frac{\sqrt{-b}\sqrt{c}x^a}{a}\right)\right)}{\sqrt{-b}\left(\sin\left(\frac{\sqrt{-b}\sqrt{c}x^a}{a}\right) + c_1 \cos\left(\frac{\sqrt{-b}\sqrt{c}x^a}{a}\right)\right)}$$

$$y(x) \to \frac{\sqrt{c}x^a \tan\left(\frac{\sqrt{-b}\sqrt{c}x^a}{a}\right)}{\sqrt{-b}}$$

12 Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29

12.1 problem Ex 1

Internal problem ID [11171]
Internal file name [`OUTPUT/10156_Saturday_December_03_2022_08_03_00_AM_96690858/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x\sqrt{1-y^2}+y\sqrt{-x^2+1}\,y'=0}$$

12.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x\sqrt{-y^2+1}}{y\sqrt{-x^2+1}} \end{aligned}$$

Where $f(x)=-\frac{x}{\sqrt{-x^2+1}}$ and $g(y)=\frac{\sqrt{-y^2+1}}{y}$. Integrating both sides gives

$$\frac{1}{\frac{\sqrt{-y^2+1}}{y}}\,dy=-\frac{x}{\sqrt{-x^2+1}}\,dx$$

$$\int \frac{1}{\frac{\sqrt{-y^2+1}}{y}}\, dy = \int -\frac{x}{\sqrt{-x^2+1}}\, dx$$
$$-\sqrt{-y^2+1} = \sqrt{-x^2+1} + c_1$$

The solution is

$$-\sqrt{1-y^2} - \sqrt{-x^2+1} - c_1 = 0$$

Summary

The solution(s) found are the following

$$-\sqrt{1-y^2} - \sqrt{-x^2+1} - c_1 = 0 \tag{1}$$

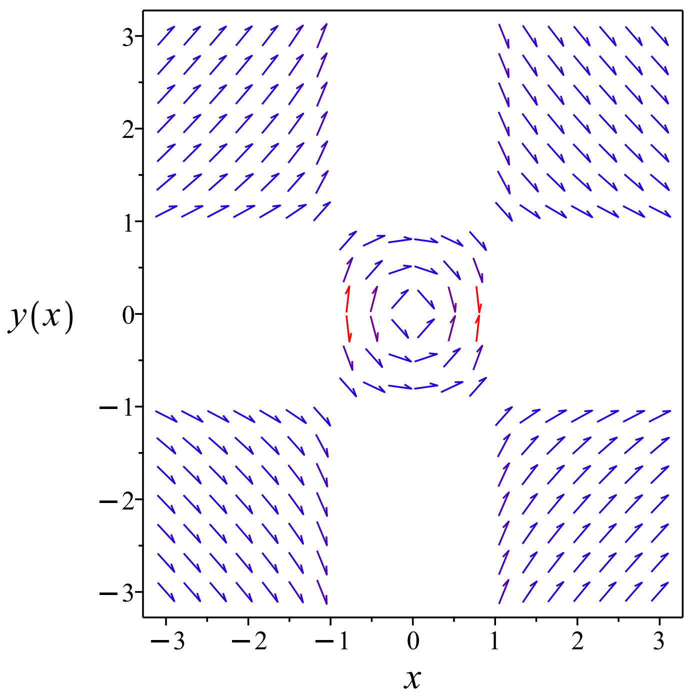


Figure 136: Slope field plot

Verification of solutions

$$-\sqrt{1-y^2} - \sqrt{-x^2+1} - c_1 = 0$$

Verified OK.

12.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{x\sqrt{-y^2+1}}{y\sqrt{-x^2+1}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 73: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{\sqrt{-x^2+1}}{x} \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{\sqrt{-x^2+1}}{x}} dx
\end{aligned}
$$

Which results in

$$
S = -\frac{(x-1)(1+x)}{\sqrt{-x^2+1}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{x\sqrt{-y^2+1}}{y\sqrt{-x^2+1}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{x}{\sqrt{-x^2+1}} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{y}{\sqrt{-y^2+1}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R}{\sqrt{-R^2+1}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{(R-1)(R+1)}{\sqrt{-R^2+1}} + c_1 \tag{4}
$$

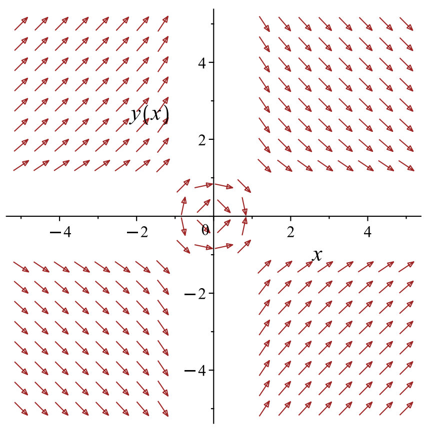
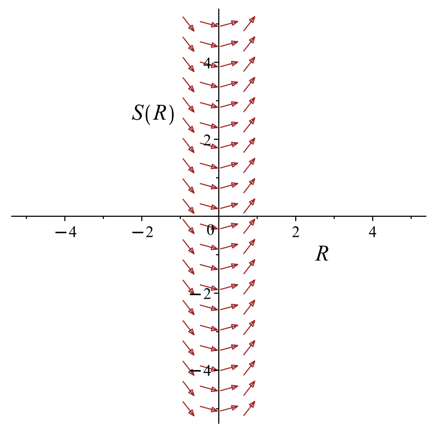
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\sqrt{-x^2+1} = \frac{(y-1)(y+1)}{\sqrt{1-y^2}} + c_1
$$

Which simplifies to

$$
\sqrt{-x^2+1} = \frac{(y-1)(y+1)}{\sqrt{1-y^2}} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x\sqrt{-y^2+1}}{y\sqrt{-x^2+1}}$ 	$R = y$ $S = \sqrt{-x^2+1}$	$\frac{dS}{dR} = \frac{R}{\sqrt{-R^2+1}}$ 

Summary

The solution(s) found are the following

$$\sqrt{-x^2+1} = \frac{(y-1)(y+1)}{\sqrt{1-y^2}} + c_1 \tag{1}$$

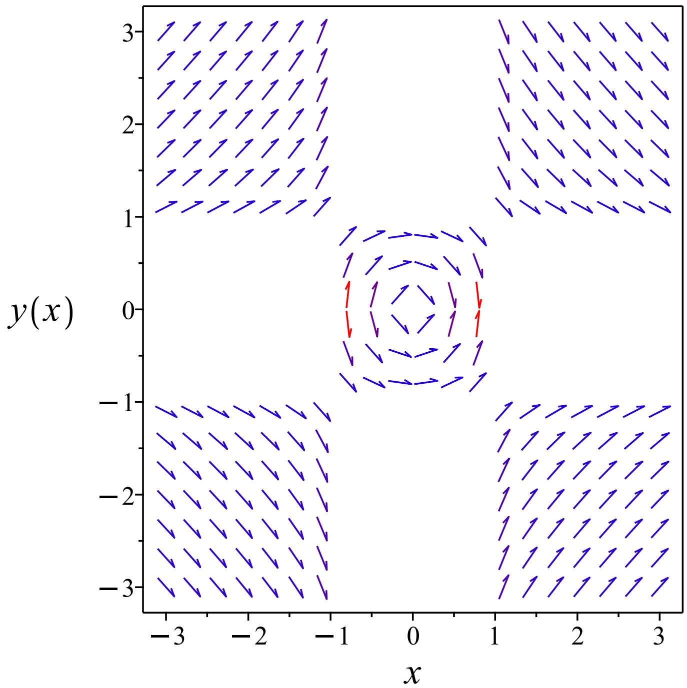


Figure 137: Slope field plot

Verification of solutions

$$\sqrt{-x^2+1} = \frac{(y-1)(y+1)}{\sqrt{1-y^2}} + c_1$$

Verified OK.

12.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{y}{\sqrt{-y^2+1}}\right) \mathrm{d}y = \left(\frac{x}{\sqrt{-x^2+1}}\right) \mathrm{d}x$$
$$\left(-\frac{x}{\sqrt{-x^2+1}}\right) \mathrm{d}x + \left(-\frac{y}{\sqrt{-y^2+1}}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x}{\sqrt{-x^2+1}}$$
$$N(x, y) = -\frac{y}{\sqrt{-y^2+1}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{\sqrt{-x^2+1}}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y}{\sqrt{-y^2+1}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M && (1) \\
\frac{\partial \phi}{\partial y} &= N && (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{x}{\sqrt{-x^2+1}} \,\mathrm{d}x \\
\phi &= \sqrt{-x^2+1} + f(y) && (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y}{\sqrt{-y^2+1}}$. Therefore equation (4) becomes

$$
-\frac{y}{\sqrt{-y^2+1}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{y}{\sqrt{-y^2+1}}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\frac{y}{\sqrt{-y^2+1}}\right) \mathrm{d}y \\
f(y) &= -\frac{(y-1)(y+1)}{\sqrt{-y^2+1}} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \sqrt{-x^2+1} - \frac{(y-1)(y+1)}{\sqrt{-y^2+1}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sqrt{-x^2+1} - \frac{(y-1)(y+1)}{\sqrt{-y^2+1}}$$

Summary

The solution(s) found are the following

$$\sqrt{-x^2+1} - \frac{(y-1)(y+1)}{\sqrt{1-y^2}} = c_1 \tag{1}$$

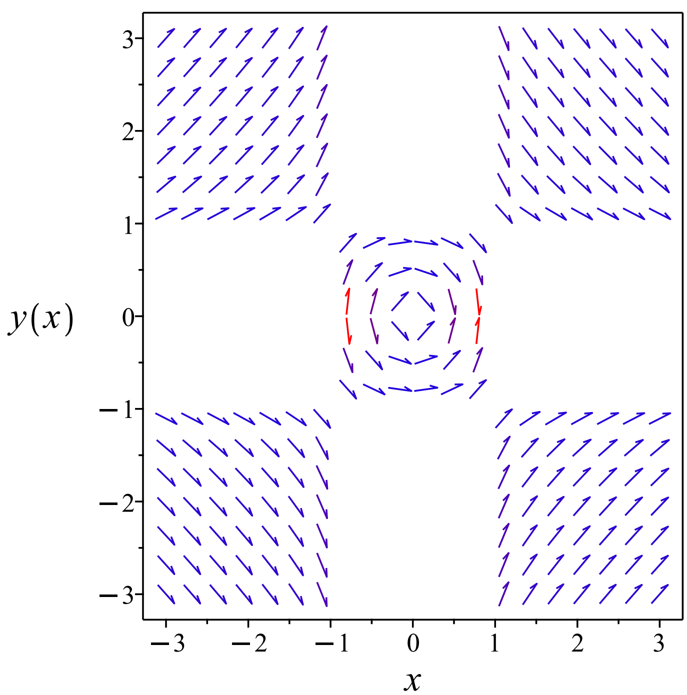


Figure 138: Slope field plot

Verification of solutions

$$\sqrt{-x^2+1} - \frac{(y-1)(y+1)}{\sqrt{1-y^2}} = c_1$$

Verified OK.

12.1.4 Maple step by step solution

Let's solve

$x\sqrt{1-y^2}+y\sqrt{-x^2+1}\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{yy'}{\sqrt{1-y^2}}=-\frac{x}{\sqrt{-x^2+1}}$

- Integrate both sides with respect to x

  $\int \frac{yy'}{\sqrt{1-y^2}}dx=\int -\frac{x}{\sqrt{-x^2+1}}dx+c_1$

- Evaluate integral

  $-\sqrt{1-y^2}=-\frac{(x-1)(1+x)}{\sqrt{-x^2+1}}+c_1$

- Solve for y

  $\left\{y=\sqrt{-2c_1\sqrt{-x^2+1}-c_1^2+x^2}, y=-\sqrt{-2c_1\sqrt{-x^2+1}-c_1^2+x^2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
dsolve(x*sqrt(1-y(x)^2)+y(x)*sqrt(1-x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{(-1+x)(1+x)}{\sqrt{-x^2+1}}+\frac{(-1+y(x))(y(x)+1)}{\sqrt{1-y(x)^2}}+c_1=0$$

✓ Solution by Mathematica

Time used: 3.778 (sec). Leaf size: 77

```
DSolve[x*Sqrt[1-y[x]^2]+y[x]*Sqrt[1-x^2]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x^2 - c_1 \left(2\sqrt{1-x^2} + c_1\right)}$$
$$y(x) \to \sqrt{x^2 - c_1 \left(2\sqrt{1-x^2} + c_1\right)}$$
$$y(x) \to -1$$
$$y(x) \to 1$$

12.2 problem Ex 2

Internal problem ID [11172]
Internal file name [`OUTPUT/10157_Saturday_December_03_2022_08_03_02_AM_46698786/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\sqrt{1-y^2}+\sqrt{-x^2+1}\, y' = 0}$$

12.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\sqrt{-y^2+1}}{\sqrt{-x^2+1}} \end{aligned}$$

Where $f(x) = -\frac{1}{\sqrt{-x^2+1}}$ and $g(y) = \sqrt{-y^2+1}$. Integrating both sides gives

$$\frac{1}{\sqrt{-y^2+1}}\, dy = -\frac{1}{\sqrt{-x^2+1}}\, dx$$

$$\int \frac{1}{\sqrt{-y^2+1}}dy = \int -\frac{1}{\sqrt{-x^2+1}}dx$$

$$\arcsin(y) = \frac{\sqrt{-(x-1)^2-2x+2}}{2} - \arcsin(x) - \frac{\sqrt{-(1+x)^2+2x+2}}{2} + c_1$$

Which results in

$$y = \sin(-\arcsin(x) + c_1)$$

Summary
The solution(s) found are the following

$$y = \sin(-\arcsin(x) + c_1) \tag{1}$$

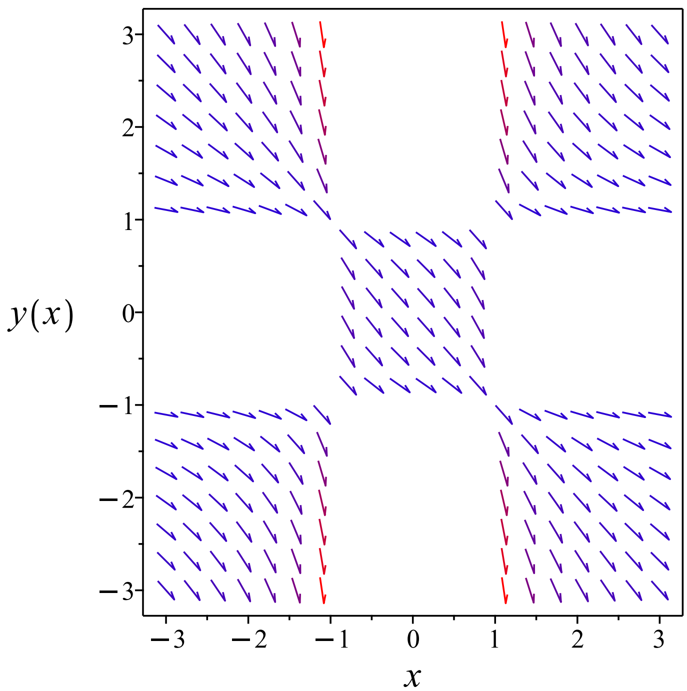


Figure 139: Slope field plot

Verification of solutions

$$y = \sin(-\arcsin(x) + c_1)$$

Verified OK.

12.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\sqrt{-y^2+1}}{\sqrt{-x^2+1}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 76: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\sqrt{-x^2+1} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\sqrt{-x^2+1}} dx
\end{aligned}
$$

Which results in

$$
S = -\arcsin(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{\sqrt{-y^2+1}}{\sqrt{-x^2+1}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{1}{\sqrt{-x^2+1}} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{\sqrt{-y^2+1}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \arcsin(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\arcsin(x) = \arcsin(y) + c_1
$$

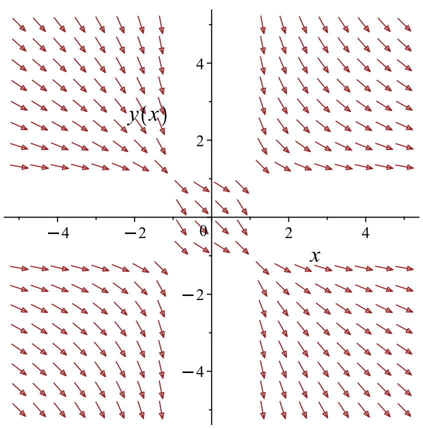
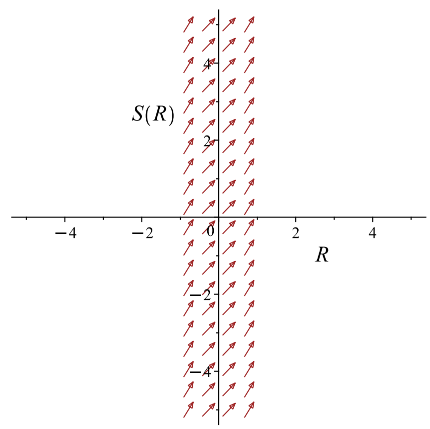
Which simplifies to

$$
-\arcsin(x) = \arcsin(y) + c_1
$$

Which gives

$$
y = -\sin(\arcsin(x) + c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sqrt{-y^2+1}}{\sqrt{-x^2+1}}$ 	$R = y$ $S = -\arcsin(x)$	$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}$ 

Summary
The solution(s) found are the following

$$y = -\sin(\arcsin(x) + c_1) \tag{1}$$

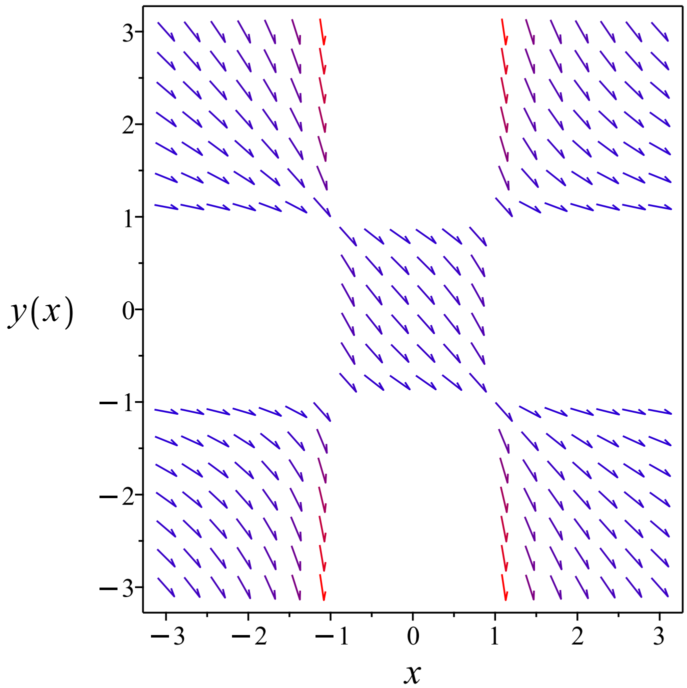


Figure 140: Slope field plot

Verification of solutions

$$y = -\sin\left(\arcsin\left(x\right) + c_1\right)$$

Verified OK.

12.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{\sqrt{-y^2+1}}\right) \mathrm{d}y &= \left(\frac{1}{\sqrt{-x^2+1}}\right) \mathrm{d}x \\
\left(-\frac{1}{\sqrt{-x^2+1}}\right) \mathrm{d}x + \left(-\frac{1}{\sqrt{-y^2+1}}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{\sqrt{-x^2+1}} \\
N(x, y) &= -\frac{1}{\sqrt{-y^2+1}}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{\sqrt{-x^2+1}}\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{\sqrt{-y^2+1}}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{\sqrt{-x^2+1}}\,\mathrm{d}x
\end{aligned}$$

$$\phi = -\arcsin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{\sqrt{-y^2+1}}$. Therefore equation (4) becomes

$$-\frac{1}{\sqrt{-y^2+1}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{\sqrt{-y^2+1}}$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{\sqrt{-y^2+1}}\right)\mathrm{d}y \\
f(y) &= -\arcsin(y) + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arcsin(x) - \arcsin(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arcsin(x) - \arcsin(y)$$

Summary

The solution(s) found are the following

$$-\arcsin(x) - \arcsin(y) = c_1 \tag{1}$$

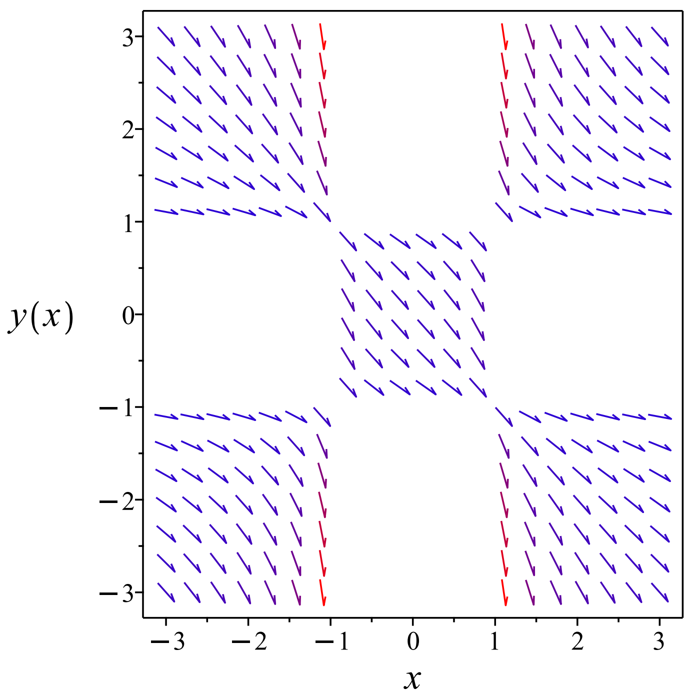


Figure 141: Slope field plot

Verification of solutions

$$-\arcsin(x) - \arcsin(y) = c_1$$

Verified OK.

12.2.4 Maple step by step solution

Let's solve

$\sqrt{1-y^2}+\sqrt{-x^2+1}\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}}=-\frac{1}{\sqrt{-x^2+1}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y^2}}dx=\int -\frac{1}{\sqrt{-x^2+1}}dx+c_1$

- Evaluate integral

  $\arcsin(y)=-\arcsin(x)+c_1$

- Solve for y

  $y=\sin(-\arcsin(x)+c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(sqrt(1-y(x)^2)+sqrt(1-x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=-\sin(\arcsin(x)+c_1)$$

✓ Solution by Mathematica

Time used: 0.496 (sec). Leaf size: 47

```
DSolve[Sqrt[1-y[x]^2]+Sqrt[1-x^2]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos\left(2\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right)+c_1\right)$$
$$y(x) \to -1$$
$$y(x) \to 1$$
$$y(x) \to \text{Interval}[\{-1,1\}]$$

12.3 problem Ex 3

Internal problem ID [11173]
Internal file name [`OUTPUT/10158_Saturday_December_03_2022_08_03_03_AM_39861667/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - x^2 y = x^5}$$

12.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -x^2$$
$$q(x) = x^5$$

Hence the ode is

$$y' - x^2 y = x^5$$

The integrating factor μ is

$$\mu = e^{\int -x^2 dx}$$
$$= e^{-\frac{x^3}{3}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x^5\right)$$
$$\frac{d}{dx}\left(e^{-\frac{x^3}{3}} y\right) = \left(e^{-\frac{x^3}{3}}\right)\left(x^5\right)$$
$$d\left(e^{-\frac{x^3}{3}} y\right) = \left(x^5 e^{-\frac{x^3}{3}}\right) dx$$

Integrating gives

$$e^{-\frac{x^3}{3}} y = \int x^5 e^{-\frac{x^3}{3}} dx$$
$$e^{-\frac{x^3}{3}} y = -\left(x^3 + 3\right) e^{-\frac{x^3}{3}} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x^3}{3}}$ results in

$$y = -e^{\frac{x^3}{3}}\left(x^3 + 3\right) e^{-\frac{x^3}{3}} + c_1 e^{\frac{x^3}{3}}$$

which simplifies to

$$y = -x^3 - 3 + c_1 e^{\frac{x^3}{3}}$$

Summary

The solution(s) found are the following

$$y = -x^3 - 3 + c_1 e^{\frac{x^3}{3}} \tag{1}$$

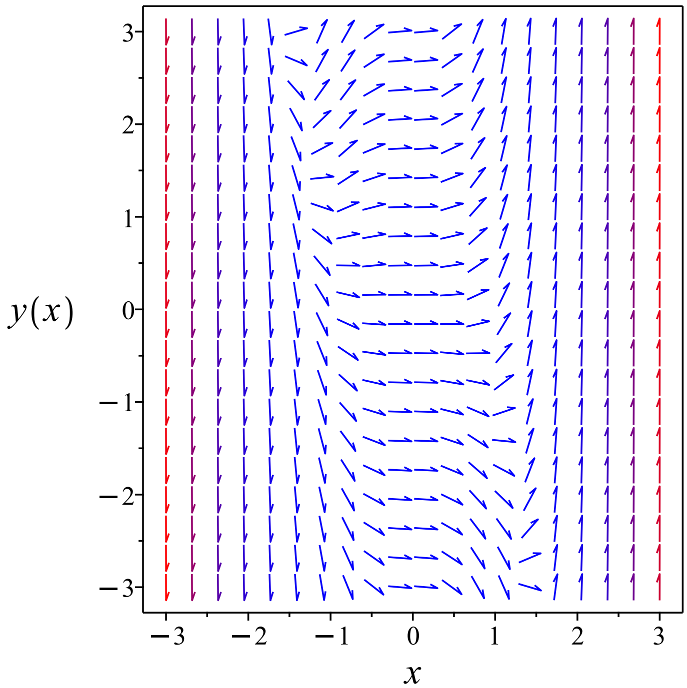


Figure 142: Slope field plot

Verification of solutions

$$y = -x^3 - 3 + c_1 e^{\frac{x^3}{3}}$$

Verified OK.

12.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= x^5 + x^2 y \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 79: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= e^{\frac{x^3}{3}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{x^3}{3}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{x^3}{3}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x^5 + x^2 y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -x^2 e^{-\frac{x^3}{3}} y \\ S_y &= e^{-\frac{x^3}{3}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^5 e^{-\frac{x^3}{3}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^5 e^{-\frac{R^3}{3}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\left(R^3 + 3\right) \mathrm{e}^{-\frac{R^3}{3}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-\frac{x^3}{3}} y = -\left(x^3 + 3\right) \mathrm{e}^{-\frac{x^3}{3}} + c_1$$

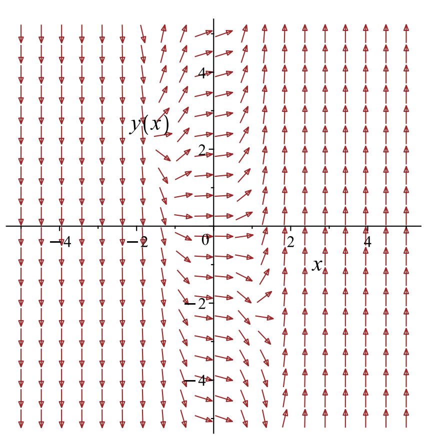
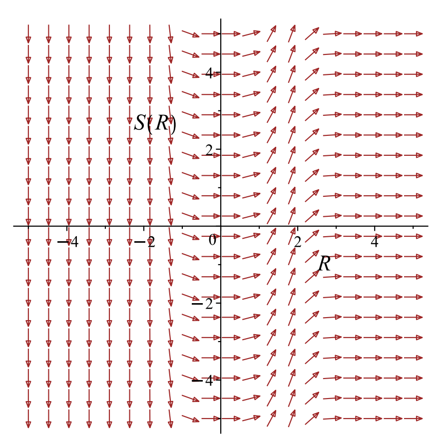
Which simplifies to

$$\left(x^3 + y + 3\right) \mathrm{e}^{-\frac{x^3}{3}} - c_1 = 0$$

Which gives

$$y = -\left(\mathrm{e}^{-\frac{x^3}{3}} x^3 + 3\,\mathrm{e}^{-\frac{x^3}{3}} - c_1\right) \mathrm{e}^{\frac{x^3}{3}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^5 + x^2 y$ 	$R = x$ $S = \mathrm{e}^{-\frac{x^3}{3}} y$	$\frac{dS}{dR} = R^5 \mathrm{e}^{-\frac{R^3}{3}}$ 

Summary

The solution(s) found are the following

$$y = -\left(\mathrm{e}^{-\frac{x^3}{3}} x^3 + 3\,\mathrm{e}^{-\frac{x^3}{3}} - c_1\right) \mathrm{e}^{\frac{x^3}{3}} \tag{1}$$

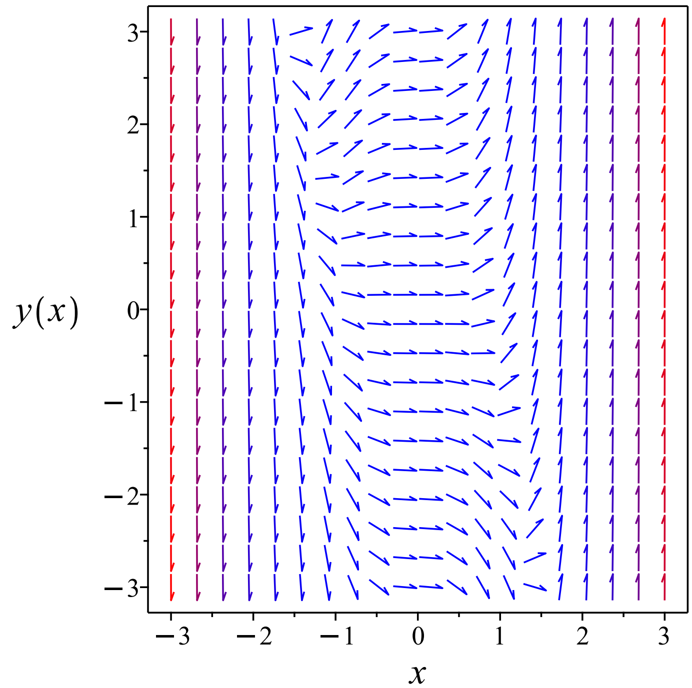


Figure 143: Slope field plot

Verification of solutions

$$y = -\left(e^{-\frac{x^3}{3}} x^3 + 3\,e^{-\frac{x^3}{3}} - c_1\right) e^{\frac{x^3}{3}}$$

Verified OK.

12.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(x^5 + x^2 y\right) \mathrm{d}x \\
\left(-x^5 - x^2 y\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x^5 - x^2 y \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^5 - x^2 y\right) \\
&= -x^2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-x^2\right) - (0)\right) \\
&= -x^2
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -x^2 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{x^3}{3}} \\
&= \mathrm{e}^{-\frac{x^3}{3}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-\frac{x^3}{3}}\left(-x^5 - x^2 y\right) \\
&= -x^2\left(x^3 + y\right)\mathrm{e}^{-\frac{x^3}{3}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\frac{x^3}{3}}(1) \\
&= \mathrm{e}^{-\frac{x^3}{3}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-x^2\left(x^3 + y\right)\mathrm{e}^{-\frac{x^3}{3}}\right) + \left(\mathrm{e}^{-\frac{x^3}{3}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -x^2 (x^3 + y) \,\mathrm{e}^{-\frac{x^3}{3}} \,\mathrm{d}x$$

$$\phi = (x^3 + y + 3) \,\mathrm{e}^{-\frac{x^3}{3}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{x^3}{3}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{x^3}{3}}$. Therefore equation (4) becomes

$$\mathrm{e}^{-\frac{x^3}{3}} = \mathrm{e}^{-\frac{x^3}{3}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (x^3 + y + 3) \,\mathrm{e}^{-\frac{x^3}{3}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (x^3 + y + 3) \,\mathrm{e}^{-\frac{x^3}{3}}$$

The solution becomes

$$y = -\left(\mathrm{e}^{-\frac{x^3}{3}} x^3 + 3\,\mathrm{e}^{-\frac{x^3}{3}} - c_1\right) \mathrm{e}^{\frac{x^3}{3}}$$

Summary

The solution(s) found are the following

$$y = -\left(\mathrm{e}^{-\frac{x^3}{3}} x^3 + 3\,\mathrm{e}^{-\frac{x^3}{3}} - c_1\right) \mathrm{e}^{\frac{x^3}{3}} \tag{1}$$

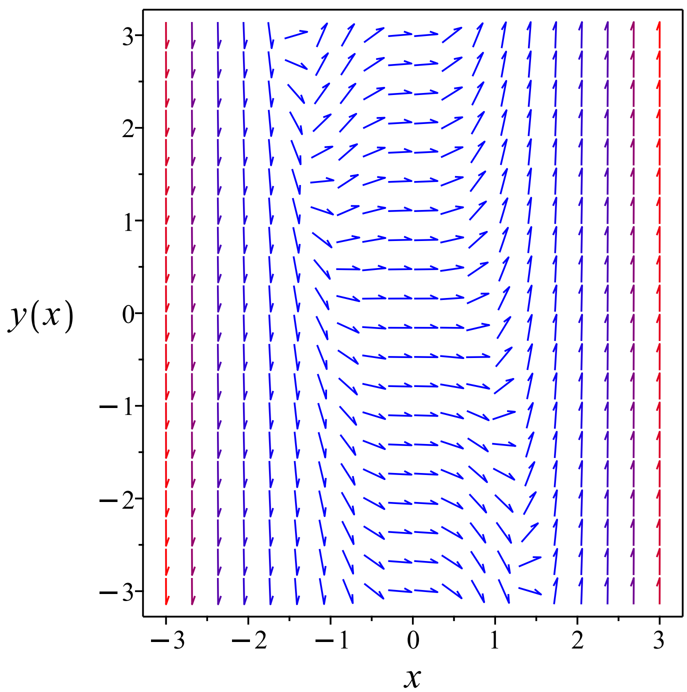


Figure 144: Slope field plot

Verification of solutions

$$y = -\left(\mathrm{e}^{-\frac{x^3}{3}} x^3 + 3\,\mathrm{e}^{-\frac{x^3}{3}} - c_1\right) \mathrm{e}^{\frac{x^3}{3}}$$

Verified OK.

12.3.4 Maple step by step solution

Let's solve

$y' - x^2 y = x^5$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = x^2 y + x^5$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - x^2 y = x^5$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - x^2 y\right) = \mu(x)\, x^5$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - x^2 y\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)\, x^2$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-\frac{x^3}{3}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^5 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^5 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^5 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{-\frac{x^3}{3}}$

  $y = \frac{\int x^5 \mathrm{e}^{-\frac{x^3}{3}} dx + c_1}{\mathrm{e}^{-\frac{x^3}{3}}}$

- Evaluate the integrals on the rhs

  $y = \frac{-(x^3+3)\mathrm{e}^{-\frac{x^3}{3}} + c_1}{\mathrm{e}^{-\frac{x^3}{3}}}$

- Simplify

  $$y = -x^3 - 3 + c_1 e^{\frac{x^3}{3}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)-x^2*y(x)=x^5,y(x), singsol=all)
```

$$y(x) = -x^3 - 3 + e^{\frac{x^3}{3}} c_1$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 24

```
DSolve[y'[x]-x^2*y[x]==x^5,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^3 + c_1 e^{\frac{x^3}{3}} - 3$$

12.4 problem Ex 4

Internal problem ID [11174]
Internal file name [OUTPUT/10159_Saturday_December_03_2022_08_03_03_AM_61324329/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{(y-x)^2 y' = 1}$$

12.4.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{1}{x^2 - 2xy + y^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{b_3 - a_2}{x^2 - 2xy + y^2} - \frac{a_3}{(x^2 - 2xy + y^2)^2} + \frac{(2x - 2y)(xa_2 + ya_3 + a_1)}{(x^2 - 2xy + y^2)^2} + \frac{(-2x + 2y)(xb_2 + yb_3 + b_1)}{(x^2 - 2xy + y^2)^2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^4b_2 - 4x^3yb_2 + 6x^2y^2b_2 - 4x\,y^3b_2 + y^4b_2 + x^2a_2 - 2x^2b_2 + x^2b_3 + 2xya_3 + 2xyb_2 - 4xyb_3 - y^2a_2 - 2y^2a_3}{(x^2 - 2xy + y^2)^2} = 0$$

Setting the numerator to zero gives

$$x^4b_2 - 4x^3yb_2 + 6x^2y^2b_2 - 4x\,y^3b_2 + y^4b_2 + x^2a_2 - 2x^2b_2 + x^2b_3 + 2xya_3 + 2xyb_2 - 4xyb_3 - y^2a_2 - 2y^2a_3 + 3y^2b_3 + 2xa_1 - 2xb_1 - 2ya_1 + 2yb_1 - a_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$b_2v_1^4 - 4b_2v_1^3v_2 + 6b_2v_1^2v_2^2 - 4b_2v_1v_2^3 + b_2v_2^4 + a_2v_1^2 - a_2v_2^2 + 2a_3v_1v_2 - 2a_3v_2^2 - 2b_2v_1^2 + 2b_2v_1v_2 + b_3v_1^2 - 4b_3v_1v_2 + 3b_3v_2^2 + 2a_1v_1 - 2a_1v_2 - 2b_1v_1 + 2b_1v_2 - a_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$b_2 v_1^4 - 4b_2 v_1^3 v_2 + 6b_2 v_1^2 v_2^2 + (a_2 - 2b_2 + b_3) v_1^2 - 4b_2 v_1 v_2^3 + (2a_3 + 2b_2 - 4b_3) v_1 v_2 + (2a_1 - 2b_1) v_1 + b_2 v_2^4 + (-a_2 - 2a_3 + 3b_3) v_2^2 + (-2a_1 + 2b_1) v_2 - a_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ -a_3 &= 0 \\ -4b_2 &= 0 \\ 6b_2 &= 0 \\ -2a_1 + 2b_1 &= 0 \\ 2a_1 - 2b_1 &= 0 \\ -a_2 - 2a_3 + 3b_3 &= 0 \\ a_2 - 2b_2 + b_3 &= 0 \\ 2a_3 + 2b_2 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\,\xi \\ &= 1 - \left(\frac{1}{x^2 - 2xy + y^2}\right)(1) \\ &= \frac{x^2 - 2xy + y^2 - 1}{x^2 - 2xy + y^2} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2-2xy+y^2-1}{x^2-2xy+y^2}} dy \end{aligned}$$

Which results in

$$S = y - \frac{\ln(y - x + 1)}{2} + \frac{\ln(y - 1 - x)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1}{x^2 - 2xy + y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{(x - y + 1)(x - y - 1)} \\ S_y &= \frac{(-y + x)^2}{(x - y + 1)(x - y - 1)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

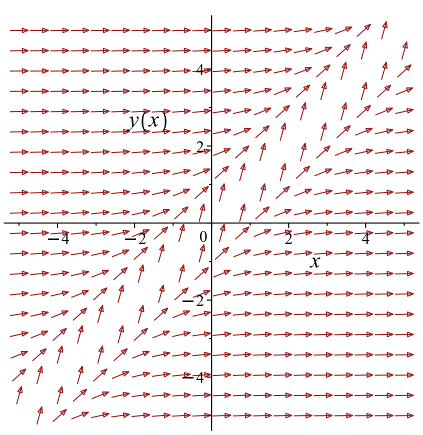
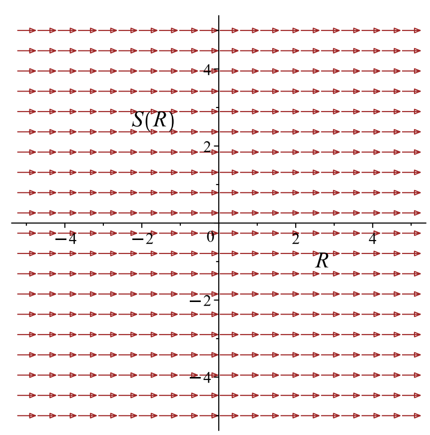
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y - \frac{\ln(y - x + 1)}{2} + \frac{\ln(y - 1 - x)}{2} = c_1$$

Which simplifies to

$$y - \frac{\ln(y - x + 1)}{2} + \frac{\ln(y - 1 - x)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1}{x^2 - 2xy + y^2}$ 	$R = x$ $S = y - \frac{\ln(y - x + 1)}{2} +$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y-\frac{\ln \left(y-x+1\right)}{2}+\frac{\ln \left(y-1-x\right)}{2}=c_1 \tag{1}$$

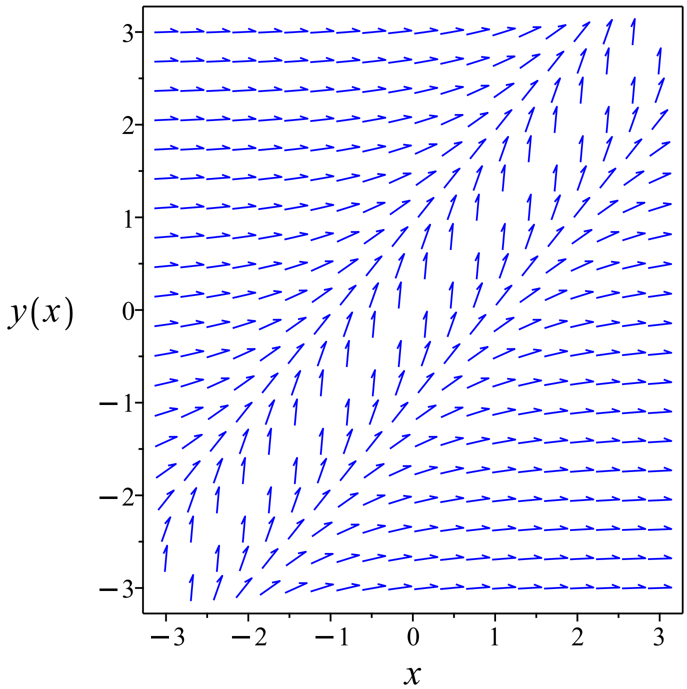


Figure 145: Slope field plot

Verification of solutions

$$y-\frac{\ln \left(y-x+1\right)}{2}+\frac{\ln \left(y-1-x\right)}{2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 29

```
dsolve((y(x)-x)^2*diff(y(x),x)=1,y(x), singsol=all)
```

$$y(x)+\frac{\ln\left(y(x)-x-1\right)}{2}-\frac{\ln\left(y(x)-x+1\right)}{2}-c_1=0$$

✓ Solution by Mathematica

Time used: 0.23 (sec). Leaf size: 33

```
DSolve[(y[x]-x)^2*y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[y(x)+\frac{1}{2}\log(-y(x)+x+1)-\frac{1}{2}\log(y(x)-x+1)=c_1,y(x)\right]$$

12.5 problem Ex 5

Internal problem ID [11175]
Internal file name [`OUTPUT/10160_Saturday_December_03_2022_08_03_04_AM_17239589/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y'x + y + e^x x^4 y^4 = 0}$$

12.5.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(e^x x^4 y^3 + 1)}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 82: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^4 x^3
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^4 x^3} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{3x^3 y^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(\mathrm{e}^x x^4 y^3 + 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x^4 y^3} \\ S_y &= \frac{1}{x^3 y^4} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\mathrm{e}^x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\mathrm{e}^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -e^R + c_1 \tag{4}$$

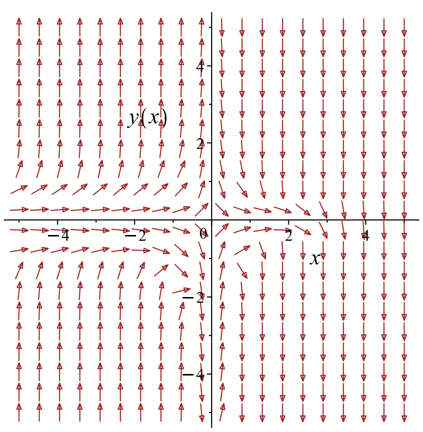
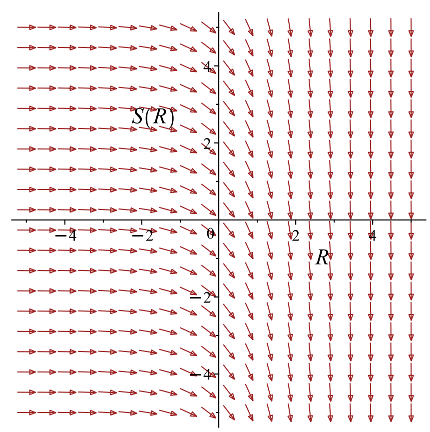
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{3x^3y^3} = -e^x + c_1$$

Which simplifies to

$$-\frac{1}{3x^3y^3} = -e^x + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(e^x x^4 y^3+1)}{x}$ 	$R = x$ $S = -\frac{1}{3x^3y^3}$	$\frac{dS}{dR} = -e^R$ 

Summary
The solution(s) found are the following

$$-\frac{1}{3x^3y^3} = -e^x + c_1 \tag{1}$$

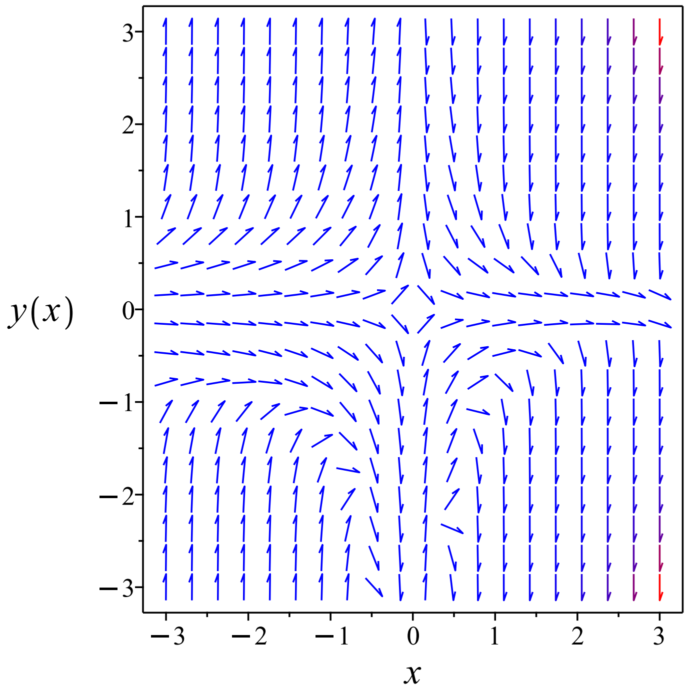


Figure 146: Slope field plot

Verification of solutions

$$-\frac{1}{3x^3y^3} = -e^x + c_1$$

Verified OK.

12.5.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(e^x x^4 y^3 + 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - x^3 e^x y^4 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= -x^3 e^x \\
n &= 4
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^4$ gives

$$
y' \frac{1}{y^4} = -\frac{1}{x\, y^3} - x^3 e^x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^3}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{3}{y^4} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{3} &= -\frac{w(x)}{x} - x^3 e^x \\
w' &= \frac{3w}{x} + 3x^3 e^x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= 3x^3 e^x
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{3w(x)}{x} = 3x^3 e^x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{3}{x}dx} \\ &= \frac{1}{x^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(3x^3 e^x\right) \\ \frac{d}{dx}\left(\frac{w}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(3x^3 e^x\right) \\ d\left(\frac{w}{x^3}\right) &= \left(3\, e^x\right)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^3} &= \int 3\, e^x\, dx \\ \frac{w}{x^3} &= 3\, e^x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$w(x) = 3x^3 e^x + c_1 x^3$$

which simplifies to

$$w(x) = x^3(3\, e^x + c_1)$$

Replacing w in the above by $\frac{1}{y^3}$ using equation (5) gives the final solution.

$$\frac{1}{y^3} = x^3(3\, e^x + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{1}{\left(3\, e^x + c_1\right)^{\frac{1}{3}}\, x} \\ y(x) &= \frac{i\sqrt{3} - 1}{2\left(3\, e^x + c_1\right)^{\frac{1}{3}}\, x} \\ y(x) &= -\frac{1 + i\sqrt{3}}{2\left(3\, e^x + c_1\right)^{\frac{1}{3}}\, x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\left(3\,\mathrm{e}^{x}+c_1\right)^{\frac{1}{3}} x} \tag{1}$$

$$y = \frac{i\sqrt{3}-1}{2\left(3\,\mathrm{e}^{x}+c_1\right)^{\frac{1}{3}} x} \tag{2}$$

$$y = -\frac{1+i\sqrt{3}}{2\left(3\,\mathrm{e}^{x}+c_1\right)^{\frac{1}{3}} x} \tag{3}$$

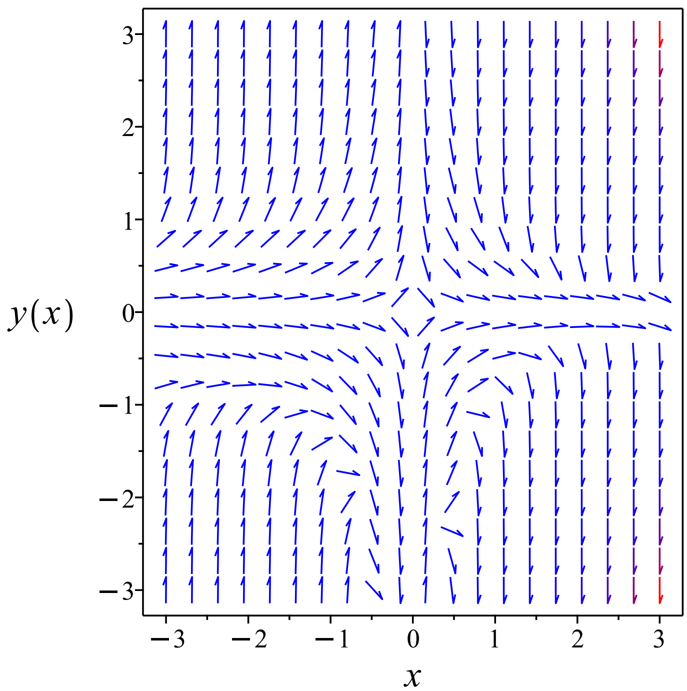


Figure 147: Slope field plot

Verification of solutions

$$y = \frac{1}{\left(3\,\mathrm{e}^{x}+c_1\right)^{\frac{1}{3}} x}$$

Verified OK.

$$y = \frac{i\sqrt{3}-1}{2\left(3\,\mathrm{e}^{x}+c_1\right)^{\frac{1}{3}} x}$$

Verified OK.

$$y = -\frac{1+i\sqrt{3}}{2\left(3\,\mathrm{e}^{x}+c_1\right)^{\frac{1}{3}} x}$$

Verified OK.

12.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(-y - \mathrm{e}^{x}x^4y^4\right)\mathrm{d}x\\ \left(y + \mathrm{e}^{x}x^4y^4\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y + \mathrm{e}^{x}x^4y^4\\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y + \mathrm{e}^{x}x^4y^4\right)\\ &= 1 + 4\,\mathrm{e}^{x}x^4y^3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}\left(\left(1 + 4\,\mathrm{e}^{x}x^4y^3\right) - (1)\right)\\ &= 4\,\mathrm{e}^{x}x^3y^3\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y\left(e^{x}x^{4}y^{3}+1\right)}\left((1) - \left(1 + 4\,e^{x}x^{4}y^{3}\right)\right) \\
&= -\frac{4y^{2}e^{x}x^{4}}{e^{x}x^{4}y^{3}+1}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - \left(1 + 4\,e^{x}x^{4}y^{3}\right)}{x\left(y + e^{x}x^{4}y^{4}\right) - y\left(x\right)} \\
&= -\frac{4}{xy}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{4}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R\,\mathrm{d}t} \\
&= e^{\int \left(-\frac{4}{t}\right)\mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-4\ln(t)} \\
&= \frac{1}{t^{4}}
\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^{4}y^{4}}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^4 y^4} \left(y + \mathrm{e}^x x^4 y^4\right) \\
&= \frac{\mathrm{e}^x x^4 y^3 + 1}{x^4 y^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^4 y^4} (x) \\
&= \frac{1}{x^3 y^4}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{\mathrm{e}^x x^4 y^3 + 1}{x^4 y^3}\right) + \left(\frac{1}{x^3 y^4}\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \frac{\mathrm{e}^x x^4 y^3 + 1}{x^4 y^3} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{3\,\mathrm{e}^x x^3 y^3 - 1}{3 x^3 y^3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{3\,\mathrm{e}^{x}}{y}-\frac{3\,\mathrm{e}^{x} x^{3} y^{3}-1}{x^{3} y^{4}}+f'(y) \qquad (4)\\
&= \frac{1}{x^{3} y^{4}}+f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{1}{x^{3} y^{4}}$. Therefore equation (4) becomes

$$
\frac{1}{x^{3} y^{4}} = \frac{1}{x^{3} y^{4}}+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{3\,\mathrm{e}^{x} x^{3} y^{3}-1}{3 x^{3} y^{3}}+c_1
$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{3\,\mathrm{e}^{x} x^{3} y^{3}-1}{3 x^{3} y^{3}}
$$

Summary

The solution(s) found are the following

$$
\frac{3\,\mathrm{e}^{x} x^{3} y^{3}-1}{3 x^{3} y^{3}} = c_1 \tag{1}
$$

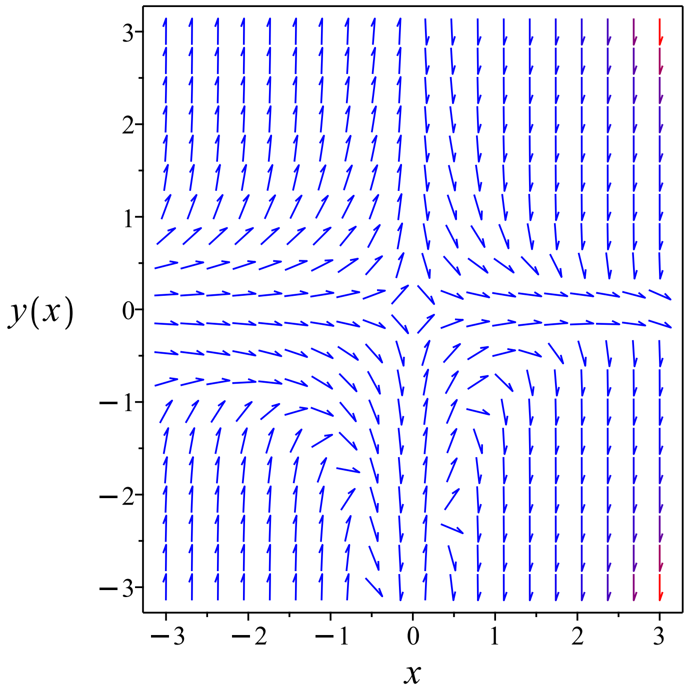


Figure 148: Slope field plot

Verification of solutions

$$\frac{3\,\mathrm{e}^{x} x^{3} y^{3}-1}{3 x^{3} y^{3}} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 64

```
dsolve(x*diff(y(x),x)+y(x)+x^4*y(x)^4*exp(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{(3\,\mathrm{e}^{x}+c_1)^{\frac{1}{3}}\,x}$$
$$y(x) = -\frac{1+i\sqrt{3}}{2\,(3\,\mathrm{e}^{x}+c_1)^{\frac{1}{3}}\,x}$$
$$y(x) = \frac{i\sqrt{3}-1}{2\,(3\,\mathrm{e}^{x}+c_1)^{\frac{1}{3}}\,x}$$

✓ Solution by Mathematica

Time used: 11.276 (sec). Leaf size: 79

```
DSolve[x*y'[x]+y[x]+x^4*y[x]^4*Exp[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{\sqrt[3]{x^3\,(3e^x+c_1)}}$$
$$y(x) \to -\frac{\sqrt[3]{-1}}{\sqrt[3]{x^3\,(3e^x+c_1)}}$$
$$y(x) \to \frac{(-1)^{2/3}}{\sqrt[3]{x^3\,(3e^x+c_1)}}$$
$$y(x) \to 0$$

12.6 problem Ex 6

Internal problem ID [11176]
Internal file name [`OUTPUT/10161_Saturday_December_03_2022_08_03_05_AM_61746487/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(1-x)\,y+(1-y)\,xy'=0}$$

12.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= -\frac{y(x-1)}{(y-1)\,x} \end{aligned}$$

Where $f(x)=-\frac{x-1}{x}$ and $g(y)=\frac{y}{y-1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{y-1}}\,dy=-\frac{x-1}{x}\,dx$$

$$\int \frac{1}{\frac{y}{y-1}} dy = \int -\frac{x-1}{x} dx$$

$$y - \ln(y) = -x + \ln(x) + c_1$$

Which results in

$$y = -\operatorname{LambertW}\left(-\frac{e^{x-c_1}}{x}\right)$$

Since c_1 is constant, then exponential powers of this constant are constants also, and these can be simplified to just c_1 in the above solution. Which simplifies to

$$y = -\operatorname{LambertW}\left(-\frac{e^{x-c_1}}{x}\right)$$

gives

$$y = -\operatorname{LambertW}\left(-\frac{e^{x}}{c_1 x}\right)$$

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{e^{x}}{c_1 x}\right) \tag{1}$$

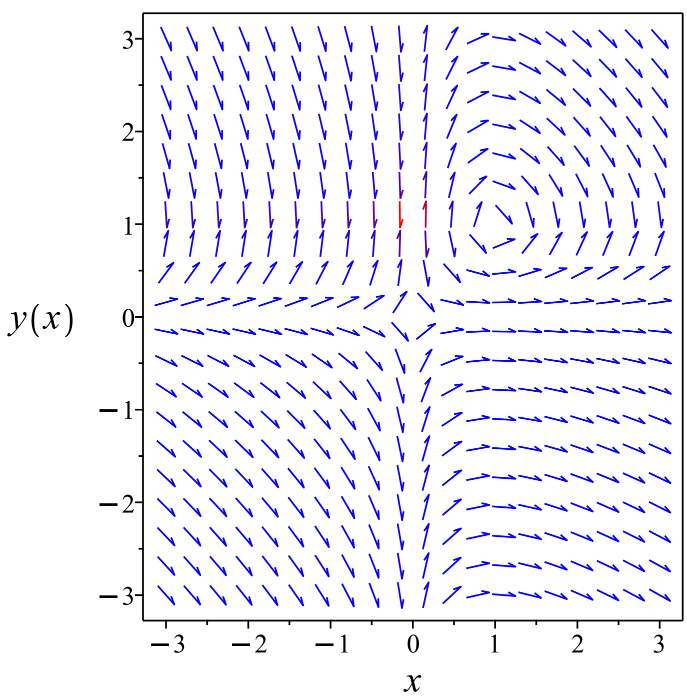


Figure 149: Slope field plot

Verification of solutions

$$
y = -\operatorname{LambertW}\left(-\frac{\mathrm{e}^{x}}{c_1 x}\right)
$$

Verified OK.

12.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y(x-1)}{(y-1)\,x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 84: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{x}{x-1} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{x}{x-1}} dx \end{aligned}$$

Which results in

$$S = -x + \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(x-1)}{(y-1)\,x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -1 + \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y-1}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R-1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R - \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-x + \ln(x) = y - \ln(y) + c_1$$

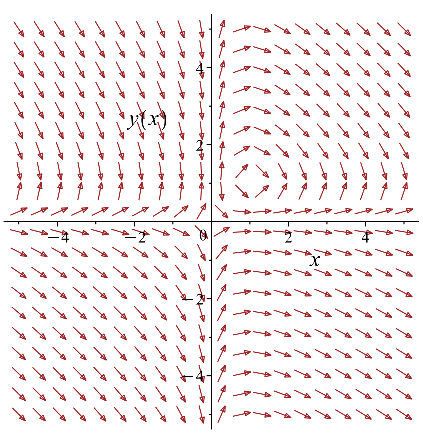
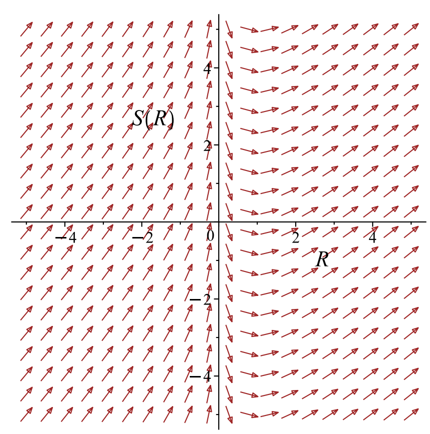
Which simplifies to

$$-x + \ln(x) = y - \ln(y) + c_1$$

Which gives

$$y = -\operatorname{LambertW}\left(-\frac{e^{x+c_1}}{x}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(x-1)}{(y-1)x}$ 	$R = y$ $S = -x + \ln(x)$	$\frac{dS}{dR} = \frac{R-1}{R}$ 

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{e^{x+c_1}}{x}\right) \tag{1}$$

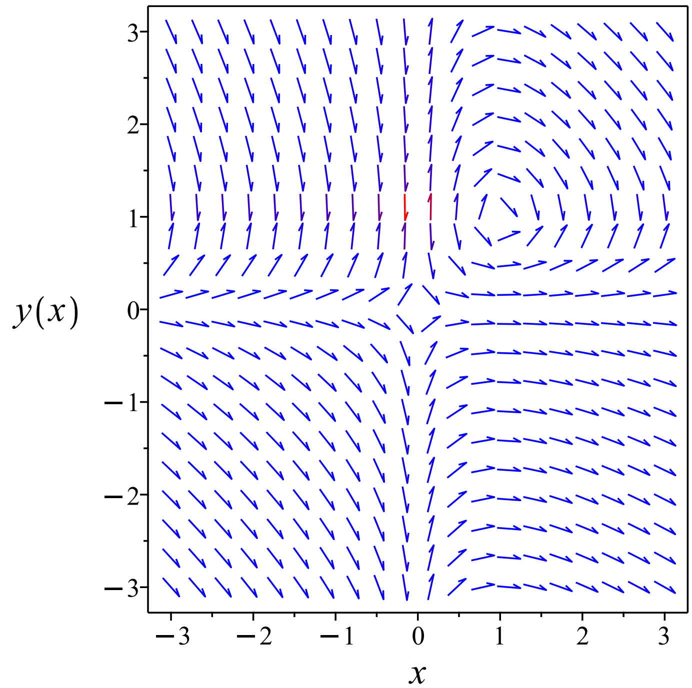


Figure 150: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-\frac{e^{x+c_1}}{x}\right)$$

Verified OK.

12.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{y-1}{y}\right)\mathrm{d}y &= \left(\frac{x-1}{x}\right)\mathrm{d}x \\ \left(-\frac{x-1}{x}\right)\mathrm{d}x + \left(-\frac{y-1}{y}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{x-1}{x} \\ N(x,y) &= -\frac{y-1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x-1}{x}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y-1}{y}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x-1}{x}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -x + \ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y-1}{y}$. Therefore equation (4) becomes

$$
-\frac{y-1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{y-1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1-y}{y}\right)\mathrm{d}y\\
f(y) &= -y + \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \ln(x) - y + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \ln(x) - y + \ln(y)$$

The solution becomes

$$y = -\operatorname{LambertW}\left(-\frac{e^{x+c_1}}{x}\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{e^{x+c_1}}{x}\right) \tag{1}$$

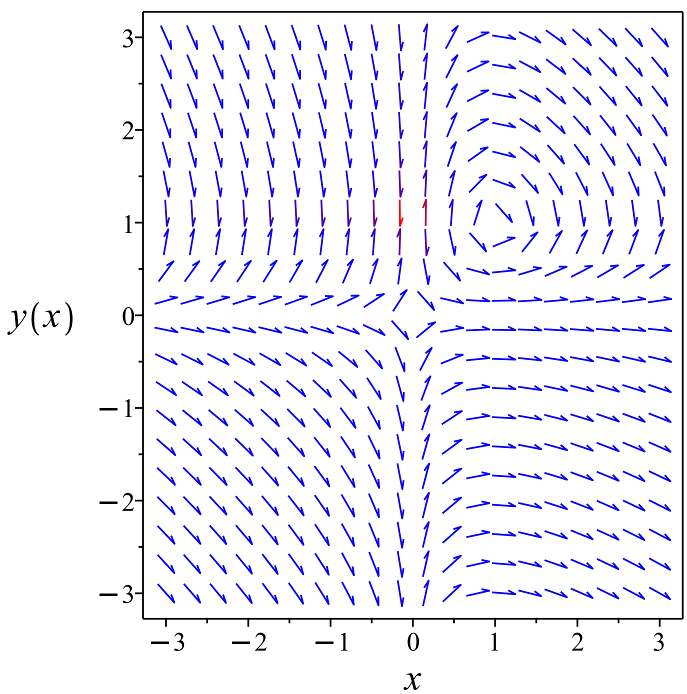


Figure 151: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-\frac{e^{x+c_1}}{x}\right)$$

Verified OK.

12.6.4 Maple step by step solution

Let's solve

$(1-x)\,y + (1-y)\,xy' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(1-y)}{y} = -\frac{1-x}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'(1-y)}{y}dx = \int -\frac{1-x}{x}dx + c_1$

- Evaluate integral

  $-y + \ln(y) = x - \ln(x) + c_1$

- Solve for y

  $y = -\mathit{LambertW}\left(-\frac{e^{x+c_1}}{x}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve((1-x)*y(x)+(1-y(x))*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}\left(-\frac{c_1 e^x}{x}\right)$$

✓ Solution by Mathematica

Time used: 4.764 (sec). Leaf size: 26

```
DSolve[(1-x)*y[x]+(1-y[x])*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -W\left(-\frac{e^{x-c_1}}{x}\right)$$
$$y(x) \to 0$$

12.7 problem Ex 7

Internal problem ID [11177]
Internal file name [`OUTPUT/10162_Saturday_December_03_2022_08_03_06_AM_83404105/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(y-x)\,y'+y=0}$$

12.7.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u(x)\,x-x)\,(u'(x)\,x+u(x))+u(x)\,x=0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{u^2}{x\,(u-1)}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{u-1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u^2}{u-1}}\, du = \int -\frac{1}{x}\, dx$$

$$\frac{1}{u} + \ln(u) = -\ln(x) + c_2$$

The solution is

$$\frac{1}{u(x)} + \ln(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{x}{y} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

$$\frac{x}{y} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{x}{y} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

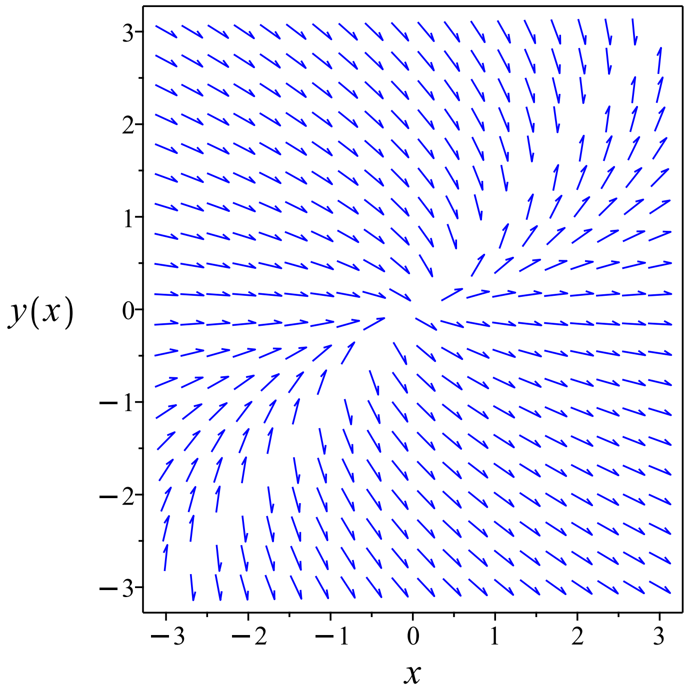


Figure 152: Slope field plot

Verification of solutions

$$\frac{x}{y} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Verified OK.

12.7.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y}{y - x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(b_3 - a_2)}{y - x} - \frac{y^2 a_3}{(y - x)^2} + \frac{y(x a_2 + y a_3 + a_1)}{(y - x)^2} - \left(\frac{y}{(y - x)^2} - \frac{1}{y - x}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2xyb_2 - y^2 a_2 - y^2 b_2 + y^2 b_3 + x b_1 - y a_1}{(-y + x)^2} = 0$$

Setting the numerator to zero gives

$$-2xyb_2 + y^2 a_2 + y^2 b_2 - y^2 b_3 - x b_1 + y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$a_2 v_2^2 - 2b_2 v_1 v_2 + b_2 v_2^2 - b_3 v_2^2 + a_1 v_2 - b_1 v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-2b_2 v_1 v_2 - b_1 v_1 + (a_2 + b_2 - b_3) v_2^2 + a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
-b_1 &= 0 \\
-2b_2 &= 0 \\
a_2 + b_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{y}{y - x}\right)(x) \\
&= -\frac{y^2}{-y + x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{y^2}{-y+x}} dy \end{aligned}$$

Which results in

$$S = \frac{x}{y} + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y}{y - x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y} \\ S_y &= \frac{y - x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)\, y + x}{y} = c_1$$

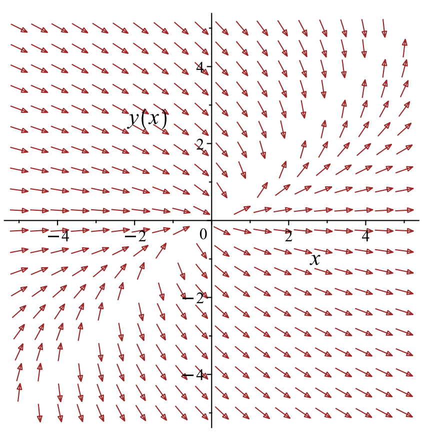
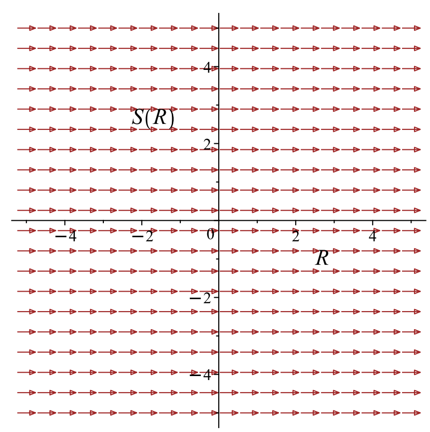
Which simplifies to

$$\frac{\ln(y)\, y + x}{y} = c_1$$

Which gives

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)+c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{y-x}$ 	$R = x$ $S = \frac{\ln(y)\, y + x}{y}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)+c_1} \tag{1}$$

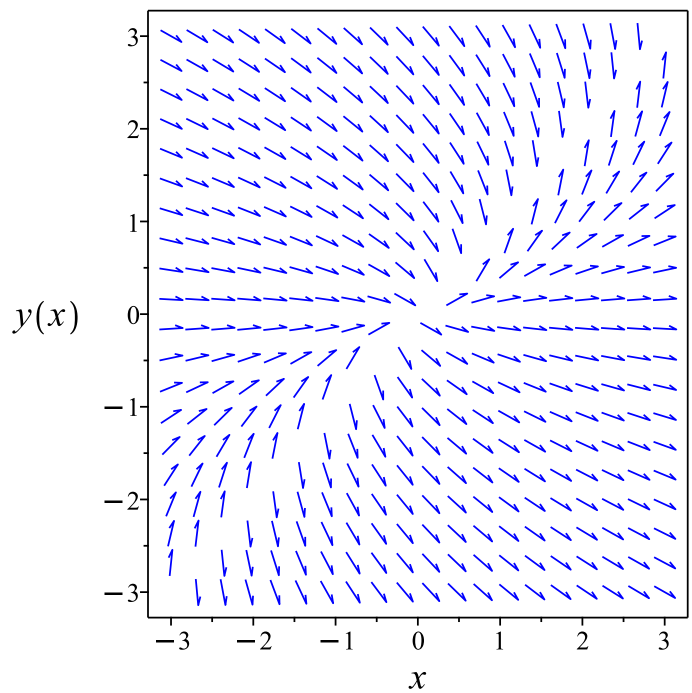


Figure 153: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{\mathrm{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)+c_1}
$$

Verified OK.

12.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(y - x) \, \mathrm{d}y &= (-y) \, \mathrm{d}x \\
(y) \, \mathrm{d}x + (y - x) \, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y \\
N(x, y) &= y - x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y - x) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{y-x}((1) - (-1))\\ &= -\frac{2}{-y+x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= \frac{1}{y}((-1) - (1))\\ &= -\frac{2}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y}\\ &= e^{\int -\frac{2}{y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(y)}\\ &= \frac{1}{y^2}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{y^2}(y)\\ &= \frac{1}{y}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{y^2}(y-x)\\ &= \frac{y-x}{y^2}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{1}{y}\right) + \left(\frac{y-x}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{1}{y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x}{y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y-x}{y^2}$. Therefore equation (4) becomes

$$
\frac{y-x}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right)\mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x}{y} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x}{y} + \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)+c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)+c_1} \tag{1}$$

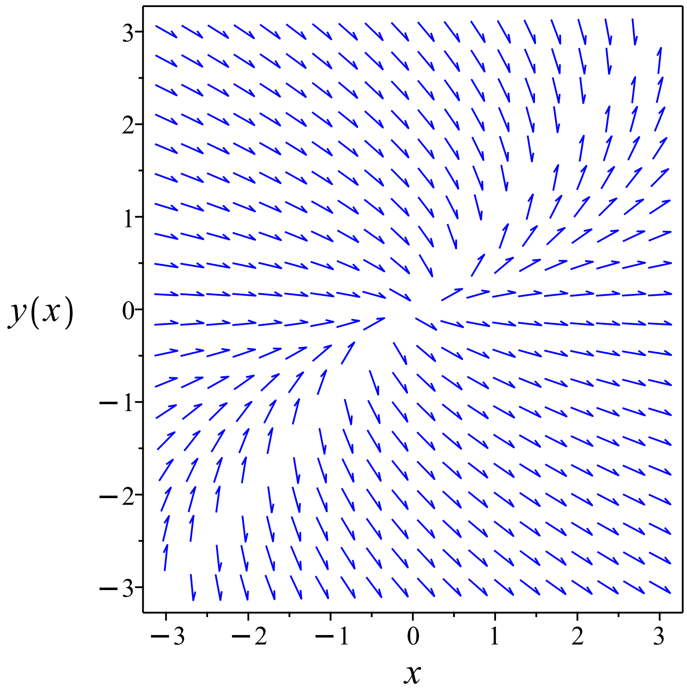


Figure 154: Slope field plot

Verification of solutions

$$y = e^{\mathrm{LambertW}(-x\,e^{-c_1})+c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve((y(x)-x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{\operatorname{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)}$$

✓ Solution by Mathematica

Time used: 5.289 (sec). Leaf size: 25

```
DSolve[(y[x]-x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{W\left(-e^{-c_1}x\right)}$$
$$y(x) \to 0$$

12.8 problem Ex 8

Internal problem ID [11178]
Internal file name [OUTPUT/10163_Saturday_December_03_2022_08_03_07_AM_37995883/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y'x - y - \sqrt{x^2 + y^2} = 0}$$

12.8.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y + \sqrt{x^2 + y^2}}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{\left(y+\sqrt{x^2+y^2}\right)(b_3-a_2)}{x} - \frac{\left(y+\sqrt{x^2+y^2}\right)^2 a_3}{x^2} \\
&- \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{y+\sqrt{x^2+y^2}}{x^2}\right)(xa_2+ya_3+a_1) \\
&- \frac{\left(1+\frac{y}{\sqrt{x^2+y^2}}\right)(xb_2+yb_3+b_1)}{x} = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\left(x^2+y^2\right)^{\frac{3}{2}} a_3 + x^3a_2 - x^3b_3 + 2x^2ya_3 + x^2yb_2 + y^3a_3 + \sqrt{x^2+y^2}\, xb_1 - \sqrt{x^2+y^2}\, ya_1 + xyb_1 - y^2a_1}{\sqrt{x^2+y^2}\, x^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-\left(x^2+y^2\right)^{\frac{3}{2}} a_3 - x^3a_2 + x^3b_3 - 2x^2ya_3 - x^2yb_2 - y^3a_3 \\
&- \sqrt{x^2+y^2}\, xb_1 + \sqrt{x^2+y^2}\, ya_1 - xyb_1 + y^2a_1 = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}
&-\left(x^2+y^2\right)^{\frac{3}{2}} a_3 + \left(x^2+y^2\right) xb_3 - \left(x^2+y^2\right) ya_3 - x^3a_2 - x^2ya_3 - x^2yb_2 \\
&- x\, y^2b_3 + \left(x^2+y^2\right) a_1 - \sqrt{x^2+y^2}\, xb_1 + \sqrt{x^2+y^2}\, ya_1 - x^2a_1 - xyb_1 = 0
\end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
&-x^3a_2 + x^3b_3 - x^2\sqrt{x^2+y^2}\, a_3 - 2x^2ya_3 - x^2yb_2 - \sqrt{x^2+y^2}\, y^2a_3 \\
&- y^3a_3 - \sqrt{x^2+y^2}\, xb_1 - xyb_1 + \sqrt{x^2+y^2}\, ya_1 + y^2a_1 = 0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \sqrt{x^2+y^2}=v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-v_1^3 a_2-2 v_1^2 v_2 a_3-v_1^2 v_3 a_3-v_2^3 a_3-v_3 v_2^2 a_3-v_1^2 v_2 b_2 \\
&+v_1^3 b_3+v_2^2 a_1+v_3 v_2 a_1-v_1 v_2 b_1-v_3 v_1 b_1=0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(b_3-a_2) v_1^3+(-2 a_3-b_2) v_1^2 v_2-v_1^2 v_3 a_3-v_1 v_2 b_1 \\
&-v_3 v_1 b_1-v_2^3 a_3-v_3 v_2^2 a_3+v_2^2 a_1+v_3 v_2 a_1=0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_1 &= 0 \\
-a_3 &= 0 \\
-b_1 &= 0 \\
-2 a_3-b_2 &= 0 \\
b_3-a_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y + \sqrt{x^2+y^2}}{x}\right)(x) \\
&= -\sqrt{x^2+y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{x^2+y^2}} dy
\end{aligned}$$

Which results in

$$S = -\ln\left(y + \sqrt{x^2+y^2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y+\sqrt{x^2+y^2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= -\frac{x}{\sqrt{x^2+y^2}\,\left(y+\sqrt{x^2+y^2}\right)}\\
S_y &= -\frac{1}{\sqrt{x^2+y^2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2\left(\sqrt{x^2+y^2}\,y+x^2+y^2\right)}{x\sqrt{x^2+y^2}\,\left(y+\sqrt{x^2+y^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(y+\sqrt{x^2+y^2}\right) = -2\ln(x) + c_1$$

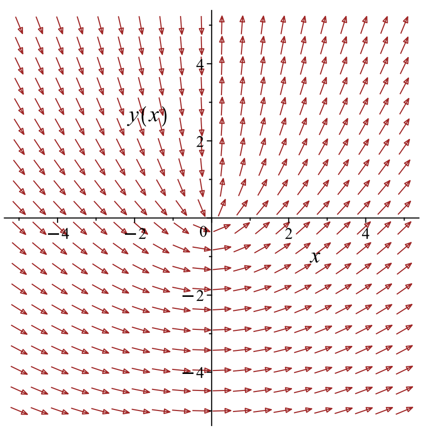
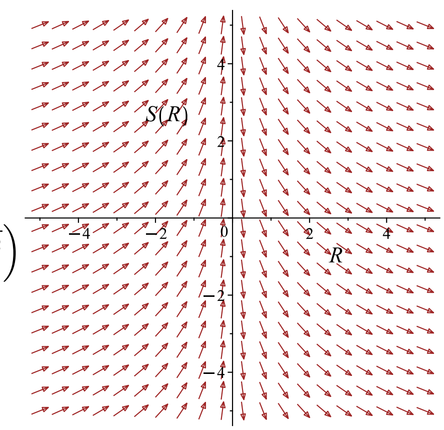
Which simplifies to

$$-\ln\left(y+\sqrt{x^2+y^2}\right) = -2\ln(x) + c_1$$

Which gives

$$y = -\frac{\mathrm{e}^{-c_1}\left(\mathrm{e}^{2c_1}-x^2\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$ 	$R = x$ $S = -\ln\left(y + \sqrt{x^2 + y^2}\right)$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{e^{-c_1}\left(e^{2c_1} - x^2\right)}{2} \tag{1}$$

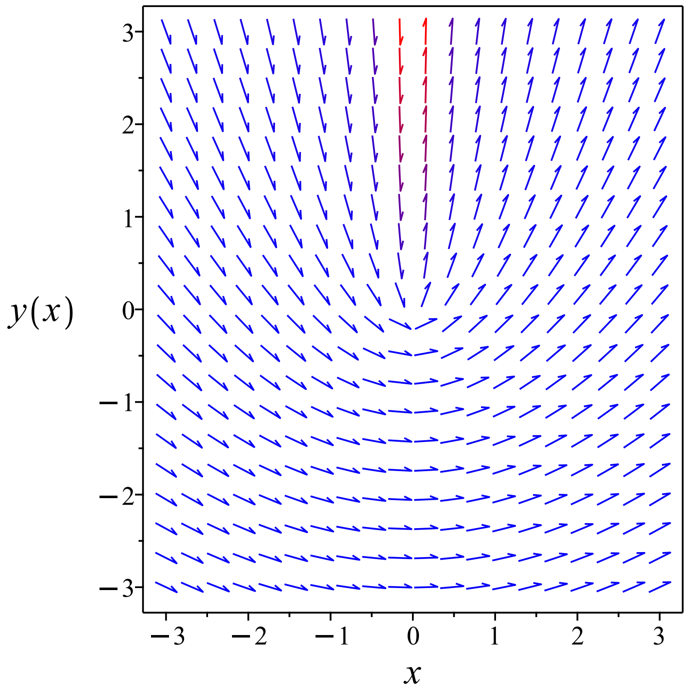


Figure 155: Slope field plot

Verification of solutions

$$y = -\frac{e^{-c_1}(e^{2c_1} - x^2)}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(x*diff(y(x),x)-y(x)=sqrt(x^2+y(x)^2),y(x), singsol=all)
```

$$\frac{-c_1 x^2 + y(x) + \sqrt{y(x)^2 + x^2}}{x^2} = 0$$

✓ Solution by Mathematica

Time used: 0.582 (sec). Leaf size: 27

```
DSolve[x*y'[x]-y[x]==Sqrt[x^2+y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-c_1} \left(-1 + e^{2c_1} x^2\right)$$

12.9 problem Ex 10

Internal problem ID [11179]
Internal file name [`OUTPUT/10164_Saturday_December_03_2022_08_03_08_AM_53634759/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y'x - y - \sqrt{x^2 - y^2} = 0}$$

12.9.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y + \sqrt{x^2 - y^2}}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(y+\sqrt{x^2-y^2}\right)(b_3-a_2)}{x} - \frac{\left(y+\sqrt{x^2-y^2}\right)^2 a_3}{x^2} - \left(\frac{1}{\sqrt{x^2-y^2}} - \frac{y+\sqrt{x^2-y^2}}{x^2}\right)(xa_2+ya_3+a_1) - \frac{\left(1-\frac{y}{\sqrt{x^2-y^2}}\right)(xb_2+yb_3+b_1)}{x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(x^2-y^2)^{\frac{3}{2}} a_3 + x^3a_2 - x^3b_3 + 2x^2ya_3 - x^2yb_2 - y^3a_3 + \sqrt{x^2-y^2}\, xb_1 - \sqrt{x^2-y^2}\, ya_1 - xyb_1 + y^2a_1}{\sqrt{x^2-y^2}\, x^2} = 0$$

Setting the numerator to zero gives

$$-(x^2-y^2)^{\frac{3}{2}} a_3 - x^3a_2 + x^3b_3 - 2x^2ya_3 + x^2yb_2 + y^3a_3 - \sqrt{x^2-y^2}\, xb_1 + \sqrt{x^2-y^2}\, ya_1 + xyb_1 - y^2a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-(x^2-y^2)^{\frac{3}{2}} a_3 + (x^2-y^2)\, xb_3 - (x^2-y^2)\, ya_3 - x^3a_2 - x^2ya_3 + x^2yb_2 + x\,y^2b_3 + (x^2-y^2)\, a_1 - \sqrt{x^2-y^2}\, xb_1 + \sqrt{x^2-y^2}\, ya_1 - x^2a_1 + xyb_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-x^3a_2 + x^3b_3 - x^2\sqrt{x^2-y^2}\, a_3 - 2x^2ya_3 + x^2yb_2 + \sqrt{x^2-y^2}\, y^2a_3 + y^3a_3 - \sqrt{x^2-y^2}\, xb_1 + xyb_1 + \sqrt{x^2-y^2}\, ya_1 - y^2a_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2-y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2 - y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -v_1^3 a_2 - 2v_1^2 v_2 a_3 - v_1^2 v_3 a_3 + v_2^3 a_3 + v_3 v_2^2 a_3 + v_1^2 v_2 b_2 \\ + v_1^3 b_3 - v_2^2 a_1 + v_3 v_2 a_1 + v_1 v_2 b_1 - v_3 v_1 b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} (b_3 - a_2) v_1^3 + (-2a_3 + b_2) v_1^2 v_2 - v_1^2 v_3 a_3 + v_1 v_2 b_1 \\ - v_3 v_1 b_1 + v_2^3 a_3 + v_3 v_2^2 a_3 - v_2^2 a_1 + v_3 v_2 a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ -a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -2a_3 + b_2 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{y + \sqrt{x^2 - y^2}}{x}\right)(x) \\
&= -\sqrt{x^2 - y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{x^2 - y^2}} dy
\end{aligned}$$

Which results in

$$S = -\arctan\left(\frac{y}{\sqrt{x^2 - y^2}}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y+\sqrt{x^2-y^2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{\sqrt{x^2-y^2}\, x} \\ S_y &= -\frac{1}{\sqrt{x^2-y^2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\arctan\left(\frac{y}{\sqrt{x^2-y^2}}\right) = -\ln(x) + c_1$$

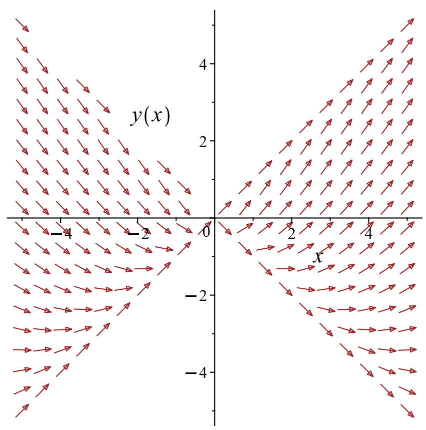
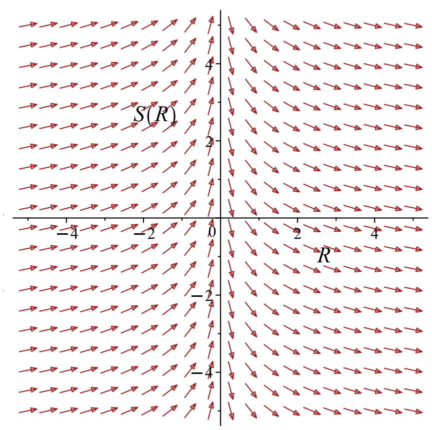
Which simplifies to

$$-\arctan\left(\frac{y}{\sqrt{x^2-y^2}}\right) = -\ln(x) + c_1$$

Which gives

$$y = -\tan(-\ln(x) + c_1)\sqrt{\frac{x^2}{\tan(-\ln(x)+c_1)^2+1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\sqrt{x^2-y^2}}{x}$ 	$R = x$ $S = -\arctan\left(\frac{y}{\sqrt{x^2-y^2}}\right)$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = -\tan\left(-\ln\left(x\right)+c_1\right)\sqrt{\frac{x^2}{\tan\left(-\ln\left(x\right)+c_1\right)^2+1}} \tag{1}$$

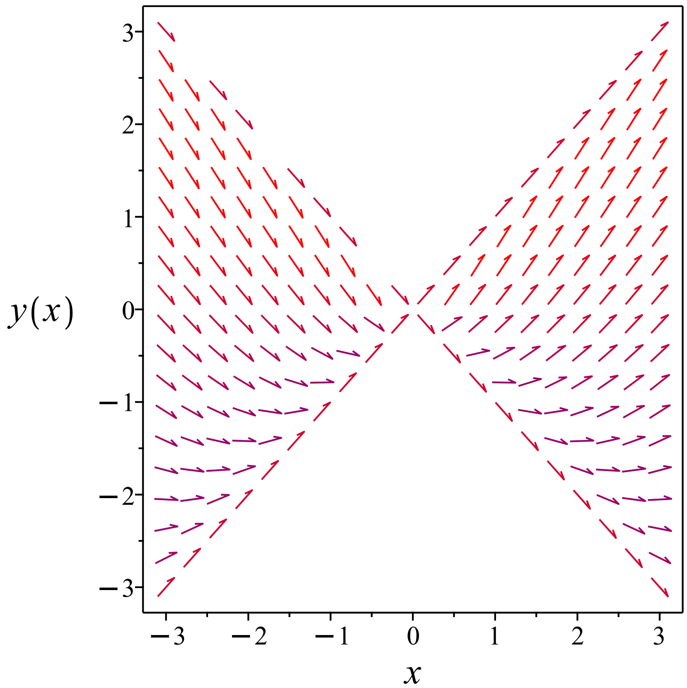


Figure 156: Slope field plot

Verification of solutions

$$y = -\tan(-\ln(x) + c_1)\sqrt{\frac{x^2}{\tan(-\ln(x) + c_1)^2 + 1}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(x*diff(y(x),x)-y(x)=sqrt(x^2-y(x)^2),y(x), singsol=all)
```

$$-\arctan\left(\frac{y(x)}{\sqrt{x^2-y(x)^2}}\right)+\ln(x)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.395 (sec). Leaf size: 18

```
DSolve[x*y'[x]-y[x]==Sqrt[x^2-y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -x\cosh(i\log(x)+c_1)$$

12.10 problem Ex 11

Internal problem ID [11180]
Internal file name [`OUTPUT/10165_Saturday_December_03_2022_08_03_10_AM_21603529/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$
\boxed{x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right) + x \cos\left(\frac{y}{x}\right) y' = 0}
$$

12.10.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$
x \sin(u(x)) - u(x)\, x \cos(u(x)) + x \cos(u(x))\, (u'(x)\, x + u(x)) = 0
$$

In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= -\frac{\tan(u)}{x}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \tan(u)$. Integrating both sides gives

$$\frac{1}{\tan(u)}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{\tan(u)}\,du = \int -\frac{1}{x}\,dx$$
$$\ln(\sin(u)) = -\ln(x) + c_2$$

Raising both side to exponential gives

$$\sin(u) = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sin(u) = \frac{c_3}{x}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x \arcsin\left(\frac{c_3 \mathrm{e}^{c_2}}{x}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = x \arcsin\left(\frac{c_3 \mathrm{e}^{c_2}}{x}\right) \tag{1}$$

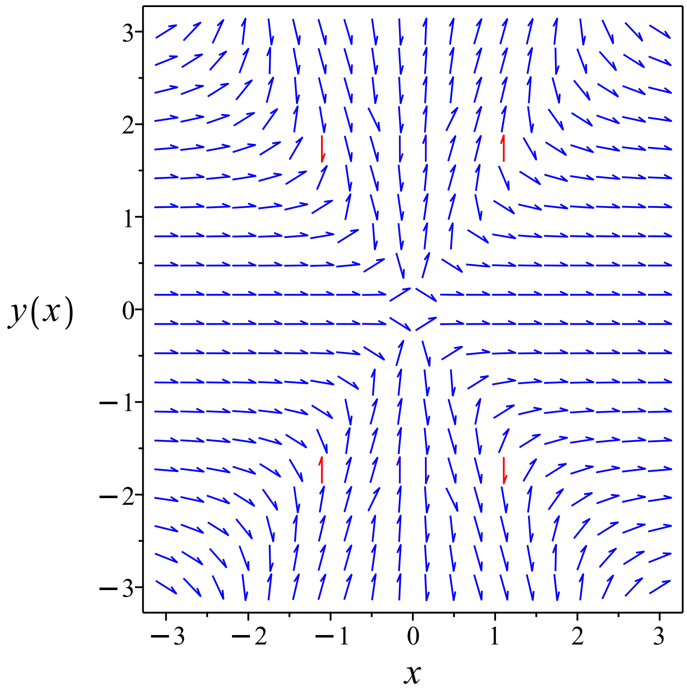


Figure 157: Slope field plot

Verification of solutions

$$y = x \arcsin\left(\frac{c_3 e^{c_2}}{x}\right)$$

Verified OK.

12.10.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)}{x \cos\left(\frac{y}{x}\right)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{\left(x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)\right)(b_3 - a_2)}{x\cos\left(\frac{y}{x}\right)} - \frac{\left(x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)\right)^2 a_3}{x^2\cos\left(\frac{y}{x}\right)^2} - \left(-\frac{\sin\left(\frac{y}{x}\right) - \frac{y\cos\left(\frac{y}{x}\right)}{x} - \frac{y^2\sin\left(\frac{y}{x}\right)}{x^2}}{x\cos\left(\frac{y}{x}\right)} + \frac{x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)}{x^2\cos\left(\frac{y}{x}\right)} + \frac{\left(x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)\right)y\sin\left(\frac{y}{x}\right)}{x^3\cos\left(\frac{y}{x}\right)^2}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{y\sin\left(\frac{y}{x}\right)}{x^2\cos\left(\frac{y}{x}\right)} - \frac{\left(x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)\right)\sin\left(\frac{y}{x}\right)}{x^2\cos\left(\frac{y}{x}\right)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{b_2x^2\cos\left(\frac{y}{x}\right)^2 - \cos\left(\frac{y}{x}\right)^2 xya_2 + \cos\left(\frac{y}{x}\right)^2 xyb_3 - \cos\left(\frac{y}{x}\right)^2 y^2a_3 + \cos\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)x^2a_2 - \cos\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)x^2b_3}{}$$
$$= 0$$

Setting the numerator to zero gives

$$b_2x^2\cos\left(\frac{y}{x}\right)^2 - \cos\left(\frac{y}{x}\right)^2 xya_2 + \cos\left(\frac{y}{x}\right)^2 xyb_3 - \cos\left(\frac{y}{x}\right)^2 y^2a_3 + \cos\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)x^2a_2 - \cos\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)x^2b_3 + 2\cos\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)xya_3 - \sin\left(\frac{y}{x}\right)^2 x^2a_3 + \sin\left(\frac{y}{x}\right)^2 x^2b_2 - \sin\left(\frac{y}{x}\right)^2 xya_2 + \sin\left(\frac{y}{x}\right)^2 xyb_3 - \sin\left(\frac{y}{x}\right)^2 y^2a_3 + \sin\left(\frac{y}{x}\right)^2 xb_1 - \sin\left(\frac{y}{x}\right)^2 ya_1 = 0 \tag{6E}$$

Simplifying the above gives

$$\frac{x\left(x^2a_3\cos\left(\frac{2y}{x}\right) + x^2a_2\sin\left(\frac{2y}{x}\right) - x^2b_3\sin\left(\frac{2y}{x}\right) + 2xya_3\sin\left(\frac{2y}{x}\right) - xb_1\cos\left(\frac{2y}{x}\right) + ya_1\cos\left(\frac{2y}{x}\right) - x^2a_3 + 2b_2\right.}{2} \tag{6E}$$
$$= 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \cos\left(\frac{2y}{x}\right), \sin\left(\frac{2y}{x}\right)\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \cos\left(\frac{2y}{x}\right) = v_3, \sin\left(\frac{2y}{x}\right) = v_4\right\}$$

The above PDE (6E) now becomes

$$\frac{v_1(v_1^2a_2v_4 + v_1^2a_3v_3 + 2v_1v_2a_3v_4 - v_1^2b_3v_4 + v_2a_1v_3 - 2v_1v_2a_2 - v_1^2a_3 - 2v_2^2a_3 - v_1b_1v_3 + 2b_2v_1^2 + 2v_1v_2b_3 -}{2} \tag{7E}$$
$$= 0$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned}&\left(-\frac{a_3}{2} + b_2\right)v_1^3 + \frac{a_3v_3v_1^3}{2} + \left(\frac{a_2}{2} - \frac{b_3}{2}\right)v_4v_1^3 + \frac{b_1v_1^2}{2} + (b_3 - a_2)\,v_2v_1^2 \\ &- \frac{b_1v_3v_1^2}{2} + a_3v_4v_2v_1^2 - \frac{a_1v_2v_1}{2} + \frac{a_1v_2v_3v_1}{2} - a_3v_2^2v_1 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_3 &= 0 \\
-\frac{a_1}{2} &= 0 \\
\frac{a_1}{2} &= 0 \\
-a_3 &= 0 \\
\frac{a_3}{2} &= 0 \\
-\frac{b_1}{2} &= 0 \\
\frac{b_1}{2} &= 0 \\
\frac{a_2}{2} - \frac{b_3}{2} &= 0 \\
-\frac{a_3}{2} + b_2 &= 0 \\
b_3 - a_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)}{x \cos\left(\frac{y}{x}\right)}\right)(x) \\
&= \frac{\sin\left(\frac{y}{x}\right) x}{\cos\left(\frac{y}{x}\right)} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{\sin\left(\frac{y}{x}\right) x}{\cos\left(\frac{y}{x}\right)}} dy \end{aligned}$$

Which results in

$$S = \ln\left(\sin\left(\frac{y}{x}\right)\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)}{x \cos\left(\frac{y}{x}\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{\cot\left(\frac{y}{x}\right) y}{x^2} \\ S_y &= \frac{\cot\left(\frac{y}{x}\right)}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(\sin\left(\frac{y}{x}\right)\right) = -\ln(x) + c_1$$

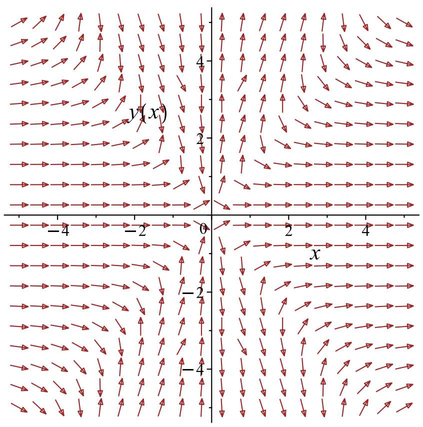
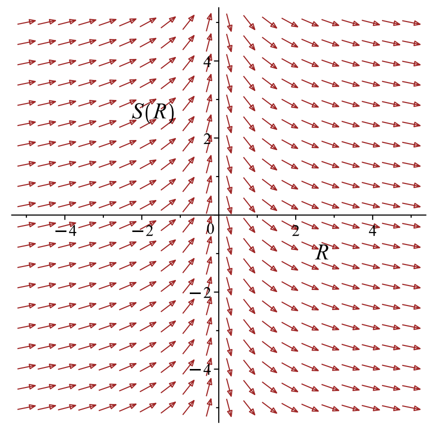
Which simplifies to

$$\ln\left(\sin\left(\frac{y}{x}\right)\right) = -\ln(x) + c_1$$

Which gives

$$y = x \arcsin\left(\frac{e^{c_1}}{x}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)}{x \cos\left(\frac{y}{x}\right)}$ 	$R = x$ $S = \ln\left(\sin\left(\frac{y}{x}\right)\right)$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = x \arcsin\left(\frac{e^{c_1}}{x}\right) \tag{1}$$

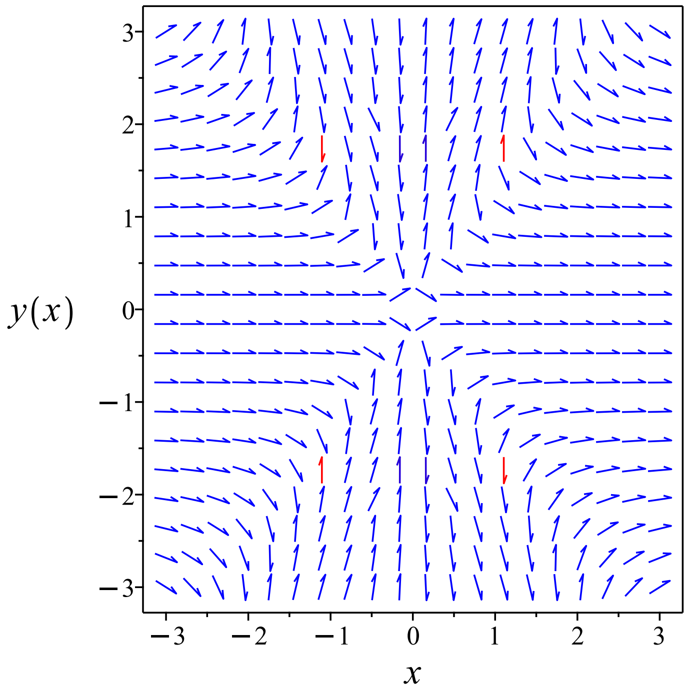


Figure 158: Slope field plot

<u>Verification of solutions</u>

$$y = x \arcsin\left(\frac{e^{c_1}}{x}\right)$$

Verified OK.

12.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(x \cos\left(\frac{y}{x}\right)\right) \mathrm{d}y &= \left(-x \sin\left(\frac{y}{x}\right) + y \cos\left(\frac{y}{x}\right)\right) \mathrm{d}x \\
\left(x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)\right) \mathrm{d}x + \left(x \cos\left(\frac{y}{x}\right)\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right) \\
N(x, y) &= x \cos\left(\frac{y}{x}\right)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)\right) \\
&= \frac{y \sin\left(\frac{y}{x}\right)}{x}
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\cos\left(\frac{y}{x}\right)\right)\\ &= \cos\left(\frac{y}{x}\right)+\frac{y\sin\left(\frac{y}{x}\right)}{x}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \frac{\sec\left(\frac{y}{x}\right)}{x}\left(\left(\frac{y\sin\left(\frac{y}{x}\right)}{x}\right)-\left(\cos\left(\frac{y}{x}\right)+\frac{y\sin\left(\frac{y}{x}\right)}{x}\right)\right)\\ &= -\frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{1}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(x)}\\ &= \frac{1}{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x}\left(x\sin\left(\frac{y}{x}\right)-y\cos\left(\frac{y}{x}\right)\right)\\ &= \frac{x\sin\left(\frac{y}{x}\right)-y\cos\left(\frac{y}{x}\right)}{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x}\left(x\cos\left(\frac{y}{x}\right)\right)\\ &= \cos\left(\frac{y}{x}\right)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)}{x}\right) + \left(\cos\left(\frac{y}{x}\right)\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)}{x}\,\mathrm{d}x$$
$$\phi = x\sin\left(\frac{y}{x}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \cos\left(\frac{y}{x}\right) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \cos\left(\frac{y}{x}\right)$. Therefore equation (4) becomes

$$\cos\left(\frac{y}{x}\right) = \cos\left(\frac{y}{x}\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x \sin\left(\frac{y}{x}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x \sin\left(\frac{y}{x}\right)$$

Summary

The solution(s) found are the following

$$x \sin\left(\frac{y}{x}\right) = c_1 \tag{1}$$

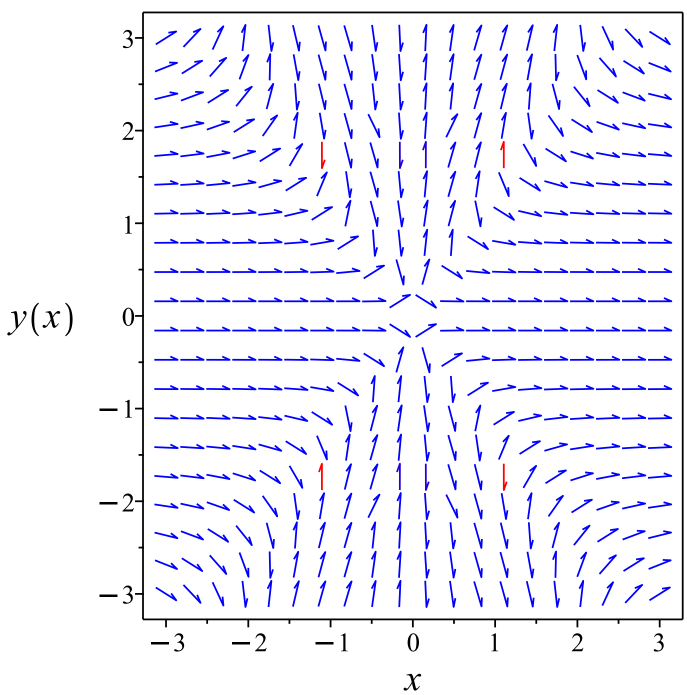


Figure 159: Slope field plot

Verification of solutions

$$x \sin\left(\frac{y}{x}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve((x*sin(y(x)/x)-y(x)*cos(y(x)/x))+x*cos(y(x)/x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x \arcsin\left(\frac{1}{x c_1}\right)$$

✓ Solution by Mathematica

Time used: 15.438 (sec). Leaf size: 21

```
DSolve[(x*Sin[y[x]/x]-y[x]*Cos[y[x]/x])+x*Cos[y[x]/x]*y'[x]==0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to x \arcsin\left(\frac{e^{c_1}}{x}\right)$$
$$y(x) \to 0$$

12.11 problem Ex 12

Internal problem ID [11181]
Internal file name [`OUTPUT/10166_Saturday_December_03_2022_08_03_11_AM_60289939/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-2y + (2x - y + 4) y' = -x - 5}$$

12.11.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{-X - x_0 + 2Y(X) + 2y_0 - 5}{-2X - 2x_0 + Y(X) + y_0 - 4}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -1$$
$$y_0 = 2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{-X + 2Y(X)}{-2X + Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X,Y) \\
&= -\frac{-X+2Y}{-2X+Y}
\end{aligned}
\tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X,Y)
$$

In this case, it can be seen that both $M = -X + 2Y$ and $N = 2X - Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-2u+1}{u-2} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-2u(X)+1}{u(X)-2} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{-2u(X)+1}{u(X)-2} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX}u(X)\right) X u(X) - 2\left(\frac{d}{dX}u(X)\right) X + u(X)^2 - 1 = 0
$$

Or

$$
(u(X) - 2)\, X \left(\frac{d}{dX}u(X)\right) + u(X)^2 - 1 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= -\frac{u^2-1}{(u-2)\, X}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-1}{u-2}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{u^2-1}{u-2}}\, du &= -\frac{1}{X}\, dX \\
\int \frac{1}{\frac{u^2-1}{u-2}}\, du &= \int -\frac{1}{X}\, dX \\
-\frac{\ln(u-1)}{2} + \frac{3\ln(u+1)}{2} &= -\ln(X) + c_2
\end{aligned}$$

The above can be written as

$$\begin{aligned}
\frac{-\ln(u-1) + 3\ln(u+1)}{2} &= -\ln(X) + c_2 \\
-\ln(u-1) + 3\ln(u+1) &= (2)(-\ln(X) + c_2) \\
&= -2\ln(X) + 2c_2
\end{aligned}$$

Raising both side to exponential gives

$$e^{-\ln(u-1)+3\ln(u+1)} = e^{-2\ln(X)+2c_2}$$

Which simplifies to

$$\begin{aligned}
\frac{(u+1)^3}{u-1} &= \frac{2c_2}{X^2} \\
&= \frac{c_3}{X^2}
\end{aligned}$$

Which simplifies to

$$\frac{(u(X)+1)^3}{u(X)-1} = \frac{c_3 e^{2c_2}}{X^2}$$

The solution is

$$\frac{(u(X)+1)^3}{u(X)-1} = \frac{c_3 e^{2c_2}}{X^2}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\left(\frac{Y(X)}{X}+1\right)^3}{\frac{Y(X)}{X}-1} = \frac{c_3 e^{2c_2}}{X^2}$$

Which simplifies to

$$-\frac{(Y(X)+X)^3}{-Y(X)+X} = c_3 e^{2c_2}$$

Using the solution for $Y(X)$

$$-\frac{(Y(X)+X)^3}{-Y(X)+X} = c_3 e^{2c_2}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 2$$
$$X = x - 1$$

Then the solution in y becomes

$$-\frac{(y-1+x)^3}{-y+3+x} = c_3 e^{2c_2}$$

Summary

The solution(s) found are the following

$$-\frac{(y-1+x)^3}{-y+3+x} = c_3 e^{2c_2} \tag{1}$$

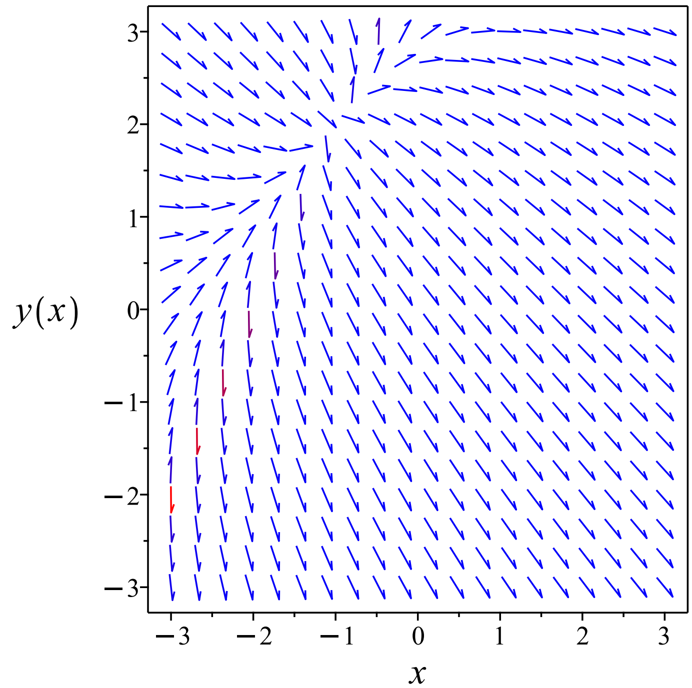


Figure 160: Slope field plot

Verification of solutions

$$-\frac{(y-1+x)^3}{-y+3+x} = c_3 \mathrm{e}^{2c_2}$$

Verified OK.

12.11.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{-x+2y-5}{-2x+y-4}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(-x+2y-5)(b_3-a_2)}{-2x+y-4} - \frac{(-x+2y-5)^2 a_3}{(-2x+y-4)^2} \\ & - \left(\frac{1}{-2x+y-4} - \frac{2(-x+2y-5)}{(-2x+y-4)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{2}{-2x+y-4} + \frac{-x+2y-5}{(-2x+y-4)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2a_2 - x^2a_3 + x^2b_2 - 2x^2b_3 - 2xya_2 + 4xya_3 - 4xyb_2 + 2xyb_3 + 2y^2a_2 - y^2a_3 + y^2b_2 - 2y^2b_3 + 8xa_2 - \cdots}{(2x \cdots}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 2x^2a_2 - x^2a_3 + x^2b_2 - 2x^2b_3 - 2xya_2 + 4xya_3 - 4xyb_2 + 2xyb_3 + 2y^2a_2 \\ & - y^2a_3 + y^2b_2 - 2y^2b_3 + 8xa_2 - 10xa_3 - 3xb_1 + 13xb_2 - 14xb_3 + 3ya_1 \\ & - 13ya_2 + 14ya_3 - 8yb_2 + 10yb_3 - 6a_1 + 20a_2 - 25a_3 - 3b_1 + 16b_2 - 20b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 2a_2v_1^2 - 2a_2v_1v_2 + 2a_2v_2^2 - a_3v_1^2 + 4a_3v_1v_2 - a_3v_2^2 + b_2v_1^2 - 4b_2v_1v_2 + b_2v_2^2 \\ & - 2b_3v_1^2 + 2b_3v_1v_2 - 2b_3v_2^2 + 3a_1v_2 + 8a_2v_1 - 13a_2v_2 - 10a_3v_1 + 14a_3v_2 - 3b_1v_1 \\ & + 13b_2v_1 - 8b_2v_2 - 14b_3v_1 + 10b_3v_2 - 6a_1 + 20a_2 - 25a_3 - 3b_1 + 16b_2 - 20b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(2a_2 - a_3 + b_2 - 2b_3)\, v_1^2 + (-2a_2 + 4a_3 - 4b_2 + 2b_3)\, v_1 v_2 \\&+ (8a_2 - 10a_3 - 3b_1 + 13b_2 - 14b_3)\, v_1 + (2a_2 - a_3 + b_2 - 2b_3)\, v_2^2 \\&+ (3a_1 - 13a_2 + 14a_3 - 8b_2 + 10b_3)\, v_2 - 6a_1 + 20a_2 - 25a_3 - 3b_1 + 16b_2 - 20b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-2a_2 + 4a_3 - 4b_2 + 2b_3 &= 0\\2a_2 - a_3 + b_2 - 2b_3 &= 0\\3a_1 - 13a_2 + 14a_3 - 8b_2 + 10b_3 &= 0\\8a_2 - 10a_3 - 3b_1 + 13b_2 - 14b_3 &= 0\\-6a_1 + 20a_2 - 25a_3 - 3b_1 + 16b_2 - 20b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -2b_2 + b_3\\a_2 &= b_3\\a_3 &= b_2\\b_1 &= b_2 - 2b_3\\b_2 &= b_2\\b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 1 + x\\\eta &= y - 2\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi\\&= y - 2 - \left(-\frac{-x + 2y - 5}{-2x + y - 4}\right)(1 + x)\\&= \frac{x^2 - y^2 + 2x + 4y - 3}{2x - y + 4}\\\xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2 - y^2 + 2x + 4y - 3}{2x - y + 4}} dy \end{aligned}$$

Which results in

$$S = \frac{3 \ln(x - 1 + y)}{2} - \frac{\ln(y - 3 - x)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-x + 2y - 5}{-2x + y - 4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x - 2y + 5}{(x - 1 + y)(x + 3 - y)} \\ S_y &= \frac{2x - y + 4}{(x - 1 + y)(x + 3 - y)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

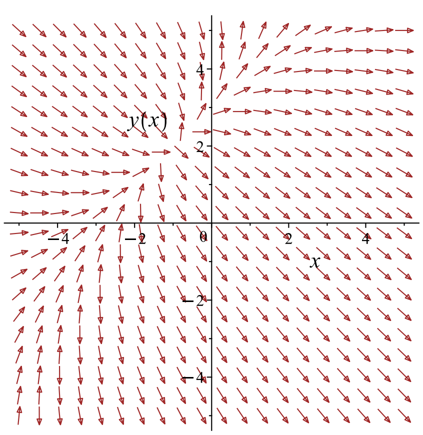
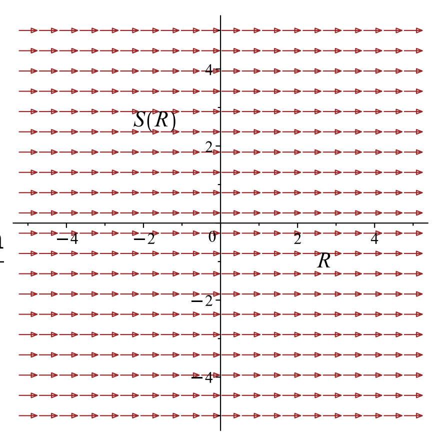
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3\ln(y-1+x)}{2} - \frac{\ln(y-3-x)}{2} = c_1$$

Which simplifies to

$$\frac{3\ln(y-1+x)}{2} - \frac{\ln(y-3-x)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x+2y-5}{-2x+y-4}$ 	$R = x$ $S = \frac{3\ln(x-1+y)}{2} - \ln$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{3 \ln \left(y-1+x\right)}{2}-\frac{\ln \left(y-3-x\right)}{2}=c_1 \tag{1}$$

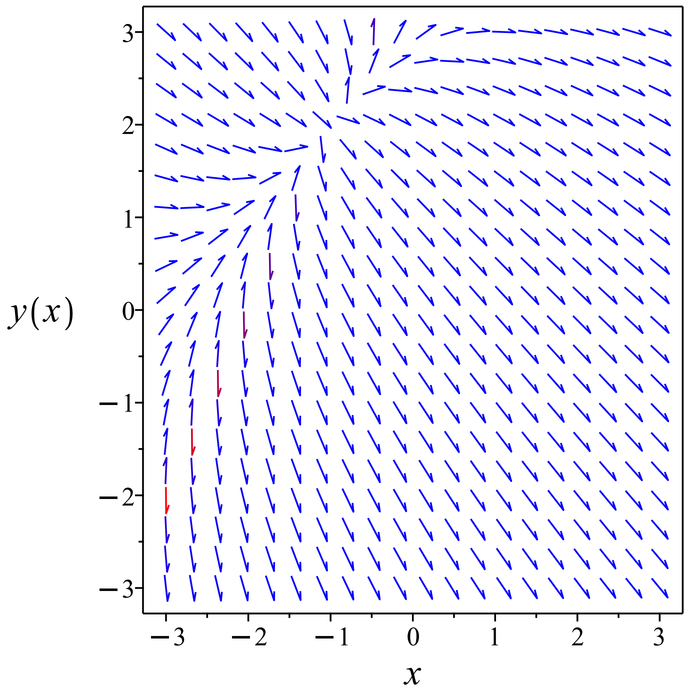


Figure 161: Slope field plot

Verification of solutions

$$\frac{3 \ln \left(y-1+x\right)}{2}-\frac{\ln \left(y-3-x\right)}{2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.766 (sec). Leaf size: 117

```
dsolve((x-2*y(x)+5)+(2*x-y(x)+4)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) =$$

$$-\frac{\left(i\sqrt{3}-1\right)\left(3\sqrt{3}\sqrt{27c_1^2\left(1+x\right)^2-1}+27c_1(1+x)\right)^{\frac{2}{3}}-3i\sqrt{3}-3+6\left(3\sqrt{3}\sqrt{27c_1^2\left(1+x\right)^2-1+2\cdots}\right.}{6\left(3\sqrt{3}\sqrt{27c_1^2\left(1+x\right)^2-1}+27c_1\left(1+x\right)\right)^{\frac{1}{3}}c_1}$$

✓ Solution by Mathematica

Time used: 60.282 (sec). Leaf size: 1601

```
DSolve[(x-2*y[x]+5)+(2*x-y[x]+4)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

12.12 problem Ex 13

Internal problem ID [11182]
Internal file name [`OUTPUT/10167_Saturday_December_03_2022_08_03_13_AM_33812630/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{(-x^2+1)^{\frac{3}{2}}} = \frac{x+\sqrt{-x^2+1}}{(-x^2+1)^2}}$$

12.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{(-x^2+1)^{\frac{3}{2}}} \\ q(x) &= -\frac{x^2-\sqrt{-x^2+1}\,x-1}{(-x^2+1)^{\frac{5}{2}}} \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{(-x^2+1)^{\frac{3}{2}}} = -\frac{x^2 - \sqrt{-x^2+1}\, x - 1}{(-x^2+1)^{\frac{5}{2}}}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{1}{(-x^2+1)^{\frac{3}{2}}} dx} \\
&= e^{\frac{x}{\sqrt{-x^2+1}}}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(-\frac{x^2 - \sqrt{-x^2+1}\, x - 1}{(-x^2+1)^{\frac{5}{2}}}\right) \\
\frac{d}{dx}\left(e^{\frac{x}{\sqrt{-x^2+1}}} y\right) &= \left(e^{\frac{x}{\sqrt{-x^2+1}}}\right)\left(-\frac{x^2 - \sqrt{-x^2+1}\, x - 1}{(-x^2+1)^{\frac{5}{2}}}\right) \\
d\left(e^{\frac{x}{\sqrt{-x^2+1}}} y\right) &= \left(-\frac{\left(x^2 - \sqrt{-x^2+1}\, x - 1\right) e^{\frac{x}{\sqrt{-x^2+1}}}}{(-x^2+1)^{\frac{5}{2}}}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{\frac{x}{\sqrt{-x^2+1}}} y &= \int -\frac{\left(x^2 - \sqrt{-x^2+1}\, x - 1\right) e^{\frac{x}{\sqrt{-x^2+1}}}}{(-x^2+1)^{\frac{5}{2}}} dx \\
e^{\frac{x}{\sqrt{-x^2+1}}} y &= \int -\frac{\left(x^2 - \sqrt{-x^2+1}\, x - 1\right) e^{\frac{x}{\sqrt{-x^2+1}}}}{(-x^2+1)^{\frac{5}{2}}} dx + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x}{\sqrt{-x^2+1}}}$ results in

$$y = e^{-\frac{x}{\sqrt{-x^2+1}}} \left(\int -\frac{\left(x^2 - \sqrt{-x^2+1}\, x - 1\right) e^{\frac{x}{\sqrt{-x^2+1}}}}{(-x^2+1)^{\frac{5}{2}}} dx\right) + c_1 e^{-\frac{x}{\sqrt{-x^2+1}}}$$

which simplifies to

$$y = e^{-\frac{x}{\sqrt{-x^2+1}}} \left(\int -\frac{\left(x^2 - \sqrt{-x^2+1}\, x - 1\right) e^{\frac{x}{\sqrt{-x^2+1}}}}{(-x^2+1)^{\frac{5}{2}}} dx + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{\sqrt{-x^2+1}}} \left(\int -\frac{\left(x^2 - \sqrt{-x^2+1}\, x - 1\right) e^{\frac{x}{\sqrt{-x^2+1}}}}{(-x^2+1)^{\frac{5}{2}}} dx + c_1\right) \tag{1}$$

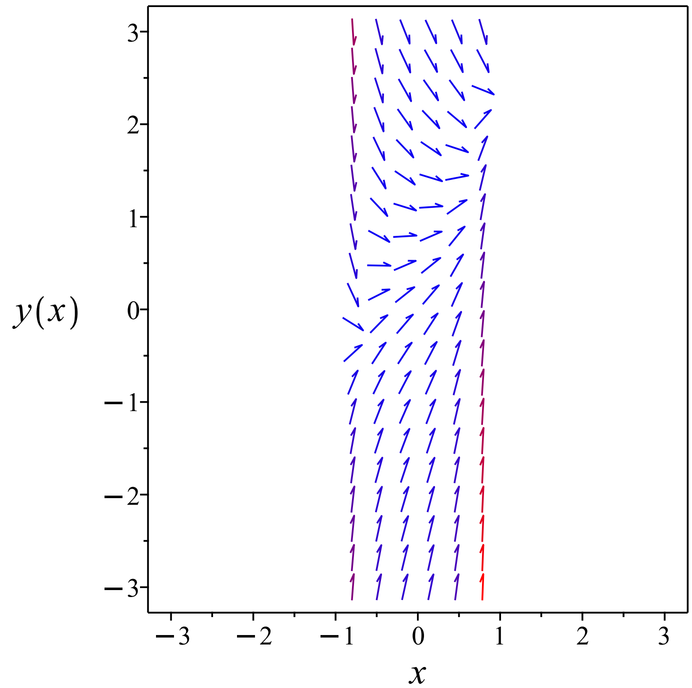


Figure 162: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{x}{\sqrt{-x^2+1}}}\left(\int -\frac{\left(x^2-\sqrt{-x^2+1}\,x-1\right)\mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}}{\left(-x^2+1\right)^{\frac{5}{2}}}dx + c_1\right)$$

Verified OK.

12.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-x^4y+\left(-x^2+1\right)^{\frac{3}{2}}x+x^4+2x^2y-2x^2-y+1}{\left(-x^2+1\right)^{\frac{3}{2}}\left(x^2-1\right)^2}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 87: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\frac{(x-1)(1+x)x}{\left(-x^2+1\right)^{\frac{3}{2}}}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\frac{(x-1)(1+x)x}{(-x^2+1)^{\frac{3}{2}}}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-\frac{(x-1)(1+x)x}{(-x^2+1)^{\frac{3}{2}}}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-x^4 y + (-x^2+1)^{\frac{3}{2}} x + x^4 + 2x^2 y - 2x^2 - y + 1}{(-x^2+1)^{\frac{3}{2}} (x^2-1)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\mathrm{e}^{\frac{-x^3+x}{(-x^2+1)^{\frac{3}{2}}}} y}{(-x^2+1)^{\frac{3}{2}}} \\ S_y &= \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\left(x^2 - \sqrt{-x^2+1}\, x - 1\right) \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}}{(-x^2+1)^{\frac{5}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{\left(R^2 - \sqrt{-R^2+1}\, R - 1\right) e^{\frac{R}{\sqrt{-R^2+1}}}}{\left(-R^2+1\right)^{\frac{5}{2}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{\left(-R^2 + \sqrt{-R^2+1}\, R + 1\right) e^{\frac{R}{\sqrt{-R^2+1}}}}{\left(-R^2+1\right)^{\frac{5}{2}}} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\frac{x}{\sqrt{-x^2+1}}} y = \int \frac{\left(-x^2 + \sqrt{-x^2+1}\, x + 1\right) e^{\frac{x}{\sqrt{-x^2+1}}}}{\left(-x^2+1\right)^{\frac{5}{2}}} dx + c_1$$

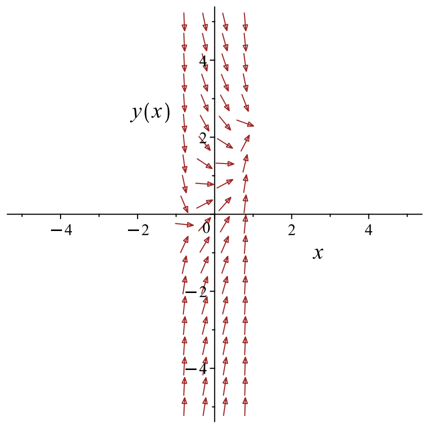
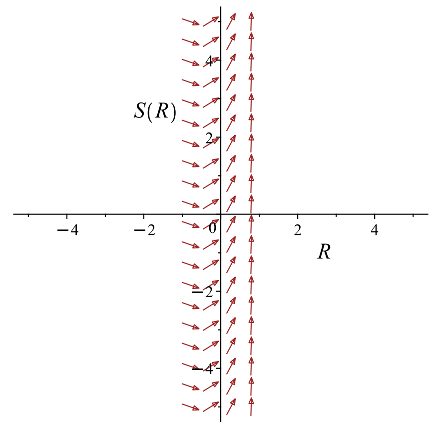
Which simplifies to

$$e^{\frac{x}{\sqrt{-x^2+1}}} y = \int \frac{\left(-x^2 + \sqrt{-x^2+1}\, x + 1\right) e^{\frac{x}{\sqrt{-x^2+1}}}}{\left(-x^2+1\right)^{\frac{5}{2}}} dx + c_1$$

Which gives

$$y = \left(\int \frac{\left(-x^2 + \sqrt{-x^2+1}\, x + 1\right) e^{\frac{x}{\sqrt{-x^2+1}}}}{\left(-x^2+1\right)^{\frac{5}{2}}} dx + c_1\right) e^{-\frac{x}{\sqrt{-x^2+1}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x^4 y+(-x^2+1)^{\frac{3}{2}} x+x^4+2x^2 y-2x^2-y+1}{(-x^2+1)^{\frac{3}{2}} (x^2-1)^2}$ 	$R = x$ $S = e^{\frac{x}{\sqrt{-x^2+1}}} y$	$\frac{dS}{dR} = -\frac{\left(R^2-\sqrt{-R^2+1}\, R-1\right) e^{\frac{R}{\sqrt{-R^2+1}}}}{(-R^2+1)^{\frac{5}{2}}}$ 

Summary

The solution(s) found are the following

$$y = \left(\int \frac{\left(-x^2+\sqrt{-x^2+1}\, x+1\right) e^{\frac{x}{\sqrt{-x^2+1}}}}{(-x^2+1)^{\frac{5}{2}}} dx + c_1\right) e^{-\frac{x}{\sqrt{-x^2+1}}} \tag{1}$$

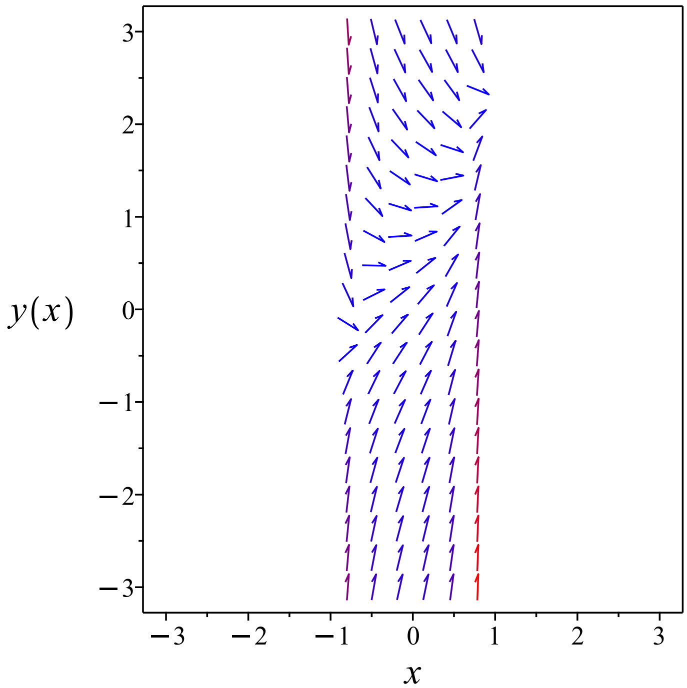


Figure 163: Slope field plot

Verification of solutions

$$y=\left(\int \frac{\left(-x^{2}+\sqrt{-x^{2}+1}\, x+1\right) \mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}}}{\left(-x^{2}+1\right)^{\frac{5}{2}}}d x+c_{1}\right) \mathrm{e}^{-\frac{x}{\sqrt{-x^{2}+1}}}$$

Verified OK.

12.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\frac{y}{(-x^2+1)^{\frac{3}{2}}} + \frac{x+\sqrt{-x^2+1}}{(-x^2+1)^2}\right)\mathrm{d}x$$

$$\left(\frac{y}{(-x^2+1)^{\frac{3}{2}}} - \frac{x+\sqrt{-x^2+1}}{(-x^2+1)^2}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{y}{(-x^2+1)^{\frac{3}{2}}} - \frac{x+\sqrt{-x^2+1}}{(-x^2+1)^2}$$
$$N(x,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{(-x^2+1)^{\frac{3}{2}}} - \frac{x+\sqrt{-x^2+1}}{(-x^2+1)^2}\right)\\ &= \frac{1}{(-x^2+1)^{\frac{3}{2}}}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{1}{(-x^2+1)^{\frac{3}{2}}}\right) - (0)\right) \\
&= \frac{1}{(-x^2+1)^{\frac{3}{2}}}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{(-x^2+1)^{\frac{3}{2}}} \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{(x-1)(1+x)x}{(-x^2+1)^{\frac{3}{2}}}} \\
&= \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}\left(\frac{y}{(-x^2+1)^{\frac{3}{2}}} - \frac{x+\sqrt{-x^2+1}}{(-x^2+1)^2}\right) \\
&= -\frac{\mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}\left(\sqrt{-x^2+1}\,x + (y-1)\,x^2 - y + 1\right)}{(-x^2+1)^{\frac{5}{2}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}(1) \\
&= \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M}+\overline{N}\frac{dy}{dx}=0$$
$$\left(-\frac{e^{\frac{x}{\sqrt{-x^2+1}}}\left(\sqrt{-x^2+1}\,x+(y-1)\,x^2-y+1\right)}{(-x^2+1)^{\frac{5}{2}}}\right)+\left(e^{\frac{x}{\sqrt{-x^2+1}}}\right)\frac{dy}{dx}=0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,dx=\int\overline{M}\,dx$$
$$\int\frac{\partial\phi}{\partial x}\,dx=\int-\frac{e^{\frac{x}{\sqrt{-x^2+1}}}\left(\sqrt{-x^2+1}\,x+(y-1)\,x^2-y+1\right)}{(-x^2+1)^{\frac{5}{2}}}\,dx$$
$$\phi=\int^x-\frac{e^{\frac{\_a}{\sqrt{-\_a^2+1}}}\left(\sqrt{-\_a^2+1}\,\_a+(y-1)\,\_a^2-y+1\right)}{(-\_a^2+1)^{\frac{5}{2}}}d\_a+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=-\left(\int^x\frac{e^{\frac{\_a}{\sqrt{-\_a^2+1}}}\left(\_a^2-1\right)}{(-\_a^2+1)^{\frac{5}{2}}}d\_a\right)+f'(y) \tag{4}$$
$$=\int^x\frac{e^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{(-\_a^2+1)^{\frac{3}{2}}}d\_a+f'(y)$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=e^{\frac{x}{\sqrt{-x^2+1}}}$. Therefore equation (4) becomes

$$e^{\frac{x}{\sqrt{-x^2+1}}}=\int^x\frac{e^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{(-\_a^2+1)^{\frac{3}{2}}}d\_a+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\left(\int^{x} \frac{\mathrm{e}^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{\left(-\_a^2+1\right)^{\frac{3}{2}}} d\_a\right) + \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\left(\int^{x} \frac{\mathrm{e}^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{\left(-\_a^2+1\right)^{\frac{3}{2}}} d\_a\right) + \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}\right) \mathrm{d}y$$

$$f(y) = \left(-\left(\int^{x} \frac{\mathrm{e}^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{\left(-\_a^2+1\right)^{\frac{3}{2}}} d\_a\right) + \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}\right) y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^{x} -\frac{\mathrm{e}^{\frac{\_a}{\sqrt{-\_a^2+1}}} \left(\sqrt{-\_a^2+1}\,\_a + (y-1)\_a^2 - y + 1\right)}{\left(-\_a^2+1\right)^{\frac{5}{2}}} d\_a + \left(-\left(\int^{x} \frac{\mathrm{e}^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{\left(-\_a^2+1\right)^{\frac{3}{2}}} d\_a\right) + \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}\right) y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^{x} -\frac{\mathrm{e}^{\frac{\_a}{\sqrt{-\_a^2+1}}} \left(\sqrt{-\_a^2+1}\,\_a + (y-1)\_a^2 - y + 1\right)}{\left(-\_a^2+1\right)^{\frac{5}{2}}} d\_a + \left(-\left(\int^{x} \frac{\mathrm{e}^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{\left(-\_a^2+1\right)^{\frac{3}{2}}} d\_a\right) + \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}\right) y$$

Summary
The solution(s) found are the following

$$\int^{x} -\frac{e^{\frac{\_a}{\sqrt{-\_a^{2}+1}}} \left(\sqrt{-\_a^{2}+1}\,\_a+(y-1)\,\_a^{2}-y+1\right)}{\left(-\_a^{2}+1\right)^{\frac{5}{2}}}d\_a + \left(-\left(\int^{x} \frac{e^{\frac{\_a}{\sqrt{-\_a^{2}+1}}}}{\left(-\_a^{2}+1\right)^{\frac{3}{2}}}d\_a\right)+e^{\frac{x}{\sqrt{-x^{2}+1}}}\right) y = c_{1} \tag{1}$$

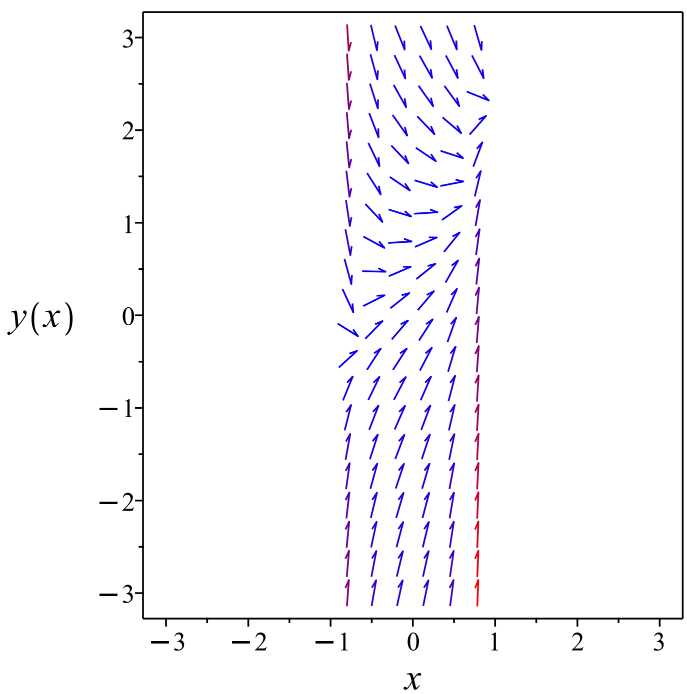


Figure 164: Slope field plot

Verification of solutions

$$\int^{x} -\frac{e^{\frac{\_a}{\sqrt{-\_a^{2}+1}}} \left(\sqrt{-\_a^{2}+1}\,\_a+(y-1)\,\_a^{2}-y+1\right)}{\left(-\_a^{2}+1\right)^{\frac{5}{2}}}d\_a + \left(-\left(\int^{x} \frac{e^{\frac{\_a}{\sqrt{-\_a^{2}+1}}}}{\left(-\_a^{2}+1\right)^{\frac{3}{2}}}d\_a\right)+e^{\frac{x}{\sqrt{-x^{2}+1}}}\right) y = c_{1}$$

Verified OK.

12.12.4 Maple step by step solution

Let's solve

$y' + \frac{y}{(-x^2+1)^{\frac{3}{2}}} = \frac{x+\sqrt{-x^2+1}}{(-x^2+1)^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{(-x^2+1)^{\frac{3}{2}}} + \frac{x+\sqrt{-x^2+1}}{(x^2-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{(-x^2+1)^{\frac{3}{2}}} = \frac{x+\sqrt{-x^2+1}}{(x^2-1)^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{(-x^2+1)^{\frac{3}{2}}}\right) = \frac{\mu(x)\left(x+\sqrt{-x^2+1}\right)}{(x^2-1)^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{(-x^2+1)^{\frac{3}{2}}}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{(-x^2+1)^{\frac{3}{2}}}$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\frac{x}{\sqrt{-(x-1)(1+x)}}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)\left(x+\sqrt{-x^2+1}\right)}{(x^2-1)^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)\left(x+\sqrt{-x^2+1}\right)}{(x^2-1)^2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\left(x+\sqrt{-x^2+1}\right)}{\left(x^2-1\right)^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{\frac{x}{\sqrt{-(x-1)(1+x)}}}$

$$y = \frac{\int \frac{e^{\frac{x}{\sqrt{-(x-1)(1+x)}}}\left(x+\sqrt{-x^2+1}\right)}{\left(x^2-1\right)^2}dx+c_1}{e^{\frac{x}{\sqrt{-(x-1)(1+x)}}}}$$

- Simplify

$$y = \left(\int \frac{e^{\frac{x}{\sqrt{-x^2+1}}}\left(x+\sqrt{-x^2+1}\right)}{\left(x^2-1\right)^2}dx + c_1\right) e^{-\frac{x}{\sqrt{-x^2+1}}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 56

```
dsolve(diff(y(x),x)+y(x)/(1-x^2)^(3/2)= (x+(1-x^2)^(1/2))/(1-x^2)^2,y(x), singsol=all)
```

$$y(x) = \left(\int \frac{e^{\frac{x}{\sqrt{-x^2+1}}}\left(x+\sqrt{-x^2+1}\right)}{\left(-1+x\right)^2\left(1+x\right)^2}dx + c_1\right) e^{-\frac{x}{\sqrt{-x^2+1}}}$$

✓ Solution by Mathematica

Time used: 0.358 (sec). Leaf size: 38

```
DSolve[y'[x]+y[x]/(1-x^2)^(3/2)== (x+(1-x^2)^(1/2))/(1-x^2)^2,y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{x}{\sqrt{1-x^2}} + c_1 e^{-\frac{x}{\sqrt{1-x^2}}}$$

12.13 problem Ex 14

Internal problem ID [11183]
Internal file name [`OUTPUT/10168_Saturday_December_03_2022_08_03_14_AM_84129772/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(-x^2+1)\, y' - yx - y^2ax = 0}$$

12.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{xy(ay+1)}{x^2-1} \end{aligned}$$

Where $f(x)=-\frac{x}{x^2-1}$ and $g(y)=y(ay+1)$. Integrating both sides gives

$$\frac{1}{y\left(ay+1\right)}\,dy=-\frac{x}{x^2-1}\,dx$$

$$\int\frac{1}{y\left(ay+1\right)}\,dy=\int-\frac{x}{x^2-1}\,dx$$

$$-\ln\left(ay+1\right)+\ln\left(y\right)=-\frac{\ln\left(x-1\right)}{2}-\frac{\ln\left(1+x\right)}{2}+c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{-\ln(ay+1)+\ln(y)}=\mathrm{e}^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}+c_1}$$

Which simplifies to

$$\frac{y}{ay+1}=c_2\mathrm{e}^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}}$$

Which simplifies to

$$y=-\frac{c_2}{\sqrt{x-1}\,\sqrt{1+x}\,\left(-1+\frac{c_2a}{\sqrt{x-1}\,\sqrt{1+x}}\right)}$$

Summary

The solution(s) found are the following

$$y=-\frac{c_2}{\sqrt{x-1}\,\sqrt{1+x}\,\left(-1+\frac{c_2a}{\sqrt{x-1}\,\sqrt{1+x}}\right)} \tag{1}$$

Verification of solutions

$$y=-\frac{c_2}{\sqrt{x-1}\,\sqrt{1+x}\,\left(-1+\frac{c_2a}{\sqrt{x-1}\,\sqrt{1+x}}\right)}$$

Verified OK.

12.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{xy(ay+1)}{x^2-1}\\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 90: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{x^2 - 1}{x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{x^2-1}{x}} dx
\end{aligned}
$$

Which results in

$$
S = -\frac{\ln(x - 1)}{2} - \frac{\ln(1 + x)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{xy(ay + 1)}{x^2 - 1}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{x}{x^2-1} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y\left(ay+1\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R\left(Ra+1\right)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln\left(Ra+1\right) + \ln\left(R\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln\left(x-1\right)}{2} - \frac{\ln\left(1+x\right)}{2} = -\ln\left(ay+1\right) + \ln\left(y\right) + c_1$$

Which simplifies to

$$-\frac{\ln\left(x-1\right)}{2} - \frac{\ln\left(1+x\right)}{2} = -\ln\left(ay+1\right) + \ln\left(y\right) + c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln\left(x-1\right)}{2} - \frac{\ln\left(1+x\right)}{2} = -\ln\left(ay+1\right) + \ln\left(y\right) + c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln\left(x-1\right)}{2} - \frac{\ln\left(1+x\right)}{2} = -\ln\left(ay+1\right) + \ln\left(y\right) + c_1$$

Verified OK.

12.13.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{xy(ay+1)}{x^2-1} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{x}{x^2-1}y - \frac{ax}{x^2-1}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{x}{x^2-1} \\ f_1(x) &= -\frac{ax}{x^2-1} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{x}{(x^2-1)\,y} - \frac{ax}{x^2-1} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{align*}
-w'(x) &= -\frac{x w(x)}{x^2-1} - \frac{a x}{x^2-1} \\
w' &= \frac{x w}{x^2-1} + \frac{a x}{x^2-1} \tag{7}
\end{align*}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{align*}
p(x) &= -\frac{x}{x^2-1} \\
q(x) &= \frac{a x}{x^2-1}
\end{align*}
$$

Hence the ode is

$$
w'(x) - \frac{x w(x)}{x^2-1} = \frac{a x}{x^2-1}
$$

The integrating factor μ is

$$
\begin{align*}
\mu &= \mathrm{e}^{\int -\frac{x}{x^2-1}dx} \\
&= \mathrm{e}^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}}
\end{align*}
$$

Which simplifies to

$$
\mu = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}
$$

The ode becomes

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\frac{a x}{x^2-1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{\sqrt{x-1}\,\sqrt{1+x}}\right) &= \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)\left(\frac{a x}{x^2-1}\right) \\
\mathrm{d}\left(\frac{w}{\sqrt{x-1}\,\sqrt{1+x}}\right) &= \left(\frac{a x}{(x^2-1)\sqrt{x-1}\,\sqrt{1+x}}\right)\mathrm{d}x
\end{align*}
$$

Integrating gives

$$
\begin{align*}
\frac{w}{\sqrt{x-1}\,\sqrt{1+x}} &= \int \frac{a x}{(x^2-1)\sqrt{x-1}\,\sqrt{1+x}}\,\mathrm{d}x \\
\frac{w}{\sqrt{x-1}\,\sqrt{1+x}} &= -\frac{\sqrt{x-1}\,\sqrt{1+x}\,a}{x^2-1} + c_1
\end{align*}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$ results in

$$w(x) = -\frac{(x-1)\,(1+x)\,a}{x^2-1} + c_1\sqrt{x-1}\,\sqrt{1+x}$$

which simplifies to

$$w(x) = -a + c_1\sqrt{x-1}\,\sqrt{1+x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = -a + c_1\sqrt{x-1}\,\sqrt{1+x}$$

Or

$$y = \frac{1}{-a + c_1\sqrt{x-1}\,\sqrt{1+x}}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{-a + c_1\sqrt{x-1}\,\sqrt{1+x}} \tag{1}$$

Verification of solutions

$$y = \frac{1}{-a + c_1\sqrt{x-1}\,\sqrt{1+x}}$$

Verified OK.

12.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, dx + N(x, y) \, dy = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y (ay + 1)}\right) dy = \left(\frac{x}{x^2 - 1}\right) dx$$
$$\left(-\frac{x}{x^2 - 1}\right) dx + \left(-\frac{1}{y (ay + 1)}\right) dy = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x}{x^2 - 1}$$
$$N(x, y) = -\frac{1}{y (ay + 1)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{x}{x^2 - 1}\right)$$
$$= 0$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y\,(ay+1)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{x}{x^2-1}\,\mathrm{d}x$$

$$\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y(ay+1)}$. Therefore equation (4) becomes

$$-\frac{1}{y\,(ay+1)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y\,(ay+1)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y\,(ay+1)}\right)\mathrm{d}y$$

$$f(y) = \ln(ay+1) - \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2} + \ln(ay+1) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2} + \ln(ay+1) - \ln(y)$$

Summary

The solution(s) found are the following

$$-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2} + \ln(ay+1) - \ln(y) = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2} + \ln(ay+1) - \ln(y) = c_1$$

Verified OK.

12.13.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{xy(ay+1)}{x^2-1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{x\,y^2 a}{x^2-1} - \frac{xy}{x^2-1}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{x}{x^2-1}$ and $f_2(x) = -\frac{ax}{x^2-1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{axu}{x^2-1}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{a}{x^2-1} + \frac{2a\,x^2}{(x^2-1)^2} \\ f_1 f_2 &= \frac{a\,x^2}{(x^2-1)^2} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{axu''(x)}{x^2-1} - \left(-\frac{a}{x^2-1} + \frac{3a\,x^2}{(x^2-1)^2}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{\sqrt{x^2-1}}$$

The above shows that

$$u'(x) = -\frac{c_2 x}{(x^2-1)^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{\sqrt{x^2-1}\,a\left(c_1 + \frac{c_2}{\sqrt{x^2-1}}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{a\left(c_3\sqrt{x^2-1}+1\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{a\left(c_3\sqrt{x^2-1}+1\right)} \tag{1}$$

Verification of solutions

$$y = -\frac{1}{a\left(c_3\sqrt{x^2-1}+1\right)}$$

Verified OK.

12.13.6 Maple step by step solution

Let's solve

$(-x^2+1)\, y' - yx - y^2 a x = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(ay+1)} = -\frac{x}{(x-1)(1+x)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(ay+1)} dx = \int -\frac{x}{(x-1)(1+x)} dx + c_1$

- Evaluate integral

  $-\ln(ay+1) + \ln(y) = -\frac{\ln((x-1)(1+x))}{2} + c_1$

- Solve for y

  $\left\{y = \frac{-e^{2c_1}a+\sqrt{x^2 e^{2c_1}-e^{2c_1}}}{e^{2c_1}a^2-x^2+1}, y = -\frac{e^{2c_1}a+\sqrt{x^2 e^{2c_1}-e^{2c_1}}}{e^{2c_1}a^2-x^2+1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve((1-x^2)*diff(y(x),x)-x*y(x)=a*x*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{-1+x}\,\sqrt{1+x}\,c_1 - a}$$

✓ Solution by Mathematica

Time used: 4.13 (sec). Leaf size: 47

```
DSolve[(1-x^2)*y'[x]-x*y[x]==a*x*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{c_1}}{-\sqrt{1-x^2}+ae^{c_1}}$$
$$y(x) \to 0$$
$$y(x) \to -\frac{1}{a}$$

12.14 problem Ex 15

Internal problem ID [11184]
Internal file name [`OUTPUT/10169_Saturday_December_03_2022_08_03_16_AM_61179337/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{xy^2(3y + y'x) - 2y + y'x = 0}$$

12.14.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(3x\,y^2 - 2)}{x\,(x\,y^2 + 1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{y(3x\,y^2-2)\,(b_3-a_2)}{x\,(x\,y^2+1)} - \frac{y^2(3x\,y^2-2)^2\,a_3}{x^2\,(x\,y^2+1)^2} \\ & - \left(-\frac{3y^3}{x\,(x\,y^2+1)} + \frac{y(3x\,y^2-2)}{x^2\,(x\,y^2+1)} + \frac{y^3(3x\,y^2-2)}{x\,(x\,y^2+1)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{3x\,y^2-2}{x\,(x\,y^2+1)} - \frac{6y^2}{x\,y^2+1} + \frac{2y^2(3x\,y^2-2)}{(x\,y^2+1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{4x^4y^4b_2 - 12x^2y^6a_3 + 3x^3y^4b_1 - 3x^2y^5a_1 + 13x^3y^2b_2 + 5x^2y^3a_2 + 10x^2y^3b_3 + 16x\,y^4a_3 + 11x^2y^2b_1 + 4x\,y}{x^2\,(x\,y^2+1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 4x^4y^4b_2 - 12x^2y^6a_3 + 3x^3y^4b_1 - 3x^2y^5a_1 + 13x^3y^2b_2 + 5x^2y^3a_2 + 10x^2y^3b_3 \\ & + 16x\,y^4a_3 + 11x^2y^2b_1 + 4x\,y^3a_1 - b_2x^2 - 2y^2a_3 - 2xb_1 + 2ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -12a_3v_1^2v_2^6 + 4b_2v_1^4v_2^4 - 3a_1v_1^2v_2^5 + 3b_1v_1^3v_2^4 + 5a_2v_1^2v_2^3 + 16a_3v_1v_2^4 + 13b_2v_1^3v_2^2 \\ & + 10b_3v_1^2v_2^3 + 4a_1v_1v_2^3 + 11b_1v_1^2v_2^2 - 2a_3v_2^2 - b_2v_1^2 + 2a_1v_2 - 2b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$4b_2v_1^4v_2^4 + 3b_1v_1^3v_2^4 + 13b_2v_1^3v_2^2 - 12a_3v_1^2v_2^6 - 3a_1v_1^2v_2^5 + (5a_2 + 10b_3)\,v_1^2v_2^3 + 11b_1v_1^2v_2^2 - b_2v_1^2 + 16a_3v_1v_2^4 + 4a_1v_1v_2^3 - 2b_1v_1 - 2a_3v_2^2 + 2a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-3a_1 &= 0 \\
2a_1 &= 0 \\
4a_1 &= 0 \\
-12a_3 &= 0 \\
-2a_3 &= 0 \\
16a_3 &= 0 \\
-2b_1 &= 0 \\
3b_1 &= 0 \\
11b_1 &= 0 \\
-b_2 &= 0 \\
4b_2 &= 0 \\
13b_2 &= 0 \\
5a_2 + 10b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= -2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -2x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y(3x\,y^2-2)}{x\,(x\,y^2+1)}\right)(-2x) \\
&= \frac{-5x\,y^3+5y}{x\,y^2+1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-5x\,y^3+5y}{x\,y^2+1}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{5} - \frac{\ln(x\,y^2-1)}{5}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(3x\,y^2-2)}{x\,(x\,y^2+1)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y^2}{5x\,y^2 - 5} \\
S_y &= \frac{1}{5y} - \frac{2xy}{5x\,y^2 - 5}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{2}{5x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{2}{5R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{2\ln(R)}{5} + c_1 \tag{4}
$$

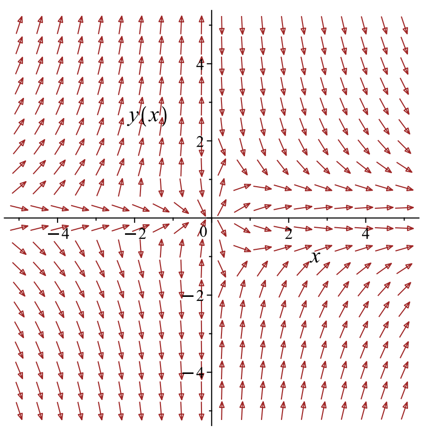
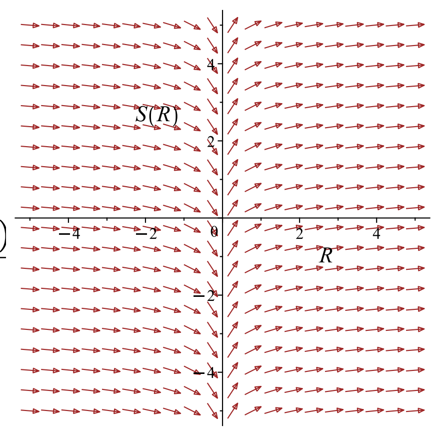
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(y)}{5} - \frac{\ln(y^2 x - 1)}{5} = \frac{2\ln(x)}{5} + c_1
$$

Which simplifies to

$$
\frac{\ln(y)}{5} - \frac{\ln(y^2 x - 1)}{5} = \frac{2\ln(x)}{5} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(3x\,y^2-2)}{x(x\,y^2+1)}$ 	$R = x$ $S = \frac{\ln(y)}{5} - \frac{\ln(x\,y^2-1)}{5}$	$\frac{dS}{dR} = \frac{2}{5R}$ 

Summary
The solution(s) found are the following

$$\frac{\ln(y)}{5} - \frac{\ln(y^2x-1)}{5} = \frac{2\ln(x)}{5} + c_1 \tag{1}$$

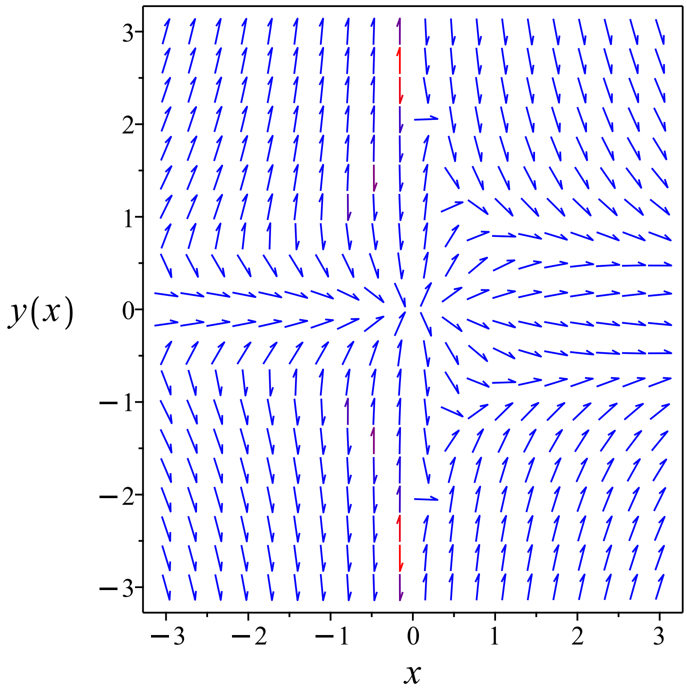


Figure 165: Slope field plot

Verification of solutions

$$\frac{\ln (y)}{5}-\frac{\ln (y^2x-1)}{5}=\frac{2\ln (x)}{5}+c_1$$

Verified OK.

12.14.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(x^2 y^2 + x\right) \mathrm{d}y &= \left(-3x\, y^3 + 2y\right) \mathrm{d}x \\
\left(3x\, y^3 - 2y\right) \mathrm{d}x + \left(x^2 y^2 + x\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 3x\, y^3 - 2y \\
N(x, y) &= x^2 y^2 + x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3x\, y^3 - 2y\right) \\
&= 9x\, y^2 - 2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 y^2 + x\right) \\
&= 2x\, y^2 + 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{x}{y^2}$ is an integrating factor. Therefore by multiplying $M = 3y^3x - 2y$ and $N = x^2y^2 + x$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{x(3y^3x - 2y)}{y^2}$$
$$N = \frac{x(x^2y^2 + x)}{y^2}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{x(x^2y^2+x)}{y^2}\right)\mathrm{d}y = \left(-\frac{x(3x\,y^3-2y)}{y^2}\right)\mathrm{d}x$$

$$\left(\frac{x(3x\,y^3-2y)}{y^2}\right)\mathrm{d}x + \left(\frac{x(x^2y^2+x)}{y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{x(3x\,y^3-2y)}{y^2}$$
$$N(x,y) = \frac{x(x^2y^2+x)}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x(3x\,y^3-2y)}{y^2}\right) = \frac{3x^2y^2+2x}{y^2}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x(x^2y^2+x)}{y^2}\right) = \frac{3x^2y^2+2x}{y^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{x(3x\,y^3-2y)}{y^2}\,\mathrm{d}x$$
$$\phi = \frac{x^2(x\,y^2-1)}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= -\frac{x^2(x\,y^2-1)}{y^2} + 2x^3 + f'(y) \\ &= \frac{x^2(x\,y^2+1)}{y^2} + f'(y) \end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x(x^2y^2+x)}{y^2}$. Therefore equation (4) becomes

$$\frac{x(x^2y^2+x)}{y^2} = \frac{x^2(x\,y^2+1)}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2(x\,y^2-1)}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2(x\,y^2-1)}{y}$$

Summary

The solution(s) found are the following

$$\frac{x^2(y^2x-1)}{y} = c_1 \tag{1}$$

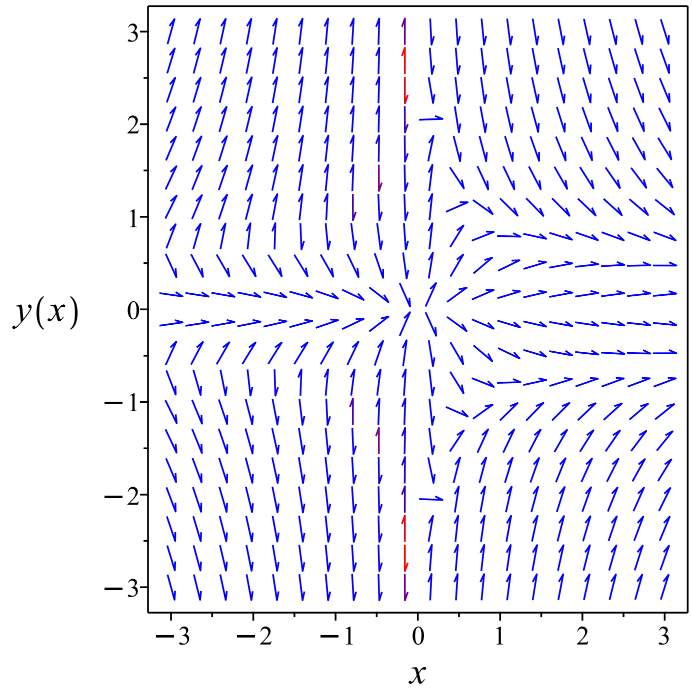


Figure 166: Slope field plot

Verification of solutions

$$\frac{x^2(y^2x-1)}{y}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 45

```
dsolve((x*y(x)^2)*(3*y(x)+x*diff(y(x),x))-(2*y(x)-x*diff(y(x),x))=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 - \sqrt{4x^5 + c_1^2}}{2x^3}$$
$$y(x) = \frac{c_1 + \sqrt{4x^5 + c_1^2}}{2x^3}$$

✓ Solution by Mathematica

Time used: 1.836 (sec). Leaf size: 75

```
DSolve[(x*y[x]^2)*(3*y[x]+x*y'[x])-(2*y[x]-x*y'[x])==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to -\frac{\sqrt{4x^5 + e^{5c_1}} + e^{\frac{5c_1}{2}}}{2x^3}$$
$$y(x) \to \frac{\sqrt{4x^5 + e^{5c_1}} - e^{\frac{5c_1}{2}}}{2x^3}$$

12.15 problem Ex 16

Internal problem ID [11185]
Internal file name [`OUTPUT/10170_Saturday_December_03_2022_08_03_18_AM_95752159/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x^2+1)\,y'+y=\arctan(x)}$$

12.15.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{1}{x^2+1}$$
$$q(x)=\frac{\arctan(x)}{x^2+1}$$

Hence the ode is

$$y'+\frac{y}{x^2+1}=\frac{\arctan(x)}{x^2+1}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{x^2+1}dx}$$
$$= e^{\arctan(x)}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\arctan(x)}{x^2+1}\right)$$
$$\frac{d}{dx}\left(e^{\arctan(x)}y\right) = \left(e^{\arctan(x)}\right)\left(\frac{\arctan(x)}{x^2+1}\right)$$
$$d\left(e^{\arctan(x)}y\right) = \left(\frac{\arctan(x)\, e^{\arctan(x)}}{x^2+1}\right) dx$$

Integrating gives

$$e^{\arctan(x)}y = \int \frac{\arctan(x)\, e^{\arctan(x)}}{x^2+1}\, dx$$
$$e^{\arctan(x)}y = \arctan(x)\, e^{\arctan(x)} - e^{\arctan(x)} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\arctan(x)}$ results in

$$y = e^{-\arctan(x)}\left(\arctan(x)\, e^{\arctan(x)} - e^{\arctan(x)}\right) + c_1 e^{-\arctan(x)}$$

which simplifies to

$$y = \arctan(x) - 1 + c_1 e^{-\arctan(x)}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \arctan(x) - 1 + c_1 e^{-\arctan(x)} \tag{1}$$

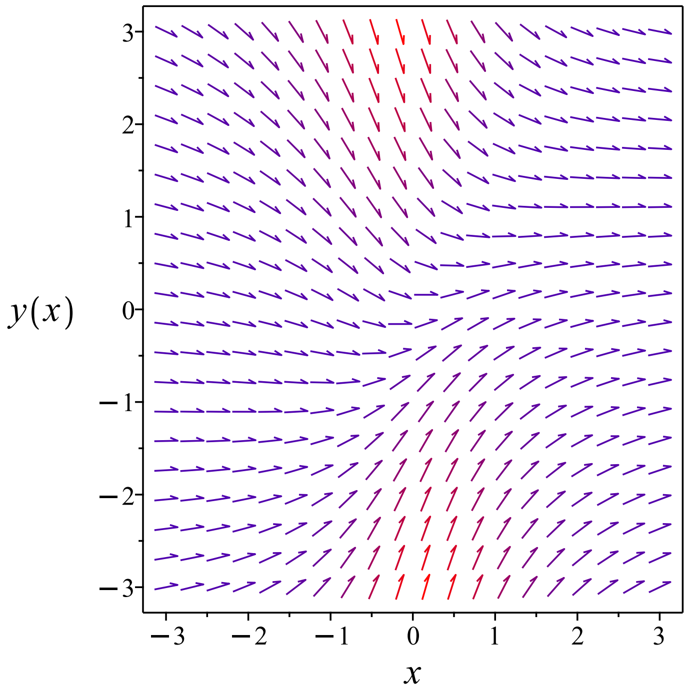


Figure 167: Slope field plot

Verification of solutions

$$y = \arctan(x) - 1 + c_1 e^{-\arctan(x)}$$

Verified OK.

12.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-y + \arctan(x)}{x^2 + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 93: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \mathrm{e}^{-\arctan(x)} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\arctan(x)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\arctan(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-y + \arctan(x)}{x^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\mathrm{e}^{\arctan(x)} y}{x^2 + 1} \\ S_y &= \mathrm{e}^{\arctan(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\arctan(x) \, \mathrm{e}^{\arctan(x)}}{x^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\arctan(R) \, \mathrm{e}^{\arctan(R)}}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R) e^{\arctan(R)} - e^{\arctan(R)} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\arctan(x)} y = \arctan(x) e^{\arctan(x)} - e^{\arctan(x)} + c_1$$

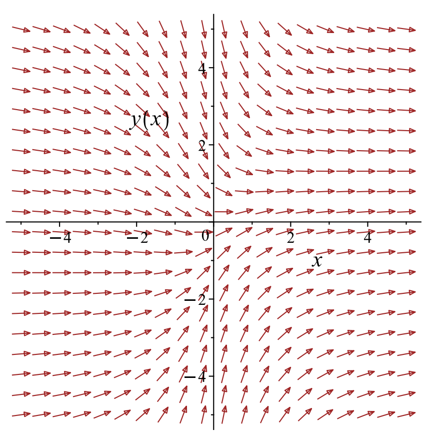
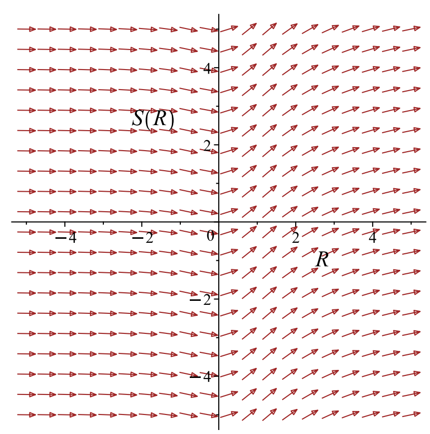
Which simplifies to

$$(y - \arctan(x) + 1) e^{\arctan(x)} - c_1 = 0$$

Which gives

$$y = \left(\arctan(x) e^{\arctan(x)} - e^{\arctan(x)} + c_1\right) e^{-\arctan(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y+\arctan(x)}{x^2+1}$ 	$R = x$ $S = e^{\arctan(x)} y$	$\frac{dS}{dR} = \frac{\arctan(R) e^{\arctan(R)}}{R^2+1}$ 

Summary

The solution(s) found are the following

$$y = \left(\arctan(x) e^{\arctan(x)} - e^{\arctan(x)} + c_1\right) e^{-\arctan(x)} \tag{1}$$

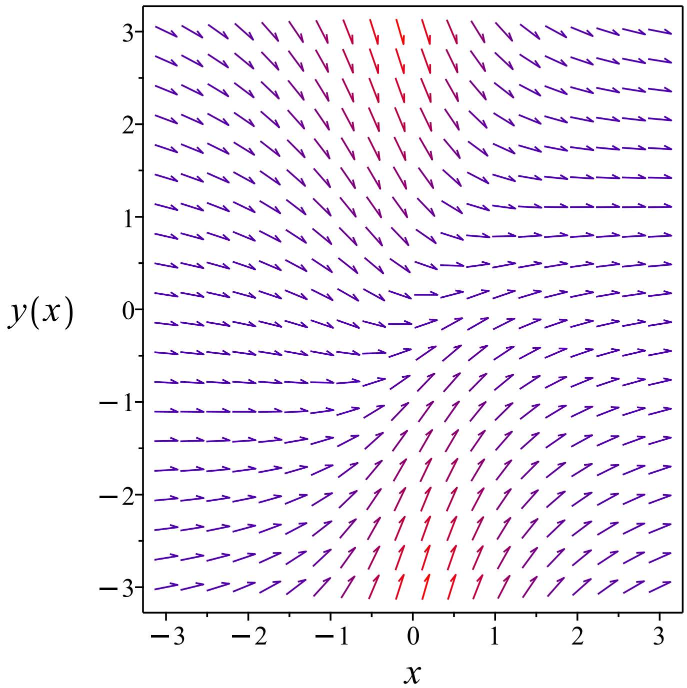


Figure 168: Slope field plot

Verification of solutions

$$y = \left(\arctan(x)\, e^{\arctan(x)} - e^{\arctan(x)} + c_1\right) e^{-\arctan(x)}$$

Verified OK.

12.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
\left(x^2 + 1\right) \mathrm{d}y &= (-y + \arctan(x))\,\mathrm{d}x \\
(y - \arctan(x))\,\mathrm{d}x + \left(x^2 + 1\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y - \arctan(x) \\
N(x, y) &= x^2 + 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \arctan(x)) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 + 1\right) \\
&= 2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^2+1}((1) - (2x)) \\
&= \frac{-2x+1}{x^2+1}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{-2x+1}{x^2+1} \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\ln(x^2+1)+\arctan(x)} \\
&= \frac{\mathrm{e}^{\arctan(x)}}{x^2+1}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\mathrm{e}^{\arctan(x)}}{x^2+1}(y - \arctan(x)) \\
&= \frac{(y - \arctan(x))\,\mathrm{e}^{\arctan(x)}}{x^2+1}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\mathrm{e}^{\arctan(x)}}{x^2+1}(x^2+1) \\
&= \mathrm{e}^{\arctan(x)}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{(y - \arctan(x))\,\mathrm{e}^{\arctan(x)}}{x^2+1}\right) + \left(\mathrm{e}^{\arctan(x)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{(y - \arctan(x)) \, \mathrm{e}^{\arctan(x)}}{x^2 + 1} \, \mathrm{d}x$$
$$\phi = (y - \arctan(x) + 1) \, \mathrm{e}^{\arctan(x)} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{\arctan(x)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\arctan(x)}$. Therefore equation (4) becomes

$$\mathrm{e}^{\arctan(x)} = \mathrm{e}^{\arctan(x)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (y - \arctan(x) + 1) \, \mathrm{e}^{\arctan(x)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y - \arctan(x) + 1) \, \mathrm{e}^{\arctan(x)}$$

Summary

The solution(s) found are the following

$$(y-\arctan(x)+1)\,e^{\arctan(x)} = c_1 \tag{1}$$

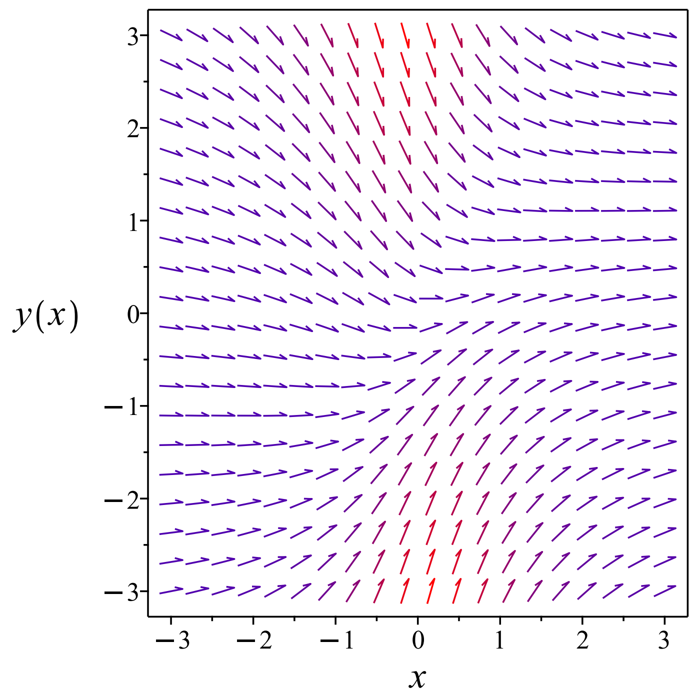


Figure 169: Slope field plot

Verification of solutions

$$(y-\arctan(x)+1)\,e^{\arctan(x)} = c_1$$

Verified OK.

12.15.4 Maple step by step solution

Let's solve

$(x^2+1)\,y'+y = \arctan(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x^2+1}+\frac{\arctan(x)}{x^2+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x^2+1} = \frac{\arctan(x)}{x^2+1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x^2+1}\right) = \frac{\mu(x)\arctan(x)}{x^2+1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{x^2+1}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x^2+1}$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\arctan(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)\arctan(x)}{x^2+1} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)\arctan(x)}{x^2+1} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\arctan(x)}{x^2+1} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{\arctan(x)}$

  $y = \frac{\int \frac{\arctan(x)\mathrm{e}^{\arctan(x)}}{x^2+1} dx + c_1}{\mathrm{e}^{\arctan(x)}}$

- Evaluate the integrals on the rhs

  $y = \frac{\arctan(x)\mathrm{e}^{\arctan(x)} - \mathrm{e}^{\arctan(x)} + c_1}{\mathrm{e}^{\arctan(x)}}$

- Simplify

  $y = \arctan(x) - 1 + c_1 \mathrm{e}^{-\arctan(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((1+x^2)*diff(y(x),x)+y(x)=arctan(x),y(x), singsol=all)
```

$$y(x) = \arctan(x) - 1 + e^{-\arctan(x)} c_1$$

✓ Solution by Mathematica

Time used: 0.23 (sec). Leaf size: 18

```
DSolve[(1+x^2)*y'[x]+y[x]==ArcTan[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arctan(x) + c_1 e^{-\arctan(x)} - 1$$

12.16 problem Ex 17

Internal problem ID [11186]
Internal file name [OUTPUT/10171_Saturday_December_03_2022_08_03_19_AM_4662408/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{5yx - 3y^3 + (3x^2 - 7y^2x)\, y' = 0}$$

12.16.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(3y^2 - 5x)}{x\,(7y^2 - 3x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(3y^2-5x)(b_3-a_2)}{x(7y^2-3x)} - \frac{y^2(3y^2-5x)^2 a_3}{x^2(7y^2-3x)^2} - \left(\frac{5y}{x(7y^2-3x)} + \frac{y(3y^2-5x)}{x^2(7y^2-3x)} - \frac{3y(3y^2-5x)}{x(7y^2-3x)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{3y^2-5x}{x(7y^2-3x)} - \frac{6y^2}{x(7y^2-3x)} + \frac{14y^2(3y^2-5x)}{x(7y^2-3x)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{70x^2y^4b_2 - 30y^6a_3 - 34x^3y^2b_2 - 26x^2y^3a_2 + 52x^2y^3b_3 + 48x\,y^4a_3 + 21x\,y^4b_1 - 21y^5a_1 + 24x^4b_2 - 40x^2y^2}{x^2(-7y^2+3x)^2} = 0$$

Setting the numerator to zero gives

$$70x^2y^4b_2 - 30y^6a_3 - 34x^3y^2b_2 - 26x^2y^3a_2 + 52x^2y^3b_3 + 48x\,y^4a_3 + 21x\,y^4b_1 - 21y^5a_1 + 24x^4b_2 - 40x^2y^2a_3 + 8x^2y^2b_1 + 18x\,y^3a_1 + 15x^3b_1 - 15x^2ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-30a_3v_2^6 + 70b_2v_1^2v_2^4 - 21a_1v_2^5 - 26a_2v_1^2v_2^3 + 48a_3v_1v_2^4 + 21b_1v_1v_2^4 - 34b_2v_1^3v_2^2 + 52b_3v_1^2v_2^3 + 18a_1v_1v_2^3 - 40a_3v_1^2v_2^2 + 8b_1v_1^2v_2^2 + 24b_2v_1^4 - 15a_1v_1^2v_2 + 15b_1v_1^3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&24b_2v_1^4 - 34b_2v_1^3v_2^2 + 15b_1v_1^3 + 70b_2v_1^2v_2^4 \\
&+ (-26a_2 + 52b_3)\, v_1^2v_2^3 + (-40a_3 + 8b_1)\, v_1^2v_2^2 - 15a_1v_1^2v_2 \\
&+ (48a_3 + 21b_1)\, v_1v_2^4 + 18a_1v_1v_2^3 - 30a_3v_2^6 - 21a_1v_2^5 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-21a_1 &= 0 \\
-15a_1 &= 0 \\
18a_1 &= 0 \\
-30a_3 &= 0 \\
15b_1 &= 0 \\
-34b_2 &= 0 \\
24b_2 &= 0 \\
70b_2 &= 0 \\
-26a_2 + 52b_3 &= 0 \\
-40a_3 + 8b_1 &= 0 \\
48a_3 + 21b_1 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 2x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y(3y^2-5x)}{x\,(7y^2-3x)}\right)(2x) \\
&= \frac{-13y^3+13xy}{-7y^2+3x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-13y^3+13xy}{-7y^2+3x}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{3\ln(y)}{13} + \frac{2\ln(y^2-x)}{13}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y(3y^2-5x)}{x\,(7y^2-3x)}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2}{-13y^2 + 13x} \\
S_y &= \frac{3}{13y} - \frac{4y}{-13y^2 + 13x}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{3}{13x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{3}{13R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{3\ln(R)}{13} + c_1 \tag{4}$$

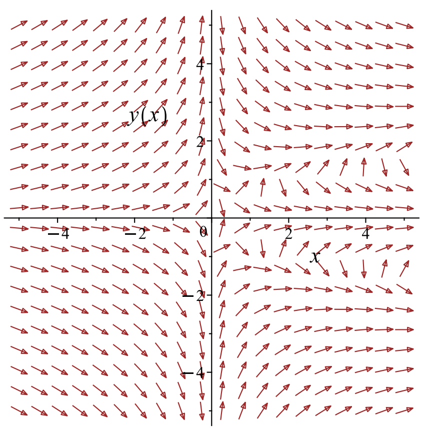
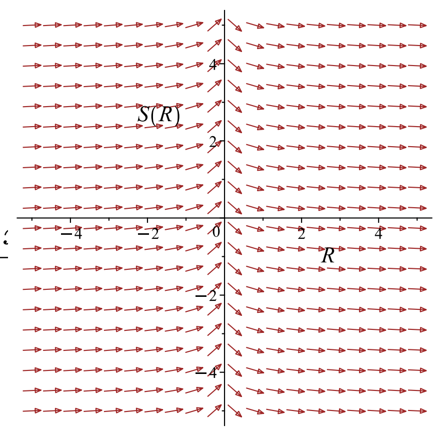
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3\ln(y)}{13} + \frac{2\ln(-x + y^2)}{13} = -\frac{3\ln(x)}{13} + c_1$$

Which simplifies to

$$\frac{3\ln(y)}{13} + \frac{2\ln(-x + y^2)}{13} = -\frac{3\ln(x)}{13} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(3y^2-5x)}{x(7y^2-3x)}$ 	$R = x$ $S = \frac{3\ln(y)}{13} + \frac{2\ln(y^2 -}{13}$	$\frac{dS}{dR} = -\frac{3}{13R}$ 

Summary

The solution(s) found are the following

$$\frac{3\ln(y)}{13} + \frac{2\ln(-x+y^2)}{13} = -\frac{3\ln(x)}{13} + c_1 \tag{1}$$

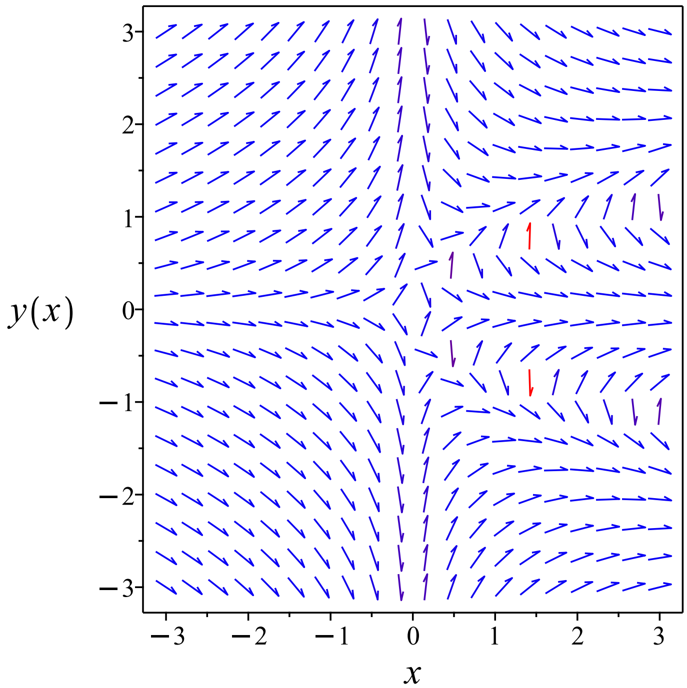


Figure 170: Slope field plot

Verification of solutions

$$\frac{3\ln\left(y\right)}{13}+\frac{2\ln\left(-x+y^{2}\right)}{13}=-\frac{3\ln\left(x\right)}{13}+c_{1}$$

Verified OK.

12.16.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-7x\,y^2 + 3x^2\right)\mathrm{d}y &= \left(3y^3 - 5xy\right)\mathrm{d}x \\
\left(-3y^3 + 5xy\right)\mathrm{d}x + \left(-7x\,y^2 + 3x^2\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -3y^3 + 5xy \\
N(x, y) &= -7x\,y^2 + 3x^2
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3y^3 + 5xy\right) \\
&= -9y^2 + 5x
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-7x\,y^2 + 3x^2\right) \\
&= -7y^2 + 6x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{-7x\,y^2 + 3x^2}\left(\left(-9y^2 + 5x\right) - \left(-7y^2 + 6x\right)\right) \\
&= \frac{-2y^2 - x}{-7x\,y^2 + 3x^2}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{-3y^3 + 5xy}\left(\left(-7y^2 + 6x\right) - \left(-9y^2 + 5x\right)\right) \\
&= \frac{2y^2 + x}{-3y^3 + 5xy}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{\left(-7y^2 + 6x\right) - \left(-9y^2 + 5x\right)}{x\left(-3y^3 + 5xy\right) - y\left(-7x\,y^2 + 3x^2\right)} \\
&= \frac{1}{2xy}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = \frac{1}{2t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(\frac{1}{2t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$\mu = e^{\frac{\ln(t)}{2}}$$
$$= \sqrt{t}$$

Now t is replaced back with xy giving

$$\mu = \sqrt{xy}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \sqrt{xy}\left(-3y^3 + 5xy\right) \\ &= y\left(-3y^2 + 5x\right)\sqrt{xy}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \sqrt{xy}\left(-7x\,y^2 + 3x^2\right) \\ &= x\left(-7y^2 + 3x\right)\sqrt{xy}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{dy}{dx} &= 0 \\ \left(y\left(-3y^2 + 5x\right)\sqrt{xy}\right) + \left(x\left(-7y^2 + 3x\right)\sqrt{xy}\right)\frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,dx &= \int \overline{M}\,dx \\ \int \frac{\partial \phi}{\partial x}\,dx &= \int y\left(-3y^2 + 5x\right)\sqrt{xy}\,dx\end{aligned}$$
$$\phi = 2x\left(-y^2 + x\right)y\sqrt{xy} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -4x\,y^2\sqrt{xy} + 2x\left(-y^2+x\right)\sqrt{xy} + \frac{x^2(-y^2+x)\,y}{\sqrt{xy}} + f'(y) \tag{4}$$

$$= \frac{y\,x^2(-7y^2+3x)}{\sqrt{xy}} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x(-7y^2+3x)\sqrt{xy}$. Therefore equation (4) becomes

$$x\left(-7y^2+3x\right)\sqrt{xy} = \frac{y\,x^2(-7y^2+3x)}{\sqrt{xy}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 2x\left(-y^2+x\right)y\sqrt{xy} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2x\left(-y^2+x\right)y\sqrt{xy}$$

<u>Summary</u>
The solution(s) found are the following

$$2x\left(-y^2+x\right)y\sqrt{yx} = c_1 \tag{1}$$

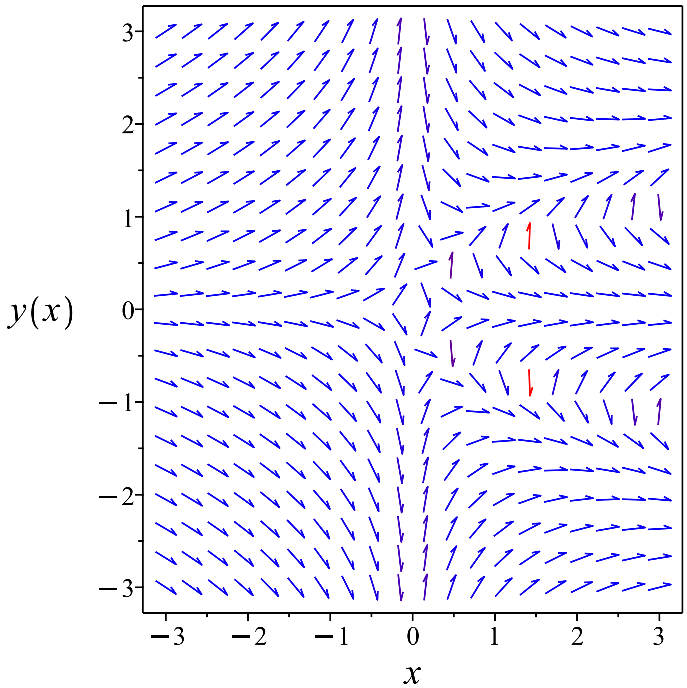


Figure 171: Slope field plot

Verification of solutions

$$2x\left(-y^2+x\right)y\sqrt{yx}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.453 (sec). Leaf size: 49

```
dsolve((5*x*y(x)-3*y(x)^3)+(3*x^2-7*x*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \operatorname{RootOf}\left(x^{\frac{3}{2}}\_Z^7 - \_Z^3 x^{\frac{5}{2}} - c_1\right)^2$$
$$y(x) = \operatorname{RootOf}\left(x^{\frac{3}{2}}\_Z^7 - \_Z^3 x^{\frac{5}{2}} + c_1\right)^2$$

✓ Solution by Mathematica

Time used: 7.756 (sec). Leaf size: 288

```
DSolve[(5*x*y[x]-3*y[x]^3)+(3*x^2-7*x*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \text{Root}\left[4\#1^7x^3 - 8\#1^5x^4 + 4\#1^3x^5 - c_1{}^2\&, 1\right]$$
$$y(x) \to \text{Root}\left[4\#1^7x^3 - 8\#1^5x^4 + 4\#1^3x^5 - c_1{}^2\&, 2\right]$$
$$y(x) \to \text{Root}\left[4\#1^7x^3 - 8\#1^5x^4 + 4\#1^3x^5 - c_1{}^2\&, 3\right]$$
$$y(x) \to \text{Root}\left[4\#1^7x^3 - 8\#1^5x^4 + 4\#1^3x^5 - c_1{}^2\&, 4\right]$$
$$y(x) \to \text{Root}\left[4\#1^7x^3 - 8\#1^5x^4 + 4\#1^3x^5 - c_1{}^2\&, 5\right]$$
$$y(x) \to \text{Root}\left[4\#1^7x^3 - 8\#1^5x^4 + 4\#1^3x^5 - c_1{}^2\&, 6\right]$$
$$y(x) \to \text{Root}\left[4\#1^7x^3 - 8\#1^5x^4 + 4\#1^3x^5 - c_1{}^2\&, 7\right]$$

12.17 problem Ex 18

Internal problem ID [11187]
Internal file name [`OUTPUT/10172_Saturday_December_03_2022_08_03_20_AM_89388370/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cos(x) = \frac{\sin(2x)}{2}}$$

12.17.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cos(x)$$
$$q(x) = \frac{\sin(2x)}{2}$$

Hence the ode is

$$y' + y\cos(x) = \frac{\sin(2x)}{2}$$

The integrating factor μ is

$$\mu = e^{\int \cos(x)dx} \\ = e^{\sin(x)}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\sin(2x)}{2}\right)$$
$$\frac{d}{dx}\left(e^{\sin(x)}y\right) = \left(e^{\sin(x)}\right)\left(\frac{\sin(2x)}{2}\right)$$
$$d\left(e^{\sin(x)}y\right) = \left(\frac{\sin(2x)\,e^{\sin(x)}}{2}\right) dx$$

Integrating gives

$$e^{\sin(x)}y = \int \frac{\sin(2x)\,e^{\sin(x)}}{2}\, dx$$
$$e^{\sin(x)}y = \sin(x)\,e^{\sin(x)} - e^{\sin(x)} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\sin(x)}$ results in

$$y = e^{-\sin(x)}\left(\sin(x)\,e^{\sin(x)} - e^{\sin(x)}\right) + c_1 e^{-\sin(x)}$$

which simplifies to

$$y = \sin(x) - 1 + c_1 e^{-\sin(x)}$$

Summary
The solution(s) found are the following

$$y = \sin(x) - 1 + c_1 e^{-\sin(x)} \tag{1}$$

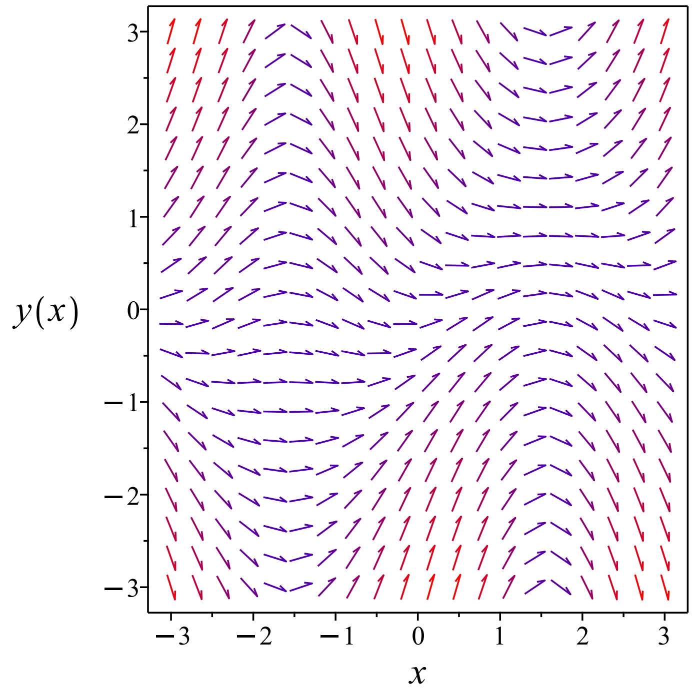


Figure 172: Slope field plot

Verification of solutions

$$y = \sin(x) - 1 + c_1 e^{-\sin(x)}$$

Verified OK.

12.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -y\cos(x) + \frac{\sin(2x)}{2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 96: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-\sin(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\sin(x)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\sin(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y \cos(x) + \frac{\sin(2x)}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x)\, \mathrm{e}^{\sin(x)} y \\ S_y &= \mathrm{e}^{\sin(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sin(2x)\, \mathrm{e}^{\sin(x)}}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\sin(2R)\, \mathrm{e}^{\sin(R)}}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 + e^{\sin(R)}(-1 + \sin(R)) \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\sin(x)} y = e^{\sin(x)}(-1 + \sin(x)) + c_1$$

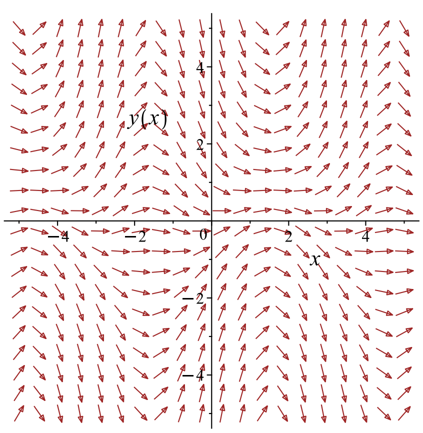
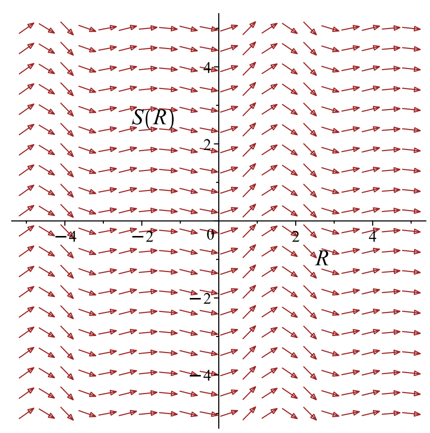
Which simplifies to

$$e^{\sin(x)} y = e^{\sin(x)}(-1 + \sin(x)) + c_1$$

Which gives

$$y = e^{-\sin(x)}\left(\sin(x) e^{\sin(x)} - e^{\sin(x)} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y \cos(x) + \frac{\sin(2x)}{2}$ 	$R = x$ $S = e^{\sin(x)} y$	$\frac{dS}{dR} = \frac{\sin(2R) e^{\sin(R)}}{2}$ 

Summary

The solution(s) found are the following

$$y = e^{-\sin(x)}\left(\sin(x) e^{\sin(x)} - e^{\sin(x)} + c_1\right) \tag{1}$$

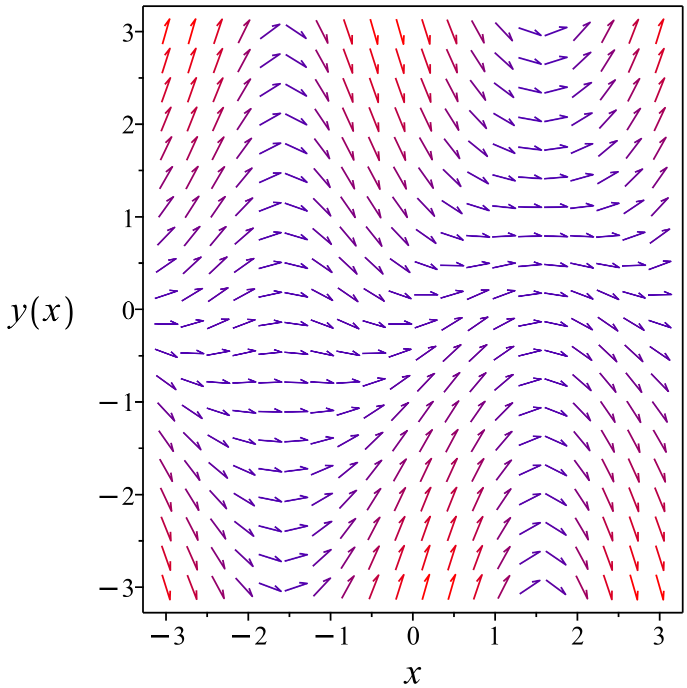


Figure 173: Slope field plot

Verification of solutions

$$y = e^{-\sin(x)}\left(\sin(x)\, e^{\sin(x)} - e^{\sin(x)} + c_1\right)$$

Verified OK.

12.17.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, dx + N(x, y) \, dy = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
dy &= \left(-y \cos(x) + \frac{\sin(2x)}{2}\right) dx \\
\left(y \cos(x) - \frac{\sin(2x)}{2}\right) dx + dy &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y \cos(x) - \frac{\sin(2x)}{2} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y \cos(x) - \frac{\sin(2x)}{2}\right) \\
&= \cos(x)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((\cos(x)) - (0)) \\
&= \cos(x)
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int \cos(x) \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\sin(x)} \\
&= \mathrm{e}^{\sin(x)}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{\sin(x)}\left(y\cos(x) - \frac{\sin(2x)}{2}\right) \\
&= \cos(x)\,(-\sin(x) + y)\,\mathrm{e}^{\sin(x)}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\sin(x)}(1) \\
&= \mathrm{e}^{\sin(x)}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\cos(x)\,(-\sin(x) + y)\,\mathrm{e}^{\sin(x)}\right) + \left(\mathrm{e}^{\sin(x)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \cos(x) (-\sin(x) + y) \,\mathrm{e}^{\sin(x)} \,\mathrm{d}x$$
$$\phi = (y - \sin(x) + 1) \,\mathrm{e}^{\sin(x)} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{\sin(x)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\sin(x)}$. Therefore equation (4) becomes

$$\mathrm{e}^{\sin(x)} = \mathrm{e}^{\sin(x)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (y - \sin(x) + 1) \,\mathrm{e}^{\sin(x)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y - \sin(x) + 1) \,\mathrm{e}^{\sin(x)}$$

The solution becomes

$$y = e^{-\sin(x)} \left(\sin(x) e^{\sin(x)} - e^{\sin(x)} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{-\sin(x)} \left(\sin(x) e^{\sin(x)} - e^{\sin(x)} + c_1\right) \tag{1}$$

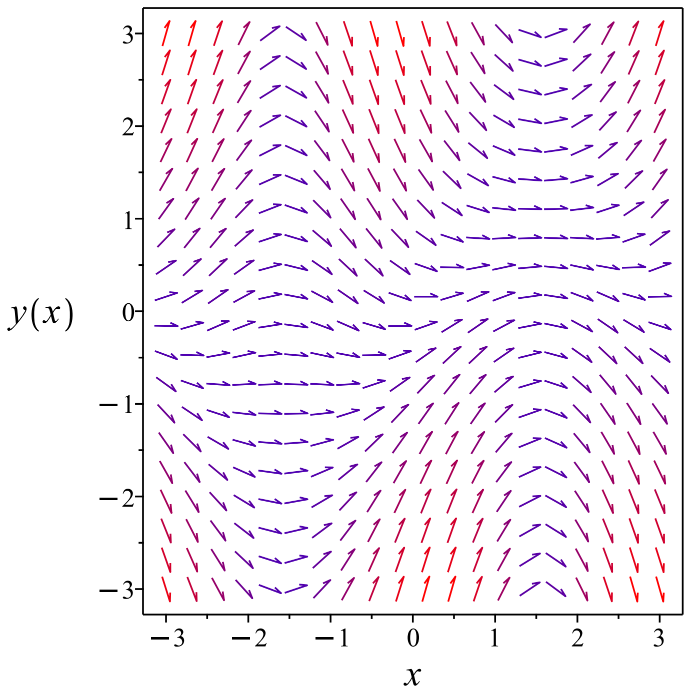


Figure 174: Slope field plot

Verification of solutions

$$y = e^{-\sin(x)} \left(\sin(x) e^{\sin(x)} - e^{\sin(x)} + c_1\right)$$

Verified OK.

12.17.4 Maple step by step solution

Let's solve

$y' + y\cos(x) = \frac{\sin(2x)}{2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y\cos(x) + \frac{\sin(2x)}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y\cos(x) = \frac{\sin(2x)}{2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + y\cos(x)\right) = \frac{\mu(x)\sin(2x)}{2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + y\cos(x)\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\cos(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\sin(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)\sin(2x)}{2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)\sin(2x)}{2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\sin(2x)}{2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{\sin(x)}$

  $y = \frac{\int \frac{\sin(2x)\mathrm{e}^{\sin(x)}}{2} dx + c_1}{\mathrm{e}^{\sin(x)}}$

- Evaluate the integrals on the rhs

  $y = \frac{\sin(x)\mathrm{e}^{\sin(x)} - \mathrm{e}^{\sin(x)} + c_1}{\mathrm{e}^{\sin(x)}}$

- Simplify

$$y = \sin(x) - 1 + c_1 e^{-\sin(x)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)+y(x)*cos(x)=1/2*sin(2*x),y(x), singsol=all)
```

$$y(x) = \sin(x) - 1 + e^{-\sin(x)} c_1$$

✓ Solution by Mathematica

Time used: 0.089 (sec). Leaf size: 18

```
DSolve[y'[x]+y[x]*Cos[x]==1/2*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x) + c_1 e^{-\sin(x)} - 1$$

12.18 problem Ex 19

Internal problem ID [11188]
Internal file name [`OUTPUT/10173_Saturday_December_03_2022_08_03_22_AM_10088190/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Bernoulli]
```

$$\boxed{y^2x + y - y'x = 0}$$

12.18.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^3 + u(x)\,x - (u'(x)\,x + u(x))\,x = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= u^2x\end{aligned}$$

Where $f(x) = x$ and $g(u) = u^2$. Integrating both sides gives

$$\frac{1}{u^2}\, du = x\, dx$$
$$\int \frac{1}{u^2}\, du = \int x\, dx$$
$$-\frac{1}{u} = \frac{x^2}{2} + c_2$$

The solution is

$$-\frac{1}{u(x)} - \frac{x^2}{2} - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x}{y} - \frac{x^2}{2} - c_2 = 0$$
$$-\frac{x}{y} - \frac{x^2}{2} - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\frac{x}{y} - \frac{x^2}{2} - c_2 = 0 \tag{1}$$

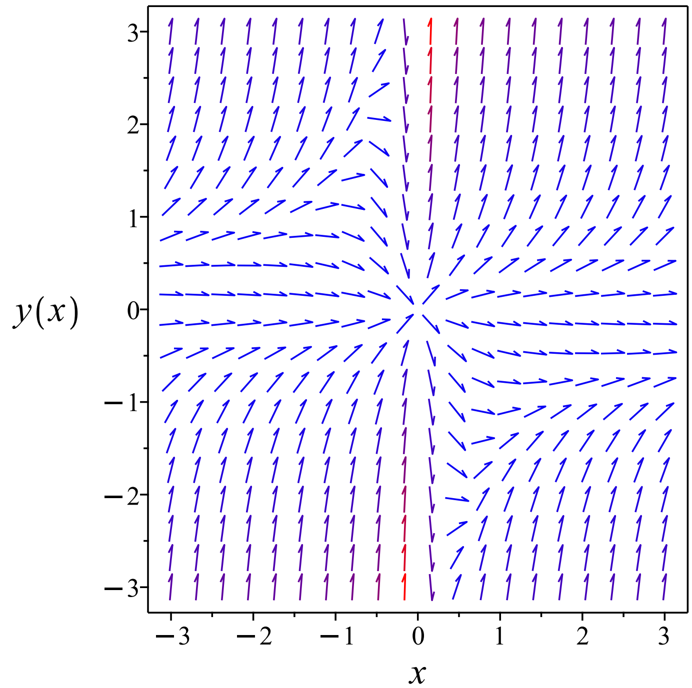


Figure 175: Slope field plot

Verification of solutions

$$-\frac{x}{y} - \frac{x^2}{2} - c_2 = 0$$

Verified OK.

12.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(xy+1)}{x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 99: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{y^2}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(xy+1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{y} \\ S_y &= \frac{x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = \frac{x^2}{2} + c_1$$

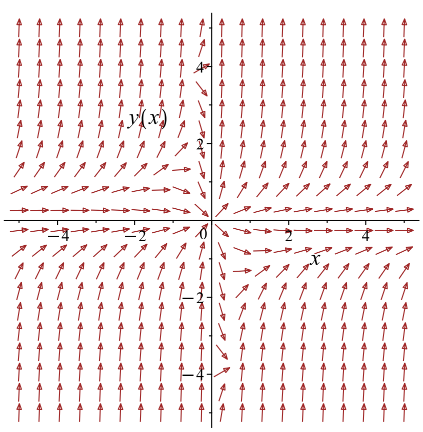
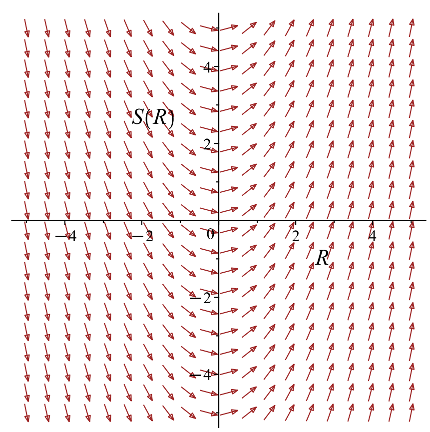
Which simplifies to

$$-\frac{x}{y} = \frac{x^2}{2} + c_1$$

Which gives

$$y = -\frac{2x}{x^2 + 2c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(xy+1)}{x}$ 	$R = x$ $S = -\frac{x}{y}$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$y = -\frac{2x}{x^2 + 2c_1} \tag{1}$$

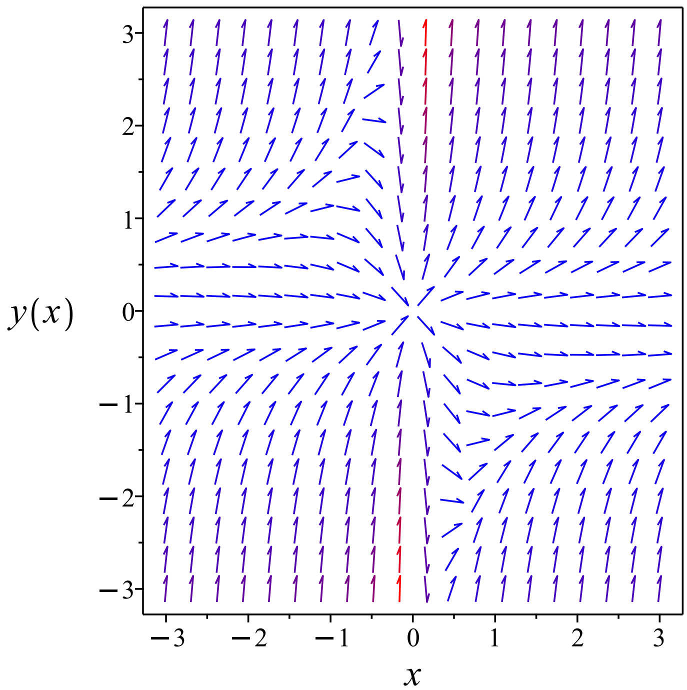


Figure 176: Slope field plot

Verification of solutions

$$y = -\frac{2x}{x^2 + 2c_1}$$

Verified OK.

12.18.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(xy + 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y + y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= 1 \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{xy} + 1 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} + 1 \\ w' &= -\frac{w}{x} - 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -1
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{w(x)}{x} = -1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x} dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)(-1) \\
\frac{d}{dx}(wx) &= (x)(-1) \\
d(wx) &= (-x)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
wx &= \int -x\, dx \\
wx &= -\frac{x^2}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
w(x) = -\frac{x}{2} + \frac{c_1}{x}
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = -\frac{x}{2} + \frac{c_1}{x}
$$

Or

$$
y = \frac{1}{-\frac{x}{2} + \frac{c_1}{x}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{1}{-\frac{x}{2} + \frac{c_1}{x}} \tag{1}
$$

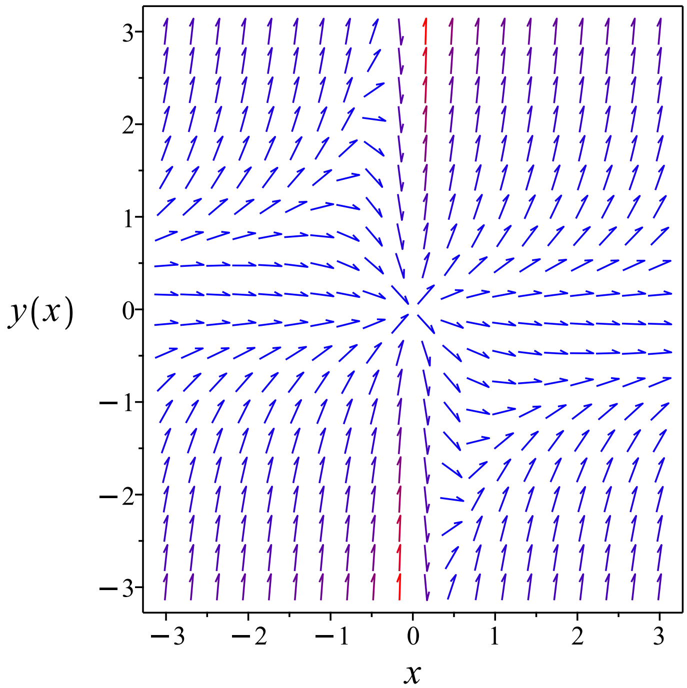


Figure 177: Slope field plot

Verification of solutions

$$y = \frac{1}{-\frac{x}{2} + \frac{c_1}{x}}$$

Verified OK.

12.18.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(-x)\, \mathrm{d}y &= \left(-x\, y^2 - y\right) \mathrm{d}x \\
\left(x\, y^2 + y\right) \mathrm{d}x + (-x)\, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= x\, y^2 + y \\
N(x, y) &= -x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x\, y^2 + y\right) \\
&= 2xy + 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{x}((2xy+1)-(-1)) \\ &= \frac{-2xy-2}{x} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{x\,y^2+y}((-1)-(2xy+1)) \\ &= -\frac{2}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{2}{y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(y)} \\ &= \frac{1}{y^2} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^2}(x\,y^2+y) \\ &= \frac{xy+1}{y} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{y^2}(-x) \\ &= -\frac{x}{y^2} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{xy+1}{y}\right) + \left(-\frac{x}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{xy+1}{y}\,\mathrm{d}x$$
$$\phi = \frac{(xy+2)\,x}{2y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x^2}{2y} - \frac{(xy+2)\,x}{2y^2} + f'(y) \tag{4}$$
$$= -\frac{x}{y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x}{y^2}$. Therefore equation (4) becomes

$$-\frac{x}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(xy + 2)\, x}{2y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(xy + 2)\, x}{2y}$$

The solution becomes

$$y = \frac{2x}{-x^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{2x}{-x^2 + 2c_1} \tag{1}$$

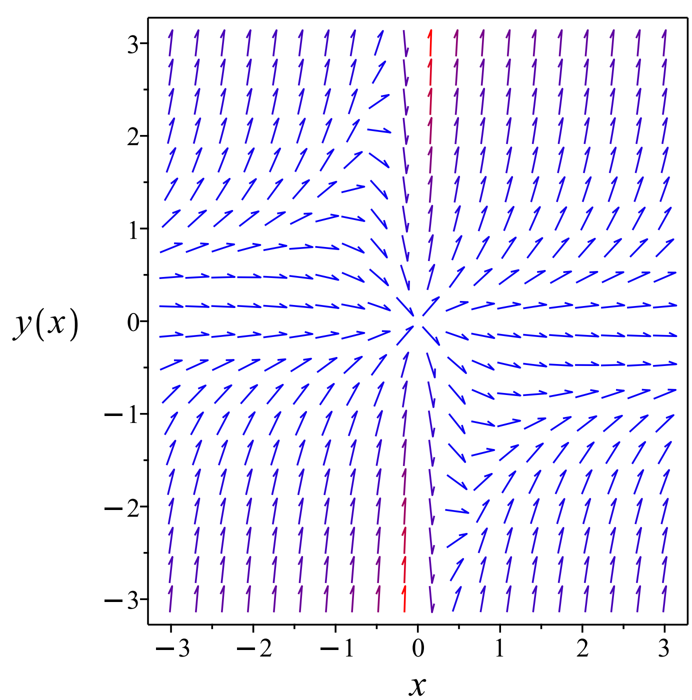


Figure 178: Slope field plot

Verification of solutions

$$y = \frac{2x}{-x^2 + 2c_1}$$

Verified OK.

12.18.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(xy+1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \frac{1}{x} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \frac{u'(x)}{x} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 x^2 + c_1$$

The above shows that

$$u'(x) = 2c_2 x$$

Using the above in (1) gives the solution

$$y = -\frac{2c_2 x}{c_2 x^2 + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{2x}{x^2 + c_3}$$

Summary
The solution(s) found are the following

$$y = -\frac{2x}{x^2 + c_3} \tag{1}$$

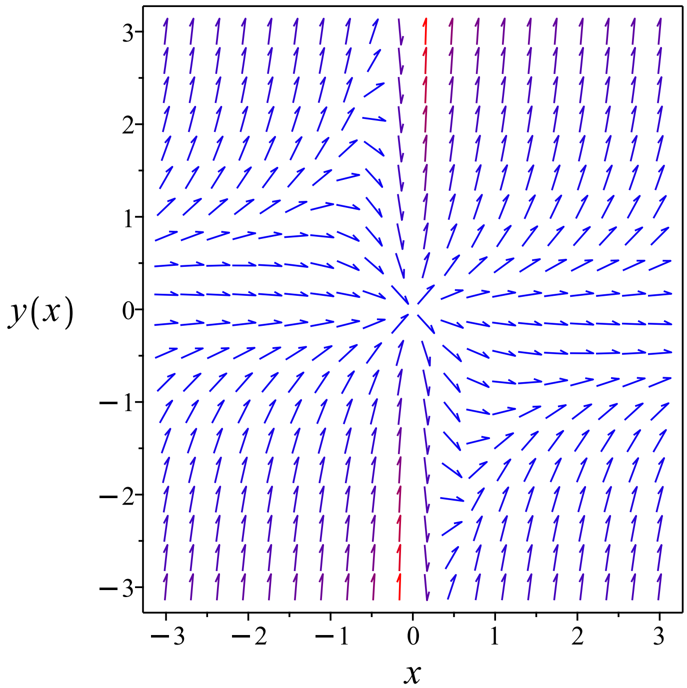


Figure 179: Slope field plot

Verification of solutions

$$y = -\frac{2x}{x^2 + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve((x*y(x)^2+y(x))-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{2x}{x^2 - 2c_1}$$

✓ Solution by Mathematica

Time used: 0.207 (sec). Leaf size: 23

```
DSolve[(x*y[x]^2+y[x])-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2x}{x^2 - 2c_1}$$
$$y(x) \to 0$$

12.19 problem Ex 20

Internal problem ID [11189]
Internal file name [`OUTPUT/10174_Saturday_December_03_2022_08_03_23_AM_75506164/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(1-x)\,y-(y+1)\,xy'=0}$$

12.19.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y(x-1)}{(y+1)\,x} \end{aligned}$$

Where $f(x)=-\frac{x-1}{x}$ and $g(y)=\frac{y}{y+1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{y+1}}\,dy=-\frac{x-1}{x}\,dx$$

$$\int \frac{1}{\frac{y}{y+1}} dy = \int -\frac{x-1}{x} dx$$

$$y + \ln(y) = -x + \ln(x) + c_1$$

Which results in

$$y = \text{LambertW}\left(e^{-x+c_1} x\right)$$

Since c_1 is constant, then exponential powers of this constant are constants also, and these can be simplified to just c_1 in the above solution. Which simplifies to

$$y = \text{LambertW}\left(e^{-x+c_1} x\right)$$

gives

$$y = \text{LambertW}\left(e^{-x} c_1 x\right)$$

Summary

The solution(s) found are the following

$$y = \text{LambertW}\left(e^{-x} c_1 x\right) \tag{1}$$

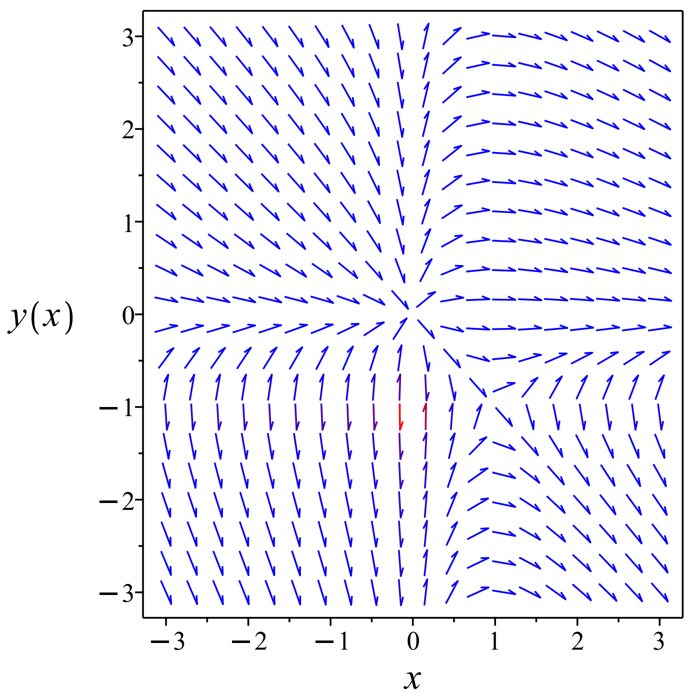


Figure 180: Slope field plot

Verification of solutions

$$y = \text{LambertW}\left(\mathrm{e}^{-x} c_1 x\right)$$

Verified OK.

12.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(x-1)}{(y+1)\,x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 101: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{x}{x-1} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{x}{x-1}} dx \end{aligned}$$

Which results in

$$S = -x + \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(x-1)}{(y+1)\, x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -1 + \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y+1}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R+1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-x + \ln(x) = y + \ln(y) + c_1$$

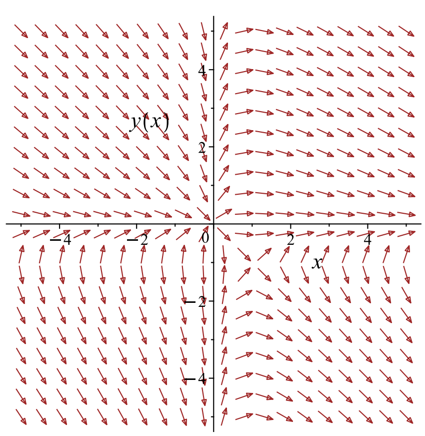
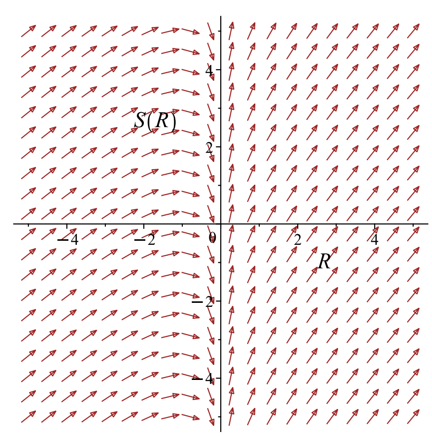
Which simplifies to

$$-x + \ln(x) = y + \ln(y) + c_1$$

Which gives

$$y = \text{LambertW}\left(e^{-x-c_1} x\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(x-1)}{(y+1)x}$ 	$R = y$ $S = -x + \ln(x)$	$\frac{dS}{dR} = \frac{R+1}{R}$ 

Summary

The solution(s) found are the following

$$y = \text{LambertW}\left(e^{-x-c_1} x\right) \tag{1}$$

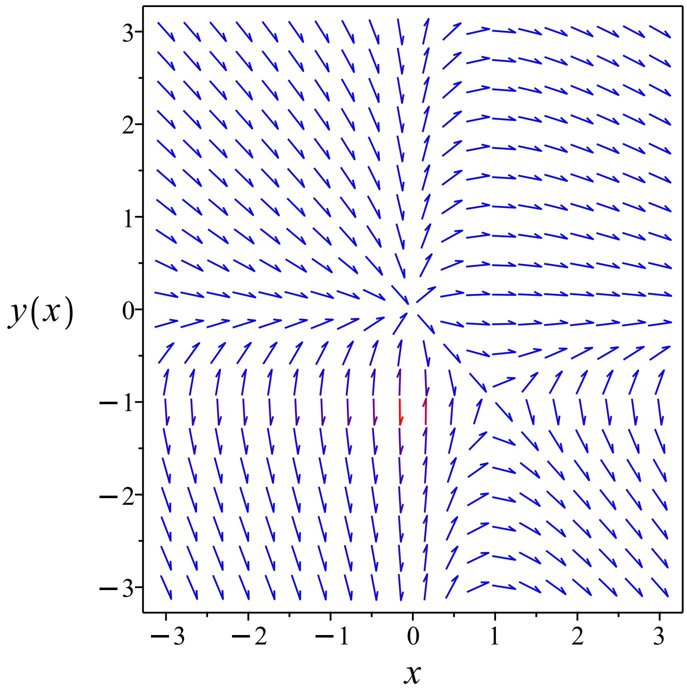


Figure 181: Slope field plot

Verification of solutions

$$y = \text{LambertW}\left(\mathrm{e}^{-x-c_1}x\right)$$

Verified OK.

12.19.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{y+1}{y}\right) \mathrm{d}y = \left(\frac{x-1}{x}\right) \mathrm{d}x$$
$$\left(-\frac{x-1}{x}\right) \mathrm{d}x + \left(-\frac{y+1}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x-1}{x}$$
$$N(x, y) = -\frac{y+1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x-1}{x}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y+1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{x-1}{x} \,\mathrm{d}x \\
\phi &= -x + \ln(x) + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y+1}{y}$. Therefore equation (4) becomes

$$
-\frac{y+1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{y+1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{-y-1}{y}\right) \mathrm{d}y \\
f(y) &= -y - \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \ln(x) - y - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \ln(x) - y - \ln(y)$$

The solution becomes

$$y = \text{LambertW}\left(e^{-x-c_1}x\right)$$

Summary

The solution(s) found are the following

$$y = \text{LambertW}\left(e^{-x-c_1}x\right) \tag{1}$$

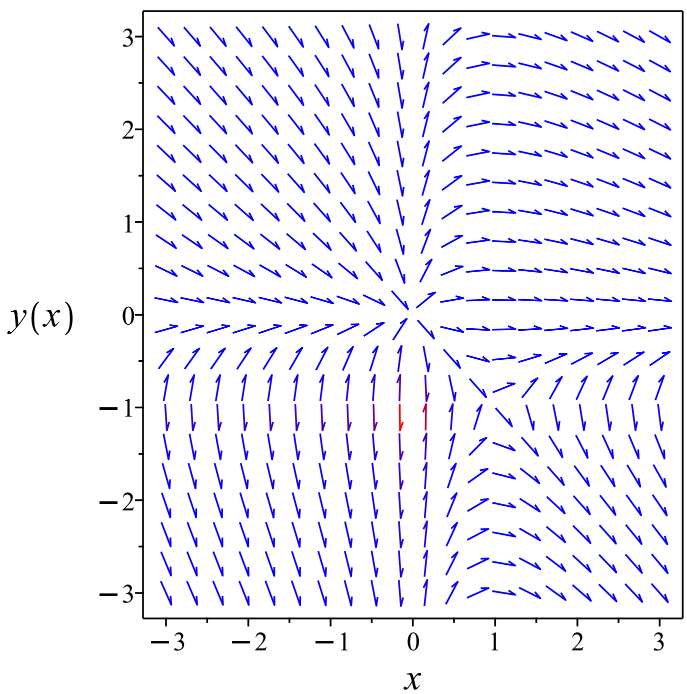


Figure 182: Slope field plot

Verification of solutions

$$y = \text{LambertW}\left(e^{-x-c_1}x\right)$$

Verified OK.

12.19.4 Maple step by step solution

Let's solve

$(1-x)\,y-(y+1)\,xy'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(y+1)}{y}=\frac{1-x}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'(y+1)}{y}dx=\int \frac{1-x}{x}dx+c_1$

- Evaluate integral

  $y+\ln(y)=-x+\ln(x)+c_1$

- Solve for y

  $y=\mathit{LambertW}(\mathrm{e}^{-x+c_1}x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve((1-x)*y(x)-(1+y(x))*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=\mathrm{LambertW}\left(\frac{\mathrm{e}^{-x}x}{c_1}\right)$$

✓ Solution by Mathematica

Time used: 5.134 (sec). Leaf size: 21

```
DSolve[(1-x)*y[x]-(1+y[x])*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to W\left(xe^{-x+c_1}\right)$$
$$y(x) \to 0$$

12.20 problem Ex 21

Internal problem ID [11190]
Internal file name [`OUTPUT/10175_Saturday_December_03_2022_08_03_24_AM_39182422/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{3x^2y + \left(x^3 + y^2x^3\right) y' = 0}$$

12.20.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{3y}{x\left(y^2+1\right)} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(y) = \frac{y}{y^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{y^2+1}}\,dy = -\frac{3}{x}\,dx$$

$$\int \frac{1}{\frac{y}{y^2+1}} dy = \int -\frac{3}{x} dx$$

$$\frac{y^2}{2} + \ln(y) = -3\ln(x) + c_1$$

Which results in

$$y = \frac{1}{\sqrt{\frac{1}{\mathrm{LambertW}\left(\frac{e^{2c_1}}{x^6}\right)}}}$$

Since c_1 is constant, then exponential powers of this constant are constants also, and these can be simplified to just c_1 in the above solution. Which simplifies to

$$y = \frac{1}{\sqrt{\frac{1}{\mathrm{LambertW}\left(\frac{e^{2c_1}}{x^6}\right)}}}$$

gives

$$y = \frac{1}{\sqrt{\frac{1}{\mathrm{LambertW}\left(\frac{c_1^2}{x^6}\right)}}}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{\frac{1}{\mathrm{LambertW}\left(\frac{c_1^2}{x^6}\right)}}} \tag{1}$$

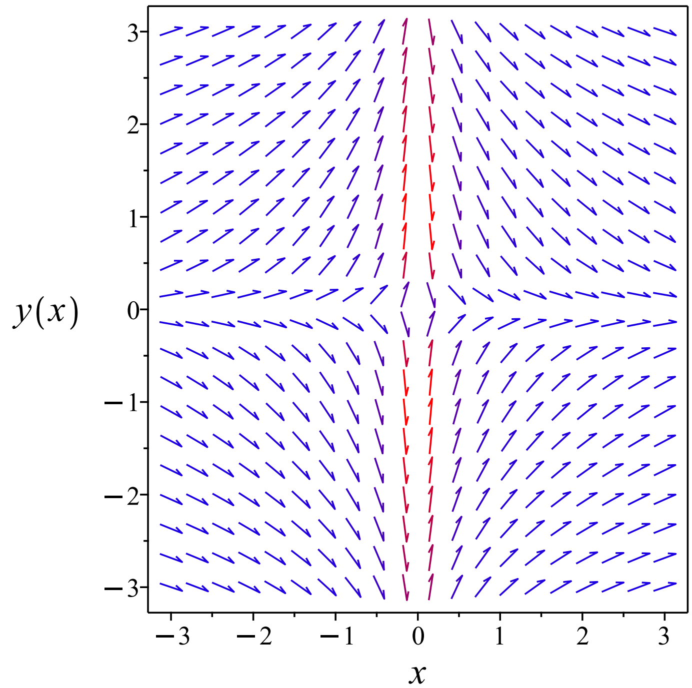


Figure 183: Slope field plot

Verification of solutions

$$
y = \frac{1}{\sqrt{\frac{1}{\operatorname{LambertW}\left(\frac{c_1^2}{x^6}\right)}}}
$$

Verified OK.

12.20.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{3y}{x\,(y^2+1)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 104: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{x}{3} \\
\eta(x,y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{x}{3}} dx \end{aligned}$$

Which results in

$$S = -3 \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3y}{x(y^2 + 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{3}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y^2 + 1}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^2 + 1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-3\ln(x) = \frac{y^2}{2} + \ln(y) + c_1$$

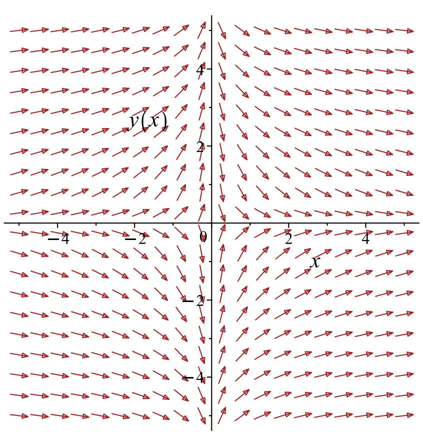
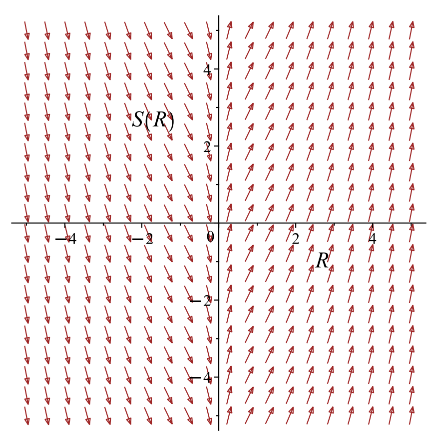
Which simplifies to

$$-3\ln(x) = \frac{y^2}{2} + \ln(y) + c_1$$

Which gives

$$y = \frac{1}{\sqrt{\frac{1}{\operatorname{LambertW}\left(\frac{e^{-2c_1}}{x^6}\right)}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y}{x(y^2+1)}$ 	$R = y$ $S = -3\ln(x)$	$\frac{dS}{dR} = \frac{R^2+1}{R}$ 

Summary
The solution(s) found are the following

$$y = \frac{1}{\sqrt{\frac{1}{\text{LambertW}\left(\frac{e^{-2c_1}}{x^6}\right)}}} \tag{1}$$

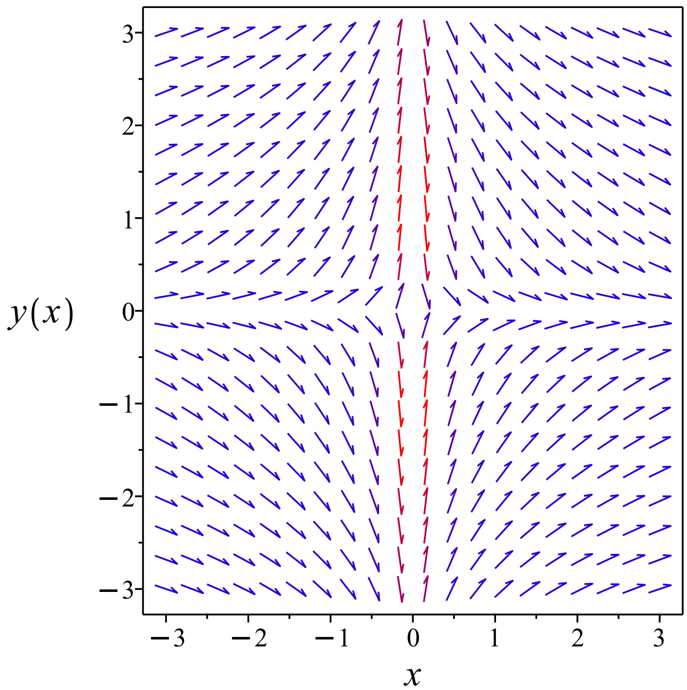


Figure 184: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{\frac{1}{\text{LambertW}\left(\frac{e^{-2c_1}}{x^6}\right)}}}$$

Verified OK.

12.20.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{y^2+1}{3y}\right)\mathrm{d}y&=\left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x+\left(-\frac{y^2+1}{3y}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x} \\
N(x,y) &= -\frac{y^2+1}{3y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y^2+1}{3y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{1}{x} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y^2+1}{3y}$. Therefore equation (4) becomes

$$-\frac{y^2+1}{3y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{y^2+1}{3y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{-y^2-1}{3y}\right)\mathrm{d}y$$
$$f(y) = -\frac{y^2}{6} - \frac{\ln(y)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{y^2}{6} - \frac{\ln(y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{y^2}{6} - \frac{\ln(y)}{3}$$

The solution becomes

$$y = \frac{1}{\sqrt{\frac{1}{\mathrm{LambertW}\left(\frac{\mathrm{e}^{-6c_1}}{x^6}\right)}}}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{\frac{1}{\mathrm{LambertW}\left(\frac{\mathrm{e}^{-6c_1}}{x^6}\right)}}} \tag{1}$$

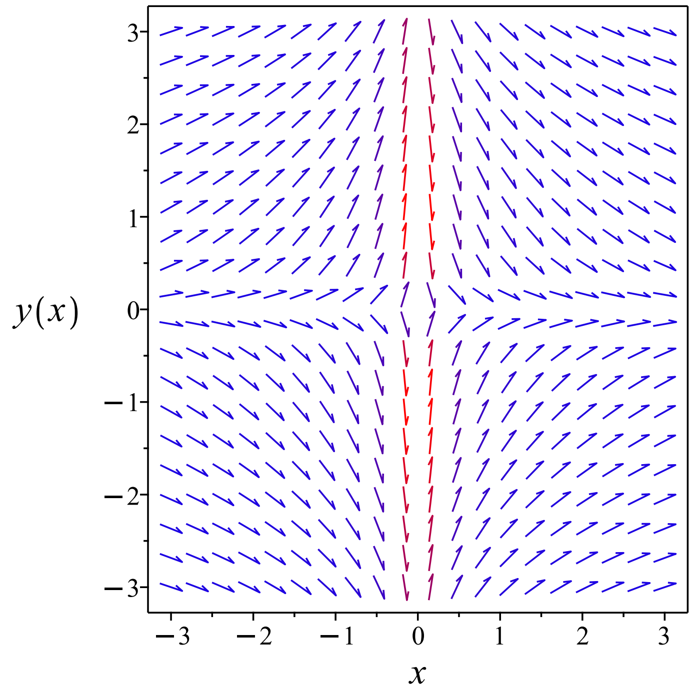


Figure 185: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{\frac{1}{\text{LambertW}\left(\frac{e^{-6c_1}}{x^6}\right)}}}$$

Verified OK.

12.20.4 Maple step by step solution

Let's solve

$3x^2y + (x^3 + y^2x^3)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(1+y^2)}{y} = -\frac{3}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'(1+y^2)}{y} dx = \int -\frac{3}{x} dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} + \ln(y) = -3\ln(x) + c_1$

- Solve for y

  $y = \frac{e^{-\frac{\mathit{LambertW}\left(\frac{e^{2c_1}}{x^6}\right)}{2} + c_1}}{x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(3*x^2*y(x)+(x^3+x^3*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{\frac{1}{\operatorname{LambertW}\left(\frac{c_1}{x^6}\right)}}}$$

✓ Solution by Mathematica

Time used: 6.245 (sec). Leaf size: 46

```
DSolve[3*x^2*y[x]+(x^3+x^3*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{W\left(\frac{e^{2c_1}}{x^6}\right)}$$
$$y(x) \to \sqrt{W\left(\frac{e^{2c_1}}{x^6}\right)}$$
$$y(x) \to 0$$

12.21 problem Ex 22

Internal problem ID [11191]
Internal file name [`OUTPUT/10176_Saturday_December_03_2022_08_03_25_AM_40769649/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational]`

Unable to solve or complete the solution.

$$\boxed{(x^2+y^2)(x+yy')-(x^2+y^2+x)(y'x-y)=0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2`[0, (x^4+2*y^2*x^2+y^4+1/2*x^3-1/2*y*x^2+1/2*y^2*
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
dsolve((x^2+y(x)^2)*(x+y(x)*diff(y(x),x))=(x^2+y(x)^2+x)*(x*diff(y(x),x)-y(x)),y(x), singsol
```

$$y(x) = -\cot\left(\operatorname{RootOf}\left(-2\_Z + 2\ln\left(2\csc(\_Z)^2 x^2 + \cot(\_Z)x + x\right) - \ln\left(\csc(\_Z)^2 x^2\right) + 2c_1\right)\right)x$$

✓ Solution by Mathematica

Time used: 0.548 (sec). Leaf size: 53

```
DSolve[(x^2+y[x]^2)*(x+y[x]*y'[x])==(x^2+y[x]^2+x)*(x*y'[x]-y[x]),y[x],x,IncludeSingularSolu
```

$$\text{Solve}\left[\frac{1}{2}\arctan\left(\frac{x}{y(x)}\right) - \frac{1}{4}\log\left(x^2+y(x)^2\right) + \frac{1}{2}\log\left(2x^2+2y(x)^2-y(x)+x\right) = c_1, y(x)\right]$$

12.22 problem Ex 23

Internal problem ID [11192]
Internal file name [`OUTPUT/10177_Saturday_December_03_2022_08_03_26_AM_82484991/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{3y + (2x + 3y - 5)\, y' = -2x + 1}$$

12.22.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2x + 3y - 1}{2x + 3y - 5}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(2x+3y-1)(b_3-a_2)}{2x+3y-5} - \frac{(2x+3y-1)^2 a_3}{(2x+3y-5)^2} - \left(-\frac{2}{2x+3y-5} + \frac{4x+6y-2}{(2x+3y-5)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{3}{2x+3y-5} + \frac{6x+9y-3}{(2x+3y-5)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{4x^2a_2 - 4x^2a_3 + 4x^2b_2 - 4x^2b_3 + 12xya_2 - 12xya_3 + 12xyb_2 - 12xyb_3 + 9y^2a_2 - 9y^2a_3 + 9y^2b_2 - 9y^2b_3 -}{(2x+3y} = 0$$

Setting the numerator to zero gives

$$4x^2a_2 - 4x^2a_3 + 4x^2b_2 - 4x^2b_3 + 12xya_2 - 12xya_3 + 12xyb_2 - 12xyb_3 + 9y^2a_2 - 9y^2a_3 + 9y^2b_2 - 9y^2b_3 - 20xa_2 + 4xa_3 - 32xb_2 + 12xb_3 - 18ya_2 - 2ya_3 - 30yb_2 + 6yb_3 - 8a_1 + 5a_2 - a_3 - 12b_1 + 25b_2 - 5b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$4a_2v_1^2 + 12a_2v_1v_2 + 9a_2v_2^2 - 4a_3v_1^2 - 12a_3v_1v_2 - 9a_3v_2^2 + 4b_2v_1^2 + 12b_2v_1v_2 + 9b_2v_2^2 - 4b_3v_1^2 - 12b_3v_1v_2 - 9b_3v_2^2 - 20a_2v_1 - 18a_2v_2 + 4a_3v_1 - 2a_3v_2 - 32b_2v_1 - 30b_2v_2 + 12b_3v_1 + 6b_3v_2 - 8a_1 + 5a_2 - a_3 - 12b_1 + 25b_2 - 5b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(4a_2 - 4a_3 + 4b_2 - 4b_3)\, v_1^2 + (12a_2 - 12a_3 + 12b_2 - 12b_3)\, v_1 v_2 \\ + (-20a_2 + 4a_3 - 32b_2 + 12b_3)\, v_1 + (9a_2 - 9a_3 + 9b_2 - 9b_3)\, v_2^2 \\ + (-18a_2 - 2a_3 - 30b_2 + 6b_3)\, v_2 - 8a_1 + 5a_2 - a_3 - 12b_1 + 25b_2 - 5b_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -20a_2 + 4a_3 - 32b_2 + 12b_3 &= 0 \\ -18a_2 - 2a_3 - 30b_2 + 6b_3 &= 0 \\ 4a_2 - 4a_3 + 4b_2 - 4b_3 &= 0 \\ 9a_2 - 9a_3 + 9b_2 - 9b_3 &= 0 \\ 12a_2 - 12a_3 + 12b_2 - 12b_3 &= 0 \\ -8a_1 + 5a_2 - a_3 - 12b_1 + 25b_2 - 5b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= \frac{7b_2}{4} - \frac{3b_1}{2} \\ a_2 &= -b_2 \\ a_3 &= -\frac{3b_2}{2} \\ b_1 &= b_1 \\ b_2 &= b_2 \\ b_3 &= \frac{3b_2}{2} \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -\frac{3}{2} \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 1 - \left(-\frac{2x+3y-1}{2x+3y-5}\right)\left(-\frac{3}{2}\right) \\
&= \frac{-2x-3y-7}{4x+6y-10} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2x-3y-7}{4x+6y-10}} dy
\end{aligned}
$$

Which results in

$$
S = -2y + 8\ln(2x+3y+7)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{2x+3y-1}{2x+3y-5}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{16}{2x + 3y + 7} \\
S_y &= -2 + \frac{24}{2x + 3y + 7}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 2 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 2
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = 2R + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-2y + 8\ln(2x + 3y + 7) = 2x + c_1
$$

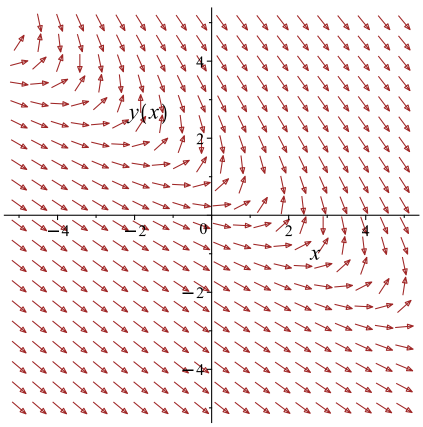
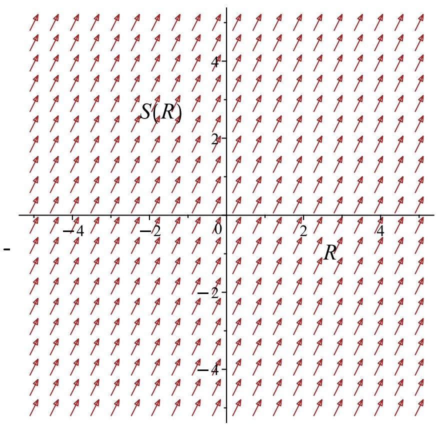
Which simplifies to

$$
-2y + 8\ln(2x + 3y + 7) = 2x + c_1
$$

Which gives

$$
y = -4\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{x}{12} + \frac{c_1}{8} - \frac{7}{12}}}{12}\right) - \frac{2x}{3} - \frac{7}{3}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+3y-1}{2x+3y-5}$ 	$R = x$ $S = -2y + 8\ln(2x + 3y -$	$\frac{dS}{dR} = 2$ 

Summary

The solution(s) found are the following

$$y = -4\,\text{LambertW}\left(-\frac{e^{\frac{x}{12}+\frac{c_1}{8}-\frac{7}{12}}}{12}\right) - \frac{2x}{3} - \frac{7}{3} \tag{1}$$

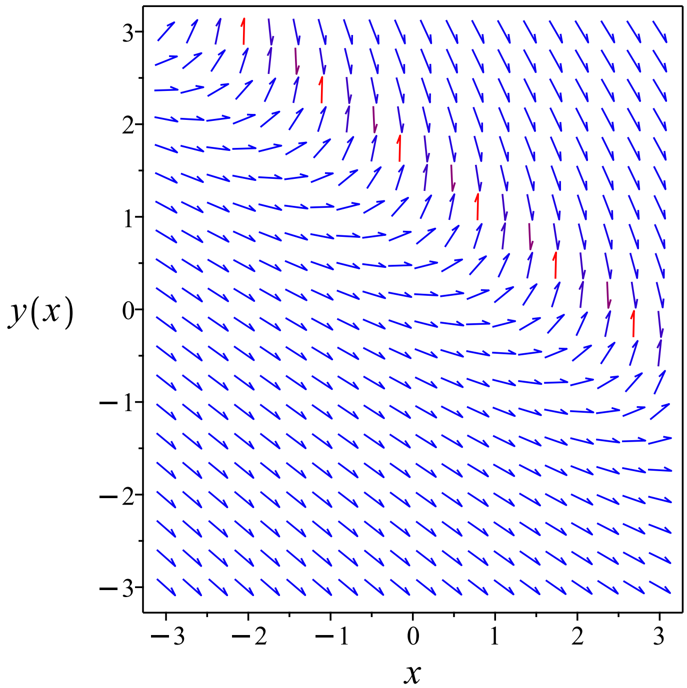


Figure 186: Slope field plot

Verification of solutions

$$y = -4\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{x}{12}+\frac{c_1}{8}-\frac{7}{12}}}{12}\right)-\frac{2x}{3}-\frac{7}{3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -2/3, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve((2*x+3*y(x)-1)+(2*x+3*y(x)-5)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{2x}{3} - 4\,\mathrm{LambertW}\left(-\frac{c_1 \mathrm{e}^{\frac{x}{12}-\frac{7}{12}}}{12}\right) - \frac{7}{3}$$

✓ Solution by Mathematica

Time used: 5.457 (sec). Leaf size: 43

```
DSolve[(2*x+3*y[x]-1)+(2*x+3*y[x]-5)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -4W\left(-e^{\frac{x}{12}-1+c_1}\right) - \frac{2x}{3} - \frac{7}{3}$$
$$y(x) \to \frac{1}{3}(-2x-7)$$

12.23 problem Ex 24

Internal problem ID [11193]
Internal file name [`OUTPUT/10178_Saturday_December_03_2022_08_03_27_AM_22224770/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y^3 - 2x^2y + \left(2y^2x - x^3\right) y' = 0}$$

12.23.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^3\,x^3 - 2x^3u(x) + \left(2u(x)^2\,x^3 - x^3\right)\left(u'(x)\,x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3(u^3 - u)}{x\,(2u^2 - 1)} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = \frac{u^3-u}{2u^2-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^3-u}{2u^2-1}}\, du = -\frac{3}{x}\, dx$$

$$\int \frac{1}{\frac{u^3-u}{2u^2-1}}\, du = \int -\frac{3}{x}\, dx$$

$$\frac{\ln (u-1)}{2} + \frac{\ln (u+1)}{2} + \ln (u) = -3\ln (x) + c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln(u-1)}{2}+\frac{\ln(u+1)}{2}+\ln(u)} = e^{-3\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u-1}\,\sqrt{u+1}\,u = \frac{c_3}{x^3}$$

The solution is

$$\sqrt{u(x)-1}\,\sqrt{u(x)+1}\,u(x) = \frac{c_3}{x^3}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\sqrt{\frac{y}{x}-1}\,\sqrt{\frac{y}{x}+1}\,y}{x} = \frac{c_3}{x^3}$$

$$\frac{\sqrt{\frac{y-x}{x}}\,\sqrt{\frac{y+x}{x}}\,y}{x} = \frac{c_3}{x^3}$$

Which simplifies to

$$\sqrt{\frac{y-x}{x}}\,\sqrt{\frac{y+x}{x}}\,y = \frac{c_3}{x^2}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{y-x}{x}}\,\sqrt{\frac{y+x}{x}}\,y = \frac{c_3}{x^2} \tag{1}$$

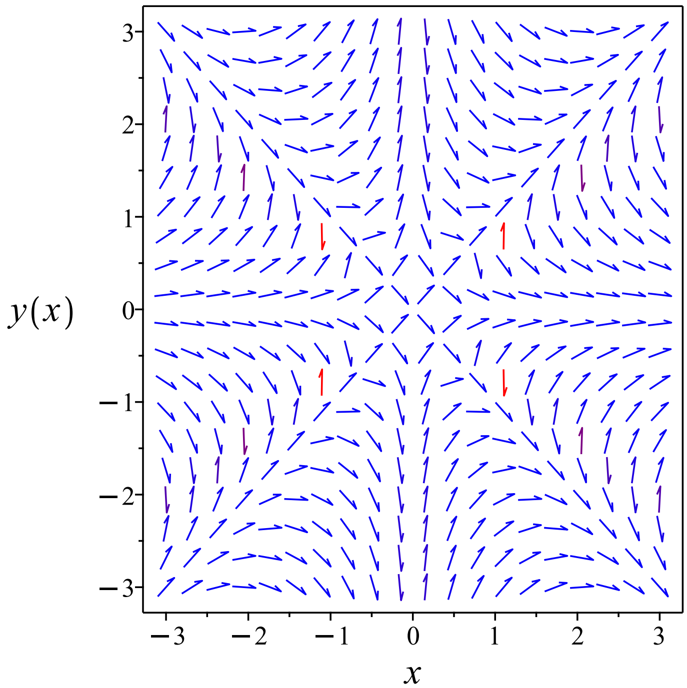


Figure 187: Slope field plot

Verification of solutions

$$\sqrt{\frac{y-x}{x}}\,\sqrt{\frac{y+x}{x}}\,y=\frac{c_3}{x^2}$$

Verified OK.

12.23.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{y(-2x^2+y^2)}{x\,(-x^2+2y^2)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$

$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{y(-2x^2+y^2)\,(b_3-a_2)}{x\,(-x^2+2y^2)} - \frac{y^2(-2x^2+y^2)^2\,a_3}{x^2\,(-x^2+2y^2)^2} \\
&- \left(\frac{4y}{-x^2+2y^2} + \frac{y(-2x^2+y^2)}{x^2\,(-x^2+2y^2)} - \frac{2y(-2x^2+y^2)}{(-x^2+2y^2)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{-2x^2+y^2}{x\,(-x^2+2y^2)} - \frac{2y^2}{x\,(-x^2+2y^2)}\right. \\
&\left.+ \frac{4y^2(-2x^2+y^2)}{x\,(-x^2+2y^2)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
\frac{3x^6b_2 - 6x^4y^2a_3 - 3x^4y^2b_2 - 6x^3y^3a_2 + 6x^3y^3b_3 + 3x^2y^4a_3 + 6x^2y^4b_2 - 3y^6a_3 + 2x^5b_1 - 2x^4ya_1 + x^3y^2b_1}{(x^2-2y^2)^2\,x^2}
$$
$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&3x^6b_2 - 6x^4y^2a_3 - 3x^4y^2b_2 - 6x^3y^3a_2 + 6x^3y^3b_3 + 3x^2y^4a_3 + 6x^2y^4b_2 \\
&- 3y^6a_3 + 2x^5b_1 - 2x^4ya_1 + x^3y^2b_1 - x^2y^3a_1 + 2x\,y^4b_1 - 2y^5a_1 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-6a_2v_1^3v_2^3 - 6a_3v_1^4v_2^2 + 3a_3v_1^2v_2^4 - 3a_3v_2^6 + 3b_2v_1^6 - 3b_2v_1^4v_2^2 + 6b_2v_1^2v_2^4 \\
&+ 6b_3v_1^3v_2^3 - 2a_1v_1^4v_2 - a_1v_1^2v_2^3 - 2a_1v_2^5 + 2b_1v_1^5 + b_1v_1^3v_2^2 + 2b_1v_1v_2^4 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$3b_2v_1^6 + 2b_1v_1^5 + (-6a_3 - 3b_2)\, v_1^4v_2^2 - 2a_1v_1^4v_2 + (-6a_2 + 6b_3)\, v_1^3v_2^3 + b_1v_1^3v_2^2 + (3a_3 + 6b_2)\, v_1^2v_2^4 - a_1v_1^2v_2^3 + 2b_1v_1v_2^4 - 3a_3v_2^6 - 2a_1v_2^5 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_1 &= 0 \\
-2a_1 &= 0 \\
-a_1 &= 0 \\
-3a_3 &= 0 \\
2b_1 &= 0 \\
3b_2 &= 0 \\
-6a_2 + 6b_3 &= 0 \\
-6a_3 - 3b_2 &= 0 \\
3a_3 + 6b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y(-2x^2+y^2)}{x\,(-x^2+2y^2)}\right)(x) \\
&= \frac{3x^2y - 3y^3}{x^2 - 2y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3x^2y-3y^3}{x^2-2y^2}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(y-x)}{6} + \frac{\ln(y)}{3} + \frac{\ln(y+x)}{6}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y(-2x^2+y^2)}{x\,(-x^2+2y^2)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x}{3x^2 - 3y^2} \\
S_y &= \frac{x^2 - 2y^2}{3x^2 y - 3y^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{3x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{3R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln (R)}{3} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln (y - x)}{6} + \frac{\ln (y)}{3} + \frac{\ln (y + x)}{6} = -\frac{\ln (x)}{3} + c_1
$$

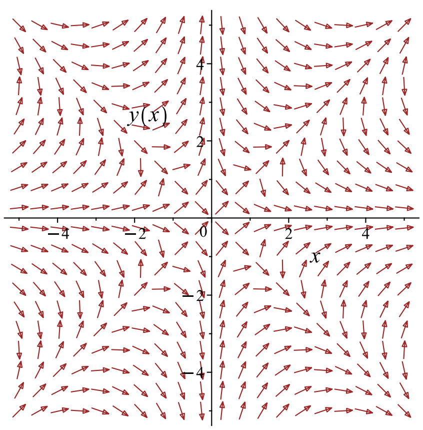
Which simplifies to

$$
\frac{\ln (y - x)}{6} + \frac{\ln (y)}{3} + \frac{\ln (y + x)}{6} = -\frac{\ln (x)}{3} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{y(-2x^2+y^2)}{x(-x^2+2y^2)}$



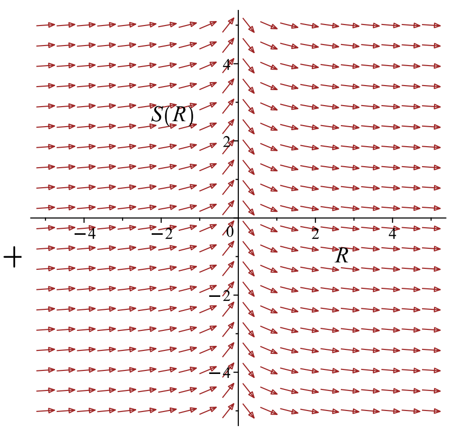
</td><td>

$R = x$

$S = \frac{\ln(y-x)}{6} + \frac{\ln(y)}{3} +$

</td><td>

$\frac{dS}{dR} = -\frac{1}{3R}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{\ln(y-x)}{6} + \frac{\ln(y)}{3} + \frac{\ln(y+x)}{6} = -\frac{\ln(x)}{3} + c_1 \tag{1}$$

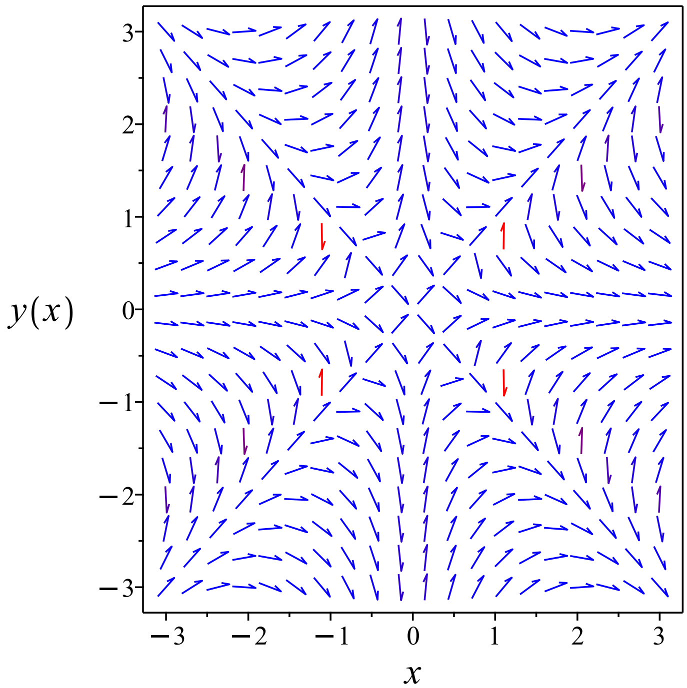


Figure 188: Slope field plot

Verification of solutions

$$\frac{\ln(y-x)}{6}+\frac{\ln(y)}{3}+\frac{\ln(y+x)}{6}=-\frac{\ln(x)}{3}+c_1$$

Verified OK.

12.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (-x^3 + 2x\, y^2)\, \mathrm{d}y &= (2x^2 y - y^3)\, \mathrm{d}x \\ (-2x^2 y + y^3)\, \mathrm{d}x + (-x^3 + 2x\, y^2)\, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -2x^2 y + y^3$$
$$N(x, y) = -x^3 + 2x\, y^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2x^2 y + y^3) \\ &= -2x^2 + 3y^2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^3 + 2x\, y^2) \\ &= -3x^2 + 2y^2 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{-x^3 + 2x\,y^2}\left(\left(-2x^2 + 3y^2\right) - \left(-3x^2 + 2y^2\right)\right) \\ &= \frac{-x^2 - y^2}{x\left(x^2 - 2y^2\right)} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{-2x^2y + y^3}\left(\left(-3x^2 + 2y^2\right) - \left(-2x^2 + 3y^2\right)\right) \\ &= \frac{x^2 + y^2}{2x^2y - y^3} \end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{\left(-3x^2 + 2y^2\right) - \left(-2x^2 + 3y^2\right)}{x\left(-2x^2y + y^3\right) - y\left(-x^3 + 2x\,y^2\right)} \\ &= \frac{1}{xy} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{1}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int \left(\frac{1}{t}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(t)} \\
&= t
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = xy
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= xy\left(-2x^2 y + y^3\right) \\
&= -2\left(x^2 - \frac{y^2}{2}\right) x\, y^2
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= xy\left(-x^3 + 2x\, y^2\right) \\
&= -y\left(x^2 - 2y^2\right) x^2
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-2\left(x^2 - \frac{y^2}{2}\right) x\, y^2\right) + \left(-y\left(x^2 - 2y^2\right) x^2\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -2\left(x^2 - \frac{y^2}{2}\right) x\, y^2\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\left(2x^2 - y^2\right)^2 y^2}{8} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{(2x^2 - y^2)\, y^3}{2} - \frac{(2x^2 - y^2)^2\, y}{4} + f'(y) \qquad (4)\\
&= 2y^3x^2 - \frac{3}{4}y^5 - x^4y + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y(x^2 - 2y^2)\, x^2$. Therefore equation (4) becomes

$$
-y(x^2 - 2y^2)\, x^2 = 2y^3x^2 - \frac{3}{4}y^5 - x^4y + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{3y^5}{4}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\, \mathrm{d}y &= \int \left(\frac{3y^5}{4}\right) \mathrm{d}y\\
f(y) &= \frac{y^6}{8} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{(2x^2 - y^2)^2\, y^2}{8} + \frac{y^6}{8} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{(2x^2 - y^2)^2\, y^2}{8} + \frac{y^6}{8}
$$

Summary

The solution(s) found are the following

$$
-\frac{(2x^2 - y^2)^2\, y^2}{8} + \frac{y^6}{8} = c_1 \qquad (1)
$$

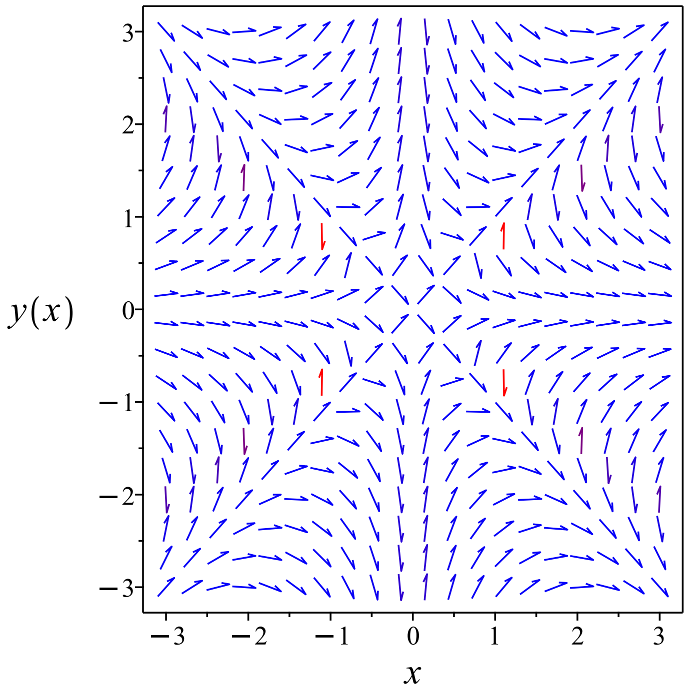


Figure 189: Slope field plot

Verification of solutions

$$-\frac{\left(2x^2-y^2\right)^2 y^2}{8}+\frac{y^6}{8}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.266 (sec). Leaf size: 71

```
dsolve((y(x)^3-2*x^2*y(x))+(2*x*y(x)^2-x^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{\frac{2c_1x^3-2\sqrt{c_1^2x^6+4}}{c_1x^3}}\, x}{2}$$

$$y(x) = \frac{\sqrt{2}\sqrt{\frac{c_1x^3+\sqrt{c_1^2x^6+4}}{c_1x^3}}\, x}{2}$$

✓ Solution by Mathematica

Time used: 15.638 (sec). Leaf size: 277

```
DSolve[(y[x]^3-2*x^2*y[x])+(2*x*y[x]^2-x^3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\frac{\sqrt{x^2-\frac{\sqrt{x^6-4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{x^2-\frac{\sqrt{x^6-4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \to -\frac{\sqrt{\frac{x^3+\sqrt{x^6-4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{\frac{x^3+\sqrt{x^6-4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \to -\frac{\sqrt{x^2-\frac{\sqrt{x^6}}{x}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{x^2-\frac{\sqrt{x^6}}{x}}}{\sqrt{2}}$$

$$y(x) \to -\frac{\sqrt{\frac{\sqrt{x^6}+x^3}{x}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{\frac{\sqrt{x^6}+x^3}{x}}}{\sqrt{2}}$$

12.24 problem Ex 25

Internal problem ID [11194]
Internal file name [`OUTPUT/10179_Saturday_December_03_2022_08_03_29_AM_70315647/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{2y^2x^3 - y + (2y^3x^2 - x)\, y' = 0}$$

12.24.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(2y^3x^2 - x\right)\mathrm{d}y &= \left(-2y^2x^3 + y\right)\mathrm{d}x\\ \left(2y^2x^3 - y\right)\mathrm{d}x + \left(2y^3x^2 - x\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2y^2x^3 - y\\ N(x, y) &= 2y^3x^2 - x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y^2x^3 - y\right)\\ &= 4y\,x^3 - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2y^3x^2 - x\right)\\ &= 4x\,y^3 - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2y^3x^2 - x}\left(\left(4y\,x^3 - 1\right) - \left(4x\,y^3 - 1\right)\right)\\ &= \frac{4x^2y - 4y^3}{2x\,y^3 - 1}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{2y^2x^3 - y}\left(\left(4x\, y^3 - 1\right) - \left(4y\, x^3 - 1\right)\right) \\
&= \frac{-4x^3 + 4x\, y^2}{2y\, x^3 - 1}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{\left(4x\, y^3 - 1\right) - \left(4y\, x^3 - 1\right)}{x\left(2y^2x^3 - y\right) - y\left(2y^3x^2 - x\right)} \\
&= -\frac{2}{xy}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{2}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{x^2y^2}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2 y^2}\left(2y^2 x^3 - y\right) \\
&= \frac{2y\, x^3 - 1}{y\, x^2}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2 y^2}\left(2y^3 x^2 - x\right) \\
&= \frac{2x\, y^3 - 1}{x\, y^2}
\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2y\, x^3 - 1}{y\, x^2}\right) + \left(\frac{2x\, y^3 - 1}{x\, y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int \overline{M}\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int \frac{2y\, x^3 - 1}{y\, x^2}\, \mathrm{d}x \\
\phi &= \frac{y\, x^3 + 1}{xy} + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= \frac{x^2}{y} - \frac{y\,x^3+1}{x\,y^2} + f'(y) \\ &= -\frac{1}{x\,y^2} + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2x\,y^3-1}{x\,y^2}$. Therefore equation (4) becomes

$$\frac{2x\,y^3-1}{x\,y^2} = -\frac{1}{x\,y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int (2y)\,\mathrm{d}y \\ f(y) &= y^2 + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y\,x^3+1}{xy} + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y\,x^3+1}{xy} + y^2$$

Summary

The solution(s) found are the following

$$\frac{yx^3+1}{xy} + y^2 = c_1 \tag{1}$$

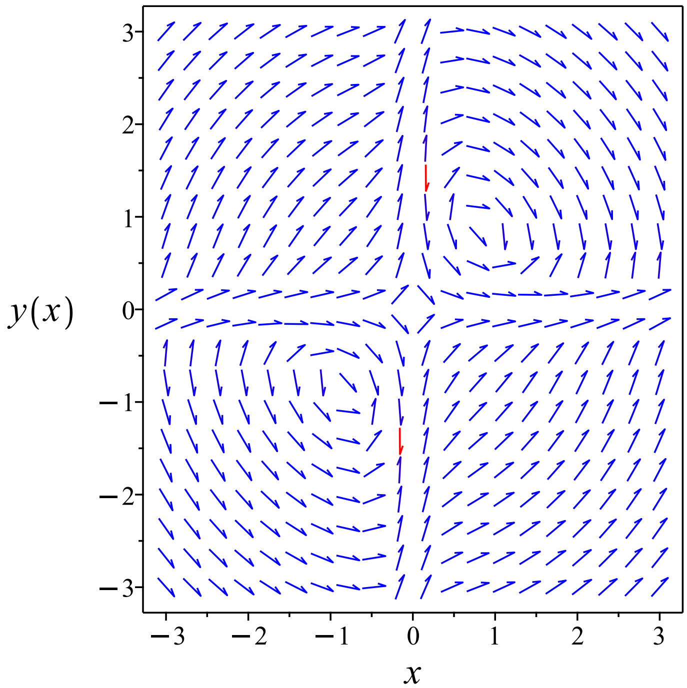


Figure 190: Slope field plot

Verification of solutions

$$\frac{yx^3+1}{xy}+y^2=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2`[0, y^2*x/(2*x*y^3-1)], [0, (x^4*y^2+x^2*y^4+x*y)
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 361

```
dsolve((2*x^3*y(x)^2-y(x))+(2*x^2*y(x)^3-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\left(-\left(\left(-9+\sqrt{12x^8-36c_1x^6+36c_1^2x^4-12c_1^3x^2+81}\right)x^2\right)^{\frac{2}{3}}+x^2 12^{\frac{1}{3}}(x^2-c_1)\right)12^{\frac{1}{3}}}{6\left(\left(-9+\sqrt{12x^8-36c_1x^6+36c_1^2x^4-12c_1^3x^2+81}\right)x^2\right)^{\frac{1}{3}}x}$$

$$y(x) = -\frac{\left(\left(1+i\sqrt{3}\right)\left(\left(-9+\sqrt{12x^8-36c_1x^6+36c_1^2x^4-12c_1^3x^2+81}\right)x^2\right)^{\frac{2}{3}}+\left(i3^{\frac{5}{6}}-3^{\frac{1}{3}}\right)2^{\frac{2}{3}}x^2(x^2-c_1)\right)2}{12\left(\left(-9+\sqrt{12x^8-36c_1x^6+36c_1^2x^4-12c_1^3x^2+81}\right)x^2\right)^{\frac{1}{3}}x}$$

$$y(x) = \frac{2^{\frac{2}{3}}3^{\frac{1}{3}}\left(\left(i\sqrt{3}-1\right)\left(\left(-9+\sqrt{12x^8-36c_1x^6+36c_1^2x^4-12c_1^3x^2+81}\right)x^2\right)^{\frac{2}{3}}+2^{\frac{2}{3}}x^2(x^2-c_1)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}\right)\right)}{12\left(\left(-9+\sqrt{12x^8-36c_1x^6+36c_1^2x^4-12c_1^3x^2+81}\right)x^2\right)^{\frac{1}{3}}x}$$

✓ Solution by Mathematica

Time used: 46.278 (sec). Leaf size: 358

```
DSolve[(2*x^3*y[x]^2-y[x])+(2*x^2*y[x]^3-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{\sqrt[3]{2}(-x^3+c_1x)}{\sqrt[3]{-27x^2+\sqrt{729x^4+108x^3\left(x^3-c_1x\right)^3}}} + \frac{\sqrt[3]{-27x^2+\sqrt{729x^4+108x^3\left(x^3-c_1x\right)^3}}}{3\sqrt[3]{2}x}$$

$$y(x) \to \frac{\left(1+i\sqrt{3}\right)\left(x^3-c_1x\right)}{2^{2/3}\sqrt[3]{-27x^2+\sqrt{729x^4+108x^3\left(x^3-c_1x\right)^3}}} - \frac{\left(1-i\sqrt{3}\right)\sqrt[3]{-27x^2+\sqrt{729x^4+108x^3\left(x^3-c_1x\right)^3}}}{6\sqrt[3]{2}x}$$

$$y(x) \to \frac{\left(1-i\sqrt{3}\right)\left(x^3-c_1x\right)}{2^{2/3}\sqrt[3]{-27x^2+\sqrt{729x^4+108x^3\left(x^3-c_1x\right)^3}}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{-27x^2+\sqrt{729x^4+108x^3\left(x^3-c_1x\right)^3}}}{6\sqrt[3]{2}x}$$

12.25 problem Ex 26

Internal problem ID [11195]
Internal file name [OUTPUT/10180_Saturday_December_03_2022_08_03_30_AM_86763740/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^2+y^2\right)(x+yy')+\sqrt{1+x^2+y^2}\,(y-y'x)=0}$$

12.25.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x^3+x\,y^2+y\sqrt{x^2+y^2+1}}{-x^2y-y^3+\sqrt{x^2+y^2+1}\,x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(x^3 + x\,y^2 + y\sqrt{x^2+y^2+1}\right)(b_3 - a_2)}{-x^2y - y^3 + \sqrt{x^2+y^2+1}\,x} - \frac{\left(x^3 + x\,y^2 + y\sqrt{x^2+y^2+1}\right)^2 a_3}{\left(-x^2y - y^3 + \sqrt{x^2+y^2+1}\,x\right)^2} - \left(\frac{3x^2 + y^2 + \frac{yx}{\sqrt{x^2+y^2+1}}}{-x^2y - y^3 + \sqrt{x^2+y^2+1}\,x} - \frac{\left(x^3 + x\,y^2 + y\sqrt{x^2+y^2+1}\right)\left(-2xy + \frac{x^2}{\sqrt{x^2+y^2+1}} + \sqrt{x^2+y^2+1}\right)}{\left(-x^2y - y^3 + \sqrt{x^2+y^2+1}\,x\right)^2}\right)(xa_2 + ya_3 + a_1) - \left(\frac{2xy + \sqrt{x^2+y^2+1} + \frac{y^2}{\sqrt{x^2+y^2+1}}}{-x^2y - y^3 + \sqrt{x^2+y^2+1}\,x} - \frac{\left(x^3 + x\,y^2 + y\sqrt{x^2+y^2+1}\right)\left(-x^2 - 3y^2 + \frac{yx}{\sqrt{x^2+y^2+1}}\right)}{\left(-x^2y - y^3 + \sqrt{x^2+y^2+1}\,x\right)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{3x^5ya_3 + 3x^5yb_2 + 3x^4y^2a_2 + 6x^3y^3a_3 + 6x^3y^3b_2 + 3x^2y^4b_3 + 3x\,y^5a_3 + 3x\,y^5b_2 + x^4yb_1 + 2x^3y^2a_1 + 2x}{}$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-3x^5 y a_3 - 3x^5 y b_2 - 3x^4 y^2 a_2 - 6x^3 y^3 a_3 - 6x^3 y^3 b_2 \\
&- 3x^2 y^4 b_3 - 3x\, y^5 a_3 - 3x\, y^5 b_2 - x^4 y b_1 - 2x^3 y^2 a_1 - 2x^2 y^3 b_1 \\
&- x\, y^4 a_1 - \sqrt{x^2+y^2+1}\, x^6 a_3 - \sqrt{x^2+y^2+1}\, x^6 b_2 \\
&+ \sqrt{x^2+y^2+1}\, y^6 a_3 + \sqrt{x^2+y^2+1}\, y^6 b_2 - \sqrt{x^2+y^2+1}\, x^5 b_1 \\
&+ \sqrt{x^2+y^2+1}\, y^5 a_1 - \left(x^2+y^2+1\right)^{\frac{3}{2}} x b_1 + \left(x^2+y^2+1\right)^{\frac{3}{2}} y a_1 \\
&- 3x^4 a_2 + x^4 b_3 + y^4 a_2 - 3y^4 b_3 - 2x^3 a_1 - 2y^3 b_1 - 2x^6 a_2 \\
&- 2y^6 b_3 - x^5 a_1 - y^5 b_1 + x^6 b_3 + y^6 a_2 - 4x^3 y a_3 - 4x^3 y b_2 \\
&- 2x^2 y^2 a_2 - 2x^2 y^2 b_3 - 4x\, y^3 a_3 - 4x\, y^3 b_2 - 2x^2 y b_1 \\
&- 2x\, y^2 a_1 + 2\sqrt{x^2+y^2+1}\, x^5 y a_2 - 2\sqrt{x^2+y^2+1}\, x^5 y b_3 \\
&- \sqrt{x^2+y^2+1}\, x^4 y^2 a_3 - \sqrt{x^2+y^2+1}\, x^4 y^2 b_2 \\
&+ 4\sqrt{x^2+y^2+1}\, x^3 y^3 a_2 - 4\sqrt{x^2+y^2+1}\, x^3 y^3 b_3 \\
&+ \sqrt{x^2+y^2+1}\, x^2 y^4 a_3 + \sqrt{x^2+y^2+1}\, x^2 y^4 b_2 \\
&+ 2\sqrt{x^2+y^2+1}\, x\, y^5 a_2 - 2\sqrt{x^2+y^2+1}\, x\, y^5 b_3 \\
&+ \sqrt{x^2+y^2+1}\, x^4 y a_1 - 2\sqrt{x^2+y^2+1}\, x^3 y^2 b_1 \\
&+ 2\sqrt{x^2+y^2+1}\, x^2 y^3 a_1 - \sqrt{x^2+y^2+1}\, x\, y^4 b_1 = 0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&x^5 y a_3+x^5 y b_2+2x^4y^2a_2+x^4y^2b_3+2x^3y^3a_3+2x^3y^3b_2\\
&+x^2y^4a_2+2x^2y^4b_3+x\,y^5a_3+x\,y^5b_2+x^4yb_1+2x^3y^2a_1\\
&+2x^2y^3b_1+x\,y^4a_1-\sqrt{x^2+y^2+1}\,x^6a_3-\sqrt{x^2+y^2+1}\,x^6b_2\\
&+\sqrt{x^2+y^2+1}\,y^6a_3+\sqrt{x^2+y^2+1}\,y^6b_2-3\left(x^2+y^2+1\right)x^4a_2\\
&+\left(x^2+y^2+1\right)x^4b_3+\left(x^2+y^2+1\right)y^4a_2-3\left(x^2+y^2+1\right)y^4b_3\\
&-\sqrt{x^2+y^2+1}\,x^5b_1+\sqrt{x^2+y^2+1}\,y^5a_1\\
&-2\left(x^2+y^2+1\right)x^3a_1-2\left(x^2+y^2+1\right)y^3b_1\\
&-\left(x^2+y^2+1\right)^{\frac{3}{2}}xb_1+\left(x^2+y^2+1\right)^{\frac{3}{2}}ya_1+x^6a_2+y^6b_3\\
&+x^5a_1+y^5b_1+2\sqrt{x^2+y^2+1}\,x^5ya_2-2\sqrt{x^2+y^2+1}\,x^5yb_3\\
&-\sqrt{x^2+y^2+1}\,x^4y^2a_3-\sqrt{x^2+y^2+1}\,x^4y^2b_2\\
&+4\sqrt{x^2+y^2+1}\,x^3y^3a_2-4\sqrt{x^2+y^2+1}\,x^3y^3b_3\\
&+\sqrt{x^2+y^2+1}\,x^2y^4a_3+\sqrt{x^2+y^2+1}\,x^2y^4b_2\\
&+2\sqrt{x^2+y^2+1}\,x\,y^5a_2-2\sqrt{x^2+y^2+1}\,x\,y^5b_3\\
&-4\left(x^2+y^2+1\right)x^3ya_3-4\left(x^2+y^2+1\right)x^3yb_2\\
&-2\left(x^2+y^2+1\right)x^2y^2a_2-2\left(x^2+y^2+1\right)x^2y^2b_3\\
&-4\left(x^2+y^2+1\right)x\,y^3a_3-4\left(x^2+y^2+1\right)x\,y^3b_2\\
&+\sqrt{x^2+y^2+1}\,x^4ya_1-2\sqrt{x^2+y^2+1}\,x^3y^2b_1\\
&+2\sqrt{x^2+y^2+1}\,x^2y^3a_1-\sqrt{x^2+y^2+1}\,x\,y^4b_1\\
&-2\left(x^2+y^2+1\right)x^2yb_1-2\left(x^2+y^2+1\right)x\,y^2a_1=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&x^{2} \sqrt{x^{2}+y^{2}+1}\, y a_{1}-x \sqrt{x^{2}+y^{2}+1}\, y^{2} b_{1}+\sqrt{x^{2}+y^{2}+1}\, y^{3} a_{1} \\
&-\sqrt{x^{2}+y^{2}+1}\, x b_{1}+\sqrt{x^{2}+y^{2}+1}\, y a_{1}-x^{3} \sqrt{x^{2}+y^{2}+1}\, b_{1} \\
&-3 x^{5} y a_{3}-3 x^{5} y b_{2}-3 x^{4} y^{2} a_{2}-6 x^{3} y^{3} a_{3}-6 x^{3} y^{3} b_{2}-3 x^{2} y^{4} b_{3}-3 x\, y^{5} a_{3} \\
&-3 x\, y^{5} b_{2}-x^{4} y b_{1}-2 x^{3} y^{2} a_{1}-2 x^{2} y^{3} b_{1}-x\, y^{4} a_{1}-\sqrt{x^{2}+y^{2}+1}\, x^{6} a_{3} \\
&-\sqrt{x^{2}+y^{2}+1}\, x^{6} b_{2}+\sqrt{x^{2}+y^{2}+1}\, y^{6} a_{3}+\sqrt{x^{2}+y^{2}+1}\, y^{6} b_{2} \\
&-\sqrt{x^{2}+y^{2}+1}\, x^{5} b_{1}+\sqrt{x^{2}+y^{2}+1}\, y^{5} a_{1}-3 x^{4} a_{2}+x^{4} b_{3}+y^{4} a_{2} \\
&-3 y^{4} b_{3}-2 x^{3} a_{1}-2 y^{3} b_{1}-2 x^{6} a_{2}-2 y^{6} b_{3}-x^{5} a_{1}-y^{5} b_{1}+x^{6} b_{3} \\
&+y^{6} a_{2}-4 x^{3} y a_{3}-4 x^{3} y b_{2}-2 x^{2} y^{2} a_{2}-2 x^{2} y^{2} b_{3}-4 x\, y^{3} a_{3}-4 x\, y^{3} b_{2} \\
&-2 x^{2} y b_{1}-2 x\, y^{2} a_{1}+2 \sqrt{x^{2}+y^{2}+1}\, x^{5} y a_{2}-2 \sqrt{x^{2}+y^{2}+1}\, x^{5} y b_{3} \\
&-\sqrt{x^{2}+y^{2}+1}\, x^{4} y^{2} a_{3}-\sqrt{x^{2}+y^{2}+1}\, x^{4} y^{2} b_{2} \\
&+4 \sqrt{x^{2}+y^{2}+1}\, x^{3} y^{3} a_{2}-4 \sqrt{x^{2}+y^{2}+1}\, x^{3} y^{3} b_{3} \\
&+\sqrt{x^{2}+y^{2}+1}\, x^{2} y^{4} a_{3}+\sqrt{x^{2}+y^{2}+1}\, x^{2} y^{4} b_{2} \\
&+2 \sqrt{x^{2}+y^{2}+1}\, x\, y^{5} a_{2}-2 \sqrt{x^{2}+y^{2}+1}\, x\, y^{5} b_{3} \\
&+\sqrt{x^{2}+y^{2}+1}\, x^{4} y a_{1}-2 \sqrt{x^{2}+y^{2}+1}\, x^{3} y^{2} b_{1} \\
&+2 \sqrt{x^{2}+y^{2}+1}\, x^{2} y^{3} a_{1}-\sqrt{x^{2}+y^{2}+1}\, x\, y^{4} b_{1}=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^{2}+y^{2}+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_{1}, y=v_{2}, \sqrt{x^{2}+y^{2}+1}=v_{3}\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2 v_{3} v_{1}^{5} v_{2} a_{2}+4 v_{3} v_{1}^{3} v_{2}^{3} a_{2}+2 v_{3} v_{1} v_{2}^{5} a_{2}-v_{3} v_{1}^{6} a_{3}-v_{3} v_{1}^{4} v_{2}^{2} a_{3}+v_{3} v_{1}^{2} v_{2}^{4} a_{3} \\
&+v_{3} v_{2}^{6} a_{3}-v_{3} v_{1}^{6} b_{2}-v_{3} v_{1}^{4} v_{2}^{2} b_{2}+v_{3} v_{1}^{2} v_{2}^{4} b_{2}+v_{3} v_{2}^{6} b_{2}-2 v_{3} v_{1}^{5} v_{2} b_{3} \\
&-4 v_{3} v_{1}^{3} v_{2}^{3} b_{3}-2 v_{3} v_{1} v_{2}^{5} b_{3}+v_{3} v_{1}^{4} v_{2} a_{1}+2 v_{3} v_{1}^{2} v_{2}^{3} a_{1}+v_{3} v_{2}^{5} a_{1}-2 v_{1}^{6} a_{2} \\
&-3 v_{1}^{4} v_{2}^{2} a_{2}+v_{2}^{6} a_{2}-3 v_{1}^{5} v_{2} a_{3}-6 v_{1}^{3} v_{2}^{3} a_{3}-3 v_{1} v_{2}^{5} a_{3}-v_{3} v_{1}^{5} b_{1} \\
&-2 v_{3} v_{1}^{3} v_{2}^{2} b_{1}-v_{3} v_{1} v_{2}^{4} b_{1}-3 v_{1}^{5} v_{2} b_{2}-6 v_{1}^{3} v_{2}^{3} b_{2}-3 v_{1} v_{2}^{5} b_{2}+v_{1}^{6} b_{3} \\
&-3 v_{1}^{2} v_{2}^{4} b_{3}-2 v_{2}^{6} b_{3}-v_{1}^{5} a_{1}-2 v_{1}^{3} v_{2}^{2} a_{1}-v_{1} v_{2}^{4} a_{1}-v_{1}^{4} v_{2} b_{1}-2 v_{1}^{2} v_{2}^{3} b_{1} \\
&-v_{2}^{5} b_{1}+v_{1}^{2} v_{3} v_{2} a_{1}+v_{3} v_{2}^{3} a_{1}-3 v_{1}^{4} a_{2}-2 v_{1}^{2} v_{2}^{2} a_{2}+v_{2}^{4} a_{2}-4 v_{1}^{3} v_{2} a_{3} \\
&-4 v_{1} v_{2}^{3} a_{3}-v_{1}^{3} v_{3} b_{1}-v_{1} v_{3} v_{2}^{2} b_{1}-4 v_{1}^{3} v_{2} b_{2}-4 v_{1} v_{2}^{3} b_{2}+v_{1}^{4} b_{3}-2 v_{1}^{2} v_{2}^{2} b_{3} \\
&-3 v_{2}^{4} b_{3}-2 v_{1}^{3} a_{1}-2 v_{1} v_{2}^{2} a_{1}-2 v_{1}^{2} v_{2} b_{1}-2 v_{2}^{3} b_{1}+v_{3} v_{2} a_{1}-v_{3} v_{1} b_{1}=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(2a_2 - 2b_3)\, v_1^5 v_2 v_3 + (-a_3 - b_2)\, v_1^4 v_2^2 v_3 + (4a_2 - 4b_3)\, v_1^3 v_2^3 v_3 \\
&+ (a_3 + b_2)\, v_1^2 v_2^4 v_3 + (2a_2 - 2b_3)\, v_1 v_2^5 v_3 + (-2a_2 - 2b_3)\, v_1^2 v_2^2 \\
&+ (-3a_3 - 3b_2)\, v_1 v_2^5 + (-4a_3 - 4b_2)\, v_1 v_2^3 + (a_3 + b_2)\, v_2^6 v_3 \\
&+ (-a_3 - b_2)\, v_1^6 v_3 + (-3a_3 - 3b_2)\, v_1^5 v_2 + (-6a_3 - 6b_2)\, v_1^3 v_2^3 \\
&+ (-4a_3 - 4b_2)\, v_1^3 v_2 - v_1 v_3 v_2^2 b_1 + v_3 v_1^4 v_2 a_1 - 2 v_3 v_1^3 v_2^2 b_1 + 2 v_3 v_1^2 v_2^3 a_1 \\
&- v_3 v_1 v_2^4 b_1 - 2 v_1^3 a_1 - 2 v_2^3 b_1 - v_1^5 a_1 - v_2^5 b_1 + v_1^2 v_3 v_2 a_1 + v_3 v_2^3 a_1 \\
&- v_3 v_1 b_1 + v_3 v_2 a_1 - v_1^3 v_3 b_1 - 3 v_1^4 v_2^2 a_2 - 3 v_1^2 v_2^4 b_3 - v_1^4 v_2 b_1 - 2 v_1^3 v_2^2 a_1 \\
&- 2 v_1^2 v_2^3 b_1 - v_1 v_2^4 a_1 - v_3 v_1^5 b_1 + v_3 v_2^5 a_1 - 2 v_1^2 v_2 b_1 - 2 v_1 v_2^2 a_1 \\
&+ (-2a_2 + b_3)\, v_1^6 + (-3a_2 + b_3)\, v_1^4 + (a_2 - 2b_3)\, v_2^6 + (a_2 - 3b_3)\, v_2^4 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
-2a_1 &= 0 \\
-a_1 &= 0 \\
2a_1 &= 0 \\
-3a_2 &= 0 \\
-2b_1 &= 0 \\
-b_1 &= 0 \\
-3b_3 &= 0 \\
-3a_2 + b_3 &= 0 \\
-2a_2 - 2b_3 &= 0 \\
-2a_2 + b_3 &= 0 \\
a_2 - 3b_3 &= 0 \\
a_2 - 2b_3 &= 0 \\
2a_2 - 2b_3 &= 0 \\
4a_2 - 4b_3 &= 0 \\
-6a_3 - 6b_2 &= 0 \\
-4a_3 - 4b_2 &= 0 \\
-3a_3 - 3b_2 &= 0 \\
-a_3 - b_2 &= 0 \\
a_3 + b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= -b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -y \\
\eta &= x
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= x - \left(\frac{x^3 + x\, y^2 + y\sqrt{x^2+y^2+1}}{-x^2 y - y^3 + \sqrt{x^2+y^2+1}\, x}\right)(-y) \\
&= \frac{x^2\sqrt{x^2+y^2+1} + y^2\sqrt{x^2+y^2+1}}{-x^2 y - y^3 + \sqrt{x^2+y^2+1}\, x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2\sqrt{x^2+y^2+1} + y^2\sqrt{x^2+y^2+1}}{-x^2 y - y^3 + \sqrt{x^2+y^2+1}\, x}} dy
\end{aligned}
$$

Which results in

$$
S = \arctan\left(\frac{y}{x}\right) - \sqrt{x^2+y^2+1}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{x^3 + x\, y^2 + y\sqrt{x^2+y^2+1}}{-x^2 y - y^3 + \sqrt{x^2+y^2+1}\, x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y}{x^2+y^2} - \frac{x}{\sqrt{x^2+y^2+1}} \\
S_y &= \frac{x}{x^2+y^2} - \frac{y}{\sqrt{x^2+y^2+1}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

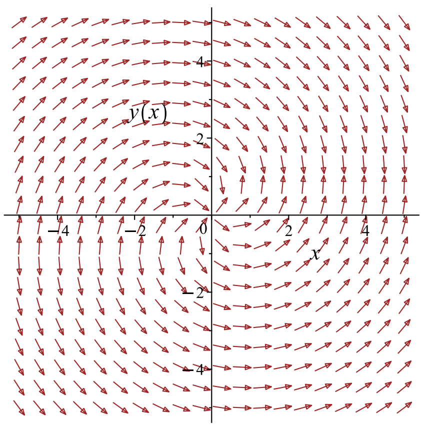
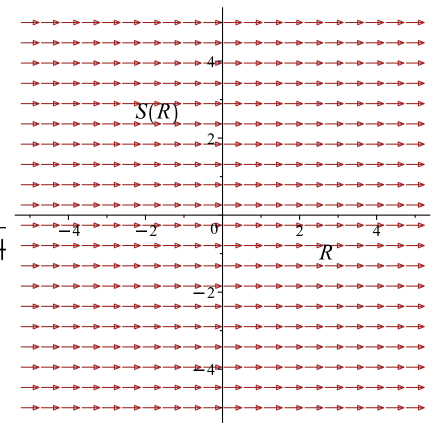
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\arctan\left(\frac{y}{x}\right) - \sqrt{1+x^2+y^2} = c_1
$$

Which simplifies to

$$
\arctan\left(\frac{y}{x}\right) - \sqrt{1+x^2+y^2} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^{3}+x\,y^{2}+y\sqrt{x^{2}+y^{2}+1}}{-x^{2}y-y^{3}+\sqrt{x^{2}+y^{2}+1}\,x}$ 	$R = x$ $S = \arctan\left(\frac{y}{x}\right) - \sqrt{x^{2} +}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\arctan\left(\frac{y}{x}\right) - \sqrt{1 + x^{2} + y^{2}} = c_1 \tag{1}$$

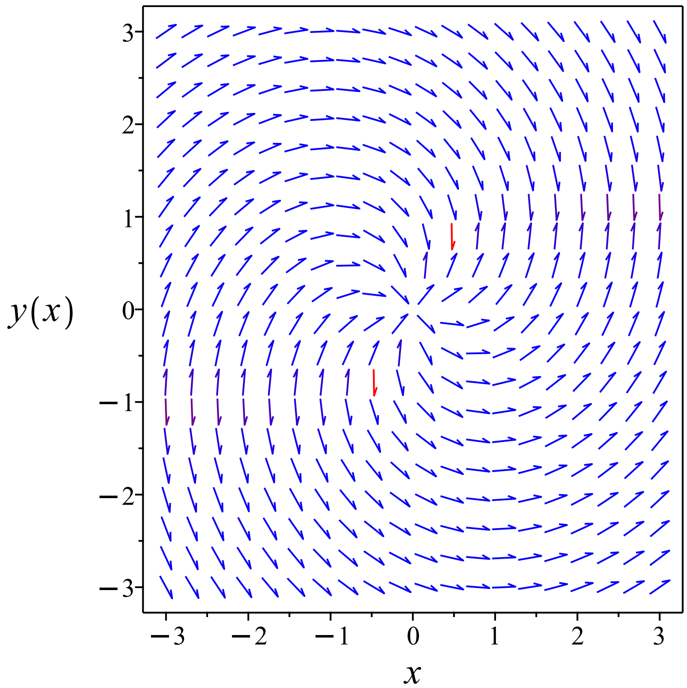


Figure 191: Slope field plot

Verification of solutions

$$\arctan\left(\frac{y}{x}\right) - \sqrt{1 + x^2 + y^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 25

```
dsolve((x^2+y(x)^2)*(x+y(x)*diff(y(x),x))+(1+x^2+y(x)^2)^(1/2)*(y(x)-x*diff(y(x),x))=0,y(x),
```

$$\arctan\left(\frac{x}{y\left(x\right)}\right)+\sqrt{1+x^{2}+y\left(x\right)^{2}}-c_{1}=0$$

✓ Solution by Mathematica

Time used: 0.454 (sec). Leaf size: 27

```
DSolve[(x^2+y[x]^2)*(x+y[x]*y'[x])+(1+x^2+y[x]^2)^(1/2)*(y[x]-x*y'[x])==0,y[x],x,IncludeSing
```

$$\text{Solve}\left[\arctan\left(\frac{x}{y(x)}\right)+\sqrt{x^{2}+y(x)^{2}+1}=c_{1},y(x)\right]$$

12.26 problem Ex 27

Internal problem ID [11196]
Internal file name [`OUTPUT/10181_Saturday_December_03_2022_08_03_31_AM_44121981/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{e^{\frac{y}{x}}+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)y'=-1}$$

12.26.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$e^{u(x)}+e^{\frac{1}{u(x)}}\left(1-\frac{1}{u(x)}\right)\left(u'(x)\,x+u(x)\right)=-1$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{u\left(e^{-\frac{1}{u}+u}+e^{-\frac{1}{u}}+u-1\right)}{x\,(u-1)}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{\left(\mathrm{e}^{-\frac{1}{u}+u}+\mathrm{e}^{-\frac{1}{u}}+u-1\right)u}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{\left(\mathrm{e}^{-\frac{1}{u}+u}+\mathrm{e}^{-\frac{1}{u}}+u-1\right)u}{u-1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{\left(\mathrm{e}^{-\frac{1}{u}+u}+\mathrm{e}^{-\frac{1}{u}}+u-1\right)u}{u-1}}\, du = \int -\frac{1}{x}\, dx$$

$$\int^{u} \frac{\_a-1}{\left(\mathrm{e}^{-\frac{1}{\_a}+\_a}+\mathrm{e}^{-\frac{1}{\_a}}+\_a-1\right)\_a} d\_a = -\ln(x) + c_2$$

Which results in

$$\int^{u} \frac{\_a-1}{\left(\mathrm{e}^{-\frac{1}{\_a}+\_a}+\mathrm{e}^{-\frac{1}{\_a}}+\_a-1\right)\_a} d\_a = -\ln(x) + c_2$$

The solution is

$$\int^{u(x)} \frac{\_a-1}{\left(\mathrm{e}^{-\frac{1}{\_a}+\_a}+\mathrm{e}^{-\frac{1}{\_a}}+\_a-1\right)\_a} d\_a + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\int^{\frac{y}{x}} \frac{\_a-1}{\left(\mathrm{e}^{-\frac{1}{\_a}+\_a}+\mathrm{e}^{-\frac{1}{\_a}}+\_a-1\right)\_a} d\_a + \ln(x) - c_2 = 0$$

$$\int^{\frac{y}{x}} \frac{\_a-1}{\left(\mathrm{e}^{\frac{(\_a-1)(\_a+1)}{\_a}}+\mathrm{e}^{-\frac{1}{\_a}}+\_a-1\right)\_a} d\_a + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\int^{\frac{y}{x}} \frac{\_a-1}{\left(\mathrm{e}^{\frac{(\_a-1)(\_a+1)}{\_a}}+\mathrm{e}^{-\frac{1}{\_a}}+\_a-1\right)\_a} d\_a + \ln(x) - c_2 = 0 \tag{1}$$

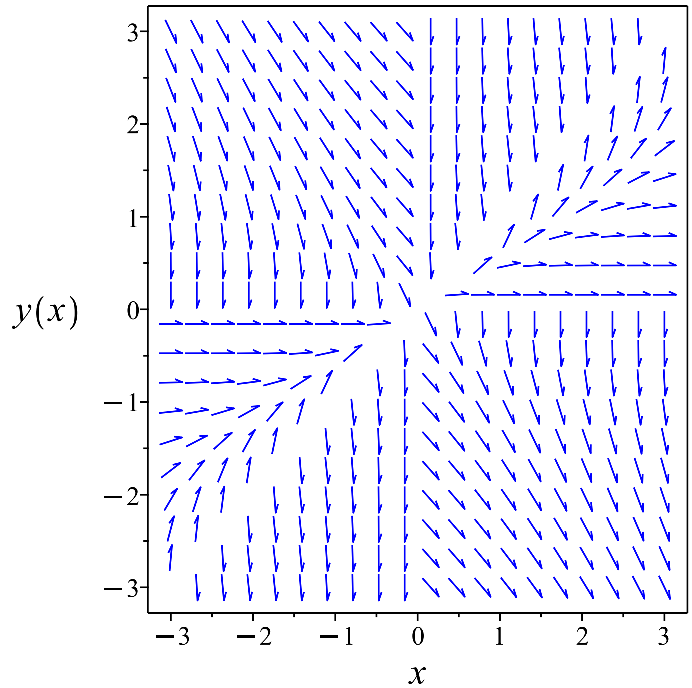


Figure 192: Slope field plot

<u>Verification of solutions</u>

$$\int^{\frac{y}{x}} \frac{\_a-1}{\left(\mathrm{e}^{\frac{(\_a-1)(\_a+1)}{\_a}}+\mathrm{e}^{-\frac{1}{\_a}}+\_a-1\right)\_a}d\_a+\ln\left(x\right)-c_{2}=0$$

Verified OK.

12.26.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y\left(\mathrm{e}^{\frac{y}{x}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{y-x}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{y\left(\mathrm{e}^{\frac{y}{x}}+1\right)\mathrm{e}^{-\frac{x}{y}}(b_3-a_2)}{y-x} - \frac{y^2\left(\mathrm{e}^{\frac{y}{x}}+1\right)^2\mathrm{e}^{-\frac{2x}{y}}a_3}{(y-x)^2} \\ & - \left(\frac{y^2\mathrm{e}^{\frac{y}{x}}\mathrm{e}^{-\frac{x}{y}}}{x^2(y-x)} + \frac{\left(\mathrm{e}^{\frac{y}{x}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{y-x} - \frac{y\left(\mathrm{e}^{\frac{y}{x}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{(y-x)^2}\right)(x a_2 + y a_3 + a_1) \\ & - \left(-\frac{\left(\mathrm{e}^{\frac{y}{x}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{y-x} - \frac{y\,\mathrm{e}^{\frac{y}{x}}\mathrm{e}^{-\frac{x}{y}}}{x(y-x)} - \frac{\left(\mathrm{e}^{\frac{y}{x}}+1\right)\mathrm{e}^{-\frac{x}{y}}x}{y(y-x)} \right. \\ & \left. + \frac{y\left(\mathrm{e}^{\frac{y}{x}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{(y-x)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{\left(-\mathrm{e}^{\frac{x}{y}}x^5b_2 - \mathrm{e}^{\frac{x}{y}}x^4b_1 + \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^4ya_2 - \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^4yb_3 - \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^3y^2a_2 + \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^3y^2a_3 - \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^3y^2b_2 + \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^3y^2b_3 - \cdots}{\cdots}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -\mathrm{e}^{\frac{x}{y}}x^5b_2 - \mathrm{e}^{\frac{x}{y}}x^4b_1 + \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^4ya_2 - \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^4yb_3 - \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^3y^2a_2 \\ & + \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^3y^2a_3 - \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^3y^2b_2 + \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^3y^2b_3 + 2\,\mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^2y^3a_2 + \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^2y^3b_2 \\ & - 2\,\mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^2y^3b_3 - \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x\,y^4a_2 + \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x\,y^4a_3 + \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x\,y^4b_3 + \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^3ya_1 \\ & - \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^2y^2b_1 + \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x\,y^3a_1 + \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x\,y^3b_1 + \mathrm{e}^{\frac{2x}{y}}x^4yb_2 - 2\,\mathrm{e}^{\frac{2x}{y}}x^3y^2b_2 \\ & + \mathrm{e}^{\frac{2x}{y}}x^2y^3b_2 - \mathrm{e}^{\frac{2y}{x}}x^2y^3a_3 - y^3a_3x^2 - \mathrm{e}^{\frac{x}{y}}x^3y^2a_2 + \mathrm{e}^{\frac{x}{y}}x^3y^2a_3 + \mathrm{e}^{\frac{x}{y}}x^3y^2b_3 \\ & + \mathrm{e}^{\frac{x}{y}}x^2y^3a_2 - \mathrm{e}^{\frac{x}{y}}x^2y^3b_3 - 2\,\mathrm{e}^{\frac{y}{x}}x^2y^3a_3 + \mathrm{e}^{\frac{x}{y}}x^3ya_1 - \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^5b_2 \\ & - \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}y^5a_3 - \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}x^4b_1 - \mathrm{e}^{\frac{x}{y}}\mathrm{e}^{\frac{y}{x}}y^4a_1 + \mathrm{e}^{\frac{x}{y}}x^4ya_2 - \mathrm{e}^{\frac{x}{y}}x^4yb_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&-\mathrm{e}^{\frac{x}{y}} x^5 b_2-\mathrm{e}^{\frac{x}{y}} x^4 b_1-x\,y^4 a_2 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}+x\,y^4 a_3 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}+x\,y^4 b_3 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}} \\
&+x^3 y a_1 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}-x^2 y^2 b_1 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}+x\,y^3 a_1 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}+x\,y^3 b_1 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}+x^4 y a_2 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}} \\
&-x^4 y b_3 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}-x^3 y^2 a_2 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}+x^3 y^2 a_3 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}-x^3 y^2 b_2 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}+x^3 y^2 b_3 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}} \\
&+2 x^2 y^3 a_2 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}+x^2 y^3 b_2 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}-2 x^2 y^3 b_3 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}+\mathrm{e}^{\frac{2x}{y}} x^4 y b_2-2\,\mathrm{e}^{\frac{2x}{y}} x^3 y^2 b_2 \\
&+\mathrm{e}^{\frac{2x}{y}} x^2 y^3 b_2-\mathrm{e}^{\frac{2y}{x}} x^2 y^3 a_3-x^5 b_2 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}-y^5 a_3 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}-x^4 b_1 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}} \\
&-y^4 a_1 \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}-y^3 a_3 x^2-\mathrm{e}^{\frac{x}{y}} x^3 y^2 a_2+\mathrm{e}^{\frac{x}{y}} x^3 y^2 a_3+\mathrm{e}^{\frac{x}{y}} x^3 y^2 b_3+\mathrm{e}^{\frac{x}{y}} x^2 y^3 a_2 \\
&-\mathrm{e}^{\frac{x}{y}} x^2 y^3 b_3-2\,\mathrm{e}^{\frac{y}{x}} x^2 y^3 a_3+\mathrm{e}^{\frac{x}{y}} x^3 y a_1+\mathrm{e}^{\frac{x}{y}} x^4 y a_2-\mathrm{e}^{\frac{x}{y}} x^4 y b_3=0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x, y, \mathrm{e}^{\frac{y}{x}}, \mathrm{e}^{\frac{x}{y}}, \mathrm{e}^{\frac{2y}{x}}, \mathrm{e}^{\frac{2x}{y}}, \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_1, y=v_2, \mathrm{e}^{\frac{y}{x}}=v_3, \mathrm{e}^{\frac{x}{y}}=v_4, \mathrm{e}^{\frac{2y}{x}}=v_5, \mathrm{e}^{\frac{2x}{y}}=v_6, \mathrm{e}^{\frac{x}{y}+\frac{y}{x}}=v_7\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&v_4 v_1^4 v_2 a_2+v_1^4 v_2 a_2 v_7-v_4 v_1^3 v_2^2 a_2-v_1^3 v_2^2 a_2 v_7+v_4 v_1^2 v_2^3 a_2+2 v_1^2 v_2^3 a_2 v_7 \\
&-v_1 v_2^4 a_2 v_7+v_4 v_1^3 v_2^2 a_3+v_1^3 v_2^2 a_3 v_7-2 v_3 v_1^2 v_2^3 a_3-v_5 v_1^2 v_2^3 a_3+v_1 v_2^4 a_3 v_7 \\
&-v_2^5 a_3 v_7-v_4 v_1^5 b_2-v_1^5 b_2 v_7+v_6 v_1^4 v_2 b_2-2 v_6 v_1^3 v_2^2 b_2-v_1^3 v_2^2 b_2 v_7 \\
&+v_6 v_1^2 v_2^3 b_2+v_1^2 v_2^3 b_2 v_7-v_4 v_1^4 v_2 b_3-v_1^4 v_2 b_3 v_7+v_4 v_1^3 v_2^2 b_3+v_1^3 v_2^2 b_3 v_7 \\
&-v_4 v_1^2 v_2^3 b_3-2 v_1^2 v_2^3 b_3 v_7+v_1 v_2^4 b_3 v_7+v_4 v_1^3 v_2 a_1+v_1^3 v_2 a_1 v_7+v_1 v_2^3 a_1 v_7 \\
&-v_2^4 a_1 v_7-v_2^3 a_3 v_1^2-v_4 v_1^4 b_1-v_1^4 b_1 v_7-v_1^2 v_2^2 b_1 v_7+v_1 v_2^3 b_1 v_7=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-v_4 v_1^5 b_2-v_1^5 b_2 v_7+(-b_3+a_2) v_1^4 v_2 v_4+v_6 v_1^4 v_2 b_2+(-b_3+a_2) v_1^4 v_2 v_7 \\
&-v_4 v_1^4 b_1-v_1^4 b_1 v_7+(-a_2+a_3+b_3) v_1^3 v_2^2 v_4-2 v_6 v_1^3 v_2^2 b_2 \\
&+(-a_2+a_3-b_2+b_3) v_1^3 v_2^2 v_7+v_4 v_1^3 v_2 a_1+v_1^3 v_2 a_1 v_7-2 v_3 v_1^2 v_2^3 a_3 \\
&+(-b_3+a_2) v_1^2 v_2^3 v_4-v_5 v_1^2 v_2^3 a_3+v_6 v_1^2 v_2^3 b_2+(2 a_2+b_2-2 b_3) v_1^2 v_2^3 v_7-v_2^3 a_3 v_1^2 \\
&-v_1^2 v_2^2 b_1 v_7+(-a_2+a_3+b_3) v_1 v_2^4 v_7+(a_1+b_1) v_1 v_2^3 v_7-v_2^5 a_3 v_7-v_2^4 a_1 v_7=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
b_2 &= 0 \\
-a_1 &= 0 \\
-2a_3 &= 0 \\
-a_3 &= 0 \\
-b_1 &= 0 \\
-2b_2 &= 0 \\
-b_2 &= 0 \\
a_1 + b_1 &= 0 \\
-b_3 + a_2 &= 0 \\
-a_2 + a_3 + b_3 &= 0 \\
2a_2 + b_2 - 2b_3 &= 0 \\
-a_2 + a_3 - b_2 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{y}{x}\\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{x}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= \ln(x)\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y\left(e^{\frac{y}{x}}+1\right)e^{-\frac{x}{y}}}{y-x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{y}{x^2} \\
R_y &= \frac{1}{x} \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{x\,\mathrm{e}^{\frac{x}{y}}(-y+x)}{y\left(\mathrm{e}^{\frac{x}{y}}\left(y-x\right)+x\left(\mathrm{e}^{\frac{y}{x}}+1\right)\right)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{\mathrm{e}^{\frac{1}{R}}(R-1)}{R\left((R-1)\,\mathrm{e}^{\frac{1}{R}}+\mathrm{e}^{R}+1\right)}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \int -\frac{\mathrm{e}^{\frac{1}{R}}(R-1)}{R\left(\mathrm{e}^{\frac{1}{R}}R+\mathrm{e}^{R}-\mathrm{e}^{\frac{1}{R}}+1\right)}dR + c_1 \tag{4}
$$

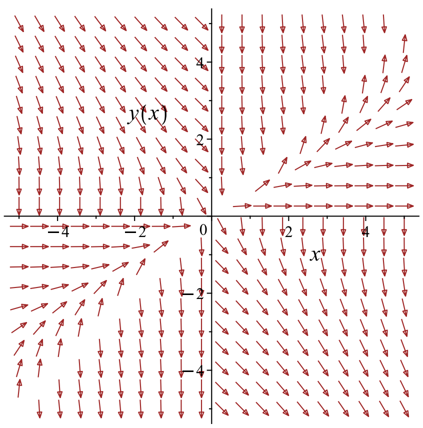
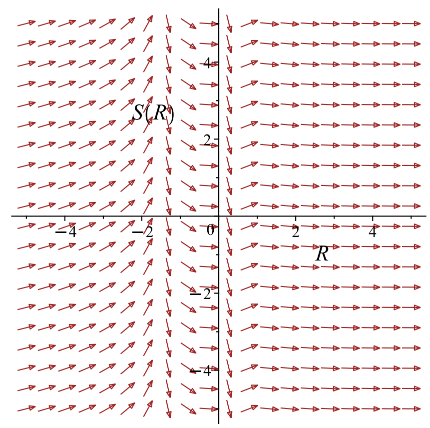
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln\left(x\right) = \int^{\frac{y}{x}} -\frac{\mathrm{e}^{\frac{1}{\_a}}(\_a-1)}{\_a\left(\mathrm{e}^{\frac{1}{\_a}}\_a+\mathrm{e}^{\_a}-\mathrm{e}^{\frac{1}{\_a}}+1\right)}d\_a + c_1
$$

Which simplifies to

$$
\ln\left(x\right) + \int^{\frac{y}{x}} \frac{\mathrm{e}^{\frac{1}{\_a}}(\_a-1)}{\_a\left(\mathrm{e}^{\frac{1}{\_a}}\left(\_a-1\right)+\mathrm{e}^{\_a}+1\right)}d\_a - c_1 = 0
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y\left(e^{\frac{y}{x}}+1\right)e^{-\frac{x}{y}}}{y-x}$ 	$R = \frac{y}{x}$ $S = \ln(x)$	$\frac{dS}{dR} = -\frac{e^{\frac{1}{R}}(R-1)}{R\left((R-1)e^{\frac{1}{R}}+e^{R}+1\right)}$ 

Summary

The solution(s) found are the following

$$\ln(x) + \int^{\frac{y}{x}} \frac{e^{\frac{1}{\_a}}(\_a - 1)}{\_a\left(e^{\frac{1}{\_a}}(\_a - 1) + e^{\_a} + 1\right)} d\_a - c_1 = 0 \tag{1}$$

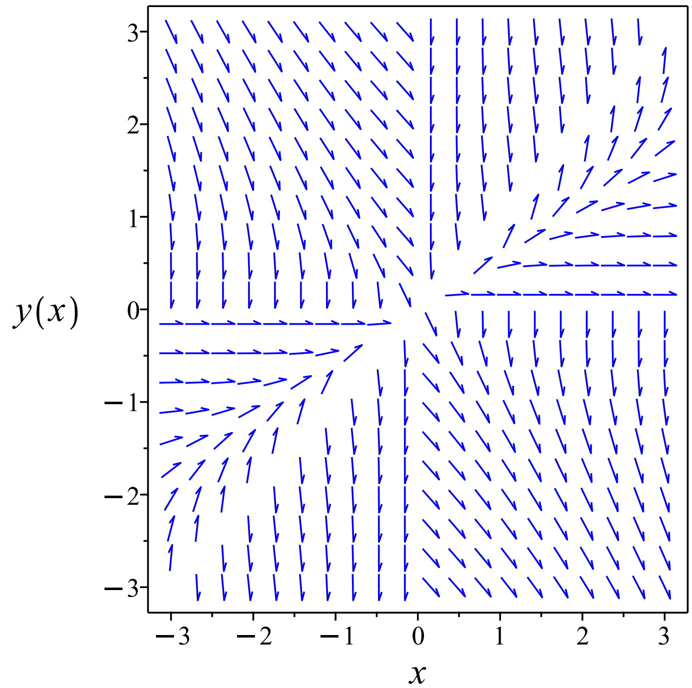


Figure 193: Slope field plot

Verification of solutions

$$\ln(x)+\int^{\frac{y}{x}}\frac{e^{\frac{1}{\_a}}(\_a-1)}{\_a\left(e^{\frac{1}{\_a}}(\_a-1)+e^{\_a}+1\right)}d\_a-c_1=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve((1+exp(y(x)/x))+exp(x/y(x))*(1-x/y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \text{RootOf}\left(\int^{\_Z} \frac{\mathrm{e}^{\frac{1}{\_a}}(\_a-1)}{\_a\left(\_a\,\mathrm{e}^{\frac{1}{\_a}}+\mathrm{e}^{\_a}-\mathrm{e}^{\frac{1}{\_a}}+1\right)}d\_a+\ln(x)+c_1\right)x$$

✓ Solution by Mathematica

Time used: 0.429 (sec). Leaf size: 63

```
DSolve[(1+Exp[y[x]/x])+Exp[x/y[x]]*(1-x/y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$\text{Solve}\left[\int_1^{\frac{y(x)}{x}} \frac{e^{\frac{1}{K[1]}}(K[1]-1)}{K[1]\left(e^{\frac{1}{K[1]}}K[1]+e^{K[1]}-e^{\frac{1}{K[1]}}+1\right)}dK[1] = -\log(x)+c_1, y(x)\right]$$

12.27 problem Ex 28

Internal problem ID [11197]
Internal file name [`OUTPUT/10182_Saturday_December_03_2022_08_03_33_AM_61242619/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y'x + y - y^2 \ln(x) = 0}$$

12.27.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(\ln(x)\, y - 1)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 107: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x\, y^2 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{x\, y^2} dy
\end{aligned}$$

Which results in

$$S = -\frac{1}{xy}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(\ln(x)\, y - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{x^2 y} \\
S_y &= \frac{1}{x\, y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\ln(x)}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\ln(R)}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{R} - \frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{yx} = -\frac{\ln(x)}{x} - \frac{1}{x} + c_1$$

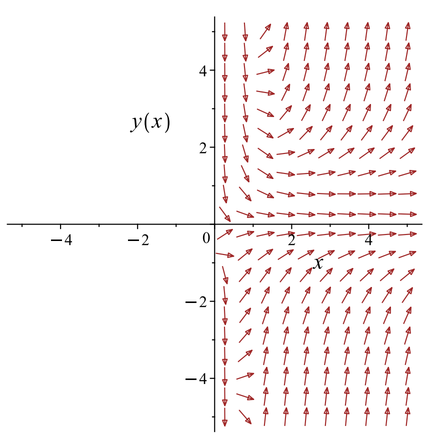
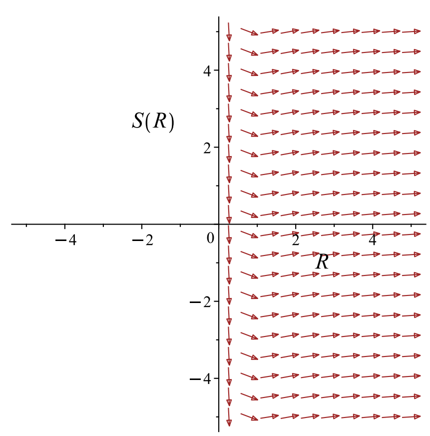
Which simplifies to

$$\frac{-c_1 xy + y\ln(x) + y - 1}{xy} = 0$$

Which gives

$$y = \frac{1}{-c_1 x + \ln(x) + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(\ln(x)y-1)}{x}$ 	$R = x$ $S = -\frac{1}{xy}$	$\frac{dS}{dR} = \frac{\ln(R)}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{1}{-c_1 x + \ln(x) + 1} \tag{1}$$

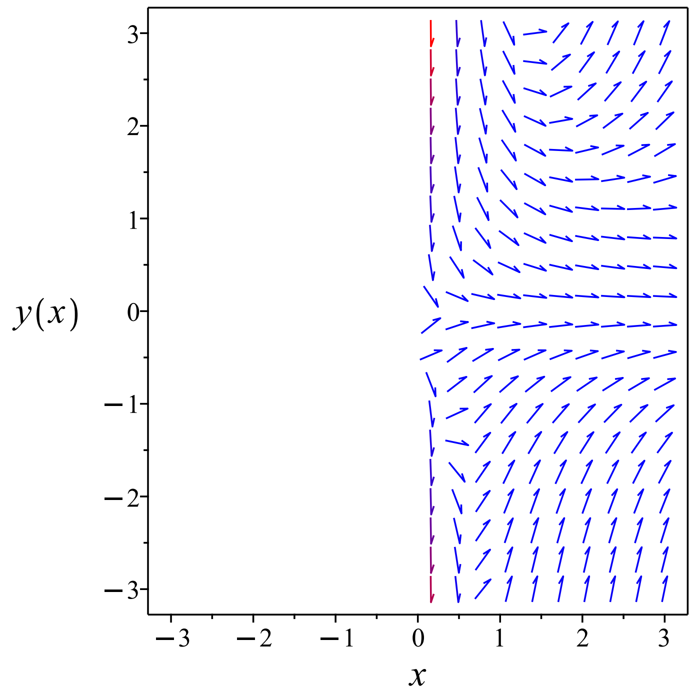


Figure 194: Slope field plot

Verification of solutions

$$y = \frac{1}{-c_1 x + \ln(x) + 1}$$

Verified OK.

12.27.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(\ln(x)\, y - 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x} y + \frac{\ln(x)}{x} y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \frac{\ln(x)}{x} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{xy} + \frac{\ln(x)}{x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{w(x)}{x} + \frac{\ln(x)}{x} \\ w' &= \frac{w}{x} - \frac{\ln(x)}{x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{1}{x}\\ q(x) &= -\frac{\ln(x)}{x}\end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = -\frac{\ln(x)}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx}\\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{\ln(x)}{x}\right)\\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(-\frac{\ln(x)}{x}\right)\\ d\left(\frac{w}{x}\right) &= \left(-\frac{\ln(x)}{x^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int -\frac{\ln(x)}{x^2} dx\\ \frac{w}{x} &= \frac{\ln(x)}{x} + \frac{1}{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = x\left(\frac{\ln(x)}{x} + \frac{1}{x}\right) + c_1 x$$

which simplifies to

$$w(x) = c_1 x + \ln(x) + 1$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = c_1 x + \ln(x) + 1$$

Or

$$y = \frac{1}{c_1 x + \ln(x) + 1}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{1}{c_1 x + \ln(x) + 1} \tag{1}$$

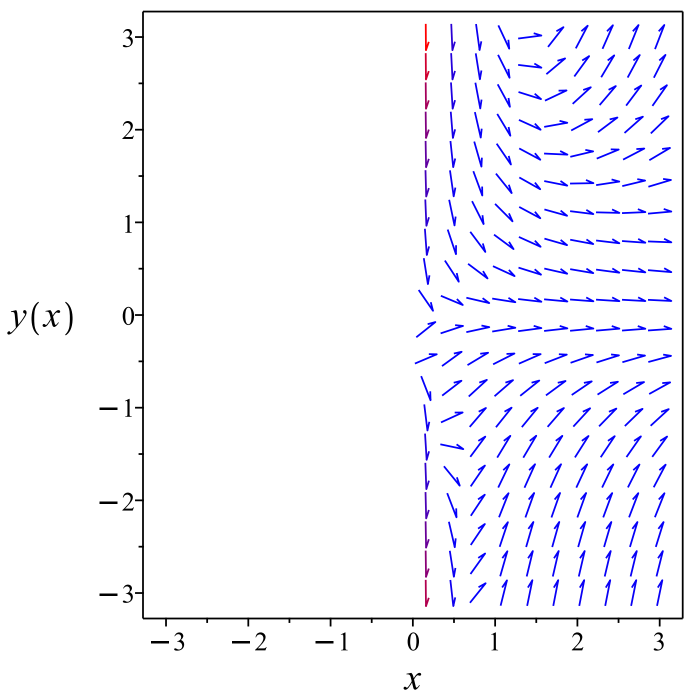


Figure 195: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1}{c_1 x + \ln(x) + 1}$$

Verified OK.

12.27.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(-y+y^2\ln(x)\right)\mathrm{d}x\\ \left(-y^2\ln(x)+y\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y^2\ln(x)+y\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y^2 \ln(x) + y\right) \\
&= -2\ln(x)\, y + 1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((-2\ln(x)\, y + 1) - (1)) \\
&= -\frac{2\ln(x)\, y}{x}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{y(\ln(x)\, y - 1)}((1) - (-2\ln(x)\, y + 1)) \\
&= -\frac{2\ln(x)}{\ln(x)\, y - 1}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(1) - (-2\ln(x)\,y + 1)}{x(-y^2\ln(x) + y) - y(x)} \\ &= -\frac{2}{xy} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R\,\mathrm{d}t} \\ &= e^{\int \left(-\frac{2}{t}\right)\mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(t)} \\ &= \frac{1}{t^2} \end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^2y^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^2y^2}\left(-y^2\ln(x) + y\right) \\ &= \frac{-\ln(x)\,y + 1}{x^2y} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^2y^2}(x) \\ &= \frac{1}{x\,y^2} \end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-\ln(x)\, y + 1}{x^2 y}\right) + \left(\frac{1}{x\, y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-\ln(x)\, y + 1}{x^2 y}\,\mathrm{d}x$$
$$\phi = \frac{\ln(x)\, y + y - 1}{xy} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{\ln(x) + 1}{xy} - \frac{\ln(x)\, y + y - 1}{x\, y^2} + f'(y) \tag{4}$$
$$= \frac{1}{x\, y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x\, y^2}$. Therefore equation (4) becomes

$$\frac{1}{x\, y^2} = \frac{1}{x\, y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x) y + y - 1}{xy} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x) y + y - 1}{xy}$$

The solution becomes

$$y = \frac{1}{-c_1 x + \ln(x) + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{-c_1 x + \ln(x) + 1} \tag{1}$$

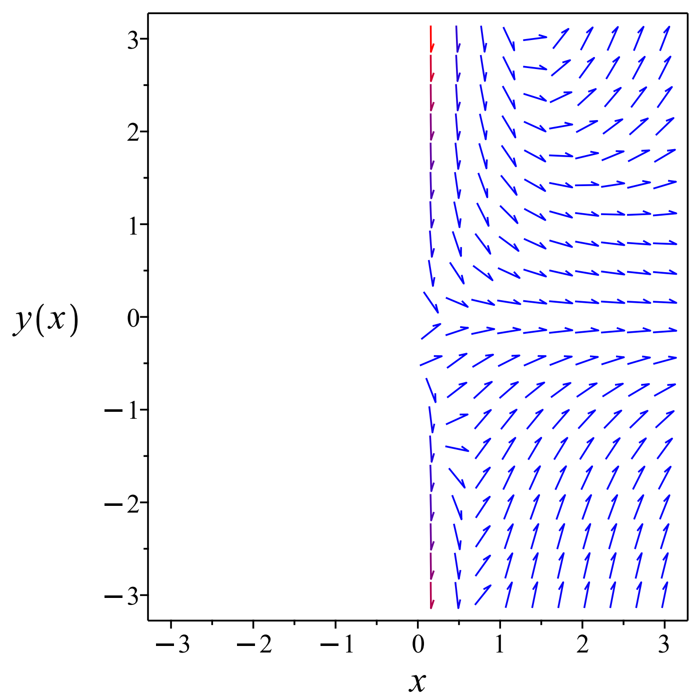


Figure 196: Slope field plot

Verification of solutions

$$y = \frac{1}{-c_1 x + \ln(x) + 1}$$

Verified OK.

12.27.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(\ln(x) y - 1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2 \ln(x)}{x} - \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = \frac{\ln(x)}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{\ln(x)u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{\ln(x)}{x^2} + \frac{1}{x^2} \\ f_1 f_2 &= -\frac{\ln(x)}{x^2} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\ln(x) u''(x)}{x} - \left(-\frac{2\ln(x)}{x^2} + \frac{1}{x^2}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{-c_2 \ln(x) + c_1 x - c_2}{x}$$

The above shows that

$$u'(x) = \frac{c_2 \ln(x)}{x^2}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{-c_2 \ln(x) + c_1 x - c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{1}{-c_3 x + \ln(x) + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{-c_3 x + \ln(x) + 1} \tag{1}$$

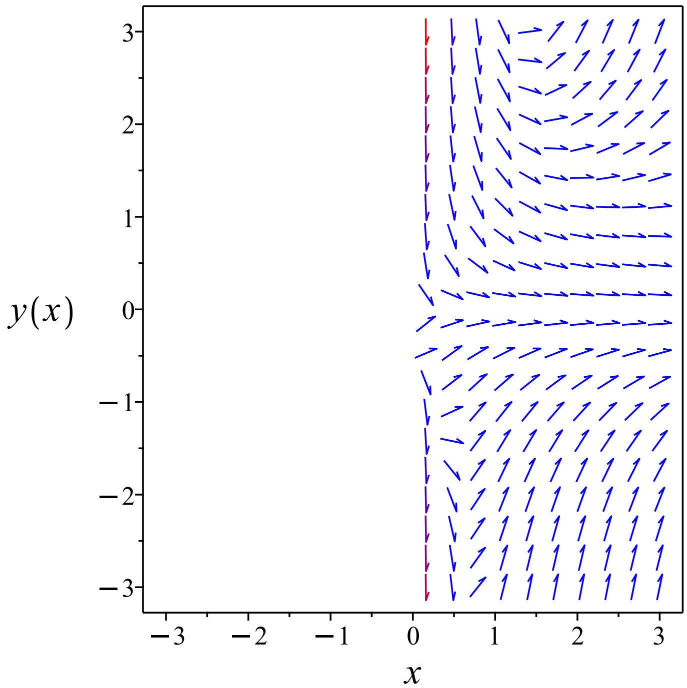


Figure 197: Slope field plot

Verification of solutions

$$y = \frac{1}{-c_3x + \ln(x) + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(x*diff(y(x),x)+y(x)-y(x)^2*ln(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{1 + c_1 x + \ln(x)}$$

✓ Solution by Mathematica

Time used: 0.233 (sec). Leaf size: 20

```
DSolve[x*y'[x]+y[x]-y[x]^2*Log[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{\log(x) + c_1 x + 1}$$
$$y(x) \to 0$$

12.28 problem Ex 29

Internal problem ID [11198]
Internal file name [OUTPUT/10183_Saturday_December_03_2022_08_03_34_AM_20909120/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational]`

Unable to solve or complete the solution.

$$\boxed{x^3y^4 + y^3x^2 + y^2x + y + \left(y^3x^4 - y^2x^3 - yx^3 + x\right) y' = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve((x^3*y(x)^4+x^2*y(x)^3+x*y(x)^2+y(x))+(x^4*y(x)^3-x^3*y(x)^2-x^3*y(x)+x)*diff(y(x),x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(x^3*y[x]^4+x^2*y[x]^3+x*y[x]^2+y[x])+(x^4*y[x]^3-x^3*y[x]^2-x^3*y[x]+x)*y'[x]==0,y[x
```

Not solved

12.29 problem Ex 30

Internal problem ID [11199]
Internal file name [OUTPUT/10184_Saturday_December_03_2022_08_03_35_AM_35982736/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter 2, differential equations of the first order and the first degree. Article 19. Summary. Page 29
Problem number: Ex 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{(2\sqrt{yx}-x)\, y'+y=0}$$

12.29.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= -\frac{y}{2\sqrt{xy}-x} \\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(b_3 - a_2)}{2\sqrt{xy} - x} - \frac{y^2 a_3}{\left(2\sqrt{xy} - x\right)^2} - \frac{y\left(\frac{y}{\sqrt{xy}} - 1\right)(x a_2 + y a_3 + a_1)}{\left(2\sqrt{xy} - x\right)^2} - \left(-\frac{1}{2\sqrt{xy} - x} + \frac{yx}{\left(2\sqrt{xy} - x\right)^2 \sqrt{xy}}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{4(xy)^{\frac{3}{2}} b_2 - 3x^2 y b_2 + x\,y^2 a_2 - x\,y^2 b_3 - y^3 a_3 + xy b_1 - \sqrt{xy}\,x b_1 + \sqrt{xy}\,y a_1 - y^2 a_1}{\left(2\sqrt{xy} - x\right)^2 \sqrt{xy}} = 0$$

Setting the numerator to zero gives

$$4(xy)^{\frac{3}{2}} b_2 - 3x^2 y b_2 + x\,y^2 a_2 - x\,y^2 b_3 - y^3 a_3 + xy b_1 - \sqrt{xy}\,x b_1 + \sqrt{xy}\,y a_1 - y^2 a_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-3x^2 y b_2 + 4xy\sqrt{xy}\,b_2 + x\,y^2 a_2 - x\,y^2 b_3 - y^3 a_3 - \sqrt{xy}\,x b_1 + xy b_1 + \sqrt{xy}\,y a_1 - y^2 a_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{xy}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{xy} = v_3\}$$

The above PDE (6E) now becomes

$$v_1 v_2^2 a_2 - v_2^3 a_3 - 3v_1^2 v_2 b_2 + 4v_1 v_2 v_3 b_2 - v_1 v_2^2 b_3 - v_2^2 a_1 + v_3 v_2 a_1 + v_1 v_2 b_1 - v_3 v_1 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$-3v_1^2v_2b_2 + (-b_3 + a_2)\, v_1v_2^2 + 4v_1v_2v_3b_2 + v_1v_2b_1 - v_3v_1b_1 - v_2^3a_3 - v_2^2a_1 + v_3v_2a_1 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ -a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -3b_2 &= 0 \\ 4b_2 &= 0 \\ -b_3 + a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(-\frac{y}{2\sqrt{xy} - x}\right)(x) \\ &= \frac{2y\sqrt{xy}}{2\sqrt{xy} - x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2y\sqrt{xy}}{2\sqrt{xy}-x}} dy \end{aligned}$$

Which results in

$$S = \ln(y) + \frac{x}{\sqrt{xy}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y}{2\sqrt{xy} - x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{2\sqrt{x}\,\sqrt{y}} \\ S_y &= -\frac{-2\sqrt{y} + \sqrt{x}}{2y^{\frac{3}{2}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sqrt{x}\,\sqrt{y} - \sqrt{xy}}{\sqrt{x}\,\sqrt{y}\,\left(-2\sqrt{xy} + x\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

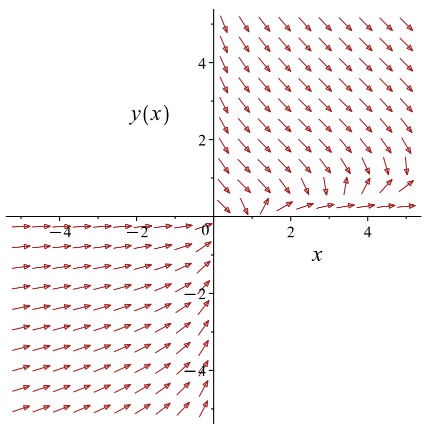
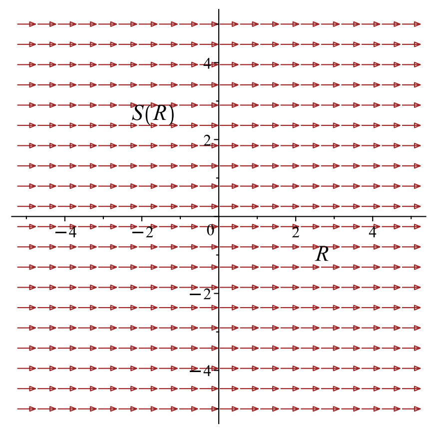
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)\sqrt{y} + \sqrt{x}}{\sqrt{y}} = c_1$$

Which simplifies to

$$\frac{\ln(y)\sqrt{y} + \sqrt{x}}{\sqrt{y}} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{2\sqrt{xy}-x}$ 	$R = x$ $S = \frac{\ln(y)\sqrt{y}+\sqrt{x}}{\sqrt{y}}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y)\sqrt{y}+\sqrt{x}}{\sqrt{y}} = c_1 \tag{1}$$

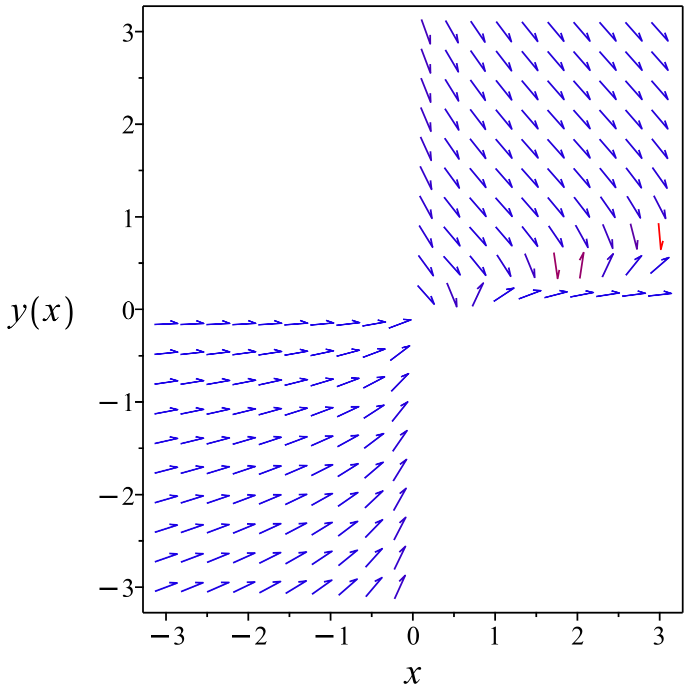


Figure 198: Slope field plot

Verification of solutions

$$\frac{\ln(y)\sqrt{y}+\sqrt{x}}{\sqrt{y}} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve((2*sqrt(x*y(x))-x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$\ln(y(x)) + \frac{x}{\sqrt{y(x)\, x}} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.376 (sec). Leaf size: 33

```
DSolve[(2*Sqrt[x*y[x]]-x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{2}{\sqrt{\frac{y(x)}{x}}} + 2\log\left(\frac{y(x)}{x}\right) = -2\log(x) + c_1, y(x)\right]$$

13 Chapter IV, differential equations of the first order and higher degree than the first. Article 24. Equations solvable for p. Page 49

13.1 problem Ex 1

Internal problem ID [11200]
Internal file name [OUTPUT/10185_Tuesday_December_06_2022_03_59_36_AM_9550685/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 24. Equations solvable for p. Page 49
Problem number: Ex 1.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y'^2 + (y + x) y' + yx = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -x \tag{1}$$
$$y' = -y \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int -x \, \mathrm{d}x \\ &= -\frac{x^2}{2} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{x^2}{2} + c_1 \tag{1}$$

Verification of solutions

$$y = -\frac{x^2}{2} + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{y}dy = \int dx$$
$$-\ln(y) = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{y} = e^{x+c_2}$$

Which simplifies to

$$\frac{1}{y} = c_3 e^x$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-x}}{c_3} \tag{1}$$

Verification of solutions

$$y = \frac{e^{-x}}{c_3}$$

Verified OK.

13.1.1 Maple step by step solution

Let's solve

$y'^2 + (y + x) y' + yx = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

$$\int \left(y'^2 + (y + x) y' + yx\right) dx = \int 0 dx + c_1$$

- Cannot compute integral

$$\int \left(y'^2 + (y + x) y' + yx\right) dx = c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x)^2+(x+y(x))*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x^2}{2} + c_1$$
$$y(x) = c_1 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 32

```
DSolve[(y'[x])^2+(x+y[x])*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x}$$
$$y(x) \to -\frac{x^2}{2} + c_1$$
$$y(x) \to 0$$

13.2 problem Ex 2

Internal problem ID [11201]
Internal file name [OUTPUT/10186_Tuesday_December_06_2022_03_59_37_AM_40386715/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 24. Equations solvable for p. Page 49
Problem number: Ex 2.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy'^2 - 2yy' = x}$$

13.2.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$x\,p^2 - 2yp = x$$

Solving for y from the above results in

$$y = \frac{x(p^2 - 1)}{2p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p^2 - 1}{2p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p^2 - 1}{2p} = x\left(1 - \frac{p^2 - 1}{2p^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p^2 - 1}{2p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -ix \\
y &= ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)^2 - 1}{2p(x)}}{x\left(1 - \frac{p(x)^2 - 1}{2p(x)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= 0
\end{aligned}
$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}\mu p &= 0 \\ \frac{d}{dx}\left(\frac{p}{x}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Summary
The solution(s) found are the following

$$y = -ix \tag{1}$$
$$y = ix \tag{2}$$
$$y = \frac{c_1^2 x^2 - 1}{2c_1} \tag{3}$$

Verification of solutions

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 32

```
dsolve(x*diff(y(x),x)^2-2*y(x)*diff(y(x),x)-x=0,y(x), singsol=all)
```

$$y(x) = -ix$$
$$y(x) = ix$$
$$y(x) = \frac{-c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.225 (sec). Leaf size: 71

```
DSolve[x*(y'[x])^2-2*y[x]*y'[x]-x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \frac{1}{2}e^{-c_1}\left(-x^2+e^{2c_1}\right) \\
y(x) &\to \frac{1}{2}e^{-c_1}\left(-1+e^{2c_1}x^2\right) \\
y(x) &\to -ix \\
y(x) &\to ix
\end{aligned}
$$

13.3 problem Ex 3

Internal problem ID [11202]
Internal file name [`OUTPUT/10187_Tuesday_December_06_2022_03_59_38_AM_34633178/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 24. Equations solvable for p. Page 49
Problem number: Ex 3.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y^2 + y'^2 = 1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{1 - y^2} \tag{1}$$
$$y' = -\sqrt{1 - y^2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2 + 1}} dy = x + c_1$$
$$\arcsin(y) = x + c_1$$

Solving for y gives these solutions

$$y_1 = \sin(x + c_1)$$

Summary
The solution(s) found are the following

$$y = \sin(x + c_1) \tag{1}$$

Verification of solutions

$$y = \sin(x + c_1)$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-y^2+1}}dy = x + c_2$$
$$-\arcsin(y) = x + c_2$$

Solving for y gives these solutions

$$y_1 = -\sin(x + c_2)$$

Summary
The solution(s) found are the following

$$y = -\sin(x + c_2) \tag{1}$$

Verification of solutions

$$y = -\sin(x + c_2)$$

Verified OK.

13.3.1 Maple step by step solution

Let's solve

$y^2 + y'^2 = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}} = 1$

- Integrate both sides with respect to x

$$\int \frac{y'}{\sqrt{1-y^2}} dx = \int 1 dx + c_1$$

- Evaluate integral

$\arcsin(y) = x + c_1$

- Solve for y

$y = \sin(x + c_1)$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   <- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 29

```
dsolve(y(x)^2+diff(y(x),x)^2=1,y(x), singsol=all)
```

$$y(x) = -1$$
$$y(x) = 1$$
$$y(x) = -\sin(c_1 - x)$$
$$y(x) = \sin(c_1 - x)$$

✓ Solution by Mathematica

Time used: 0.211 (sec). Leaf size: 39

```
DSolve[y[x]^2+(y'[x])^2==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x + c_1)$$
$$y(x) \to \cos(x - c_1)$$
$$y(x) \to -1$$
$$y(x) \to 1$$
$$y(x) \to \text{Interval}[\{-1, 1\}]$$

13.4 problem Ex 4

Internal problem ID [11203]
Internal file name [`OUTPUT/10188_Tuesday_December_06_2022_03_59_39_AM_88112184/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 24. Equations solvable for p. Page 49
Problem number: Ex 4.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(2y'x - y)^2 = 8x^3}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{y + 2\sqrt{2}\, x^{\frac{3}{2}}}{2x} \tag{1}$$

$$y' = \frac{y - 2\sqrt{2}\, x^{\frac{3}{2}}}{2x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{2x}$$
$$q(x) = \sqrt{x}\,\sqrt{2}$$

Hence the ode is

$$y' - \frac{y}{2x} = \sqrt{x}\,\sqrt{2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{2x}dx} \\ &= \frac{1}{\sqrt{x}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\sqrt{x}\,\sqrt{2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{x}}\right) &= \left(\frac{1}{\sqrt{x}}\right)\left(\sqrt{x}\,\sqrt{2}\right) \\ \mathrm{d}\left(\frac{y}{\sqrt{x}}\right) &= \sqrt{2}\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{\sqrt{x}} &= \int \sqrt{2}\,\mathrm{d}x \\ \frac{y}{\sqrt{x}} &= \sqrt{2}\,x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x}}$ results in

$$y = \sqrt{2}\,x^{\frac{3}{2}} + c_1\sqrt{x}$$

which simplifies to

$$y = \sqrt{x}\left(\sqrt{2}\,x + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \sqrt{x}\left(\sqrt{2}\,x + c_1\right) \tag{1}$$

Verification of solutions

$$y = \sqrt{x}\left(\sqrt{2}\,x + c_1\right)$$

Verified OK.

Solving equation (2)

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\sqrt{x}\,\sqrt{2}$$

Hence the ode is

$$y' - \frac{y}{2x} = -\sqrt{x}\,\sqrt{2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{2x}dx}$$
$$= \frac{1}{\sqrt{x}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-\sqrt{x}\,\sqrt{2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{\sqrt{x}}\right) = \left(\frac{1}{\sqrt{x}}\right)\left(-\sqrt{x}\,\sqrt{2}\right)$$
$$d\left(\frac{y}{\sqrt{x}}\right) = \left(-\sqrt{2}\right)\,dx$$

Integrating gives

$$\frac{y}{\sqrt{x}} = \int -\sqrt{2}\,dx$$
$$\frac{y}{\sqrt{x}} = -\sqrt{2}\,x + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x}}$ results in

$$y = -\sqrt{2}\,x^{\frac{3}{2}} + c_2\sqrt{x}$$

Summary

The solution(s) found are the following

$$y = -\sqrt{2}\,x^{\frac{3}{2}} + c_2\sqrt{x} \tag{1}$$

Verification of solutions

$$y = -\sqrt{2}\,x^{\frac{3}{2}} + c_2\sqrt{x}$$

Verified OK.

13.4.1 Maple step by step solution

Let's solve

$(2y'x - y)^2 = 8x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{2x} + \sqrt{x}\,\sqrt{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{2x} = \sqrt{x}\,\sqrt{2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{2x}\right) = \mu(x)\sqrt{x}\,\sqrt{2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{y}{2x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{2x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{\sqrt{x}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\sqrt{x}\,\sqrt{2}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\sqrt{x}\,\sqrt{2}dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\sqrt{x}\,\sqrt{2}dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\sqrt{x}}$

  $y = \sqrt{x}\left(\int \sqrt{2}dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = \sqrt{x}\left(\sqrt{2}\,x + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 30

```
dsolve((2*x*diff(y(x),x)-y(x))^2=8*x^3,y(x), singsol=all)
```

$$y(x) = \left(-x\sqrt{2} + c_1\right)\sqrt{x}$$
$$y(x) = \left(x\sqrt{2} + c_1\right)\sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.128 (sec). Leaf size: 42

```
DSolve[(2*x*y'[x]-y[x])^2==8*x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x}\left(-\sqrt{2}x + c_1\right)$$
$$y(x) \to \sqrt{x}\left(\sqrt{2}x + c_1\right)$$

13.5 problem Ex 5

Internal problem ID [11204]
Internal file name [`OUTPUT/10189_Tuesday_December_06_2022_03_59_40_AM_68924892/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 24. Equations solvable for p. Page 49
Problem number: Ex 5.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{(x^2+1)\, y'^2 = 1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1}{\sqrt{x^2+1}} \tag{1}$$

$$y' = -\frac{1}{\sqrt{x^2+1}} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{\sqrt{x^2+1}}\, \mathrm{d}x \\ &= \operatorname{arcsinh}(x) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \operatorname{arcsinh}(x) + c_1 \tag{1}$$

Verification of solutions

$$y = \operatorname{arcsinh}(x) + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x \\ &= -\operatorname{arcsinh}(x) + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\operatorname{arcsinh}(x) + c_2 \tag{1}$$

Verification of solutions

$$y = -\operatorname{arcsinh}(x) + c_2$$

Verified OK.

13.5.1 Maple step by step solution

Let's solve

$(x^2+1)\,y'^2 = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (x^2+1)\,y'^2 dx = \int 1 dx + c_1$

- Cannot compute integral

  $\int (x^2+1)\,y'^2 dx = x + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 17

```
dsolve((1+x^2)*diff(y(x),x)^2=1,y(x), singsol=all)
```

$$y(x) = \operatorname{arcsinh}(x) + c_1$$
$$y(x) = -\operatorname{arcsinh}(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 45

```
DSolve[(1+x^2)*(y'[x])^2==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\log\left(\sqrt{x^2+1} - x\right) + c_1$$
$$y(x) \to \log\left(\sqrt{x^2+1} - x\right) + c_1$$

13.6 problem Ex 6

Internal problem ID [11205]
Internal file name [OUTPUT/10190_Tuesday_December_06_2022_03_59_41_AM_54585174/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 24. Equations solvable for p. Page 49
Problem number: Ex 6.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"exact", "linear", "quadrature", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^3 - (2x + y^2)\, y'^2 + (x^2 - y^2 + 2y^2x)\, y' - (x^2 - y^2)\, y^2 = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = y^2 \tag{1}$$
$$y' = -y + x \tag{2}$$
$$y' = y + x \tag{3}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\int \frac{1}{y^2} dy = x + c_1$$
$$-\frac{1}{y} = x + c_1$$

Solving for y gives these solutions

$$y_1 = -\frac{1}{x + c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x + c_1} \tag{1}$$

Verification of solutions

$$y = -\frac{1}{x + c_1}$$

Verified OK.

Solving equation (2)

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x \end{aligned}$$

Hence the ode is

$$y' + y = x$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)(x) \\ \frac{d}{dx}(e^x y) &= (e^x)(x) \\ d(e^x y) &= (x\, e^x)\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^x y &= \int x\, e^x\, dx \\ e^x y &= (x - 1)\, e^x + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$y = \mathrm{e}^{-x}(x-1)\,\mathrm{e}^{x} + c_2\mathrm{e}^{-x}$$

which simplifies to

$$y = x - 1 + c_2\mathrm{e}^{-x}$$

Summary
The solution(s) found are the following

$$y = x - 1 + c_2\mathrm{e}^{-x} \tag{1}$$

Verification of solutions

$$y = x - 1 + c_2\mathrm{e}^{-x}$$

Verified OK.
Solving equation (3)

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= x \end{aligned}$$

Hence the ode is

$$y' - y = x$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int (-1)dx} \\ &= \mathrm{e}^{-x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x}y\right) &= \left(\mathrm{e}^{-x}\right)(x) \\ \mathrm{d}\left(\mathrm{e}^{-x}y\right) &= \left(x\,\mathrm{e}^{-x}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$
\begin{aligned}
e^{-x}y &= \int x\,e^{-x}\,dx \\
e^{-x}y &= -(1+x)\,e^{-x} + c_3
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$
y = -e^{x}(1+x)\,e^{-x} + c_3 e^{x}
$$

which simplifies to

$$
y = -x - 1 + c_3 e^{x}
$$

Summary
The solution(s) found are the following

$$
y = -x - 1 + c_3 e^{x} \tag{1}
$$

Verification of solutions

$$
y = -x - 1 + c_3 e^{x}
$$

Verified OK.

13.6.1 Maple step by step solution

Let's solve
$y'^3 - (2x + y^2)\,y'^2 + (x^2 - y^2 + 2y^2x)\,y' - (x^2 - y^2)\,y^2 = 0$

- Highest derivative means the order of the ODE is 1
  y'
- Separate variables
  $\frac{y'}{y^2} = 1$
- Integrate both sides with respect to x
  $\int \frac{y'}{y^2}dx = \int 1dx + c_1$
- Evaluate integral
  $-\frac{1}{y} = x + c_1$
- Solve for y
  $y = -\frac{1}{x+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x)^3-(2*x+y(x)^2)*diff(y(x),x)^2+(x^2-y(x)^2+2*x*y(x)^2)*diff(y(x),x)-(x^2-
```

$$y(x) = \frac{1}{c_1 - x}$$
$$y(x) = -x - 1 + c_1 \mathrm{e}^{x}$$
$$y(x) = x - 1 + c_1 \mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.276 (sec). Leaf size: 48

```
DSolve[(y'[x])^3-(2*x+y[x]^2)*(y'[x])^2+(x^2-y[x]^2+2*x*y[x]^2)*y'[x]-(x^2-y[x]^2)*y[x]^2==0
```

$$y(x) \to -\frac{1}{x + c_1}$$
$$y(x) \to x + c_1 e^{-x} - 1$$
$$y(x) \to -x + c_1 e^{x} - 1$$
$$y(x) \to 0$$

14 Chapter IV, differential equations of the first order and higher degree than the first. Article 25. Equations solvable for y. Page 52

14.1 problem Ex 1

Internal problem ID [11206]
Internal file name [OUTPUT/10191_Tuesday_December_06_2022_03_59_42_AM_50556870/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 25. Equations solvable for y. Page 52
Problem number: Ex 1.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _dAlembert]
```

$$\boxed{2y'x - y + \ln(y') = 0}$$

14.1.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$2px - y + \ln(p) = 0$$

Solving for y from the above results in

$$y = 2px + \ln(p) \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= 2p \\
g &= \ln(p)
\end{aligned}
$$

Hence (2) becomes

$$
-p = \left(2x + \frac{1}{p}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-p = 0
$$

Solving for p from the above gives

$$
p = 0
$$

Substituting these in (1A) gives

$$
y = -\infty
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = -\frac{p(x)}{2x + \frac{1}{p(x)}} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = -\frac{2x(p) + \frac{1}{p}}{p} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= \frac{2}{p} \\
q(p) &= -\frac{1}{p^2}
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p} = -\frac{1}{p^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{p}dp} \\ &= p^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(-\frac{1}{p^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}p}(p^2 x) &= (p^2)\left(-\frac{1}{p^2}\right) \\ \mathrm{d}(p^2 x) &= -1\,\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}p^2 x &= \int -1\,\mathrm{d}p \\ p^2 x &= -p + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^2$ results in

$$x(p) = -\frac{1}{p} + \frac{c_1}{p^2}$$

which simplifies to

$$x(p) = \frac{-p + c_1}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = e^{-\operatorname{LambertW}(2x\,e^y)+y}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{2(-2c_1 x + \operatorname{LambertW}(2x\,e^y))\,x}{\operatorname{LambertW}(2x\,e^y)^2}$$

Summary

The solution(s) found are the following

$$y = -\infty \tag{1}$$

$$x = -\frac{2(-2c_1x + \text{LambertW}\left(2x\,\mathrm{e}^{y}\right))\,x}{\text{LambertW}\left(2x\,\mathrm{e}^{y}\right)^2} \tag{2}$$

Verification of solutions

$$y = -\infty$$

Warning, solution could not be verified

$$x = -\frac{2(-2c_1x + \text{LambertW}\left(2x\,\mathrm{e}^{y}\right))\,x}{\text{LambertW}\left(2x\,\mathrm{e}^{y}\right)^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
dsolve(2*diff(y(x),x)*x-y(x)+ln(diff(y(x),x))=0,y(x), singsol=all)
```

$$y(x) = -1 + \sqrt{4c_1x + 1} - \ln(2) + \ln\left(\frac{-1 + \sqrt{4c_1x + 1}}{x}\right)$$

$$y(x) = -1 - \sqrt{4c_1x + 1} - \ln(2) + \ln\left(\frac{-1 - \sqrt{4c_1x + 1}}{x}\right)$$

✓ Solution by Mathematica

Time used: 0.157 (sec). Leaf size: 32

```
DSolve[2*y'[x]*x-y[x]+Log[y'[x]]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[W\left(2xe^{y(x)}\right)-\log\left(W\left(2xe^{y(x)}\right)+2\right)-y(x)=c_1,y(x)\right]$$

14.2 problem Ex 2

Internal problem ID [11207]
Internal file name [`OUTPUT/10192_Tuesday_December_06_2022_03_59_45_AM_82210746/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 25. Equations solvable for y. Page 52
Problem number: Ex 2.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{4xy'^2 + 2y'x - y = 0}$$

14.2.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$4x\,p^2 + 2px - y = 0$$

Solving for y from the above results in

$$y = \left(4p^2 + 2p\right)x \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 4p^2 + 2p$$
$$g = 0$$

Hence (2) becomes

$$-4p^2 - p = x(8p + 2)\, p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-4p^2 - p = 0$$

Solving for p from the above gives

$$p = 0$$
$$p = -\frac{1}{4}$$

Substituting these in (1A) gives

$$y = 0$$
$$y = -\frac{x}{4}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{-4p(x)^2 - p(x)}{x\,(8p\,(x) + 2)} \tag{3}$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = \frac{1}{2x}$$
$$q(x) = 0$$

Hence the ode is

$$p'(x) + \frac{p(x)}{2x} = 0$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{2x} dx}$$
$$= \sqrt{x}$$

The ode becomes

$$\frac{d}{dx}\mu p = 0$$
$$\frac{d}{dx}(\sqrt{x}\, p) = 0$$

Integrating gives

$$\sqrt{x}\, p = c_1$$

Dividing both sides by the integrating factor $\mu = \sqrt{x}$ results in

$$p(x) = \frac{c_1}{\sqrt{x}}$$

Substituing the above solution for p in (2A) gives

$$y = \left(\frac{4c_1^2}{x} + \frac{2c_1}{\sqrt{x}}\right) x$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$
$$y = -\frac{x}{4} \tag{2}$$
$$y = \left(\frac{4c_1^2}{x} + \frac{2c_1}{\sqrt{x}}\right) x \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = -\frac{x}{4}$$

Verified OK.

$$y = \left(\frac{4c_1^2}{x} + \frac{2c_1}{\sqrt{x}}\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 35

```
dsolve(4*x*diff(y(x),x)^2+2*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{4}$$
$$y(x) = 4c_1 + 2\sqrt{c_1 x}$$
$$y(x) = 4c_1 - 2\sqrt{c_1 x}$$

✓ Solution by Mathematica

Time used: 0.196 (sec). Leaf size: 72

```
DSolve[4*x*(y'[x])^2+2*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{2c_1}\left(-2\sqrt{x} + e^{2c_1}\right)$$
$$y(x) \to \frac{1}{4}e^{-4c_1}\left(1 + 2e^{2c_1}\sqrt{x}\right)$$
$$y(x) \to 0$$
$$y(x) \to -\frac{x}{4}$$

14.3 problem Ex 3

Internal problem ID [11208]
Internal file name [OUTPUT/10193_Tuesday_December_06_2022_03_59_47_AM_42602599/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 25. Equations solvable for y. Page 52
Problem number: Ex 3.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy'^2 - 2yy' = x}$$

14.3.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$x\,p^2 - 2yp = x$$

Solving for y from the above results in

$$y = \frac{x(p^2 - 1)}{2p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p^2 - 1}{2p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p^2 - 1}{2p} = x\left(1 - \frac{p^2 - 1}{2p^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p^2 - 1}{2p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -ix \\
y &= ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)^2 - 1}{2p(x)}}{x\left(1 - \frac{p(x)^2 - 1}{2p(x)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= 0
\end{aligned}
$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu p &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{x}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Summary

The solution(s) found are the following

$$y = -ix \tag{1}$$

$$y = ix \tag{2}$$

$$y = \frac{c_1^2 x^2 - 1}{2c_1} \tag{3}$$

Verification of solutions

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 32

```
dsolve(x*diff(y(x),x)^2-2*y(x)*diff(y(x),x)-x=0,y(x), singsol=all)
```

$$y(x) = -ix$$
$$y(x) = ix$$
$$y(x) = \frac{-c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.186 (sec). Leaf size: 71

```
DSolve[x*(y'[x])^2-2*y[x]*y'[x]-x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-c_1}\left(-x^2 + e^{2c_1}\right)$$
$$y(x) \to \frac{1}{2}e^{-c_1}\left(-1 + e^{2c_1}x^2\right)$$
$$y(x) \to -ix$$
$$y(x) \to ix$$

14.4 problem Ex 4

Internal problem ID [11209]
Internal file name [`OUTPUT/10194_Tuesday_December_06_2022_03_59_48_AM_53887226/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 25. Equations solvable for y. Page 52
Problem number: Ex 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _Riccati]
```

$$\boxed{y' + 2yx - y^2 = x^2}$$

14.4.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= x^2 - 2xy + y^2 \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + (x^2 - 2xy + y^2)(b_3 - a_2) - (x^2 - 2xy + y^2)^2 a_3 - (2x - 2y)(xa_2 + ya_3 + a_1) - (-2x + 2y)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-x^4a_3 + 4x^3ya_3 - 6x^2y^2a_3 + 4x\,y^3a_3 - y^4a_3 - 3x^2a_2 + 2x^2b_2 + x^2b_3 + 4xya_2 - 2xya_3 - 2xyb_2 - y^2a_2 + 2y^2a_3 - y^2b_3 - 2xa_1 + 2xb_1 + 2ya_1 - 2yb_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-x^4a_3 + 4x^3ya_3 - 6x^2y^2a_3 + 4x\,y^3a_3 - y^4a_3 - 3x^2a_2 + 2x^2b_2 + x^2b_3 + 4xya_2 - 2xya_3 - 2xyb_2 - y^2a_2 + 2y^2a_3 - y^2b_3 - 2xa_1 + 2xb_1 + 2ya_1 - 2yb_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_3v_1^4 + 4a_3v_1^3v_2 - 6a_3v_1^2v_2^2 + 4a_3v_1v_2^3 - a_3v_2^4 - 3a_2v_1^2 + 4a_2v_1v_2 - a_2v_2^2 - 2a_3v_1v_2 + 2a_3v_2^2 + 2b_2v_1^2 - 2b_2v_1v_2 + b_3v_1^2 - b_3v_2^2 - 2a_1v_1 + 2a_1v_2 + 2b_1v_1 - 2b_1v_2 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-a_3 v_1^4 + 4a_3 v_1^3 v_2 - 6a_3 v_1^2 v_2^2 + (-3a_2 + 2b_2 + b_3) v_1^2 \\
&+ 4a_3 v_1 v_2^3 + (4a_2 - 2a_3 - 2b_2) v_1 v_2 + (-2a_1 + 2b_1) v_1 \\
&- a_3 v_2^4 + (-a_2 + 2a_3 - b_3) v_2^2 + (2a_1 - 2b_1) v_2 + b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-6a_3 &= 0 \\
-a_3 &= 0 \\
4a_3 &= 0 \\
-2a_1 + 2b_1 &= 0 \\
2a_1 - 2b_1 &= 0 \\
-3a_2 + 2b_2 + b_3 &= 0 \\
-a_2 + 2a_3 - b_3 &= 0 \\
4a_2 - 2a_3 - 2b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= b_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= 1 - \left(x^2 - 2xy + y^2\right)(1) \\
&= -x^2 + 2xy - y^2 + 1 \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-x^2 + 2xy - y^2 + 1} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(y - x + 1)}{2} - \frac{\ln(y - 1 - x)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x^2 - 2xy + y^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{(x - y + 1)(x - y - 1)} \\ S_y &= -\frac{1}{(x - y + 1)(x - y - 1)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y - x + 1)}{2} - \frac{\ln(y - 1 - x)}{2} = -x + c_1$$

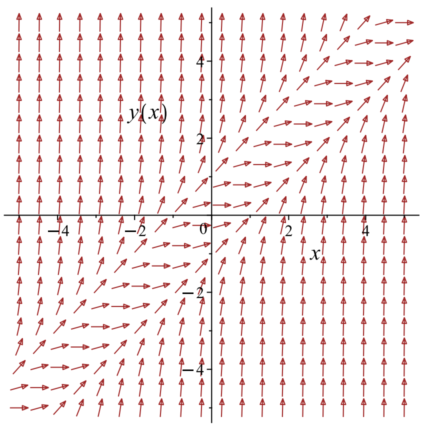
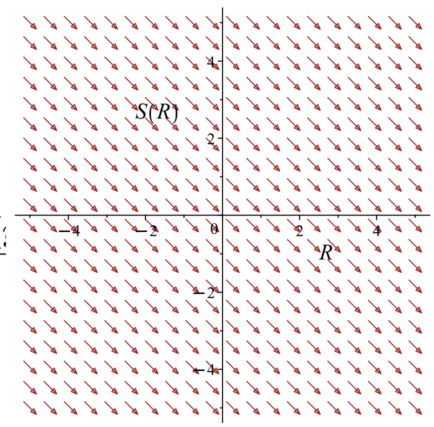
Which simplifies to

$$\frac{\ln(y - x + 1)}{2} - \frac{\ln(y - 1 - x)}{2} = -x + c_1$$

Which gives

$$y = \frac{x\,\mathrm{e}^{-2x+2c_1} + \mathrm{e}^{-2x+2c_1} - x + 1}{-1 + \mathrm{e}^{-2x+2c_1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2 - 2xy + y^2$ 	$R = x$ $S = \frac{\ln(y - x + 1)}{2} - \ln($	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = \frac{x\,e^{-2x+2c_1} + e^{-2x+2c_1} - x + 1}{-1 + e^{-2x+2c_1}} \tag{1}$$

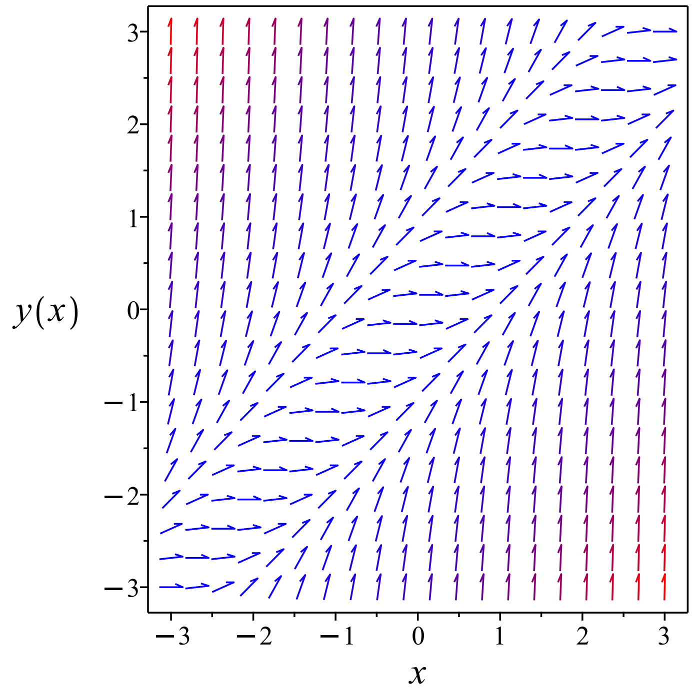


Figure 199: Slope field plot

Verification of solutions

$$y = \frac{x\,\mathrm{e}^{-2x+2c_1} + \mathrm{e}^{-2x+2c_1} - x + 1}{-1 + \mathrm{e}^{-2x+2c_1}}$$

Verified OK.

14.4.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x^2 - 2xy + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 - 2xy + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2$, $f_1(x) = -2x$ and $f_2(x) = 1$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{u}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= -2x \\
f_2^2 f_0 &= x^2
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
u''(x) + 2xu'(x) + x^2 u(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 \mathrm{e}^{-\frac{x(x-2)}{2}} + c_2 \mathrm{e}^{-\frac{x(x+2)}{2}}
$$

The above shows that

$$
u'(x) = -c_1(x-1)\,\mathrm{e}^{-\frac{x(x-2)}{2}} - c_2 \mathrm{e}^{-\frac{x(x+2)}{2}}(1+x)
$$

Using the above in (1) gives the solution

$$
y = -\frac{-c_1(x-1)\,\mathrm{e}^{-\frac{x(x-2)}{2}} - c_2 \mathrm{e}^{-\frac{x(x+2)}{2}}(1+x)}{c_1 \mathrm{e}^{-\frac{x(x-2)}{2}} + c_2 \mathrm{e}^{-\frac{x(x+2)}{2}}}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{c_3(x-1)\,\mathrm{e}^{-\frac{x(x-2)}{2}} + \mathrm{e}^{-\frac{x(x+2)}{2}}(1+x)}{c_3 \mathrm{e}^{-\frac{x(x-2)}{2}} + \mathrm{e}^{-\frac{x(x+2)}{2}}}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_3(x-1)\,e^{-\frac{x(x-2)}{2}} + e^{-\frac{x(x+2)}{2}}(1+x)}{c_3 e^{-\frac{x(x-2)}{2}} + e^{-\frac{x(x+2)}{2}}} \tag{1}$$

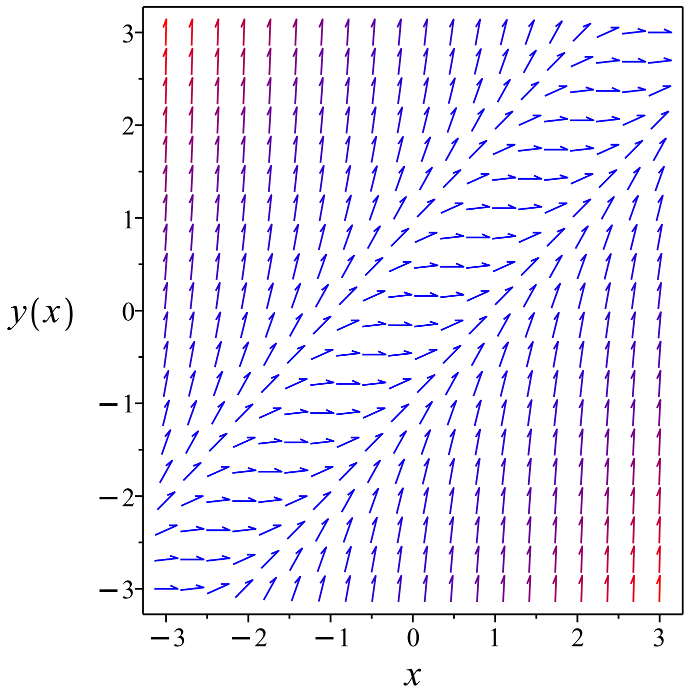


Figure 200: Slope field plot

Verification of solutions

$$y = \frac{c_3(x-1)\,e^{-\frac{x(x-2)}{2}} + e^{-\frac{x(x+2)}{2}}(1+x)}{c_3 e^{-\frac{x(x-2)}{2}} + e^{-\frac{x(x+2)}{2}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 29

```
dsolve(diff(y(x),x)+2*x*y(x)=x^2+y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{c_1(-1+x)\,e^{2x} - x - 1}{-1 + e^{2x}c_1}$$

✓ Solution by Mathematica

Time used: 0.208 (sec). Leaf size: 29

```
DSolve[y'[x]+2*x*y[x]==x^2+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + \frac{1}{\frac{1}{2} + c_1 e^{2x}} - 1$$
$$y(x) \to x - 1$$

14.5 problem Ex 5

Internal problem ID [11210]
Internal file name [`OUTPUT/10195_Tuesday_December_06_2022_03_59_50_AM_21624321/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 25. Equations solvable for y. Page 52
Problem number: Ex 5.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y + y'x - x^4 y'^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1 + \sqrt{1 + 4x^2 y}}{2x^3} \tag{1}$$

$$y' = -\frac{-1 + \sqrt{1 + 4x^2 y}}{2x^3} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{1 + \sqrt{4x^2 y + 1}}{2x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(1+\sqrt{4x^2y+1}\right)(b_3 - a_2)}{2x^3} - \frac{\left(1+\sqrt{4x^2y+1}\right)^2 a_3}{4x^6} - \left(-\frac{3\left(1+\sqrt{4x^2y+1}\right)}{2x^4} + \frac{2y}{x^2\sqrt{4x^2y+1}}\right)(xa_2 + ya_3 + a_1) - \frac{xb_2 + yb_3 + b_1}{x\sqrt{4x^2y+1}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4b_2x^6\sqrt{4x^2y+1} + 4x^6b_2 - 8x^5ya_2 - 4x^5yb_3 - 16x^4y^2a_3 + 4x^5b_1 - 16x^4ya_1 - 4\sqrt{4x^2y+1}\,x^3a_2 - 2\sqrt{4}}{} = 0$$

Setting the numerator to zero gives

$$4b_2x^6\sqrt{4x^2y+1} - 4x^6b_2 + 8x^5ya_2 + 4x^5yb_3 + 16x^4y^2a_3 - 4x^5b_1 + 16x^4ya_1 + 4\sqrt{4x^2y+1}\,x^3a_2 + 2\sqrt{4x^2y+1}\,x^3b_3 + 6\sqrt{4x^2y+1}\,x^2ya_3 - \left(4x^2y+1\right)^{\frac{3}{2}} a_3 + 6\sqrt{4x^2y+1}\,x^2a_1 + 4x^3a_2 + 2x^3b_3 - 2x^2ya_3 + 6x^2a_1 - a_3\sqrt{4x^2y+1} - 2a_3 = 0 \tag{6E}$$

Simplifying the above gives

$$4b_2x^6\sqrt{4x^2y+1} - 4x^6b_2 - 8x^5ya_2 - 4x^5yb_3 - 8x^4y^2a_3 + 4\left(4x^2y+1\right)x^3a_2 + 2\left(4x^2y+1\right)x^3b_3 + 6\left(4x^2y+1\right)x^2ya_3 - 4x^5b_1 - 8x^4ya_1 + 6\left(4x^2y+1\right)x^2a_1 + 4\sqrt{4x^2y+1}\,x^3a_2 + 2\sqrt{4x^2y+1}\,x^3b_3 + 6\sqrt{4x^2y+1}\,x^2ya_3 - \left(4x^2y+1\right)^{\frac{3}{2}} a_3 + 6\sqrt{4x^2y+1}\,x^2a_1 - 2\left(4x^2y+1\right)a_3 - a_3\sqrt{4x^2y+1} = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}&4b_2x^6\sqrt{4x^2y+1}-4x^6b_2+8x^5ya_2+4x^5yb_3+16x^4y^2a_3-4x^5b_1+16x^4ya_1\\&+4\sqrt{4x^2y+1}\,x^3a_2+2\sqrt{4x^2y+1}\,x^3b_3+2\sqrt{4x^2y+1}\,x^2ya_3+4x^3a_2\\&+2x^3b_3+6\sqrt{4x^2y+1}\,x^2a_1-2x^2ya_3+6x^2a_1-2a_3\sqrt{4x^2y+1}-2a_3=0\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{4x^2y+1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \sqrt{4x^2y+1}=v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&4b_2v_1^6v_3+8v_1^5v_2a_2+16v_1^4v_2^2a_3-4v_1^6b_2+4v_1^5v_2b_3+16v_1^4v_2a_1\\&-4v_1^5b_1+4v_3v_1^3a_2+2v_3v_1^2v_2a_3+2v_3v_1^3b_3+6v_3v_1^2a_1\\&+4v_1^3a_2-2v_1^2v_2a_3+2v_1^3b_3+6v_1^2a_1-2a_3v_3-2a_3=0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}&4b_2v_1^6v_3-4v_1^6b_2+(8a_2+4b_3)\,v_1^5v_2-4v_1^5b_1+16v_1^4v_2^2a_3\\&+16v_1^4v_2a_1+(4a_2+2b_3)\,v_1^3v_3+(4a_2+2b_3)\,v_1^3+2v_3v_1^2v_2a_3\\&-2v_1^2v_2a_3+6v_3v_1^2a_1+6v_1^2a_1-2a_3v_3-2a_3=0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}6a_1 &= 0\\16a_1 &= 0\\-2a_3 &= 0\\2a_3 &= 0\\16a_3 &= 0\\-4b_1 &= 0\\-4b_2 &= 0\\4b_2 &= 0\\4a_2 + 2b_3 &= 0\\8a_2 + 4b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\a_2 &= a_2\\a_3 &= 0\\b_1 &= 0\\b_2 &= 0\\b_3 &= -2a_2\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\\eta &= -2y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi\\&= -2y - \left(\frac{1+\sqrt{4x^2y+1}}{2x^3}\right)(x)\\&= \frac{-4x^2y - \sqrt{4x^2y+1} - 1}{2x^2}\\\xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-4x^2y - \sqrt{4x^2y+1} - 1}{2x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{4x^2y+1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1 + \sqrt{4x^2y+1}}{2x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x\sqrt{4x^2y+1}} \\ S_y &= \frac{-1 + \frac{1}{\sqrt{4x^2y+1}}}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{-1+\sqrt{4x^2y+1}}{2x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{\left(-1+\sqrt{4x^2y+1}\right)(b_3-a_2)}{2x^3} - \frac{\left(-1+\sqrt{4x^2y+1}\right)^2 a_3}{4x^6} - \left(-\frac{2y}{x^2\sqrt{4x^2y+1}} + \frac{-\frac{3}{2}+\frac{3\sqrt{4x^2y+1}}{2}}{x^4}\right)(x a_2 + y a_3 + a_1) + \frac{x b_2 + y b_3 + b_1}{x\sqrt{4x^2y+1}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4b_2x^6\sqrt{4x^2y+1} - 4x^6b_2 + 8x^5ya_2 + 4x^5yb_3 + 16x^4y^2a_3 - 4x^5b_1 + 16x^4ya_1 - 4\sqrt{4x^2y+1}\,x^3a_2 - 2\sqrt{4}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$4b_2x^6\sqrt{4x^2y+1} + 4x^6b_2 - 8x^5ya_2 - 4x^5yb_3 - 16x^4y^2a_3 + 4x^5b_1 - 16x^4ya_1 + 4\sqrt{4x^2y+1}\,x^3a_2 + 2\sqrt{4x^2y+1}\,x^3b_3 + 6\sqrt{4x^2y+1}\,x^2ya_3 - \left(4x^2y+1\right)^{\frac{3}{2}} a_3 + 6\sqrt{4x^2y+1}\,x^2a_1 - 4x^3a_2 - 2x^3b_3 + 2x^2ya_3 - 6x^2a_1 - a_3\sqrt{4x^2y+1} + 2a_3 = 0 \tag{6E}$$

Simplifying the above gives

$$4b_2x^6\sqrt{4x^2y+1} + 4x^6b_2 + 8x^5ya_2 + 4x^5yb_3 + 8x^4y^2a_3 - 4\left(4x^2y+1\right)x^3a_2 - 2\left(4x^2y+1\right)x^3b_3 - 6\left(4x^2y+1\right)x^2ya_3 + 4x^5b_1 + 8x^4ya_1 - 6\left(4x^2y+1\right)x^2a_1 + 4\sqrt{4x^2y+1}\,x^3a_2 + 2\sqrt{4x^2y+1}\,x^3b_3 + 6\sqrt{4x^2y+1}\,x^2ya_3 - \left(4x^2y+1\right)^{\frac{3}{2}} a_3 + 6\sqrt{4x^2y+1}\,x^2a_1 + 2\left(4x^2y+1\right)a_3 - a_3\sqrt{4x^2y+1} = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}&4b_2x^6\sqrt{4x^2y+1}+4x^6b_2-8x^5ya_2-4x^5yb_3-16x^4y^2a_3+4x^5b_1-16x^4ya_1\\&+4\sqrt{4x^2y+1}\,x^3a_2+2\sqrt{4x^2y+1}\,x^3b_3+2\sqrt{4x^2y+1}\,x^2ya_3-4x^3a_2\\&-2x^3b_3+6\sqrt{4x^2y+1}\,x^2a_1+2x^2ya_3-6x^2a_1-2a_3\sqrt{4x^2y+1}+2a_3=0\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{4x^2y+1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{4x^2y+1} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&4b_2v_1^6v_3-8v_1^5v_2a_2-16v_1^4v_2^2a_3+4v_1^6b_2-4v_1^5v_2b_3-16v_1^4v_2a_1\\&+4v_1^5b_1+4v_3v_1^3a_2+2v_3v_1^2v_2a_3+2v_3v_1^3b_3+6v_3v_1^2a_1\\&-4v_1^3a_2+2v_1^2v_2a_3-2v_1^3b_3-6v_1^2a_1-2a_3v_3+2a_3=0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}&4b_2v_1^6v_3+4v_1^6b_2+(-8a_2-4b_3)\,v_1^5v_2+4v_1^5b_1-16v_1^4v_2^2a_3\\&-16v_1^4v_2a_1+(4a_2+2b_3)\,v_1^3v_3+(-4a_2-2b_3)\,v_1^3\\&+2v_3v_1^2v_2a_3+2v_1^2v_2a_3+6v_3v_1^2a_1-6v_1^2a_1-2a_3v_3+2a_3=0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-16a_1 &= 0 \\
-6a_1 &= 0 \\
6a_1 &= 0 \\
-16a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
4b_1 &= 0 \\
4b_2 &= 0 \\
-8a_2 - 4b_3 &= 0 \\
-4a_2 - 2b_3 &= 0 \\
4a_2 + 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -2y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -2y - \left(-\frac{-1 + \sqrt{4x^2y + 1}}{2x^3}\right)(x) \\
&= \frac{-4x^2y + \sqrt{4x^2y + 1} - 1}{2x^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-4x^2y + \sqrt{4x^2y+1} - 1}{2x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{4x^2y + 1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-1 + \sqrt{4x^2y + 1}}{2x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{x\sqrt{4x^2y + 1}} \\ S_y &= \frac{-\frac{1}{\sqrt{4x^2y+1}} - 1}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 81

```
dsolve(y(x)=-x*diff(y(x),x)+x^4*diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{1}{4x^2}$$
$$y(x) = \frac{-c_1 i - x}{c_1^2 x}$$
$$y(x) = \frac{c_1 i - x}{x\, c_1^2}$$
$$y(x) = \frac{c_1 i - x}{x\, c_1^2}$$
$$y(x) = \frac{-c_1 i - x}{c_1^2 x}$$

✓ Solution by Mathematica

Time used: 0.809 (sec). Leaf size: 123

```
DSolve[y[x]==-x*y'[x]+x^4*(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{x\sqrt{4x^2y(x)+1}\text{arctanh}\left(\sqrt{4x^2y(x)+1}\right)}{\sqrt{4x^4y(x)+x^2}} - \frac{1}{2}\log(y(x)) = c_1, y(x)\right]$$
$$\text{Solve}\left[\frac{x\sqrt{4x^2y(x)+1}\text{arctanh}\left(\sqrt{4x^2y(x)+1}\right)}{\sqrt{4x^4y(x)+x^2}} - \frac{1}{2}\log(y(x)) = c_1, y(x)\right]$$
$$y(x) \to 0$$

14.6 problem Ex 6

Internal problem ID [11211]
Internal file name [`OUTPUT/10196_Tuesday_December_06_2022_03_59_52_AM_83900819/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 25. Equations solvable for y. Page 52
Problem number: Ex 6.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{y'^2 + 2y'x - y = 0}$$

14.6.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 + 2px - y = 0$$

Solving for y from the above results in

$$y = p^2 + 2px \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 2p$$
$$g = p^2$$

Hence (2) becomes

$$-p = (2x + 2p)\, p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{p(x)}{2x + 2p(x)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{2x(p) + 2p}{p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{2}{p}$$
$$q(p) = -2$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p} = -2$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{p}dp}$$
$$= p^2$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) = (\mu)(-2)$$
$$\frac{\mathrm{d}}{\mathrm{d}p}(p^2 x) = (p^2)(-2)$$
$$\mathrm{d}(p^2 x) = (-2p^2)\ \mathrm{d}p$$

Integrating gives

$$p^2 x = \int -2p^2\,\mathrm{d}p$$
$$p^2 x = -\frac{2p^3}{3} + c_1$$

Dividing both sides by the integrating factor $\mu = p^2$ results in

$$x(p) = -\frac{2p}{3} + \frac{c_1}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = -x + \sqrt{x^2 + y}$$
$$p = -x - \sqrt{x^2 + y}$$

Substituting the above in the solution for x found above gives

$$x = \frac{(-8x^2 - 2y)\sqrt{x^2 + y} + 8x^3 + 6yx + 3c_1}{3\left(x - \sqrt{x^2 + y}\right)^2}$$
$$x = \frac{(8x^2 + 2y)\sqrt{x^2 + y} + 8x^3 + 6yx + 3c_1}{3\left(x + \sqrt{x^2 + y}\right)^2}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{(-8x^2 - 2y)\sqrt{x^2+y} + 8x^3 + 6yx + 3c_1}{3\left(x - \sqrt{x^2+y}\right)^2} \tag{2}$$

$$x = \frac{(8x^2 + 2y)\sqrt{x^2+y} + 8x^3 + 6yx + 3c_1}{3\left(x + \sqrt{x^2+y}\right)^2} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{(-8x^2 - 2y)\sqrt{x^2+y} + 8x^3 + 6yx + 3c_1}{3\left(x - \sqrt{x^2+y}\right)^2}$$

Verified OK.

$$x = \frac{(8x^2 + 2y)\sqrt{x^2+y} + 8x^3 + 6yx + 3c_1}{3\left(x + \sqrt{x^2+y}\right)^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 650

```
dsolve(diff(y(x),x)^2+2*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(x^2 - x\left(-x^3 + 2\sqrt{3}\,\sqrt{-c_1\,(x^3 - 3c_1)} + 6c_1\right)^{\frac{1}{3}} + \left(-x^3 + 2\sqrt{3}\,\sqrt{-c_1\,(x^3 - 3c_1)} + 6c_1\right)^{\frac{2}{3}}\right)\left(x^2 + 3x\left(-\right.\right.}{4\left(-x^3 + 2\sqrt{3}\,\sqrt{-c_1\,(x^3 - 3c_1)}\right.}$$

$$y(x) = \frac{\left(i\left(-x^3 + 2\sqrt{3}\,\sqrt{-c_1\,(x^3 - 3c_1)} + 6c_1\right)^{\frac{2}{3}}\sqrt{3} - i\sqrt{3}\,x^2 + \left(-x^3 + 2\sqrt{3}\,\sqrt{-c_1\,(x^3 - 3c_1)} + 6c_1\right)^{\frac{2}{3}} + 2x\left(\right.\right.}{}$$

$$y(x) = \frac{\left(i\sqrt{3}\,x^2 - i\left(-x^3 + 2\sqrt{3}\,\sqrt{-c_1\,(x^3 - 3c_1)} + 6c_1\right)^{\frac{2}{3}}\sqrt{3} + x^2 + 2x\left(-x^3 + 2\sqrt{3}\,\sqrt{-c_1\,(x^3 - 3c_1)} + 6c_1\right)\right.}{}$$

✓ Solution by Mathematica

Time used: 60.154 (sec). Leaf size: 931

```
DSolve[(y'[x])^2+2*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(-x^2 + \frac{x(x^3+8e^{3c_1})}{\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}} + \sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)$$

$$y(x) \to \frac{1}{72}\left(-18x^2 - \frac{9i(\sqrt{3}-i)\,x(x^3+8e^{3c_1})}{\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}} + 9i\left(\sqrt{3}+i\right)\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)$$

$$y(x) \to \frac{1}{72}\left(-18x^2 + \frac{9i(\sqrt{3}+i)\,x(x^3+8e^{3c_1})}{\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}} - 9\left(1+i\sqrt{3}\right)\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)$$

$$y(x) \to \frac{1}{4}\left(-x^2 + \frac{x(x^3-8e^{3c_1})}{\sqrt[3]{-x^6-20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(x^3+e^{3c_1}\right)^3}+8e^{6c_1}}} + \sqrt[3]{-x^6-20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)$$

$$y(x) \to \frac{1}{72}\left(-18x^2 + \frac{9(1+i\sqrt{3})\,x(-x^3+8e^{3c_1})}{\sqrt[3]{-x^6-20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(x^3+e^{3c_1}\right)^3}+8e^{6c_1}}} + 9i\left(\sqrt{3}+i\right)\sqrt[3]{-x^6-20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)$$

$$y(x) \to \frac{1}{72}\left(-18x^2 + \frac{9i(\sqrt{3}+i)\,x(x^3-8e^{3c_1})}{\sqrt[3]{-x^6-20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(x^3+e^{3c_1}\right)^3}+8e^{6c_1}}} - 9\left(1+i\sqrt{3}\right)\sqrt[3]{-x^6-20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)$$

15 Chapter IV, differential equations of the first order and higher degree than the first. Article 26. Equations solvable for x. Page 55

15.1 problem Ex 1

Internal problem ID [11212]
Internal file name [OUTPUT/10197_Tuesday_December_06_2022_03_59_53_AM_86153348/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 26. Equations solvable for x. Page 55
Problem number: Ex 1.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y'y\left(2y'^2+3\right)=-x}$$

15.1.1 Solving as dAlembert ode

Let $p=y'$ the ode becomes

$$py\left(2p^2+3\right)=-x$$

Solving for y from the above results in

$$y=-\frac{x}{p\left(2p^2+3\right)} \tag{1A}$$

This has the form

$$y=xf(p)+g(p) \tag{*}$$

Where f,g are functions of $p=y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p=f+(xf'+g')\frac{dp}{dx}$$
$$p-f=(xf'+g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= -\frac{1}{2p^3 + 3p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p + \frac{1}{2p^3 + 3p} = \frac{x(6p^2 + 3)\, p'(x)}{(2p^3 + 3p)^2} \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p + \frac{1}{2p^3 + 3p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= \frac{i\sqrt{2}}{2} \\
p &= -\frac{i\sqrt{2}}{2} \\
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -ix \\
y &= ix \\
y &= -\frac{i\sqrt{2}\, x}{2} \\
y &= \frac{i\sqrt{2}\, x}{2}
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{\left(p(x) + \frac{1}{2p(x)^3 + 3p(x)}\right) \left(2p(x)^3 + 3p(x)\right)^2}{x \left(6p\left(x\right)^2 + 3\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\,(6p^2 + 3)}{(2p^3 + 3p)^2 \left(p + \frac{1}{2p^3 + 3p}\right)} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{3}{2p^5 + 5p^3 + 3p}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{3x(p)}{2p^5 + 5p^3 + 3p} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{2p^5+5p^3+3p}dp}$$
$$= e^{-\ln(2p^2+3) - \ln(p) + \frac{3\ln(p^2+1)}{2}}$$

Which simplifies to

$$\mu = \frac{(p^2+1)^{\frac{3}{2}}}{2p^3 + 3p}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{(p^2+1)^{\frac{3}{2}}x}{2p^3+3p}\right) = 0$$

Integrating gives

$$\frac{(p^2+1)^{\frac{3}{2}}x}{2p^3+3p} = c_3$$

Dividing both sides by the integrating factor $\mu = \frac{(p^2+1)^{\frac{3}{2}}}{2p^3+3p}$ results in

$$x(p) = \frac{c_3(2p^3+3p)}{(p^2+1)^{\frac{3}{2}}}$$

which simplifies to

$$x(p) = \frac{c_3 p(2p^2+3)}{(p^2+1)^{\frac{3}{2}}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{\left(\left(-2x+2\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}}}{2y} - \frac{y}{\left(\left(-2x+2\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}}}$$

$$p = -\frac{\left(\left(-2x+2\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}}}{4y} + \frac{y}{2\left(\left(-2x+2\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(\left(-2x+2\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}}}{2y} + \frac{}{\left(\left(-2x+\right.\right.}\right.}{2}$$

$$p = -\frac{\left(\left(-2x+2\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}}}{4y} + \frac{y}{2\left(\left(-2x+2\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(\left(-2x+2\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}}}{2y} + \frac{}{\left(\left(-2x+\right.\right.}\right.}{2}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{\left(-2^{\frac{1}{3}}y^2 + \left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}\right)2^{\frac{5}{6}}c_3\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}\left(-2\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}+2\right.}{\sqrt{-\frac{2^{\frac{1}{3}}\left(x-\sqrt{x^2+2y^2}\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}}-2y^2}{\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}}}\left(2^{\frac{1}{3}}\left(x-\sqrt{x^2+2y^2}\right)\left(\left(-x+\sqrt{x^2+2y\right.\right.\right.}$$

$$x = \frac{c_3\left(\left(\sqrt{3}+i\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}} - y^2 2^{\frac{1}{3}}\left(i-\sqrt{3}\right)\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}\left(4\left(\left(-x+\sqrt{x^2+2y^2}\right.\right.\right.}{\left(x-\sqrt{x^2+2y^2}\right)\sqrt{\frac{\left(x-\sqrt{x^2+2y^2}\right)2^{\frac{1}{3}}\left(1+i\sqrt{3}\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}}+2y^2\left(i\sqrt{3}-1\right)}{\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}}}\,y^3\left(2^{\frac{1}{3}}\left(i-\right.\right.}$$

$$x = \frac{\left(-4\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}} + \left(i\sqrt{3}-1\right)2^{\frac{2}{3}}\left(x-\sqrt{x^2+2y^2}\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}} + 2y^2\left(1+i\sqrt{3}\right.\right.}{\sqrt{\frac{\left(1-i\sqrt{3}\right)2^{\frac{1}{3}}\left(x-\sqrt{x^2+2y^2}\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}}+2\left(-i\sqrt{3}-1\right)y^2}{\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}}}\,y^3\left(x-\sqrt{x^2+2y^2}\right)\left(2^{\frac{1}{3}}\left(x\right.\right.}$$

Summary

The solution(s) found are the following

$$y = -ix \tag{1}$$

$$y = ix \tag{2}$$

$$y = -\frac{i\sqrt{2}\, x}{2} \tag{3}$$

$$y = \frac{i\sqrt{2}\, x}{2} \tag{4}$$

$$x = \tag{5}$$

$$-\frac{\left(-2^{\frac{1}{3}} y^2 + \left(\left(-x+\sqrt{x^2+2y^2}\right) y^2\right)^{\frac{2}{3}}\right) 2^{\frac{5}{6}} c_3 \left(\left(-x+\sqrt{x^2+2y^2}\right) y^2\right)^{\frac{2}{3}} \left(-2\left(\left(-x+\sqrt{x^2+2y^2}\right) y^2\right)^{\frac{2}{3}} + 2\right.}{\sqrt{-\frac{2^{\frac{1}{3}}\left(x-\sqrt{x^2+2y^2}\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}} - 2y^2}{\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}}}\ \left(2^{\frac{1}{3}}\left(x-\sqrt{x^2+2y^2}\right)\left(\left(-x+\sqrt{x^2+2y\right.\right.\right.}$$

$$x \tag{6}$$

$$= \frac{c_3\left(\left(\sqrt{3}+i\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}} - y^2 2^{\frac{1}{3}}\left(i-\sqrt{3}\right)\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}\left(4\left(\left(-x+\sqrt{x^2+2y^2}\right)\right.\right.}{\left(x-\sqrt{x^2+2y^2}\right)\sqrt{\frac{\left(x-\sqrt{x^2+2y^2}\right)2^{\frac{1}{3}}\left(1+i\sqrt{3}\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}} + 2y^2\left(i\sqrt{3}-1\right)}{\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}}}\ y^3\left(2^{\frac{1}{3}}\left(i-\right.\right.}$$

$$x \tag{7}$$

$$= \frac{\left(-4\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}} + \left(i\sqrt{3}-1\right)2^{\frac{2}{3}}\left(x-\sqrt{x^2+2y^2}\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}} + 2y^2\left(1+i\sqrt{3}\right.\right.}{\sqrt{\frac{\left(1-i\sqrt{3}\right)2^{\frac{1}{3}}\left(x-\sqrt{x^2+2y^2}\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}} + 2\left(-i\sqrt{3}-1\right)y^2}{\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}}}\ y^3\left(x-\sqrt{x^2+2y^2}\right)\left(2^{\frac{1}{3}}\left(x\right.\right.}$$

Verification of solutions

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$y = -\frac{i\sqrt{2}\,x}{2}$$

Verified OK.

$$y = \frac{i\sqrt{2}\,x}{2}$$

Verified OK.

$$x = -\frac{\left(-2^{\frac{1}{3}}y^2 + \left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}\right)2^{\frac{5}{6}}c_3\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}\left(-2\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}} + 2\right.}{\sqrt{-\frac{2^{\frac{1}{3}}\left(x-\sqrt{x^2+2y^2}\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}}-2y^2}{\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}}}\ \left(2^{\frac{1}{3}}\left(x-\sqrt{x^2+2y^2}\right)\left(\left(-x+\sqrt{x^2+2y\right.\right.\right.}}$$

Warning, solution could not be verified

$$x = \frac{c_3\left(\left(\sqrt{3}+i\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}} - y^2 2^{\frac{1}{3}}\left(i-\sqrt{3}\right)\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}\left(4\left(\left(-x+\sqrt{x^2+2y^2}\right)\right.\right.}{\left(x-\sqrt{x^2+2y^2}\right)\sqrt{\frac{\left(x-\sqrt{x^2+2y^2}\right)2^{\frac{1}{3}}\left(1+i\sqrt{3}\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}}+2y^2\left(i\sqrt{3}-1\right)}{\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}}}\ y^3\left(2^{\frac{1}{3}}\left(i-\right.\right.}$$

Warning, solution could not be verified

$$x = \frac{\left(-4\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}} + \left(i\sqrt{3}-1\right)2^{\frac{2}{3}}\left(x-\sqrt{x^2+2y^2}\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}} + 2y^2\left(1+i\sqrt{3}\right.\right.}{\sqrt{\frac{\left(1-i\sqrt{3}\right)2^{\frac{1}{3}}\left(x-\sqrt{x^2+2y^2}\right)\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{1}{3}}+2\left(-i\sqrt{3}-1\right)y^2}{\left(\left(-x+\sqrt{x^2+2y^2}\right)y^2\right)^{\frac{2}{3}}}}\ y^3\left(x-\sqrt{x^2+2y^2}\right)\left(2^{\frac{1}{3}}\left(x\right.\right.}$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   <- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 776

```
dsolve(x+diff(y(x),x)*y(x)*(2*diff(y(x),x)^2+3)=0,y(x), singsol=all)
```

$$y(x) = -\frac{i\sqrt{2}\,x}{2}$$

$$y(x) = \frac{i\sqrt{2}\,x}{2}$$

$$y(x) = \mathrm{RootOf}\left(-\ln(x) + \int^{\_Z} -\frac{-2\left(\frac{\left(\_a^2-\sqrt{2\_a^2+1}+1\right)\_a}{(2\_a^2+1)^{\frac{3}{2}}}\right)^{\frac{2}{3}}\_a^2 + 2\left(\frac{\left(\_a^2-\sqrt{2\_a^2+1}+1\right)\_a}{(2\_a^2+1)^{\frac{3}{2}}}\right)^{\frac{1}{3}}\_a^3 - \left(\frac{\left(\_a^2-\sqrt{2\_a^2+1}+1\right)\_a}{(2\_a^2+1)^{\frac{3}{2}}}\right)^{\frac{2}{3}} + \_}{\left(\frac{\left(\_a^2-\sqrt{2\_a^2+1}+1\right)\_a}{(2\_a^2+1)^{\frac{3}{2}}}\right)^{\frac{1}{3}}(2\_a^4+3\_a^2+1)} + c_1\right)x$$

$$y(x) = \mathrm{RootOf}\left(-2\ln(x) + \int^{\_Z} \frac{2i\left(\frac{\left(\_a^2-\sqrt{2\_a^2+1}+1\right)\_a}{(2\_a^2+1)^{\frac{3}{2}}}\right)^{\frac{2}{3}}\sqrt{3}\,\_a^2 + i\left(\frac{\left(\_a^2-\sqrt{2\_a^2+1}+1\right)\_a}{(2\_a^2+1)^{\frac{3}{2}}}\right)^{\frac{2}{3}}\sqrt{3} - 2\left(\frac{\left(\_a^2-\sqrt{2\_a^2+1}+1\right)\_a}{(2\_a^2+1)^{\frac{3}{2}}}\right.}{\left(\frac{\left(\_a^2-\sqrt{\right.}{(2\_}\right.} + 2c_1\right)x$$

$$y(x) = \mathrm{RootOf}\left(-2\ln(x) - \left(\int^{\_Z} \frac{2i\left(\frac{\left(\_a^2-\sqrt{2\_a^2+1}+1\right)\_a}{(2\_a^2+1)^{\frac{3}{2}}}\right)^{\frac{2}{3}}\sqrt{3}\,\_a^2 + i\left(\frac{\left(\_a^2-\sqrt{2\_a^2+1}+1\right)\_a}{(2\_a^2+1)^{\frac{3}{2}}}\right)^{\frac{2}{3}}\sqrt{3} + 2\left(\frac{\left(\_a^2-\sqrt{2\_a^2+1}+1\right)\_}{(2\_a^2+1)^{\frac{3}{2}}}\right.}{}\right.\right.$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x+y'[x]*y[x]*(2*(y'[x])^2+3)==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

15.2 problem Ex 2

Internal problem ID [11213]
Internal file name [OUTPUT/10198_Tuesday_December_06_2022_04_00_03_AM_63551345/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 26. Equations solvable for x. Page 55
Problem number: Ex 2.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{a^2 y y'^2 - 2y'x + y = 0}$$

15.2.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$a^2 y\, p^2 - 2px + y = 0$$

Solving for y from the above results in

$$y = \frac{2px}{a^2p^2 + 1} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{2p}{a^2p^2+1} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{2p}{a^2p^2+1} = x\left(\frac{2}{a^2p^2+1} - \frac{4p^2a^2}{\left(a^2p^2+1\right)^2}\right)p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{2p}{a^2p^2+1} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 0 \\
p &= \frac{1}{a} \\
p &= -\frac{1}{a}
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= 0 \\
y &= \frac{x}{a} \\
y &= -\frac{x}{a}
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{2p(x)}{a^2p(x)^2+1}}{x\left(\frac{2}{a^2p(x)^2+1} - \frac{4p(x)^2a^2}{\left(a^2p(x)^2+1\right)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{2}{a^2p^2+1} - \frac{4p^2a^2}{\left(a^2p^2+1\right)^2}\right)}{p - \frac{2p}{a^2p^2+1}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$\begin{aligned} p(p) &= \frac{2}{p\left(a^2p^2+1\right)} \\ q(p) &= 0 \end{aligned}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p\left(a^2p^2+1\right)} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{2}{p\left(a^2p^2+1\right)}dp} \\ &= \mathrm{e}^{-\ln\left(a^2p^2+1\right)+2\ln(p)} \end{aligned}$$

Which simplifies to

$$\mu = \frac{p^2}{a^2p^2+1}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}p}\mu x &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{p^2x}{a^2p^2+1}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{p^2x}{a^2p^2+1} = c_3$$

Dividing both sides by the integrating factor $\mu = \frac{p^2}{a^2p^2+1}$ results in

$$x(p) = \frac{c_3(a^2p^2+1)}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{x + \sqrt{x^2 - y^2 a^2}}{y a^2}$$
$$p = -\frac{-x + \sqrt{x^2 - y^2 a^2}}{y a^2}$$

Substituting the above in the solution for x found above gives

$$x = \frac{2 c_3 x\, a^2}{x + \sqrt{x^2 - y^2 a^2}}$$
$$x = -\frac{2 c_3 x\, a^2}{-x + \sqrt{x^2 - y^2 a^2}}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$
$$y = \frac{x}{a} \tag{2}$$
$$y = -\frac{x}{a} \tag{3}$$
$$x = \frac{2 c_3 x\, a^2}{x + \sqrt{x^2 - y^2 a^2}} \tag{4}$$
$$x = -\frac{2 c_3 x\, a^2}{-x + \sqrt{x^2 - y^2 a^2}} \tag{5}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = \frac{x}{a}$$

Verified OK.

$$y = -\frac{x}{a}$$

Verified OK.

$$x = \frac{2c_3 x\, a^2}{x + \sqrt{x^2 - y^2 a^2}}$$

Verified OK.

$$x = -\frac{2c_3 x\, a^2}{-x + \sqrt{x^2 - y^2 a^2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   <- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 51

```
dsolve(a^2*y(x)*diff(y(x),x)^2-2*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{a}$$
$$y(x) = \frac{x}{a}$$
$$y(x) = 0$$
$$y(x) = \mathrm{e}^{\mathrm{RootOf}\left(\mathrm{e}^{2\_Z}\sinh(-\_Z+c_1-\ln(x))^2a^2+1\right)}x$$

✓ Solution by Mathematica

Time used: 30.099 (sec). Leaf size: 244

```
DSolve[a^2*y[x]*(y'[x])^2-2*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\left(\cosh\left(\frac{a^2c_1}{2}\right)+\sinh\left(\frac{a^2c_1}{2}\right)\right)\sqrt{\cosh(a^2c_1)+\sinh(a^2c_1)-8ix}}{4a}$$
$$y(x) \to \frac{\left(\cosh\left(\frac{a^2c_1}{2}\right)+\sinh\left(\frac{a^2c_1}{2}\right)\right)\sqrt{\cosh(a^2c_1)+\sinh(a^2c_1)-8ix}}{4a}$$
$$y(x) \to -\frac{\left(\cosh\left(\frac{a^2c_1}{2}\right)+\sinh\left(\frac{a^2c_1}{2}\right)\right)\sqrt{\cosh(a^2c_1)+\sinh(a^2c_1)+8ix}}{4a}$$
$$y(x) \to \frac{\left(\cosh\left(\frac{a^2c_1}{2}\right)+\sinh\left(\frac{a^2c_1}{2}\right)\right)\sqrt{\cosh(a^2c_1)+\sinh(a^2c_1)+8ix}}{4a}$$
$$y(x) \to -\frac{x}{a}$$
$$y(x) \to \frac{x}{a}$$

15.3 problem Ex 3

Internal problem ID [11214]
Internal file name [OUTPUT/10199_Tuesday_December_06_2022_04_00_07_AM_253671/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 26. Equations solvable for x. Page 55
Problem number: Ex 3.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy'^2 - 2yy' = x}$$

15.3.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$x\,p^2 - 2yp = x$$

Solving for y from the above results in

$$y = \frac{x(p^2 - 1)}{2p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p^2 - 1}{2p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p^2 - 1}{2p} = x\left(1 - \frac{p^2 - 1}{2p^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p^2 - 1}{2p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -ix \\
y &= ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)^2 - 1}{2p(x)}}{x\left(1 - \frac{p(x)^2 - 1}{2p(x)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= 0
\end{aligned}
$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}\mu p &= 0 \\ \frac{d}{dx}\left(\frac{p}{x}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Summary

The solution(s) found are the following

$$y = -ix \tag{1}$$

$$y = ix \tag{2}$$

$$y = \frac{c_1^2 x^2 - 1}{2c_1} \tag{3}$$

Verification of solutions

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
dsolve(x*diff(y(x),x)^2-2*y(x)*diff(y(x),x)-x=0,y(x), singsol=all)
```

$$y(x) = -ix$$
$$y(x) = ix$$
$$y(x) = \frac{-c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.213 (sec). Leaf size: 71

```
DSolve[x*(y'[x])^2-2*y[x]*y'[x]-x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-c_1}\left(-x^2+e^{2c_1}\right)$$
$$y(x) \to \frac{1}{2}e^{-c_1}\left(-1+e^{2c_1}x^2\right)$$
$$y(x) \to -ix$$
$$y(x) \to ix$$

15.4 problem Ex 4

Internal problem ID [11215]
Internal file name [`OUTPUT/10200_Tuesday_December_06_2022_04_00_09_AM_117444/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 26. Equations solvable for x. Page 55
Problem number: Ex 4.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{y'^3 - 4xyy' + 8y^2 = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}{3} + \frac{4yx}{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}} \tag{1}$$

$$y' = -\frac{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}{6} - \frac{2yx}{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}} + i\sqrt{3}\left(\frac{\left(-108y^2 + 12\sqrt{-\right.}}{}\right. \tag{2}$$

$$y' = -\frac{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}{6} - \frac{2yx}{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}} - i\sqrt{3}\left(\frac{\left(-108y^2 + 12\sqrt{-\right.}}{}\right. \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{2}{3}} + 12xy}{3\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\frac{\left(\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)(b_3-a_2)}{3\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}-\frac{\left(\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)^2 a_3}{9\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}}-\left(\frac{-\frac{144y^3x^2}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12y^3x^3+81y^4}}+12y}{3\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}+\frac{24\left(\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)y^3x^2}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{4}{3}}\sqrt{-12y^3x^3+81y^4}}\right)(xa_2+ya_3+a_1)-\left(\frac{\frac{-144y+\frac{2\left(-216y^2x^3+1944y^3\right)}{3\sqrt{-12y^3x^3+81y^4}}}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}+12x}{3\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}-\frac{\left(\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)\left(-216y+\frac{-216y^2x^3+1944y^3}{\sqrt{-12y^3x^3+81y^4}}\right)}{9\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{4}{3}}}\right)(xb_2+yb_3+b_1)=0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-2592\sqrt{-12y^3x^3+81y^4}\,x\,y^3a_2-72\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}x^4y^2b_2-72\left(-108y^2+12\sqrt{-12\right.}}{}=0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&2592\sqrt{-12y^3x^3+81y^4}\,x\,y^3a_2 \\
&+72\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}x^4y^2b_2 \\
&+72\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}x^3y^3a_2 \\
&+72\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}x^3y^3b_3 \\
&+72\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}x^2y^4a_3 \\
&+72\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}x^3y^2b_1 \\
&+72\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}x^2y^3a_1 \\
&-648\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}x\,y^3b_2 \\
&+72\Big(-108y^2 \\
&+12\sqrt{-12y^3x^3+81y^4}\Big)^{\frac{2}{3}}\sqrt{-12y^3x^3+81y^4}\,y^2b_3 \\
&+432\sqrt{-12y^3x^3+81y^4}\,x^2y^2b_2 \\
&-864\sqrt{-12y^3x^3+81y^4}\,x\,y^3b_3+72\Big(-108y^2 \\
&+12\sqrt{-12y^3x^3+81y^4}\Big)^{\frac{2}{3}}\sqrt{-12y^3x^3+81y^4}\,yb_1 \\
&+432\sqrt{-12y^3x^3+81y^4}\,x\,y^2b_1-8\Big(-108y^2 \\
&+12\sqrt{-12y^3x^3+81y^4}\Big)^{\frac{4}{3}}\sqrt{-12y^3x^3+81y^4}\,xya_3 \\
&-48\Big(-108y^2 \\
&+12\sqrt{-12y^3x^3+81y^4}\Big)^{\frac{2}{3}}\sqrt{-12y^3x^3+81y^4}\,x^2y^2a_3 \\
&+72\Big(-108y^2 \\
&+12\sqrt{-12y^3x^3+81y^4}\Big)^{\frac{2}{3}}\sqrt{-12y^3x^3+81y^4}\,xyb_2 \\
&-48\left(-12y^3x^3+81y^4\right)^{\frac{3}{2}}a_3+58320y^6a_3 \\
&-11664y^5a_1-23328x\,y^5a_2+3b_2\Big(-108y^2 \\
&+12\sqrt{-12y^3x^3+81y^4}\Big)^{\frac{4}{3}}\sqrt{-12y^3x^3+81y^4} \\
&-648\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}y^3b_1 \\
&-2592\sqrt{-12y^3x^3+81y^4}\,y^4a_3 \\
&+1296\sqrt{-12y^3x^3+81y^4}\,y^3a_1+2592x^4y^4a_2 \\
&-9504x^3y^5a_3+864x^3y^4a_1+864x^5y^3b_2 \\
&-864x^4y^4b_3+864x^4y^3b_1-3888x^2y^4b_2
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-y^3(4x^3-27y)}, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3(4x^3-27y)}\right)^{\frac{1}{3}}, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3(4x^3-27y)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-y^3(4x^3-27y)} = v_3, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3(4x^3-27y)}\right)^{\frac{1}{3}} = v_4, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3(4x^3-27y)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-36v_2\Big(-4860v_2^5a_3 + 972v_2^4a_1 - 6v_5v_1^4v_2b_2 - 18v_5v_1^3v_2^2a_2 \\
&+ 6v_5v_1^3v_2^2b_3 - 6v_5v_1^2v_2^3a_3 - 6v_5v_1^3v_2b_1 - 6v_5v_1^2v_2^2a_1 + 36v_4v_1^3v_2^2b_2 \\
&+ 648v_4v_1v_2^4a_3 + 54v_5v_1v_2^2b_2 + 540\sqrt{3}\,v_3v_2^3a_3 - 6v_5\sqrt{3}\,v_3b_1 \\
&- 108\sqrt{3}\,v_3v_2^2a_1 - 96v_4v_1^4v_2^3a_3 - 48\sqrt{3}\,v_3v_1^3v_2^2a_3 - 6v_5\sqrt{3}\,v_3v_1b_2 \\
&- 9v_5\sqrt{3}\,v_3v_2a_2 + 3v_5\sqrt{3}\,v_3v_2b_3 - 36\sqrt{3}\,v_3v_1^2v_2b_2 - 216\sqrt{3}\,v_3v_1v_2^2a_2 \\
&+ 72\sqrt{3}\,v_3v_1v_2^2b_3 + 27v_4\sqrt{3}\,v_3v_2b_2 - 36\sqrt{3}\,v_3v_1v_2b_1 + 4v_5\sqrt{3}\,v_3v_1^2v_2a_3 \\
&- 72v_4\sqrt{3}\,v_3v_1v_2^2a_3 + 324v_2^3v_1^2b_2 + 1944v_2^4v_1a_2 + 324v_2^3v_1b_1 - 72v_1^5v_2^2b_2 \\
&- 216v_1^4v_2^3a_2 + 72v_1^4v_2^3b_3 + 792v_1^3v_2^4a_3 - 72v_1^4v_2^2b_1 - 72v_1^3v_2^3a_1 \\
&- 648v_1v_2^4b_3 + 81v_5v_2^3a_2 - 27v_5v_2^3b_3 + 54v_5v_2^2b_1 - 243v_4v_2^3b_2\Big) = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&3456a_3v_4v_1^4v_2^4+(7776a_2-2592b_3)\,v_1^4v_2^4+216b_2v_5v_1^4v_2^2-1296b_2v_4v_1^3v_2^3\\
&+(648a_2-216b_3)\,v_5v_1^3v_2^3+216b_1v_5v_1^3v_2^2+216a_3v_5v_1^2v_2^4+216a_1v_5v_1^2v_2^3\\
&-23328a_3v_4v_1v_2^5+(-69984a_2+23328b_3)\,v_1v_2^5+174960a_3v_2^6\\
&-34992a_1v_2^5+\left(7776\sqrt{3}\,a_2-2592\sqrt{3}\,b_3\right)v_3v_1v_2^3-1944b_2v_5v_1v_2^3\\
&-19440\sqrt{3}\,a_3v_3v_2^4+3888\sqrt{3}\,a_1v_3v_2^3+\left(324\sqrt{3}\,a_2-108\sqrt{3}\,b_3\right)v_3v_5v_2^2\\
&+1728\sqrt{3}\,a_3v_3v_1^3v_2^3+1296\sqrt{3}\,b_2v_3v_1^2v_2^2+1296\sqrt{3}\,b_1v_3v_1v_2^2\\
&-972\sqrt{3}\,b_2v_3v_4v_2^2+216v_5\sqrt{3}\,v_3b_1v_2-144\sqrt{3}\,a_3v_3v_5v_1^2v_2^2\\
&+2592\sqrt{3}\,a_3v_3v_4v_1v_2^3+216\sqrt{3}\,b_2v_3v_5v_1v_2-11664b_2v_1^2v_2^4\\
&-11664b_1v_1v_2^4+8748b_2v_4v_2^4+(-2916a_2+972b_3)\,v_5v_2^4-1944b_1v_5v_2^3\\
&+2592v_2^3b_2v_1^5+2592b_1v_1^4v_2^3-28512a_3v_1^3v_2^5+2592a_1v_1^3v_2^4=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-34992a_1 &= 0 \\
216a_1 &= 0 \\
2592a_1 &= 0 \\
-28512a_3 &= 0 \\
-23328a_3 &= 0 \\
216a_3 &= 0 \\
3456a_3 &= 0 \\
174960a_3 &= 0 \\
-11664b_1 &= 0 \\
-1944b_1 &= 0 \\
216b_1 &= 0 \\
2592b_1 &= 0 \\
-11664b_2 &= 0 \\
-1944b_2 &= 0 \\
-1296b_2 &= 0 \\
216b_2 &= 0 \\
2592b_2 &= 0 \\
8748b_2 &= 0 \\
3888\sqrt{3}\, a_1 &= 0 \\
-19440\sqrt{3}\, a_3 &= 0 \\
-144\sqrt{3}\, a_3 &= 0 \\
1728\sqrt{3}\, a_3 &= 0 \\
2592\sqrt{3}\, a_3 &= 0 \\
216\sqrt{3}\, b_1 &= 0 \\
1296\sqrt{3}\, b_1 &= 0 \\
-972\sqrt{3}\, b_2 &= 0 \\
216\sqrt{3}\, b_2 &= 0 \\
1296\sqrt{3}\, b_2 &= 0 \\
-69984a_2 + 23328b_3 &= 0 \\
-2916a_2 + 972b_3 &= 0 \\
648a_2 - 216b_3 &= 0 \\
7776a_2 - 2592b_3 &= 0 \\
324\sqrt{3}\, a_2 - 108\sqrt{3}\, b_3 &= 0 \\
7776\sqrt{3}\, a_2 - 2592\sqrt{3}\, b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{y}{\frac{x}{3}} \\
&= \frac{3y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x^3
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x^3}
$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{\frac{x}{3}}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= 3\ln(x)\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{2}{3}} + 12xy}{3\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{3y}{x^4}\\ R_y &= \frac{1}{x^3}\\ S_x &= \frac{3}{x}\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{9x^3\left(-108y^2 + 12\sqrt{3}\,\sqrt{-4y^3x^3 + 27y^4}\right)^{\frac{1}{3}}}{\left(-108y^2 + 12\sqrt{3}\,\sqrt{-4y^3x^3 + 27y^4}\right)^{\frac{2}{3}} x + 12x^2y - 9y\left(-108y^2 + 12\sqrt{3}\,\sqrt{-4y^3x^3 + 27y^4}\right)^{\frac{1}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{9\,12^{\frac{1}{3}}\left(\sqrt{3}\,\sqrt{27R-4} - 9\sqrt{R}\right)^{\frac{1}{3}}}{\sqrt{R}\left(12^{\frac{2}{3}}\left(\sqrt{3}\,\sqrt{27R-4} - 9\sqrt{R}\right)^{\frac{2}{3}} - 9\sqrt{R}\,12^{\frac{1}{3}}\left(\sqrt{3}\,\sqrt{27R-4} - 9\sqrt{R}\right)^{\frac{1}{3}} + 12\right)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{9\left(12\sqrt{81R-12}-108\sqrt{R}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(-\sqrt{81R-12}+9\sqrt{R}\right)^{2}\right)^{\frac{1}{3}}-9\sqrt{R}\,\left(12\sqrt{81R-12}-108\sqrt{R}\right)^{\frac{1}{3}}+12\right)\sqrt{R}}dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$3\ln(x) = \int^{\frac{y}{x^3}} \frac{9\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(-\sqrt{81\_a-12}+9\sqrt{\_a}\right)^{2}\right)^{\frac{1}{3}}-9\sqrt{\_a}\,\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}+12\right)\sqrt{\_a}}$$

Which simplifies to

$$3\ln(x) = \int^{\frac{y}{x^3}} \frac{9\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(-\sqrt{81\_a-12}+9\sqrt{\_a}\right)^{2}\right)^{\frac{1}{3}}-9\sqrt{\_a}\,\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}+12\right)\sqrt{\_a}}$$

Summary

The solution(s) found are the following

$$3\ln(x) = \int^{\frac{y}{x^3}} \frac{9\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(-\sqrt{81\_a-12}+9\sqrt{\_a}\right)^{2}\right)^{\frac{1}{3}}-9\sqrt{\_a}\,\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}+12\right)\sqrt{\_a}}d\_a + c_1 \tag{1}$$

Verification of solutions

$$3\ln(x) = \int^{\frac{y}{x^3}} \frac{9\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(-\sqrt{81\_a-12}+9\sqrt{\_a}\right)^{2}\right)^{\frac{1}{3}}-9\sqrt{\_a}\,\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}+12\right)\sqrt{\_a}}d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx-\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12xy}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 \tag{5E}$$

$$+\frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx-\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12xy\right)(b_3-}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}$$

$$-\frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx-\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12xy\right)^2 a_3}{36\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}}$$

$$-\left(\frac{-\frac{144i\sqrt{3}\,y^3x^2}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12y^3x^3+81y^4}}-12i\sqrt{3}\,y+\frac{144y^3x^2}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12y^3x^3+81y^4}}-12y}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}\right.$$

$$+\frac{12\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx-\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12xy\right)y^3}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{4}{3}}\sqrt{-12y^3x^3+81y^4}}$$

$$+ya_3+a_1)$$

$$-\left(\frac{\frac{2i\sqrt{3}\left(-216y+\frac{-216y^2x^3+1944y^3}{\sqrt{-12y^3x^3+81y^4}}\right)}{3\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}-12i\sqrt{3}\,x-\frac{2\left(-216y+\frac{-216y^2x^3+1944y^3}{\sqrt{-12y^3x^3+81y^4}}\right)}{3\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}-12x}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}\right.$$

$$-\frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx-\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12xy\right)\left(-\right.}{18\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{4}{3}}}$$

$$+yb_3+b_1)=0$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-y^3\,(4x^3-27y)}, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3\,(4x^3-27y)}\right)^{\frac{1}{3}}, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3\,(4x^3-27y)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-y^3\,(4x^3-27y)} = v_3, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3\,(4x^3-27y)}\right)^{\frac{1}{3}} = v_4, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3\,(4x^3-27y)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&72v_2\Big(-6v_5v_1^4v_2b_2-18v_5v_1^3v_2^2a_2+6v_5v_1^3v_2^2b_3-6v_5v_1^2v_2^3a_3\\
&-6v_5v_1^3v_2b_1-6v_5v_1^2v_2^2a_1-72v_4v_1^3v_2^2b_2-1296v_4v_1v_2^4a_3\\
&+540\sqrt{3}\,v_3v_2^3a_3+54v_5v_1v_2^2b_2-6\sqrt{3}\,v_5v_3b_1\\
&-108\sqrt{3}\,v_3v_2^2a_1-4860i\sqrt{3}\,v_2^5a_3+972i\sqrt{3}\,v_2^4a_1\\
&+1620iv_3v_2^3a_3+18iv_5v_3b_1-324iv_3v_2^2a_1+324v_2^3v_1^2b_2\\
&+1944v_2^4v_1a_2+324v_2^3v_1b_1-72v_1^5v_2^2b_2-216v_1^4v_2^3a_2\\
&+72v_1^4v_2^3b_3+792v_1^3v_2^4a_3-72v_1^4v_2^2b_1-72v_1^3v_2^3a_1\\
&-648v_1v_2^4b_3+81v_5v_2^3a_2-27v_5v_2^3b_3+4\sqrt{3}\,v_5v_3v_1^2v_2a_3\\
&+144\sqrt{3}\,v_4v_3v_1v_2^2a_3-144iv_3v_1^3v_2^2a_3-54i\sqrt{3}\,v_5v_2^2b_1\\
&+324i\sqrt{3}\,v_2^3v_1b_1+18iv_5v_3v_1b_2+27iv_5v_3v_2a_2-9iv_5v_3v_2b_3\\
&-108iv_3v_1^2v_2b_2-648iv_3v_1v_2^2a_2+216iv_3v_1v_2^2b_3\\
&-108iv_3v_1v_2b_1-72i\sqrt{3}\,v_1^5v_2^2b_2-216i\sqrt{3}\,v_1^4v_2^3a_2\\
&+72i\sqrt{3}\,v_1^4v_2^3b_3+792i\sqrt{3}\,v_1^3v_2^4a_3-72i\sqrt{3}\,v_1^4v_2^2b_1\\
&-72i\sqrt{3}\,v_1^3v_2^3a_1-81i\sqrt{3}\,v_5v_2^3a_2+27i\sqrt{3}\,v_5v_2^3b_3\\
&+324i\sqrt{3}\,v_2^3v_1^2b_2+1944i\sqrt{3}\,v_2^4v_1a_2-648i\sqrt{3}\,v_1v_2^4b_3\\
&+6i\sqrt{3}\,v_5v_1^4v_2b_2+18i\sqrt{3}\,v_5v_1^3v_2^2a_2-6i\sqrt{3}\,v_5v_1^3v_2^2b_3\\
&+6i\sqrt{3}\,v_5v_1^2v_2^3a_3+6i\sqrt{3}\,v_5v_1^3v_2b_1+6i\sqrt{3}\,v_5v_1^2v_2^2a_1\\
&-54i\sqrt{3}\,v_5v_1v_2^2b_2-12iv_5v_3v_1^2v_2a_3+54v_5v_2^2b_1+486v_4v_2^3b_2\\
&+192v_4v_1^4v_2^3a_3-48\sqrt{3}\,v_3v_1^3v_2^2a_3-6\sqrt{3}\,v_5v_3v_1b_2\\
&-9\sqrt{3}\,v_5v_3v_2a_2+3\sqrt{3}\,v_5v_3v_2b_3-36\sqrt{3}\,v_3v_1^2v_2b_2\\
&-216\sqrt{3}\,v_3v_1v_2^2a_2+72\sqrt{3}\,v_3v_1v_2^2b_3-54\sqrt{3}\,v_4v_3v_2b_2\\
&-36\sqrt{3}\,v_3v_1v_2b_1-4860v_2^5a_3+972v_2^4a_1\Big)=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2,v_3,v_4,v_5\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-93312a_3v_4v_1v_2^5 + 10368\sqrt{3}\,a_3v_3v_4v_1v_2^3 \\
&- 5184b_2v_4v_1^3v_2^3 + 13824a_3v_4v_1^4v_2^4 + 34992b_2v_4v_2^4 \\
&+ \Big(139968i\sqrt{3}\,a_2 - 46656i\sqrt{3}\,b_3 + 139968a_2 \\
&- 46656b_3\Big)\, v_1v_2^5 + \Big(23328i\sqrt{3}\,b_1 + 23328b_1\Big)\, v_1v_2^4 \\
&+ \Big(57024i\sqrt{3}\,a_3 + 57024a_3\Big)\, v_1^3v_2^5 \\
&+ \Big(-5184i\sqrt{3}\,a_1 - 5184a_1\Big)\, v_1^3v_2^4 \\
&+ \Big(-5184i\sqrt{3}\,b_2 - 5184b_2\Big)\, v_1^5v_2^3 \\
&+ \Big(-15552i\sqrt{3}\,a_2 + 5184i\sqrt{3}\,b_3 - 15552a_2 + 5184b_3\Big)\, v_1^4v_2^4 \\
&+ \Big(-5184i\sqrt{3}\,b_1 - 5184b_1\Big)\, v_1^4v_2^3 \\
&+ \Big(38880\sqrt{3}\,a_3 + 116640ia_3\Big)\, v_3v_2^4 \\
&+ \Big(-5832i\sqrt{3}\,a_2 + 1944i\sqrt{3}\,b_3 + 5832a_2 - 1944b_3\Big)\, v_5v_2^4 \\
&+ \Big(-7776\sqrt{3}\,a_1 - 23328ia_1\Big)\, v_3v_2^3 \\
&+ \Big(-3888i\sqrt{3}\,b_1 + 3888b_1\Big)\, v_5v_2^3 \\
&+ \Big(23328i\sqrt{3}\,b_2 + 23328b_2\Big)\, v_1^2v_2^4 \\
&+ \Big(69984i\sqrt{3}\,a_1 + 69984a_1\Big)\, v_2^5 \\
&+ \Big(-349920i\sqrt{3}\,a_3 - 349920a_3\Big)\, v_2^6 \\
&+ \Big(-15552\sqrt{3}\,a_2 + 5184\sqrt{3}\,b_3 - 46656ia_2 \\
&+ 15552ib_3\Big)\, v_3v_1v_2^3 + \Big(-3888i\sqrt{3}\,b_2 + 3888b_2\Big)\, v_5v_1v_2^3 \\
&+ \Big(-2592\sqrt{3}\,b_1 - 7776ib_1\Big)\, v_3v_1v_2^2 \\
&+ \Big(-3456\sqrt{3}\,a_3 - 10368ia_3\Big)\, v_3v_1^3v_2^3 \\
&+ \Big(1296i\sqrt{3}\,a_2 - 432i\sqrt{3}\,b_3 - 1296a_2 + 432b_3\Big)\, v_5v_1^3v_2^3 \\
&+ \Big(432i\sqrt{3}\,b_1 - 432b_1\Big)\, v_5v_1^3v_2^2 \\
&+ \Big(432i\sqrt{3}\,b_2 - 432b_2\Big)\, v_5v_1^4v_2^2 \\
&+ \Big(432i\sqrt{3}\,a_3 - 432a_3\Big)\, v_5v_1^2v_2^4 \\
&+ \Big(432i\sqrt{3}\,a_1 - 432a_1\Big)\, v_5v_1^2v_2^3 \\
&+ \Big(-2592\sqrt{3}\,b_2 - 7776ib_2\Big)\, v_3v_1^2v_2^2 \\
&+ \Big(-648\sqrt{3}\,a_2 + 216\sqrt{3}\,b_3 + 1944ia_2 - 648ib_3\Big)\, v_3v_5v_2^2 \\
&+ \Big(-432\sqrt{3}\,b_1 + 1296ib_1\Big)\, v_3v_5v_2
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-93312a_3 &= 0\\
13824a_3 &= 0\\
-5184b_2 &= 0\\
34992b_2 &= 0\\
10368\sqrt{3}\, a_3 &= 0\\
-3888\sqrt{3}\, b_2 &= 0\\
-7776\sqrt{3}\, a_1 - 23328ia_1 &= 0\\
-3456\sqrt{3}\, a_3 - 10368ia_3 &= 0\\
288\sqrt{3}\, a_3 - 864ia_3 &= 0\\
38880\sqrt{3}\, a_3 + 116640ia_3 &= 0\\
-2592\sqrt{3}\, b_1 - 7776ib_1 &= 0\\
-432\sqrt{3}\, b_1 + 1296ib_1 &= 0\\
-2592\sqrt{3}\, b_2 - 7776ib_2 &= 0\\
-432\sqrt{3}\, b_2 + 1296ib_2 &= 0\\
-349920i\sqrt{3}\, a_3 - 349920a_3 &= 0\\
-5184i\sqrt{3}\, a_1 - 5184a_1 &= 0\\
-5184i\sqrt{3}\, b_1 - 5184b_1 &= 0\\
-5184i\sqrt{3}\, b_2 - 5184b_2 &= 0\\
-3888i\sqrt{3}\, b_1 + 3888b_1 &= 0\\
-3888i\sqrt{3}\, b_2 + 3888b_2 &= 0\\
432i\sqrt{3}\, a_1 - 432a_1 &= 0\\
432i\sqrt{3}\, a_3 - 432a_3 &= 0\\
432i\sqrt{3}\, b_1 - 432b_1 &= 0\\
432i\sqrt{3}\, b_2 - 432b_2 &= 0\\
23328i\sqrt{3}\, b_1 + 23328b_1 &= 0\\
23328i\sqrt{3}\, b_2 + 23328b_2 &= 0\\
57024i\sqrt{3}\, a_3 + 57024a_3 &= 0\\
69984i\sqrt{3}\, a_1 + 69984a_1 &= 0\\
-15552\sqrt{3}\, a_2 + 5184\sqrt{3}\, b_3 - 46656ia_2 + 15552ib_3 &= 0\\
-648\sqrt{3}\, a_2 + 216\sqrt{3}\, b_3 + 1944ia_2 - 648ib_3 &= 0\\
-15552i\sqrt{3}\, a_2 + 5184i\sqrt{3}\, b_3 - 15552a_2 + 5184b_3 &= 0\\
-5832i\sqrt{3}\, a_2 + 1944i\sqrt{3}\, b_3 + 5832a_2 - 1944b_3 &= 0\\
1296i\sqrt{3}\, a_2 - 432i\sqrt{3}\, b_3 - 1296a_2 + 432b_3 &= 0\\
139968i\sqrt{3}\, a_2 - 46656i\sqrt{3}\, b_3 + 139968a_2 - 46656b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= a_2\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= 3a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x\\
\eta &= 3y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$
y' = -\frac{i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} - 12i\sqrt{3}\,yx + \left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} + 12xy}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}
$$

$$
y' = \omega(x,y)
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 \tag{5E}$$

$$- \frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} - 12i\sqrt{3}\,yx + \left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} + 12xy\right)(b_3}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}$$

$$- \frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} - 12i\sqrt{3}\,yx + \left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} + 12xy\right)^2 a_3}{36\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}}$$

$$- \left(-\frac{-\frac{144i\sqrt{3}\,y^3x^2}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12y^3x^3+81y^4}} - 12i\sqrt{3}\,y - \frac{144y^3x^2}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12y^3x^3+81y^4}} + 12y}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}\right.$$

$$- \frac{12\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} - 12i\sqrt{3}\,yx + \left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} + 12xy\right)y}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{4}{3}}\sqrt{-12y^3x^3+81y^4}}$$

$$+ ya_3 + a_1)$$

$$- \left(-\frac{\frac{2i\sqrt{3}\left(-216y+\frac{-216y^2x^3+1944y^3}{\sqrt{-12y^3x^3+81y^4}}\right)}{3\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}} - 12i\sqrt{3}\,x + \frac{-144y+\frac{2\left(-216y^2x^3+1944y^3\right)}{3\sqrt{-12y^3x^3+81y^4}}}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}} + 12x}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}\right.$$

$$+ \frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} - 12i\sqrt{3}\,yx + \left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} + 12xy\right)\left(-2\right.}{18\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{4}{3}}}$$

$$+ yb_3 + b_1) = 0$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-y^3\left(4x^3-27y\right)}, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3\left(4x^3-27y\right)}\right)^{\frac{1}{3}}, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3\left(4x^3-27y\right)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-y^3\left(4x^3-27y\right)} = v_3, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3\left(4x^3-27y\right)}\right)^{\frac{1}{3}} = v_4, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3\left(4x^3-27y\right)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
72v_2\Big(&-36v_3\sqrt{3}\,v_1^2v_2b_2-216v_3\sqrt{3}\,v_1v_2^2a_2+72v_3\sqrt{3}\,v_1v_2^2b_3\\
&-54v_3\sqrt{3}\,v_4v_2b_2-36v_3\sqrt{3}\,v_1v_2b_1+72i\sqrt{3}\,v_1^5v_2^2b_2\\
&+216i\sqrt{3}\,v_1^4v_2^3a_2-72i\sqrt{3}\,v_1^4v_2^3b_3-792i\sqrt{3}\,v_1^3v_2^4a_3\\
&+72i\sqrt{3}\,v_1^4v_2^2b_1+72i\sqrt{3}\,v_1^3v_2^3a_1-6v_5v_1^3v_2b_1-18iv_3v_5b_1\\
&+324iv_3v_2^2a_1-6v_5v_1^2v_2^2a_1-72v_4v_1^3v_2^2b_2-1296v_4v_1v_2^4a_3\\
&+540v_3\sqrt{3}\,v_2^3a_3+54v_5v_1v_2^2b_2-6v_3\sqrt{3}\,v_5b_1-108v_3\sqrt{3}\,v_2^2a_1\\
&+192v_4v_1^4v_2^3a_3-6v_5v_1^4v_2b_2-18v_5v_1^3v_2^2a_2+6v_5v_1^3v_2^2b_3\\
&-6v_5v_1^2v_2^3a_3+4860i\sqrt{3}\,v_2^5a_3-972i\sqrt{3}\,v_2^4a_1-1620iv_3v_2^3a_3\\
&-6i\sqrt{3}\,v_5v_1^2v_2^2a_1+12iv_3v_5v_1^2v_2a_3+54i\sqrt{3}\,v_5v_1v_2^2b_2\\
&+792v_1^3v_2^4a_3-72v_1^4v_2^2b_1-72v_1^3v_2^3a_1-648v_1v_2^4b_3+81v_5v_2^3a_2\\
&-27v_5v_2^3b_3+54v_5v_2^2b_1+486v_4v_2^3b_2-72v_1^5v_2^2b_2-216v_1^4v_2^3a_2\\
&+72v_1^4v_2^3b_3+324v_2^3v_1^2b_2+1944v_2^4v_1a_2+324v_2^3v_1b_1\\
&-48v_3\sqrt{3}\,v_1^3v_2^2a_3-6v_3\sqrt{3}\,v_5v_1b_2-9v_3\sqrt{3}\,v_5v_2a_2\\
&+3v_3\sqrt{3}\,v_5v_2b_3+144iv_3v_1^3v_2^2a_3+81i\sqrt{3}\,v_5v_2^3a_2\\
&-27i\sqrt{3}\,v_5v_2^3b_3-324i\sqrt{3}\,v_2^3v_1^2b_2-1944i\sqrt{3}\,v_2^4v_1a_2\\
&+648i\sqrt{3}\,v_1v_2^4b_3+54i\sqrt{3}\,v_5v_2^2b_1-324i\sqrt{3}\,v_2^3v_1b_1\\
&-18iv_3v_5v_1b_2-27iv_3v_5v_2a_2+9iv_3v_5v_2b_3+108iv_3v_1^2v_2b_2\\
&+648iv_3v_1v_2^2a_2-216iv_3v_1v_2^2b_3+108iv_3v_1v_2b_1\\
&+4v_3\sqrt{3}\,v_5v_1^2v_2a_3+144v_3\sqrt{3}\,v_4v_1v_2^2a_3-6i\sqrt{3}\,v_5v_1^4v_2b_2\\
&-18i\sqrt{3}\,v_5v_1^3v_2^2a_2+6i\sqrt{3}\,v_5v_1^3v_2^2b_3-6i\sqrt{3}\,v_5v_1^2v_2^3a_3\\
&-6i\sqrt{3}\,v_5v_1^3v_2b_1-4860v_2^5a_3+972v_2^4a_1\Big)=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3,v_4,v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(864ia_3 + 288\sqrt{3}\,a_3\right) v_3v_5v_1^2v_2^2 + \left(-1296ib_2 - 432\sqrt{3}\,b_2\right) v_3v_5v_1v_2 \\
&+ \left(-1944ia_2 + 648ib_3 - 648\sqrt{3}\,a_2 + 216\sqrt{3}\,b_3\right) v_3v_5v_2^2 + \left(-1296ib_1 - 432\sqrt{3}\,b_1\right) v_3v_5v_2 \\
&+ 34992b_2v_4v_2^4 - 5184b_2v_4v_1^3v_2^3 + \left(10368ia_3 - 3456\sqrt{3}\,a_3\right) v_3v_1^3v_2^3 \\
&+ \left(-1296i\sqrt{3}\,a_2 + 432i\sqrt{3}\,b_3 - 1296a_2 + 432b_3\right) v_5v_1^3v_2^3 + \left(-432i\sqrt{3}\,b_1 - 432b_1\right) v_5v_1^3v_2^2 \\
&+ \left(46656ia_2 - 15552ib_3 - 15552\sqrt{3}\,a_2 + 5184\sqrt{3}\,b_3\right) v_3v_1v_2^3 \\
&+ \left(3888i\sqrt{3}\,b_2 + 3888b_2\right) v_5v_1v_2^3 + \left(7776ib_1 - 2592\sqrt{3}\,b_1\right) v_3v_1v_2^2 \\
&+ \left(-432i\sqrt{3}\,b_2 - 432b_2\right) v_5v_1^4v_2^2 + \left(-432i\sqrt{3}\,a_3 - 432a_3\right) v_5v_1^2v_2^4 \\
&+ \left(-432i\sqrt{3}\,a_1 - 432a_1\right) v_5v_1^2v_2^3 + \left(7776ib_2 - 2592\sqrt{3}\,b_2\right) v_3v_1^2v_2^2 - 93312a_3v_4v_1v_2^5 \\
&+ 13824a_3v_4v_1^4v_2^4 + \left(349920i\sqrt{3}\,a_3 - 349920a_3\right) v_2^6 + \left(-69984i\sqrt{3}\,a_1 + 69984a_1\right) v_2^5 \\
&+ \left(5184i\sqrt{3}\,b_2 - 5184b_2\right) v_1^5v_2^3 + \left(-116640ia_3 + 38880\sqrt{3}\,a_3\right) v_3v_2^4 \\
&+ \left(5832i\sqrt{3}\,a_2 - 1944i\sqrt{3}\,b_3 + 5832a_2 - 1944b_3\right) v_5v_2^4 \\
&+ \left(23328ia_1 - 7776\sqrt{3}\,a_1\right) v_3v_2^3 + \left(3888i\sqrt{3}\,b_1 + 3888b_1\right) v_5v_2^3 \\
&+ \left(-57024i\sqrt{3}\,a_3 + 57024a_3\right) v_1^3v_2^5 + \left(5184i\sqrt{3}\,a_1 - 5184a_1\right) v_1^3v_2^4 \\
&+ \left(-139968i\sqrt{3}\,a_2 + 46656i\sqrt{3}\,b_3 + 139968a_2 - 46656b_3\right) v_1v_2^5 \\
&+ \left(-23328i\sqrt{3}\,b_1 + 23328b_1\right) v_1v_2^4 + \left(15552i\sqrt{3}\,a_2 - 5184i\sqrt{3}\,b_3 - 15552a_2 + 5184b_3\right) v_1^4v_2^4 \\
&+ \left(5184i\sqrt{3}\,b_1 - 5184b_1\right) v_1^4v_2^3 + \left(-23328i\sqrt{3}\,b_2 + 23328b_2\right) v_1^2v_2^4 \\
&- 3888\sqrt{3}\,b_2v_3v_4v_2^2 + 10368\sqrt{3}\,a_3v_3v_4v_1v_2^3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-93312a_3 = 0$$
$$13824a_3 = 0$$
$$-5184b_2 = 0$$
$$34992b_2 = 0$$
$$10368\sqrt{3}\, a_3 = 0$$
$$-3888\sqrt{3}\, b_2 = 0$$
$$-116640ia_3 + 38880\sqrt{3}\, a_3 = 0$$
$$-1296ib_1 - 432\sqrt{3}\, b_1 = 0$$
$$-1296ib_2 - 432\sqrt{3}\, b_2 = 0$$
$$864ia_3 + 288\sqrt{3}\, a_3 = 0$$
$$7776ib_1 - 2592\sqrt{3}\, b_1 = 0$$
$$7776ib_2 - 2592\sqrt{3}\, b_2 = 0$$
$$10368ia_3 - 3456\sqrt{3}\, a_3 = 0$$
$$23328ia_1 - 7776\sqrt{3}\, a_1 = 0$$
$$-69984i\sqrt{3}\, a_1 + 69984a_1 = 0$$
$$-57024i\sqrt{3}\, a_3 + 57024a_3 = 0$$
$$-23328i\sqrt{3}\, b_1 + 23328b_1 = 0$$
$$-23328i\sqrt{3}\, b_2 + 23328b_2 = 0$$
$$-432i\sqrt{3}\, a_1 - 432a_1 = 0$$
$$-432i\sqrt{3}\, a_3 - 432a_3 = 0$$
$$-432i\sqrt{3}\, b_1 - 432b_1 = 0$$
$$-432i\sqrt{3}\, b_2 - 432b_2 = 0$$
$$3888i\sqrt{3}\, b_1 + 3888b_1 = 0$$
$$3888i\sqrt{3}\, b_2 + 3888b_2 = 0$$
$$5184i\sqrt{3}\, a_1 - 5184a_1 = 0$$
$$5184i\sqrt{3}\, b_1 - 5184b_1 = 0$$
$$5184i\sqrt{3}\, b_2 - 5184b_2 = 0$$
$$349920i\sqrt{3}\, a_3 - 349920a_3 = 0$$
$$-1944ia_2 + 648ib_3 - 648\sqrt{3}\, a_2 + 216\sqrt{3}\, b_3 = 0$$
$$46656ia_2 - 15552ib_3 - 15552\sqrt{3}\, a_2 + 5184\sqrt{3}\, b_3 = 0$$
$$-139968i\sqrt{3}\, a_2 + 46656i\sqrt{3}\, b_3 + 139968a_2 - 46656b_3 = 0$$
$$-1296i\sqrt{3}\, a_2 + 432i\sqrt{3}\, b_3 - 1296a_2 + 432b_3 = 0$$
$$5832i\sqrt{3}\, a_2 - 1944i\sqrt{3}\, b_3 + 5832a_2 - 1944b_3 = 0$$
$$15552i\sqrt{3}\, a_2 - 5184i\sqrt{3}\, b_3 - 15552a_2 + 5184b_3 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 3a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 3y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 2
   -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
   trying dAlembert
   -> Calling odsolve with the ODE`, diff(y(x), x) = (2*y(x)*x^3-8*y(x)^3)/(x^4-4*y(x)^2*x),
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 29

```
dsolve(diff(y(x),x)^3-4*x*y(x)*diff(y(x),x)+8*y(x)^2=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= \frac{4x^3}{27} \\
y(x) &= 0 \\
y(x) &= \frac{(4c_1x-1)^2}{64c_1^3}
\end{aligned}
$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(y'[x])^3-4*x*y[x]*y'[x]+8*y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

16 Chapter IV, differential equations of the first order and higher degree than the first. Article 27. Clairaut equation. Page 56

16.1 problem Ex 1

Internal problem ID [11216]
Internal file name [OUTPUT/10201_Tuesday_December_06_2022_04_00_10_AM_84846167/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 27. Clairaut equation. Page 56
Problem number: Ex 1.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational, _Clairaut]
```

$$\boxed{(y'x-y)^2-y'^2=1}$$

16.1.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y=y'x+g(y')$$

Where g is function of $y'(x)$. Let $p=y'$ the ode becomes

$$(px-y)^2-p^2=1$$

Solving for y from the above results in

$$y=px+\sqrt{p^2+1} \tag{1A}$$
$$y=px-\sqrt{p^2+1} \tag{2A}$$

Each of the above ode's is a Clairaut ode which is now solved. Solving ode 1A We start by replacing y' by p which gives

$$\begin{aligned} y &= px+\sqrt{p^2+1} \\ &= px+\sqrt{p^2+1} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = \sqrt{p^2 + 1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \sqrt{c_1^2 + 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \sqrt{p^2 + 1}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x + \frac{p}{\sqrt{p^2 + 1}} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = -x\sqrt{-\frac{1}{x^2-1}}$$

Substituting the above back in (1) results in

$$y_1 = \left(-x^2+1\right)\sqrt{-\frac{1}{x^2-1}}$$

Solving ode 2A We start by replacing y' by p which gives

$$\begin{aligned} y &= px - \sqrt{p^2+1} \\ &= px - \sqrt{p^2+1} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = -\sqrt{p^2+1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp+g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x+g')\frac{dp}{dx} \\ 0 &= (x+g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_2 x - \sqrt{c_2^2 + 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -\sqrt{p^2 + 1}$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x - \frac{p}{\sqrt{p^2 + 1}} \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = x\sqrt{-\frac{1}{x^2 - 1}}$$

Substituting the above back in (1) results in

$$y_1 = \sqrt{-\frac{1}{x^2 - 1}}\,(x^2 - 1)$$

Summary

The solution(s) found are the following

$$y = c_1 x + \sqrt{c_1^2 + 1} \tag{1}$$

$$y = (-x^2 + 1)\sqrt{-\frac{1}{x^2 - 1}} \tag{2}$$

$$y = c_2 x - \sqrt{c_2^2 + 1} \tag{3}$$

$$y = \sqrt{-\frac{1}{x^2 - 1}}\,(x^2 - 1) \tag{4}$$

Verification of solutions

$$y = c_1 x + \sqrt{c_1^2 + 1}$$

Verified OK.

$$y = (-x^2 + 1)\sqrt{-\frac{1}{x^2 - 1}}$$

Verified OK.

$$y = c_2 x - \sqrt{c_2^2 + 1}$$

Verified OK.

$$y = \sqrt{-\frac{1}{x^2 - 1}}\,(x^2 - 1)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 57

```
dsolve((diff(y(x),x)*x-y(x))^2=diff(y(x),x)^2+1,y(x), singsol=all)
```

$$y(x) = \sqrt{-x^2+1}$$
$$y(x) = -\sqrt{-x^2+1}$$
$$y(x) = c_1 x - \sqrt{c_1^2+1}$$
$$y(x) = c_1 x + \sqrt{c_1^2+1}$$

✓ Solution by Mathematica

Time used: 0.192 (sec). Leaf size: 73

```
DSolve[(y'[x]*x-y[x])^2==(y'[x])^2+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \sqrt{1+c_1{}^2}$$
$$y(x) \to c_1 x + \sqrt{1+c_1{}^2}$$
$$y(x) \to -\sqrt{1-x^2}$$
$$y(x) \to \sqrt{1-x^2}$$

16.2 problem Ex 2

Internal problem ID [11217]
Internal file name [OUTPUT/10202_Tuesday_December_06_2022_04_00_12_AM_28627475/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 27. Clairaut equation. Page 56
Problem number: Ex 2.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(y)]`]]
```

Unable to solve or complete the solution.

$$\boxed{4\,\mathrm{e}^{2y}{y'}^{2}+2y'x=1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\mathrm{e}^{-2y}\left(-x+\sqrt{x^{2}+4\,\mathrm{e}^{2y}}\right)}{4} \tag{1}$$

$$y' = -\frac{\left(x+\sqrt{x^{2}+4\,\mathrm{e}^{2y}}\right)\mathrm{e}^{-2y}}{4} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Unable to determine ODE type.

Unable to determine ODE type.

Solving equation (2)

Unable to determine ODE type.

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 2
-> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
trying dAlembert
-> Calling odsolve with the ODE`, diff(y(x), x) = (-4*exp(2*y(x))*x^2-1)/(8*exp(2*y(x))*x^3+
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   <- quadrature successful
<- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 23

```
dsolve(4*exp(2*y(x))*diff(y(x),x)^2+2*x*diff(y(x),x)-1=0,y(x), singsol=all)
```

$$y(x) = \frac{\ln(2)}{2} - \frac{\ln\left(\frac{1}{2\,e^{2c_1}+x}\right)}{2} + c_1$$

✓ Solution by Mathematica

Time used: 12.616 (sec). Leaf size: 119

```
DSolve[4*Exp[2*y[x]]*(y'[x])^2+2*x*y'[x]-1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log \left(-e^{\frac{c_1}{2}}\sqrt{-x+e^{c_1}}\right)$$
$$y(x) \to \log \left(e^{\frac{c_1}{2}}\sqrt{-x+e^{c_1}}\right)$$
$$y(x) \to \log \left(-e^{\frac{c_1}{2}}\sqrt{x+e^{c_1}}\right)$$
$$y(x) \to \log \left(e^{\frac{c_1}{2}}\sqrt{x+e^{c_1}}\right)$$
$$y(x) \to \frac{1}{2}\log \left(-\frac{x^2}{4}\right)$$

16.3 problem Ex 3

Internal problem ID [11218]
Internal file name [`OUTPUT/10203_Tuesday_December_06_2022_04_00_14_AM_37147350/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 27. Clairaut equation. Page 56
Problem number: Ex 3.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{4\,\mathrm{e}^{2y}{y'}^{2}+2\,\mathrm{e}^{2x}y'=\mathrm{e}^{2x}}$$

16.3.1 Solving as dAlembert ode

Let $p=y'$ the ode becomes

$$4\,\mathrm{e}^{2y}p^{2}+2\,\mathrm{e}^{2x}p=\mathrm{e}^{2x}$$

Solving for y from the above results in

$$y=\frac{\ln\left(-\frac{2p-1}{4p^{2}}\right)}{2}+x \tag{1A}$$

This has the form

$$y=xf(p)+g(p) \tag{*}$$

Where f,g are functions of $p=y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p=f+(xf'+g')\frac{dp}{dx}$$
$$p-f=(xf'+g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= 1 \\
g &= -\ln(2) + \frac{\ln\left(\frac{-2p+1}{p^2}\right)}{2}
\end{aligned}
$$

Hence (2) becomes

$$
p - 1 = \frac{\left(-\frac{2}{p^2} - \frac{2(-2p+1)}{p^3}\right) p^2 p'(x)}{-4p + 2} \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - 1 = 0
$$

Solving for p from the above gives

$$
p = 1
$$

Substituting these in (1A) gives

$$
y = x - \ln(2) + \frac{i\pi}{2}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{2(p(x) - 1)(-2p(x) + 1)}{\left(-\frac{2}{p(x)^2} - \frac{2(-2p(x)+1)}{p(x)^3}\right) p(x)^2} \tag{3}
$$

This ODE is now solved for $p(x)$. Integrating both sides gives

$$
\begin{aligned}
\int -\frac{1}{(2p-1)\,p} dp &= \int dx \\
\ln(p) - \ln(2p - 1) &= x + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{\ln(p) - \ln(2p-1)} = \mathrm{e}^{x + c_1}
$$

Which simplifies to

$$
\frac{p}{2p - 1} = c_2 \mathrm{e}^{x}
$$

Substituing the above solution for p in (2A) gives

$$y = x - \ln(2) + \frac{\ln\left(\frac{\left(-\frac{2c_2 e^x}{-1+2c_2 e^x}+1\right)e^{-2x}(-1+2c_2 e^x)^2}{c_2^2}\right)}{2}$$

Summary

The solution(s) found are the following

$$y = x - \ln(2) + \frac{i\pi}{2} \tag{1}$$

$$y = x - \ln(2) + \frac{\ln\left(\frac{\left(-\frac{2c_2 e^x}{-1+2c_2 e^x}+1\right)e^{-2x}(-1+2c_2 e^x)^2}{c_2^2}\right)}{2} \tag{2}$$

Verification of solutions

$$y = x - \ln(2) + \frac{i\pi}{2}$$

Verified OK.

$$y = x - \ln(2) + \frac{\ln\left(\frac{\left(-\frac{2c_2 e^x}{-1+2c_2 e^x}+1\right)e^{-2x}(-1+2c_2 e^x)^2}{c_2^2}\right)}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous C
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous C
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.171 (sec). Leaf size: 87

```
dsolve(4*exp(2*y(x))*diff(y(x),x)^2+2*exp(2*x)*diff(y(x),x)-exp(2*x)=0,y(x), singsol=all)
```

$$y(x) = \operatorname{arctanh}\left(\operatorname{RootOf}\left(-1+\left(e^{4}+4\,e^{\operatorname{RootOf}\left(-4\,e^{-Z}\sinh\left(-\frac{\_Z}{2}+2+c_1-x\right)^{2}+e^{4}\right)}\right)\_Z^{2}\right)e^{2}\right)+c_1$$

$$y(x) = -\operatorname{arctanh}\left(\operatorname{RootOf}\left(-1+\left(e^{4}+4\,e^{\operatorname{RootOf}\left(-4\,e^{-Z}\sinh\left(-\frac{\_Z}{2}+2+c_1-x\right)^{2}+e^{4}\right)}\right)\_Z^{2}\right)e^{2}\right)+c_1$$

✓ Solution by Mathematica

Time used: 2.772 (sec). Leaf size: 332

```
DSolve[4*Exp[2*y[x]]*(y'[x])^2+2*Exp[2*x]*y'[x]-Exp[2*x]==0,y[x],x,IncludeSingularSolutions
```

$$\text{Solve}\left[-\frac{2e^{-x}\sqrt{4e^{2(y(x)+x)}+e^{4x}}\operatorname{arctanh}\left(\frac{-\sqrt{4e^{2y(x)}+e^{2x}}+e^{x}+1}{\sqrt{4e^{2y(x)}+e^{2x}}-e^{x}+1}\right)}{\sqrt{4e^{2y(x)}+e^{2x}}} - \frac{e^{-x}\sqrt{4e^{2(y(x)+x)}+e^{4x}}\,y(x)}{\sqrt{4e^{2y(x)}+e^{2x}}}+y(x)=c_1,y(x)\right]$$

$$\text{Solve}\left[\frac{2e^{-x}\sqrt{4e^{2(y(x)+x)}+e^{4x}}\operatorname{arctanh}\left(\frac{-\sqrt{4e^{2y(x)}+e^{2x}}+e^{x}+1}{\sqrt{4e^{2y(x)}+e^{2x}}-e^{x}+1}\right)}{\sqrt{4e^{2y(x)}+e^{2x}}} + \frac{e^{-x}\sqrt{4e^{2(y(x)+x)}+e^{4x}}\,y(x)}{\sqrt{4e^{2y(x)}+e^{2x}}}+y(x)=c_1,y(x)\right]$$

$$y(x)\to\frac{1}{2}\left(\log\left(-\frac{e^{4x}}{4}\right)-2x\right)$$

16.4 problem Ex 4

Internal problem ID [11219]
Internal file name [`OUTPUT/10204_Tuesday_December_06_2022_04_00_16_AM_95578631/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 27. Clairaut equation. Page 56
Problem number: Ex 4.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`` [`y=_G(x,y')`] ``

Unable to solve or complete the solution.

$$
\boxed{e^{2y}{y'}^{3}+\left(e^{2x}+e^{3x}\right)y' = e^{3x}}
$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{e^{-2y}\left(\left(12\sqrt{3}\,\sqrt{\left(27\,e^{6x}e^{2y} + 4\,e^{6x} + 12\,e^{4x}e^{3x} + 12\,e^{2x}e^{6x} + 4\,e^{9x}\right)e^{-2y}} + 108\,e^{3x}\right)e^{4y}\right)^{\frac{1}{3}}}{6} - \frac{}{\left(\left(12\sqrt{3}\,\sqrt{\cdots}\right.\right.} \tag{1}$$

$$y' = -\frac{e^{-2y}\left(\left(12\sqrt{3}\,\sqrt{\left(27\,e^{6x}e^{2y} + 4\,e^{6x} + 12\,e^{4x}e^{3x} + 12\,e^{2x}e^{6x} + 4\,e^{9x}\right)e^{-2y}} + 108\,e^{3x}\right)e^{4y}\right)^{\frac{1}{3}}}{12} + \frac{}{\left(\left(12\sqrt{3}\cdots\right.\right.} \tag{2}$$

$$y' = -\frac{e^{-2y}\left(\left(12\sqrt{3}\,\sqrt{\left(27\,e^{6x}e^{2y} + 4\,e^{6x} + 12\,e^{4x}e^{3x} + 12\,e^{2x}e^{6x} + 4\,e^{9x}\right)e^{-2y}} + 108\,e^{3x}\right)e^{4y}\right)^{\frac{1}{3}}}{12} + \frac{}{\left(\left(12\sqrt{3}\cdots\right.\right.} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Unable to determine ODE type.

Unable to determine ODE type.

Solving equation (2)

Unable to determine ODE type.

Unable to determine ODE type.

Solving equation (3)

Unable to determine ODE type.

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 2
-> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
trying dAlembert
-> Calling odsolve with the ODE`, diff(y(x), x) = (-2*x*exp(2*y(x))-2*x*exp(3*y(x))+3*exp(3*
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   <- quadrature successful
<- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 1.188 (sec). Leaf size: 24

```
dsolve(exp(2*y(x))*diff(y(x),x)^3+(exp(2*x)+exp(3*x))*diff(y(x),x)-exp(3*x)=0,y(x), singsol=
```

$$y(x) = \frac{\ln\left(-(c_1+1)\left(c_1 e^{-x}-1\right)^2\right)}{2} + x$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[Exp[2*y[x]]*(y'[x])^3+(Exp[2*x]+Exp[3*x])*y'[x]-Exp[3*x]==0,y[x],x,IncludeSingularSol
```

Timed out

16.5 problem Ex 5

Internal problem ID [11220]
Internal file name [`OUTPUT/10205_Wednesday_December_07_2022_01_20_24_PM_9550685/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 27. Clairaut equation. Page 56
Problem number: Ex 5.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{xy^2y'^2 - y^3y' = -x}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{y^2 + \sqrt{y^4 - 4x^2}}{2xy} \tag{1}$$
$$y' = \frac{y^2 - \sqrt{y^4 - 4x^2}}{2xy} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Writing the ode as

$$y' = \frac{y^2 + \sqrt{y^4 - 4x^2}}{2xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(y^2 + \sqrt{y^4 - 4x^2}\right)(b_3 - a_2)}{2xy} - \frac{\left(y^2 + \sqrt{y^4 - 4x^2}\right)^2 a_3}{4x^2 y^2} - \left(-\frac{y^2 + \sqrt{y^4 - 4x^2}}{2x^2 y} - \frac{2}{\sqrt{y^4 - 4x^2}\, y}\right)(xa_2 + ya_3 + a_1) - \left(\frac{2y + \frac{2y^3}{\sqrt{y^4 - 4x^2}}}{2xy} - \frac{y^2 + \sqrt{y^4 - 4x^2}}{2x\, y^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2y^4b_2 - 2b_2x^2y^2\sqrt{y^4 - 4x^2} - \sqrt{y^4 - 4x^2}\, y^4 a_3 + 2x\, y^4 b_1 - 2y^5 a_1 + 2\sqrt{y^4 - 4x^2}\, x\, y^2 b_1 - 2\sqrt{y^4 - 4x^2}\, y^3}{4x^2y^2\sqrt{y^4 - 4x^2}} = 0$$

Setting the numerator to zero gives

$$-2x^2y^4b_2 + 2b_2x^2y^2\sqrt{y^4 - 4x^2} + \sqrt{y^4 - 4x^2}\, y^4 a_3 - 2x\, y^4 b_1 + 2y^5 a_1 - 2\sqrt{y^4 - 4x^2}\, x\, y^2 b_1 + 2\sqrt{y^4 - 4x^2}\, y^3 a_1 - 8x^4 b_2 + 8x^3 y a_2 - 16x^3 y b_3 + 8x^2 y^2 a_3 - \left(y^4 - 4x^2\right)^{\frac{3}{2}} a_3 - 8x^3 b_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-4x^2y^4b_2 - 4x\, y^5 b_3 + 2b_2x^2y^2\sqrt{y^4 - 4x^2} + \sqrt{y^4 - 4x^2}\, y^4 a_3 - 4x\, y^4 b_1 + 2\left(y^4 - 4x^2\right) x^2 b_2 + 4\left(y^4 - 4x^2\right) xyb_3 - 2\sqrt{y^4 - 4x^2}\, x\, y^2 b_1 + 2\sqrt{y^4 - 4x^2}\, y^3 a_1 + 8x^3 y a_2 + 8x^2 y^2 a_3 - \left(y^4 - 4x^2\right)^{\frac{3}{2}} a_3 + 2\left(y^4 - 4x^2\right) xb_1 + 2\left(y^4 - 4x^2\right) ya_1 + 8x^2 y a_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-2x^2y^4b_2 + 2b_2x^2y^2\sqrt{y^4-4x^2} - 2x\,y^4b_1 + 2y^5a_1 - 8x^4b_2 + 8x^3ya_2 - 16x^3yb_3 + 8x^2y^2a_3 - 2\sqrt{y^4-4x^2}\,x\,y^2b_1 + 2\sqrt{y^4-4x^2}\,y^3a_1 - 8x^3b_1 + 4x^2\sqrt{y^4-4x^2}\,a_3 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{y^4-4x^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{y^4-4x^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$-2v_1^2v_2^4b_2 + 2v_2^5a_1 - 2v_1v_2^4b_1 + 2b_2v_1^2v_2^2v_3 + 2v_3v_2^3a_1 + 8v_1^3v_2a_2 + 8v_1^2v_2^2a_3 - 2v_3v_1v_2^2b_1 - 8v_1^4b_2 - 16v_1^3v_2b_3 + 4v_1^2v_3a_3 - 8v_1^3b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$-8v_1^4b_2 + (8a_2 - 16b_3)\,v_1^3v_2 - 8v_1^3b_1 - 2v_1^2v_2^4b_2 + 2b_2v_1^2v_2^2v_3 + 8v_1^2v_2^2a_3 + 4v_1^2v_3a_3 - 2v_1v_2^4b_1 - 2v_3v_1v_2^2b_1 + 2v_2^5a_1 + 2v_3v_2^3a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
2a_1 &= 0\\
4a_3 &= 0\\
8a_3 &= 0\\
-8b_1 &= 0\\
-2b_1 &= 0\\
-8b_2 &= 0\\
-2b_2 &= 0\\
2b_2 &= 0\\
8a_2 - 16b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{y^2 + \sqrt{y^4 - 4x^2}}{2xy}\right)(2x) \\
&= -\frac{\sqrt{y^4 - 4x^2}}{y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{\sqrt{y^4 - 4x^2}}{y}} dy
\end{aligned}
$$

Which results in

$$S = -\frac{\ln\left(y^2 + \sqrt{y^4 - 4x^2}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^2 + \sqrt{y^4 - 4x^2}}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x}{\sqrt{y^4 - 4x^2}\,\left(y^2 + \sqrt{y^4 - 4x^2}\right)} \\ S_y &= -\frac{y}{\sqrt{y^4 - 4x^2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{-y^4 - \sqrt{y^4 - 4x^2}\, y^2 + 4x^2}{x\sqrt{y^4 - 4x^2}\,\left(y^2 + \sqrt{y^4 - 4x^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln\left(y^2 + \sqrt{y^4 - 4x^2}\right)}{2} = -\ln(x) + c_1$$

Which simplifies to

$$-\frac{\ln\left(y^2+\sqrt{y^4-4x^2}\right)}{2} = -\ln\left(x\right)+c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln\left(y^2+\sqrt{y^4-4x^2}\right)}{2} = -\ln\left(x\right)+c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln\left(y^2+\sqrt{y^4-4x^2}\right)}{2} = -\ln\left(x\right)+c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{y^2-\sqrt{y^4-4x^2}}{2xy}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 + \frac{\left(y^2 - \sqrt{y^4 - 4x^2}\right)(b_3 - a_2)}{2xy} - \frac{\left(y^2 - \sqrt{y^4 - 4x^2}\right)^2 a_3}{4x^2 y^2} \\
& - \left(-\frac{y^2 - \sqrt{y^4 - 4x^2}}{2x^2 y} + \frac{2}{\sqrt{y^4 - 4x^2}\, y}\right)(x a_2 + y a_3 + a_1) \\
& - \left(\frac{2y - \frac{2y^3}{\sqrt{y^4-4x^2}}}{2xy} - \frac{y^2 - \sqrt{y^4 - 4x^2}}{2x\, y^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-2x^2 y^4 b_2 - 2b_2 x^2 y^2 \sqrt{y^4 - 4x^2} - \sqrt{y^4 - 4x^2}\, y^4 a_3 - 2x\, y^4 b_1 + 2y^5 a_1 + 2\sqrt{y^4 - 4x^2}\, x\, y^2 b_1 - 2\sqrt{y^4 - 4x^2}}{4x^2 y^2 \sqrt{y^4 - 4x^2}} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& 2x^2 y^4 b_2 + 2b_2 x^2 y^2 \sqrt{y^4 - 4x^2} + \sqrt{y^4 - 4x^2}\, y^4 a_3 + 2x\, y^4 b_1 \\
& - 2y^5 a_1 - 2\sqrt{y^4 - 4x^2}\, x\, y^2 b_1 + 2\sqrt{y^4 - 4x^2}\, y^3 a_1 + 8x^4 b_2 \\
& - 8x^3 y a_2 + 16x^3 y b_3 - 8x^2 y^2 a_3 - \left(y^4 - 4x^2\right)^{\frac{3}{2}} a_3 + 8x^3 b_1 = 0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
& 4x^2 y^4 b_2 + 4x\, y^5 b_3 + 2b_2 x^2 y^2 \sqrt{y^4 - 4x^2} + \sqrt{y^4 - 4x^2}\, y^4 a_3 + 4x\, y^4 b_1 \\
& - 2\left(y^4 - 4x^2\right) x^2 b_2 - 4\left(y^4 - 4x^2\right) xy b_3 - 2\sqrt{y^4 - 4x^2}\, x\, y^2 b_1 \\
& + 2\sqrt{y^4 - 4x^2}\, y^3 a_1 - 8x^3 y a_2 - 8x^2 y^2 a_3 - \left(y^4 - 4x^2\right)^{\frac{3}{2}} a_3 \\
& - 2\left(y^4 - 4x^2\right) x b_1 - 2\left(y^4 - 4x^2\right) y a_1 - 8x^2 y a_1 = 0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
& 2x^2 y^4 b_2 + 2b_2 x^2 y^2 \sqrt{y^4 - 4x^2} + 2x\, y^4 b_1 - 2y^5 a_1 + 8x^4 b_2 - 8x^3 y a_2 + 16x^3 y b_3 \\
& - 8x^2 y^2 a_3 - 2\sqrt{y^4 - 4x^2}\, x\, y^2 b_1 + 2\sqrt{y^4 - 4x^2}\, y^3 a_1 + 8x^3 b_1 + 4x^2 \sqrt{y^4 - 4x^2}\, a_3 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{y^4 - 4x^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{y^4 - 4x^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$2v_1^2v_2^4b_2 - 2v_2^5a_1 + 2v_1v_2^4b_1 + 2b_2v_1^2v_2^2v_3 + 2v_3v_2^3a_1 - 8v_1^3v_2a_2 \\ - 8v_1^2v_2^2a_3 - 2v_3v_1v_2^2b_1 + 8v_1^4b_2 + 16v_1^3v_2b_3 + 4v_1^2v_3a_3 + 8v_1^3b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$8v_1^4b_2 + (-8a_2 + 16b_3)\, v_1^3v_2 + 8v_1^3b_1 + 2v_1^2v_2^4b_2 + 2b_2v_1^2v_2^2v_3 \\ - 8v_1^2v_2^2a_3 + 4v_1^2v_3a_3 + 2v_1v_2^4b_1 - 2v_3v_1v_2^2b_1 - 2v_2^5a_1 + 2v_3v_2^3a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_1 &= 0 \\ 2a_1 &= 0 \\ -8a_3 &= 0 \\ 4a_3 &= 0 \\ -2b_1 &= 0 \\ 2b_1 &= 0 \\ 8b_1 &= 0 \\ 2b_2 &= 0 \\ 8b_2 &= 0 \\ -8a_2 + 16b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 2x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y^2 - \sqrt{y^4 - 4x^2}}{2xy}\right)(2x) \\
&= \frac{\sqrt{y^4 - 4x^2}}{y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{\sqrt{y^4 - 4x^2}}{y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(y^2 + \sqrt{y^4 - 4x^2}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2 - \sqrt{y^4 - 4x^2}}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2x}{\sqrt{y^4 - 4x^2}\,\left(y^2 + \sqrt{y^4 - 4x^2}\right)} \\
S_y &= \frac{y}{\sqrt{y^4 - 4x^2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(y^2 + \sqrt{y^4 - 4x^2}\right)}{2} = c_1$$

Which simplifies to

$$\frac{\ln\left(y^2 + \sqrt{y^4 - 4x^2}\right)}{2} = c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(y^2 + \sqrt{y^4 - 4x^2}\right)}{2} = c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln\left(y^2+\sqrt{y^4-4x^2}\right)}{2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      trying an integrating factor from the invariance group
      <- integrating factor successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      trying an integrating factor from the invariance group
      <- integrating factor successful
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.188 (sec). Leaf size: 140

```
dsolve(x*y(x)^2*diff(y(x),x)^2-y(x)^3*diff(y(x),x)+x=0,y(x), singsol=all)
```

$$y(x) = \sqrt{2}\,\sqrt{-x}$$
$$y(x) = -\sqrt{2}\,\sqrt{-x}$$
$$y(x) = \sqrt{2}\,\sqrt{x}$$
$$y(x) = -\sqrt{2}\,\sqrt{x}$$
$$y(x) = \frac{e^{\frac{c_1}{2}+\frac{\operatorname{RootOf}\left(16x\,e^{2\_Z+2c_1}+e^{2\_Z}x^3-4\,e^{3\_Z+2c_1}\right)}{2}}}{\sqrt{x}}$$
$$y(x) = \sqrt{x}\,e^{-\frac{c_1}{2}+\frac{\operatorname{RootOf}\left(x^2\left(16x^2e^{2\_Z-2c_1}+e^{2\_Z}-4\,e^{3\_Z-2c_1}x\right)\right)}{2}}$$

✓ Solution by Mathematica

Time used: 6.367 (sec). Leaf size: 187

```
DSolve[x*y[x]^2*(y'[x])^2-y[x]^3*y'[x]+x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-2e^{-c_1}x^2-\frac{e^{c_1}}{2}}$$
$$y(x) \to \sqrt{-2e^{-c_1}x^2-\frac{e^{c_1}}{2}}$$
$$y(x) \to -\frac{\sqrt{4e^{-c_1}x^2+e^{c_1}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{4e^{-c_1}x^2+e^{c_1}}}{\sqrt{2}}$$
$$y(x) \to -\sqrt{2}\sqrt{x}$$
$$y(x) \to -i\sqrt{2}\sqrt{x}$$
$$y(x) \to i\sqrt{2}\sqrt{x}$$
$$y(x) \to \sqrt{2}\sqrt{x}$$

16.6 problem Ex 6

Internal problem ID [11221]
Internal file name [OUTPUT/10206_Wednesday_December_07_2022_01_20_25_PM_31494725/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 27. Clairaut equation. Page 56
Problem number: Ex 6.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_homogeneous, `class A`], _dAlembert]`

$$\boxed{(x^2+y^2)\left(1+y'\right)^2-2(y+x)\left(1+y'\right)\left(x+yy'\right)+\left(x+yy'\right)^2=0}$$

16.6.1 Solving as dAlembert ode

Let $p=y'$ the ode becomes

$$(x^2+y^2)(1+p)^2-2(y+x)(1+p)(yp+x)+(yp+x)^2=0$$

Solving for y from the above results in

$$y=\left(p^2+p+1+\sqrt{p^4+2p^3+2p^2+2p+1}\right)x \tag{1A}$$

$$y=\left(p^2+p+1-\sqrt{p^4+2p^3+2p^2+2p+1}\right)x \tag{2A}$$

This has the form

$$y=xf(p)+g(p) \tag{*}$$

Where f,g are functions of $p=y'(x)$. Each of the above ode's is dAlembert ode which is now solved. Solving ode 1A Taking derivative of (*) w.r.t. x gives

$$p=f+(xf'+g')\frac{dp}{dx}$$

$$p-f=(xf'+g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= p^2 + p + 1 + \sqrt{(p^2+1)(1+p)^2} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
-p^2 - 1 - \sqrt{(p^2+1)(1+p)^2} = x\left(2p + 1 + \frac{2p(1+p)^2 + 2(p^2+1)(1+p)}{2\sqrt{(p^2+1)(1+p)^2}}\right)p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-p^2 - 1 - \sqrt{(p^2+1)(1+p)^2} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -ix \\
y &= ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{-p(x)^2 - 1 - \sqrt{\left(p(x)^2+1\right)(1+p(x))^2}}{x\left(2p(x) + 1 + \frac{2p(x)(1+p(x))^2 + 2\left(p(x)^2+1\right)(1+p(x))}{2\sqrt{\left(p(x)^2+1\right)(1+p(x))^2}}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(2p + 1 + \frac{2p(1+p)^2 + 2(p^2+1)(1+p)}{2\sqrt{(p^2+1)(1+p)^2}}\right)}{-p^2 - 1 - \sqrt{(p^2+1)(1+p)^2}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{(-2p-1)\sqrt{(p^2+1)(1+p)^2} - 2p^3 - 3p^2 - 2p - 1}{\sqrt{(p^2+1)(1+p)^2}\left(p^2+1+\sqrt{(p^2+1)(1+p)^2}\right)}$$

$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{\left((-2p-1)\sqrt{(p^2+1)(1+p)^2} - 2p^3 - 3p^2 - 2p - 1\right)x(p)}{\sqrt{(p^2+1)(1+p)^2}\left(p^2+1+\sqrt{(p^2+1)(1+p)^2}\right)} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{(-2p-1)\sqrt{(p^2+1)(1+p)^2}-2p^3-3p^2-2p-1}{\sqrt{(p^2+1)(1+p)^2}\left(p^2+1+\sqrt{(p^2+1)(1+p)^2}\right)}dp}$$

The ode becomes

$$\frac{d}{dp}\mu x = 0$$

$$\frac{d}{dp}\left(e^{\int -\frac{(-2p-1)\sqrt{(p^2+1)(1+p)^2}-2p^3-3p^2-2p-1}{\sqrt{(p^2+1)(1+p)^2}\left(p^2+1+\sqrt{(p^2+1)(1+p)^2}\right)}dp}x\right) = 0$$

Integrating gives

$$e^{\int -\frac{(-2p-1)\sqrt{(p^2+1)(1+p)^2}-2p^3-3p^2-2p-1}{\sqrt{(p^2+1)(1+p)^2}\left(p^2+1+\sqrt{(p^2+1)(1+p)^2}\right)}dp}x = c_2$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{(-2p-1)\sqrt{(p^2+1)(1+p)^2}-2p^3-3p^2-2p-1}{\sqrt{(p^2+1)(1+p)^2}\left(p^2+1+\sqrt{(p^2+1)(1+p)^2}\right)}dp}$ results in

$$x(p) = c_2 e^{-\left(\int \frac{(2p+1)\sqrt{(p^2+1)(1+p)^2}+2p^3+3p^2+2p+1}{\sqrt{(p^2+1)(1+p)^2}\left(p^2+1+\sqrt{(p^2+1)(1+p)^2}\right)}dp\right)}$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Solving ode 2A Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= p^2 + p + 1 - \sqrt{(p^2+1)(1+p)^2} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
-p^2 - 1 + \sqrt{(p^2+1)(1+p)^2} = x\left(2p + 1 - \frac{2p(1+p)^2 + 2(p^2+1)(1+p)}{2\sqrt{(p^2+1)(1+p)^2}}\right)p'(x) \qquad (2A)
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-p^2 - 1 + \sqrt{(p^2+1)(1+p)^2} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 0 \\
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= 0 \\
y &= -ix \\
y &= ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{-p(x)^2 - 1 + \sqrt{(p(x)^2+1)(1+p(x))^2}}{x\left(2p(x) + 1 - \frac{2p(x)(1+p(x))^2 + 2\left(p(x)^2+1\right)(1+p(x))}{2\sqrt{\left(p(x)^2+1\right)(1+p(x))^2}}\right)} \qquad (3)
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\left(2p+1-\frac{2p(1+p)^2+2(p^2+1)(1+p)}{2\sqrt{(p^2+1)(1+p)^2}}\right)}{-p^2-1+\sqrt{(p^2+1)(1+p)^2}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{-2p^3+2p\sqrt{(p^2+1)(1+p)^2}-3p^2+\sqrt{(p^2+1)(1+p)^2}-2p-1}{\left(-p^2-1+\sqrt{(p^2+1)(1+p)^2}\right)\sqrt{(p^2+1)(1+p)^2}}$$

$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{x(p)\left(-2p^3+2p\sqrt{(p^2+1)(1+p)^2}-3p^2+\sqrt{(p^2+1)(1+p)^2}-2p-1\right)}{\sqrt{(p^2+1)(1+p)^2}\left(-p^2-1+\sqrt{(p^2+1)(1+p)^2}\right)} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-2p^3+2p\sqrt{(p^2+1)(1+p)^2}-3p^2+\sqrt{(p^2+1)(1+p)^2}-2p-1}{\left(-p^2-1+\sqrt{(p^2+1)(1+p)^2}\right)\sqrt{(p^2+1)(1+p)^2}}dp}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(e^{\int -\frac{-2p^3+2p\sqrt{(p^2+1)(1+p)^2}-3p^2+\sqrt{(p^2+1)(1+p)^2}-2p-1}{\left(-p^2-1+\sqrt{(p^2+1)(1+p)^2}\right)\sqrt{(p^2+1)(1+p)^2}}dp}x\right) = 0$$

Integrating gives

$$e^{\int -\frac{-2p^3+2p\sqrt{(p^2+1)(1+p)^2}-3p^2+\sqrt{(p^2+1)(1+p)^2}-2p-1}{\left(-p^2-1+\sqrt{(p^2+1)(1+p)^2}\right)\sqrt{(p^2+1)(1+p)^2}}dp}x = c_4$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{-2p^3+2p\sqrt{(p^2+1)(1+p)^2}-3p^2+\sqrt{(p^2+1)(1+p)^2}-2p-1}{\left(-p^2-1+\sqrt{(p^2+1)(1+p)^2}\right)\sqrt{(p^2+1)(1+p)^2}}dp}$ results in

$$x(p) = c_4 \mathrm{e}^{\int \frac{-2p^3+2p\sqrt{(p^2+1)(1+p)^2}-3p^2+\sqrt{(p^2+1)(1+p)^2}-2p-1}{\left(-p^2-1+\sqrt{(p^2+1)(1+p)^2}\right)\sqrt{(p^2+1)(1+p)^2}}dp}$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Summary

The solution(s) found are the following

$$y = -ix \tag{1}$$
$$y = ix \tag{2}$$
$$y = 0 \tag{3}$$
$$y = -ix \tag{4}$$
$$y = ix \tag{5}$$

Verification of solutions

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$y = 0$$

Verified OK.

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   <- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 103

```
dsolve((x^2+y(x)^2)*(1+diff(y(x),x))^2-2*(x+y(x))*(1+diff(y(x),x))*(x+y(x)*diff(y(x),x))+(x+
```

$$y(x) = 0$$

$$y(x) = \text{RootOf}\left(-2\ln(x) - \left(\int^{\_Z} \frac{2\_a^2 + \sqrt{2}\sqrt{\_a(\_a-1)^2}}{\_a(\_a^2+1)} d\_a\right) + 2c_1\right) x$$

$$y(x) = \text{RootOf}\left(-2\ln(x) + \int^{\_Z} \frac{\sqrt{2}\sqrt{\_a(\_a-1)^2} - 2\_a^2}{\_a(\_a^2+1)} d\_a + 2c_1\right) x$$

✓ Solution by Mathematica

Time used: 7.379 (sec). Leaf size: 167

```
DSolve[(x^2+y[x]^2)*(1+y'[x])^2-2*(x+y[x])*(1+y'[x])*(x+y[x]*y'[x])+(x+y[x]*y'[x])^2==0,y[x]
```

$$y(x) \to -\sqrt{-x\left(x+2e^{\frac{c_1}{2}}\right)} - e^{\frac{c_1}{2}}$$
$$y(x) \to \sqrt{-x\left(x+2e^{\frac{c_1}{2}}\right)} - e^{\frac{c_1}{2}}$$
$$y(x) \to e^{\frac{c_1}{2}} - \sqrt{x\left(-x+2e^{\frac{c_1}{2}}\right)}$$
$$y(x) \to \sqrt{x\left(-x+2e^{\frac{c_1}{2}}\right)} + e^{\frac{c_1}{2}}$$
$$y(x) \to -\sqrt{-x^2}$$
$$y(x) \to \sqrt{-x^2}$$

16.7 problem Ex 7

Internal problem ID [11222]
Internal file name [`OUTPUT/10207_Wednesday_December_07_2022_01_20_29_PM_51821075/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 27. Clairaut equation. Page 56
Problem number: Ex 7.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{y - 2y'x - y^2 {y'}^3 = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}{6y} - \frac{4x}{y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}} \tag{1}$$

$$y' = -\frac{\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}{12y} + \frac{2x}{y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right.}{6y}\right.}{} \tag{2}$$

$$y' = -\frac{\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}{12y} + \frac{2x}{y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right.}{6y}\right.}{} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}} - 24x}{6y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\frac{\left(\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)(b_3-a_2)}{6y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}$$
$$-\frac{\left(\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)^2 a_3}{36y^2\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}}$$
$$-\left(\frac{\frac{384\sqrt{3}\,x^2}{\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}\sqrt{27y^4+32x^3}}-24}{6y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}\right.$$
$$\left.-\frac{32\left(\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)\sqrt{3}\,x^2}{y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}}\sqrt{27y^4+32x^3}}\right)(xa_2+ya_3+a_1) \tag{5E}$$
$$-\left(\frac{216y+\frac{648\sqrt{3}\,y^3}{\sqrt{27y^4+32x^3}}}{9y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}}\right.$$
$$-\frac{\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x}{6y^2\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}$$
$$\left.-\frac{\left(\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)\left(216y+\frac{648\sqrt{3}\,y^3}{\sqrt{27y^4+32x^3}}\right)}{18y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}}}\right)(xb_2$$
$$+yb_3+b_1)=0$$

Putting the above in normal form gives

$$-\frac{72(27y^4+32x^3)^{\frac{3}{2}}a_3+216\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}\sqrt{3}\,y^5b_3-\sqrt{27y^4+32x^3}\left(108y^2+12\sqrt{3}\,\sqrt{27\right.}{}$$

$$=0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-72\left(27y^4+32x^3\right)^{\frac{3}{2}} a_3 \\
&-216\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} \sqrt{3}\,y^5 b_3 \\
&+\sqrt{27y^4+32x^3}\,\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{5}{3}} x b_2 \\
&-\sqrt{27y^4+32x^3}\,\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{5}{3}} y a_2 \\
&+2\sqrt{27y^4+32x^3}\,\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{5}{3}} y b_3 \\
&+6b_2 y^2\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}} \sqrt{27y^4+32x^3} \\
&-216\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} \sqrt{3}\,y^4 b_1 \\
&-12960\sqrt{3}\,x^2y^4b_2-20736\sqrt{3}\,x\,y^5b_3 \\
&+8\sqrt{27y^4+32x^3}\,\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}} x a_3 \\
&-72\sqrt{27y^4+32x^3}\,\left(108y^2\right. \\
&\left.+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} y^3 b_3 \\
&-216\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} \sqrt{3}\,x\,y^4 b_2 \\
&-192\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} \sqrt{3}\,x^3 y a_2 \\
&-192\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} \sqrt{3}\,x^2 y^2 a_3 \\
&-72\sqrt{27y^4+32x^3}\,\left(108y^2\right. \\
&\left.+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} x\,y^2 b_2 \\
&-192\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} \sqrt{3}\,x^2 y a_1 \\
&+13824\sqrt{3}\,x^4ya_2-9216\sqrt{3}\,x^3y^2a_3 \\
&-12960\sqrt{3}\,x\,y^4b_1-96\sqrt{27y^4+32x^3}\,\left(108y^2\right. \\
&\left.+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} x^2 a_3 \\
&-72\sqrt{27y^4+32x^3}\,\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} y^2 b_1 \\
&-4320\sqrt{27y^4+32x^3}\,x^2y^2b_2-6912\sqrt{27y^4+32x^3}\,x\,y^3b_3 \\
&+4608\sqrt{3}\,x^3ya_1-4320\sqrt{27y^4+32x^3}\,x\,y^2b_1 \\
&+\sqrt{27y^4+32x^3}\,\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{5}{3}} b_1 \\
&+648\sqrt{27y^4+32x^3}\,y^4a_3+2592\sqrt{27y^4+32x^3}\,y^3a_1 \\
&-3888\sqrt{3}\,y^6a_3-9216\sqrt{3}\,x^5b_2+7776\sqrt{3}\,y^5a_1 \\
&-9216\sqrt{3}\,x^4b_1+5184\sqrt{27y^4+32x^3}\,x\,y^3a_2
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}, \left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}, \sqrt{27y^4+32x^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}} = v_3, \left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} = v_4, \sqrt{27y^4+32x^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&11664\sqrt{3}\,v_3v_2^6b_2 + 2304\sqrt{3}\,v_1^4v_4b_2 + 18432\sqrt{3}\,v_1^4v_3a_3 \\
&+ 2304\sqrt{3}\,v_1^3v_4b_1 + 2592v_4\sqrt{3}\,v_2^5b_3 + 648v_4\sqrt{3}\,v_2^4b_1 \\
&- 77760\sqrt{3}\,v_1^2v_2^4b_2 - 124416\sqrt{3}\,v_1v_2^5b_3 + 864v_5v_4v_2^3b_3 \\
&+ 82944\sqrt{3}\,v_1^4v_2a_2 - 55296\sqrt{3}\,v_1^3v_2^2a_3 - 77760\sqrt{3}\,v_1v_2^4b_1 \\
&- 576v_5v_4v_1^2a_3 + 216v_5v_4v_2^2b_1 - 25920v_5v_1^2v_2^2b_2 - 41472v_5v_1v_2^3b_3 \\
&+ 27648\sqrt{3}\,v_1^3v_2a_1 - 25920v_5v_1v_2^2b_1 + 31104v_5v_1v_2^3a_2 \\
&+ 93312\sqrt{3}\,v_1v_2^5a_2 - 110592\sqrt{3}\,v_1^4v_2b_3 - 648v_5v_4v_2^3a_2 + 3888v_5v_3v_2^4b_2 \\
&- 7776v_5v_2^4a_3 + 15552v_5v_2^3a_1 - 23328\sqrt{3}\,v_2^6a_3 - 55296\sqrt{3}\,v_1^5b_2 \\
&+ 46656\sqrt{3}\,v_2^5a_1 - 55296\sqrt{3}\,v_1^4b_1 - 13824v_1^3v_5a_3 - 1944\sqrt{3}\,v_4v_2^5a_2 \\
&+ 648v_4\sqrt{3}\,v_1v_2^4b_2 - 3456v_4\sqrt{3}\,v_1^3v_2a_2 - 1152v_4\sqrt{3}\,v_1^2v_2^2a_3 \\
&+ 216v_5v_4v_1v_2^2b_2 - 1152v_4\sqrt{3}\,v_1^2v_2a_1 + 4608\sqrt{3}\,v_1^3v_4v_2b_3 \\
&+ 13824\sqrt{3}\,v_1^3v_3v_2^2b_2 + 15552\sqrt{3}\,v_1v_3v_2^4a_3 + 5184v_1v_5v_3v_2^2a_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}
&11664\sqrt{3}\,v_3v_2^6b_2 + 2304\sqrt{3}\,v_1^4v_4b_2 + 18432\sqrt{3}\,v_1^4v_3a_3\\
&+ 2304\sqrt{3}\,v_1^3v_4b_1 + 648v_4\sqrt{3}\,v_2^4b_1 - 77760\sqrt{3}\,v_1^2v_2^4b_2\\
&- 55296\sqrt{3}\,v_1^3v_2^2a_3 - 77760\sqrt{3}\,v_1v_2^4b_1 - 576v_5v_4v_1^2a_3\\
&+ 216v_5v_4v_2^2b_1 - 25920v_5v_1^2v_2^2b_2 + 27648\sqrt{3}\,v_1^3v_2a_1\\
&- 25920v_5v_1v_2^2b_1 + 3888v_5v_3v_2^4b_2 - 7776v_5v_2^4a_3 + 15552v_5v_2^3a_1\\
&- 23328\sqrt{3}\,v_2^6a_3 - 55296\sqrt{3}\,v_1^5b_2 + 46656\sqrt{3}\,v_2^5a_1 - 55296\sqrt{3}\,v_1^4b_1\\
&- 13824v_1^3v_5a_3 + \left(82944\sqrt{3}\,a_2 - 110592\sqrt{3}\,b_3\right)v_1^4v_2\\
&+ \left(93312\sqrt{3}\,a_2 - 124416\sqrt{3}\,b_3\right)v_1v_2^5 + 648v_4\sqrt{3}\,v_1v_2^4b_2\\
&- 1152v_4\sqrt{3}\,v_1^2v_2^2a_3 + 216v_5v_4v_1v_2^2b_2 - 1152v_4\sqrt{3}\,v_1^2v_2a_1\\
&+ 13824\sqrt{3}\,v_1^3v_3v_2^2b_2 + 15552\sqrt{3}\,v_1v_3v_2^4a_3\\
&+ 5184v_1v_5v_3v_2^2a_3 + \left(-1944\sqrt{3}\,a_2 + 2592\sqrt{3}\,b_3\right)v_2^5v_4\\
&+ \left(-648a_2 + 864b_3\right)v_2^3v_4v_5 + \left(-3456\sqrt{3}\,a_2 + 4608\sqrt{3}\,b_3\right)v_1^3v_2v_4\\
&+ \left(31104a_2 - 41472b_3\right)v_1v_2^3v_5 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
15552a_1 &= 0\\
-13824a_3 &= 0\\
-7776a_3 &= 0\\
-576a_3 &= 0\\
5184a_3 &= 0\\
-25920b_1 &= 0\\
216b_1 &= 0\\
-25920b_2 &= 0\\
216b_2 &= 0\\
3888b_2 &= 0\\
-1152\sqrt{3}\,a_1 &= 0\\
27648\sqrt{3}\,a_1 &= 0\\
46656\sqrt{3}\,a_1 &= 0\\
-55296\sqrt{3}\,a_3 &= 0\\
-23328\sqrt{3}\,a_3 &= 0\\
-1152\sqrt{3}\,a_3 &= 0\\
15552\sqrt{3}\,a_3 &= 0\\
18432\sqrt{3}\,a_3 &= 0\\
-77760\sqrt{3}\,b_1 &= 0\\
-55296\sqrt{3}\,b_1 &= 0\\
648\sqrt{3}\,b_1 &= 0\\
2304\sqrt{3}\,b_1 &= 0\\
-77760\sqrt{3}\,b_2 &= 0\\
-55296\sqrt{3}\,b_2 &= 0\\
648\sqrt{3}\,b_2 &= 0\\
2304\sqrt{3}\,b_2 &= 0\\
11664\sqrt{3}\,b_2 &= 0\\
13824\sqrt{3}\,b_2 &= 0\\
-648a_2 + 864b_3 &= 0\\
31104a_2 - 41472b_3 &= 0\\
-3456\sqrt{3}\,a_2 + 4608\sqrt{3}\,b_3 &= 0\\
-1944\sqrt{3}\,a_2 + 2592\sqrt{3}\,b_3 &= 0\\
82944\sqrt{3}\,a_2 - 110592\sqrt{3}\,b_3 &= 0\\
93312\sqrt{3}\,a_2 - 124416\sqrt{3}\,b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{4b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{4x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{y}{\frac{4x}{3}} \\
&= \frac{3y}{4x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x^{\frac{3}{4}}
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x^{\frac{3}{4}}}
$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{\frac{4x}{3}} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \frac{3\ln(x)}{4} \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}} - 24x}{6y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{3y}{4x^{\frac{7}{4}}} \\ R_y &= \frac{1}{x^{\frac{3}{4}}} \\ S_x &= \frac{3}{4x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{9x^{\frac{3}{4}}y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}{2\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}} x - 9y^2\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}} - 48x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{9R12^{\frac{1}{3}}\left(\sqrt{3}\,\sqrt{27R^4+32}+9R^2\right)^{\frac{1}{3}}}{2\,12^{\frac{2}{3}}\left(\sqrt{3}\,\sqrt{27R^4+32}+9R^2\right)^{\frac{2}{3}} - 9\,12^{\frac{1}{3}}\left(\sqrt{3}\,\sqrt{27R^4+32}+9R^2\right)^{\frac{1}{3}}R^2 - 48}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{9R\left(12\sqrt{81R^4+96}+108R^2\right)^{\frac{1}{3}}}{4\,18^{\frac{1}{3}}\left(\left(\sqrt{81R^4+96}+9R^2\right)^2\right)^{\frac{1}{3}} - 9R^2\left(12\sqrt{81R^4+96}+108R^2\right)^{\frac{1}{3}} - 48}dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}}}{4\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a^4+96}+9\_a^2\right)^2\right)^{\frac{1}{3}} - 9\_a^2\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}} - 48}d\_a + c_1$$

Which simplifies to

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}}}{4\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a^4+96}+9\_a^2\right)^2\right)^{\frac{1}{3}} - 9\_a^2\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}} - 48}d\_a + c_1$$

Summary

The solution(s) found are the following

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}}}{4\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a^4+96}+9\_a^2\right)^2\right)^{\frac{1}{3}} - 9\_a^2\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}} - 48}d\_a + c_1 \tag{1}$$

Verification of solutions

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}}}{4\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a^4+96}+9\_a^2\right)^2\right)^{\frac{1}{3}}-9\_a^2\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}}-48}d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24x}{12y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(5E)}\\
&+\frac{\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24x\right)(b_3-a_2)}{12y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}\\
&-\frac{\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24x\right)^2 a_3}{144y^2\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}}\\
&-\left(\frac{\frac{1152ix^2}{\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}\sqrt{27y^4+32x^3}}+24i\sqrt{3}-\frac{384\sqrt{3}\,x^2}{\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}\sqrt{27y^4+32x^3}}+24}{12y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}\right.\\
&\left.-\frac{16\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24x\right)\sqrt{3}\,x^2}{y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}}\sqrt{27y^4+32x^3}}\right)\\
&+ya_3+a_1)-\left(\frac{\frac{2i\sqrt{3}\left(216y+\frac{648\sqrt{3}\,y^3}{\sqrt{27y^4+32x^3}}\right)}{3\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}-\frac{2\left(216y+\frac{648\sqrt{3}\,y^3}{\sqrt{27y^4+32x^3}}\right)}{3\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}}{12y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}\right.\\
&-\frac{i\sqrt{3}\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24x}{12y^2\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}\\
&-\frac{\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24x\right)\left(216y+\frac{\cdots}{\sqrt{\cdots}}\right.}{36y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}}}\\
&+yb_3+b_1)=0
\end{aligned}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}, \left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}, \sqrt{27y^4+32x^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}} = v_3, \left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} = v_4, \sqrt{27y^4+32x^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&331776iv_1^4b_1 + 466560iv_1^2v_2^4b_2 + 3888iv_4v_2^4b_1\\
&+ 13824iv_1^3v_4b_1 - 559872iv_1v_2^5a_2 + 746496iv_1v_2^5b_3\\
&- 497664iv_1^4v_2a_2 + 663552iv_1^4v_2b_3 + 331776iv_1^3v_2^2a_3\\
&+ 466560iv_1v_2^4b_1 - 165888iv_1^3v_2a_1 - 11664iv_4v_2^5a_2\\
&+ 15552iv_4v_2^5b_3 + 13824iv_1^4v_4b_2 + 51840iv_5\sqrt{3}\,v_1^2v_2^2b_2\\
&- 62208iv_5\sqrt{3}\,v_1v_2^3a_2 + 82944iv_5\sqrt{3}\,v_1v_2^3b_3\\
&+ 51840iv_5\sqrt{3}\,v_1v_2^2b_1 - 1152iv_5v_4\sqrt{3}\,v_1^2a_3\\
&+ 46656\sqrt{3}\,v_2^6a_3 + 110592\sqrt{3}\,v_1^5b_2 - 93312\sqrt{3}\,v_2^5a_1\\
&+ 110592\sqrt{3}\,v_1^4b_1 + 15552v_5v_2^4a_3 - 31104v_5v_2^3a_1\\
&+ 27648v_1^3v_5a_3 - 1296i\sqrt{3}\,v_5v_4v_2^3a_2\\
&+ 1728iv_5v_4\sqrt{3}\,v_2^3b_3 + 432iv_5v_4\sqrt{3}\,v_2^2b_1 + 139968iv_2^6a_3\\
&+ 331776iv_1^5b_2 - 279936iv_2^5a_1 - 9216\sqrt{3}\,v_1^3v_4v_2b_3\\
&+ 55296\sqrt{3}\,v_1^3v_3v_2^2b_2 + 62208\sqrt{3}\,v_1v_3v_2^4a_3\\
&+ 20736v_1v_5v_3v_2^2a_3 - 1296v_4\sqrt{3}\,v_1v_2^4b_2\\
&+ 6912v_4\sqrt{3}\,v_1^3v_2a_2 + 2304v_4\sqrt{3}\,v_1^2v_2^2a_3\\
&- 432v_5v_4v_1v_2^2b_2 + 2304v_4\sqrt{3}\,v_1^2v_2a_1 + 3888iv_4v_1v_2^4b_2\\
&- 20736iv_4v_1^3v_2a_2 + 27648iv_1^3v_4v_2b_3 - 31104iv_5\sqrt{3}\,v_2^3a_1\\
&+ 15552iv_5\sqrt{3}\,v_2^4a_3 + 27648i\sqrt{3}\,v_1^3v_5a_3\\
&- 6912iv_4v_1^2v_2^2a_3 - 6912iv_4v_1^2v_2a_1 + 432iv_5v_4\sqrt{3}\,v_1v_2^2b_2\\
&+ 1152v_5v_4v_1^2a_3 - 5184v_4\sqrt{3}\,v_2^5b_3 - 1296v_4\sqrt{3}\,v_2^4b_1\\
&+ 155520\sqrt{3}\,v_1^2v_2^4b_2 + 248832\sqrt{3}\,v_1v_2^5b_3 - 1728v_5v_4v_2^3b_3\\
&- 165888\sqrt{3}\,v_1^4v_2a_2 + 110592\sqrt{3}\,v_1^3v_2^2a_3\\
&- 186624\sqrt{3}\,v_1v_2^5a_2 + 221184\sqrt{3}\,v_1^4v_2b_3\\
&+ 1296v_5v_4v_2^3a_2 + 15552v_5v_3v_2^4b_2 + 155520\sqrt{3}\,v_1v_2^4b_1\\
&- 432v_5v_4v_2^2b_1 + 51840v_5v_1^2v_2^2b_2 + 82944v_5v_1v_2^3b_3\\
&- 55296\sqrt{3}\,v_1^3v_2a_1 + 51840v_5v_1v_2^2b_1 - 62208v_5v_1v_2^3a_2\\
&+ 3888\sqrt{3}\,v_4v_2^5a_2 + 46656\sqrt{3}\,v_3v_2^6b_2 - 4608\sqrt{3}\,v_1^4v_4b_2\\
&+ 73728\sqrt{3}\,v_1^4v_3a_3 - 4608\sqrt{3}\,v_1^3v_4b_1 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\Big(-497664ia_2 + 663552ib_3 - 165888\sqrt{3}\,a_2 \\
&+ 221184\sqrt{3}\,b_3\Big)\, v_1^4 v_2 + \Big(13824ib_2 - 4608\sqrt{3}\,b_2\Big)\, v_1^4 v_4 \\
&+ \Big(331776ia_3 + 110592\sqrt{3}\,a_3\Big)\, v_1^3 v_2^2 \\
&+ \Big(-165888ia_1 - 55296\sqrt{3}\,a_1\Big)\, v_1^3 v_2 \\
&+ \Big(13824ib_1 - 4608\sqrt{3}\,b_1\Big)\, v_1^3 v_4 \\
&+ \Big(27648i\sqrt{3}\,a_3 + 27648a_3\Big)\, v_1^3 v_5 \\
&+ \Big(466560ib_2 + 155520\sqrt{3}\,b_2\Big)\, v_1^2 v_2^4 \\
&+ \Big(-559872ia_2 + 746496ib_3 - 186624\sqrt{3}\,a_2 \\
&+ 248832\sqrt{3}\,b_3\Big)\, v_1 v_2^5 + \Big(466560ib_1 + 155520\sqrt{3}\,b_1\Big)\, v_1 v_2^4 \\
&+ \Big(-11664ia_2 + 15552ib_3 + 3888\sqrt{3}\,a_2 - 5184\sqrt{3}\,b_3\Big)\, v_2^5 v_4 \\
&+ \Big(3888ib_1 - 1296\sqrt{3}\,b_1\Big)\, v_2^4 v_4 \\
&+ \Big(15552i\sqrt{3}\,a_3 + 15552a_3\Big)\, v_2^4 v_5 \\
&+ \Big(-31104i\sqrt{3}\,a_1 - 31104a_1\Big)\, v_2^3 v_5 \\
&+ \Big(331776ib_2 + 110592\sqrt{3}\,b_2\Big)\, v_1^5 \\
&+ \Big(331776ib_1 + 110592\sqrt{3}\,b_1\Big)\, v_1^4 \\
&+ \Big(139968ia_3 + 46656\sqrt{3}\,a_3\Big)\, v_2^6 \\
&+ \Big(-279936ia_1 - 93312\sqrt{3}\,a_1\Big)\, v_2^5 \\
&+ \Big(432i\sqrt{3}\,b_2 - 432b_2\Big)\, v_1 v_2^2 v_4 v_5 + 55296\sqrt{3}\, v_1^3 v_3 v_2^2 b_2 \\
&+ 62208\sqrt{3}\, v_1 v_3 v_2^4 a_3 + 20736 v_1 v_5 v_3 v_2^2 a_3 \\
&+ 15552 v_5 v_3 v_2^4 b_2 + 46656\sqrt{3}\, v_3 v_2^6 b_2 + 73728\sqrt{3}\, v_1^4 v_3 a_3 \\
&+ \Big(-20736ia_2 + 27648ib_3 + 6912\sqrt{3}\,a_2 \\
&- 9216\sqrt{3}\,b_3\Big)\, v_1^3 v_2 v_4 + \Big(-6912ia_3 + 2304\sqrt{3}\,a_3\Big)\, v_1^2 v_2^2 v_4 \\
&+ \Big(51840i\sqrt{3}\,b_2 + 51840b_2\Big)\, v_1^2 v_2^2 v_5 \\
&+ \Big(-6912ia_1 + 2304\sqrt{3}\,a_1\Big)\, v_1^2 v_2 v_4 \\
&+ \Big(-1152i\sqrt{3}\,a_3 + 1152a_3\Big)\, v_1^2 v_4 v_5 \\
&+ \Big(3888ib_2 - 1296\sqrt{3}\,b_2\Big)\, v_1 v_2^4 v_4 \\
&+ \Big(-62208i\sqrt{3}\,a_2 + 82944i\sqrt{3}\,b_3 - 62208a_2 \\
&+ 82944b_3\Big)\, v_1 v_2^3 v_5 + \Big(51840i\sqrt{3}\,b_1 + 51840b_1\Big)\, v_1 v_2^2 v_5
\end{aligned}
\qquad (8E)
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
20736a_3 &= 0\\
15552b_2 &= 0\\
62208\sqrt{3}\, a_3 &= 0\\
73728\sqrt{3}\, a_3 &= 0\\
46656\sqrt{3}\, b_2 &= 0\\
55296\sqrt{3}\, b_2 &= 0\\
-279936ia_1 - 93312\sqrt{3}\, a_1 &= 0\\
-165888ia_1 - 55296\sqrt{3}\, a_1 &= 0\\
-6912ia_1 + 2304\sqrt{3}\, a_1 &= 0\\
-6912ia_3 + 2304\sqrt{3}\, a_3 &= 0\\
3888ib_1 - 1296\sqrt{3}\, b_1 &= 0\\
3888ib_2 - 1296\sqrt{3}\, b_2 &= 0\\
13824ib_1 - 4608\sqrt{3}\, b_1 &= 0\\
13824ib_2 - 4608\sqrt{3}\, b_2 &= 0\\
139968ia_3 + 46656\sqrt{3}\, a_3 &= 0\\
331776ia_3 + 110592\sqrt{3}\, a_3 &= 0\\
331776ib_1 + 110592\sqrt{3}\, b_1 &= 0\\
331776ib_2 + 110592\sqrt{3}\, b_2 &= 0\\
466560ib_1 + 155520\sqrt{3}\, b_1 &= 0\\
466560ib_2 + 155520\sqrt{3}\, b_2 &= 0\\
-31104i\sqrt{3}\, a_1 - 31104a_1 &= 0\\
-1152i\sqrt{3}\, a_3 + 1152a_3 &= 0\\
432i\sqrt{3}\, b_1 - 432b_1 &= 0\\
432i\sqrt{3}\, b_2 - 432b_2 &= 0\\
15552i\sqrt{3}\, a_3 + 15552a_3 &= 0\\
27648i\sqrt{3}\, a_3 + 27648a_3 &= 0\\
51840i\sqrt{3}\, b_1 + 51840b_1 &= 0\\
51840i\sqrt{3}\, b_2 + 51840b_2 &= 0\\
-559872ia_2 + 746496ib_3 - 186624\sqrt{3}\, a_2 + 248832\sqrt{3}\, b_3 &= 0\\
-497664ia_2 + 663552ib_3 - 165888\sqrt{3}\, a_2 + 221184\sqrt{3}\, b_3 &= 0\\
-20736ia_2 + 27648ib_3 + 6912\sqrt{3}\, a_2 - 9216\sqrt{3}\, b_3 &= 0\\
-11664ia_2 + 15552ib_3 + 3888\sqrt{3}\, a_2 - 5184\sqrt{3}\, b_3 &= 0\\
-62208i\sqrt{3}\, a_2 + 82944i\sqrt{3}\, b_3 - 62208a_2 + 82944b_3 &= 0\\
-1296i\sqrt{3}\, a_2 + 1728i\sqrt{3}\, b_3 + 1296a_2 - 1728b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= \frac{4b_3}{3} \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= \frac{4x}{3} \\ \eta &= y \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$\begin{aligned} y' &= -\frac{i\sqrt{3}\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}} + 24i\sqrt{3}\,x + \left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}} - 24x}{12y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 & \qquad\qquad (5E)\\
&-\frac{\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x+\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)(b_3-a_2)}{12y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}\\
&-\frac{\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x+\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)^2 a_3}{144y^2\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}}\\
&-\left(-\frac{-\frac{1152ix^2}{\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}\sqrt{27y^4+32x^3}}+24i\sqrt{3}+\frac{384\sqrt{3}\,x^2}{\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}\sqrt{27y^4+32x^3}}-24}{12y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}\right.\\
&\left.+\frac{16\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x+\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)\sqrt{3}\,x^2}{y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}}\sqrt{27y^4+32x^3}}\right)(\\
&+ya_3+a_1)-\left(-\frac{\frac{2i\sqrt{3}\left(216y+\frac{648\sqrt{3}\,y^3}{\sqrt{27y^4+32x^3}}\right)}{3\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}+\frac{144y+\frac{432\sqrt{3}\,y^3}{\sqrt{27y^4+32x^3}}}{\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}}{12y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}\right.\\
&+\frac{i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x+\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x}{12y^2\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}\\
&+\frac{\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x+\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)\left(216y+\right.}{36y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}}}\\
&+yb_3+b_1)=0
\end{aligned}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}, \left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}, \sqrt{27y^4+32x^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}} = v_3, \left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} = v_4, \sqrt{27y^4+32x^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-4608\sqrt{3}\,v_1^4v_4b_2 - 432iv_5v_4\sqrt{3}\,v_1v_2^2b_2 \\
&+ 73728\sqrt{3}\,v_1^4v_3a_3 - 4608\sqrt{3}\,v_1^3v_4b_1 - 186624\sqrt{3}\,v_1v_2^5a_2 \\
&+ 1152v_5v_4v_1^2a_3 - 5184v_4\sqrt{3}\,v_2^5b_3 - 1296v_4\sqrt{3}\,v_2^4b_1 \\
&+ 155520\sqrt{3}\,v_1^2v_2^4b_2 + 248832\sqrt{3}\,v_1v_2^5b_3 - 1728v_5v_4v_2^3b_3 \\
&- 165888\sqrt{3}\,v_1^4v_2a_2 + 110592\sqrt{3}\,v_1^3v_2^2a_3 \\
&+ 155520\sqrt{3}\,v_1v_2^4b_1 - 432v_5v_4v_2^2b_1 + 51840v_5v_1^2v_2^2b_2 \\
&+ 82944v_5v_1v_2^3b_3 - 55296\sqrt{3}\,v_1^3v_2a_1 + 51840v_5v_1v_2^2b_1 \\
&+ 3888\sqrt{3}\,v_4v_2^5a_2 + 46656\sqrt{3}\,v_3v_2^6b_2 + 221184\sqrt{3}\,v_1^4v_2b_3 \\
&+ 1296v_5v_4v_2^3a_2 + 15552v_5v_3v_2^4b_2 - 62208v_5v_1v_2^3a_2 \\
&- 1296v_4\sqrt{3}\,v_1v_2^4b_2 + 6912v_4\sqrt{3}\,v_1^3v_2a_2 \\
&+ 2304v_4\sqrt{3}\,v_1^2v_2^2a_3 - 432v_5v_4v_1v_2^2b_2 + 2304v_4\sqrt{3}\,v_1^2v_2a_1 \\
&+ 20736v_1v_5v_3v_2^2a_3 - 9216\sqrt{3}\,v_1^3v_4v_2b_3 \\
&+ 55296\sqrt{3}\,v_1^3v_3v_2^2b_2 + 62208\sqrt{3}\,v_1v_3v_2^4a_3 \\
&+ 1296i\sqrt{3}\,v_5v_4v_2^3a_2 - 1728iv_5v_4\sqrt{3}\,v_2^3b_3 \\
&- 432iv_5v_4\sqrt{3}\,v_2^2b_1 - 51840iv_5\sqrt{3}\,v_1^2v_2^2b_2 \\
&+ 62208iv_5\sqrt{3}\,v_1v_2^3a_2 - 82944iv_5\sqrt{3}\,v_1v_2^3b_3 \\
&- 51840iv_5\sqrt{3}\,v_1v_2^2b_1 + 1152iv_5v_4\sqrt{3}\,v_1^2a_3 \\
&+ 6912iv_4v_1^2v_2^2a_3 + 6912iv_4v_1^2v_2a_1 - 15552iv_5\sqrt{3}\,v_2^4a_3 \\
&- 27648i\sqrt{3}\,v_1^3v_5a_3 + 31104iv_5\sqrt{3}\,v_2^3a_1 \\
&- 3888iv_4v_1v_2^4b_2 + 20736iv_4v_1^3v_2a_2 - 27648iv_1^3v_4v_2b_3 \\
&- 331776iv_1^4b_1 - 139968iv_2^6a_3 - 331776iv_1^5b_2 \\
&+ 279936iv_2^5a_1 + 27648v_1^3v_5a_3 + 46656\sqrt{3}\,v_2^6a_3 \\
&+ 110592\sqrt{3}\,v_1^5b_2 - 93312\sqrt{3}\,v_2^5a_1 + 110592\sqrt{3}\,v_1^4b_1 \\
&+ 15552v_5v_2^4a_3 - 31104v_5v_2^3a_1 + 11664iv_4v_2^5a_2 \\
&- 15552iv_4v_2^5b_3 - 13824iv_1^4v_4b_2 - 3888iv_4v_2^4b_1 \\
&- 13824iv_1^3v_4b_1 + 165888iv_1^3v_2a_1 - 466560iv_1^2v_2^4b_2 \\
&+ 559872iv_1v_2^5a_2 - 746496iv_1v_2^5b_3 + 497664iv_1^4v_2a_2 \\
&- 663552iv_1^4v_2b_3 - 331776iv_1^3v_2^2a_3 - 466560iv_1v_2^4b_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&73728\sqrt{3}\,v_1^4v_3a_3+46656\sqrt{3}\,v_3v_2^6b_2+15552v_5v_3v_2^4b_2\\
&+\left(20736ia_2-27648ib_3+6912\sqrt{3}\,a_2-9216\sqrt{3}\,b_3\right)v_1^3v_2v_4\\
&+\left(6912ia_3+2304\sqrt{3}\,a_3\right)v_1^2v_2^2v_4\\
&+\left(-51840i\sqrt{3}\,b_2+51840b_2\right)v_1^2v_2^2v_5\\
&+\left(6912ia_1+2304\sqrt{3}\,a_1\right)v_1^2v_2v_4\\
&+\left(1152i\sqrt{3}\,a_3+1152a_3\right)v_1^2v_4v_5\\
&+\left(-3888ib_2-1296\sqrt{3}\,b_2\right)v_1v_2^4v_4+\left(62208i\sqrt{3}\,a_2\right.\\
&\left.-82944i\sqrt{3}\,b_3-62208a_2+82944b_3\right)v_1v_2^3v_5\\
&+\left(-51840i\sqrt{3}\,b_1+51840b_1\right)v_1v_2^2v_5\\
&+\left(1296i\sqrt{3}\,a_2-1728i\sqrt{3}\,b_3+1296a_2-1728b_3\right)v_2^3v_4v_5\\
&+\left(-432i\sqrt{3}\,b_1-432b_1\right)v_2^2v_4v_5\\
&+\left(-331776ib_2+110592\sqrt{3}\,b_2\right)v_1^5\\
&+\left(-331776ib_1+110592\sqrt{3}\,b_1\right)v_1^4\\
&+\left(-139968ia_3+46656\sqrt{3}\,a_3\right)v_2^6\\
&+\left(279936ia_1-93312\sqrt{3}\,a_1\right)v_2^5+20736v_1v_5v_3v_2^2a_3\\
&+55296\sqrt{3}\,v_1^3v_3v_2^2b_2+62208\sqrt{3}\,v_1v_3v_2^4a_3\\
&+\left(-432i\sqrt{3}\,b_2-432b_2\right)v_1v_2^2v_4v_5+\left(497664ia_2\right.\\
&\left.-663552ib_3-165888\sqrt{3}\,a_2+221184\sqrt{3}\,b_3\right)v_1^4v_2\\
&+\left(-13824ib_2-4608\sqrt{3}\,b_2\right)v_1^4v_4\\
&+\left(-331776ia_3+110592\sqrt{3}\,a_3\right)v_1^3v_2^2\\
&+\left(165888ia_1-55296\sqrt{3}\,a_1\right)v_1^3v_2\\
&+\left(-13824ib_1-4608\sqrt{3}\,b_1\right)v_1^3v_4\\
&+\left(-27648i\sqrt{3}\,a_3+27648a_3\right)v_1^3v_5\\
&+\left(-466560ib_2+155520\sqrt{3}\,b_2\right)v_1^2v_2^4+\left(559872ia_2\right.\\
&\left.-746496ib_3-186624\sqrt{3}\,a_2+248832\sqrt{3}\,b_3\right)v_1v_2^5\\
&+\left(-466560ib_1+155520\sqrt{3}\,b_1\right)v_1v_2^4\\
&+\left(11664ia_2-15552ib_3+3888\sqrt{3}\,a_2-5184\sqrt{3}\,b_3\right)v_2^5v_4\\
&+\left(-3888ib_1-1296\sqrt{3}\,b_1\right)v_2^4v_4
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
20736a_3 &= 0\\
15552b_2 &= 0\\
62208\sqrt{3}\,a_3 &= 0\\
73728\sqrt{3}\,a_3 &= 0\\
46656\sqrt{3}\,b_2 &= 0\\
55296\sqrt{3}\,b_2 &= 0\\
-466560ib_1 + 155520\sqrt{3}\,b_1 &= 0\\
-466560ib_2 + 155520\sqrt{3}\,b_2 &= 0\\
-331776ia_3 + 110592\sqrt{3}\,a_3 &= 0\\
-331776ib_1 + 110592\sqrt{3}\,b_1 &= 0\\
-331776ib_2 + 110592\sqrt{3}\,b_2 &= 0\\
-139968ia_3 + 46656\sqrt{3}\,a_3 &= 0\\
-13824ib_1 - 4608\sqrt{3}\,b_1 &= 0\\
-13824ib_2 - 4608\sqrt{3}\,b_2 &= 0\\
-3888ib_1 - 1296\sqrt{3}\,b_1 &= 0\\
-3888ib_2 - 1296\sqrt{3}\,b_2 &= 0\\
6912ia_1 + 2304\sqrt{3}\,a_1 &= 0\\
6912ia_3 + 2304\sqrt{3}\,a_3 &= 0\\
165888ia_1 - 55296\sqrt{3}\,a_1 &= 0\\
279936ia_1 - 93312\sqrt{3}\,a_1 &= 0\\
-51840i\sqrt{3}\,b_1 + 51840b_1 &= 0\\
-51840i\sqrt{3}\,b_2 + 51840b_2 &= 0\\
-27648i\sqrt{3}\,a_3 + 27648a_3 &= 0\\
-15552i\sqrt{3}\,a_3 + 15552a_3 &= 0\\
-432i\sqrt{3}\,b_1 - 432b_1 &= 0\\
-432i\sqrt{3}\,b_2 - 432b_2 &= 0\\
1152i\sqrt{3}\,a_3 + 1152a_3 &= 0\\
31104i\sqrt{3}\,a_1 - 31104a_1 &= 0\\
11664ia_2 - 15552ib_3 + 3888\sqrt{3}\,a_2 - 5184\sqrt{3}\,b_3 &= 0\\
20736ia_2 - 27648ib_3 + 6912\sqrt{3}\,a_2 - 9216\sqrt{3}\,b_3 &= 0\\
497664ia_2 - 663552ib_3 - 165888\sqrt{3}\,a_2 + 221184\sqrt{3}\,b_3 &= 0\\
559872ia_2 - 746496ib_3 - 186624\sqrt{3}\,a_2 + 248832\sqrt{3}\,b_3 &= 0\\
1296i\sqrt{3}\,a_2 - 1728i\sqrt{3}\,b_3 + 1296a_2 - 1728b_3 &= 0\\
62208i\sqrt{3}\,a_2 - 82944i\sqrt{3}\,b_3 - 62208a_2 + 82944b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{4b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{4x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 2
   -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
   trying dAlembert
   -> Calling odsolve with the ODE`, diff(y(x), x) = (-2*y(x)^2*x^3-y(x))/(2*y(x)*x^4+x), y(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 97

```
dsolve(y(x)=2*diff(y(x),x)*x+y(x)^2*diff(y(x),x)^3,y(x), singsol=all)
```

$$y(x) = -\frac{2(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$
$$y(x) = \frac{2(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$
$$y(x) = -\frac{2i(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$
$$y(x) = \frac{2i(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$
$$y(x) = 0$$
$$y(x) = \sqrt{c_1 (c_1^2 + 2x)}$$
$$y(x) = -\sqrt{c_1 (c_1^2 + 2x)}$$

✓ Solution by Mathematica

Time used: 0.183 (sec). Leaf size: 119

```
DSolve[y[x]==2*y'[x]*x+y[x]^2*(y'[x])^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2c_1 x + c_1{}^3}$$
$$y(x) \to \sqrt{2c_1 x + c_1{}^3}$$
$$y(x) \to (-1-i) \left(\frac{2}{3}\right)^{3/4} x^{3/4}$$
$$y(x) \to (1-i) \left(\frac{2}{3}\right)^{3/4} x^{3/4}$$
$$y(x) \to (-1+i) \left(\frac{2}{3}\right)^{3/4} x^{3/4}$$
$$y(x) \to (1+i) \left(\frac{2}{3}\right)^{3/4} x^{3/4}$$

16.8 problem Ex 8

Internal problem ID [11223]
Internal file name [OUTPUT/10208_Wednesday_December_07_2022_01_20_31_PM_36766337/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 27. Clairaut equation. Page 56
Problem number: Ex 8.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{a^2yy'^2 - 2y'x + y = 0}$$

16.8.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$a^2y\,p^2 - 2px + y = 0$$

Solving for y from the above results in

$$y = \frac{2px}{a^2p^2 + 1} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{2p}{a^2p^2 + 1} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{2p}{a^2p^2 + 1} = x\left(\frac{2}{a^2p^2 + 1} - \frac{4p^2a^2}{(a^2p^2 + 1)^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{2p}{a^2p^2 + 1} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 0 \\
p &= \frac{1}{a} \\
p &= -\frac{1}{a}
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= 0 \\
y &= \frac{x}{a} \\
y &= -\frac{x}{a}
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{2p(x)}{a^2p(x)^2+1}}{x\left(\frac{2}{a^2p(x)^2+1} - \frac{4p(x)^2a^2}{\left(a^2p(x)^2+1\right)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{2}{a^2p^2+1} - \frac{4p^2a^2}{(a^2p^2+1)^2}\right)}{p - \frac{2p}{a^2p^2+1}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$\begin{aligned} p(p) &= \frac{2}{p\,(a^2p^2+1)} \\ q(p) &= 0 \end{aligned}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p\,(a^2p^2+1)} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{2}{p\left(a^2p^2+1\right)}dp} \\ &= \mathrm{e}^{-\ln\left(a^2p^2+1\right)+2\ln(p)} \end{aligned}$$

Which simplifies to

$$\mu = \frac{p^2}{a^2p^2+1}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}p}\mu x &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{p^2x}{a^2p^2+1}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{p^2x}{a^2p^2+1} = c_3$$

Dividing both sides by the integrating factor $\mu = \frac{p^2}{a^2p^2+1}$ results in

$$x(p) = \frac{c_3(a^2p^2+1)}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{x+\sqrt{x^2-y^2a^2}}{y a^2}$$
$$p = -\frac{-x+\sqrt{x^2-y^2a^2}}{y a^2}$$

Substituting the above in the solution for x found above gives

$$x = \frac{2c_3 x\, a^2}{x+\sqrt{x^2-y^2a^2}}$$
$$x = -\frac{2c_3 x\, a^2}{-x+\sqrt{x^2-y^2a^2}}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$
$$y = \frac{x}{a} \tag{2}$$
$$y = -\frac{x}{a} \tag{3}$$
$$x = \frac{2c_3 x\, a^2}{x+\sqrt{x^2-y^2a^2}} \tag{4}$$
$$x = -\frac{2c_3 x\, a^2}{-x+\sqrt{x^2-y^2a^2}} \tag{5}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = \frac{x}{a}$$

Verified OK.

$$y = -\frac{x}{a}$$

Verified OK.

$$x = \frac{2c_3 x\, a^2}{x + \sqrt{x^2 - y^2 a^2}}$$

Verified OK.

$$x = -\frac{2c_3 x\, a^2}{-x + \sqrt{x^2 - y^2 a^2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   <- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 51

```
dsolve(a^2*y(x)*diff(y(x),x)^2-2*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{a}$$
$$y(x) = \frac{x}{a}$$
$$y(x) = 0$$
$$y(x) = \mathrm{e}^{\mathrm{RootOf}\left(\mathrm{e}^{2\_Z}\sinh(-\_Z+c_1-\ln(x))^2a^2+1\right)}x$$

✓ Solution by Mathematica

Time used: 31.661 (sec). Leaf size: 244

```
DSolve[a^2*y[x]*(y'[x])^2-2*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\left(\cosh\left(\frac{a^2c_1}{2}\right)+\sinh\left(\frac{a^2c_1}{2}\right)\right)\sqrt{\cosh\left(a^2c_1\right)+\sinh\left(a^2c_1\right)-8ix}}{4a}$$
$$y(x) \to \frac{\left(\cosh\left(\frac{a^2c_1}{2}\right)+\sinh\left(\frac{a^2c_1}{2}\right)\right)\sqrt{\cosh\left(a^2c_1\right)+\sinh\left(a^2c_1\right)-8ix}}{4a}$$
$$y(x) \to -\frac{\left(\cosh\left(\frac{a^2c_1}{2}\right)+\sinh\left(\frac{a^2c_1}{2}\right)\right)\sqrt{\cosh\left(a^2c_1\right)+\sinh\left(a^2c_1\right)+8ix}}{4a}$$
$$y(x) \to \frac{\left(\cosh\left(\frac{a^2c_1}{2}\right)+\sinh\left(\frac{a^2c_1}{2}\right)\right)\sqrt{\cosh\left(a^2c_1\right)+\sinh\left(a^2c_1\right)+8ix}}{4a}$$
$$y(x) \to -\frac{x}{a}$$
$$y(x) \to \frac{x}{a}$$

16.9 problem Ex 9

Internal problem ID [11224]
Internal file name [`OUTPUT/10209_Wednesday_December_07_2022_01_20_34_PM_29877762/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 27. Clairaut equation. Page 56
Problem number: Ex 9.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`` [`y=_G(x,y')`] ``

Unable to solve or complete the solution.

$$\boxed{(x - y' - y)^2 - x^2(2yx - x^2y') = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{x^4}{2} - y + x + \frac{\sqrt{x^8 + 4yx^4 - 4x^5 + 8yx^3}}{2} \tag{1}$$

$$y' = -\frac{x^4}{2} - y + x - \frac{\sqrt{x^8 + 4yx^4 - 4x^5 + 8yx^3}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Unable to determine ODE type.

Unable to determine ODE type.

Solving equation (2)

Unable to determine ODE type.

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 2
-> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
trying dAlembert
-> Calling odsolve with the ODE`, diff(y(x), x) = (1/2)*(-2*(y(x)^3*(y(x)^3-y(x)*x-2*x+2*y(x
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 2nd trial
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
`, `-> Computing symmetries using: way = 5`
```

✗ Solution by Maple

```
dsolve((x-diff(y(x),x)-y(x))^2=x^2*(2*x*y(x)-x^2*diff(y(x),x)),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(x-y'[x]-y[x])^2==x^2*(2*x*y[x]-x^2*y'[x]),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

17 Chapter IV, differential equations of the first order and higher degree than the first. Article 28. Summary. Page 59

17.1 problem Ex 1

Internal problem ID [11225]
Internal file name [`OUTPUT/10210_Wednesday_December_07_2022_01_20_36_PM_8466414/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 28. Summary. Page 59
Problem number: Ex 1.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y^2\left(y'^2+1\right)=a^2}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y'=\frac{\sqrt{-y^2+a^2}}{y} \tag{1}$$
$$y'=-\frac{\sqrt{-y^2+a^2}}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{y}{\sqrt{a^2-y^2}}dy=\int dx$$
$$-\frac{(a-y)\,(y+a)}{\sqrt{-y^2+a^2}}=x+c_1$$

Summary
The solution(s) found are the following

$$-\frac{(a-y)(y+a)}{\sqrt{-y^2+a^2}} = x+c_1 \tag{1}$$

Verification of solutions

$$-\frac{(a-y)(y+a)}{\sqrt{-y^2+a^2}} = x+c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y}{\sqrt{a^2-y^2}}dy = \int dx$$
$$\frac{(a-y)(y+a)}{\sqrt{-y^2+a^2}} = x+c_2$$

Summary
The solution(s) found are the following

$$\frac{(a-y)(y+a)}{\sqrt{-y^2+a^2}} = x+c_2 \tag{1}$$

Verification of solutions

$$\frac{(a-y)(y+a)}{\sqrt{-y^2+a^2}} = x+c_2$$

Verified OK.

17.1.1 Maple step by step solution

Let's solve

$y^2\left(y'^2+1\right)=a^2$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{\sqrt{-y^2+a^2}}=1$

- Integrate both sides with respect to x

  $\int \frac{y'y}{\sqrt{-y^2+a^2}}dx = \int 1dx + c_1$

- Evaluate integral

  $-\sqrt{-y^2+a^2} = x + c_1$

- Solve for y

  $\left\{y = \sqrt{-c_1^2 - 2c_1x + a^2 - x^2}, y = -\sqrt{-c_1^2 - 2c_1x + a^2 - x^2}\right\}$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   <- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 54

```
dsolve(y(x)^2*(1+diff(y(x),x)^2)=a^2,y(x), singsol=all)
```

$$y(x) = -a$$
$$y(x) = a$$
$$y(x) = \sqrt{a^2 - c_1^2 + 2c_1x - x^2}$$
$$y(x) = -\sqrt{(a + x - c_1)(c_1 + a - x)}$$

✓ Solution by Mathematica

Time used: 0.344 (sec). Leaf size: 101

```
DSolve[y[x]^2*(1+(y'[x])^2)==a^2,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -\sqrt{a^2-(x+c_1)^2}\\
y(x) &\to \sqrt{a^2-(x+c_1)^2}\\
y(x) &\to -\sqrt{a^2-(x-c_1)^2}\\
y(x) &\to \sqrt{a^2-(x-c_1)^2}\\
y(x) &\to -a\\
y(x) &\to a
\end{aligned}
$$

17.2 problem Ex 2

Internal problem ID [11226]
Internal file name [`OUTPUT/10211_Wednesday_December_07_2022_01_20_37_PM_96663236/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 28. Summary. Page 59
Problem number: Ex 2.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{yy' - (x - b)\, y'^2 = a}$$

17.2.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = y'x + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$yp - (x - b)\, p^2 = a$$

Solving for y from the above results in

$$y = \frac{-p^2 b + p^2 x + a}{p} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= px + \frac{-p^2 b + a}{p} \\ &= px + \frac{-p^2 b + a}{p} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = \frac{-p^2 b + a}{p}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g' \frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \frac{-b\,c_1^2 + a}{c_1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{-p^2 b + a}{p}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - 2b - \frac{-p^2 b + a}{p^2} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{\sqrt{-\left(-x+b\right) a}}{-x+b}$$
$$p_2 = -\frac{\sqrt{-\left(-x+b\right) a}}{-x+b}$$

Substituting the above back in (1) results in

$$y_1 = \frac{2(-x+b)\, a}{\sqrt{-\left(-x+b\right) a}}$$
$$y_2 = -\frac{2(-x+b)\, a}{\sqrt{-\left(-x+b\right) a}}$$

Summary
The solution(s) found are the following

$$y = c_1 x + \frac{-b\, c_1^2 + a}{c_1} \tag{1}$$
$$y = \frac{2(-x+b)\, a}{\sqrt{-\left(-x+b\right) a}} \tag{2}$$
$$y = -\frac{2(-x+b)\, a}{\sqrt{-\left(-x+b\right) a}} \tag{3}$$

Verification of solutions

$$y = c_1 x + \frac{-b\, c_1^2 + a}{c_1}$$

Verified OK.

$$y = \frac{2(-x+b)\, a}{\sqrt{-\left(-x+b\right) a}}$$

Verified OK.

$$y = -\frac{2(-x+b)\, a}{\sqrt{-\left(-x+b\right) a}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 49

```
dsolve(y(x)*diff(y(x),x)=(x-b)*diff(y(x),x)^2+a,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= -2\sqrt{-a\,(b-x)} \\
y(x) &= 2\sqrt{-a\,(b-x)} \\
y(x) &= \frac{(-b+x)\,c_1^2+a}{c_1}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.1 (sec). Leaf size: 59

```
DSolve[y[x]*y'[x]==(x-b)*(y'[x])^2+a,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \frac{a}{c_1}+c_1(x-b) \\
y(x) &\to \text{Indeterminate} \\
y(x) &\to -2\sqrt{a(x-b)} \\
y(x) &\to 2\sqrt{a(x-b)}
\end{aligned}
$$

17.3 problem Ex 3

Internal problem ID [11227]
Internal file name [`OUTPUT/10212_Wednesday_December_07_2022_01_20_39_PM_22353921/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 28. Summary. Page 59
Problem number: Ex 3.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{x^3 {y'}^2 + x^2 y y' = -1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{yx - \sqrt{x^2y^2 - 4x}}{2x^2} \tag{1}$$
$$y' = -\frac{yx + \sqrt{x^2y^2 - 4x}}{2x^2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$\begin{aligned}
y' &= -\frac{xy - \sqrt{x^2y^2 - 4x}}{2x^2} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{\left(xy - \sqrt{x^2y^2 - 4x}\right)(b_3 - a_2)}{2x^2} - \frac{\left(xy - \sqrt{x^2y^2 - 4x}\right)^2 a_3}{4x^4} - \left(-\frac{y - \frac{2x\,y^2 - 4}{2\sqrt{x^2y^2-4x}}}{2x^2} + \frac{xy - \sqrt{x^2y^2 - 4x}}{x^3}\right)(xa_2 + ya_3 + a_1) + \frac{\left(x - \frac{x^2y}{\sqrt{x^2y^2-4x}}\right)(xb_2 + yb_3 + b_1)}{2x^2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^5yb_2 - 4x^3y^3a_3 - 6b_2x^4\sqrt{x^2y^2 - 4x} + 3\sqrt{x^2y^2 - 4x}\,x^2y^2a_3 + 2x^4yb_1 - 2x^3y^2a_1 - 2\sqrt{x^2y^2 - 4x}\,x^3b_1 +}{4x^4\sqrt{x^2y^2 - 4x}} = 0$$

Setting the numerator to zero gives

$$-2x^5yb_2 + 4x^3y^3a_3 + 6b_2x^4\sqrt{x^2y^2 - 4x} - 3\sqrt{x^2y^2 - 4x}\,x^2y^2a_3 - 2x^4yb_1 + 2x^3y^2a_1 + 2\sqrt{x^2y^2 - 4x}\,x^3b_1 - 2\sqrt{x^2y^2 - 4x}\,x^2ya_1 - \left(x^2y^2 - 4x\right)^{\frac{3}{2}} a_3 - 4x^3a_2 - 8x^3b_3 - 20x^2ya_3 - 12x^2a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-2x^5yb_2 - 2x^4y^2a_2 - 2x^4y^2b_3 - 2x^3y^3a_3 + 6b_2x^4\sqrt{x^2y^2 - 4x} - 3\sqrt{x^2y^2 - 4x}\,x^2y^2a_3 - 2x^4yb_1 - 2x^3y^2a_1 + 2\left(x^2y^2 - 4x\right)x^2a_2 + 2\left(x^2y^2 - 4x\right)x^2b_3 + 6\left(x^2y^2 - 4x\right)xya_3 + 2\sqrt{x^2y^2 - 4x}\,x^3b_1 - 2\sqrt{x^2y^2 - 4x}\,x^2ya_1 - \left(x^2y^2 - 4x\right)^{\frac{3}{2}} a_3 + 4\left(x^2y^2 - 4x\right)xa_1 + 4x^3a_2 + 4x^2ya_3 + 4x^2a_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$2x\left(-x^4yb_2+2x^2y^3a_3+3\sqrt{x\,(x\,y^2-4)}\,x^3b_2-2\sqrt{x\,(x\,y^2-4)}\,x\,y^2a_3 \\ -x^3yb_1+x^2y^2a_1+\sqrt{x\,(x\,y^2-4)}\,x^2b_1-\sqrt{x\,(x\,y^2-4)}\,xya_1 \\ -2x^2a_2-4x^2b_3-10xya_3+2\sqrt{x\,(x\,y^2-4)}\,a_3-6xa_1\right)=0$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,\sqrt{x\,(x\,y^2-4)}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\left\{x=v_1,y=v_2,\sqrt{x\,(x\,y^2-4)}=v_3\right\}$$

The above PDE (6E) now becomes

$$2v_1\left(2v_1^2v_2^3a_3-v_1^4v_2b_2+v_1^2v_2^2a_1-2v_3v_1v_2^2a_3-v_1^3v_2b_1+3v_3v_1^3b_2 \\ -v_3v_1v_2a_1+v_3v_1^2b_1-2v_1^2a_2-10v_1v_2a_3-4v_1^2b_3-6v_1a_1+2v_3a_3\right)=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3\}$$

Equation (7E) now becomes

$$-2b_2v_2v_1^5-2b_1v_2v_1^4+6b_2v_3v_1^4+4a_3v_2^3v_1^3+2a_1v_2^2v_1^3+2b_1v_3v_1^3 \\ +(-4a_2-8b_3)\,v_1^3-4a_3v_2^2v_3v_1^2-2a_1v_2v_3v_1^2-20a_3v_2v_1^2-12a_1v_1^2+4v_3a_3v_1=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-12a_1 &= 0 \\
-2a_1 &= 0 \\
2a_1 &= 0 \\
-20a_3 &= 0 \\
-4a_3 &= 0 \\
4a_3 &= 0 \\
-2b_1 &= 0 \\
2b_1 &= 0 \\
-2b_2 &= 0 \\
6b_2 &= 0 \\
-4a_2 - 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{xy - \sqrt{x^2y^2 - 4x}}{2x^2}\right)(-2x) \\
&= \frac{\sqrt{x^2y^2 - 4x}}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{\sqrt{x^2y^2-4x}}{x}} dy \end{aligned}$$

Which results in

$$S = \frac{x \ln \left(\frac{x^2 y}{\sqrt{x^2}} + \sqrt{x^2 y^2 - 4x}\right)}{\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{xy - \sqrt{x^2 y^2 - 4x}}{2x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y\sqrt{x}\,\sqrt{x\,y^2 - 4} + x\,y^2 - 2}{\sqrt{x}\,\sqrt{x\,y^2 - 4}\,\left(xy + \sqrt{x}\,\sqrt{x\,y^2 - 4}\right)} \\ S_y &= \frac{x\left(y\sqrt{x} + \sqrt{x\,y^2 - 4}\right)}{\sqrt{x\,y^2 - 4}\,\left(xy + \sqrt{x}\,\sqrt{x\,y^2 - 4}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left(xy + \sqrt{x}\,\sqrt{x\,y^2 - 4}\right)\sqrt{x\,(x\,y^2 - 4)} + x^2y^2 + x^{\frac{3}{2}}\sqrt{x\,y^2 - 4}\,y - 4x}{\sqrt{x\,y^2 - 4}\,x^{\frac{3}{2}}\left(2xy + 2\sqrt{x}\,\sqrt{x\,y^2 - 4}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(yx + \sqrt{x}\,\sqrt{y^2x - 4}\right) = \ln(x) + c_1$$

Which simplifies to

$$\ln\left(yx + \sqrt{x}\,\sqrt{y^2x - 4}\right) = \ln(x) + c_1$$

Which gives

$$y = \frac{(x\,e^{2c_1} + 4)\,e^{-c_1}}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{(x\,e^{2c_1} + 4)\,e^{-c_1}}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{(x\,e^{2c_1} + 4)\,e^{-c_1}}{2x}$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{xy + \sqrt{x^2y^2 - 4x}}{2x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{\left(xy + \sqrt{x^2y^2 - 4x}\right)(b_3 - a_2)}{2x^2} - \frac{\left(xy + \sqrt{x^2y^2 - 4x}\right)^2 a_3}{4x^4} - \left(-\frac{y + \frac{2x\,y^2 - 4}{2\sqrt{x^2y^2 - 4x}}}{2x^2} + \frac{xy + \sqrt{x^2y^2 - 4x}}{x^3}\right)(xa_2 + ya_3 + a_1) + \frac{\left(x + \frac{x^2y}{\sqrt{x^2y^2 - 4x}}\right)(xb_2 + yb_3 + b_1)}{2x^2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-2x^5yb_2 + 4x^3y^3a_3 - 6b_2x^4\sqrt{x^2y^2 - 4x} + 3\sqrt{x^2y^2 - 4x}\,x^2y^2a_3 - 2x^4yb_1 + 2x^3y^2a_1 - 2\sqrt{x^2y^2 - 4x}\,x^3b_1}{4x^4\sqrt{x^2y^2 - 4x}} = 0$$

Setting the numerator to zero gives

$$2x^5yb_2 - 4x^3y^3a_3 + 6b_2x^4\sqrt{x^2y^2 - 4x} - 3\sqrt{x^2y^2 - 4x}\,x^2y^2a_3 + 2x^4yb_1 - 2x^3y^2a_1 + 2\sqrt{x^2y^2 - 4x}\,x^3b_1 - 2\sqrt{x^2y^2 - 4x}\,x^2ya_1 - \left(x^2y^2 - 4x\right)^{\frac{3}{2}} a_3 + 4x^3a_2 + 8x^3b_3 + 20x^2ya_3 + 12x^2a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&2x^5yb_2 + 2x^4y^2a_2 + 2x^4y^2b_3 + 2x^3y^3a_3 + 6b_2x^4\sqrt{x^2y^2-4x} \\
&- 3\sqrt{x^2y^2-4x}\,x^2y^2a_3 + 2x^4yb_1 + 2x^3y^2a_1 \\
&- 2(x^2y^2-4x)\,x^2a_2 - 2(x^2y^2-4x)\,x^2b_3 - 6(x^2y^2-4x)\,xya_3 \\
&+ 2\sqrt{x^2y^2-4x}\,x^3b_1 - 2\sqrt{x^2y^2-4x}\,x^2ya_1 - (x^2y^2-4x)^{\frac{3}{2}}\,a_3 \\
&- 4(x^2y^2-4x)\,xa_1 - 4x^3a_2 - 4x^2ya_3 - 4x^2a_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&2x\Big(x^4yb_2 - 2x^2y^3a_3 + 3\sqrt{x\,(x\,y^2-4)}\,x^3b_2 - 2\sqrt{x\,(x\,y^2-4)}\,x\,y^2a_3 \\
&+ x^3yb_1 - x^2y^2a_1 + \sqrt{x\,(x\,y^2-4)}\,x^2b_1 - \sqrt{x\,(x\,y^2-4)}\,xya_1 \\
&+ 2x^2a_2 + 4x^2b_3 + 10xya_3 + 2\sqrt{x\,(x\,y^2-4)}\,a_3 + 6xa_1\Big) = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x\,(x\,y^2-4)}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{x\,(x\,y^2-4)} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2v_1\big(-2v_1^2v_2^3a_3 + v_1^4v_2b_2 - v_1^2v_2^2a_1 - 2v_3v_1v_2^2a_3 + v_1^3v_2b_1 + 3v_3v_1^3b_2 \\
&- v_3v_1v_2a_1 + v_3v_1^2b_1 + 2v_1^2a_2 + 10v_1v_2a_3 + 4v_1^2b_3 + 6v_1a_1 + 2v_3a_3\big) = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&2b_2v_2v_1^5 + 2b_1v_2v_1^4 + 6b_2v_3v_1^4 - 4a_3v_2^3v_1^3 - 2a_1v_2^2v_1^3 + 2b_1v_3v_1^3 + (4a_2+8b_3)\,v_1^3 \\
&- 4a_3v_2^2v_3v_1^2 - 2a_1v_2v_3v_1^2 + 20a_3v_2v_1^2 + 12a_1v_1^2 + 4v_3a_3v_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_1 &= 0 \\
12a_1 &= 0 \\
-4a_3 &= 0 \\
4a_3 &= 0 \\
20a_3 &= 0 \\
2b_1 &= 0 \\
2b_2 &= 0 \\
6b_2 &= 0 \\
4a_2 + 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{xy + \sqrt{x^2y^2 - 4x}}{2x^2}\right)(-2x) \\
&= -\frac{\sqrt{x^2y^2 - 4x}}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{\sqrt{x^2y^2-4x}}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x \ln\left(\frac{x^2y}{\sqrt{x^2}} + \sqrt{x^2y^2 - 4x}\right)}{\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{xy + \sqrt{x^2y^2 - 4x}}{2x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y\sqrt{x}\,\sqrt{x\,y^2 - 4} + x\,y^2 - 2}{\sqrt{x}\,\sqrt{x\,y^2 - 4}\,\left(xy + \sqrt{x}\,\sqrt{x\,y^2 - 4}\right)} \\ S_y &= \frac{\left(-y\sqrt{x} - \sqrt{x\,y^2 - 4}\right) x}{\sqrt{x\,y^2 - 4}\,\left(xy + \sqrt{x}\,\sqrt{x\,y^2 - 4}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\left(-xy-\sqrt{x}\,\sqrt{x\,y^{2}-4}\right)\sqrt{x\,(x\,y^{2}-4)}+x^{2}y^{2}+x^{\frac{3}{2}}\sqrt{x\,y^{2}-4}\,y-4x}{\sqrt{x\,y^{2}-4}\,x^{\frac{3}{2}}\left(2xy+2\sqrt{x}\,\sqrt{x\,y^{2}-4}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(yx+\sqrt{x}\,\sqrt{y^{2}x-4}\right) = c_1$$

Which simplifies to

$$-\ln\left(yx+\sqrt{x}\,\sqrt{y^{2}x-4}\right) = c_1$$

Which gives

$$y = \frac{\left(4x\,\mathrm{e}^{2c_1}+1\right)\mathrm{e}^{-c_1}}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(4x\,\mathrm{e}^{2c_1}+1\right)\mathrm{e}^{-c_1}}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{\left(4x\,\mathrm{e}^{2c_1}+1\right)\mathrm{e}^{-c_1}}{2x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.281 (sec). Leaf size: 53

```
dsolve(x^3*diff(y(x),x)^2+x^2*y(x)*diff(y(x),x)+1=0,y(x), singsol=all)
```

$$y(x) = -\frac{2}{\sqrt{x}}$$
$$y(x) = \frac{2}{\sqrt{x}}$$
$$y(x) = \frac{c_1^2 x + 4}{2c_1 x}$$
$$y(x) = \frac{c_1^2 + 4x}{2c_1 x}$$

✓ Solution by Mathematica

Time used: 0.934 (sec). Leaf size: 77

```
DSolve[x^3*(y'[x])^2+x^2*y[x]*y'[x]+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{-\frac{c_1}{2}}(x + 16e^{c_1})}{4x}$$
$$y(x) \to \frac{e^{-\frac{c_1}{2}}(x + 16e^{c_1})}{4x}$$
$$y(x) \to -\frac{2}{\sqrt{x}}$$
$$y(x) \to \frac{2}{\sqrt{x}}$$

17.4 problem Ex 4

Internal problem ID [11228]
Internal file name [`OUTPUT/10213_Wednesday_December_07_2022_01_20_41_PM_101706/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 28. Summary. Page 59
Problem number: Ex 4.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{3xy'^2 - 6yy' + 2y = -x}$$

17.4.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$3x\,p^2 - 6yp + 2y = -x$$

Solving for y from the above results in

$$y = \frac{x(3p^2+1)}{6p-2} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{3p^2+1}{6p-2} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p - \frac{3p^2+1}{6p-2} = x\left(\frac{6p}{6p-2} - \frac{6(3p^2+1)}{(6p-2)^2}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{3p^2+1}{6p-2} = 0$$

Solving for p from the above gives

$$\begin{aligned} p &= 1 \\ p &= -\frac{1}{3} \end{aligned}$$

Substituting these in (1A) gives

$$\begin{aligned} y &= x \\ y &= -\frac{x}{3} \end{aligned}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{3p(x)^2+1}{6p(x)-2}}{x\left(\frac{6p(x)}{6p(x)-2} - \frac{6\left(3p(x)^2+1\right)}{(6p(x)-2)^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -\frac{1}{3x} \end{aligned}$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = -\frac{1}{3x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu p) &= (\mu)\left(-\frac{1}{3x}\right) \\ \frac{d}{dx}\left(\frac{p}{x}\right) &= \left(\frac{1}{x}\right)\left(-\frac{1}{3x}\right) \\ d\left(\frac{p}{x}\right) &= \left(-\frac{1}{3x^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{p}{x} &= \int -\frac{1}{3x^2} dx \\ \frac{p}{x} &= \frac{1}{3x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = \frac{1}{3} + c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{3\left(\frac{1}{3} + c_1 x\right)^2 + 1}{6c_1}$$

Summary

The solution(s) found are the following

$$y = x \tag{1}$$

$$y = -\frac{x}{3} \tag{2}$$

$$y = \frac{3\left(\frac{1}{3} + c_1 x\right)^2 + 1}{6c_1} \tag{3}$$

Verification of solutions

$$y = x$$

Verified OK.

$$y = -\frac{x}{3}$$

Verified OK.

$$y = \frac{3\left(\frac{1}{3}+c_1 x\right)^2+1}{6c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 32

```
dsolve(3*x*diff(y(x),x)^2-6*y(x)*diff(y(x),x)+x+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = x$$
$$y(x) = -\frac{x}{3}$$
$$y(x) = \frac{4c_1^2+2c_1 x+x^2}{6c_1}$$

✓ Solution by Mathematica

Time used: 0.505 (sec). Leaf size: 67

```
DSolve[3*x*(y'[x])^2-6*y[x]*y'[x]+x+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{3}x\left(-1 + 2\cosh\left(-\log(x) + \sqrt{3}c_1\right)\right)$$
$$y(x) \to -\frac{1}{3}x\left(-1 + 2\cosh\left(\log(x) + \sqrt{3}c_1\right)\right)$$
$$y(x) \to -\frac{x}{3}$$
$$y(x) \to x$$

17.5 problem Ex 5

Internal problem ID [11229]
Internal file name [`OUTPUT/10214_Wednesday_December_07_2022_01_20_43_PM_35591546/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 28. Summary. Page 59
Problem number: Ex 5.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert", "first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, _dAlembert]
```

$$\boxed{y - {y'}^2(1+x) = 0}$$

17.5.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{1+x}, g = y$. Hence the ode is

$$(y')^2 = \frac{y}{1+x}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{1+x} > 0$$
$$y > 0$$

Under the above assumption the differential equations become separable and can be written as

$$y' = \sqrt{f}\,\sqrt{g}$$
$$y' = -\sqrt{f}\,\sqrt{g}$$

Therefore

$$\frac{1}{\sqrt{g}}\,dy = \left(\sqrt{f}\right)\,dx$$
$$-\frac{1}{\sqrt{g}}\,dy = \left(\sqrt{f}\right)\,dx$$

Replacing $f(x), g(y)$ by their values gives

$$\frac{1}{\sqrt{y}}\,dy = \left(\sqrt{\frac{1}{1+x}}\right)\,dx$$
$$-\frac{1}{\sqrt{y}}\,dy = \left(\sqrt{\frac{1}{1+x}}\right)\,dx$$

Integrating now gives the solutions.

$$\int \frac{1}{\sqrt{y}}dy = \int \sqrt{\frac{1}{1+x}}dx + c_1$$
$$\int -\frac{1}{\sqrt{y}}dy = \int \sqrt{\frac{1}{1+x}}dx + c_1$$

Integrating gives

$$2\sqrt{y} = 2(1+x)\sqrt{\frac{1}{1+x}} + c_1$$
$$-2\sqrt{y} = 2(1+x)\sqrt{\frac{1}{1+x}} + c_1$$

Therefore

$$y = \sqrt{\frac{1}{1+x}}\,xc_1 + \sqrt{\frac{1}{1+x}}\,c_1 + \frac{c_1^2}{4} + x + 1$$

$$y = \sqrt{\frac{1}{1+x}}\,xc_1 + \sqrt{\frac{1}{1+x}}\,c_1 + \frac{c_1^2}{4} + x + 1$$

Summary
The solution(s) found are the following

$$y = \sqrt{\frac{1}{1+x}}\, x c_1 + \sqrt{\frac{1}{1+x}}\, c_1 + \frac{c_1^2}{4} + x + 1 \tag{1}$$

$$y = \sqrt{\frac{1}{1+x}}\, x c_1 + \sqrt{\frac{1}{1+x}}\, c_1 + \frac{c_1^2}{4} + x + 1 \tag{2}$$

Verification of solutions

$$y = \sqrt{\frac{1}{1+x}}\, x c_1 + \sqrt{\frac{1}{1+x}}\, c_1 + \frac{c_1^2}{4} + x + 1$$

Verified OK. `{0 < y, 0 < 1/(1+x)}`

$$y = \sqrt{\frac{1}{1+x}}\, x c_1 + \sqrt{\frac{1}{1+x}}\, c_1 + \frac{c_1^2}{4} + x + 1$$

Verified OK. `{0 < y, 0 < 1/(1+x)}`

17.5.2 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y - p^2(1+x) = 0$$

Solving for y from the above results in

$$y = p^2 x + p^2 \tag{1A}$$

This has the form

$$y = x f(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (x f' + g') \frac{dp}{dx} \\ p - f &= (x f' + g') \frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= p^2 \\ g &= p^2 \end{aligned}$$

Hence (2) becomes

$$-p^2 + p = (2xp + 2p)\, p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p^2 + p = 0$$

Solving for p from the above gives

$$\begin{aligned} p &= 0 \\ p &= 1 \end{aligned}$$

Substituting these in (1A) gives

$$\begin{aligned} y &= 0 \\ y &= 1 + x \end{aligned}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{-p(x)^2 + p(x)}{2p(x)\, x + 2p(x)} \tag{3}$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{2 + 2x} \\ q(x) &= \frac{1}{2 + 2x} \end{aligned}$$

Hence the ode is

$$p'(x) + \frac{p(x)}{2 + 2x} = \frac{1}{2 + 2x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{1}{2+2x} dx} \\ &= \sqrt{1 + x} \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu p) = (\mu)\left(\frac{1}{2+2x}\right)$$
$$\frac{d}{dx}\left(\sqrt{1+x}\,p\right) = \left(\sqrt{1+x}\right)\left(\frac{1}{2+2x}\right)$$
$$d\left(\sqrt{1+x}\,p\right) = \left(\frac{1}{2\sqrt{1+x}}\right) dx$$

Integrating gives

$$\sqrt{1+x}\,p = \int \frac{1}{2\sqrt{1+x}}\, dx$$
$$\sqrt{1+x}\,p = \sqrt{1+x} + c_1$$

Dividing both sides by the integrating factor $\mu = \sqrt{1+x}$ results in

$$p(x) = 1 + \frac{c_1}{\sqrt{1+x}}$$

Substituing the above solution for p in (2A) gives

$$y = \left(1 + \frac{c_1}{\sqrt{1+x}}\right)^2 x + \left(1 + \frac{c_1}{\sqrt{1+x}}\right)^2$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$
$$y = 1 + x \tag{2}$$
$$y = \left(1 + \frac{c_1}{\sqrt{1+x}}\right)^2 x + \left(1 + \frac{c_1}{\sqrt{1+x}}\right)^2 \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK. `{0 < y, 0 < 1/(1+x)}`

$$y = 1 + x$$

Verified OK. `{0 < y, 0 < 1/(1+x)}`

$$y = \left(1 + \frac{c_1}{\sqrt{1+x}}\right)^2 x + \left(1 + \frac{c_1}{\sqrt{1+x}}\right)^2$$

Verified OK. `{0 < y, 0 < 1/(1+x)}`

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 53

```
dsolve(y(x)=diff(y(x),x)^2*(x+1),y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = \frac{\left(x+1+\sqrt{(1+x)\,(c_1+1)}\right)^2}{1+x}$$

$$y(x) = \frac{\left(-x-1+\sqrt{(1+x)\,(c_1+1)}\right)^2}{1+x}$$

✓ Solution by Mathematica

Time used: 0.1 (sec). Leaf size: 57

```
DSolve[y[x]==(y'[x])^2*(x+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - c_1\sqrt{x+1} + 1 + \frac{c_1{}^2}{4}$$

$$y(x) \to x + c_1\sqrt{x+1} + 1 + \frac{c_1{}^2}{4}$$

$$y(x) \to 0$$

17.6 problem Ex 6

Internal problem ID [11230]
Internal file name [`OUTPUT/10215_Wednesday_December_07_2022_01_21_18_PM_32489092/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 28. Summary. Page 59
Problem number: Ex 6.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{(y'x - y)(x + yy') - a^2y' = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{y^2 + a^2 - x^2 + \sqrt{y^4 + 2y^2a^2 + 2x^2y^2 + a^4 - 2a^2x^2 + x^4}}{2yx} \tag{1}$$

$$y' = \frac{y^2 + a^2 - x^2 - \sqrt{y^4 + 2y^2a^2 + 2x^2y^2 + a^4 - 2a^2x^2 + x^4}}{2yx} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{y^2 + a^2 - x^2 + \sqrt{a^4 - 2a^2x^2 + 2y^2a^2 + x^4 + 2x^2y^2 + y^4}}{2yx}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 3 to use as anstaz gives

$$\xi = x^3 a_7 + x^2 y a_8 + x\,y^2 a_9 + y^3 a_{10} + x^2 a_4 + x y a_5 + y^2 a_6 + x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x^3 b_7 + x^2 y b_8 + x\,y^2 b_9 + y^3 b_{10} + x^2 b_4 + x y b_5 + y^2 b_6 + x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&3x^2 b_7 + 2xyb_8 + y^2 b_9 + 2xb_4 + yb_5 + b_2 \\
&+ \frac{\left(y^2 + a^2 - x^2 + \sqrt{a^4 - 2a^2x^2 + 2y^2a^2 + x^4 + 2x^2y^2 + y^4}\right)\left(-3x^2 a_7 + x^2 b_8 - 2xya_8 + 2xyb_9 - y^2 a_9 + 3y\right.}{2yx} \\
&- \frac{\left(y^2 + a^2 - x^2 + \sqrt{a^4 - 2a^2x^2 + 2y^2a^2 + x^4 + 2x^2y^2 + y^4}\right)^2 \left(x^2 a_8 + 2xya_9 + 3y^2 a_{10} + xa_5 + 2ya_6 + a_3\right)}{4y^2x^2} \\
&- \left(-\frac{y^2 + a^2 - x^2 + \sqrt{a^4 - 2a^2x^2 + 2y^2a^2 + x^4 + 2x^2y^2 + y^4}}{2y\,x^2} \right. \\
&\left. + \frac{-2x + \frac{-4a^2x + 4x^3 + 4x\,y^2}{2\sqrt{a^4 - 2a^2x^2 + 2y^2a^2 + x^4 + 2x^2y^2 + y^4}}}{2yx}\right)\left(x^3 a_7 + x^2 y a_8\right. \\
&\left. + x\,y^2 a_9 + y^3 a_{10} + x^2 a_4 + xya_5 + y^2 a_6 + xa_2 + ya_3 + a_1\right) \\
&- \left(-\frac{y^2 + a^2 - x^2 + \sqrt{a^4 - 2a^2x^2 + 2y^2a^2 + x^4 + 2x^2y^2 + y^4}}{2y^2x} \right. \\
&\left. + \frac{2y + \frac{4a^2y + 4x^2y + 4y^3}{2\sqrt{a^4 - 2a^2x^2 + 2y^2a^2 + x^4 + 2x^2y^2 + y^4}}}{2yx}\right)\left(x^3 b_7 + x^2 y b_8\right. \\
&\left. + x\,y^2 b_9 + y^3 b_{10} + x^2 b_4 + xyb_5 + y^2 b_6 + xb_2 + yb_3 + b_1\right) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{a^4 - 2a^2x^2 + 2y^2a^2 + x^4 + 2x^2y^2 + y^4}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{a^4 - 2a^2x^2 + 2y^2a^2 + x^4 + 2x^2y^2 + y^4} = v_3\right\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\text{Expression too large to display} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_6 &= 0\\
-4a_{10} &= 0\\
-14a^2a_{10} &= 0\\
-10a^2a_{10} &= 0\\
-2a^4a_3 &= 0\\
-2a^6a_3 &= 0\\
-2a_4 - 4a_6 &= 0\\
-2a_5 - 2b_4 &= 0\\
-2a_5 + 2b_6 &= 0\\
-2a_5 + 6b_6 &= 0\\
2a_5 + 2b_4 &= 0\\
-4a_9 + 4b_{10} &= 0\\
-2b_7 - 2a_8 &= 0\\
2b_7 + 2a_8 &= 0\\
-8a_4 + 2a_6 + 4b_5 &= 0\\
-6a_4 + 4a_6 + 4b_5 &= 0\\
-2a_4 + 2a_6 + 4b_5 &= 0\\
6a_4 - 4a_6 - 4b_5 &= 0\\
-6a_5 - 2b_4 + 8b_6 &= 0\\
4a_5 + 6b_4 - 6b_6 &= 0\\
-4a_7 - 8a_9 + 12b_{10} &= 0\\
-4a_8 - 6a_{10} + 2b_9 &= 0\\
-4a_9 + 8a_7 - 4b_8 &= 0\\
4a_9 + 4b_8 - 8a_7 &= 0\\
2a_{10} + 6b_9 - 4a_8 &= 0\\
6a_{10} + 6b_9 - 4a_8 &= 0\\
8b_{10} - 12a_7 + 4b_8 &= 0\\
-10a_8 + 4a_{10} - 2b_7 + 8b_9 &= 0\\
-6a_{10} + 10b_7 + 6a_8 - 6b_9 &= 0\\
8b_8 - 4a_7 + 4a_9 - 8b_{10} &= 0\\
-8a^2a_6 + 2a_1 &= 0\\
-6a^2a_6 + 2a_1 &= 0\\
-10a^4a_6 + 4a^2a_1 &= 0\\
-4a^4a_6 + 2a^2a_1 &= 0\\
-4a^6a_6 + 2a^4a_1 &= 0\\
-16a^4a_{10} - 2a^2a_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= -a^2 b_{10} \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
a_7 &= b_{10} \\
a_8 &= 0 \\
a_9 &= b_{10} \\
a_{10} &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= a^2 b_{10} \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= 0 \\
b_7 &= 0 \\
b_8 &= b_{10} \\
b_9 &= 0 \\
b_{10} &= b_{10}
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -a^2 x + x^3 + x\,y^2 \\
\eta &= a^2 y + x^2 y + y^3
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}\right) S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{a^2 y + x^2 y + y^3}{-a^2 x + x^3 + x\,y^2}\\ &= -\frac{y(a^2+x^2+y^2)}{x\,(a^2-x^2-y^2)}\end{aligned}$$

This is easily solved to give

$$\frac{1}{\frac{1}{y^2}-\frac{1}{a^2-x^2}} = -\frac{\sqrt{x-a}\,\sqrt{x+a}\,x}{\sqrt{c_1+8a^2\left(\frac{1}{4a(x-a)}-\frac{1}{4a(x+a)}\right)}} - \frac{(a-x)\,(x+a)}{2}$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{4a^4-8a^2x^2+8y^2a^2+4x^4+8x^2y^2+4y^4}{a^4-2a^2x^2+2y^2a^2+x^4-2x^2y^2+y^4}$$

Since ξ depends on y and η depends on x then we can use either one to find S. Let us use

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{-a^2x+x^3+x\,y^2}\end{aligned}$$

But we have now to replace y in ξ from its value from the solution of $\frac{dy}{dx}=\frac{\eta}{\xi}$ found above. This results in

$$\xi = -a^2x+x^3+x\left(-\frac{\sqrt{x-a}\,\sqrt{x+a}\,x}{\sqrt{c_1+8a^2\left(\frac{1}{4a(x-a)}-\frac{1}{4a(x+a)}\right)}}-\frac{(a-x)\,(x+a)}{2}\right)^2$$

Integrating gives

$$\begin{aligned}S &= \frac{dx}{-a^2x+x^3+x\left(-\frac{\sqrt{x-a}\,\sqrt{x+a}\,x}{\sqrt{c_1+8a^2\left(\frac{1}{4a(x-a)}-\frac{1}{4a(x+a)}\right)}}-\frac{(a-x)(x+a)}{2}\right)^2}\\ &= \text{Expression too large to display}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Replacing back $c_1 = \frac{4a^4-8a^2x^2+8y^2a^2+4x^4+8x^2y^2+4y^4}{a^4-2a^2x^2+2y^2a^2+x^4-2x^2y^2+y^4}$ then the above becomes

$$S = \text{Expression too large to display}$$

Unable to determine ODE type.

Solving equation (2)

Writing the ode as

$$y' = -\frac{-a^2+x^2-y^2+\sqrt{a^4-2a^2x^2+2y^2a^2+x^4+2x^2y^2+y^4}}{2xy}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 3 to use as anstaz gives

$$\xi = x^3a_7 + x^2ya_8 + x\,y^2a_9 + y^3a_{10} + x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = x^3b_7 + x^2yb_8 + x\,y^2b_9 + y^3b_{10} + x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&3x^2b_7 + 2xyb_8 + y^2b_9 + 2xb_4 + yb_5 + b_2 \qquad (5E)\\
&- \frac{\left(-a^2+x^2-y^2+\sqrt{a^4-2a^2x^2+2y^2a^2+x^4+2x^2y^2+y^4}\right)\left(-3x^2a_7+x^2b_8-2xya_8+2xyb_9-y^2a_9+3\right.}{2xy}\\
&- \frac{\left(-a^2+x^2-y^2+\sqrt{a^4-2a^2x^2+2y^2a^2+x^4+2x^2y^2+y^4}\right)^2\left(x^2a_8+2xya_9+3y^2a_{10}+xa_5+2ya_6+a_3\right.}{4x^2y^2}\\
&- \left(-\frac{2x+\frac{-4a^2x+4x^3+4x\,y^2}{2\sqrt{a^4-2a^2x^2+2y^2a^2+x^4+2x^2y^2+y^4}}}{2xy}\right.\\
&\left.+ \frac{-a^2+x^2-y^2+\sqrt{a^4-2a^2x^2+2y^2a^2+x^4+2x^2y^2+y^4}}{2x^2y}\right)\left(x^3a_7\right.\\
&\left.+ x^2ya_8 + x\,y^2a_9 + y^3a_{10} + x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1\right)\\
&- \left(-\frac{-2y+\frac{4a^2y+4x^2y+4y^3}{2\sqrt{a^4-2a^2x^2+2y^2a^2+x^4+2x^2y^2+y^4}}}{2xy}\right.\\
&\left.+ \frac{-a^2+x^2-y^2+\sqrt{a^4-2a^2x^2+2y^2a^2+x^4+2x^2y^2+y^4}}{2x\,y^2}\right)\left(x^3b_7\right.\\
&\left.+ x^2yb_8 + x\,y^2b_9 + y^3b_{10} + x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1\right) = 0
\end{aligned}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{a^4 - 2a^2x^2 + 2y^2a^2 + x^4 + 2x^2y^2 + y^4}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{a^4 - 2a^2x^2 + 2y^2a^2 + x^4 + 2x^2y^2 + y^4} = v_3\right\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\text{Expression too large to display} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_6 &= 0 \\
2a_6 &= 0 \\
-4a_{10} &= 0 \\
4a_{10} &= 0 \\
-10a^2 a_{10} &= 0 \\
14a^2 a_{10} &= 0 \\
-2a^4 a_3 &= 0 \\
2a^6 a_3 &= 0 \\
2a_4 + 4a_6 &= 0 \\
-2a_5 - 2b_4 &= 0 \\
-2a_5 + 2b_6 &= 0 \\
2a_5 - 6b_6 &= 0 \\
2a_5 - 2b_6 &= 0 \\
-2a_8 - 2b_7 &= 0 \\
-4a_9 + 4b_{10} &= 0 \\
4a_9 - 4b_{10} &= 0 \\
-2a_4 + 2a_6 + 4b_5 &= 0 \\
6a_4 - 4a_6 - 4b_5 &= 0 \\
8a_4 - 2a_6 - 4b_5 &= 0 \\
4a_5 + 6b_4 - 6b_6 &= 0 \\
6a_5 + 2b_4 - 8b_6 &= 0 \\
4a_7 + 8a_9 - 12b_{10} &= 0 \\
-4a_8 + 2a_{10} + 6b_9 &= 0 \\
4a_8 + 6a_{10} - 2b_9 &= 0 \\
-4b_8 - 4a_9 + 8a_7 &= 0 \\
-6b_9 - 6a_{10} + 4a_8 &= 0 \\
-8b_{10} + 12a_7 - 4b_8 &= 0 \\
-4a_7 + 4a_9 - 8b_{10} + 8b_8 &= 0 \\
6a_8 - 6b_9 - 6a_{10} + 10b_7 &= 0 \\
10a_8 - 4a_{10} + 2b_7 - 8b_9 &= 0 \\
-6a^2 a_6 + 2a_1 &= 0 \\
8a^2 a_6 - 2a_1 &= 0 \\
-4a^4 a_6 + 2a^2 a_1 &= 0 \\
10a^4 a_6 - 4a^2 a_1 &= 0 \\
4a^6 a_6 - 2a^4 a_1 &= 0 \\
-6a^4 a_{10} - 2a^2 a_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= -a^2 b_{10}\\
a_3 &= 0\\
a_4 &= 0\\
a_5 &= 0\\
a_6 &= 0\\
a_7 &= b_{10}\\
a_8 &= 0\\
a_9 &= b_{10}\\
a_{10} &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= a^2 b_{10}\\
b_4 &= 0\\
b_5 &= 0\\
b_6 &= 0\\
b_7 &= 0\\
b_8 &= b_{10}\\
b_9 &= 0\\
b_{10} &= b_{10}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -a^2 x + x^3 + x\,y^2\\
\eta &= a^2 y + x^2 y + y^3
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi\\
&= a^2 y + x^2 y + y^3 - \left(-\frac{-a^2 + x^2 - y^2 + \sqrt{a^4 - 2a^2x^2 + 2y^2a^2 + x^4 + 2x^2y^2 + y^4}}{2xy}\right)\left(-a^2 x + x^3 + x\,y^2\right)\\
&= \frac{a^4 x - 2x^3a^2 + 2a^2 x\,y^2 + x^5 + 2y^2x^3 + x\,y^4 - \sqrt{a^4 - 2a^2x^2 + 2y^2a^2 + x^4 + 2x^2y^2 + y^4}\,a^2 x + x^3\sqrt{a^4 -}}{2xy}
\end{aligned}
$$

$$\xi = 0$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{a^4x-2x^3a^2+2a^2x\,y^2+x^5+2y^2x^3+x\,y^4-\sqrt{a^4-2a^2x^2+2y^2a^2+x^4+2x^2y^2+y^4}\,a^2x+x^3\sqrt{a^4-2a^2x^2+2y^2a^2+x^4+2x^2y^2+y^4}+x\,y^2\sqrt{a^4-}}{2xy}} \end{aligned}$$

Which results in

$$S = -\frac{\ln\left(\frac{2a^4-4a^2x^2+2x^4+(2a^2+2x^2)y^2+2\sqrt{(-a^2+x^2)^2}\,\sqrt{y^4+(2a^2+2x^2)y^2+a^4-2a^2x^2+x^4}}{y^2}\right)}{4\sqrt{(-a^2+x^2)^2}} + \frac{\ln(a^2-2ax+x^2+y^2)}{16ax} -$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-a^2 + x^2 - y^2 + \sqrt{a^4 - 2a^2x^2 + 2y^2a^2 + x^4 + 2x^2y^2 + y^4}}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x}{\sqrt{(a^2-2ax+x^2+y^2)(a^2+2ax+x^2+y^2)}\left(a^2+x^2+y^2+\sqrt{(a^2-2ax+x^2+y^2)(a^2+2ax+x}\right.} \\ S_y &= \frac{(x^4+(-2a^2+y^2)x^2+a^4)\sqrt{(a^2-2ax+x^2+y^2)(a^2+2ax+x^2+y^2)}-x^6+(3a^2-2y^2)x^4}{\sqrt{(a^2-2ax+x^2+y^2)(a^2+2ax+x^2+y^2)}\left((a^2-x^2)\sqrt{(a^2-2ax+x^2+y^2)(a^2+2ax+x^2+y}\right.} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{-\ln\left(\left(a^2-x^2\right)\sqrt{\left(y^2+\left(x+a\right)^2\right)\left(y^2+\left(a-x\right)^2\right)}+a^4+\left(y^2-2x^2\right)a^2+x^4+x^2y^2\right)+4\ln\left(y\right)-\ln\left(a^2\right.}{4a^2}$$

Which simplifies to

$$\frac{-\ln\left(\left(a^2-x^2\right)\sqrt{\left(y^2+\left(x+a\right)^2\right)\left(y^2+\left(a-x\right)^2\right)}+a^4+\left(y^2-2x^2\right)a^2+x^4+x^2y^2\right)+4\ln\left(y\right)-\ln\left(a^2\right.}{4a^2}$$

Summary

The solution(s) found are the following

$$\frac{-\ln\left(\left(a^2-x^2\right)\sqrt{\left(y^2+\left(x+a\right)^2\right)\left(y^2+\left(a-x\right)^2\right)}+a^4+\left(y^2-2x^2\right)a^2+x^4+x^2y^2\right)+4\ln\left(y\right)-\ln\left(a^2\right.}{4a^2} \tag{1}$$

$= c_1$

Verification of solutions

$$\frac{-\ln\left(\left(a^2-x^2\right)\sqrt{\left(y^2+\left(x+a\right)^2\right)\left(y^2+\left(a-x\right)^2\right)}+a^4+\left(y^2-2x^2\right)a^2+x^4+x^2y^2\right)+4\ln\left(y\right)-\ln\left(a^2\right.}{4a^2}$$

$= c_1$

Verified OK.

✗ Solution by Maple

```
dsolve((diff(y(x),x)*x-y(x))*(diff(y(x),x)*y(x)+x)=a^2*diff(y(x),x),y(x), singsol=all)
```

No solution found

✓ Solution by Mathematica

Time used: 0.6 (sec). Leaf size: 75

```
DSolve[(y'[x]*x-y[x])*(y'[x]*y[x]+x)==a^2*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \sqrt{c_1 \left(x^2 - \frac{a^2}{1 + c_1}\right)} \\
y(x) &\to -i(a - x) \\
y(x) &\to i(a - x) \\
y(x) &\to -i(a + x) \\
y(x) &\to i(a + x)
\end{aligned}
$$

17.7 problem Ex 7

Internal problem ID [11231]
Internal file name [OUTPUT/10216_Wednesday_December_07_2022_01_21_21_PM_8757104/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 28. Summary. Page 59
Problem number: Ex 7.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y'^2 + 2y'y\cot(x) - y^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(-1 + \sqrt{\tan(x)^2 + 1}\right) y}{\tan(x)} \tag{1}$$

$$y' = -\frac{\left(1 + \sqrt{\tan(x)^2 + 1}\right) y}{\tan(x)} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\left(-1+\sqrt{\tan(x)^2+1}\right)y}{\tan(x)} \end{aligned}$$

Where $f(x) = \frac{-1+\sqrt{\tan(x)^2+1}}{\tan(x)}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= \frac{-1+\sqrt{\tan(x)^2+1}}{\tan(x)}\,dx \\ \int \frac{1}{y}\,dy &= \int \frac{-1+\sqrt{\tan(x)^2+1}}{\tan(x)}\,dx \\ \ln(y) &= \cos(x)\ln(\csc(x)-\cot(x))\sqrt{\sec(x)^2} - \ln(\csc(x)-\cot(x)) + \ln\left(\frac{2}{\cos(x)+1}\right) + c_1 \\ y &= e^{\cos(x)\ln(\csc(x)-\cot(x))\sqrt{\sec(x)^2}-\ln(\csc(x)-\cot(x))+\ln\left(\frac{2}{\cos(x)+1}\right)+c_1} \\ &= c_1 e^{\cos(x)\ln(\csc(x)-\cot(x))\sqrt{\sec(x)^2}-\ln(\csc(x)-\cot(x))+\ln\left(\frac{2}{\cos(x)+1}\right)} \end{aligned}$$

Which simplifies to

$$y = \frac{2c_1(\csc(x)-\cot(x))^{\cos(x)\sqrt{\sec(x)^2}}}{(\csc(x)-\cot(x))(\cos(x)+1)}$$

Summary

The solution(s) found are the following

$$y = \frac{2c_1(\csc(x)-\cot(x))^{\cos(x)\sqrt{\sec(x)^2}}}{(\csc(x)-\cot(x))(\cos(x)+1)} \tag{1}$$

Verification of solutions

$$y = \frac{2c_1(\csc(x)-\cot(x))^{\cos(x)\sqrt{\sec(x)^2}}}{(\csc(x)-\cot(x))(\cos(x)+1)}$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= -\frac{\left(1+\sqrt{\tan\left(x\right)^{2}+1}\right)y}{\tan\left(x\right)}
\end{aligned}
$$

Where $f(x) = -\frac{1+\sqrt{\tan(x)^2+1}}{\tan(x)}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= -\frac{1+\sqrt{\tan\left(x\right)^{2}+1}}{\tan\left(x\right)}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{1+\sqrt{\tan\left(x\right)^{2}+1}}{\tan\left(x\right)}\,dx \\
\ln\left(y\right) &= -\cos\left(x\right)\ln\left(\csc\left(x\right)-\cot\left(x\right)\right)\sqrt{\sec\left(x\right)^{2}}-\ln\left(\csc\left(x\right)-\cot\left(x\right)\right)+\ln\left(\frac{2}{\cos\left(x\right)+1}\right)+c_2 \\
y &= \mathrm{e}^{-\cos(x)\ln(\csc(x)-\cot(x))\sqrt{\sec(x)^2}-\ln(\csc(x)-\cot(x))+\ln\left(\frac{2}{\cos(x)+1}\right)+c_2} \\
&= c_2\mathrm{e}^{-\cos(x)\ln(\csc(x)-\cot(x))\sqrt{\sec(x)^2}-\ln(\csc(x)-\cot(x))+\ln\left(\frac{2}{\cos(x)+1}\right)}
\end{aligned}
$$

Which simplifies to

$$
y = \frac{2c_2\left(\csc\left(x\right)-\cot\left(x\right)\right)^{-\cos(x)\sqrt{\sec(x)^2}}}{\left(\csc\left(x\right)-\cot\left(x\right)\right)\left(\cos\left(x\right)+1\right)}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2c_2\left(\csc\left(x\right)-\cot\left(x\right)\right)^{-\cos(x)\sqrt{\sec(x)^2}}}{\left(\csc\left(x\right)-\cot\left(x\right)\right)\left(\cos\left(x\right)+1\right)} \tag{1}
$$

Verification of solutions

$$
y = \frac{2c_2\left(\csc\left(x\right)-\cot\left(x\right)\right)^{-\cos(x)\sqrt{\sec(x)^2}}}{\left(\csc\left(x\right)-\cot\left(x\right)\right)\left(\cos\left(x\right)+1\right)}
$$

Verified OK.

17.7.1 Maple step by step solution

Let's solve

$y'^2 + 2y'y\cot(x) - y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{-1+\sqrt{\tan(x)^2+1}}{\tan(x)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{-1+\sqrt{\tan(x)^2+1}}{\tan(x)}dx + c_1$

- Evaluate integral

  $\ln(y) = -\operatorname{arctanh}\left(\frac{1}{\sqrt{\tan(x)^2+1}}\right) - \ln(\tan(x)) + \frac{\ln\left(\tan(x)^2+1\right)}{2} + c_1$

- Solve for y

  $y = \frac{e^{c_1}\cos(x)\left(-1+\sqrt{\frac{1}{\cos(x)^2}}\right)}{\sqrt{\sin(x)^2}\,\sin(x)}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
<- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 39

```
dsolve(diff(y(x),x)^2+2*diff(y(x),x)*y(x)*cot(x)=y(x)^2,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \frac{\operatorname{csgn}(\sin(x)) c_1}{\cos(x) + \operatorname{csgn}(\sec(x))}$$
$$y(x) = \csc(x)^2 (\cos(x) + \operatorname{csgn}(\sec(x))) \operatorname{csgn}(\sin(x)) c_1$$

✓ Solution by Mathematica

Time used: 0.241 (sec). Leaf size: 36

```
DSolve[(y'[x])^2+2*y'[x]*y[x]*Cot[x]==y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \csc^2\left(\frac{x}{2}\right)$$
$$y(x) \to c_1 \sec^2\left(\frac{x}{2}\right)$$
$$y(x) \to 0$$

17.8 problem Ex 8

Internal problem ID [11232]
Internal file name [OUTPUT/10217_Wednesday_December_07_2022_01_21_24_PM_1151142/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 28. Summary. Page 59
Problem number: Ex 8.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _rational, _Clairaut]`

$$\boxed{(x^2+1)\,y'^2 - 2xyy' + y^2 = 1}$$

17.8.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = y'x + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$(x^2+1)\,p^2 - 2xyp + y^2 = 1$$

Solving for y from the above results in

$$y = px + \sqrt{-p^2+1} \tag{1A}$$
$$y = px - \sqrt{-p^2+1} \tag{2A}$$

Each of the above ode's is a Clairaut ode which is now solved. Solving ode 1A We start by replacing y' by p which gives

$$\begin{aligned} y &= px + \sqrt{-p^2+1} \\ &= px + \sqrt{-p^2+1} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = \sqrt{-p^2 + 1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \sqrt{-c_1^2 + 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \sqrt{-p^2 + 1}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{p}{\sqrt{-p^2 + 1}} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{x}{\sqrt{x^2+1}}$$

Substituting the above back in (1) results in

$$y_1 = \sqrt{x^2+1}$$

Solving ode 2A We start by replacing y' by p which gives

$$\begin{aligned} y &= px - \sqrt{-p^2+1} \\ &= px - \sqrt{-p^2+1} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = -\sqrt{-p^2+1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp+g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x+g')\frac{dp}{dx} \\ 0 &= (x+g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_2 x - \sqrt{-c_2^2 + 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -\sqrt{-p^2 + 1}$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x + \frac{p}{\sqrt{-p^2 + 1}} \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = -\frac{x}{\sqrt{x^2 + 1}}$$

Substituting the above back in (1) results in

$$y_1 = -\sqrt{x^2 + 1}$$

Summary

The solution(s) found are the following

$$y = c_1 x + \sqrt{-c_1^2 + 1} \tag{1}$$

$$y = \sqrt{x^2 + 1} \tag{2}$$

$$y = c_2 x - \sqrt{-c_2^2 + 1} \tag{3}$$

$$y = -\sqrt{x^2 + 1} \tag{4}$$

Verification of solutions

$$y = c_1 x + \sqrt{-c_1^2 + 1}$$

Verified OK.

$$y = \sqrt{x^2 + 1}$$

Verified OK.

$$y = c_2 x - \sqrt{-c_2^2 + 1}$$

Verified OK.

$$y = -\sqrt{x^2 + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 57

```
dsolve((1+x^2)*diff(y(x),x)^2-2*x*y(x)*diff(y(x),x)+y(x)^2-1=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x^2+1}$$
$$y(x) = -\sqrt{x^2+1}$$
$$y(x) = c_1x - \sqrt{-c_1^2+1}$$
$$y(x) = c_1x + \sqrt{-c_1^2+1}$$

✓ Solution by Mathematica

Time used: 0.168 (sec). Leaf size: 73

```
DSolve[(1+x^2)*(y'[x])^2-2*x*y[x]*y'[x]+y[x]^2-1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1x - \sqrt{1-c_1{}^2}$$
$$y(x) \to c_1x + \sqrt{1-c_1{}^2}$$
$$y(x) \to -\sqrt{x^2+1}$$
$$y(x) \to \sqrt{x^2+1}$$

17.9 problem Ex 9

Internal problem ID [11233]
Internal file name [OUTPUT/10218_Wednesday_December_07_2022_01_21_26_PM_27725878/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 28. Summary. Page 59
Problem number: Ex 9.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x^2{y'}^2 - 2(yx + 2y')\,y' + y^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{y}{x+2} \tag{1}$$
$$y' = \frac{y}{x-2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{x+2} \end{aligned}$$

Where $f(x) = \frac{1}{x+2}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{x+2}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{x+2}\,dx \\
\ln(y) &= \ln(x+2) + c_1 \\
y &= e^{\ln(x+2)+c_1} \\
&= c_1(x+2)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x+2) \tag{1}
$$

Verification of solutions

$$
y = c_1(x+2)
$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{y}{x-2}
\end{aligned}
$$

Where $f(x) = \frac{1}{x-2}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{x-2}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{x-2}\,dx \\
\ln(y) &= \ln(x-2) + c_2 \\
y &= e^{\ln(x-2)+c_2} \\
&= c_2(x-2)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2(x-2) \tag{1}
$$

Verification of solutions

$$y = c_2(x - 2)$$

Verified OK.

17.9.1 Maple step by step solution

Let's solve

$x^2{y'}^2 - 2(yx + 2y')\,y' + y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x+2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{x+2}dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x+2) + c_1$

- Solve for y

  $y = \mathrm{e}^{c_1}(x+2)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x)^2-2*(x*y(x)+2*diff(y(x),x))*diff(y(x),x)+y(x)^2=0,y(x), singsol=all)
```

$$y(x) = c_1(x-2)$$
$$y(x) = c_1(x+2)$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 26

```
DSolve[x^2*(y'[x])^2-2*(x*y[x]+2*y'[x])*y'[x]+y[x]^2==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to c_1(x-2)$$
$$y(x) \to c_1(x+2)$$
$$y(x) \to 0$$

17.10 problem Ex 10

Internal problem ID [11234]
Internal file name [`OUTPUT/10219_Wednesday_December_07_2022_01_21_27_PM_89764836/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 28. Summary. Page 59
Problem number: Ex 10.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries]]`

$$\boxed{y - y'x - \frac{y{y'}^2}{x^2} = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(-x^2 + \sqrt{x^4 + 4y^2}\right) x}{2y} \tag{1}$$

$$y' = -\frac{\left(x^2 + \sqrt{x^4 + 4y^2}\right) x}{2y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{\left(-x^2 + \sqrt{x^4 + 4y^2}\right) x}{2y}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{\left(-x^2 + \sqrt{x^4 + 4y^2}\right) x (b_3 - a_2)}{2y} - \frac{\left(-x^2 + \sqrt{x^4 + 4y^2}\right)^2 x^2 a_3}{4y^2} \\ & - \left(\frac{\left(-2x + \frac{2x^3}{\sqrt{x^4+4y^2}}\right) x}{2y} + \frac{-x^2 + \sqrt{x^4 + 4y^2}}{2y} \right) (x a_2 + y a_3 + a_1) \\ & - \left(-\frac{\left(-x^2 + \sqrt{x^4 + 4y^2}\right) x}{2y^2} + \frac{2x}{\sqrt{x^4 + 4y^2}} \right) (x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-2x^8 a_3 + \sqrt{x^4 + 4y^2}\, x^6 a_3 - 2x^6 b_2 + 8x^5 y a_2 - 4x^5 y b_3 - 2x^4 y^2 a_3 + \left(x^4 + 4y^2\right)^{\frac{3}{2}} x^2 a_3 + 2\sqrt{x^4 + 4y^2}\, x^4 b_2}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 2x^8 a_3 - \sqrt{x^4 + 4y^2}\, x^6 a_3 + 2x^6 b_2 - 8x^5 y a_2 + 4x^5 y b_3 + 2x^4 y^2 a_3 \\ & - \left(x^4 + 4y^2\right)^{\frac{3}{2}} x^2 a_3 - 2\sqrt{x^4 + 4y^2}\, x^4 b_2 + 8\sqrt{x^4 + 4y^2}\, x^3 y a_2 \\ & - 4\sqrt{x^4 + 4y^2}\, x^3 y b_3 + 6\sqrt{x^4 + 4y^2}\, x^2 y^2 a_3 + 2x^5 b_1 - 6x^4 y a_1 \\ & - 2\sqrt{x^4 + 4y^2}\, x^3 b_1 + 6\sqrt{x^4 + 4y^2}\, x^2 y a_1 - 16x\, y^3 a_2 \\ & + 8x\, y^3 b_3 - 8y^4 a_3 + 4b_2 y^2 \sqrt{x^4 + 4y^2} - 8y^3 a_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&-\sqrt{x^4+4y^2}\, x^6 a_3+2\left(x^4+4y^2\right) x^4 a_3-4x^5 y a_2-4x^4 y^2 a_3\\
&-\left(x^4+4y^2\right)^{\frac{3}{2}} x^2 a_3-2\sqrt{x^4+4y^2}\, x^4 b_2+8\sqrt{x^4+4y^2}\, x^3 y a_2\\
&-4\sqrt{x^4+4y^2}\, x^3 y b_3+6\sqrt{x^4+4y^2}\, x^2 y^2 a_3-4x^4 y a_1+2\left(x^4+4y^2\right) x^2 b_2\\
&-4\left(x^4+4y^2\right) x y a_2+4\left(x^4+4y^2\right) x y b_3-2\left(x^4+4y^2\right) y^2 a_3\\
&-2\sqrt{x^4+4y^2}\, x^3 b_1+6\sqrt{x^4+4y^2}\, x^2 y a_1-8x^2 y^2 b_2-8x\, y^3 b_3\\
&+2\left(x^4+4y^2\right) x b_1-2\left(x^4+4y^2\right) y a_1+4b_2 y^2\sqrt{x^4+4y^2}-8x\, y^2 b_1=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&2x^8 a_3-2\sqrt{x^4+4y^2}\, x^6 a_3+2x^6 b_2-8x^5 y a_2+4x^5 y b_3+2x^4 y^2 a_3+2x^5 b_1\\
&-2\sqrt{x^4+4y^2}\, x^4 b_2-6x^4 y a_1+8\sqrt{x^4+4y^2}\, x^3 y a_2-4\sqrt{x^4+4y^2}\, x^3 y b_3\\
&+2\sqrt{x^4+4y^2}\, x^2 y^2 a_3-2\sqrt{x^4+4y^2}\, x^3 b_1+6\sqrt{x^4+4y^2}\, x^2 y a_1\\
&-16x\, y^3 a_2+8x\, y^3 b_3-8y^4 a_3+4b_2 y^2\sqrt{x^4+4y^2}-8y^3 a_1=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^4+4y^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{x^4+4y^2}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2v_1^8 a_3-2v_3 v_1^6 a_3-8v_1^5 v_2 a_2+2v_1^4 v_2^2 a_3+2v_1^6 b_2+4v_1^5 v_2 b_3-6v_1^4 v_2 a_1\\
&+8v_3 v_1^3 v_2 a_2+2v_3 v_1^2 v_2^2 a_3+2v_1^5 b_1-2v_3 v_1^4 b_2-4v_3 v_1^3 v_2 b_3+6v_3 v_1^2 v_2 a_1\\
&-16v_1 v_2^3 a_2-8v_2^4 a_3-2v_3 v_1^3 b_1+8v_1 v_2^3 b_3-8v_2^3 a_1+4b_2 v_2^2 v_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&2v_1^8a_3 - 2v_3v_1^6a_3 + 2v_1^6b_2 + (-8a_2 + 4b_3)\, v_1^5v_2 + 2v_1^5b_1 + 2v_1^4v_2^2a_3 \\
&- 6v_1^4v_2a_1 - 2v_3v_1^4b_2 + (8a_2 - 4b_3)\, v_1^3v_2v_3 - 2v_3v_1^3b_1 + 2v_3v_1^2v_2^2a_3 \\
&+ 6v_3v_1^2v_2a_1 + (-16a_2 + 8b_3)\, v_1v_2^3 - 8v_2^4a_3 - 8v_2^3a_1 + 4b_2v_2^2v_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-8a_1 &= 0 \\
-6a_1 &= 0 \\
6a_1 &= 0 \\
-8a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
-2b_1 &= 0 \\
2b_1 &= 0 \\
-2b_2 &= 0 \\
2b_2 &= 0 \\
4b_2 &= 0 \\
-16a_2 + 8b_3 &= 0 \\
-8a_2 + 4b_3 &= 0 \\
8a_2 - 4b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 2y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 2y - \left(\frac{\left(-x^2+\sqrt{x^4+4y^2}\right)x}{2y}\right)(x) \\
&= \frac{x^4 - \sqrt{x^4+4y^2}\,x^2 + 4y^2}{2y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta}dy \\
&= \int \frac{1}{\frac{x^4-\sqrt{x^4+4y^2}\,x^2+4y^2}{2y}}dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{2} - \frac{x^2\ln\left(\frac{2x^4+2\sqrt{x^4}\,\sqrt{x^4+4y^2}}{y}\right)}{2\sqrt{x^4}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\left(-x^2+\sqrt{x^4+4y^2}\right)x}{2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= -\frac{x^2+\sqrt{x^4+4y^2}}{x\sqrt{x^4+4y^2}}\\
S_y &= \frac{\sqrt{x^4+4y^2}\,x^2+x^4+2y^2}{y\sqrt{x^4+4y^2}\,\left(x^2+\sqrt{x^4+4y^2}\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{x^4+\sqrt{x^4+4y^2}\,x^2+4y^2}{x\sqrt{x^4+4y^2}\,\left(x^2+\sqrt{x^4+4y^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

Which gives

$$y = e^{\frac{\ln(2)}{2}+\frac{\ln\left(2\,e^{-2c_1}e^{2c_1}x^2+8\,e^{-2c_1}e^{4c_1}\right)}{2}+c_1}$$

<u>Summary</u>

The solution(s) found are the following

$$y = e^{\frac{\ln(2)}{2}+\frac{\ln\left(2\,e^{-2c_1}e^{2c_1}x^2+8\,e^{-2c_1}e^{4c_1}\right)}{2}+c_1} \tag{1}$$

<u>Verification of solutions</u>

$$y = e^{\frac{\ln(2)}{2}+\frac{\ln\left(2\,e^{-2c_1}e^{2c_1}x^2+8\,e^{-2c_1}e^{4c_1}\right)}{2}+c_1}$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{\left(x^2+\sqrt{x^4+4y^2}\right)x}{2y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2-\frac{\left(x^2+\sqrt{x^4+4y^2}\right)x(b_3-a_2)}{2y}-\frac{\left(x^2+\sqrt{x^4+4y^2}\right)^2x^2a_3}{4y^2} - \left(-\frac{\left(2x+\frac{2x^3}{\sqrt{x^4+4y^2}}\right)x}{2y}-\frac{x^2+\sqrt{x^4+4y^2}}{2y}\right)(xa_2+ya_3+a_1) - \left(-\frac{2x}{\sqrt{x^4+4y^2}}+\frac{\left(x^2+\sqrt{x^4+4y^2}\right)x}{2y^2}\right)(xb_2+yb_3+b_1)=0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^8a_3+\sqrt{x^4+4y^2}\,x^6a_3+2x^6b_2-8x^5ya_2+4x^5yb_3+2x^4y^2a_3+\left(x^4+4y^2\right)^{\frac{3}{2}}x^2a_3+2\sqrt{x^4+4y^2}\,x^4b_2-}{}$$
$$=0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-2x^{8}a_{3}-\sqrt{x^{4}+4y^{2}}\,x^{6}a_{3}-2x^{6}b_{2}+8x^{5}ya_{2}-4x^{5}yb_{3}-2x^{4}y^{2}a_{3}\\
&-\left(x^{4}+4y^{2}\right)^{\frac{3}{2}}x^{2}a_{3}-2\sqrt{x^{4}+4y^{2}}\,x^{4}b_{2}+8\sqrt{x^{4}+4y^{2}}\,x^{3}ya_{2}\\
&-4\sqrt{x^{4}+4y^{2}}\,x^{3}yb_{3}+6\sqrt{x^{4}+4y^{2}}\,x^{2}y^{2}a_{3}-2x^{5}b_{1}+6x^{4}ya_{1}\\
&-2\sqrt{x^{4}+4y^{2}}\,x^{3}b_{1}+6\sqrt{x^{4}+4y^{2}}\,x^{2}ya_{1}+16x\,y^{3}a_{2}\\
&-8x\,y^{3}b_{3}+8y^{4}a_{3}+4b_{2}y^{2}\sqrt{x^{4}+4y^{2}}+8y^{3}a_{1}=0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-\sqrt{x^{4}+4y^{2}}\,x^{6}a_{3}-2\left(x^{4}+4y^{2}\right)x^{4}a_{3}+4x^{5}ya_{2}+4x^{4}y^{2}a_{3}\\
&-\left(x^{4}+4y^{2}\right)^{\frac{3}{2}}x^{2}a_{3}-2\sqrt{x^{4}+4y^{2}}\,x^{4}b_{2}+8\sqrt{x^{4}+4y^{2}}\,x^{3}ya_{2}\\
&-4\sqrt{x^{4}+4y^{2}}\,x^{3}yb_{3}+6\sqrt{x^{4}+4y^{2}}\,x^{2}y^{2}a_{3}+4x^{4}ya_{1}-2\left(x^{4}+4y^{2}\right)x^{2}b_{2}\\
&+4\left(x^{4}+4y^{2}\right)xya_{2}-4\left(x^{4}+4y^{2}\right)xyb_{3}+2\left(x^{4}+4y^{2}\right)y^{2}a_{3}\\
&-2\sqrt{x^{4}+4y^{2}}\,x^{3}b_{1}+6\sqrt{x^{4}+4y^{2}}\,x^{2}ya_{1}+8x^{2}y^{2}b_{2}+8x\,y^{3}b_{3}\\
&-2\left(x^{4}+4y^{2}\right)xb_{1}+2\left(x^{4}+4y^{2}\right)ya_{1}+4b_{2}y^{2}\sqrt{x^{4}+4y^{2}}+8x\,y^{2}b_{1}=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-2x^{8}a_{3}-2\sqrt{x^{4}+4y^{2}}\,x^{6}a_{3}-2x^{6}b_{2}+8x^{5}ya_{2}-4x^{5}yb_{3}-2x^{4}y^{2}a_{3}-2x^{5}b_{1}\\
&-2\sqrt{x^{4}+4y^{2}}\,x^{4}b_{2}+6x^{4}ya_{1}+8\sqrt{x^{4}+4y^{2}}\,x^{3}ya_{2}-4\sqrt{x^{4}+4y^{2}}\,x^{3}yb_{3}\\
&+2\sqrt{x^{4}+4y^{2}}\,x^{2}y^{2}a_{3}-2\sqrt{x^{4}+4y^{2}}\,x^{3}b_{1}+6\sqrt{x^{4}+4y^{2}}\,x^{2}ya_{1}\\
&+16x\,y^{3}a_{2}-8x\,y^{3}b_{3}+8y^{4}a_{3}+4b_{2}y^{2}\sqrt{x^{4}+4y^{2}}+8y^{3}a_{1}=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^{4}+4y^{2}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_{1}, y=v_{2}, \sqrt{x^{4}+4y^{2}}=v_{3}\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2v_1^8a_3 - 2v_3v_1^6a_3 + 8v_1^5v_2a_2 - 2v_1^4v_2^2a_3 - 2v_1^6b_2 - 4v_1^5v_2b_3 + 6v_1^4v_2a_1 \\
&+ 8v_3v_1^3v_2a_2 + 2v_3v_1^2v_2^2a_3 - 2v_1^5b_1 - 2v_3v_1^4b_2 - 4v_3v_1^3v_2b_3 + 6v_3v_1^2v_2a_1 \\
&+ 16v_1v_2^3a_2 + 8v_2^4a_3 - 2v_3v_1^3b_1 - 8v_1v_2^3b_3 + 8v_2^3a_1 + 4b_2v_2^2v_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-2v_1^8a_3 - 2v_3v_1^6a_3 - 2v_1^6b_2 + (8a_2 - 4b_3)\, v_1^5v_2 - 2v_1^5b_1 - 2v_1^4v_2^2a_3 \\
&+ 6v_1^4v_2a_1 - 2v_3v_1^4b_2 + (8a_2 - 4b_3)\, v_1^3v_2v_3 - 2v_3v_1^3b_1 + 2v_3v_1^2v_2^2a_3 \\
&+ 6v_3v_1^2v_2a_1 + (16a_2 - 8b_3)\, v_1v_2^3 + 8v_2^4a_3 + 8v_2^3a_1 + 4b_2v_2^2v_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
6a_1 &= 0 \\
8a_1 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
8a_3 &= 0 \\
-2b_1 &= 0 \\
-2b_2 &= 0 \\
4b_2 &= 0 \\
8a_2 - 4b_3 &= 0 \\
16a_2 - 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = 2y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 2y - \left(-\frac{\left(x^2+\sqrt{x^4+4y^2}\right)x}{2y}\right)(x) \\
&= \frac{x^4+\sqrt{x^4+4y^2}\,x^2+4y^2}{2y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^4+\sqrt{x^4+4y^2}\,x^2+4y^2}{2y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{2} + \frac{x^2 \ln\left(\frac{2x^4+2\sqrt{x^4}\,\sqrt{x^4+4y^2}}{y}\right)}{2\sqrt{x^4}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\left(x^2 + \sqrt{x^4 + 4y^2}\right)x}{2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x^2 + \sqrt{x^4 + 4y^2}}{x\sqrt{x^4 + 4y^2}} \\
S_y &= \frac{2y}{\sqrt{x^4 + 4y^2}\,\left(x^2 + \sqrt{x^4 + 4y^2}\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

Summary

The solution(s) found are the following

$$\frac{\ln(2)}{2} + \ln(x) + \frac{\ln\left(x^2 + \sqrt{x^4 + 4y^2}\right)}{2} = \ln(x) + c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(2)}{2}+\ln(x)+\frac{\ln\left(x^2+\sqrt{x^4+4y^2}\right)}{2}=\ln(x)+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 3`[1/2*x, y]
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 89

```
dsolve(y(x)=x*diff(y(x),x)+y(x)*diff(y(x),x)^2/x^2,y(x), singsol=all)
```

$$y(x) = -\frac{ix^2}{2}$$
$$y(x) = \frac{ix^2}{2}$$
$$y(x) = 0$$
$$y(x) = -\frac{\sqrt{c_1(-4x^2+c_1)}}{4}$$
$$y(x) = \frac{\sqrt{c_1(-4x^2+c_1)}}{4}$$
$$y(x) = -\frac{2\sqrt{c_1x^2+4}}{c_1}$$
$$y(x) = \frac{2\sqrt{c_1x^2+4}}{c_1}$$

✓ Solution by Mathematica

Time used: 0.986 (sec). Leaf size: 244

```
DSolve[y[x]==x*y'[x]+y[x]*(y'[x])^2/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{\sqrt{x^6+4x^2y(x)^2}\log\left(\sqrt{x^4+4y(x)^2}+x^2\right)}{2x\sqrt{x^4+4y(x)^2}} + \frac{1}{2}\left(1-\frac{\sqrt{x^6+4x^2y(x)^2}}{x\sqrt{x^4+4y(x)^2}}\right)\log(y(x)) = c_1, y(x)\right]$$
$$\text{Solve}\left[\frac{1}{2}\left(\frac{\sqrt{x^6+4x^2y(x)^2}}{x\sqrt{x^4+4y(x)^2}}+1\right)\log(y(x)) - \frac{\sqrt{x^6+4x^2y(x)^2}\log\left(\sqrt{x^4+4y(x)^2}+x^2\right)}{2x\sqrt{x^4+4y(x)^2}} = c_1, y(x)\right]$$
$$y(x) \to -\frac{ix^2}{2}$$
$$y(x) \to \frac{ix^2}{2}$$

17.11 problem Ex 11

Internal problem ID [11235]
Internal file name [`OUTPUT/10220_Wednesday_December_07_2022_01_21_29_PM_54609112/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IV, differential equations of the first order and higher degree than the first. Article 28. Summary. Page 59
Problem number: Ex 11.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x),G(x)*y+H(x)]`]]
```

Unable to solve or complete the solution.

$$\boxed{x^2y'^2 - 2xyy' + y^2 - x^2y^2 = x^4}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{y + \sqrt{x^2y^2 + x^4}}{x} \tag{1}$$

$$y' = \frac{y - \sqrt{x^2y^2 + x^4}}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Unable to determine ODE type.

Unable to determine ODE type.

Solving equation (2)

Unable to determine ODE type.

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 3
   `, `-> Computing symmetries using: way = 4`[1, 1/x*y]
```

✓ Solution by Maple

Time used: 0.578 (sec). Leaf size: 58

```
dsolve(x^2*diff(y(x),x)^2-2*x*y(x)*diff(y(x),x)+y(x)^2=x^2*y(x)^2+x^4,y(x), singsol=all)
```

$$y(x) = -ix$$
$$y(x) = ix$$
$$y(x) = -\frac{x(\mathrm{e}^{x} - c_1^2\mathrm{e}^{-x})}{2c_1}$$
$$y(x) = \frac{x(c_1^2\mathrm{e}^{x} - \mathrm{e}^{-x})}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.366 (sec). Leaf size: 60

```
DSolve[x^2*(y'[x])^2-2*x*y[x]*y'[x]+y[x]^2==x^2*y[x]^2+x^4,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{1}{2}xe^{-x-c_1}\left(-1+e^{2(x+c_1)}\right)$$
$$y(x) \to \frac{1}{2}\left(xe^{-x+c_1}-xe^{x-c_1}\right)$$

18 Chapter V, Singular solutions. Article 30. Page 63

18.1 problem Ex 1

Internal problem ID [11236]
Internal file name [`OUTPUT/10221_Sunday_December_11_2022_01_20_18_AM_9550685/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter V, Singular solutions. Article 30. Page 63
Problem number: Ex 1.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Clairaut]
```

$$\boxed{y - y'x - \frac{1}{y'} = 0}$$

18.1.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = y'x + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$y - px - \frac{1}{p} = 0$$

Solving for y from the above results in

$$y = \frac{p^2x + 1}{p} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= px + \frac{1}{p} \\ &= px + \frac{1}{p} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = \frac{1}{p}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \frac{1}{c_1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{1}{p}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{1}{p^2} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{1}{\sqrt{x}}$$
$$p_2 = -\frac{1}{\sqrt{x}}$$

Substituting the above back in (1) results in

$$y_1 = 2\sqrt{x}$$
$$y_2 = -2\sqrt{x}$$

Summary

The solution(s) found are the following

$$y = c_1 x + \frac{1}{c_1} \tag{1}$$
$$y = 2\sqrt{x} \tag{2}$$
$$y = -2\sqrt{x} \tag{3}$$

Verification of solutions

$$y = c_1 x + \frac{1}{c_1}$$

Verified OK.

$$y = 2\sqrt{x}$$

Verified OK.

$$y = -2\sqrt{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 27

```
dsolve(y(x)=diff(y(x),x)*x+1/diff(y(x),x),y(x), singsol=all)
```

$$y(x) = -2\sqrt{x}$$
$$y(x) = 2\sqrt{x}$$
$$y(x) = c_1 x + \frac{1}{c_1}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 41

```
DSolve[y[x]==y'[x]*x+1/y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x + \frac{1}{c_1}$$
$$y(x) \to \text{Indeterminate}$$
$$y(x) \to -2\sqrt{x}$$
$$y(x) \to 2\sqrt{x}$$

18.2 problem Ex 2

Internal problem ID [11237]
Internal file name [OUTPUT/10222_Sunday_December_11_2022_01_20_19_AM_58281880/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter V, Singular solutions. Article 30. Page 63
Problem number: Ex 2.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy'^2 - 2yy' = x}$$

18.2.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$x\,p^2 - 2yp = x$$

Solving for y from the above results in

$$y = \frac{x(p^2 - 1)}{2p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p^2 - 1}{2p}$$
$$g = 0$$

Hence (2) becomes

$$p - \frac{p^2 - 1}{2p} = x\left(1 - \frac{p^2 - 1}{2p^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{p^2 - 1}{2p} = 0$$

Solving for p from the above gives

$$p = i$$
$$p = -i$$

Substituting these in (1A) gives

$$y = -ix$$
$$y = ix$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{p(x)^2 - 1}{2p(x)}}{x\left(1 - \frac{p(x)^2 - 1}{2p(x)^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 0$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}\mu p &= 0 \\ \frac{d}{dx}\left(\frac{p}{x}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Summary

The solution(s) found are the following

$$y = -ix \tag{1}$$

$$y = ix \tag{2}$$

$$y = \frac{c_1^2 x^2 - 1}{2c_1} \tag{3}$$

Verification of solutions

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 32

```
dsolve(x*diff(y(x),x)^2-2*y(x)*diff(y(x),x)-x=0,y(x), singsol=all)
```

$$y(x) = -ix$$
$$y(x) = ix$$
$$y(x) = \frac{-c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.213 (sec). Leaf size: 71

```
DSolve[x*(y'[x])^2-2*y[x]*y'[x]-x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \frac{1}{2}e^{-c_1}\left(-x^2+e^{2c_1}\right)\\
y(x) &\to \frac{1}{2}e^{-c_1}\left(-1+e^{2c_1}x^2\right)\\
y(x) &\to -ix\\
y(x) &\to ix
\end{aligned}
$$

19 Chapter V, Singular solutions. Article 32. Page 69

19.1 problem Ex 5

Internal problem ID [11238]
Internal file name [OUTPUT/10223_Sunday_December_11_2022_01_20_21_AM_41898295/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter V, Singular solutions. Article 32. Page 69
Problem number: Ex 5.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _Clairaut]
```

$$\boxed{x^2 {y'}^2 - 2(yx - 2)\, y' + y^2 = 0}$$

19.1.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = y'x + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$x^2 p^2 - 2(xy - 2)\, p + y^2 = 0$$

Solving for y from the above results in

$$y = px + 2\sqrt{-p} \tag{1A}$$
$$y = px - 2\sqrt{-p} \tag{2A}$$

Each of the above ode's is a Clairaut ode which is now solved. Solving ode 1A We start by replacing y' by p which gives

$$\begin{aligned} y &= px + 2\sqrt{-p} \\ &= px + 2\sqrt{-p} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = 2\sqrt{-p}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + 2\sqrt{-c_1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = 2\sqrt{-p}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{1}{\sqrt{-p}} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = -\frac{1}{x^2}$$

Substituting the above back in (1) results in

$$y_1 = \frac{2 \operatorname{csgn}\left(\frac{1}{x}\right) - 1}{x}$$

Solving ode 2A We start by replacing y' by p which gives

$$\begin{aligned} y &= px - 2\sqrt{-p} \\ &= px - 2\sqrt{-p} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = -2\sqrt{-p}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x + g')\frac{dp}{dx} \\ 0 &= (x + g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_2 x - 2\sqrt{-c_2}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -2\sqrt{-p}$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x + \frac{1}{\sqrt{-p}} \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = -\frac{1}{x^2}$$

Substituting the above back in (1) results in

$$y_1 = \frac{-2 \operatorname{csgn}\left(\frac{1}{x}\right) - 1}{x}$$

Simplifying the solution $y = \frac{2 \operatorname{csgn}\left(\frac{1}{x}\right)-1}{x}$ to $y = \frac{1}{x}$ Simplifying the solution $y = \frac{-2 \operatorname{csgn}\left(\frac{1}{x}\right)-1}{x}$ to $y = -\frac{3}{x}$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x + 2\sqrt{-c_1} \\ y &= \frac{1}{x} \\ y &= c_2 x - 2\sqrt{-c_2} \\ y &= -\frac{3}{x} \end{aligned}$$

Verification of solutions

$$y = c_1 x + 2\sqrt{-c_1}$$

Verified OK.

$$y = \frac{1}{x}$$

Verified OK.

$$y = c_2 x - 2\sqrt{-c_2}$$

Verified OK.

$$y = -\frac{3}{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 35

```
dsolve(x^2*diff(y(x),x)^2-2*(x*y(x)-2)*diff(y(x),x)+y(x)^2=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{x}$$
$$y(x) = c_1 x - 2\sqrt{-c_1}$$
$$y(x) = c_1 x + 2\sqrt{-c_1}$$

✓ <u>Solution by Mathematica</u>
Time used: 0.416 (sec). Leaf size: 43

```
DSolve[x^2*(y'[x])^2-2*(x*y[x]-2)*y'[x]+y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4(-x+c_1)}{c_1{}^2}$$
$$y(x) \to -\frac{4(x+c_1)}{c_1{}^2}$$
$$y(x) \to 0$$
$$y(x) \to \frac{1}{x}$$

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20.1 problem Ex 1

Internal problem ID [11239]
Internal file name [`OUTPUT/10224_Sunday_December_11_2022_01_20_22_AM_84656961/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter V, Singular solutions. Article 33. Page 73
Problem number: Ex 1.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x^2 {y'}^2 = (x-1)^2}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{x-1}{x} \tag{1}$$
$$y' = -\frac{x-1}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{x-1}{x}\, \mathrm{d}x \\ &= x - \ln(x) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x - \ln(x) + c_1 \tag{1}$$

Verification of solutions

$$y = x - \ln(x) + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{x-1}{x}\,\mathrm{d}x \\ &= -x + \ln(x) + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -x + \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = -x + \ln(x) + c_2$$

Verified OK.

20.1.1 Maple step by step solution

Let's solve

$x^2 {y'}^2 = (x-1)^2$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int x^2 {y'}^2 dx = \int (x-1)^2\, dx + c_1$

- Cannot compute integral

  $\int x^2 {y'}^2 dx = \frac{(x-1)^3}{3} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(x^2*diff(y(x),x)^2-(x-1)^2=0,y(x), singsol=all)
```

$$y(x) = x - \ln(x) + c_1$$
$$y(x) = -x + \ln(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 25

```
DSolve[x^2*(y'[x])^2-(x-1)^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - \log(x) + c_1$$
$$y(x) \to -x + \log(x) + c_1$$

20.2 problem Ex 2

Internal problem ID [11240]
Internal file name [OUTPUT/10225_Sunday_December_11_2022_01_20_22_AM_84043608/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter V, Singular solutions. Article 33. Page 73
Problem number: Ex 2.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{8(1+y')^3 - 27(y+x)\,(1-y')^3 = 0}$$

20.2.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$8(1+p)^3 - 27(y+x)\,(1-p)^3 = 0$$

Solving for y from the above results in

$$y = -\frac{(27p^3 - 81p^2 + 81p - 27)\,x}{27\,(-1+p)^3} - \frac{8p^3 + 24p^2 + 24p + 8}{27\,(-1+p)^3} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= -1 \\
g &= -\frac{8(1+p)^3}{27\left(-1+p\right)^3}
\end{aligned}
$$

Hence (2) becomes

$$
1 + p = \left(-\frac{8(1+p)^2}{9\left(-1+p\right)^3} + \frac{8(1+p)^3}{9\left(-1+p\right)^4}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
1 + p = 0
$$

Solving for p from the above gives

$$
p = -1
$$

Substituting these in (1A) gives

$$
y = -x
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{1 + p(x)}{-\frac{8(1+p(x))^2}{9(-1+p(x))^3} + \frac{8(1+p(x))^3}{9(-1+p(x))^4}} \tag{3}
$$

This ODE is now solved for $p(x)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{\frac{16}{9} + \frac{16p}{9}}{\left(-1+p\right)^4} dp &= x + c_1 \\
-\frac{8}{9\left(-1+p\right)^2} - \frac{32}{27\left(-1+p\right)^3} &= x + c_1
\end{aligned}
$$

Solving for p gives these solutions

$$p_1 = \frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)\left(x+c_1\right)^2\right)^{\frac{1}{3}}}{9\left(x+c_1\right)}-\frac{4}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)\left(x+c_1\right)^2\right)^{\frac{1}{3}}}+1$$

$$p_2 = -\frac{\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)\left(x+c_1\right)^2\right)^{\frac{1}{3}}}{9\left(x+c_1\right)}+\frac{2}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)\left(x+c_1\right)^2\right)^{\frac{1}{3}}}+1-\frac{i\sqrt{3}\left(\frac{2(}{}\right.}{}$$

$$p_3 = -\frac{\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)\left(x+c_1\right)^2\right)^{\frac{1}{3}}}{9\left(x+c_1\right)}+\frac{2}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)\left(x+c_1\right)^2\right)^{\frac{1}{3}}}+1+\frac{i\sqrt{3}\left(\frac{2(}{}\right.}{}$$

Substituing the above solution for p in (2A) gives

$$y = -x - \frac{8\left(2 + \frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x \cdots}{9(x+c_1)} \cdots\right.}{27\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+ \cdots}{9(x+c_1)} \cdots\right.}$$

$$y = -x - \frac{8\left(2 - \frac{\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)} + \frac{2}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54 \cdots}{9(x+c_1)} \cdots\right.}{\cdots}\right.}{27\left(-\frac{\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)} + \frac{2}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54 \cdots}{9(x+c_1)} \cdots\right.}{\cdots}\right.}$$

$$y = -x - \frac{8\left(2 - \frac{\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)} + \frac{2}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54 \cdots}{9(x+c_1)} \cdots\right.}{\cdots}\right.}{27\left(-\frac{\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)} + \frac{2}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54 \cdots}{9(x+c_1)} \cdots\right.}{\cdots}\right.}$$

Summary

The solution(s) found are the following

$$y = -x \tag{1}$$

$$y = -x - \frac{8\left(2+\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)}-\frac{4}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}\right)^3}{27\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)}-\frac{4}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}\right)^3} \tag{2}$$

$$y = -x \tag{3}$$

$$- \frac{8\left(2-\frac{\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)}+\frac{2}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)\right.}{9(x+c_1)}\right.}{}\right.}{27\left(-\frac{\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)}+\frac{2}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right.}{9(x+c_1)}\right.}{}\right.}$$

$$y = -x \tag{4}$$

$$- \frac{8\left(2-\frac{\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)}+\frac{2}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)\right.}{9(x+c_1)}\right.}{}\right.}{27\left(-\frac{\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)}+\frac{2}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right.}{9(x+c_1)}\right.}{}\right.}$$

Verification of solutions

$$y = -x$$

Verified OK.

$$y = -x - \frac{8\left(2 + \frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)} - \frac{4}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}\right)^3}{27\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)} - \frac{4}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}\right)^3}$$

Warning, solution could not be verified

$$y = -x - \frac{8\left(2 - \frac{\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)} + \frac{2}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2 \cdots}{9(x+c_1)} \cdots\right.}{\cdots}\right.}{27\left(-\frac{\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)} + \frac{2}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2 \cdots}{9(x+c_1)} \cdots\right.}{\cdots}\right.}$$

Warning, solution could not be verified

$$y = -x - \frac{8\left(2 - \frac{\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)} + \frac{2}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2 \cdots}{9(x+c_1)} \cdots\right.}{\cdots}\right.}{27\left(-\frac{\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}}{9(x+c_1)} + \frac{2}{3\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{2\left(\left(6\sqrt{3}\,\sqrt{\frac{2+27x+27c_1}{x+c_1}}-54\right)(x+c_1)^2 \cdots}{9(x+c_1)} \cdots\right.}{\cdots}\right.}$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous C
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous C
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous C
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 140

```
dsolve(8*(1+diff(y(x),x))^3=27*(x+y(x))*(1-diff(y(x),x))^3,y(x), singsol=all)
```

$$y(x) = -x$$

$$\frac{x}{2} - \frac{4\ln\left(27y(x)+27x+8\right)}{27} + \frac{4\ln\left(2+3(x+y(x))^{\frac{1}{3}}\right)}{27} + \frac{4\ln\left(9(x+y(x))^{\frac{2}{3}} - 6(x+y(x))^{\frac{1}{3}} + 4\right)}{27} - \frac{y(x)}{2} - \frac{(x+y(x))^{\frac{2}{3}}}{2} - c_1 = 0$$

$$\frac{x}{2} - \frac{y(x)}{2} - \frac{i\sqrt{3}\,(x+y(x))^{\frac{2}{3}}}{4} + \frac{(x+y(x))^{\frac{2}{3}}}{4} - c_1 = 0$$

$$\frac{x}{2} - \frac{y(x)}{2} + \frac{i\sqrt{3}\,(x+y(x))^{\frac{2}{3}}}{4} + \frac{(x+y(x))^{\frac{2}{3}}}{4} - c_1 = 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[8*(1+y'[x])^3==27*(x+y[x])*(1-y'[x])^3,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

20.3 problem Ex 3

Internal problem ID [11241]
Internal file name [OUTPUT/10226_Sunday_December_11_2022_01_26_54_AM_10557687/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter V, Singular solutions. Article 33. Page 73
Problem number: Ex 3.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{4{y'}^2 = 9x}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{3\sqrt{x}}{2} \tag{1}$$

$$y' = -\frac{3\sqrt{x}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{3\sqrt{x}}{2}\, \mathrm{d}x \\ &= x^{\frac{3}{2}} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x^{\frac{3}{2}} + c_1 \tag{1}$$

Verification of solutions

$$y = x^{\frac{3}{2}} + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{3\sqrt{x}}{2}\,\mathrm{d}x \\ &= -x^{\frac{3}{2}} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -x^{\frac{3}{2}} + c_2 \tag{1}$$

Verification of solutions

$$y = -x^{\frac{3}{2}} + c_2$$

Verified OK.

20.3.1 Maple step by step solution

Let's solve

$4y'^2 = 9x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int 4y'^2 dx = \int 9x dx + c_1$

- Cannot compute integral

  $\int 4y'^2 dx = \frac{9x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 19

```
dsolve(4*diff(y(x),x)^2=9*x,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= -x^{\frac{3}{2}} + c_1 \\
y(x) &= x^{\frac{3}{2}} + c_1
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 27

```
DSolve[4*y'[x]^2==9*x,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -x^{3/2} + c_1 \\
y(x) &\to x^{3/2} + c_1
\end{aligned}
$$

20.4 problem Ex 4

Internal problem ID [11242]
Internal file name [`OUTPUT/10227_Sunday_December_11_2022_01_26_54_AM_29869398/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter V, Singular solutions. Article 33. Page 73
Problem number: Ex 4.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y(3-4y)^2 {y'}^2+4y=4}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{2\sqrt{-y\left(y-1\right)}}{y\left(4y-3\right)} \tag{1}$$

$$y' = -\frac{2\sqrt{-y\left(y-1\right)}}{y\left(4y-3\right)} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\int \frac{y(4y-3)}{2\sqrt{-y\left(y-1\right)}}dy = \int dx$$

$$\frac{y^2(y-1)}{\sqrt{-y\left(y-1\right)}} = x+c_1$$

Summary
The solution(s) found are the following

$$\frac{y^2(y-1)}{\sqrt{-y\,(y-1)}} = x + c_1 \tag{1}$$

Verification of solutions

$$\frac{y^2(y-1)}{\sqrt{-y\,(y-1)}} = x + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y(4y-3)}{2\sqrt{-y\,(y-1)}}dy = \int dx$$
$$-\frac{y^2(y-1)}{\sqrt{-y\,(y-1)}} = x + c_2$$

Summary
The solution(s) found are the following

$$-\frac{y^2(y-1)}{\sqrt{-y\,(y-1)}} = x + c_2 \tag{1}$$

Verification of solutions

$$-\frac{y^2(y-1)}{\sqrt{-y\,(y-1)}} = x + c_2$$

Verified OK.

20.4.1 Maple step by step solution

Let's solve

$y(3-4y)^2\,y'^2 + 4y = 4$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$$\frac{y'y(4y-3)}{\sqrt{-y(y-1)}} = 2$$

- Integrate both sides with respect to x

$$\int \frac{y'y(4y-3)}{\sqrt{-y(y-1)}}dx = \int 2dx + c_1$$

- Evaluate integral

$$-2y\sqrt{-y^2+y} = 2x + c_1$$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   <- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 58

```
dsolve(y(x)*(3-4*y(x))^2*diff(y(x),x)^2=4*(1-y(x)),y(x), singsol=all)
```

$$y(x) = 1$$

$$x + \frac{y(x)^2(-1+y(x))}{\sqrt{-y(x)(-1+y(x))}} - c_1 = 0$$

$$x - \frac{y(x)^2(-1+y(x))}{\sqrt{-y(x)(-1+y(x))}} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 60.436 (sec). Leaf size: 3751

```
DSolve[y[x]*(3-4*y[x])^2*y'[x]^2==4*(1-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

21 Chapter VII, Linear differential equations with constant coefficients. Article 43. Page 92

21.1 problem Ex 1

Internal problem ID [11243]
Internal file name [`OUTPUT/10228_Sunday_December_11_2022_01_26_55_AM_44938500/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 43. Page 92
Problem number: Ex 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 3y' + 2y = 0}$$

21.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-3^2 - (4)(1)(2)} \\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= \frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= 1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x} \tag{1}$$

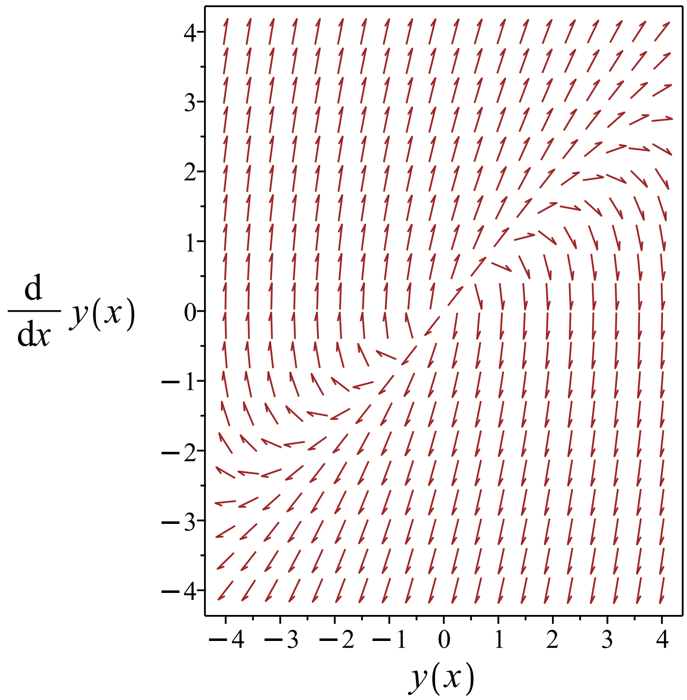

Figure 201: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{x}$$

Verified OK.

21.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 120: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\mathrm{e}^x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^x c_1 + c_2 \mathrm{e}^{2x} \tag{1}
$$

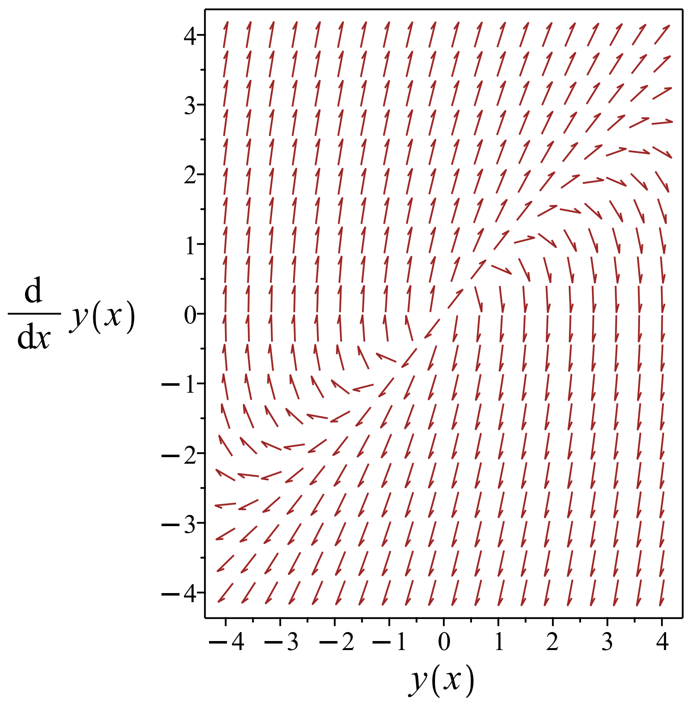

Figure 202: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^x c_1 + c_2 \mathrm{e}^{2x}
$$

Verified OK.

21.1.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the ODE

  $y_1(x) = e^x$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = e^x c_1 + c_2 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + c_2 e^{2x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 18

```
DSolve[y''[x]-3*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(c_2 e^x + c_1)$$

21.2 problem Ex 2

Internal problem ID [11244]
Internal file name [`OUTPUT/10229_Sunday_December_11_2022_01_26_56_AM_67720322/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 43. Page 92
Problem number: Ex 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 6y' + 25y = 0}$$

21.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 25$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda\, e^{\lambda x} + 25\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 25$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-6^2 - (4)(1)(25)} \\ &= 3 \pm 4i\end{aligned}$$

Hence

$$\lambda_1 = 3 + 4i$$
$$\lambda_2 = 3 - 4i$$

Which simplifies to

$$\lambda_1 = 3 + 4i$$
$$\lambda_2 = 3 - 4i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 3$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{3x}(c_1 \cos(4x) + c_2 \sin(4x))$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{3x}(c_1 \cos(4x) + c_2 \sin(4x)) \tag{1}$$

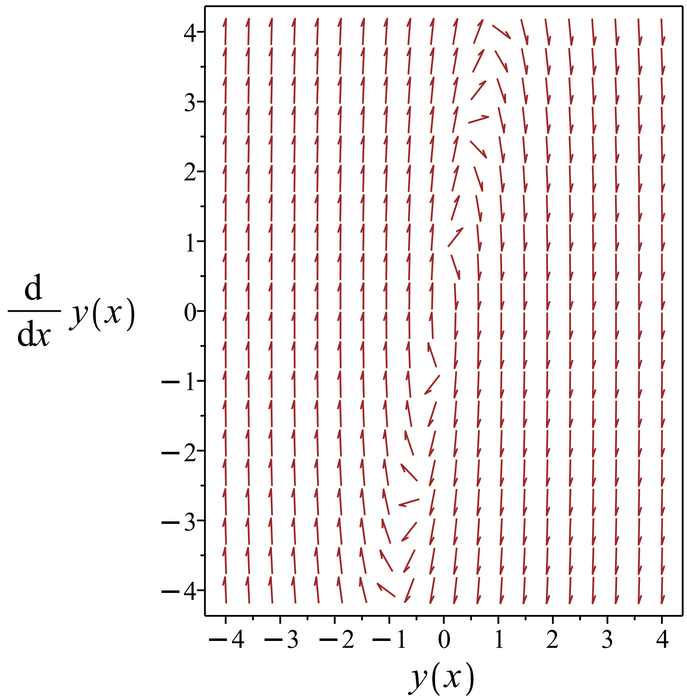

Figure 203: Slope field plot

Verification of solutions

$$y = e^{3x}(c_1 \cos(4x) + c_2 \sin(4x))$$

Verified OK.

21.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 25y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 122: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(4x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx} \\ &= z_1 e^{3x} \\ &= z_1\left(\mathrm{e}^{3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{3x}\cos(4x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{6x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(4x)}{4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{3x}\cos(4x)\right) + c_2\left(e^{3x}\cos(4x)\left(\frac{\tan(4x)}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{3x}\cos(4x) + \frac{c_2 e^{3x}\sin(4x)}{4} \tag{1}
$$

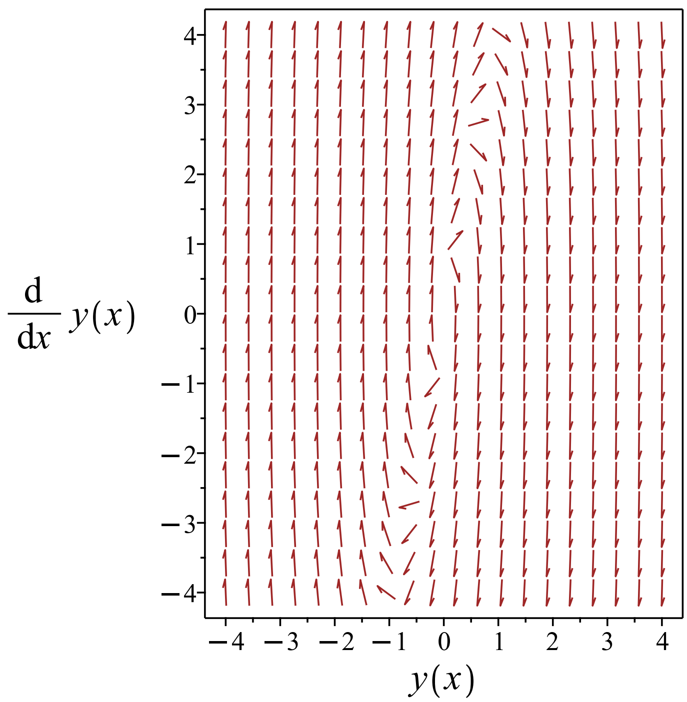

Figure 204: Slope field plot

Verification of solutions

$$
y = c_1 e^{3x}\cos(4x) + \frac{c_2 e^{3x}\sin(4x)}{4}
$$

Verified OK.

21.2.3 Maple step by step solution

Let's solve

$y'' - 6y' + 25y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 6r + 25 = 0$

- Use quadratic formula to solve for r

  $r = \frac{6 \pm (\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (3 - 4\,\mathrm{I}, 3 + 4\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{3x} \cos(4x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{3x} \sin(4x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{3x} \cos(4x) + c_2 \mathrm{e}^{3x} \sin(4x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+25*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{3x}(c_1 \sin(4x) + c_2 \cos(4x))$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 26

```
DSolve[y''[x]-6*y'[x]+25*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{3x}(c_2 \cos(4x) + c_1 \sin(4x))$$

21.3 problem Ex 3

Internal problem ID [11245]
Internal file name [`OUTPUT/10230_Sunday_December_11_2022_01_26_57_AM_51515799/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 43. Page 92
Problem number: Ex 3.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - y' = 0}$$

The characteristic equation is

$$\lambda^3 - \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + c_3 e^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = 1$$
$$y_3 = e^{x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + c_3 e^{x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 + c_3 e^{x}$$

Verified OK.

21.3.1 Maple step by step solution

Let's solve

$y''' - y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = y_2(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1 e^{-x} + c_2 + c_3 e^{x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$3)-diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 \mathrm{e}^{-x} + c_3 \mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 23

```
DSolve[y'''[x]-y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2 e^{-x} + c_3$$

21.4 problem Ex 4

Internal problem ID [11246]
Internal file name [OUTPUT/10231_Sunday_December_11_2022_01_26_58_AM_38294561/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 43. Page 92
Problem number: Ex 4.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 2y'' - y' + 2y = 0}$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{x}$$
$$y_3 = \mathrm{e}^{2x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{x} + e^{2x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{x} + e^{2x} c_3$$

Verified OK.

21.4.1 Maple step by step solution

Let's solve

$y''' - 2y'' - y' + 2y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 2y_3(x) + y_2(x) - 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 2y_3(x) + y_2(x) - 2y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + e^{2x} c_3 \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1 e^{-x} + c_2 e^{x} + \frac{e^{2x} c_3}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-2*diff(y(x),x$2)-diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + c_3 \mathrm{e}^{2x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 28

```
DSolve[y'''[x]-2*y''[x]-y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x}$$

22 Chapter VII, Linear differential equations with constant coefficients. Article 44. Roots of auxiliary equation repeated. Page 94

22.1 problem Ex 1

Internal problem ID [11247]
Internal file name [OUTPUT/10232_Sunday_December_11_2022_01_26_59_AM_92137488/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 44. Roots of auxiliary equation repeated. Page 94
Problem number: Ex 1.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{4y''' - 3y' + y = 0}$$

The characteristic equation is

$$4\lambda^3 - 3\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = \frac{1}{2}$$
$$\lambda_3 = \frac{1}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + e^{\frac{x}{2}} c_2 + x\, e^{\frac{x}{2}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{\frac{x}{2}}$$
$$y_3 = x\, e^{\frac{x}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{\frac{x}{2}} c_2 + x \,\mathrm{e}^{\frac{x}{2}} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{\frac{x}{2}} c_2 + x \,\mathrm{e}^{\frac{x}{2}} c_3$$

Verified OK.

22.1.1 Maple step by step solution

Let's solve

$4y''' - 3y' + y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{3y'}{4} - \frac{y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{3y'}{4} + \frac{y}{4} = 0$

- □ Convert linear ODE into a system of first order ODEs
  - ∘ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ∘ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ∘ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ∘ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = \frac{3y_2(x)}{4} - \frac{y_1(x)}{4}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \frac{3y_2(x)}{4} - \frac{y_1(x)}{4}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue $\frac{1}{2}$

$$\vec{y}_2(x) = \mathrm{e}^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = \frac{1}{2}$ is the eigenvalue, an

$$\vec{y}_3(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_3(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \mathrm{e}^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_3(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue $\frac{1}{2}$

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -8 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue $\frac{1}{2}$

$$\vec{y}_3(x) = e^{\frac{x}{2}} \cdot \left(x \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 0 \\ 0 \end{bmatrix} \right)$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + e^{\frac{x}{2}} c_2 \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{\frac{x}{2}} \cdot \left(x \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 0 \\ 0 \end{bmatrix} \right)$$

- First component of the vector is the solution to the ODE

$$y = 4 e^{-x} \left(((x-2) c_3 + c_2) e^{\frac{3x}{2}} + \frac{c_1}{4} \right)$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(4*diff(y(x),x$3)-3*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x} \left((c_3 x + c_2) e^{\frac{3x}{2}} + c_1 \right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 29

```
DSolve[4*y'''[x]-3*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(e^{3x/2}(c_2x + c_1) + c_3\right)$$

22.2 problem Ex 2

Internal problem ID [11248]
Internal file name [`OUTPUT/10233_Sunday_December_11_2022_01_27_00_AM_36277721/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 44. Roots of auxiliary equation repeated. Page 94
Problem number: Ex 2.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - y'' - y' + y = 0}$$

The characteristic equation is

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + x\,\mathrm{e}^{x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{x}$$
$$y_3 = x\,\mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{x} + x e^{x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{x} + x e^{x} c_3$$

Verified OK.

22.2.1 Maple step by step solution

Let's solve

$y''' - y'' - y' + y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = y_3(x) + y_2(x) - y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = y_3(x) + y_2(x) - y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

$$\vec{y}_2(x) = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\vec{y}_3(x) = e^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_3(x)$ into the homogeneous system

$$\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = \left(e^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = e^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_3(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{y}_3(x) = e^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

  $\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$

- Substitute solutions into the general solution

  $$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right)$$

- First component of the vector is the solution to the ODE

  $y = c_1 e^{-x} + ((x-1)c_3 + c_2)e^x$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)-diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x} + (c_3 x + c_2) e^x$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 25

```
DSolve[y'''[x]-y''[x]-y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x} + e^x (c_3 x + c_2)$$

22.3 problem Ex 3

Internal problem ID [11249]
Internal file name [`OUTPUT/10234_Sunday_December_11_2022_01_27_00_AM_28536283/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 44. Roots of auxiliary equation repeated. Page 94
Problem number: Ex 3.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 2y''' - 2y' - y = 0}$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 - 2\lambda - 1 = 0$$

The roots of the above equation are

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -1 \\ \lambda_3 &= -1 \\ \lambda_4 &= -1 \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + x^2 \mathrm{e}^{-x} c_3 + \mathrm{e}^{x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= x\,\mathrm{e}^{-x} \\ y_3 &= x^2 \mathrm{e}^{-x} \\ y_4 &= \mathrm{e}^{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + x\, e^{-x} c_2 + x^2 e^{-x} c_3 + e^{x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + x\, e^{-x} c_2 + x^2 e^{-x} c_3 + e^{x} c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$3)-2*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_4 x^2 + c_3 x + c_2\right) e^{-x} + c_1 e^{x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 32

```
DSolve[y''''[x]+2*y'''[x]-2*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_3 x^2 + c_2 x + c_4 e^{2x} + c_1\right)$$

22.4 problem Ex 4

Internal problem ID [11250]
Internal file name [`OUTPUT/10235_Sunday_December_11_2022_01_27_01_AM_4137081/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 44. Roots of auxiliary equation repeated. Page 94
Problem number: Ex 4.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 6y'' + 9y' = 0}$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 9\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 3$$
$$\lambda_3 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{3x}c_2 + x\,e^{3x}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{3x}$$
$$y_3 = x\,e^{3x}$$

Summary
The solution(s) found are the following

$$y = c_1 + e^{3x} c_2 + x e^{3x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 + e^{3x} c_2 + x e^{3x} c_3$$

Verified OK.

22.4.1 Maple step by step solution

Let's solve

$y''' - 6y'' + 9y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ∘ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ∘ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ∘ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ∘ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 6y_3(x) - 9y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 6y_3(x) - 9y_2(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -9 & 6 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -9 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 3

$$\overrightarrow{y}_2(x) = \mathrm{e}^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, an

$$\overrightarrow{y}_3(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_3(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_3(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -9 & 6 \end{bmatrix} - 3 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} -\frac{1}{27} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 3

$$\vec{y}_3(x) = e^{3x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{27} \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

  $\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$

- Substitute solutions into the general solution

$$\vec{y} = e^{3x}c_2 \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + c_3e^{3x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{27} \\ 0 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = \frac{((3x-1)c_3+3c_2)e^{3x}}{27} + c_1$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+9*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (c_3x + c_2)e^{3x} + c_1$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 30

```
DSolve[y'''[x]-6*y''[x]+9*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9}e^{3x}(c_2(3x-1) + 3c_1) + c_3$$

23 Chapter VII, Linear differential equations with constant coefficients. Article 45. Roots of auxiliary equation complex. Page 95

23.1 problem Ex 2

Internal problem ID [11251]
Internal file name [`OUTPUT/10236_Sunday_December_11_2022_01_27_02_AM_91030160/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 45. Roots of auxiliary equation complex. Page 95
Problem number: Ex 2.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 2y'' + y = 0}$$

The characteristic equation is

$$\lambda^4 + 2\lambda^2 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = e^{ix}c_1 + x\,e^{ix}c_2 + e^{-ix}c_3 + x\,e^{-ix}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{ix}$$
$$y_2 = x\,e^{ix}$$
$$y_3 = e^{-ix}$$
$$y_4 = x\,e^{-ix}$$

Summary

The solution(s) found are the following

$$y = e^{ix} c_1 + x\, e^{ix} c_2 + e^{-ix} c_3 + x\, e^{-ix} c_4 \tag{1}$$

Verification of solutions

$$y = e^{ix} c_1 + x\, e^{ix} c_2 + e^{-ix} c_3 + x\, e^{-ix} c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$2)+y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_4 x + c_2) \cos (x) + \sin (x) (c_3 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 26

```
DSolve[y''''[x]+2*y''[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (c_2 x + c_1) \cos(x) + (c_4 x + c_3) \sin(x)$$

23.2 problem Ex 3

Internal problem ID [11252]
Internal file name [OUTPUT/10237_Sunday_December_11_2022_01_27_03_AM_84344330/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 45. Roots of auxiliary equation complex. Page 95
Problem number: Ex 3.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - y'' + y' = 0}$$

The characteristic equation is

$$\lambda^3 - \lambda^2 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_3 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}$$
$$y_3 = e^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}$$

Summary
The solution(s) found are the following

$$y = c_1 + \mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 + \mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3$$

Verified OK.

23.2.1 Maple step by step solution

Let's solve

$y''' - y'' + y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = y_3(x) - y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = y_3(x) - y_2(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[\frac{1}{2} - \frac{I\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(\frac{1}{2} - \frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2} - \frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{1}{2} + \frac{I\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2} + \frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{2} - \frac{I\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(\frac{1}{2} - \frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2} - \frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \cdot \begin{bmatrix} \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{\frac{x}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{\frac{x}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}, \vec{y}_3(x) = e^{\frac{x}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = e^{\frac{x}{2}} c_2 \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix} + c_3 e^{\frac{x}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{e^{\frac{x}{2}} \left(-c_3\sqrt{3}+c_2\right) \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{\frac{x}{2}} \left(c_2\sqrt{3}+c_3\right) \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + c_3 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.799 (sec). Leaf size: 75

```
DSolve[y'''[x]-y''[x]+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(c_1 - \sqrt{3}c_2\right) e^{x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + \frac{1}{2}\left(\sqrt{3}c_1 + c_2\right) e^{x/2} \sin\left(\frac{\sqrt{3}x}{2}\right) + c_3$$

24 Chapter VII, Linear differential equations with constant coefficients. Article 47. Particular integral. Page 100

24.1 problem Ex 1

Internal problem ID [11253]
Internal file name [OUTPUT/10238_Sunday_December_11_2022_01_27_04_AM_22035885/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 47. Particular integral. Page 100
Problem number: Ex 1.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - y'' - 2y' = \mathrm{e}^{-x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' - 2y' = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 - 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = 1$$
$$y_3 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y''' - y'' - 2y' = e^{-x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}, e^{2x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^{-x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{-x} = e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,\mathrm{e}^{-x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^{2x} c_3\right) + \left(\frac{x\,\mathrm{e}^{-x}}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^{2x} c_3 + \frac{x\,\mathrm{e}^{-x}}{3} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^{2x} c_3 + \frac{x\,\mathrm{e}^{-x}}{3}$$

Verified OK.

24.1.1 Maple step by step solution

Let's solve

$y''' - y'' - 2y' = \mathrm{e}^{-x}$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = \mathrm{e}^{-x} + y_3(x) + 2y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \mathrm{e}^{-x} + y_3(x) + 2y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{-x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{-x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{-x} & 1 & \frac{e^{2x}}{4} \\ -e^{-x} & 0 & \frac{e^{2x}}{2} \\ e^{-x} & 0 & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & 1 & \frac{e^{2x}}{4} \\ -e^{-x} & 0 & \frac{e^{2x}}{2} \\ e^{-x} & 0 & e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & \frac{1}{4} \\ -1 & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{2e^{-x}}{3} + \frac{1}{2} + \frac{e^{2x}}{6} & \frac{e^{-x}}{3} - \frac{1}{2} + \frac{e^{2x}}{6} \\ 0 & \frac{2e^{-x}}{3} + \frac{e^{2x}}{3} & -\frac{e^{-x}}{3} + \frac{e^{2x}}{3} \\ 0 & -\frac{2e^{-x}}{3} + \frac{2e^{2x}}{3} & \frac{e^{-x}}{3} + \frac{2e^{2x}}{3} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

$\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

$\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} -\frac{1}{2}+\frac{(4+3x)\,\mathrm{e}^{-x}}{9}+\frac{\mathrm{e}^{2x}}{18} \\ \frac{(-3x-1)\,\mathrm{e}^{-x}}{9}+\frac{\mathrm{e}^{2x}}{9} \\ \frac{(3x-2)\,\mathrm{e}^{-x}}{9}+\frac{2\,\mathrm{e}^{2x}}{9} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3 + \left[\begin{array}{c} -\frac{1}{2}+\frac{(4+3x)\,\mathrm{e}^{-x}}{9}+\frac{\mathrm{e}^{2x}}{18} \\ \frac{(-3x-1)\,\mathrm{e}^{-x}}{9}+\frac{\mathrm{e}^{2x}}{9} \\ \frac{(3x-2)\,\mathrm{e}^{-x}}{9}+\frac{2\,\mathrm{e}^{2x}}{9} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = -\frac{1}{2}+\frac{(4+3x+9c_1)\,\mathrm{e}^{-x}}{9}+\frac{(9c_3+2)\,\mathrm{e}^{2x}}{36}+c_2$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = diff(_b(_a), _a)+2*_b(_a)+exp
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)-2*diff(y(x),x)=exp(-x),y(x), singsol=all)
```

$$y(x) = \frac{(2x - 6c_2 + 2)\,\mathrm{e}^{-x}}{6} + \frac{\mathrm{e}^{2x}c_1}{2} + c_3$$

✓ Solution by Mathematica

Time used: 0.166 (sec). Leaf size: 37

```
DSolve[y'''[x]-y''[x]-2*y'[x]==Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9}e^{-x}(3x + 4 - 9c_1) + \frac{1}{2}c_2e^{2x} + c_3$$

24.2 problem Ex 2

Internal problem ID [11254]
Internal file name [`OUTPUT/10239_Sunday_December_11_2022_01_27_05_AM_15795818/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 47. Particular integral. Page 100
Problem number: Ex 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = \mathrm{e}^{\mathrm{e}^x}}$$

24.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = \mathrm{e}^{\mathrm{e}^x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 3\lambda\, \mathrm{e}^{\lambda x} + 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)} \\
&= -\frac{3}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{3}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -1 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{-2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-2x} \\ \frac{d}{dx}(\mathrm{e}^{-x}) & \frac{d}{dx}(\mathrm{e}^{-2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-2x} \\ -\mathrm{e}^{-x} & -2\,\mathrm{e}^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(-2\,\mathrm{e}^{-2x}\right) - \left(\mathrm{e}^{-2x}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = -\mathrm{e}^{-x}\mathrm{e}^{-2x}$$

Which simplifies to

$$W = -e^{-3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-2x} e^{e^x}}{-e^{-3x}} \, dx$$

Which simplifies to

$$u_1 = -\int -e^{x+e^x} dx$$

Hence

$$u_1 = e^{e^x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-x} e^{e^x}}{-e^{-3x}} \, dx$$

Which simplifies to

$$u_2 = \int -e^{2x+e^x} dx$$

Hence

$$u_2 = -e^x e^{e^x} + e^{e^x}$$

Which simplifies to

$$u_1 = e^{e^x}$$
$$u_2 = -e^{x+e^x} + e^{e^x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = e^{-x} e^{e^x} + \left(-e^{x+e^x} + e^{e^x}\right) e^{-2x}$$

Which simplifies to

$$y_p(x) = e^{-2x+e^x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left(e^{-2x+e^x}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x+e^x} \tag{1}$$

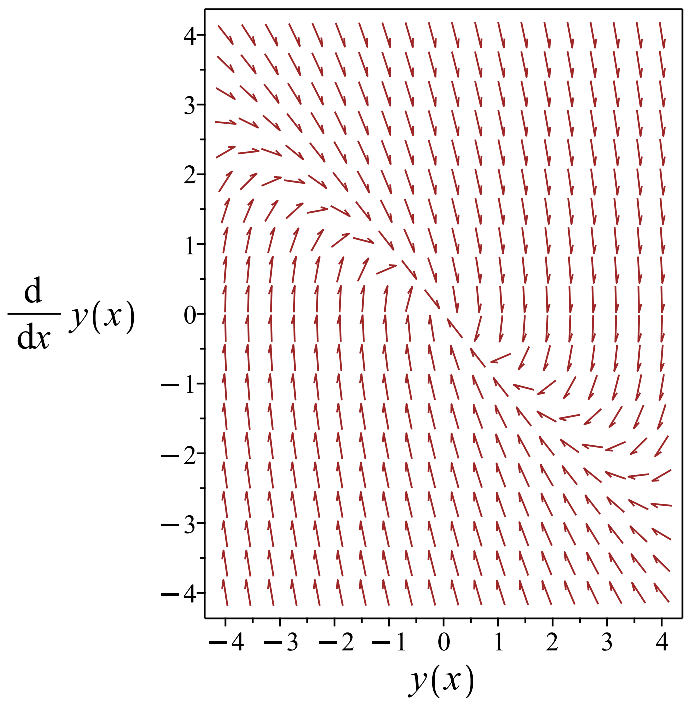

Figure 205: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x+e^x}$$

Verified OK.

24.2.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' + 3y' + 2y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 3 \tag{3} \\
C &= 2
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \,dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 1 \\
t &= 4
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A} \,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 131: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx}
\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\, dx \\
&= y_1 (\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} (\mathrm{e}^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \mathrm{e}^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-x} \\ \frac{d}{dx}(\mathrm{e}^{-2x}) & \frac{d}{dx}(\mathrm{e}^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-x} \\ -2\,\mathrm{e}^{-2x} & -\mathrm{e}^{-x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-2x}\right)\left(-\mathrm{e}^{-x}\right) - \left(\mathrm{e}^{-x}\right)\left(-2\,\mathrm{e}^{-2x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-x}\mathrm{e}^{-2x}$$

Which simplifies to

$$W = e^{-3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-x} e^{e^x}}{e^{-3x}} \, dx$$

Which simplifies to

$$u_1 = -\int e^{2x+e^x} dx$$

Hence

$$u_1 = -e^x e^{e^x} + e^{e^x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2x} e^{e^x}}{e^{-3x}} \, dx$$

Which simplifies to

$$u_2 = \int e^{x+e^x} dx$$

Hence

$$u_2 = e^{e^x}$$

Which simplifies to

$$u_1 = -e^{x+e^x} + e^{e^x}$$
$$u_2 = e^{e^x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = e^{-x} e^{e^x} + \left(-e^{x+e^x} + e^{e^x}\right) e^{-2x}$$

Which simplifies to

$$y_p(x) = e^{-2x+e^x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-2x} + c_2 e^{-x}\right) + \left(e^{-2x+e^x}\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 e^{-2x} + c_2 e^{-x} + e^{-2x+e^x} \tag{1}
$$

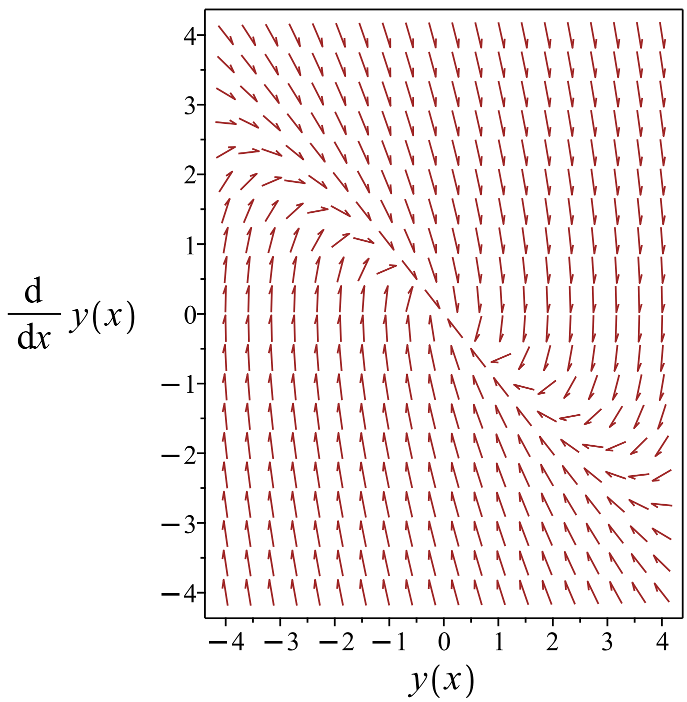

Figure 206: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 e^{-2x} + c_2 e^{-x} + e^{-2x+e^x}
$$

Verified OK.

24.2.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = e^{e^x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = e^{e^x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-3x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{-2x}\left(\int e^{2x+e^x} dx\right) + e^{-x}\left(\int e^{x+e^x} dx\right)$

- Compute integrals

  $y_p(x) = \mathrm{e}^{-2x+\mathrm{e}^x}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-x} + \mathrm{e}^{-2x+\mathrm{e}^x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=exp(exp(x)),y(x), singsol=all)
```

$$y(x) = \left(\mathrm{e}^{\mathrm{e}^x} + c_2\mathrm{e}^x - c_1\right)\mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 25

```
DSolve[y''[x]+3*y'[x]+2*y[x]==Exp[Exp[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(e^{e^x} + c_2e^x + c_1\right)$$

24.3 problem Ex 3

Internal problem ID [11255]
Internal file name [`OUTPUT/10240_Sunday_December_11_2022_01_27_06_AM_79243032/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 47. Particular integral. Page 100
Problem number: Ex 3.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''' + 3y'' + 3y' + y = 2\,\mathrm{e}^{-x} - x^2\mathrm{e}^{-x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 3y'' + 3y' + y = 0$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + x^2 \mathrm{e}^{-x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= x\,\mathrm{e}^{-x} \\ y_3 &= x^2 \mathrm{e}^{-x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' + 3y'' + 3y' + y = 2\,\mathrm{e}^{-x} - x^2 \mathrm{e}^{-x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{-x} - x^2 \mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-x}, x^2 \mathrm{e}^{-x}, \mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, x^2 \mathrm{e}^{-x}, \mathrm{e}^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}, x^2 \mathrm{e}^{-x}, x^3 \mathrm{e}^{-x}\}]$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \mathrm{e}^{-x}, x^3 \mathrm{e}^{-x}, x^4 \mathrm{e}^{-x}\}]$$

Since $x^2 \mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3 \mathrm{e}^{-x}, x^4 \mathrm{e}^{-x}, x^5 \mathrm{e}^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^3 \mathrm{e}^{-x} + A_2 x^4 \mathrm{e}^{-x} + A_3 x^5 \mathrm{e}^{-x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$24A_2 x\,e^{-x} + 60A_3 x^2 e^{-x} + 6A_1 e^{-x} = 2\,e^{-x} - x^2 e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = 0, A_3 = -\frac{1}{60}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3 e^{-x}}{3} - \frac{x^5 e^{-x}}{60}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + x\,e^{-x} c_2 + x^2 e^{-x} c_3\right) + \left(\frac{x^3 e^{-x}}{3} - \frac{x^5 e^{-x}}{60}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}\left(c_3 x^2 + c_2 x + c_1\right) + \frac{x^3 e^{-x}}{3} - \frac{x^5 e^{-x}}{60}$$

Summary

The solution(s) found are the following

$$y = e^{-x}\left(c_3 x^2 + c_2 x + c_1\right) + \frac{x^3 e^{-x}}{3} - \frac{x^5 e^{-x}}{60} \tag{1}$$

Verification of solutions

$$y = e^{-x}\left(c_3 x^2 + c_2 x + c_1\right) + \frac{x^3 e^{-x}}{3} - \frac{x^5 e^{-x}}{60}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$3)+3*diff(y(x),x$2)+3*diff(y(x),x)+y(x)=2*exp(-x)-x^2*exp(-x),y(x), sings
```

$$y(x) = -\frac{e^{-x}(x^5 - 60c_2x^2 - 20x^3 - 60c_3x - 60c_1)}{60}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 41

```
DSolve[y'''[x]+3*y''[x]+3*y'[x]+y[x]==2*Exp[-x]-x^2*Exp[-x],y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{60}e^{-x}\left(-x^5 + 20x^3 + 60c_3x^2 + 60c_2x + 60c_1\right)$$

24.4 problem Ex 4

Internal problem ID [11256]
Internal file name [`OUTPUT/10241_Sunday_December_11_2022_01_27_08_AM_75471682/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 47. Particular integral. Page 100
Problem number: Ex 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = \frac{e^{x}}{(1-x)^{2}}}$$

24.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = \frac{e^{x}}{(x-1)^{2}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\, \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)}\\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 \mathrm{e}^{x} + c_2 x\, \mathrm{e}^{x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{x} c_1 + c_2 x\, \mathrm{e}^{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \mathrm{e}^{x}\\ y_2 &= x\, \mathrm{e}^{x}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & x\,e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(x\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & x\,e^x \\ e^x & e^x + x\,e^x \end{vmatrix}$$

Therefore

$$W = (e^x)\,(e^x + x\,e^x) - (x\,e^x)\,(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\,e^{2x}}{(x-1)^2}}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{(x-1)^2} dx$$

Hence

$$u_1 = -\ln(x-1) + \frac{1}{x-1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{2x}}{(x-1)^2}}{e^{2x}} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{(x-1)^2} dx$$

Hence

$$u_2 = -\frac{1}{x-1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\ln(x-1) + \frac{1}{x-1}\right) e^x - \frac{x\,e^x}{x-1}$$

Which simplifies to

$$y_p(x) = -(\ln(x-1)+1)\,e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + c_2 x\,e^x) + (-(\ln(x-1)+1)\,e^x) \end{aligned}$$

Which simplifies to

$$y = e^x(c_2 x + c_1) - (\ln(x-1)+1)\,e^x$$

Summary

The solution(s) found are the following

$$y = e^x(c_2 x + c_1) - (\ln(x-1)+1)\,e^x \tag{1}$$

Verification of solutions

$$y = e^x(c_2 x + c_1) - (\ln(x-1)+1)\,e^x$$

Verified OK.

24.4.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x) &= e^{\frac{1}{2}\int p\,dx}\\ &= e^{\int -2\,dx}\\ &= \mathrm{e}^{-x}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}(M(x)y)'' &= \frac{\mathrm{e}^{-x}\mathrm{e}^{x}}{(x-1)^2}\\ \left(\mathrm{e}^{-x}y\right)'' &= \frac{\mathrm{e}^{-x}\mathrm{e}^{x}}{(x-1)^2}\end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-x}y\right)' = -\frac{1}{x-1} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-x}y\right) = c_1 x - \ln(x-1) + c_2$$

Hence the solution is

$$y = \frac{c_1 x - \ln(x-1) + c_2}{\mathrm{e}^{-x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} - \mathrm{e}^{x}\ln(x-1)$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} - \mathrm{e}^{x}\ln(x-1) \tag{1}$$

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} - \mathrm{e}^{x}\ln(x-1)$$

Verified OK.

24.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 133: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^x \\
&= z_1(\mathrm{e}^x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x c_1 + c_2 x\, \mathrm{e}^x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^x \\ y_2 &= x\,\mathrm{e}^x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^x & x\,\mathrm{e}^x \\ \frac{d}{dx}(\mathrm{e}^x) & \frac{d}{dx}(x\,\mathrm{e}^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^x & x\,\mathrm{e}^x \\ \mathrm{e}^x & \mathrm{e}^x + x\,\mathrm{e}^x \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^x)(\mathrm{e}^x + x\,\mathrm{e}^x) - (x\,\mathrm{e}^x)(\mathrm{e}^x)$$

Which simplifies to

$$W = \mathrm{e}^{2x}$$

Which simplifies to

$$W = \mathrm{e}^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\,e^{2x}}{(x-1)^2}}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{(x-1)^2} dx$$

Hence

$$u_1 = -\ln(x-1) + \frac{1}{x-1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{2x}}{(x-1)^2}}{e^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{(x-1)^2} dx$$

Hence

$$u_2 = -\frac{1}{x-1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\ln(x-1) + \frac{1}{x-1}\right) e^x - \frac{x\,e^x}{x-1}$$

Which simplifies to

$$y_p(x) = -(\ln(x-1) + 1)\,e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + c_2 x\,e^x) + (-(\ln(x-1) + 1)\,e^x) \end{aligned}$$

Which simplifies to

$$y = e^{x}(c_2 x + c_1) - (\ln(x-1) + 1)\, e^{x}$$

Summary
The solution(s) found are the following

$$y = e^{x}(c_2 x + c_1) - (\ln(x-1) + 1)\, e^{x} \tag{1}$$

Verification of solutions

$$y = e^{x}(c_2 x + c_1) - (\ln(x-1) + 1)\, e^{x}$$

Verified OK.

24.4.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = \frac{e^{x}}{(x-1)^2}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{x} c_1 + c_2 x\, e^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = \frac{e^x}{(x-1)^2}\right]$$

- Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & x e^x \\ e^x & e^x + x e^x \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = e^{2x}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = e^x\left(-\left(\int \frac{x}{(x-1)^2}dx\right) + \left(\int \frac{1}{(x-1)^2}dx\right)x\right)$$

- Compute integrals

  $$y_p(x) = -(\ln(x-1) + 1) e^x$$

- Substitute particular solution into general solution to ODE

  $$y = e^x c_1 + c_2 x e^x - (\ln(x-1) + 1) e^x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=exp(x)/(1-x)^2,y(x), singsol=all)
```

$$y(x) = e^x(-1 + c_1 x - \ln(-1 + x) + c_2)$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 23

```
DSolve[y''[x]-2*y'[x]+y[x]==Exp[x]/(1-x)^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(-\log(x-1) + c_2x - 1 + c_1)$$

25 Chapter VII, Linear differential equations with constant coefficients. Article 48. Page 103

25.1 problem Ex 1

Internal problem ID [11257]
Internal file name [`OUTPUT/10242_Sunday_December_11_2022_01_27_09_AM_95399680/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 48. Page 103
Problem number: Ex 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 3y' + 2y = e^x}$$

25.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-3^2 - (4)(1)(2)} \\
&= \frac{3}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{3}{2} + \frac{1}{2} \\
\lambda_2 &= \frac{3}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= 1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(2)x} + c_2 e^{(1)x}
\end{aligned}$$

Or

$$y = c_1 e^{2x} + c_2 e^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\, e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\, e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^x\right) + \left(-x\, e^x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x} - x\,\mathrm{e}^{x} \tag{1}$$

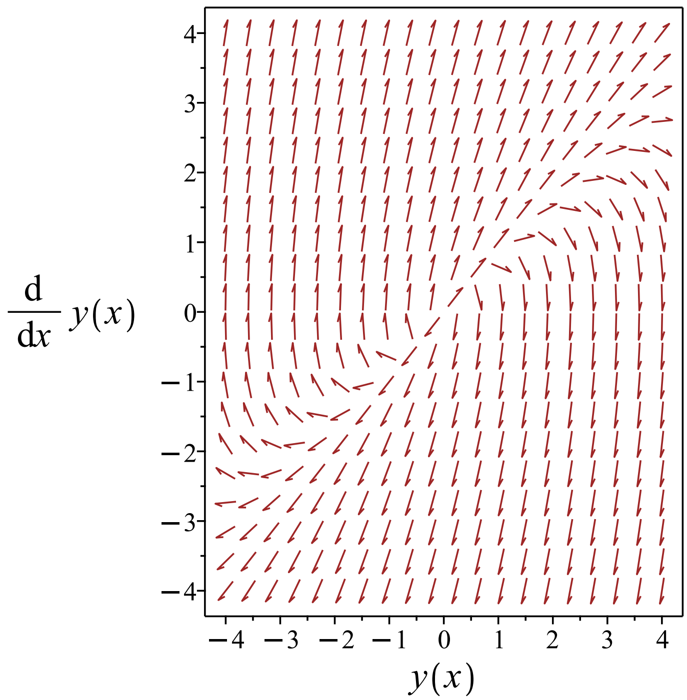

Figure 207: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x} - x\,\mathrm{e}^{x}$$

Verified OK.

25.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 135: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2}\, dx \\ &= y_1(\mathrm{e}^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\mathrm{e}^x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^x c_1 + c_2 \mathrm{e}^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\,e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + c_2 e^{2x}\right) + \left(-x\,e^x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x c_1 + c_2 e^{2x} - x\,e^x \tag{1}$$

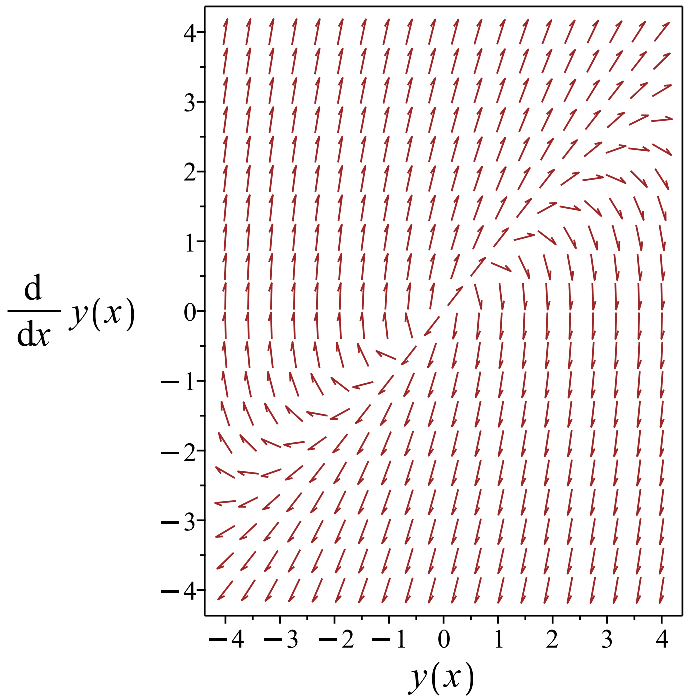

Figure 208: Slope field plot

Verification of solutions

$$y = e^{x}c_1 + c_2 e^{2x} - x\,e^{x}$$

Verified OK.

25.1.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

$r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x c_1 + c_2 e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = e^x\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} e^x & e^{2x} \\ e^x & 2\,e^{2x} \end{array}\right]$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{3x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^x\left(\int 1 dx\right) + e^{2x}\left(\int e^{-x} dx\right)$

- ○ Compute integrals

  $y_p(x) = e^x(-x - 1)$

- Substitute particular solution into general solution to ODE

  $y = e^x c_1 + c_2 e^{2x} + e^x(-x - 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = (-x + c_1 e^x + c_2) e^x$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 22

```
DSolve[y''[x]-3*y'[x]+2*y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(-x + c_2 e^x - 1 + c_1)$$

25.2 problem Ex 2

Internal problem ID [11258]
Internal file name [`OUTPUT/10243_Sunday_December_11_2022_01_27_10_AM_21577199/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 48. Page 103
Problem number: Ex 2.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' - 3y'' - y' + 3y = x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 3y'' - y' + 3y = 0$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{3x}$$

Now the particular solution to the given ODE is found

$$y''' - 3y'' - y' + 3y = x^2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_3x^2 + 3A_2x - 2xA_3 + 3A_1 - A_2 - 6A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{20}{27}, A_2 = \frac{2}{9}, A_3 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{3}x^2 + \frac{2}{9}x + \frac{20}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2 e^{x} + c_3 e^{3x}) + \left(\frac{1}{3}x^2 + \frac{2}{9}x + \frac{20}{27}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{3x} + \frac{x^2}{3} + \frac{2x}{9} + \frac{20}{27} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{3x} + \frac{x^2}{3} + \frac{2x}{9} + \frac{20}{27}$$

Verified OK.

25.2.1 Maple step by step solution

Let's solve

$y''' - 3y'' - y' + 3y = x^2$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = x^2 + 3y_3(x) + y_2(x) - 3y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x^2 + 3y_3(x) + y_2(x) - 3y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 1 & 3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x^2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 1 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\begin{bmatrix} -1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{-x} & e^{x} & \frac{e^{3x}}{9} \\ -e^{-x} & e^{x} & \frac{e^{3x}}{3} \\ e^{-x} & e^{x} & e^{3x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

  $$\Phi(x) = \begin{bmatrix} e^{-x} & e^{x} & \frac{e^{3x}}{9} \\ -e^{-x} & e^{x} & \frac{e^{3x}}{3} \\ e^{-x} & e^{x} & e^{3x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & \frac{1}{9} \\ -1 & 1 & \frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}}$$
- Evaluate and simplify to get the fundamental matrix

  $$\Phi(x) = \begin{bmatrix} \frac{3e^{-x}}{8} + \frac{3e^{x}}{4} - \frac{e^{3x}}{8} & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{e^{-x}}{8} - \frac{e^{x}}{4} + \frac{e^{3x}}{8} \\ -\frac{3e^{-x}}{8} + \frac{3e^{x}}{4} - \frac{3e^{3x}}{8} & \frac{e^{-x}}{2} + \frac{e^{x}}{2} & -\frac{e^{-x}}{8} - \frac{e^{x}}{4} + \frac{3e^{3x}}{8} \\ \frac{3e^{-x}}{8} + \frac{3e^{x}}{4} - \frac{9e^{3x}}{8} & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{e^{-x}}{8} - \frac{e^{x}}{4} + \frac{9e^{3x}}{8} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} -\frac{e^{-x}}{4} - \frac{e^x}{2} + \frac{20}{27} + \frac{x^2}{3} + \frac{2x}{9} + \frac{e^{3x}}{108} \\ \frac{e^{-x}}{4} - \frac{e^x}{2} + \frac{2}{9} + \frac{2x}{3} + \frac{e^{3x}}{36} \\ -\frac{e^{-x}}{4} - \frac{e^x}{2} + \frac{2}{3} + \frac{e^{3x}}{12} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \begin{bmatrix} -\frac{e^{-x}}{4} - \frac{e^x}{2} + \frac{20}{27} + \frac{x^2}{3} + \frac{2x}{9} + \frac{e^{3x}}{108} \\ \frac{e^{-x}}{4} - \frac{e^x}{2} + \frac{2}{9} + \frac{2x}{3} + \frac{e^{3x}}{36} \\ -\frac{e^{-x}}{4} - \frac{e^x}{2} + \frac{2}{3} + \frac{e^{3x}}{12} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(108c_1 - 27)e^{-x}}{108} + \frac{(12c_3 + 1)e^{3x}}{108} + \frac{(108c_2 - 54)e^x}{108} + \frac{x^2}{3} + \frac{2x}{9} + \frac{20}{27}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)-diff(y(x),x)+3*y(x)=x^2,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{3} + \frac{2x}{9} + \frac{20}{27} + c_1 e^x + c_2 e^{-x} + c_3 e^{3x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 42

```
DSolve[y'''[x]-3*y''[x]-y'[x]+3*y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{27}(9x^2 + 6x + 20) + c_1 e^{-x} + c_2 e^x + c_3 e^{3x}$$

25.3 problem Ex 3

Internal problem ID [11259]
Internal file name [`OUTPUT/10244_Sunday_December_11_2022_01_27_11_AM_40817107/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 48. Page 103
Problem number: Ex 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \sec(x)}$$

25.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sec(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(x) + x\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (\ln(\cos(x))\cos(x) + x\sin(x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + x \sin(x) \tag{1}$$

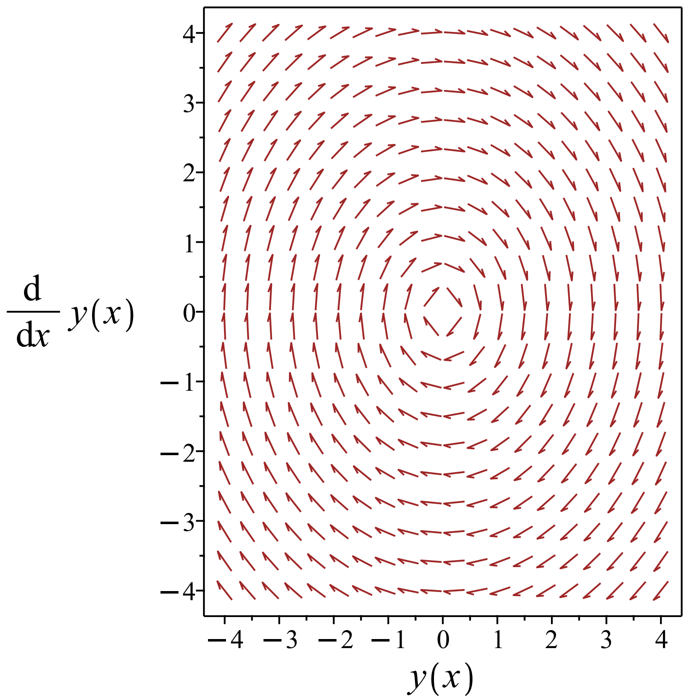

Figure 209: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + x \sin(x)$$

Verified OK.

25.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 138: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \cos(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(x) \\ y_2 &= \sin(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(x) + x\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (\ln(\cos(x))\cos(x) + x\sin(x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\cos(x) + c_2\sin(x) + \ln(\cos(x))\cos(x) + x\sin(x) \tag{1}$$

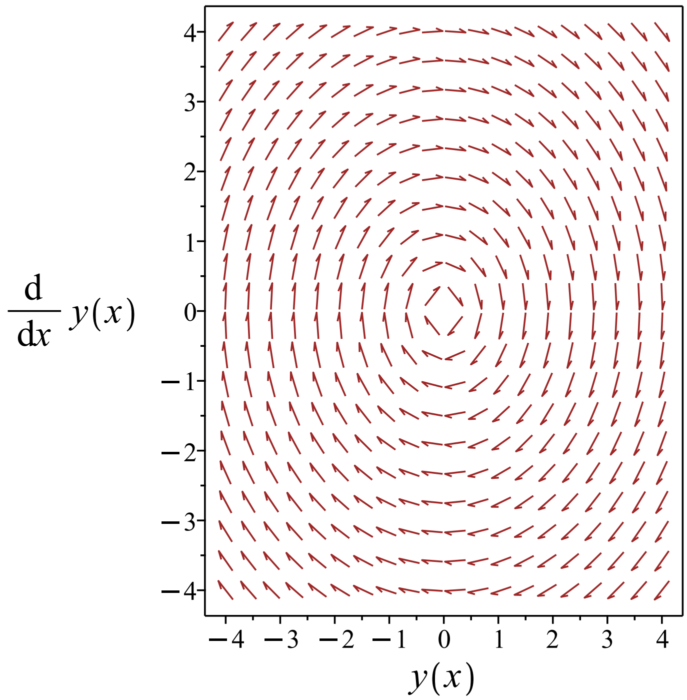

Figure 210: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + x \sin(x)$$

Verified OK.

25.3.3 Maple step by step solution

Let's solve

$y'' + y = \sec(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sec(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x)\left(\int \tan(x)\, dx\right) + \sin(x)\left(\int 1 dx\right)$

  - Compute integrals

    $y_p(x) = \ln(\cos(x)) \cos(x) + x \sin(x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + x \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+y(x)=sec(x),y(x), singsol=all)
```

$$y(x) = -\ln(\sec(x))\cos(x) + c_1 \cos(x) + \sin(x)(c_2 + x)$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 22

```
DSolve[y''[x]+y[x]==Sec[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x + c_2)\sin(x) + \cos(x)(\log(\cos(x)) + c_1)$$

25.4 problem Ex 4

Internal problem ID [11260]
Internal file name [`OUTPUT/10245_Sunday_December_11_2022_01_27_13_AM_78645479/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 48. Page 103
Problem number: Ex 4.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' - 4y'' + 5y' - 2y = x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y'' + 5y' - 2y = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + c_2 x\, e^x + e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = x\, e^x$$
$$y_3 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y''' - 4y'' + 5y' - 2y = x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^x, e^x, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_2 x - 2A_1 + 5A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{4}, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x}{2} - \frac{5}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + c_2 x\, e^x + e^{2x} c_3\right) + \left(-\frac{x}{2} - \frac{5}{4}\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x} c_3 + e^x (c_2 x + c_1) - \frac{x}{2} - \frac{5}{4}$$

Summary

The solution(s) found are the following

$$y = e^{2x} c_3 + e^x (c_2 x + c_1) - \frac{x}{2} - \frac{5}{4} \tag{1}$$

Verification of solutions

$$y = e^{2x} c_3 + e^x (c_2 x + c_1) - \frac{x}{2} - \frac{5}{4}$$

Verified OK.

25.4.1 Maple step by step solution

Let's solve

$y''' - 4y'' + 5y' - 2y = x$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = x + 4y_3(x) - 5y_2(x) + 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x + 4y_3(x) - 5y_2(x) + 2y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]\right], \left[1, \left[\begin{array}{c}0 \\ 0 \\ 0\end{array}\right]\right], \left[2, \left[\begin{array}{c}\frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]\right]$$

- First solution from eigenvalue 1

$$\overrightarrow{y}_1(x) = \mathrm{e}^x \cdot \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{y}_2(x) = e^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3 + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^x & e^x(x-1) & \frac{e^{2x}}{4} \\ e^x & x\,e^x & \frac{e^{2x}}{2} \\ e^x & x\,e^x & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix} \mathrm{e}^{x} & \mathrm{e}^{x}(x-1) & \frac{\mathrm{e}^{2x}}{4} \\ \mathrm{e}^{x} & x\,\mathrm{e}^{x} & \frac{\mathrm{e}^{2x}}{2} \\ \mathrm{e}^{x} & x\,\mathrm{e}^{x} & \mathrm{e}^{2x} \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} 1 & -1 & \frac{1}{4} \\ 1 & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} -\mathrm{e}^{x}(x-1) & -\frac{\mathrm{e}^{2x}}{2}+\frac{(3x+1)\mathrm{e}^{x}}{2} & \frac{\mathrm{e}^{2x}}{2}+\frac{\mathrm{e}^{x}(-x-1)}{2} \\ -x\,\mathrm{e}^{x} & 2\,\mathrm{e}^{x}+\frac{3x\,\mathrm{e}^{x}}{2}-\mathrm{e}^{2x} & -\mathrm{e}^{x}-\frac{x\,\mathrm{e}^{x}}{2}+\mathrm{e}^{2x} \\ -x\,\mathrm{e}^{x} & 2\,\mathrm{e}^{x}+\frac{3x\,\mathrm{e}^{x}}{2}-2\,\mathrm{e}^{2x} & -\mathrm{e}^{x}-\frac{x\,\mathrm{e}^{x}}{2}+2\,\mathrm{e}^{2x} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x)=\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot\vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_0^x\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{e^{2x}}{8} + \frac{(-4x+4)e^x}{8} - \frac{x}{4} - \frac{5}{8} \\ \frac{e^{2x}}{4} - \frac{1}{4} - \frac{x e^x}{2} \\ -\frac{x}{2} + \frac{e^{2x}}{2} - \frac{1}{2} - \frac{x e^x}{2} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3 + \begin{bmatrix} \frac{e^{2x}}{8} + \frac{(-4x+4)e^x}{8} - \frac{x}{4} - \frac{5}{8} \\ \frac{e^{2x}}{4} - \frac{1}{4} - \frac{x e^x}{2} \\ -\frac{x}{2} + \frac{e^{2x}}{2} - \frac{1}{2} - \frac{x e^x}{2} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(2c_3+1)e^{2x}}{8} + \frac{((8c_2-4)x+8c_1-8c_2+4)e^x}{8} - \frac{x}{4} - \frac{5}{8}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x$2)+5*diff(y(x),x)-2*y(x)=x,y(x), singsol=all)
```

$$y(x) = -\frac{5}{4} + c_2 e^{2x} + (c_3 x + c_1)\, e^x - \frac{x}{2}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 35

```
DSolve[y'''[x]-4*y''[x]+5*y'[x]-2*y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x + x\left(-\frac{1}{2} + c_2 e^x\right) + c_3 e^{2x} - \frac{5}{4}$$

26 Chapter VII, Linear differential equations with constant coefficients. Article 49. Variation of parameters. Page 106

26.1 problem Ex 1

Internal problem ID [11261]
Internal file name [`OUTPUT/10246_Sunday_December_11_2022_01_27_14_AM_85152406/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 49. Variation of parameters. Page 106
Problem number: Ex 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \sec(x)}$$

26.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sec(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(x) + x\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (\ln(\cos(x))\cos(x) + x\sin(x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + x \sin(x) \tag{1}$$

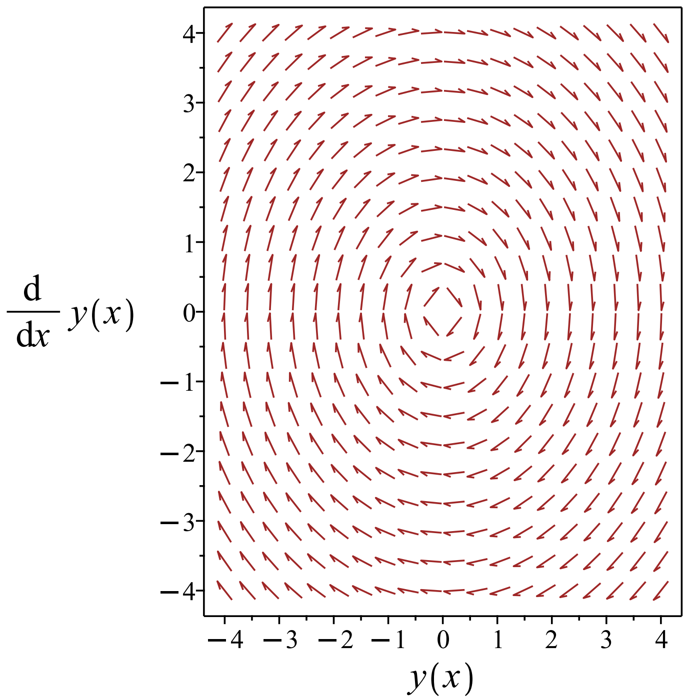

Figure 211: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + x \sin(x)$$

Verified OK.

26.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 141: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)}{1} dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\, dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)}{1} dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(x) + x\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (\ln(\cos(x))\cos(x) + x\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(x) + c_2\sin(x) + \ln(\cos(x))\cos(x) + x\sin(x) \tag{1}$$

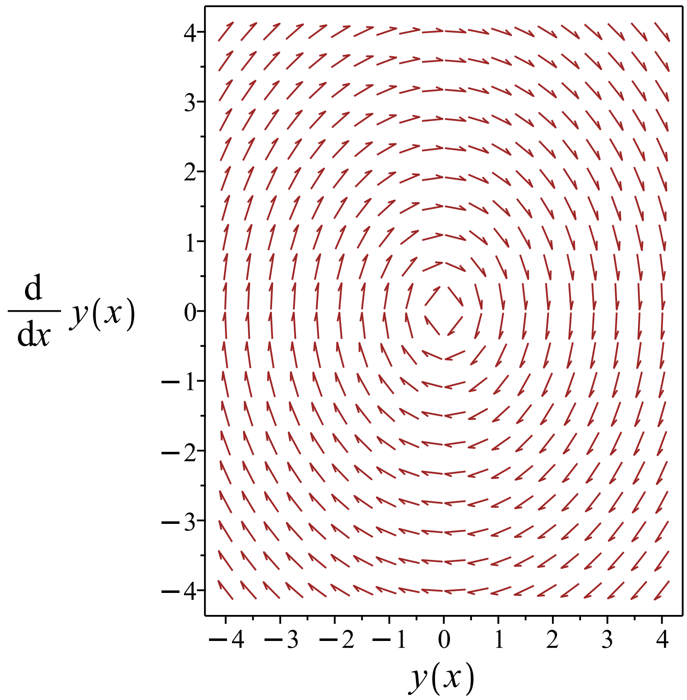

Figure 212: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + x \sin(x)$$

Verified OK.

26.1.3 Maple step by step solution

Let's solve

$y'' + y = \sec(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sec(x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x)\left(\int \tan(x)\, dx\right) + \sin(x)\left(\int 1dx\right)$

- ○ Compute integrals

  $y_p(x) = \ln(\cos(x))\cos(x) + x\sin(x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x))\cos(x) + x\sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+y(x)=sec(x),y(x), singsol=all)
```

$$y(x) = -\ln(\sec(x))\cos(x) + c_1\cos(x) + \sin(x)(c_2 + x)$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 22

```
DSolve[y''[x]+y[x]==Sec[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x + c_2)\sin(x) + \cos(x)(\log(\cos(x)) + c_1)$$

26.2 problem Ex 2

Internal problem ID [11262]
Internal file name [`OUTPUT/10247_Sunday_December_11_2022_01_27_15_AM_76481023/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 49. Variation of parameters. Page 106
Problem number: Ex 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \tan(x)}$$

26.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \tan(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\tan(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\tan(x)\,dx$$

Hence

$$u_1 = \sin(x) - \ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\tan(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int \sin(x)\,dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\sin(x) - \ln(\sec(x) + \tan(x)))\cos(x) - \sin(x)\cos(x)$$

Which simplifies to

$$y_p(x) = -\cos(x)\ln(\sec(x) + \tan(x))$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (-\cos(x) \ln(\sec(x) + \tan(x)))\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) \ln(\sec(x) + \tan(x)) \tag{1}$$

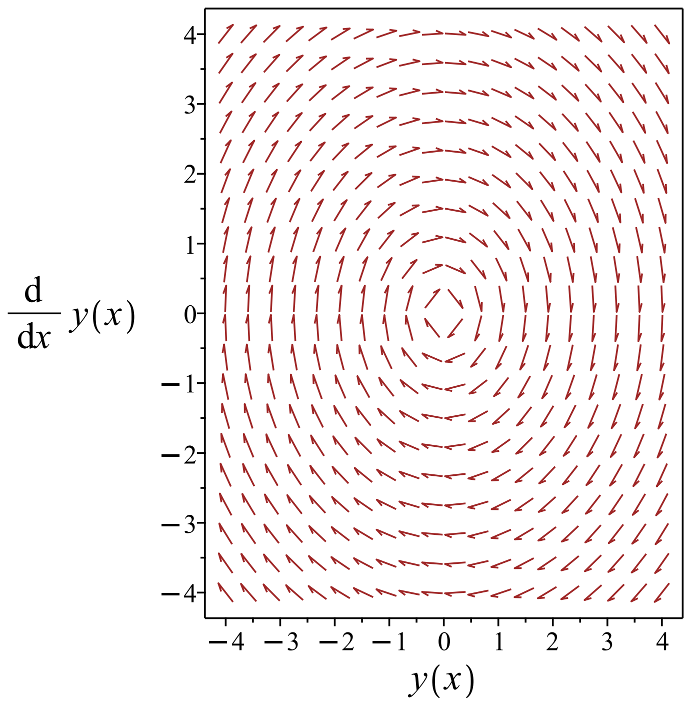

Figure 213: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) \ln(\sec(x) + \tan(x))$$

Verified OK.

26.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 143: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x)\left(\tan(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)\left(\tan(x)\right))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\tan(x)}{1}\, dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\tan(x)\, dx$$

Hence

$$u_1 = \sin(x) - \ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\tan(x)}{1}\, dx$$

Which simplifies to

$$u_2 = \int \sin(x)\, dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\sin(x) - \ln(\sec(x) + \tan(x)))\cos(x) - \sin(x)\cos(x)$$

Which simplifies to

$$y_p(x) = -\cos(x)\ln(\sec(x) + \tan(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (-\cos(x)\ln(\sec(x) + \tan(x))) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) \ln(\sec(x) + \tan(x)) \tag{1}$$

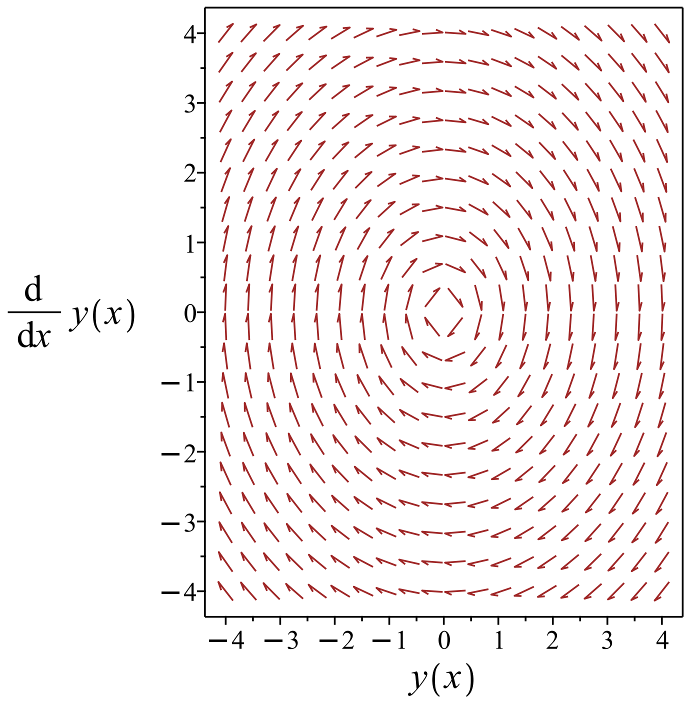

Figure 214: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) \ln(\sec(x) + \tan(x))$$

Verified OK.

26.2.3 Maple step by step solution

Let's solve

$y'' + y = \tan(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \tan(x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \sin(x) \tan(x) \, dx\right) + \sin(x) \left(\int \sin(x) \, dx\right)$
  - Compute integrals

    $y_p(x) = -\cos(x) \ln(\sec(x) + \tan(x))$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) \ln(\sec(x) + \tan(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)=tan(x),y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + c_1 \cos(x) - \cos(x) \ln(\sec(x) + \tan(x))$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 23

```
DSolve[y''[x]+y[x]==Tan[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x)(-\operatorname{arctanh}(\sin(x))) + c_1 \cos(x) + c_2 \sin(x)$$

27 Chapter VII, Linear differential equations with constant coefficients. Article 50. Method of undetermined coefficients. Page 107

27.1 problem Ex 1

Internal problem ID [11263]
Internal file name [`OUTPUT/10248_Wednesday_December_21_2022_03_46_45_PM_9550685/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 50. Method of undetermined coefficients. Page 107
Problem number: Ex 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = x^2 + \cos(x)}$$

27.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = x^2 + \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\,e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 + A_4 x + A_5 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 \cos(x) + 3A_2 \sin(x) + 2A_5 + 4A_3 + 4A_4 x + 4A_5 x^2 = x^2 + \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = 0, A_3 = -\frac{1}{8}, A_4 = 0, A_5 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{3} - \frac{1}{8} + \frac{x^2}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{\cos(x)}{3} - \frac{1}{8} + \frac{x^2}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\cos(x)}{3} - \frac{1}{8} + \frac{x^2}{4} \tag{1}$$

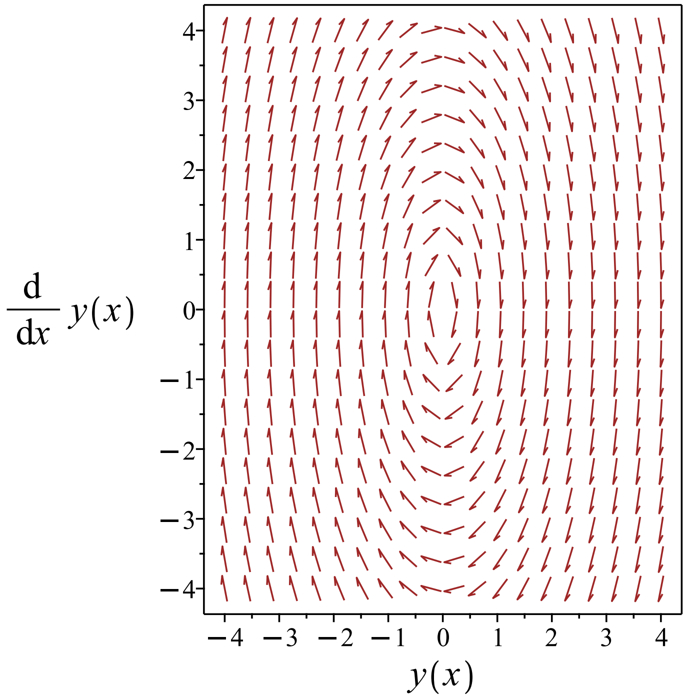

Figure 215: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\cos(x)}{3} - \frac{1}{8} + \frac{x^2}{4}$$

Verified OK.

27.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 145: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2} \, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2 \left(\cos(2x) \left(\frac{\tan(2x)}{2} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 + A_4 x + A_5 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 \cos(x) + 3A_2 \sin(x) + 2A_5 + 4A_3 + 4A_4 x + 4A_5 x^2 = x^2 + \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = 0, A_3 = -\frac{1}{8}, A_4 = 0, A_5 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{3} - \frac{1}{8} + \frac{x^2}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\frac{\cos(x)}{3} - \frac{1}{8} + \frac{x^2}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{\cos(x)}{3} - \frac{1}{8} + \frac{x^2}{4} \tag{1}$$

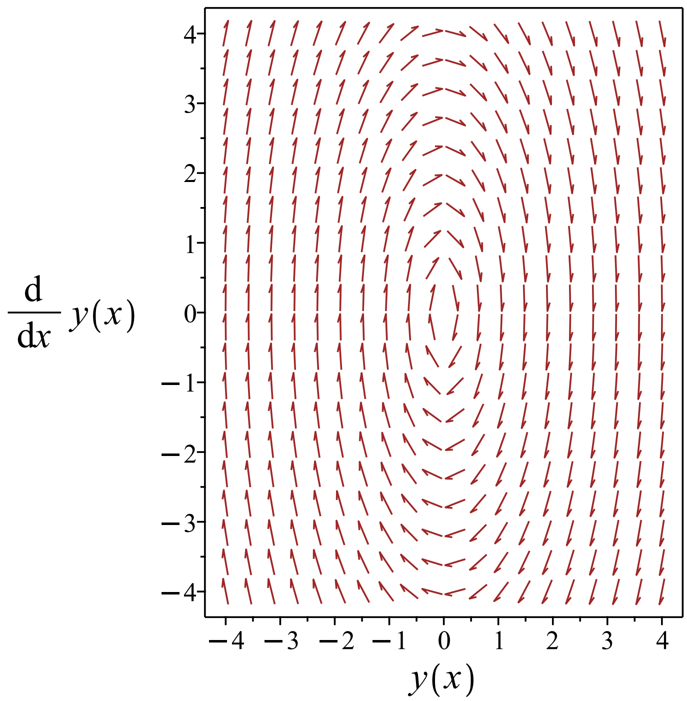

Figure 216: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{\cos(x)}{3} - \frac{1}{8} + \frac{x^2}{4}$$

Verified OK.

27.1.3 Maple step by step solution

Let's solve

$y'' + 4y = x^2 + \cos(x)$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

$r^2+4=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r=(-2\,\mathrm{I},2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x)=\cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\sin(2x)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(2x)+c_2\sin(2x)+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x)=-y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=x^2+\cos(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x),y_2(x))=\begin{bmatrix}\cos(2x) & \sin(2x)\\ -2\sin(2x) & 2\cos(2x)\end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x),y_2(x))=2$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x)=-\frac{\cos(2x)\left(\int \sin(2x)(x^2+\cos(x))dx\right)}{2}+\frac{\sin(2x)\left(\int \cos(2x)(x^2+\cos(x))dx\right)}{2}$

  ◦ Compute integrals

  $y_p(x)=\frac{\cos(x)}{3}-\frac{1}{8}+\frac{x^2}{4}$

- Substitute particular solution into general solution to ODE

  $y=c_1\cos(2x)+c_2\sin(2x)+\frac{\cos(x)}{3}-\frac{1}{8}+\frac{x^2}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+4*y(x)=x^2+cos(x),y(x), singsol=all)
```

$$y(x) = \sin(2x)\, c_2 + c_1 \cos(2x) + \frac{x^2}{4} - \frac{1}{8} + \frac{\cos(x)}{3}$$

✓ Solution by Mathematica

Time used: 0.321 (sec). Leaf size: 36

```
DSolve[y''[x]+4*y[x]==x^2+Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{4} + \frac{\cos(x)}{3} + c_1 \cos(2x) + c_2 \sin(2x) - \frac{1}{8}$$

27.2 problem Ex 2

Internal problem ID [11264]
Internal file name [`OUTPUT/10249_Wednesday_December_21_2022_03_46_46_PM_75656588/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 50. Method of undetermined coefficients. Page 107
Problem number: Ex 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = 2\,\mathrm{e}^{2x}x - \sin\left(x\right)^{2}}$$

27.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = 2\,\mathrm{e}^{2x}x - \sin\left(x\right)^{2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-2)^2 - (4)\,(1)\,(1)}\\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^{x} + c_2 x\, e^{x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{x} c_1 + c_2 x\, e^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\, e^{2x} x - \sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^{2x} x, e^{2x}\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{x}, e^{x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 e^{2x} x + A_3 e^{2x} + A_4 \cos(2x) + A_5 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &A_2 e^{2x} x + 2A_2 e^{2x} + A_3 e^{2x} - 3A_4 \cos(2x) - 3A_5 \sin(2x) \\ &+ 4A_4 \sin(2x) - 4A_5 \cos(2x) + A_1 = 2\,e^{2x} x - \sin(x)^2 \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 2, A_3 = -4, A_4 = -\frac{3}{50}, A_5 = -\frac{2}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{2} + 2\,e^{2x} x - 4\,e^{2x} - \frac{3\cos(2x)}{50} - \frac{2\sin(2x)}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + c_2 x\, e^x) + \left(-\frac{1}{2} + 2\,e^{2x} x - 4\,e^{2x} - \frac{3\cos(2x)}{50} - \frac{2\sin(2x)}{25}\right) \end{aligned}$$

Which simplifies to

$$y = e^x (c_2 x + c_1) - \frac{1}{2} + 2\,e^{2x} x - 4\,e^{2x} - \frac{3\cos(2x)}{50} - \frac{2\sin(2x)}{25}$$

Summary

The solution(s) found are the following

$$y = e^x (c_2 x + c_1) - \frac{1}{2} + 2\,e^{2x} x - 4\,e^{2x} - \frac{3\cos(2x)}{50} - \frac{2\sin(2x)}{25} \tag{1}$$

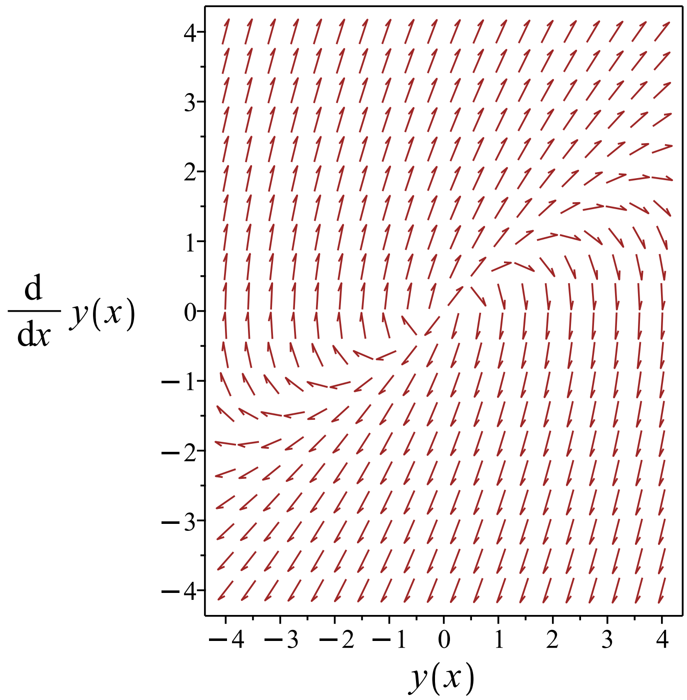

Figure 217: Slope field plot

Verification of solutions

$$y = e^{x}(c_2 x + c_1) - \frac{1}{2} + 2\,e^{2x} x - 4\,e^{2x} - \frac{3\cos(2x)}{50} - \frac{2\sin(2x)}{25}$$

Verified OK.

27.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = e^{-x}\left(2\,e^{2x}x - \sin(x)^2\right)$$
$$\left(e^{-x}y\right)'' = e^{-x}\left(2\,e^{2x}x - \sin(x)^2\right)$$

Integrating once gives

$$\left(e^{-x}y\right)' = \frac{\left(-\cos(x)^2 + 2\sin(x)\cos(x) + 3\right)e^{-x}}{5} + 2\,e^{x}(x-1) + c_1$$

Integrating again gives

$$\left(e^{-x}y\right) = \frac{\left(-25 - 3\cos(2x) - 4\sin(2x)\right)e^{-x}}{50} + 2(x-2)\,e^{x} + c_1x + c_2$$

Hence the solution is

$$y = \frac{\frac{(-25-3\cos(2x)-4\sin(2x))e^{-x}}{50} + 2(x-2)\,e^{x} + c_1x + c_2}{e^{-x}}$$

Or

$$y = c_1x\,e^{x} + 2\,e^{2x}x + c_2e^{x} - \frac{3\cos(x)^2}{25} - \frac{4\cos(x)\sin(x)}{25} - 4\,e^{2x} - \frac{11}{25}$$

Summary

The solution(s) found are the following

$$y = c_1x\,e^{x} + 2\,e^{2x}x + c_2e^{x} - \frac{3\cos(x)^2}{25} - \frac{4\cos(x)\sin(x)}{25} - 4\,e^{2x} - \frac{11}{25} \tag{1}$$

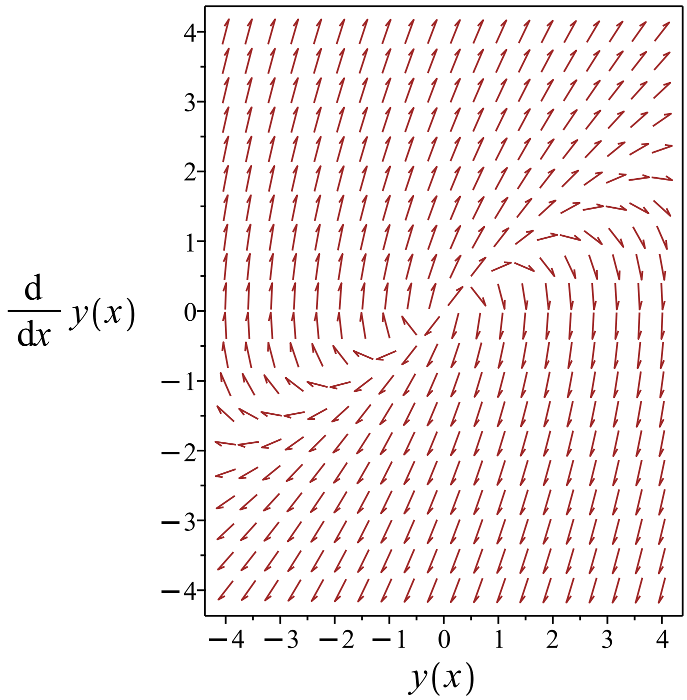

Figure 218: Slope field plot

Verification of solutions

$$y = c_1 x\,e^{x} + 2\,e^{2x}x + c_2 e^{x} - \frac{3\cos(x)^2}{25} - \frac{4\cos(x)\sin(x)}{25} - 4\,e^{2x} - \frac{11}{25}$$

Verified OK.

27.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 147: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1 (\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x c_1 + c_2 x\,\mathrm{e}^x
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2\,\mathrm{e}^{2x} x - \sin(x)^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}, \{\mathrm{e}^{2x} x, \mathrm{e}^{2x}\}, \{\cos(2x), \sin(2x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^x, \mathrm{e}^x\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 + A_2\mathrm{e}^{2x} x + A_3\mathrm{e}^{2x} + A_4\cos(2x) + A_5\sin(2x)
$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&A_2\mathrm{e}^{2x} x + 2A_2\mathrm{e}^{2x} + A_3\mathrm{e}^{2x} - 3A_4\cos(2x) - 3A_5\sin(2x) \\
&+ 4A_4\sin(2x) - 4A_5\cos(2x) + A_1 = 2\,\mathrm{e}^{2x} x - \sin(x)^2
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 2, A_3 = -4, A_4 = -\frac{3}{50}, A_5 = -\frac{2}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{2} + 2\,e^{2x}x - 4\,e^{2x} - \frac{3\cos(2x)}{50} - \frac{2\sin(2x)}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + c_2 x\, e^x) + \left(-\frac{1}{2} + 2\,e^{2x}x - 4\,e^{2x} - \frac{3\cos(2x)}{50} - \frac{2\sin(2x)}{25}\right) \end{aligned}$$

Which simplifies to

$$y = e^x(c_2 x + c_1) - \frac{1}{2} + 2\,e^{2x}x - 4\,e^{2x} - \frac{3\cos(2x)}{50} - \frac{2\sin(2x)}{25}$$

Summary

The solution(s) found are the following

$$y = e^x(c_2 x + c_1) - \frac{1}{2} + 2\,e^{2x}x - 4\,e^{2x} - \frac{3\cos(2x)}{50} - \frac{2\sin(2x)}{25} \tag{1}$$

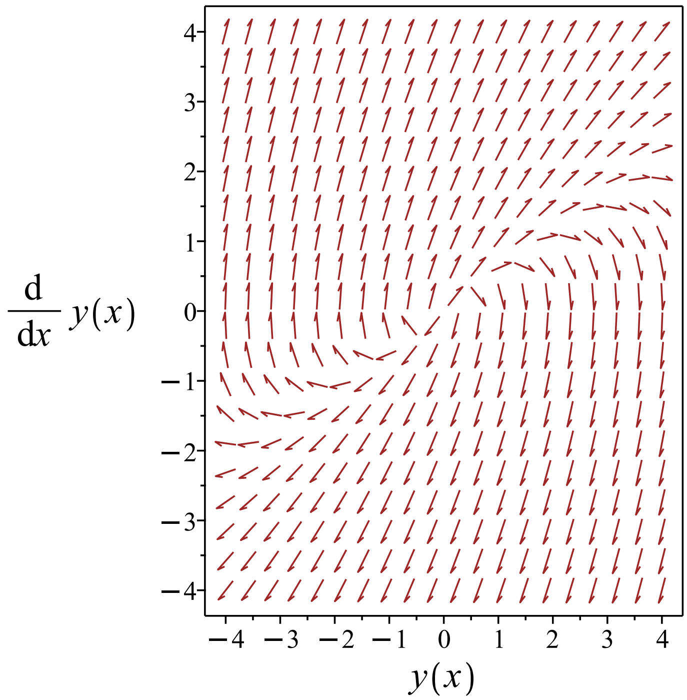

Figure 219: Slope field plot

Verification of solutions

$$y = e^{x}(c_2 x + c_1) - \frac{1}{2} + 2\,e^{2x}x - 4\,e^{2x} - \frac{3\cos(2x)}{50} - \frac{2\sin(2x)}{25}$$

Verified OK.

27.2.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = 2\,e^{2x}x - \sin(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

$r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x c_1 + c_2 x\,e^x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\,e^{2x}x - \sin(x)^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & x\,e^x \\ e^x & e^x + x\,e^x \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = e^x\left(\int x\left(\sin(x)^2 e^{-x} - 2x\,e^x\right)dx + x\left(\int \left(2x\,e^x - \sin(x)^2 e^{-x}\right)dx\right)\right)$

  - Compute integrals

    $y_p(x) = -\frac{1}{2} + 2(x-2)\,e^{2x} - \frac{3\cos(2x)}{50} - \frac{2\sin(2x)}{25}$

- Substitute particular solution into general solution to ODE

  $y = e^x c_1 + c_2 x\,e^x - \frac{1}{2} + 2(x-2)\,e^{2x} - \frac{3\cos(2x)}{50} - \frac{2\sin(2x)}{25}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=2*x*exp(2*x)-sin(x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{1}{2} + 2(x-2)\,e^{2x} - \frac{3\cos(2x)}{50} - \frac{2\sin(2x)}{25} + (c_1 x + c_2)\,e^{x}$$

✓ Solution by Mathematica

Time used: 1.17 (sec). Leaf size: 53

```
DSolve[y''[x]-2*y'[x]+y[x]==2*x*Exp[2*x]-Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2e^{2x}x - 4e^{2x} - \frac{2}{25}\sin(2x) - \frac{3}{50}\cos(2x) + c_2 e^x x + c_1 e^x - \frac{1}{2}$$

27.3 problem Ex 3

Internal problem ID [11265]
Internal file name [`OUTPUT/10250_Wednesday_December_21_2022_03_46_47_PM_48284131/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 50. Method of undetermined coefficients. Page 107
Problem number: Ex 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = 2\,e^{x} + x^3 - x}$$

27.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 2\,e^{x} + x^3 - x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{x} + x^3 - x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}, \{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{x} + A_2 + A_3 x + A_4 x^2 + A_5 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^{x} + 2A_4 + 6A_5 x + A_2 + A_3 x + A_4 x^2 + A_5 x^3 = 2\,\mathrm{e}^{x} + x^3 - x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0, A_3 = -7, A_4 = 0, A_5 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{x} - 7x + x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(\mathrm{e}^{x} - 7x + x^3\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + e^x - 7x + x^3 \tag{1}$$

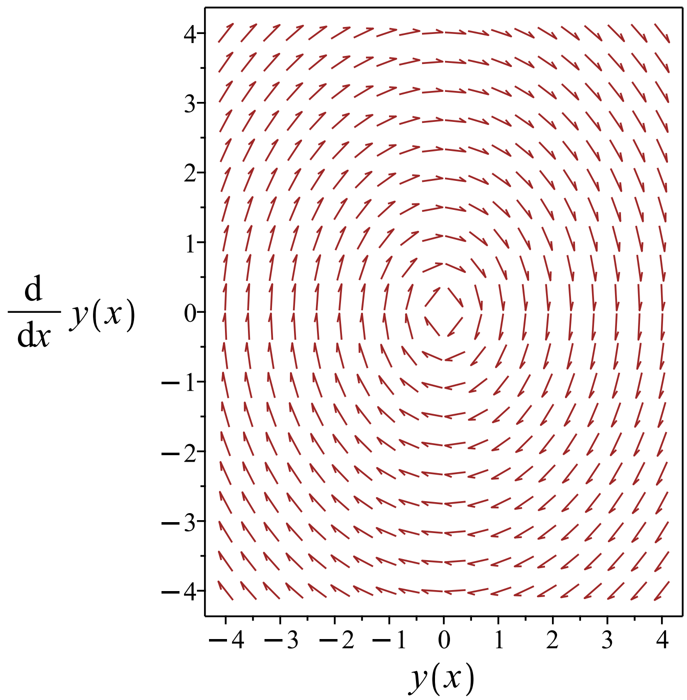

Figure 220: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + e^x - 7x + x^3$$

Verified OK.

27.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 149: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x) (\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\, e^{x} + x^3 - x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x + A_2 + A_3 x + A_4 x^2 + A_5 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x + 2A_4 + 6A_5 x + A_2 + A_3 x + A_4 x^2 + A_5 x^3 = 2\,e^x + x^3 - x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0, A_3 = -7, A_4 = 0, A_5 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^x - 7x + x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(e^x - 7x + x^3\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + e^x - 7x + x^3 \tag{1}$$

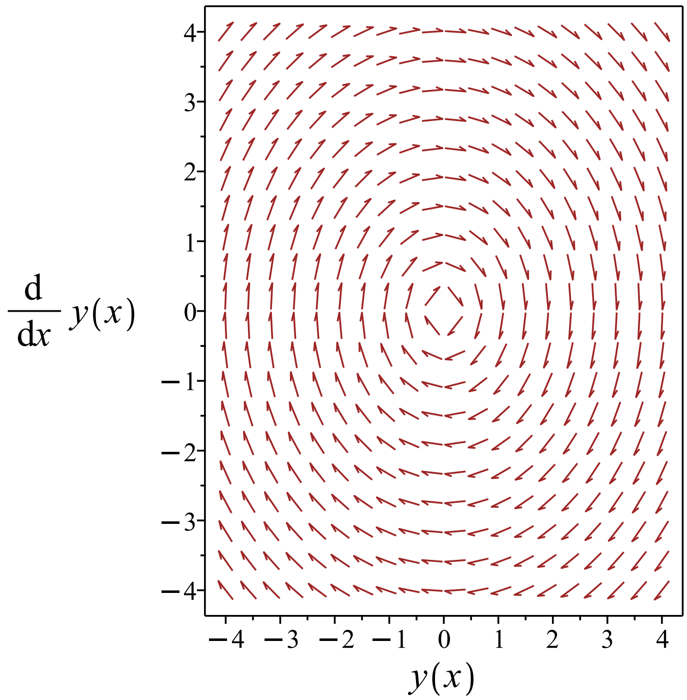

Figure 221: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + e^x - 7x + x^3$$

Verified OK.

27.3.3 Maple step by step solution

Let's solve

$y'' + y = 2\,e^x + x^3 - x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\,\mathrm{e}^x + x^3 - x\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x)\left(\int \sin(x)\left(2\,\mathrm{e}^x + x^3 - x\right) dx\right) + \sin(x)\left(\int \cos(x)\left(2\,\mathrm{e}^x + x^3 - x\right) dx\right)$

  ○ Compute integrals

  $y_p(x) = \mathrm{e}^x - 7x + x^3$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \mathrm{e}^x - 7x + x^3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+y(x)=2*exp(x)+x^3-x,y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + c_1 \cos(x) + x^3 + e^x - 7x$$

✓ Solution by Mathematica

Time used: 0.234 (sec). Leaf size: 25

```
DSolve[y''[x]+y[x]==2*Exp[x]+x^3-x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3 - 7x + e^x + c_1 \cos(x) + c_2 \sin(x)$$

27.4 problem Ex 4

Internal problem ID [11266]
Internal file name [`OUTPUT/10251_Wednesday_December_21_2022_03_46_48_PM_64705552/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 50. Method of undetermined coefficients. Page 107
Problem number: Ex 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 2y' + y = 3\,\mathrm{e}^{2x} - \cos(x)}$$

27.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = 3\,\mathrm{e}^{2x} - \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)} \\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-x} + c_2 x e^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + x e^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3 e^{2x} - \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^{-x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x} + A_2 \cos(x) + A_3 \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 e^{2x} - 2A_2 \sin(x) + 2A_3 \cos(x) = 3\,e^{2x} - \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = 0, A_3 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{2x}}{3} - \frac{\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + x\,e^{-x} c_2\right) + \left(\frac{e^{2x}}{3} - \frac{\sin(x)}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) + \frac{e^{2x}}{3} - \frac{\sin(x)}{2}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + \frac{e^{2x}}{3} - \frac{\sin(x)}{2} \tag{1}$$

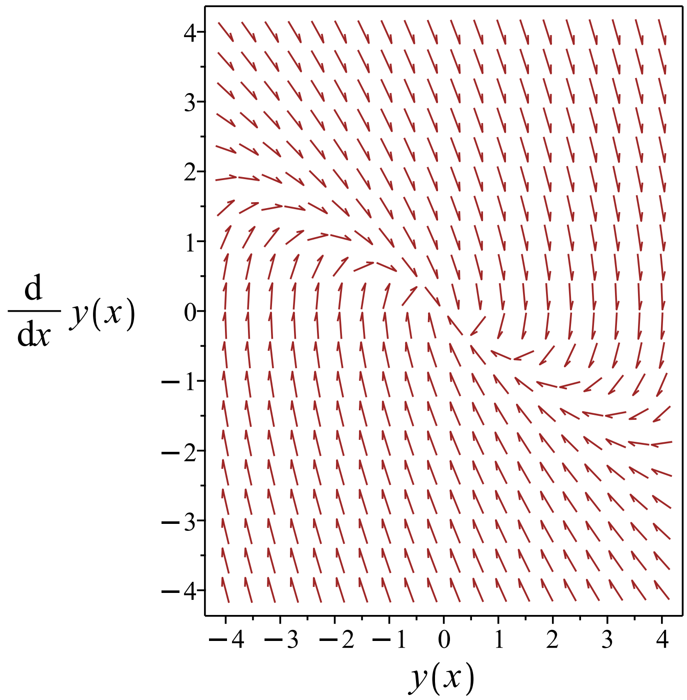

Figure 222: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + \frac{e^{2x}}{3} - \frac{\sin(x)}{2}$$

Verified OK.

27.4.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 2\, dx} \\ &= e^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = e^x (3\,e^{2x} - \cos(x))$$
$$(e^x y)'' = e^x (3\,e^{2x} - \cos(x))$$

Integrating once gives

$$(e^x y)' = -\frac{e^x(-2\,e^{2x} + \cos(x) + \sin(x))}{2} + c_1$$

Integrating again gives

$$(e^x y) = c_1 x + \frac{e^{3x}}{3} - \frac{e^x \sin(x)}{2} + c_2$$

Hence the solution is

$$y = \frac{c_1 x + \frac{e^{3x}}{3} - \frac{e^x \sin(x)}{2} + c_2}{e^x}$$

Or

$$y = \frac{e^{2x}}{3} + e^{-x} c_1 x - \frac{\sin(x)}{2} + c_2 e^{-x}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{2x}}{3} + e^{-x} c_1 x - \frac{\sin(x)}{2} + c_2 e^{-x} \tag{1}$$

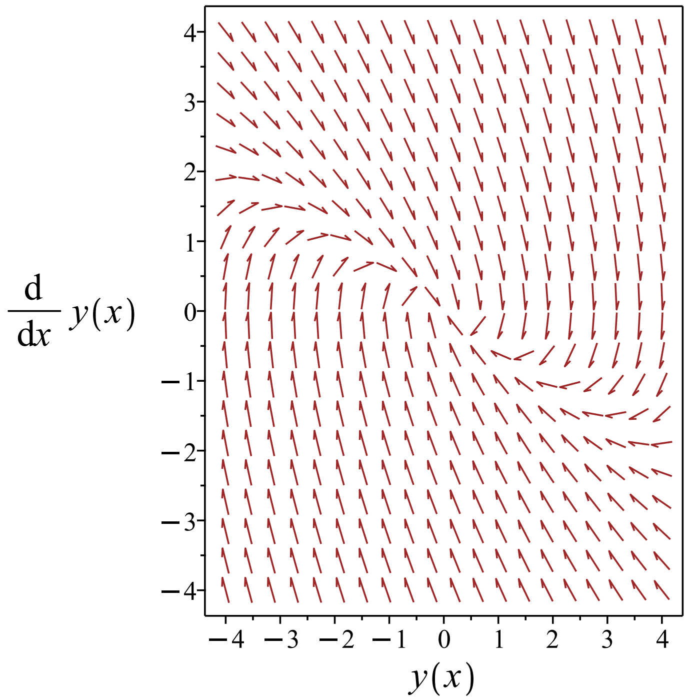

Figure 223: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{2x}}{3} + e^{-x} c_1 x - \frac{\sin(x)}{2} + c_2 e^{-x}$$

Verified OK.

27.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 151: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
3\,\mathrm{e}^{2x} - \cos(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{2x}\}, \{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{2x} + A_2 \cos(x) + A_3 \sin(x)
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
9A_1 \mathrm{e}^{2x} - 2A_2 \sin(x) + 2A_3 \cos(x) = 3\,\mathrm{e}^{2x} - \cos(x)
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = 0, A_3 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{2x}}{3} - \frac{\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + x\, e^{-x} c_2\right) + \left(\frac{e^{2x}}{3} - \frac{\sin(x)}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) + \frac{e^{2x}}{3} - \frac{\sin(x)}{2}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + \frac{e^{2x}}{3} - \frac{\sin(x)}{2} \tag{1}$$

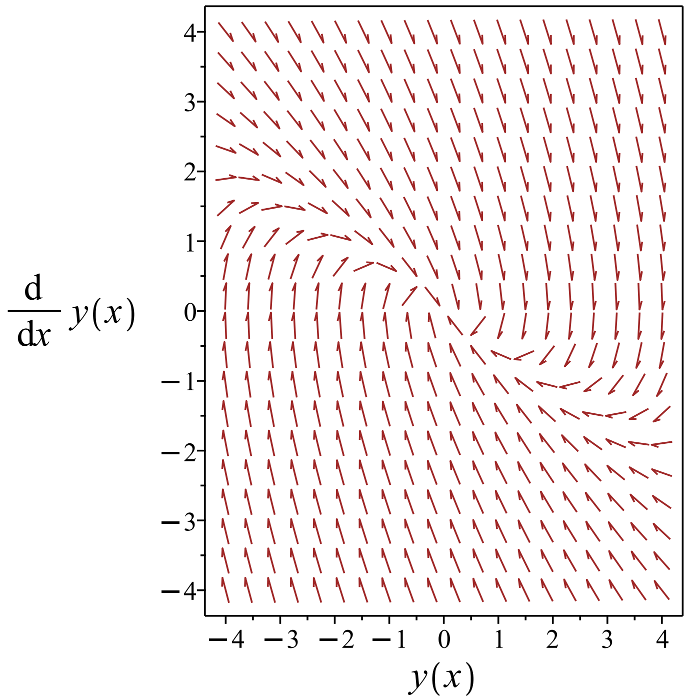

Figure 224: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2x + c_1) + \frac{e^{2x}}{3} - \frac{\sin(x)}{2}$$

Verified OK.

27.4.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = 3\,e^{2x} - \cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 3\,\mathrm{e}^{2x} - \cos(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \mathrm{e}^{-x}\left(x\left(\int -\mathrm{e}^{x}(-3\,\mathrm{e}^{2x} + \cos(x))\,dx\right) - \left(\int -(-3\,\mathrm{e}^{2x} + \cos(x))\,x\,\mathrm{e}^{x} dx\right)\right)$

  - Compute integrals

    $y_p(x) = \frac{\mathrm{e}^{2x}}{3} - \frac{\sin(x)}{2}$

- Substitute particular solution into general solution to ODE

  $y = x\,\mathrm{e}^{-x} c_2 + c_1 \mathrm{e}^{-x} + \frac{\mathrm{e}^{2x}}{3} - \frac{\sin(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=3*exp(2*x)-cos(x),y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2)\,\mathrm{e}^{-x} - \frac{\sin(x)}{2} + \frac{\mathrm{e}^{2x}}{3}$$

✓ Solution by Mathematica

Time used: 0.454 (sec). Leaf size: 38

```
DSolve[y''[x]+2*y'[x]+y[x]==3*Exp[2*x]-Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}e^{-x}\left(2e^{3x} - 3e^{x}\sin(x) + 6c_2 x + 6c_1\right)$$

27.5 problem Ex 5

Internal problem ID [11267]
Internal file name [OUTPUT/10252_Wednesday_December_21_2022_03_46_50_PM_90718951/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 50. Method of undetermined coefficients. Page 107
Problem number: Ex 5.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' - y = x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y = 0$$

The characteristic equation is

$$\lambda^3 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$
$$\lambda_3 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}$$
$$y_3 = e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}$$

Now the particular solution to the given ODE is found

$$y''' - y = x^2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^x, e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}, e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_3 x^2 - A_2 x - A_1 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3\right) + (-x^2) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^x c_1 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3 - x^2 \tag{1}$$

Verification of solutions

$$y = e^x c_1 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3 - x^2$$

Verified OK.

27.5.1 Maple step by step solution

Let's solve

$y''' - y = x^2$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = x^2 + y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x^2 + y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x^2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2} - \frac{I\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \cdot \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-\frac{x}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \mathrm{e}^{-\frac{x}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}, \vec{y}_3(x) = \mathrm{e}^{-\frac{x}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \mathrm{e}^{x} & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ \mathrm{e}^{x} & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & \mathrm{e}^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ \mathrm{e}^{x} & \mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \mathrm{e}^{x} & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ \mathrm{e}^{x} & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & \mathrm{e}^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ \mathrm{e}^{x} & \mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{\mathrm{e}^{x}}{3} + \frac{2\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} & \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} & \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\ldots\right.}{3} \\ \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} & \frac{\mathrm{e}^{x}}{3} + \frac{2\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} & \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\ldots\right.}{3} \\ \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} & \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} & \frac{\mathrm{e}^{x}}{3} + \ldots \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{2\,\mathrm{e}^{x}}{3} - x^2 - \frac{2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} - \frac{2\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \\ \frac{2\,\mathrm{e}^{x}}{3} + \frac{2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} - 2x - \frac{2\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \\ \frac{2\,\mathrm{e}^{x}}{3} - 2 + \frac{4\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \begin{bmatrix} \frac{2\,\mathrm{e}^{x}}{3} - x^2 - \frac{2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} - \frac{2\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \\ \frac{2\,\mathrm{e}^{x}}{3} + \frac{2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} - 2x - \frac{2\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \\ \frac{2\,\mathrm{e}^{x}}{3} - 2 + \frac{4\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{\left(c_3\sqrt{3}+c_2+\frac{4}{3}\right)\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\left(\left(c_2+\frac{4}{3}\right)\sqrt{3}-c_3\right)\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{(6c_1+4)\mathrm{e}^{x}}{6} - x^2$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$3)-y(x)=x^2,y(x), singsol=all)
```

$$y(x) = -x^2 + c_1 e^x + c_2 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_3 e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 59

```
DSolve[y'''[x]-y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^2 + c_1 e^x + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_3 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

27.6 problem Ex 6

Internal problem ID [11268]
Internal file name [`OUTPUT/10253_Wednesday_December_21_2022_03_46_51_PM_39861667/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 50. Method of undetermined coefficients. Page 107
Problem number: Ex 6.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 2y'' - 3y' = 3x^2 + \sin(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 2y'' - 3y' = 0$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 - 3\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 3$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = 1$$
$$y_3 = e^{3x}$$

Now the particular solution to the given ODE is found

$$y''' - 2y'' - 3y' = 3x^2 + \sin(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3x^2 + \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}, e^{3x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x), \sin(x)\}, \{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 x + A_4 x^2 + A_5 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 \sin(x) - 4A_2 \cos(x) + 6A_5 + 2A_1 \cos(x) + 2A_2 \sin(x) - 4A_4 - 12A_5 x - 3A_3 - 6A_4 x - 9A_5 x^2 = 3x^2 + \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}, A_2 = \frac{1}{10}, A_3 = -\frac{14}{9}, A_4 = \frac{2}{3}, A_5 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{5} + \frac{\sin(x)}{10} - \frac{14x}{9} + \frac{2x^2}{3} - \frac{x^3}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 + c_3 e^{3x}\right) + \left(\frac{\cos(x)}{5} + \frac{\sin(x)}{10} - \frac{14x}{9} + \frac{2x^2}{3} - \frac{x^3}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + c_3 e^{3x} + \frac{\cos(x)}{5} + \frac{\sin(x)}{10} - \frac{14x}{9} + \frac{2x^2}{3} - \frac{x^3}{3} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 + c_3 e^{3x} + \frac{\cos(x)}{5} + \frac{\sin(x)}{10} - \frac{14x}{9} + \frac{2x^2}{3} - \frac{x^3}{3}$$

Verified OK.

27.6.1 Maple step by step solution

Let's solve

$y''' - 2y'' - 3y' = 3x^2 + \sin(x)$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 3x^2 + \sin(x) + 2y_3(x) + 3y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3x^2 + \sin(x) + 2y_3(x) + 3y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 3x^2 + \sin(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 3x^2 + \sin(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c}1\\-1\\1\end{array}\right]\right], \left[0, \left[\begin{array}{c}1\\0\\0\end{array}\right]\right], \left[3, \left[\begin{array}{c}\frac{1}{9}\\\frac{1}{3}\\1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c}1\\-1\\1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \left[\begin{array}{c}1\\-1\\1\end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c}1\\0\\0\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \left[\begin{array}{c}1\\0\\0\end{array}\right]$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c}\frac{1}{9}\\\frac{1}{3}\\1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3x} \cdot \left[\begin{array}{c}\frac{1}{9}\\\frac{1}{3}\\1\end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($
  $\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{-x} & 1 & \frac{e^{3x}}{9} \\ -e^{-x} & 0 & \frac{e^{3x}}{3} \\ e^{-x} & 0 & e^{3x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t
  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & 1 & \frac{e^{3x}}{9} \\ -e^{-x} & 0 & \frac{e^{3x}}{3} \\ e^{-x} & 0 & e^{3x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & \frac{1}{9} \\ -1 & 0 & \frac{1}{3} \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{3\,e^{-x}}{4} + \frac{2}{3} + \frac{e^{3x}}{12} & \frac{e^{-x}}{4} - \frac{1}{3} + \frac{e^{3x}}{12} \\ 0 & \frac{3\,e^{-x}}{4} + \frac{e^{3x}}{4} & -\frac{e^{-x}}{4} + \frac{e^{3x}}{4} \\ 0 & -\frac{3\,e^{-x}}{4} + \frac{3\,e^{3x}}{4} & \frac{e^{-x}}{4} + \frac{3\,e^{3x}}{4} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$
  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$
- Take the derivative of the particular solution
  $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs
  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der
  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} -\frac{x^3}{3} + \frac{2x^2}{3} - \frac{14x}{9} + \frac{\sin(x)}{10} + \frac{\cos(x)}{5} + \frac{29\,\mathrm{e}^{3x}}{1080} + \frac{31}{27} - \frac{11\,\mathrm{e}^{-x}}{8} \\ \frac{29\,\mathrm{e}^{3x}}{360} - x^2 + \frac{4x}{3} - \frac{\sin(x)}{5} + \frac{\cos(x)}{10} - \frac{14}{9} + \frac{11\,\mathrm{e}^{-x}}{8} \\ \frac{29\,\mathrm{e}^{3x}}{120} - 2x - \frac{\sin(x)}{10} - \frac{\cos(x)}{5} + \frac{4}{3} - \frac{11\,\mathrm{e}^{-x}}{8} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \left[\begin{array}{c} -\frac{x^3}{3} + \frac{2x^2}{3} - \frac{14x}{9} + \frac{\sin(x)}{10} + \frac{\cos(x)}{5} + \frac{29\,\mathrm{e}^{3x}}{1080} + \frac{31}{27} - \frac{11\,\mathrm{e}^{-x}}{8} \\ \frac{29\,\mathrm{e}^{3x}}{360} - x^2 + \frac{4x}{3} - \frac{\sin(x)}{5} + \frac{\cos(x)}{10} - \frac{14}{9} + \frac{11\,\mathrm{e}^{-x}}{8} \\ \frac{29\,\mathrm{e}^{3x}}{120} - 2x - \frac{\sin(x)}{10} - \frac{\cos(x)}{5} + \frac{4}{3} - \frac{11\,\mathrm{e}^{-x}}{8} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y = \frac{(1080c_1-1485)\mathrm{e}^{-x}}{1080} + \frac{(120c_3+29)\mathrm{e}^{3x}}{1080} - \frac{x^3}{3} + \frac{2x^2}{3} - \frac{14x}{9} + c_2 + \frac{\cos(x)}{5} + \frac{\sin(x)}{10} + \frac{31}{27}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 3*_a^2+2*(diff(_b(_a), _a))+3
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$3)-2*diff(y(x),x$2)-3*diff(y(x),x)=3*x^2+sin(x),y(x), singsol=all)
```

$$y(x) = -\frac{x^3}{3} + \frac{2x^2}{3} - c_1 e^{-x} + \frac{c_2 e^{3x}}{3} + \frac{\sin(x)}{10} + \frac{\cos(x)}{5} - \frac{14x}{9} + c_3$$

✓ Solution by Mathematica

Time used: 0.68 (sec). Leaf size: 58

```
DSolve[y'''[x]-2*y''[x]-3*y'[x]==3*x^2+Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9}\left(-3x^3 + 6x^2 - 14x - 9c_1 e^{-x} + 3c_2 e^{3x} + 9c_3\right) + \frac{\sin(x)}{10} + \frac{\cos(x)}{5}$$

27.7 problem Ex 7

Internal problem ID [11269]
Internal file name [`OUTPUT/10254_Wednesday_December_21_2022_03_46_52_PM_88694947/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 50. Method of undetermined coefficients. Page 107
Problem number: Ex 7.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _with_linear_symmetries]]`

$$\boxed{y'''' - 2y'' + y = e^{x} + 4}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 2y'' + y = 0$$

The characteristic equation is

$$\lambda^4 - 2\lambda^2 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + x\,e^{-x} c_2 + c_3 e^{x} + x\,e^{x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = x\,e^{-x}$$
$$y_3 = e^{x}$$
$$y_4 = x\,e^{x}$$

Now the particular solution to the given ODE is found

$$y'''' - 2y'' + y = e^{x} + 4$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{x} + 4$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{x}, x\,e^{-x}, e^{x}, e^{-x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x\,e^{x}\}]$$

Since $x\,e^{x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x^2 e^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x^2 e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_2 e^x + A_1 = e^x + 4$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 4, A_2 = \frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4 + \frac{x^2 e^x}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + x\,e^{-x} c_2 + c_3 e^x + x\,e^x c_4\right) + \left(4 + \frac{x^2 e^x}{8}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) + e^x(c_4 x + c_3) + 4 + \frac{x^2 e^x}{8}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + e^x(c_4 x + c_3) + 4 + \frac{x^2 e^x}{8} \tag{1}$$

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + e^x(c_4 x + c_3) + 4 + \frac{x^2 e^x}{8}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$4)-2*diff(y(x),x$2)+y(x)=exp(x)+4,y(x), singsol=all)
```

$$y(x) = 4 + (c_4 x + c_2)\,e^{-x} + \frac{(3 + 2x^2 + 4(-1 + 4c_3)\,x + 16c_1)\,e^{x}}{16}$$

✓ Solution by Mathematica

Time used: 0.174 (sec). Leaf size: 47

```
DSolve[y''''[x]-2*y''[x]+y[x]==Exp[x]+4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x \left(\frac{x^2}{8} + \left(-\frac{1}{4} + c_4\right) x + \frac{3}{16} + c_3\right) + e^{-x}((2 + c_2)x + c_1) + 4$$

27.8 problem Ex 8

Internal problem ID [11270]
Internal file name [`OUTPUT/10255_Wednesday_December_21_2022_03_46_54_PM_12086574/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 50. Method of undetermined coefficients. Page 107
Problem number: Ex 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - 2y' = e^{2x} + 1}$$

27.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 0, f(x) = e^{2x} + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(0)}\\ &= 1 \pm 1\end{aligned}$$

Hence

$$\lambda_1 = 1 + 1$$
$$\lambda_2 = 1 - 1$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = 0$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(0)x}$$

Or

$$y = c_1 e^{2x} + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x} + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{e^{2x} x\}]$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}, \{e^{2x} x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x + A_2 e^{2x} x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 e^{2x} - 2A_1 = e^{2x} + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x}{2} + \frac{e^{2x} x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{2x} + c_2) + \left(-\frac{x}{2} + \frac{e^{2x} x}{2}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 - \frac{x}{2} + \frac{e^{2x} x}{2} \tag{1}$$

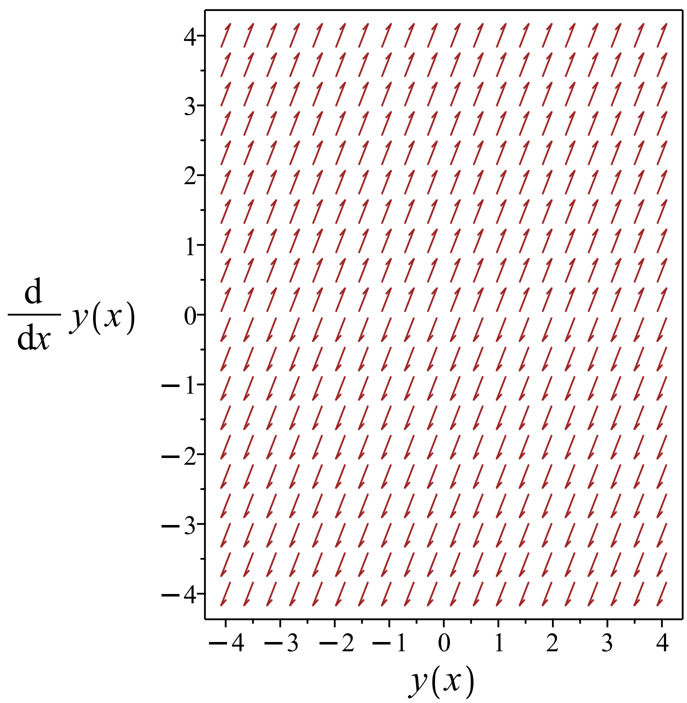

Figure 225: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{2x} + c_2 - \frac{x}{2} + \frac{e^{2x} x}{2}$$

Verified OK.

27.8.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (y'' - 2y')\, dx &= \int \left(\mathrm{e}^{2x} + 1\right) dx \\
-2y + y' &= x + \frac{\mathrm{e}^{2x}}{2} + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -2 \\
q(x) &= x + \frac{\mathrm{e}^{2x}}{2} + c_1
\end{aligned}
$$

Hence the ode is

$$
-2y + y' = x + \frac{\mathrm{e}^{2x}}{2} + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-2)dx} \\
&= \mathrm{e}^{-2x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(x + \frac{\mathrm{e}^{2x}}{2} + c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-2x} y\right) &= \left(\mathrm{e}^{-2x}\right)\left(x + \frac{\mathrm{e}^{2x}}{2} + c_1\right) \\
\mathrm{d}\left(\mathrm{e}^{-2x} y\right) &= \left(\frac{1}{2} + (x + c_1)\,\mathrm{e}^{-2x}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-2x} y &= \int \frac{1}{2} + (x + c_1)\,\mathrm{e}^{-2x}\, \mathrm{d}x \\
\mathrm{e}^{-2x} y &= \frac{x}{2} - \frac{x\,\mathrm{e}^{-2x}}{2} - \frac{\mathrm{e}^{-2x}}{4} - \frac{c_1 \mathrm{e}^{-2x}}{2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-2x}$ results in

$$y = e^{2x}\left(\frac{x}{2} - \frac{x\,e^{-2x}}{2} - \frac{e^{-2x}}{4} - \frac{c_1 e^{-2x}}{2}\right) + c_2 e^{2x}$$

which simplifies to

$$y = \frac{(2x + 4c_2)\,e^{2x}}{4} - \frac{x}{2} - \frac{c_1}{2} - \frac{1}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{(2x + 4c_2)\,e^{2x}}{4} - \frac{x}{2} - \frac{c_1}{2} - \frac{1}{4} \tag{1}$$

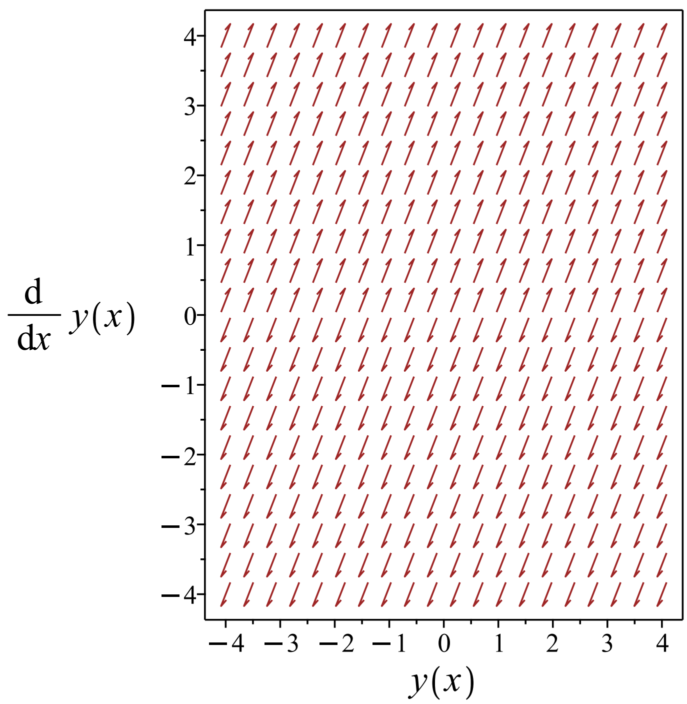

Figure 226: Slope field plot

Verification of solutions

$$y = \frac{(2x + 4c_2)\,e^{2x}}{4} - \frac{x}{2} - \frac{c_1}{2} - \frac{1}{4}$$

Verified OK.

27.8.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 2p(x) - e^{2x} - 1 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= e^{2x} + 1 \end{aligned}$$

Hence the ode is

$$p'(x) - 2p(x) = e^{2x} + 1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-2)dx} \\ &= e^{-2x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu) \left(e^{2x} + 1\right) \\ \frac{d}{dx}\left(e^{-2x} p\right) &= \left(e^{-2x}\right) \left(e^{2x} + 1\right) \\ d\left(e^{-2x} p\right) &= \left(1 + e^{-2x}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-2x} p &= \int 1 + e^{-2x} dx \\ e^{-2x} p &= x - \frac{e^{-2x}}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-2x}$ results in

$$p(x) = \mathrm{e}^{2x}\left(x - \frac{\mathrm{e}^{-2x}}{2}\right) + c_1 \mathrm{e}^{2x}$$

which simplifies to

$$p(x) = -\frac{1}{2} + (x + c_1)\,\mathrm{e}^{2x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{1}{2} + (x + c_1)\,\mathrm{e}^{2x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \mathrm{e}^{2x} x - \frac{1}{2} + c_1 \mathrm{e}^{2x}\, \mathrm{d}x \\ &= -\frac{x}{2} + \frac{c_1 \mathrm{e}^{2x}}{2} + \frac{\mathrm{e}^{2x} x}{2} - \frac{\mathrm{e}^{2x}}{4} + c_2 \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{x}{2} + \frac{c_1 \mathrm{e}^{2x}}{2} + \frac{\mathrm{e}^{2x} x}{2} - \frac{\mathrm{e}^{2x}}{4} + c_2 \tag{1}$$

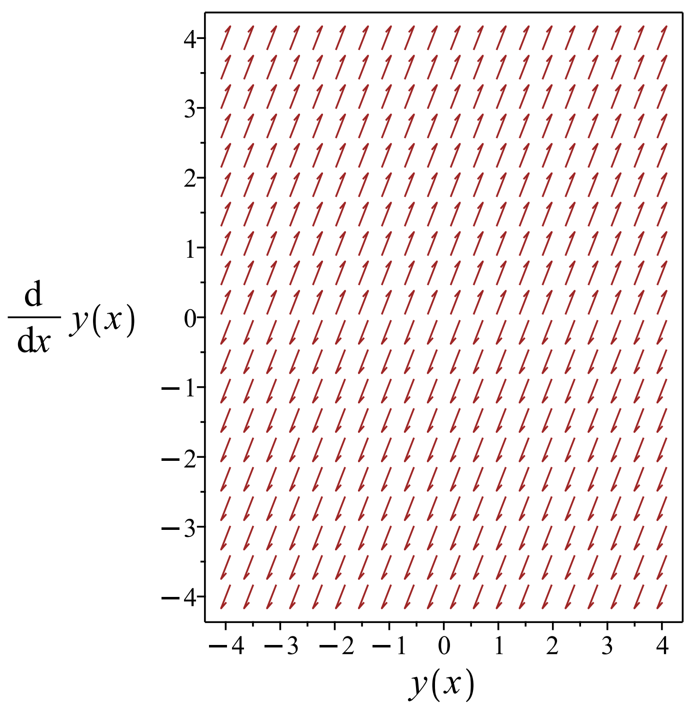

Figure 227: Slope field plot

Verification of solutions

$$y = -\frac{x}{2}+\frac{c_1 e^{2x}}{2}+\frac{e^{2x} x}{2}-\frac{e^{2x}}{4}+c_2$$

Verified OK.

27.8.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y''-2y' = e^{2x}+1$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y''-2y')\,dx = \int (e^{2x}+1)\,dx$$
$$-2y+y' = x+\frac{e^{2x}}{2}+c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$p(x) = -2$$
$$q(x) = x+\frac{e^{2x}}{2}+c_1$$

Hence the ode is

$$-2y+y' = x+\frac{e^{2x}}{2}+c_1$$

The integrating factor μ is

$$\mu = e^{\int (-2)dx}$$
$$= e^{-2x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x+\frac{e^{2x}}{2}+c_1\right)$$
$$\frac{d}{dx}\left(e^{-2x} y\right) = \left(e^{-2x}\right)\left(x+\frac{e^{2x}}{2}+c_1\right)$$
$$d\left(e^{-2x} y\right) = \left(\frac{1}{2}+(x+c_1)\,e^{-2x}\right) dx$$

Integrating gives

$$e^{-2x}y = \int \frac{1}{2} + (x + c_1)\, e^{-2x}\, dx$$

$$e^{-2x}y = \frac{x}{2} - \frac{x\, e^{-2x}}{2} - \frac{e^{-2x}}{4} - \frac{c_1 e^{-2x}}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-2x}$ results in

$$y = e^{2x}\left(\frac{x}{2} - \frac{x\, e^{-2x}}{2} - \frac{e^{-2x}}{4} - \frac{c_1 e^{-2x}}{2}\right) + c_2 e^{2x}$$

which simplifies to

$$y = \frac{(2x + 4c_2)\, e^{2x}}{4} - \frac{x}{2} - \frac{c_1}{2} - \frac{1}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{(2x + 4c_2)\, e^{2x}}{4} - \frac{x}{2} - \frac{c_1}{2} - \frac{1}{4} \tag{1}$$

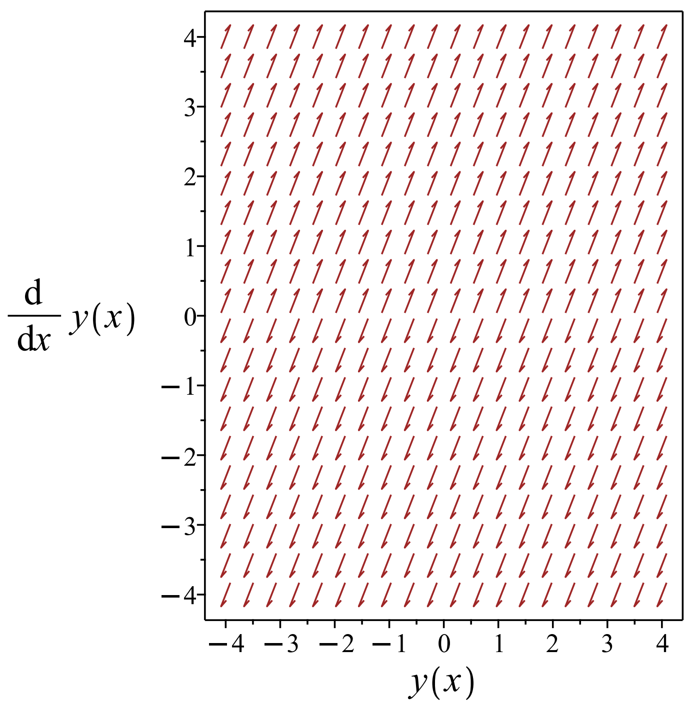

Figure 228: Slope field plot

Verification of solutions

$$y = \frac{(2x + 4c_2)\,e^{2x}}{4} - \frac{x}{2} - \frac{c_1}{2} - \frac{1}{4}$$

Verified OK.

27.8.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 155: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1 (\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + \frac{c_2 e^{2x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \frac{e^{2x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{e^{2x}}{2} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{e^{2x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{e^{2x}}{2} \\ 0 & e^{2x} \end{vmatrix}$$

Therefore

$$W = (1)\left(e^{2x}\right) - \left(\frac{e^{2x}}{2}\right)(0)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{2x}\left(e^{2x}+1\right)}{2}}{e^{2x}}\, dx$$

Which simplifies to

$$u_1 = -\int \left(\frac{e^{2x}}{2} + \frac{1}{2}\right) dx$$

Hence

$$u_1 = -\frac{x}{2} - \frac{e^{2x}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x}+1}{e^{2x}}\, dx$$

Which simplifies to

$$u_2 = \int \left(1 + e^{-2x}\right) dx$$

Hence

$$u_2 = x - \frac{e^{-2x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x}{2} - \frac{e^{2x}}{4} + \frac{e^{2x}\left(x - \frac{e^{-2x}}{2}\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{(2x-1)\,e^{2x}}{4} - \frac{x}{2} - \frac{1}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 e^{2x}}{2}\right) + \left(\frac{(2x-1)\,e^{2x}}{4} - \frac{x}{2} - \frac{1}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \frac{c_2 e^{2x}}{2} + \frac{(2x-1)\,e^{2x}}{4} - \frac{x}{2} - \frac{1}{4} \tag{1}$$

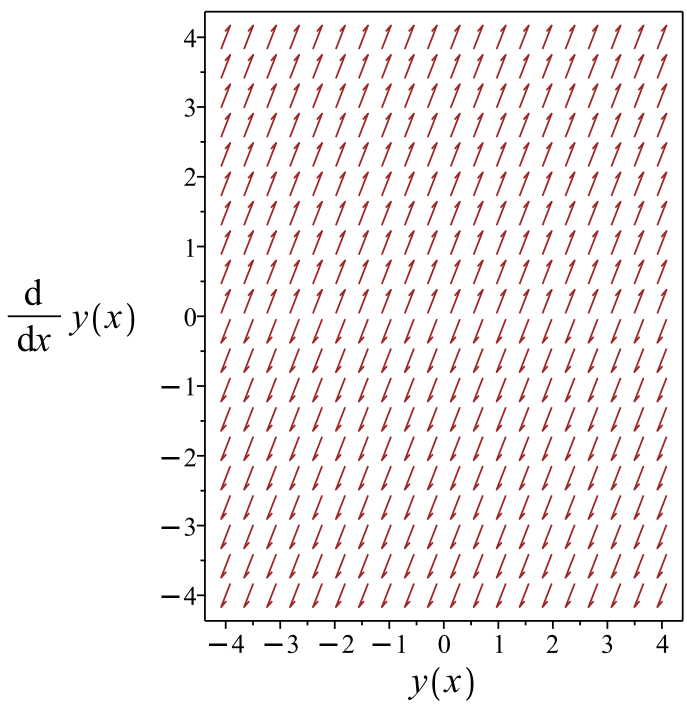

Figure 229: Slope field plot

Verification of solutions

$$y = c_1 + \frac{c_2 e^{2x}}{2} + \frac{(2x-1)\,e^{2x}}{4} - \frac{x}{2} - \frac{1}{4}$$

Verified OK.

27.8.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -2 \\ r(x) &= 0 \\ s(x) &= e^{2x} + 1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$-2y + y' = \int e^{2x} + 1\,dx$$

We now have a first order ode to solve which is

$$-2y + y' = x + \frac{e^{2x}}{2} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= x + \frac{e^{2x}}{2} + c_1 \end{aligned}$$

Hence the ode is

$$-2y + y' = x + \frac{e^{2x}}{2} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-2)dx} \\ &= e^{-2x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(x + \frac{e^{2x}}{2} + c_1\right) \\ \frac{d}{dx}(e^{-2x}y) &= (e^{-2x})\left(x + \frac{e^{2x}}{2} + c_1\right) \\ d(e^{-2x}y) &= \left(\frac{1}{2} + (x + c_1)e^{-2x}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-2x}y &= \int \frac{1}{2} + (x + c_1)e^{-2x}\, dx \\ e^{-2x}y &= \frac{x}{2} - \frac{x\,e^{-2x}}{2} - \frac{e^{-2x}}{4} - \frac{c_1 e^{-2x}}{2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-2x}$ results in

$$y = e^{2x}\left(\frac{x}{2} - \frac{x\,e^{-2x}}{2} - \frac{e^{-2x}}{4} - \frac{c_1 e^{-2x}}{2}\right) + c_2 e^{2x}$$

which simplifies to

$$y = \frac{(2x + 4c_2)\,e^{2x}}{4} - \frac{x}{2} - \frac{c_1}{2} - \frac{1}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{(2x + 4c_2)\,e^{2x}}{4} - \frac{x}{2} - \frac{c_1}{2} - \frac{1}{4} \tag{1}$$

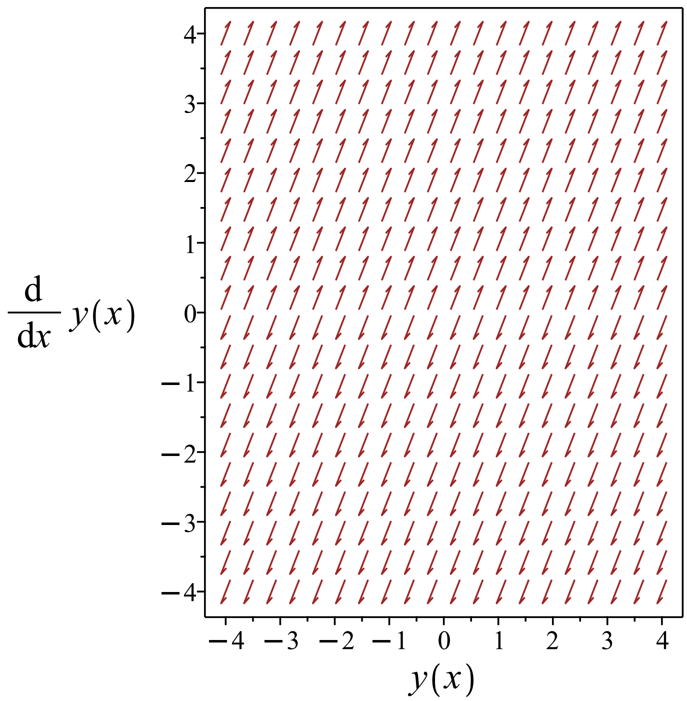

Figure 230: Slope field plot

Verification of solutions

$$y = \frac{(2x + 4c_2)\,e^{2x}}{4} - \frac{x}{2} - \frac{c_1}{2} - \frac{1}{4}$$

Verified OK.

27.8.7 Maple step by step solution

Let's solve

$y'' - 2y' = e^{2x} + 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 2r = 0$

- Factor the characteristic polynomial

  $r(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (0, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{2x} + 1\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^{2x} \\ 0 & 2\,e^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,e^{2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\left(\int (e^{2x}+1) dx\right)}{2} + \frac{e^{2x}\left(\int (1+e^{-2x}) dx\right)}{2}$

  - Compute integrals

    $y_p(x) = \frac{(2x-1)e^{2x}}{4} - \frac{x}{2} - \frac{1}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 e^{2x} + \frac{(2x-1)e^{2x}}{4} - \frac{x}{2} - \frac{1}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 2*_b(_a)+exp(2*_a)+1, _b(_a)`   *** Sub
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)=exp(2*x)+1,y(x), singsol=all)
```

$$y(x) = \frac{(2x + 2c_1 - 1)\,e^{2x}}{4} - \frac{x}{2} + c_2$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 31

```
DSolve[y''[x]-2*y'[x]==Exp[2*x]+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{2} + \frac{1}{4}e^{2x}(2x - 1 + 2c_1) + c_2$$

27.9 problem Ex 9

Internal problem ID [11271]
Internal file name [`OUTPUT/10256_Wednesday_December_21_2022_03_46_55_PM_3227727/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 50. Method of undetermined coefficients. Page 107
Problem number: Ex 9.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _linear, _nonhomogeneous]]`

$$\boxed{y'''' + 2y'' + y = \cos(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y'' + y = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^2 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = e^{ix}c_1 + x\,e^{ix}c_2 + e^{-ix}c_3 + x\,e^{-ix}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{ix}$$

$$y_2 = x\,e^{ix}$$

$$y_3 = e^{-ix}$$

$$y_4 = x\,e^{-ix}$$

Now the particular solution to the given ODE is found

$$y'''' + 2y'' + y = \cos(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{ix} & x\,e^{ix} & e^{-ix} & x\,e^{-ix} \\ ie^{ix} & e^{ix}(ix+1) & -ie^{-ix} & e^{-ix}(-ix+1) \\ -e^{ix} & e^{ix}(2i-x) & -e^{-ix} & e^{-ix}(-2i-x) \\ -ie^{ix} & -e^{ix}(ix+3) & ie^{-ix} & e^{-ix}(ix-3) \end{bmatrix}$$

$$|W| = 16\,e^{-2ix}e^{2ix}$$

The determinant simplifies to

$$|W| = 16$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} x\,e^{ix} & e^{-ix} & x\,e^{-ix} \\ e^{ix}(ix+1) & -ie^{-ix} & e^{-ix}(-ix+1) \\ e^{ix}(2i-x) & -e^{-ix} & e^{-ix}(-2i-x) \end{bmatrix}$$
$$= -4\,e^{-ix}(x-i)$$

$$W_2(x) = \det \begin{bmatrix} e^{ix} & e^{-ix} & x\,e^{-ix} \\ ie^{ix} & -ie^{-ix} & e^{-ix}(-ix+1) \\ -e^{ix} & -e^{-ix} & e^{-ix}(-2i-x) \end{bmatrix}$$
$$= -4\,e^{-ix}$$

$$W_3(x) = \det \begin{bmatrix} e^{ix} & x\,e^{ix} & x\,e^{-ix} \\ ie^{ix} & e^{ix}(ix+1) & e^{-ix}(-ix+1) \\ -e^{ix} & e^{ix}(2i-x) & e^{-ix}(-2i-x) \end{bmatrix}$$
$$= -4\,e^{ix}(x+i)$$

$$W_4(x) = \det \begin{bmatrix} e^{ix} & x\,e^{ix} & e^{-ix} \\ ie^{ix} & e^{ix}(ix+1) & -ie^{-ix} \\ -e^{ix} & e^{ix}(2i-x) & -e^{-ix} \end{bmatrix}$$
$$= -4\,e^{ix}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^3 \int \frac{(\cos(x))\left(-4\,e^{-ix}(x-i)\right)}{(1)\,(16)}\,dx \\
&= -\int \frac{-4\cos(x)\,e^{-ix}(x-i)}{16}\,dx \\
&= -\int \left(-\frac{\cos(x)\,e^{-ix}(x-i)}{4}\right)dx \\
&= -\left(\int -\frac{\cos(x)\,e^{-ix}(x-i)}{4}\,dx\right)
\end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x) W_2(x)}{a W(x)}\, dx \\
&= (-1)^2 \int \frac{(\cos (x)) \left(-4\, \mathrm{e}^{-i x}\right)}{(1)\,(16)}\, dx \\
&= \int \frac{-4 \cos (x)\, \mathrm{e}^{-i x}}{16}\, dx \\
&= \int \left(-\frac{\cos (x)\, \mathrm{e}^{-i x}}{4}\right) dx \\
&= \int -\frac{\cos (x)\, \mathrm{e}^{-i x}}{4} dx
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^1 \int \frac{(\cos (x)) \left(-4\, \mathrm{e}^{i x} (x+i)\right)}{(1)\,(16)}\, dx \\
&= -\int \frac{-4 \cos (x)\, \mathrm{e}^{i x} (x+i)}{16}\, dx \\
&= -\int \left(-\frac{\cos (x)\, \mathrm{e}^{i x} (x+i)}{4}\right) dx \\
&= \frac{x^2}{16} + \frac{i x}{8} - \frac{i (3 i+2 x)\, \mathrm{e}^{2 i x}}{32}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(\cos (x)) \left(-4\, \mathrm{e}^{i x}\right)}{(1)\,(16)}\, dx \\
&= \int \frac{-4 \cos (x)\, \mathrm{e}^{i x}}{16}\, dx \\
&= \int \left(-\frac{\cos (x)\, \mathrm{e}^{i x}}{4}\right) dx \\
&= -\frac{x}{8} + \frac{i \mathrm{e}^{2 i x}}{16}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = {} & \left(-\left(\int -\frac{\cos(x)\,\mathrm{e}^{-ix}(x-i)}{4}dx\right)\right)\left(\mathrm{e}^{ix}\right) \\
& + \left(\int -\frac{\cos(x)\,\mathrm{e}^{-ix}}{4}dx\right)\left(x\,\mathrm{e}^{ix}\right) \\
& + \left(\frac{x^2}{16}+\frac{ix}{8}-\frac{i(3i+2x)\,\mathrm{e}^{2ix}}{32}\right)\left(\mathrm{e}^{-ix}\right) \\
& + \left(-\frac{x}{8}+\frac{i\mathrm{e}^{2ix}}{16}\right)\left(x\,\mathrm{e}^{-ix}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\cos(x)\left(-4x^2+2ix+5\right)}{32} - \frac{(i-6x)\sin(x)}{32}
$$

Which simplifies to

$$
y_p = \frac{\cos(x)\left(-4x^2+2ix+5\right)}{32} - \frac{(i-6x)\sin(x)}{32}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{ix}c_1 + x\,\mathrm{e}^{ix}c_2 + \mathrm{e}^{-ix}c_3 + x\,\mathrm{e}^{-ix}c_4\right) + \left(\frac{\cos(x)\left(-4x^2+2ix+5\right)}{32} - \frac{(i-6x)\sin(x)}{32}\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{-ix}(c_4x+c_3) + (c_2x+c_1)\,\mathrm{e}^{ix} + \frac{\cos(x)\left(-4x^2+2ix+5\right)}{32} - \frac{(i-6x)\sin(x)}{32}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-ix}(c_4x+c_3) + (c_2x+c_1)\,\mathrm{e}^{ix} + \frac{\cos(x)\left(-4x^2+2ix+5\right)}{32} - \frac{(i-6x)\sin(x)}{32} \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{-ix}(c_4x+c_3) + (c_2x+c_1)\,\mathrm{e}^{ix} + \frac{\cos(x)\left(-4x^2+2ix+5\right)}{32} - \frac{(i-6x)\sin(x)}{32}
$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$2)+y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = \frac{(8c_4x - x^2 + 8c_1 + 2)\cos(x)}{8} + \left(\left(c_3 + \frac{1}{8}\right)x + c_2\right)\sin(x)$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 43

```
DSolve[y''''[x]+2*y''[x]+y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(-\frac{x^2}{8} + c_2x + \frac{5}{16} + c_1\right)\cos(x) + \frac{1}{4}(x + 4c_4x + 4c_3)\sin(x)$$

28 Chapter VII, Linear differential equations with constant coefficients. Article 51. Cauchy linear equation. Page 114

28.1 problem Ex 1

Internal problem ID [11272]
Internal file name [`OUTPUT/10257_Wednesday_December_21_2022_03_46_57_PM_78523104/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 51. Cauchy linear equation. Page 114
Problem number: Ex 1.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y''' + y'x - y = x\ln(x)}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' + y'x - y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' + y'x - y = x\ln(x)$$

gives

$$x\lambda\, x^{\lambda-1}+x^{3}\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}-x^{\lambda} = 0$$

Which simplifies to

$$\lambda\, x^{\lambda}+\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda}-x^{\lambda} = 0$$

And since $x^{\lambda}\neq 0$ then dividing through by x^{λ}, the above becomes

$$\lambda+\lambda(\lambda-1)\,(\lambda-2)-1 = 0$$

Simplifying gives the characteristic equation as

$$(\lambda-1)^{3} = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

This table summarises the result

root	multiplicity	type of root
1	3	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^{λ} basis solution. Each real root of multiplicty two, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ and $c_3x^{\lambda}\ln(x)^{2}$ basis solutions, and so on. Each complex root $\alpha\pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity two generates $\ln(x)\, x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity three generates $\ln(x)^{2} x^{\alpha}(c_1\cos(\beta\ln(x))+c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_1x+c_2\ln(x)\, x+c_3\ln(x)^{2} x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = x$$
$$y_2 = x\ln(x)$$
$$y_3 = \ln(x)^{2} x$$

Now the particular solution to the given ODE is found

$$x^3y''' + y'x - y = x\ln(x)$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i}\int \frac{F(x)W_i(x)}{aW(x)}\,dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} x & x\ln(x) & \ln(x)^2 x \\ 1 & \ln(x)+1 & \ln(x)(\ln(x)+2) \\ 0 & \frac{1}{x} & \frac{2+2\ln(x)}{x} \end{bmatrix}$$
$$|W| = 2$$

The determinant simplifies to

$$|W| = 2$$

Now we determine W_i for each U_i.

$$W_1(x) = \det\begin{bmatrix} x\ln(x) & \ln(x)^2 x \\ \ln(x)+1 & \ln(x)(\ln(x)+2) \end{bmatrix}$$
$$= \ln(x)^2 x$$

$$
\begin{aligned}
W_2(x) &= \det \begin{bmatrix} x & \ln(x)^2 x \\ 1 & \ln(x)(\ln(x)+2) \end{bmatrix} \\
&= 2x\ln(x)
\end{aligned}
$$

$$
\begin{aligned}
W_3(x) &= \det \begin{bmatrix} x & x\ln(x) \\ 1 & \ln(x)+1 \end{bmatrix} \\
&= x
\end{aligned}
$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{(x\ln(x))\left(\ln(x)^2 x\right)}{(x^3)(2)}\,dx \\
&= \int \frac{x^2\ln(x)^3}{2x^3}\,dx \\
&= \int \left(\frac{\ln(x)^3}{2x}\right) dx \\
&= \frac{\ln(x)^4}{8}
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(x\ln(x))(2x\ln(x))}{(x^3)(2)}\,dx \\
&= -\int \frac{2x^2\ln(x)^2}{2x^3}\,dx \\
&= -\int \left(\frac{\ln(x)^2}{x}\right) dx \\
&= -\frac{\ln(x)^3}{3}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(x\ln(x))(x)}{(x^3)(2)}\,dx \\
&= \int \frac{x^2\ln(x)}{2x^3}\,dx \\
&= \int \left(\frac{\ln(x)}{2x}\right) dx \\
&= \frac{\ln(x)^2}{4}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{\ln(x)^4}{8}\right)(x) \\
&+ \left(-\frac{\ln(x)^3}{3}\right)(x\ln(x)) \\
&+ \left(\frac{\ln(x)^2}{4}\right)\left(\ln(x)^2 x\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\ln(x)^4 x}{24}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 x + c_2\ln(x)\,x + c_3\ln(x)^2 x\right) + \left(\frac{\ln(x)^4 x}{24}\right)
\end{aligned}
$$

Which simplifies to

$$
y = x\left(c_1 + c_2\ln(x) + c_3\ln(x)^2\right) + \frac{\ln(x)^4 x}{24}
$$

Summary

The solution(s) found are the following

$$y = x\left(c_1 + c_2 \ln(x) + c_3 \ln(x)^2\right) + \frac{\ln(x)^4 x}{24} \tag{1}$$

Verification of solutions

$$y = x\left(c_1 + c_2 \ln(x) + c_3 \ln(x)^2\right) + \frac{\ln(x)^4 x}{24}$$

Verified OK.

28.1.1 Maple step by step solution

Let's solve

$x^3 y''' + y'x - y = x \ln(x)$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(x^3*diff(y(x),x$3)+x*diff(y(x),x)-y(x)=x*ln(x),y(x), singsol=all)
```

$$y(x) = x\left(\frac{\ln(x)^4}{24} + c_1 + c_2 \ln(x) + c_3 \ln(x)^2\right)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 33

```
DSolve[x^3*y'''[x]+x*y'[x]-y[x]==x*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{24} x \log^4(x) + c_1 x + c_3 x \log^2(x) + c_2 x \log(x)$$

28.2 problem Ex 2

Internal problem ID [11273]
Internal file name [`OUTPUT/10258_Wednesday_December_21_2022_03_46_58_PM_83170867/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 51. Cauchy linear equation. Page 114
Problem number: Ex 2.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^3y''' + 2x^2y'' + 2y = 10x + \frac{10}{x}}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' + 2x^2y'' + 2y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^3y''' + 2x^2y'' + 2y = 10x + \frac{10}{x}$$

gives

$$2x^2\lambda(\lambda-1)\,x^{\lambda-2}+x^3\lambda(\lambda-1)\,(\lambda-2)\,x^{\lambda-3}+2x^\lambda=0$$

Which simplifies to

$$2\lambda(\lambda-1)\,x^\lambda+\lambda(\lambda-1)\,(\lambda-2)\,x^\lambda+2x^\lambda=0$$

And since $x^\lambda\neq 0$ then dividing through by x^λ, the above becomes

$$2\lambda(\lambda-1)+\lambda(\lambda-1)\,(\lambda-2)+2=0$$

Simplifying gives the characteristic equation as

$$\lambda^3-\lambda^2+2=0$$

Solving the above gives the following roots

$$\lambda_1=-1$$
$$\lambda_2=1-i$$
$$\lambda_3=1+i$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
$1\pm 1i$	1	complex conjugate root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicty two, generates c_1x^λ and $c_2x^\lambda\ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^λ and $c_2x^\lambda\ln(x)$ and $c_3x^\lambda\ln(x)^2$ basis solutions, and so on. Each complex root $\alpha\pm i\beta$ of multiplicity one generates $x^\alpha(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity two generates $\ln(x)\,x^\alpha(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity three generates $\ln(x)^2\,x^\alpha(c_1\cos(\beta\ln(x))+c_2$ basis solutions. And so on. Using the above show that the solution is

$$y=\frac{c_1}{x}+x(c_2\cos(\ln(x))+c_3\sin(\ln(x)))$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \frac{1}{x} \\
y_2 &= x\cos(\ln(x)) \\
y_3 &= x\sin(\ln(x))
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
x^3y''' + 2x^2y'' + 2y = 10x + \frac{10}{x}
$$

Let the particular solution be

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i}\int \frac{F(x)W_i(x)}{aW(x)}\,dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
\begin{aligned}
W &= \begin{bmatrix} \frac{1}{x} & x\cos(\ln(x)) & x\sin(\ln(x)) \\ -\frac{1}{x^2} & \cos(\ln(x)) - \sin(\ln(x)) & \sin(\ln(x)) + \cos(\ln(x)) \\ \frac{2}{x^3} & \frac{-\sin(\ln(x)) - \cos(\ln(x))}{x} & \frac{\cos(\ln(x)) - \sin(\ln(x))}{x} \end{bmatrix} \\
|W| &= \frac{5\cos(\ln(x))^2 + 5\sin(\ln(x))^2}{x^2}
\end{aligned}
$$

The determinant simplifies to

$$|W| = \frac{5}{x^2}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} x\cos(\ln(x)) & x\sin(\ln(x)) \\ \cos(\ln(x)) - \sin(\ln(x)) & \sin(\ln(x)) + \cos(\ln(x)) \end{bmatrix}$$
$$= x$$

$$W_2(x) = \det \begin{bmatrix} \frac{1}{x} & x\sin(\ln(x)) \\ -\frac{1}{x^2} & \sin(\ln(x)) + \cos(\ln(x)) \end{bmatrix}$$
$$= \frac{2\sin(\ln(x)) + \cos(\ln(x))}{x}$$

$$W_3(x) = \det \begin{bmatrix} \frac{1}{x} & x\cos(\ln(x)) \\ -\frac{1}{x^2} & \cos(\ln(x)) - \sin(\ln(x)) \end{bmatrix}$$
$$= \frac{2\cos(\ln(x)) - \sin(\ln(x))}{x}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2 \int \frac{\left(10x + \frac{10}{x}\right)(x)}{(x^3)\left(\frac{5}{x^2}\right)}\,dx \\ &= \int \frac{\left(10x + \frac{10}{x}\right)x}{5x}\,dx \\ &= \int \left(2x + \frac{2}{x}\right)dx \\ &= x^2 + 2\ln(x) \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)} \, dx \\
&= (-1)^{1} \int \frac{\left(10x + \frac{10}{x}\right) \left(\frac{2 \sin(\ln(x)) + \cos(\ln(x))}{x}\right)}{(x^3) \left(\frac{5}{x^2}\right)} \, dx \\
&= - \int \frac{\frac{\left(10x + \frac{10}{x}\right)(2 \sin(\ln(x)) + \cos(\ln(x)))}{x}}{5x} \, dx \\
&= - \int \left(\frac{2(x^2+1)\left(2 \sin\left(\ln(x)\right) + \cos\left(\ln(x)\right)\right)}{x^3}\right) dx \\
&= -\frac{2\left(-\frac{2}{5} + \frac{2 \tan\left(\frac{\ln(x)}{2}\right)^2}{5} + \frac{2 \tan\left(\frac{\ln(x)}{2}\right)}{5}\right)}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x^2} - \frac{4\left(-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4 \tan\left(\frac{\ln(x)}{2}\right)}{5}\right)}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x^2} - 2 \sin\left(\ln(x)\right) + 4 \cos (\ln \\
&= -\frac{2\left(-\frac{2}{5} + \frac{2 \tan\left(\frac{\ln(x)}{2}\right)^2}{5} + \frac{2 \tan\left(\frac{\ln(x)}{2}\right)}{5}\right)}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x^2} - \frac{4\left(-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4 \tan\left(\frac{\ln(x)}{2}\right)}{5}\right)}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x^2} - 2 \sin\left(\ln(x)\right) + 4 \cos (\ln
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)} \, dx \\
&= (-1)^{0} \int \frac{\left(10x + \frac{10}{x}\right) \left(\frac{2 \cos(\ln(x)) - \sin(\ln(x))}{x}\right)}{(x^3) \left(\frac{5}{x^2}\right)} \, dx \\
&= \int \frac{\frac{\left(10x + \frac{10}{x}\right)(2 \cos(\ln(x)) - \sin(\ln(x)))}{x}}{5x} \, dx \\
&= \int \left(\frac{2(x^2+1)\left(2 \cos\left(\ln(x)\right) - \sin\left(\ln(x)\right)\right)}{x^3}\right) dx \\
&= \frac{-\frac{8}{5} + \frac{8 \tan\left(\frac{\ln(x)}{2}\right)^2}{5} + \frac{8 \tan\left(\frac{\ln(x)}{2}\right)}{5}}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x^2} - \frac{2\left(-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4 \tan\left(\frac{\ln(x)}{2}\right)}{5}\right)}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x^2} + 4 \sin\left(\ln(x)\right) + 2 \cos\left(\ln(x)\right) \\
&= \frac{-\frac{8}{5} + \frac{8 \tan\left(\frac{\ln(x)}{2}\right)^2}{5} + \frac{8 \tan\left(\frac{\ln(x)}{2}\right)}{5}}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x^2} - \frac{2\left(-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4 \tan\left(\frac{\ln(x)}{2}\right)}{5}\right)}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x^2} + 4 \sin\left(\ln(x)\right) + 2 \cos\left(\ln(x)\right)
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Hence

$$
\begin{aligned}
y_p &= \left(x^2 + 2\ln(x)\right)\left(\frac{1}{x}\right) \\
&+ \Bigg(-\frac{2\left(-\frac{2}{5} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)^2}{5} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)}{5}\right)}{\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)x^2} - \frac{4\left(-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{5}\right)}{\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)x^2} - 2\sin(\ln(x)) + 4\cos(\ln \\
&+ \left(\frac{-\frac{8}{5} + \frac{8\tan\left(\frac{\ln(x)}{2}\right)^2}{5} + \frac{8\tan\left(\frac{\ln(x)}{2}\right)}{5}}{\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)x^2} - \frac{2\left(-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{5}\right)}{\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)x^2} + 4\sin(\ln(x)) + 2\cos(\ln(x))\right)
\end{aligned}
$$

Therefore the particular solution is

$$y_p = \frac{25x^2 + 10\ln(x) + 8}{5x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x} + x(c_2\cos(\ln(x)) + c_3\sin(\ln(x)))\right) + \left(\frac{25x^2 + 10\ln(x) + 8}{5x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + x(c_2\cos(\ln(x)) + c_3\sin(\ln(x))) + \frac{25x^2 + 10\ln(x) + 8}{5x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + x(c_2\cos(\ln(x)) + c_3\sin(\ln(x))) + \frac{25x^2 + 10\ln(x) + 8}{5x}$$

Verified OK.

28.2.1 Maple step by step solution

Let's solve

$x^3 y''' + 2y'' x^2 + 2y = 10x + \frac{10}{x}$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = (c__1-2*_b(_a)*_a+_a^2*(diff(
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      <- LODE of Euler type successful
   <- solving first the homogeneous part of the ODE successful
<- high order exact_linear_nonhomogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 120

```
dsolve(x^3*diff(y(x),x$3)+2*x^2*diff(y(x),x$2)+2*y(x)=10*(x+1/x),y(x), singsol=all)
```

$$
\begin{aligned}
y(x) = {} & \frac{((20-10i)\ln(x)+6-8i+(2-i)c_1)(i\cos(\ln(x))-\sin(\ln(x)))x^{-1-i}}{10} \\
& + \frac{((20+10i)\ln(x)+6+8i+(2+i)c_1)(-\sin(\ln(x))-i\cos(\ln(x)))x^{-1+i}}{10} \\
& + \frac{5x^{1-i}(i\sin(\ln(x))+\cos(\ln(x)))}{2} \\
& + \frac{5(-i\sin(\ln(x))+\cos(\ln(x)))x^{1+i}}{2} + x(\cos(\ln(x))c_2+\sin(\ln(x))c_3)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.187 (sec). Leaf size: 42

```
DSolve[x^3*y'''[x]+2*x^2*y''[x]+2*y[x]==10*(x+1/x),y[x],x,IncludeSingularSolutions -> True]
```

$$
y(x) \to \frac{25x^2+10\log(x)+8+5c_3}{5x}+c_2x\cos(\log(x))+c_1x\sin(\log(x))
$$

28.3 problem Ex 3

Internal problem ID [11274]
Internal file name [`OUTPUT/10259_Wednesday_December_21_2022_03_47_00_PM_99636391/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 51. Cauchy linear equation. Page 114
Problem number: Ex 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' + 3y'x + y = \frac{1}{(1-x)^2}}$$

28.3.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 3x, C = 1, f(x) = \frac{1}{(x-1)^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 3y'x + y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 3xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 3r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 3r + 1 = 0$$

Or

$$r^2 + 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= -1 \end{aligned}$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x}$$

Next, we find the particular solution to the ODE

$$x^2y'' + 3y'x + y = \frac{1}{(x-1)^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$

$$y_2 = \frac{\ln(x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ -\frac{1}{x^2} & \frac{1}{x^2} - \frac{\ln(x)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{x^2} - \frac{\ln(x)}{x^2}\right) - \left(\frac{\ln(x)}{x}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\ln(x)}{x(x-1)^2}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{(x-1)^2}dx$$

Hence

$$u_1 = -\ln(x-1) + \frac{\ln(x)\,x}{x-1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x(x-1)^2}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{(x-1)^2}dx$$

Hence

$$u_2 = -\frac{1}{x-1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\ln(x-1) + \frac{\ln(x)x}{x-1}}{x} - \frac{\ln(x)}{(x-1)\,x}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x) - \ln(x-1)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{c_2 \ln(x) + \ln(x) - \ln(x-1) + c_1}{x} \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_2 \ln(x) + \ln(x) - \ln(x-1) + c_1}{x} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_2 \ln(x) + \ln(x) - \ln(x-1) + c_1}{x}$$

Verified OK.

28.3.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 3y'x + y = 0$$

In normal form the ode

$$x^2y'' + 3y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-(\int p(x)dx)}\,dx \\ &= \int e^{-(\int \frac{3}{x}dx)}\,dx \\ &= \int e^{-3\ln(x)}\,dx \\ &= \int \frac{1}{x^3}dx \\ &= -\frac{1}{2x^2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{x^2}}{\frac{1}{x^6}} \\ &= x^4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + x^4 y(\tau) = 0$$

But in terms of τ

$$x^4 = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{x^2}\right)\right)}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{x^2}\right)\right)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{-\frac{1}{x^2}}$$
$$y_2 = -\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\frac{1}{x^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2} \\ \frac{d}{dx}\left(\sqrt{-\frac{1}{x^2}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\frac{1}{x^2}} & -\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2} \\ \frac{1}{\sqrt{-\frac{1}{x^2}}\,x^3} & -\frac{\sqrt{2}\,\ln(2)}{2\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{\sqrt{2}\,\ln\left(-\frac{1}{x^2}\right)}{2\sqrt{-\frac{1}{x^2}}\,x^3} - \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}}{x} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-\frac{1}{x^2}}\right)\left(-\frac{\sqrt{2}\,\ln(2)}{2\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{\sqrt{2}\,\ln\left(-\frac{1}{x^2}\right)}{2\sqrt{-\frac{1}{x^2}}\,x^3} - \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}}{x}\right) - \left(-\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}\right)\left(\frac{1}{\sqrt{-\frac{1}{x^2}}\,x^3}\right)$$

Which simplifies to

$$W = \frac{\sqrt{2}}{x^3}$$

Which simplifies to

$$W = \frac{\sqrt{2}}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{-\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}}{(x-1)^2}}{\frac{\sqrt{2}}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{-\frac{1}{x^2}}\,\left(-\ln(2) + \ln\left(-\frac{1}{x^2}\right)\right)x}{2\,(x-1)^2}\,dx$$

Hence

$$u_1 = \frac{x\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2x-2} - \frac{x\sqrt{-\frac{1}{x^2}}\,(2x\ln(x) - 2\ln(x-1)\,x - 2\ln(x) + 2\ln(x-1) + \ln(2))}{2\,(x-1)}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\sqrt{-\frac{1}{x^2}}}{(x-1)^2}}{\frac{\sqrt{2}}{x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{-\frac{1}{x^2}}\, x\sqrt{2}}{2\,(x-1)^2} dx$$

Hence

$$u_2 = -\frac{\sqrt{-\frac{1}{x^2}}\, x\sqrt{2}}{2\,(x-1)}$$

Which simplifies to

$$u_1 = -\frac{x\sqrt{-\frac{1}{x^2}}\,\left(2x\ln(x) - 2\ln(x-1)\,x - 2\ln(x) + 2\ln(x-1) - \ln\left(-\frac{1}{x^2}\right) + \ln(2)\right)}{2x-2}$$

$$u_2 = -\frac{\sqrt{-\frac{1}{x^2}}\, x\sqrt{2}}{2x-2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2x\ln(x) - 2\ln(x-1)\,x - 2\ln(x) + 2\ln(x-1) - \ln\left(-\frac{1}{x^2}\right) + \ln(2)}{x\,(2x-2)} - \frac{\sqrt{-\frac{1}{x^2}}\, x\sqrt{2}\left(-\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}\right)}{2x-2}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x) - \ln(x-1)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{x^2}\right)\right)}{2}\right) + \left(\frac{\ln(x) - \ln(x-1)}{x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{x^2}\right)\right)}{2} + \frac{\ln(x) - \ln(x-1)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{x^2}\right)\right)}{2} + \frac{\ln(x) - \ln(x-1)}{x}$$

Verified OK.

28.3.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 3x, C = 1, f(x) = \frac{1}{(x-1)^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 3y'x + y = 0$$

In normal form the ode

$$x^2y'' + 3y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^2}}}{c} \qquad (6)\\ \tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{3}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= 2c\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 \qquad (7)\end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{-c\tau} c_1$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\ &= \frac{\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{c_1}{x}$$

Now the particular solution to this ODE is found

$$x^2 y'' + 3y'x + y = \frac{1}{(x-1)^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{-\frac{1}{x^2}}$$

$$y_2 = -\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\frac{1}{x^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2} \\ \frac{d}{dx}\left(\sqrt{-\frac{1}{x^2}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\frac{1}{x^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2} \\ \frac{1}{\sqrt{-\frac{1}{x^2}}\,x^3} & -\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{x^2}\right)}{2\sqrt{-\frac{1}{x^2}}\,x^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}}{x} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-\frac{1}{x^2}}\right)\left(-\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{x^2}\right)}{2\sqrt{-\frac{1}{x^2}}\,x^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}}{x}\right) - \left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2}\right)\left(\frac{1}{\sqrt{-\frac{1}{x^2}}\,x^3}\right)$$

Which simplifies to

$$W = \frac{\sqrt{2}}{x^3}$$

Which simplifies to

$$W = \frac{\sqrt{2}}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2}}{(x-1)^2}}{\frac{\sqrt{2}}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{-\frac{1}{x^2}}\left(-\ln(2) + \ln\left(-\frac{1}{x^2}\right)\right)x}{2(x-1)^2}\,dx$$

Hence

$$u_1 = \frac{x\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2x-2} - \frac{x\sqrt{-\frac{1}{x^2}}\left(2x\ln(x) - 2\ln(x-1)x - 2\ln(x) + 2\ln(x-1) + \ln(2)\right)}{2(x-1)}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\sqrt{-\frac{1}{x^2}}}{(x-1)^2}}{\frac{\sqrt{2}}{x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{-\frac{1}{x^2}}\, x\sqrt{2}}{2\,(x-1)^2} dx$$

Hence

$$u_2 = -\frac{\sqrt{-\frac{1}{x^2}}\, x\sqrt{2}}{2\,(x-1)}$$

Which simplifies to

$$u_1 = -\frac{x\sqrt{-\frac{1}{x^2}}\,\left(2x\ln(x) - 2\ln(x-1)\,x - 2\ln(x) + 2\ln(x-1) - \ln\left(-\frac{1}{x^2}\right) + \ln(2)\right)}{2x-2}$$

$$u_2 = -\frac{\sqrt{-\frac{1}{x^2}}\, x\sqrt{2}}{2x-2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2x\ln(x) - 2\ln(x-1)\,x - 2\ln(x) + 2\ln(x-1) - \ln\left(-\frac{1}{x^2}\right) + \ln(2)}{x\,(2x-2)} - \frac{\sqrt{-\frac{1}{x^2}}\, x\sqrt{2}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}\right)}{2x-2}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x) - \ln(x-1)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x}\right) + \left(\frac{\ln(x) - \ln(x-1)}{x}\right) \\ &= \frac{\ln(x) - \ln(x-1)}{x} + \frac{c_1}{x} \end{aligned}$$

Which simplifies to

$$y = \frac{\ln(x) - \ln(x-1) + c_1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x) - \ln(x-1) + c_1}{x} \tag{1}$$

Verification of solutions

$$y = \frac{\ln(x) - \ln(x-1) + c_1}{x}$$

Verified OK.

28.3.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 3x, C = 1, f(x) = \frac{1}{(x-1)^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 3y'x + y = 0$$

In normal form the ode

$$x^2y'' + 3y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{3n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \frac{c_1 \ln(x) + c_2}{x} \\
&= \frac{c_1 \ln(x) + c_2}{x}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + 3y'x + y = \frac{1}{(x-1)^2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$

$$y_2 = \frac{\ln(x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ -\frac{1}{x^2} & \frac{1}{x^2} - \frac{\ln(x)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{x^2} - \frac{\ln(x)}{x^2}\right) - \left(\frac{\ln(x)}{x}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\ln(x)}{x(x-1)^2}}{\frac{1}{x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln (x)}{(x-1)^2} dx$$

Hence

$$u_1 = -\ln (x-1) + \frac{\ln (x) x}{x-1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x(x-1)^2}}{\frac{1}{x}} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{(x-1)^2} dx$$

Hence

$$u_2 = -\frac{1}{x-1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\ln (x-1) + \frac{\ln(x)x}{x-1}}{x} - \frac{\ln (x)}{(x-1) x}$$

Which simplifies to

$$y_p(x) = \frac{\ln (x) - \ln (x-1)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \ln (x) + c_2}{x}\right) + \left(\frac{\ln (x) - \ln (x-1)}{x}\right) \\ &= \frac{\ln (x) - \ln (x-1)}{x} + \frac{c_1 \ln (x) + c_2}{x} \end{aligned}$$

Which simplifies to

$$y = \frac{c_1 \ln (x) + \ln (x) - \ln (x - 1) + c_2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln (x) + \ln (x) - \ln (x - 1) + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln (x) + \ln (x) - \ln (x - 1) + c_2}{x}$$

Verified OK.

28.3.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned} \int (x^2 y'' + 3y'x + y) \, dx &= \int \frac{1}{(x-1)^2} dx \\ x^2 y' + yx &= -\frac{1}{x-1} + c_1 \end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{c_1(x-1) - 1}{x^2 (x-1)} \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x} = \frac{c_1(x-1) - 1}{x^2 (x-1)}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x} dx} \\ &= x \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1(x-1)-1}{x^2(x-1)}\right)$$
$$\frac{d}{dx}(xy) = (x)\left(\frac{c_1(x-1)-1}{x^2(x-1)}\right)$$
$$d(xy) = \left(\frac{c_1(x-1)-1}{x(x-1)}\right) dx$$

Integrating gives

$$xy = \int \frac{c_1(x-1)-1}{x(x-1)} dx$$
$$xy = -\ln(x-1) + (c_1+1)\ln(x) + c_2$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{-\ln(x-1) + (c_1+1)\ln(x)}{x} + \frac{c_2}{x}$$

which simplifies to

$$y = \frac{c_1\ln(x) + \ln(x) - \ln(x-1) + c_2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\ln(x) + \ln(x) - \ln(x-1) + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\ln(x) + \ln(x) - \ln(x-1) + c_2}{x}$$

Verified OK.

28.3.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' + 3y'x + y = \frac{1}{(x-1)^2}$$

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int \left(x^2 y'' + 3y'x + y\right) dx &= \int \frac{1}{\left(x-1\right)^2} dx \\
x^2 y' + yx &= -\frac{1}{x-1} + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{c_1(x-1) - 1}{x^2\left(x-1\right)}
\end{aligned}
$$

Hence the ode is

$$
y' + \frac{y}{x} = \frac{c_1(x-1) - 1}{x^2\left(x-1\right)}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{x} dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu) \left(\frac{c_1(x-1) - 1}{x^2\left(x-1\right)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(xy) &= (x) \left(\frac{c_1(x-1) - 1}{x^2\left(x-1\right)}\right) \\
\mathrm{d}(xy) &= \left(\frac{c_1(x-1) - 1}{x\left(x-1\right)}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
xy &= \int \frac{c_1(x-1) - 1}{x\left(x-1\right)} \mathrm{d}x \\
xy &= -\ln\left(x-1\right) + \left(c_1+1\right)\ln\left(x\right) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
y = \frac{-\ln\left(x-1\right) + \left(c_1+1\right)\ln\left(x\right)}{x} + \frac{c_2}{x}
$$

which simplifies to

$$y = \frac{c_1 \ln(x) + \ln(x) - \ln(x-1) + c_2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(x) + \ln(x) - \ln(x-1) + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(x) + \ln(x) - \ln(x-1) + c_2}{x}$$

Verified OK.

28.3.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 3y'x + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 159: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3x}{x^2} \, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{x^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}(\ln(x))\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 3y'x + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \frac{1}{x} \\
y_2 &= \frac{\ln(x)}{x}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ -\frac{1}{x^2} & \frac{1}{x^2} - \frac{\ln(x)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{x^2} - \frac{\ln(x)}{x^2}\right) - \left(\frac{\ln(x)}{x}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\ln(x)}{x(x-1)^2}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{(x-1)^2}\,dx$$

Hence

$$u_1 = -\ln(x-1) + \frac{\ln(x)\,x}{x-1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x(x-1)^2}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{(x-1)^2} dx$$

Hence

$$u_2 = -\frac{1}{x-1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\ln(x-1) + \frac{\ln(x)x}{x-1}}{x} - \frac{\ln(x)}{(x-1)\,x}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x) - \ln(x-1)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2 \ln(x)}{x}\right) + \left(\frac{\ln(x) - \ln(x-1)}{x}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_2 \ln(x) + c_1}{x} + \frac{\ln(x) - \ln(x-1)}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2 \ln(x) + c_1}{x} + \frac{\ln(x) - \ln(x-1)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_2 \ln(x) + c_1}{x} + \frac{\ln(x) - \ln(x-1)}{x}$$

Verified OK.

28.3.8 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)$$

is exact if

$$p''(x)-q'(x)+r(x)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x)&=x^2\\q(x)&=3x\\r(x)&=1\\s(x)&=\frac{1}{(x-1)^2}\end{aligned}$$

Hence

$$\begin{aligned}p''(x)&=2\\q'(x)&=3\end{aligned}$$

Therefore (1) becomes

$$2-(3)+(1)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+(q(x)-p'(x))\,y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$x^2y'+yx=\int\frac{1}{(x-1)^2}\,dx$$

We now have a first order ode to solve which is

$$x^2y'+yx=-\frac{1}{x-1}+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{c_1(x-1)-1}{x^2(x-1)} \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x} = \frac{c_1(x-1)-1}{x^2(x-1)}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x} dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1(x-1)-1}{x^2(x-1)}\right) \\ \frac{d}{dx}(xy) &= (x)\left(\frac{c_1(x-1)-1}{x^2(x-1)}\right) \\ d(xy) &= \left(\frac{c_1(x-1)-1}{x(x-1)}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} xy &= \int \frac{c_1(x-1)-1}{x(x-1)} dx \\ xy &= -\ln(x-1) + (c_1+1)\ln(x) + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{-\ln(x-1) + (c_1+1)\ln(x)}{x} + \frac{c_2}{x}$$

which simplifies to

$$y = \frac{c_1 \ln(x) + \ln(x) - \ln(x-1) + c_2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(x) + \ln(x) - \ln(x-1) + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(x) + \ln(x) - \ln(x-1) + c_2}{x}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+y(x)=1/(1-x)^2,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \ln(x) - \ln(-1+x) + \ln(x) + c_2}{x}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 27

```
DSolve[x^2*y''[x]+3*x*y'[x]+y[x]==1/(1-x)^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-\log(1-x) + \log(x) + c_2 \log(x) + c_1}{x}$$

28.4 problem Ex 4

Internal problem ID [11275]
Internal file name [`OUTPUT/10260_Wednesday_December_21_2022_03_47_01_PM_86529830/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 51. Cauchy linear equation. Page 114
Problem number: Ex 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1+x)^2 y'' - (1+x)\, y' + 6y = x}$$

28.4.1 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = (1+x)^2, B = -x-1, C = 6, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$(1+x)^2 y'' + (-x-1)\, y' + 6y = 0$$

In normal form the ode

$$(1+x)^2 y'' + (-x-1)\, y' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{1+x} \\ q(x) &= \frac{6}{\left(1+x\right)^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\left(x\right)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}}{c} \\ \tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{(1+x)^2}}\,\left(1+x\right)^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \\ &= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{(1+x)^2}}\,(1+x)^3} - \frac{1}{1+x}\frac{\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}}{c}\right)^2} \\ &= -\frac{c\sqrt{6}}{3} \end{aligned}$$

Therefore ode (3) now becomes

$$y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2} y(\tau) - \frac{c\sqrt{6}\left(\frac{d}{d\tau} y(\tau)\right)}{3} + c^2 y(\tau) = 0 \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{\frac{\sqrt{6}\, c\tau}{6}} \left(c_1 \cos\left(\frac{c\sqrt{30}\,\tau}{6}\right) + c_2 \sin\left(\frac{c\sqrt{30}\,\tau}{6}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{6}\sqrt{\frac{1}{(1+x)^2}} dx}{c} \\
&= \frac{\sqrt{6}\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = (1+x)\left(c_1 \cos\left(\sqrt{5}\,\ln(1+x)\right) + c_2 \sin\left(\sqrt{5}\,\ln(1+x)\right)\right)$$

Now the particular solution to this ODE is found

$$(1+x)^2 y'' + (-x-1)\, y' + 6y = x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)} \cos\left(\frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}{6}\right)$$

$$y_2 = e^{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)} \sin\left(\frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}{6}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\sqrt{\frac{1}{(1+x)^2}}(1+x)\ln(1+x)} \cos\left(\frac{\sqrt{30}\sqrt{6}\sqrt{\frac{1}{(1+x)^2}}(1+x)\ln(1+x)}{6}\right) & e^{\sqrt{\frac{1}{(1+x)^2}}(1+x)\ln(1+x)} \sin\left(\frac{\sqrt{30}\sqrt{6}\sqrt{\frac{1}{(1+x)^2}}\right. \\ \frac{d}{dx}\left(e^{\sqrt{\frac{1}{(1+x)^2}}(1+x)\ln(1+x)} \cos\left(\frac{\sqrt{30}\sqrt{6}\sqrt{\frac{1}{(1+x)^2}}(1+x)\ln(1+x)}{6}\right)\right) & \frac{d}{dx}\left(e^{\sqrt{\frac{1}{(1+x)^2}}(1+x)\ln(1+x)} \sin\left(\frac{\sqrt{30}\sqrt{6}\sqrt{\frac{1}{(1+x)}}\right.\right. \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} & e^{\sqrt{\frac{1}{(1+x)^2}}(1+x)\ln(1+x} \\ \left(-\frac{\ln(1+x)}{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)^2} + \sqrt{\frac{1}{(1+x)^2}}\ln(1+x) + \sqrt{\frac{1}{(1+x)^2}}\right) e^{\sqrt{\frac{1}{(1+x)^2}}(1+x)\ln(1+x)} \cos\left(\frac{\sqrt{30}\sqrt{6}\sqrt{\frac{1}{(1+x)^2}}(1+x)\ln(1+x}{6}\right. & \end{vmatrix}$$

Therefore

$$W = \left(e^{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)} \cos\left(\frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}{6}\right)\right) \left(\left(\left(-\frac{\ln(1+x)}{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)^2} + \sqrt{\frac{1}{(1+x)^2}}\,\ln(1+x) + \sqrt{\frac{1}{(1+x)^2}}\right) e^{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)} \sin\left(\frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}{6}\right) + e^{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)} \left(-\frac{\sqrt{30}\,\sqrt{6}\,\ln(1+x)}{6\sqrt{\frac{1}{(1+x)^2}}\,(1+x)^2} + \frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}\,\ln(1+x)}{6} + \frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}}{6}\right) \cos\left(\frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}{6}\right)\right)\right) - \left(e^{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)} \sin\left(\frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}{6}\right)\right) \left(\left(\left(-\frac{\ln(1+x)}{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)^2} + \sqrt{\frac{1}{(1+x)^2}}\,\ln(1+x) + \sqrt{\frac{1}{(1+x)^2}}\right) e^{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)} \cos\left(\frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}{6}\right) - e^{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)} \left(-\frac{\sqrt{30}\,\sqrt{6}\,\ln(1+x)}{6\sqrt{\frac{1}{(1+x)^2}}\,(1+x)^2} + \frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}\,\ln(1+x)}{6} + \frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}}{6}\right) \sin\left(\frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}{6}\right)\right)\right)$$

Which simplifies to

$$W = \frac{\left(\cos\left(\frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}{6}\right)^2 + \sin\left(\frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}{6}\right)^2\right)\sqrt{6}\,\sqrt{30}\,e^{2\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}}{6\sqrt{\frac{1}{(1+x)^2}}\,(1+x)^2}$$

Which simplifies for $0 < x$ to

$$W = \sqrt{5}\,(1+x)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)} \sin\left(\frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}{6}\right) x}{(1+x)^3\,\sqrt{5}}\, dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int \frac{\sin\left(\sqrt{5}\,\ln(1+x)\right) x\sqrt{5}}{5\,(1+x)^2} dx$$

Hence

$$u_1 = -\frac{(-6x-1)\cos\left(\sqrt{5}\,\ln(1+x)\right) + \sin\left(\sqrt{5}\,\ln(1+x)\right)\sqrt{5} - 6x - 6}{30+30x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)} \cos\left(\frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}{6}\right) x}{(1+x)^3\,\sqrt{5}}\, dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{\cos\left(\sqrt{5}\,\ln(1+x)\right) x\sqrt{5}}{5\,(1+x)^2} dx$$

Hence

$$u_2 = \frac{\cos\left(\sqrt{5}\,\ln(1+x)\right)\sqrt{5} + 6\sin\left(\sqrt{5}\,\ln(1+x)\right) x + \sin\left(\sqrt{5}\,\ln(1+x)\right)}{30+30x}$$

Which simplifies to

$$u_1 = \frac{(6x+1)\cos\left(\sqrt{5}\,\ln(1+x)\right) - \sin\left(\sqrt{5}\,\ln(1+x)\right)\sqrt{5} + 6x + 6}{30+30x}$$

$$u_2 = \frac{\cos\left(\sqrt{5}\,\ln(1+x)\right)\sqrt{5} + 6\sin\left(\sqrt{5}\,\ln(1+x)\right) x + \sin\left(\sqrt{5}\,\ln(1+x)\right)}{30+30x}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) &\\
&= \frac{\left(\left(6x+1\right)\cos\left(\sqrt{5}\,\ln\left(1+x\right)\right)-\sin\left(\sqrt{5}\,\ln\left(1+x\right)\right)\sqrt{5}+6x+6\right)\mathrm{e}^{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}\cos\left(\frac{\sqrt{30}\,\sqrt{6}\,\sqrt{\frac{1}{(1+x)^2}}}{6}\right.}{30+30x} \\
&+ \frac{\left(\cos\left(\sqrt{5}\,\ln\left(1+x\right)\right)\sqrt{5}+6\sin\left(\sqrt{5}\,\ln\left(1+x\right)\right)x+\sin\left(\sqrt{5}\,\ln\left(1+x\right)\right)\right)\mathrm{e}^{\sqrt{\frac{1}{(1+x)^2}}\,(1+x)\ln(1+x)}\sin\left(\frac{\sqrt{30}}{}\right.}{30+30x}
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = \frac{1}{30}+\frac{(1+x)\cos\left(\sqrt{5}\,\ln\left(1+x\right)\right)}{5}+\frac{x}{5}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(1+x\right)\left(c_1\cos\left(\sqrt{5}\,\ln\left(1+x\right)\right)+c_2\sin\left(\sqrt{5}\,\ln\left(1+x\right)\right)\right)\right) \\
&\quad + \left(\frac{1}{30}+\frac{(1+x)\cos\left(\sqrt{5}\,\ln\left(1+x\right)\right)}{5}+\frac{x}{5}\right) \\
&= \frac{1}{30}+\frac{(1+x)\cos\left(\sqrt{5}\,\ln\left(1+x\right)\right)}{5}+\frac{x}{5} \\
&\quad + \left(1+x\right)\left(c_1\cos\left(\sqrt{5}\,\ln\left(1+x\right)\right)+c_2\sin\left(\sqrt{5}\,\ln\left(1+x\right)\right)\right)
\end{aligned}
$$

Which simplifies to

$$
y = (1+x)\left(c_1+\frac{1}{5}\right)\cos\left(\sqrt{5}\,\ln\left(1+x\right)\right)+(1+x)\,c_2\sin\left(\sqrt{5}\,\ln\left(1+x\right)\right)+\frac{x}{5}+\frac{1}{30}
$$

Summary
The solution(s) found are the following

$$
y = (1+x)\left(c_1+\frac{1}{5}\right)\cos\left(\sqrt{5}\,\ln\left(1+x\right)\right)+(1+x)\,c_2\sin\left(\sqrt{5}\,\ln\left(1+x\right)\right)+\frac{x}{5}+\frac{1}{30} \tag{1}
$$

Verification of solutions

$$
y = (1+x)\left(c_1+\frac{1}{5}\right)\cos\left(\sqrt{5}\,\ln\left(1+x\right)\right)+(1+x)\,c_2\sin\left(\sqrt{5}\,\ln\left(1+x\right)\right)+\frac{x}{5}+\frac{1}{30}
$$

Verified OK. `{0 < x}`

28.4.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
(1+x)^2 y'' + (-x-1) y' + 6y &= 0 & (1)\\
Ay'' + By' + Cy &= 0 & (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= (1+x)^2\\
B &= -x-1 \qquad (3)\\
C &= 6
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \qquad (5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-21}{4(1+x)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -21\\
t &= 4(1+x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{21}{4(1+x)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 160: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(1+x)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{21}{4(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i\sqrt{5} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i\sqrt{5}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{21}{4\left(1+x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{21}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i\sqrt{5} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i\sqrt{5}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{21}{4\left(1+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{1}{2} + i\sqrt{5}$	$\frac{1}{2} - i\sqrt{5}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2} + i\sqrt{5}$	$\frac{1}{2} - i\sqrt{5}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2} - i\sqrt{5}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - i\sqrt{5} - \left(\frac{1}{2} - i\sqrt{5}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - i\sqrt{5}}{1 + x} + (-)\,(0) \\
&= \frac{\frac{1}{2} - i\sqrt{5}}{1 + x} \\
&= -\frac{2i\sqrt{5} - 1}{2\,(1 + x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - i\sqrt{5}}{1 + x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - i\sqrt{5}}{(1 + x)^2} \right) + \left(\frac{\frac{1}{2} - i\sqrt{5}}{1 + x} \right)^2 - \left(-\frac{21}{4\,(1 + x)^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{\frac{1}{2} - i\sqrt{5}}{1 + x} dx} \\
&= (1 + x)^{\frac{1}{2} - i\sqrt{5}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x-1}{(1+x)^2}\,dx} \\
&= z_1 e^{\frac{\ln(1+x)}{2}} \\
&= z_1\left(\sqrt{1+x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (1+x)^{1-i\sqrt{5}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x-1}{(1+x)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{i\sqrt{5}\,(1+x)^{2i\sqrt{5}}}{10}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((1+x)^{1-i\sqrt{5}}\right) + c_2\left((1+x)^{1-i\sqrt{5}}\left(-\frac{i\sqrt{5}\,(1+x)^{2i\sqrt{5}}}{10}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$(1+x)^2 y''+(-x-1)\,y'+6y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1(1+x)^{1-i\sqrt{5}} - \frac{ic_2\sqrt{5}\,(1+x)^{i\sqrt{5}+1}}{10}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (1+x)^{1-i\sqrt{5}}$$

$$y_2 = -\frac{i\sqrt{5}\,(1+x)^{i\sqrt{5}+1}}{10}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (1+x)^{1-i\sqrt{5}} & -\frac{i\sqrt{5}\,(1+x)^{i\sqrt{5}+1}}{10} \\ \frac{d}{dx}\left((1+x)^{1-i\sqrt{5}}\right) & \frac{d}{dx}\left(-\frac{i\sqrt{5}\,(1+x)^{i\sqrt{5}+1}}{10}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (1+x)^{1-i\sqrt{5}} & -\frac{i\sqrt{5}\,(1+x)^{i\sqrt{5}+1}}{10} \\ \frac{(1+x)^{1-i\sqrt{5}}\left(1-i\sqrt{5}\right)}{1+x} & -\frac{i\sqrt{5}\,(1+x)^{i\sqrt{5}+1}\left(i\sqrt{5}+1\right)}{10(1+x)} \end{vmatrix}$$

Therefore

$$W = \left((1+x)^{1-i\sqrt{5}}\right)\left(-\frac{i\sqrt{5}\,(1+x)^{i\sqrt{5}+1}\,(i\sqrt{5}+1)}{10\,(1+x)}\right) - \left(-\frac{i\sqrt{5}\,(1+x)^{i\sqrt{5}+1}}{10}\right)\left(\frac{(1+x)^{1-i\sqrt{5}}\,(1-i\sqrt{5})}{1+x}\right)$$

Which simplifies to

$$W = \frac{(1+x)^{1-i\sqrt{5}}\,(1+x)^{i\sqrt{5}+1}}{1+x}$$

Which simplifies to

$$W = 1+x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{i\sqrt{5}\,(1+x)^{i\sqrt{5}+1}x}{10}}{(1+x)^3}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{ix\sqrt{5}\,(1+x)^{-2+i\sqrt{5}}}{10}dx$$

Hence

$$u_1 = -\frac{\left(1-i(2x+1)\sqrt{5}-4x\right)\left(\frac{1+x}{x}\right)^{i\sqrt{5}}x^{i\sqrt{5}} + \left(i\sqrt{5}-1\right)(1+x)}{10\,(1+x)\left(\sqrt{5}+i\right)^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(1+x)^{1-i\sqrt{5}}\,x}{(1+x)^3}\,dx$$

Which simplifies to

$$u_2 = \int x(1+x)^{-2-i\sqrt{5}}\,dx$$

Hence

$$u_2 = -\frac{\left(x^{-i\sqrt{5}}\left(x+i\sqrt{5}\,x+1\right)\left(\frac{1+x}{x}\right)^{-i\sqrt{5}} - x - 1\right)\sqrt{5}}{5\,(1+x)\left(-\sqrt{5}+i\right)}$$

Which simplifies to

$$u_1 = \frac{(1+x)^{i\sqrt{5}}\left(-1+i(2x+1)\sqrt{5}+4x\right)+\left(1-i\sqrt{5}\right)(1+x)}{10\,(1+x)\left(\sqrt{5}+i\right)^2}$$

$$u_2 = -\frac{\sqrt{5}\left((1+x)^{-i\sqrt{5}}\left(x+i\sqrt{5}\,x+1\right)-x-1\right)}{5\,(1+x)\left(-\sqrt{5}+i\right)}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left((1+x)^{i\sqrt{5}}\left(-1+i(2x+1)\sqrt{5}+4x\right)+\left(1-i\sqrt{5}\right)(1+x)\right)(1+x)^{1-i\sqrt{5}}}{10\,(1+x)\left(\sqrt{5}+i\right)^2} + \frac{i\left((1+x)^{-i\sqrt{5}}\left(x+i\sqrt{5}\,x+1\right)-x-1\right)(1+x)^{i\sqrt{5}+1}}{10\,(1+x)\left(-\sqrt{5}+i\right)}$$

Which simplifies to

$$y_p(x) = \frac{\left(\sqrt{5}-2i\right)(1+x)^{i\sqrt{5}+1}+3i(1+x)^{1-i\sqrt{5}}+(-6x-1)\sqrt{5}-6ix-i}{5\left(-\sqrt{5}+i\right)\left(\sqrt{5}+i\right)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1(1+x)^{1-i\sqrt{5}} - \frac{ic_2\sqrt{5}\,(1+x)^{i\sqrt{5}+1}}{10}\right) \\ &\quad + \left(\frac{\left(\sqrt{5}-2i\right)(1+x)^{i\sqrt{5}+1}+3i(1+x)^{1-i\sqrt{5}}+(-6x-1)\sqrt{5}-6ix-i}{5\left(-\sqrt{5}+i\right)\left(\sqrt{5}+i\right)^2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = c_1(1+x)^{1-i\sqrt{5}} - \frac{i c_2 \sqrt{5}\,(1+x)^{i\sqrt{5}+1}}{10} \\
+ \frac{\left(\sqrt{5}-2i\right)(1+x)^{i\sqrt{5}+1} + 3i(1+x)^{1-i\sqrt{5}} + (-6x-1)\sqrt{5} - 6ix - i}{5\left(-\sqrt{5}+i\right)\left(\sqrt{5}+i\right)^{2}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1(1+x)^{1-i\sqrt{5}} - \frac{i c_2 \sqrt{5}\,(1+x)^{i\sqrt{5}+1}}{10} \\
+ \frac{\left(\sqrt{5}-2i\right)(1+x)^{i\sqrt{5}+1} + 3i(1+x)^{1-i\sqrt{5}} + (-6x-1)\sqrt{5} - 6ix - i}{5\left(-\sqrt{5}+i\right)\left(\sqrt{5}+i\right)^{2}}
\end{aligned}
$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 37

```
dsolve((x+1)^2*diff(y(x),x$2)-(x+1)*diff(y(x),x)+6*y(x)=x,y(x), singsol=all)
```

$$
y(x) = (1+x)\sin\left(\sqrt{5}\,\ln(1+x)\right) c_2 + (1+x)\cos\left(\sqrt{5}\,\ln(1+x)\right) c_1 + \frac{x}{5} + \frac{1}{30}
$$

✓ Solution by Mathematica

Time used: 0.508 (sec). Leaf size: 49

```
DSolve[(x+1)^2*y''[x]-(x+1)*y'[x]+6*y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{30}(6x+1) + c_2(x+1)\cos\left(\sqrt{5}\log(x+1)\right) + c_1(x+1)\sin\left(\sqrt{5}\log(x+1)\right)$$

29 Chapter VII, Linear differential equations with constant coefficients. Article 52. Summary. Page 117

29.1 problem Ex 1

Internal problem ID [11276]
Internal file name [`OUTPUT/10261_Wednesday_December_21_2022_03_47_04_PM_69135302/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 52. Summary. Page 117
Problem number: Ex 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 5y' + 6y = \cos(x) - e^{2x}}$$

29.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -5, C = 6, f(x) = \cos(x) - e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 5\lambda e^{\lambda x} + 6 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-5^2 - (4)(1)(6)} \\ &= \frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{5}{2} + \frac{1}{2} \\ \lambda_2 &= \frac{5}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3 \\ \lambda_2 &= 2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(3)x} + c_2 e^{(2)x}\end{aligned}$$

Or

$$y = c_1 e^{3x} + c_2 e^{2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{3x} + c_2 e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x) - e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}, e^{3x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x}x\}, \{\cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{2x} x + A_2 \cos(x) + A_3 \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{2x} + 5A_2 \cos(x) + 5A_3 \sin(x) + 5A_2 \sin(x) - 5A_3 \cos(x) = \cos(x) - e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = \frac{1}{10}, A_3 = -\frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{2x} x + \frac{\cos(x)}{10} - \frac{\sin(x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + c_2 e^{2x}\right) + \left(e^{2x} x + \frac{\cos(x)}{10} - \frac{\sin(x)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{3x} + c_2 e^{2x} + e^{2x} x + \frac{\cos(x)}{10} - \frac{\sin(x)}{10} \tag{1}$$

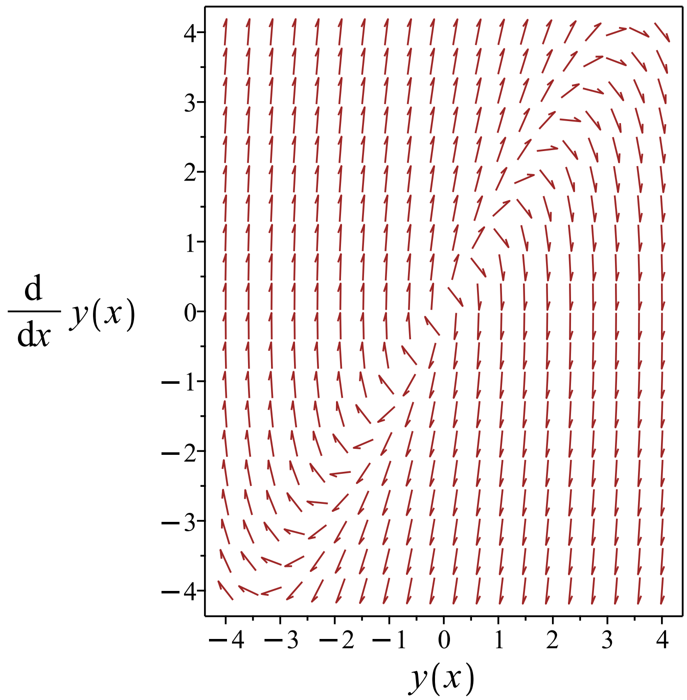

Figure 231: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{2x} + e^{2x} x + \frac{\cos(x)}{10} - \frac{\sin(x)}{10}$$

Verified OK.

29.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 6y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -5 \\
C &= 6
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 161: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dx} \\ &= z_1 e^{\frac{5x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{5x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{5x}}{(y_1)^2}\,dx \\ &= y_1(e^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{2x}\right) + c_2\left(e^{2x}(e^x)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{2x} + e^{3x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x) - e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}, e^{3x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x}x\}, \{\cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{2x} x + A_2 \cos(x) + A_3 \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{2x} + 5A_2 \cos(x) + 5A_3 \sin(x) + 5A_2 \sin(x) - 5A_3 \cos(x) = \cos(x) - e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = \frac{1}{10}, A_3 = -\frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{2x} x + \frac{\cos(x)}{10} - \frac{\sin(x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + e^{3x} c_2\right) + \left(e^{2x} x + \frac{\cos(x)}{10} - \frac{\sin(x)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + e^{3x} c_2 + e^{2x} x + \frac{\cos(x)}{10} - \frac{\sin(x)}{10} \tag{1}$$

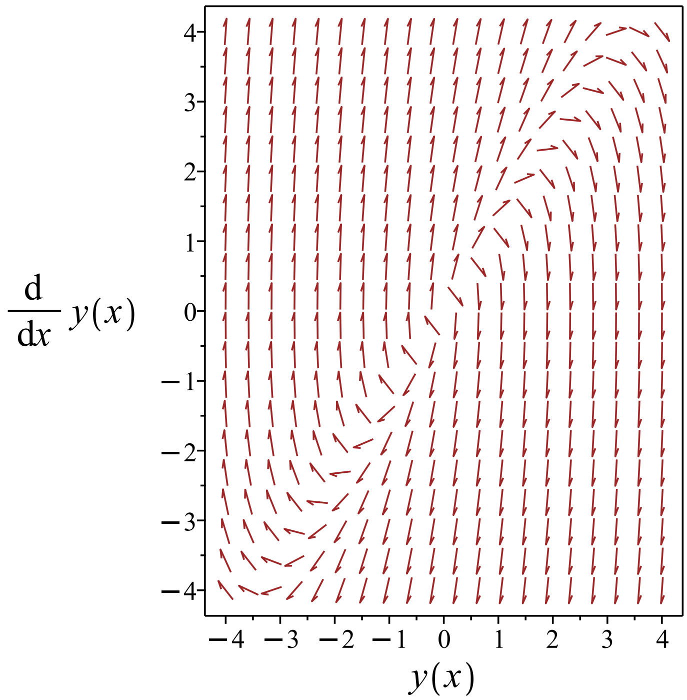

Figure 232: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + e^{3x} c_2 + e^{2x} x + \frac{\cos(x)}{10} - \frac{\sin(x)}{10}$$

Verified OK.

29.1.3 Maple step by step solution

Let's solve

$y'' - 5y' + 6y = \cos(x) - e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

$r = (2, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{2x} + e^{3x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \cos(x) - e^{2x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & e^{3x} \\ 2\,e^{2x} & 3\,e^{3x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{5x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{2x}\left(\int (\cos(x)\,e^{-2x} - 1)\,dx\right) + e^{3x}\left(\int (\cos(x) - e^{2x})\,e^{-3x}dx\right)$

  ◦ Compute integrals

  $y_p(x) = e^{2x}x + \frac{\cos(x)}{10} - \frac{\sin(x)}{10} + e^{2x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{2x} + e^{3x} c_2 + e^{2x}x + \frac{\cos(x)}{10} - \frac{\sin(x)}{10} + e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-5*diff(y(x),x)+6*y(x)=cos(x)-exp(2*x),y(x), singsol=all)
```

$$y(x) = (1 + x + c_2)\,\mathrm{e}^{2x} + c_1\mathrm{e}^{3x} + \frac{\cos(x)}{10} - \frac{\sin(x)}{10}$$

✓ Solution by Mathematica

Time used: 0.345 (sec). Leaf size: 34

```
DSolve[y''[x]-5*y'[x]+6*y[x]==Cos[x]-Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{10}\left(-\sin(x) + \cos(x) + 10e^{2x}(x + c_2e^x + 1 + c_1)\right)$$

29.2 problem Ex 2

Internal problem ID [11277]
Internal file name [OUTPUT/10262_Wednesday_December_21_2022_03_47_05_PM_14647184/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 52. Summary. Page 117
Problem number: Ex 2.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - y = \mathrm{e}^{x} \cos(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - y = 0$$

The characteristic equation is

$$\lambda^4 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + e^{ix} c_3 + e^{-ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{ix}$$
$$y_4 = e^{-ix}$$

Now the particular solution to the given ODE is found

$$y'''' - y = e^{x} \cos(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{x} \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x} \cos(x), e^{x} \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{ix}, e^{-x}, e^{-ix}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x} \cos(x) + A_2 e^{x} \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 e^{x} \cos(x) - 5A_2 e^{x} \sin(x) = e^{x} \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^x \cos(x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^x + e^{ix} c_3 + e^{-ix} c_4\right) + \left(-\frac{e^x \cos(x)}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^x + e^{ix} c_3 + e^{-ix} c_4 - \frac{e^x \cos(x)}{5} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^x + e^{ix} c_3 + e^{-ix} c_4 - \frac{e^x \cos(x)}{5}$$

Verified OK.

29.2.1 Maple step by step solution

Let's solve

$y'''' - y = e^x \cos(x)$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Define new variable $y_4(x)$

$y_4(x) = y'''$

- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = e^x \cos(x) + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = e^x \cos(x) + y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^x \cos(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^x \cos(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \left[\begin{array}{c} -I \\ -1 \\ I \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \left[\begin{array}{c} -I \\ -1 \\ I \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - I\sin(x)) \cdot \left[\begin{array}{c} -I \\ -1 \\ I \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -I(\cos(x) - I\sin(x)) \\ -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \left[\begin{array}{c} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{array}\right], \vec{y}_4(x) = \left[\begin{array}{c} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$$

☐ Fundamental matrix

  ◦ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -e^{-x} & e^{x} & -\sin(x) & -\cos(x) \\ e^{-x} & e^{x} & -\cos(x) & \sin(x) \\ -e^{-x} & e^{x} & \sin(x) & \cos(x) \\ e^{-x} & e^{x} & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t
  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -e^{-x} & e^{x} & -\sin(x) & -\cos(x) \\ e^{-x} & e^{x} & -\cos(x) & \sin(x) \\ -e^{-x} & e^{x} & \sin(x) & \cos(x) \\ e^{-x} & e^{x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} \\ \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$
  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution
  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs
  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der
  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{e^{-x}}{10} + \frac{(-2\,e^x+1)\cos(x)}{10} + \frac{3\sin(x)}{10} \\ -\frac{e^{-x}}{10} + \frac{(-2\,e^x+3)\cos(x)}{10} + \frac{e^x\sin(x)}{5} - \frac{\sin(x)}{10} \\ \frac{e^{-x}}{10} + \frac{(4\,e^x-3)\sin(x)}{10} - \frac{\cos(x)}{10} \\ -\frac{e^{-x}}{10} + \frac{(4\,e^x-3)\cos(x)}{10} + \frac{2\,e^x\sin(x)}{5} + \frac{\sin(x)}{10} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \begin{bmatrix} \frac{e^{-x}}{10} + \frac{(-2\,e^x+1)\cos(x)}{10} + \frac{3\sin(x)}{10} \\ -\frac{e^{-x}}{10} + \frac{(-2\,e^x+3)\cos(x)}{10} + \frac{e^x\sin(x)}{5} - \frac{\sin(x)}{10} \\ \frac{e^{-x}}{10} + \frac{(4\,e^x-3)\sin(x)}{10} - \frac{\cos(x)}{10} \\ -\frac{e^{-x}}{10} + \frac{(4\,e^x-3)\cos(x)}{10} + \frac{2\,e^x\sin(x)}{5} + \frac{\sin(x)}{10} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = \frac{(-10c_1+1)e^{-x}}{10} + \frac{(-10c_4-2\,e^x+1)\cos(x)}{10} + c_2 e^x + \frac{(-10c_3+3)\sin(x)}{10}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$4)-y(x)=exp(x)*cos(x),y(x), singsol=all)
```

$$y(x) = c_4 e^{-x} + \frac{(5c_1 - e^x)\cos(x)}{5} + c_2 e^x + c_3 \sin(x)$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 38

```
DSolve[y''''[x]-y[x]==Exp[x]*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x + c_3 e^{-x} + \left(-\frac{e^x}{5} + c_2\right)\cos(x) + c_4 \sin(x)$$

29.3 problem Ex 3

Internal problem ID [11278]
Internal file name [`OUTPUT/10263_Wednesday_December_21_2022_03_47_06_PM_78866000/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 52. Summary. Page 117
Problem number: Ex 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 2y' + y = 2x^3 - x\,\mathrm{e}^{3x}}$$

29.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = 2x^3 - x\,\mathrm{e}^{3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-x} + c_2 x e^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + x e^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2x^3 - x e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^{3x}, e^{3x}\}, \{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^{-x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{3x} + A_2 e^{3x} + A_3 + A_4 x + A_5 x^2 + A_6 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&8A_1 e^{3x} + 16A_1 x\,e^{3x} + 16A_2 e^{3x} + 2A_5 + 6A_6 x + 2A_4 + 4A_5 x \\ &+ 6A_6 x^2 + A_3 + A_4 x + A_5 x^2 + A_6 x^3 = 2x^3 - x\,e^{3x}\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{16}, A_2 = \frac{1}{32}, A_3 = -48, A_4 = 36, A_5 = -12, A_6 = 2\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^{3x}}{16} + \frac{e^{3x}}{32} - 48 + 36x - 12x^2 + 2x^3$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(c_1 e^{-x} + x\,e^{-x} c_2\right) + \left(-\frac{x\,e^{3x}}{16} + \frac{e^{3x}}{32} - 48 + 36x - 12x^2 + 2x^3\right)\end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) - \frac{x\,e^{3x}}{16} + \frac{e^{3x}}{32} - 48 + 36x - 12x^2 + 2x^3$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) - \frac{x\,e^{3x}}{16} + \frac{e^{3x}}{32} - 48 + 36x - 12x^2 + 2x^3 \tag{1}$$

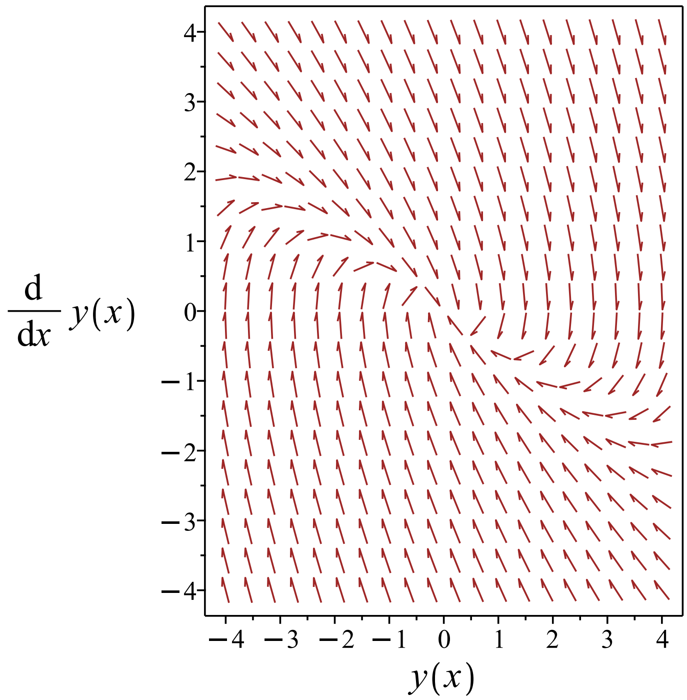

Figure 233: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) - \frac{x\, e^{3x}}{16} + \frac{e^{3x}}{32} - 48 + 36x - 12x^2 + 2x^3$$

Verified OK.

29.3.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2} \int p\, dx} \\ &= e^{\int 2\, dx} \\ &= e^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}(M(x)y)'' &= e^x(2x^3 - x\,e^{3x})\\ (e^x y)'' &= e^x(2x^3 - x\,e^{3x})\end{aligned}$$

Integrating once gives

$$(e^x y)' = \frac{(-4x+1)\,e^{4x}}{16} + 2\,e^x(x^3 - 3x^2 + 6x - 6) + c_1$$

Integrating again gives

$$(e^x y) = \frac{(-2x+1)\,e^{4x}}{32} + 2(x^3 - 6x^2 + 18x - 24)\,e^x + c_1 x + c_2$$

Hence the solution is

$$y = \frac{\frac{(-2x+1)e^{4x}}{32} + 2(x^3 - 6x^2 + 18x - 24)\,e^x + c_1 x + c_2}{e^x}$$

Or

$$y = -\frac{x\,e^{3x}}{16} + 2x^3 + \frac{e^{3x}}{32} - 12x^2 + e^{-x}c_1 x + 36x + c_2 e^{-x} - 48$$

Summary
The solution(s) found are the following

$$y = -\frac{x\,e^{3x}}{16} + 2x^3 + \frac{e^{3x}}{32} - 12x^2 + e^{-x}c_1 x + 36x + c_2 e^{-x} - 48 \tag{1}$$

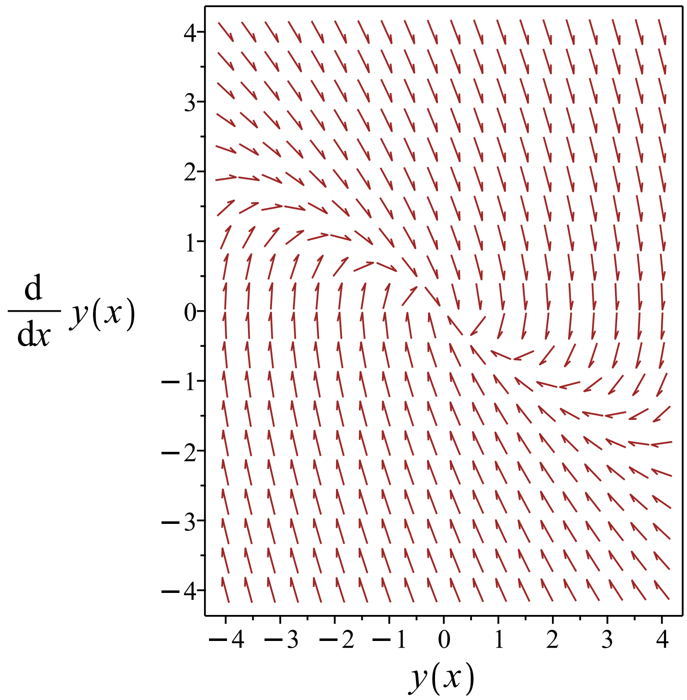

Figure 234: Slope field plot

Verification of solutions

$$y = -\frac{x\,e^{3x}}{16} + 2x^3 + \frac{e^{3x}}{32} - 12x^2 + e^{-x}c_1x + 36x + c_2e^{-x} - 48$$

Verified OK.

29.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 164: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(\mathrm{e}^{-x}) + c_2(\mathrm{e}^{-x}(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\mathrm{e}^{-x} + x\,\mathrm{e}^{-x}c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2x^3 - x\,\mathrm{e}^{3x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\,\mathrm{e}^{3x}, \mathrm{e}^{3x}\}, \{1, x, x^2, x^3\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1x\,\mathrm{e}^{3x} + A_2\mathrm{e}^{3x} + A_3 + A_4x + A_5x^2 + A_6x^3
$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&8A_1\mathrm{e}^{3x} + 16A_1x\,\mathrm{e}^{3x} + 16A_2\mathrm{e}^{3x} + 2A_5 + 6A_6x + 2A_4 + 4A_5x \\
&+ 6A_6x^2 + A_3 + A_4x + A_5x^2 + A_6x^3 = 2x^3 - x\,\mathrm{e}^{3x}
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{16}, A_2 = \frac{1}{32}, A_3 = -48, A_4 = 36, A_5 = -12, A_6 = 2\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,\mathrm{e}^{3x}}{16} + \frac{\mathrm{e}^{3x}}{32} - 48 + 36x - 12x^2 + 2x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + x\,\mathrm{e}^{-x}c_2\right) + \left(-\frac{x\,\mathrm{e}^{3x}}{16} + \frac{\mathrm{e}^{3x}}{32} - 48 + 36x - 12x^2 + 2x^3\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2x + c_1) - \frac{x\,\mathrm{e}^{3x}}{16} + \frac{\mathrm{e}^{3x}}{32} - 48 + 36x - 12x^2 + 2x^3$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2x + c_1) - \frac{x\,\mathrm{e}^{3x}}{16} + \frac{\mathrm{e}^{3x}}{32} - 48 + 36x - 12x^2 + 2x^3 \tag{1}$$

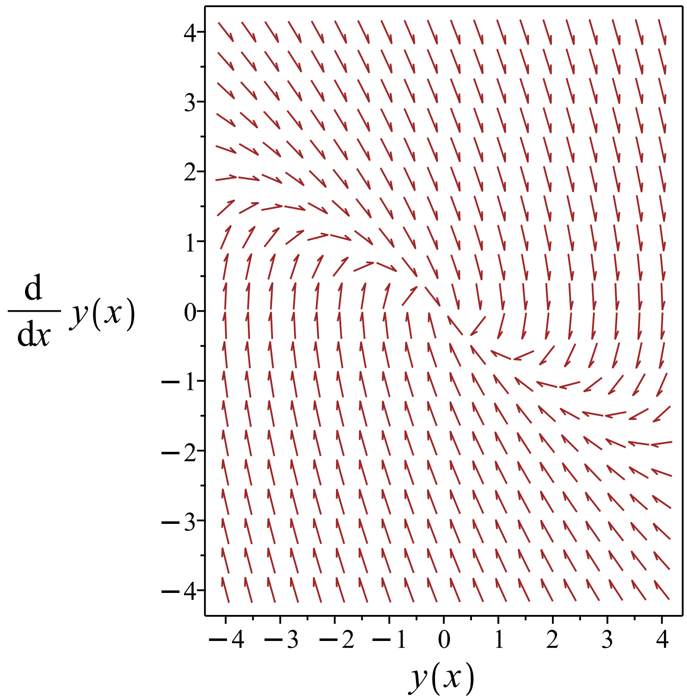

Figure 235: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) - \frac{x\, e^{3x}}{16} + \frac{e^{3x}}{32} - 48 + 36x - 12x^2 + 2x^3$$

Verified OK.

29.3.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = 2x^3 - x\, e^{3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2x^3 - x\,\mathrm{e}^{3x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \mathrm{e}^{-x}\left(\int x^2 \mathrm{e}^{x}(-2x^2 + \mathrm{e}^{3x})\,dx - \left(\int \mathrm{e}^{x} x(-2x^2 + \mathrm{e}^{3x})\,dx\right) x\right)$

  - Compute integrals

    $y_p(x) = \frac{(-2x+1)\mathrm{e}^{3x}}{32} + 2x^3 - 12x^2 + 36x - 48$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \frac{(-2x+1)\mathrm{e}^{3x}}{32} + 2x^3 - 12x^2 + 36x - 48$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=2*x^3-x*exp(3*x),y(x), singsol=all)
```

$$y(x) = -48 + (c_1 x + c_2)\,\mathrm{e}^{-x} + \frac{(1-2x)\,\mathrm{e}^{3x}}{32} + 2x^3 - 12x^2 + 36x$$

✓ Solution by Mathematica

Time used: 0.335 (sec). Leaf size: 48

```
DSolve[y''[x]+2*y'[x]+y[x]==2*x^3-x*Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\left(x^3 - 6x^2 + 18x - 24\right) + \frac{1}{32}e^{3x}(1-2x) + e^{-x}(c_2 x + c_1)$$

29.4 problem Ex 5

Internal problem ID [11279]
Internal file name [`OUTPUT/10264_Wednesday_December_21_2022_03_47_08_PM_10831701/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 52. Summary. Page 117
Problem number: Ex 5.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 4y' = x^2 - 3\,e^{2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y' = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{-2x} + e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{-2x}$$
$$y_3 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y''' - 4y' = x^2 - 3\,e^{2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 - 3\,e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-2x}, e^{2x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x}\}, \{x, x^2, x^3\}]$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x} x\}, \{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{2x} x + A_2 x + A_3 x^2 + A_4 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 e^{2x} + 6A_4 - 4A_2 - 8A_3 x - 12A_4 x^2 = x^2 - 3\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{8}, A_2 = -\frac{1}{8}, A_3 = 0, A_4 = -\frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\,e^{2x} x}{8} - \frac{x}{8} - \frac{x^3}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 e^{-2x} + e^{2x} c_3\right) + \left(-\frac{3\,e^{2x} x}{8} - \frac{x}{8} - \frac{x^3}{12}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-2x} + e^{2x} c_3 - \frac{3\,e^{2x} x}{8} - \frac{x}{8} - \frac{x^3}{12} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^{-2x} + e^{2x} c_3 - \frac{3\,e^{2x} x}{8} - \frac{x}{8} - \frac{x^3}{12}$$

Verified OK.

29.4.1 Maple step by step solution

Let's solve

$y''' - 4y' = x^2 - 3\,e^{2x}$

- Highest derivative means the order of the ODE is 3

  y'''

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = x^2 - 3\,\mathrm{e}^{2x} + 4y_2(x)$

    Convert linear ODE into a system of first order ODEs

    $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = x^2 - 3\,\mathrm{e}^{2x} + 4y_2(x)]$
- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$
- System to solve

  $$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x^2 - 3\,\mathrm{e}^{2x} \end{bmatrix}$$
- Define the forcing function

  $$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x^2 - 3\,\mathrm{e}^{2x} \end{bmatrix}$$
- Define the coefficient matrix

  $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$
- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right], \left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

  $\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3 + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & 1 & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & 0 & \frac{e^{2x}}{2} \\ e^{-2x} & 0 & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & 1 & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & 0 & \frac{e^{2x}}{2} \\ e^{-2x} & 0 & e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{e^{-2x}}{4} + \frac{e^{2x}}{4} & \frac{e^{-2x}}{8} - \frac{1}{4} + \frac{e^{2x}}{8} \\ 0 & \frac{e^{-2x}}{2} + \frac{e^{2x}}{2} & -\frac{e^{-2x}}{4} + \frac{e^{2x}}{4} \\ 0 & -e^{-2x} + e^{2x} & \frac{e^{-2x}}{2} + \frac{e^{2x}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A\cdot\Phi(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$$

- Cancel like terms

$$\Phi(x)\cdot\vec{v}'(x)=\vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot\vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x)=\left[\begin{array}{c} -\frac{3}{8}+\frac{(5-6x)\mathrm{e}^{2x}}{16}-\frac{x^3}{12}-\frac{x}{8}+\frac{\mathrm{e}^{-2x}}{16} \\ \frac{(-6x+2)\mathrm{e}^{2x}}{8}-\frac{x^2}{4}-\frac{\mathrm{e}^{-2x}}{8}-\frac{1}{8} \\ \frac{(-6x-1)\mathrm{e}^{2x}}{4}-\frac{x}{2}+\frac{\mathrm{e}^{-2x}}{4} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3+\left[\begin{array}{c} -\frac{3}{8}+\frac{(5-6x)\mathrm{e}^{2x}}{16}-\frac{x^3}{12}-\frac{x}{8}+\frac{\mathrm{e}^{-2x}}{16} \\ \frac{(-6x+2)\mathrm{e}^{2x}}{8}-\frac{x^2}{4}-\frac{\mathrm{e}^{-2x}}{8}-\frac{1}{8} \\ \frac{(-6x-1)\mathrm{e}^{2x}}{4}-\frac{x}{2}+\frac{\mathrm{e}^{-2x}}{4} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=-\frac{3}{8}+\frac{(5-6x+4c_3)\mathrm{e}^{2x}}{16}+\frac{(1+4c_1)\mathrm{e}^{-2x}}{16}-\frac{x^3}{12}-\frac{x}{8}+c_2$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _a^2+4*_b(_a)-3*exp(2*_a), _b
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x)=x^2-3*exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(9 - 12x + 16c_1)\,\mathrm{e}^{2x}}{32} - \frac{x^3}{12} - \frac{\mathrm{e}^{-2x}c_2}{2} - \frac{x}{8} + c_3$$

✓ Solution by Mathematica

Time used: 0.311 (sec). Leaf size: 49

```
DSolve[y'''[x]-4*y'[x]==x^2-3*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^3}{12} - \frac{x}{8} + \frac{1}{32}e^{2x}(-12x + 9 + 16c_1) - \frac{1}{2}c_2e^{-2x} + c_3$$

29.5 problem Ex 6

Internal problem ID [11280]
Internal file name [OUTPUT/10265_Wednesday_December_21_2022_03_47_09_PM_57161094/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 52. Summary. Page 117
Problem number: Ex 6.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 2y'' + y = \cos(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 2y'' + y = 0$$

The characteristic equation is

$$\lambda^4 - 2\lambda^2 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + x\,e^{-x} c_2 + c_3 e^{x} + x\,e^{x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= x\,e^{-x} \\ y_3 &= e^{x} \\ y_4 &= x\,e^{x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 2y'' + y = \cos(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{x}, x\,e^{-x}, e^{x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 \cos(x) + 4A_2 \sin(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + x\,e^{-x} c_2 + c_3 e^{x} + x\,e^{x} c_4\right) + \left(\frac{\cos(x)}{4}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) + e^{x}(c_4 x + c_3) + \frac{\cos(x)}{4}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + e^{x}(c_4 x + c_3) + \frac{\cos(x)}{4} \tag{1}$$

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + e^{x}(c_4 x + c_3) + \frac{\cos(x)}{4}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$4)-2*diff(y(x),x$2)+y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = (c_4x + c_2)\,\mathrm{e}^{-x} + (c_3x + c_1)\,\mathrm{e}^{x} + \frac{\cos(x)}{4}$$

✓ Solution by Mathematica

Time used: 0.131 (sec). Leaf size: 42

```
DSolve[y''''[x]-2*y''[x]+y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\cos(x)}{4} + e^{-x}\left(c_2x + c_3e^{2x} + c_4e^{2x}x + c_1\right)$$

29.6 problem Ex 7

Internal problem ID [11281]
Internal file name [`OUTPUT/10266_Wednesday_December_21_2022_03_47_10_PM_87781503/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 52. Summary. Page 117
Problem number: Ex 7.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^4 y'''' + 6x^3 y''' + 9x^2 y'' + 3y'x + y = (\ln(x) + 1)^2}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^4 y'''' + 6x^3 y''' + 9x^2 y'' + 3y'x + y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\
y'''' &= \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4}
\end{aligned}$$

Substituting these back into

$$x^4 y'''' + 6x^3 y''' + 9x^2 y'' + 3y'x + y = (\ln(x) + 1)^2$$

gives

$$3x\lambda\,x^{\lambda-1}+9x^2\lambda(\lambda-1)\,x^{\lambda-2}+6x^3\lambda(\lambda-1)\,(\lambda-2)\,x^{\lambda-3}$$
$$+x^4\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\,x^{\lambda-4}+x^{\lambda}=0$$

Which simplifies to

$$3\lambda\,x^{\lambda}+9\lambda(\lambda-1)\,x^{\lambda}+6\lambda(\lambda-1)\,(\lambda-2)\,x^{\lambda}+\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\,x^{\lambda}+x^{\lambda}=0$$

And since $x^{\lambda}\neq 0$ then dividing through by x^{λ}, the above becomes

$$3\lambda+9\lambda(\lambda-1)+6\lambda(\lambda-1)\,(\lambda-2)+\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)+1=0$$

Simplifying gives the characteristic equation as

$$\left(\lambda^2+1\right)^2=0$$

Solving the above gives the following roots

$$\lambda_1=i$$
$$\lambda_2=-i$$
$$\lambda_3=i$$
$$\lambda_4=-i$$

This table summarises the result

root	multiplicity	type of root
$\pm 1i$	2	complex conjugate root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^{λ} basis solution. Each real root of multiplicty two, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ and $c_3x^{\lambda}\ln(x)^2$ basis solutions, and so on. Each complex root $\alpha\pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity two generates $\ln(x)\,x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity three generates $\ln(x)^2\,x^{\alpha}(c_1\cos(\beta\ln(x))+c_2$ basis solutions. And so on. Using the above show that the solution is

$$y=c_1\cos(\ln(x))+c_2\sin(\ln(x))+\ln(x)\,(c_3\cos(\ln(x))+c_4\sin(\ln(x)))$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \cos(\ln(x)) \\
y_2 &= \sin(\ln(x)) \\
y_3 &= \ln(x)\cos(\ln(x)) \\
y_4 &= \ln(x)\sin(\ln(x))
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
x^4 y'''' + 6x^3 y''' + 9x^2 y'' + 3y'x + y = (\ln(x) + 1)^2
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
W = \begin{bmatrix} \cos(\ln(x)) & \sin(\ln(x)) & \ln(x)\cos(\ln(x)) & \ln(x)\sin(\ln(\cdots \\ -\frac{\sin(\ln(x))}{x} & \frac{\cos(\ln(x))}{x} & \frac{\cos(\ln(x)) - \ln(x)\sin(\ln(x))}{x} & \frac{\sin(\ln(x)) + \ln(x)\cos(\cdots}{x} \\ \frac{-\cos(\ln(x)) + \sin(\ln(x))}{x^2} & \frac{-\sin(\ln(x)) - \cos(\ln(x))}{x^2} & \frac{(-\ln(x) - 1)\cos(\ln(x)) + \sin(\ln(x))(\ln(x) - 2)}{x^2} & \frac{(-\ln(x) + 2)\cos(\ln(x)) - \sin(\ln\cdots}{x^2} \\ \frac{3\cos(\ln(x)) - \sin(\ln(x))}{x^3} & \frac{3\sin(\ln(x)) + \cos(\ln(x))}{x^3} & \frac{(3\ln(x) - 1)\cos(\ln(x)) - \sin(\ln(x))(\ln(x) - 6)}{x^3} & \frac{(\ln(x) - 6)\cos(\ln(x)) + \sin(\ln(\cdots}{x^3} \end{bmatrix}
$$

$$
|W| = \frac{4\sin(\ln(x))^4 + 8\sin(\ln(x))^2\cos(\ln(x))^2 + 4\cos(\ln(x))^4}{x^6}
$$

The determinant simplifies to

$$|W| = \frac{4}{x^6}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} \sin(\ln(x)) & \ln(x)\cos(\ln(x)) & \ln(x)\sin(\ln(x)) \\ \frac{\cos(\ln(x))}{x} & \frac{\cos(\ln(x))-\ln(x)\sin(\ln(x))}{x} & \frac{\sin(\ln(x))+\ln(x)\cos(\ln(x))}{x} \\ \frac{-\sin(\ln(x))-\cos(\ln(x))}{x^2} & \frac{(-\ln(x)-1)\cos(\ln(x))+\sin(\ln(x))(\ln(x)-2)}{x^2} & \frac{(-\ln(x)+2)\cos(\ln(x))-\sin(\ln(x))(\ln(x)+1)}{x^2} \end{bmatrix}$$
$$= \frac{-2\ln(x)\cos(\ln(x)) + 2\sin(\ln(x))}{x^3}$$

$$W_2(x) = \det \begin{bmatrix} \cos(\ln(x)) & \ln(x)\cos(\ln(x)) & \ln(x)\sin(\ln(x)) \\ -\frac{\sin(\ln(x))}{x} & \frac{\cos(\ln(x))-\ln(x)\sin(\ln(x))}{x} & \frac{\sin(\ln(x))+\ln(x)\cos(\ln(x))}{x} \\ \frac{-\cos(\ln(x))+\sin(\ln(x))}{x^2} & \frac{(-\ln(x)-1)\cos(\ln(x))+\sin(\ln(x))(\ln(x)-2)}{x^2} & \frac{(-\ln(x)+2)\cos(\ln(x))-\sin(\ln(x))(\ln(x)+1)}{x^2} \end{bmatrix}$$
$$= \frac{2\ln(x)\sin(\ln(x)) + 2\cos(\ln(x))}{x^3}$$

$$W_3(x) = \det \begin{bmatrix} \cos(\ln(x)) & \sin(\ln(x)) & \ln(x)\sin(\ln(x)) \\ -\frac{\sin(\ln(x))}{x} & \frac{\cos(\ln(x))}{x} & \frac{\sin(\ln(x))+\ln(x)\cos(\ln(x))}{x} \\ \frac{-\cos(\ln(x))+\sin(\ln(x))}{x^2} & \frac{-\sin(\ln(x))-\cos(\ln(x))}{x^2} & \frac{(-\ln(x)+2)\cos(\ln(x))-\sin(\ln(x))(\ln(x)+1)}{x^2} \end{bmatrix}$$
$$= \frac{2\cos(\ln(x))}{x^3}$$

$$W_4(x) = \det \begin{bmatrix} \cos(\ln(x)) & \sin(\ln(x)) & \ln(x)\cos(\ln(x)) \\ -\frac{\sin(\ln(x))}{x} & \frac{\cos(\ln(x))}{x} & \frac{\cos(\ln(x))-\ln(x)\sin(\ln(x))}{x} \\ \frac{-\cos(\ln(x))+\sin(\ln(x))}{x^2} & \frac{-\sin(\ln(x))-\cos(\ln(x))}{x^2} & \frac{(-\ln(x)-1)\cos(\ln(x))+\sin(\ln(x))(\ln(x)-2)}{x^2} \end{bmatrix}$$
$$= -\frac{2\sin(\ln(x))}{x^3}$$

Now we are ready to evaluate each $U_i(x)$.

$$U_1 = (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx$$
$$= (-1)^3 \int \frac{\left((\ln(x)+1)^2\right)\left(\frac{-2\ln(x)\cos(\ln(x))+2\sin(\ln(x))}{x^3}\right)}{(x^4)\left(\frac{4}{x^6}\right)}\,dx$$
$$= -\int \frac{\frac{2(\ln(x)+1)^2(-\ln(x)\cos(\ln(x))+\sin(\ln(x)))}{x^3}}{\frac{4}{x^2}}\,dx$$
$$= -\int \left(\frac{(\ln(x)+1)^2(-\ln(x)\cos(\ln(x))+\sin(\ln(x)))}{2x}\right)dx$$
$$= \frac{\sin(\ln(x))\ln(x)^3}{2} + 2\cos(\ln(x))\ln(x)^2 - 3\cos(\ln(x)) - \frac{7\ln(x)\sin(\ln(x))}{2} + \ln(x)^2\sin(\ln(x)) -$$

$$U_2 = (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx$$
$$= (-1)^2 \int \frac{\left((\ln(x)+1)^2\right)\left(\frac{2\ln(x)\sin(\ln(x))+2\cos(\ln(x))}{x^3}\right)}{(x^4)\left(\frac{4}{x^6}\right)}\,dx$$
$$= \int \frac{\frac{2(\ln(x)+1)^2(\ln(x)\sin(\ln(x))+\cos(\ln(x)))}{x^3}}{\frac{4}{x^2}}\,dx$$
$$= \int \left(\frac{(\ln(x)+1)^2(\ln(x)\sin(\ln(x))+\cos(\ln(x)))}{2x}\right)dx$$
$$= 2\ln(x)^2\sin(\ln(x)) - 3\sin(\ln(x)) + \frac{7\ln(x)\cos(\ln(x))}{2} + 3\cos(\ln(x)) + 3\ln(x)\sin(\ln(x)) - \frac{\ln(x}{}$$

$$U_3 = (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx$$
$$= (-1)^1 \int \frac{\left((\ln(x)+1)^2\right)\left(\frac{2\cos(\ln(x))}{x^3}\right)}{(x^4)\left(\frac{4}{x^6}\right)}\,dx$$
$$= -\int \frac{\frac{2(\ln(x)+1)^2\cos(\ln(x))}{x^3}}{\frac{4}{x^2}}\,dx$$
$$= -\int \left(\frac{(\ln(x)+1)^2\cos(\ln(x))}{2x}\right)dx$$
$$= -\frac{\ln(x)^2\sin(\ln(x))}{2} + \frac{\sin(\ln(x))}{2} - \ln(x)\cos(\ln(x)) - \cos(\ln(x)) - \ln(x)\sin(\ln(x))$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{\left((\ln(x)+1)^2\right)\left(-\frac{2\sin(\ln(x))}{x^3}\right)}{(x^4)\left(\frac{4}{x^6}\right)}\,dx \\
&= \int \frac{-\frac{2(\ln(x)+1)^2\sin(\ln(x))}{x^3}}{\frac{4}{x^2}}\,dx \\
&= \int \left(-\frac{(\ln(x)+1)^2\sin(\ln(x))}{2x}\right)dx \\
&= \frac{\cos(\ln(x))\ln(x)^2}{2} - \frac{\cos(\ln(x))}{2} - \ln(x)\sin(\ln(x)) + \ln(x)\cos(\ln(x)) - \sin(\ln(x))
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{\sin(\ln(x))\ln(x)^3}{2} + 2\cos(\ln(x))\ln(x)^2 - 3\cos(\ln(x)) - \frac{7\ln(x)\sin(\ln(x))}{2} + \ln(x)^2\sin(\ln(x))\right. \\
&+ \left(2\ln(x)^2\sin(\ln(x)) - 3\sin(\ln(x)) + \frac{7\ln(x)\cos(\ln(x))}{2} + 3\cos(\ln(x)) + 3\ln(x)\sin(\ln(x)) - \frac{\ln(}{}\right. \\
&+ \left(-\frac{\ln(x)^2\sin(\ln(x))}{2} + \frac{\sin(\ln(x))}{2} - \ln(x)\cos(\ln(x)) - \cos(\ln(x)) - \ln(x)\sin(\ln(x))\right)(\ln(x)\cos \\
&+ \left(\frac{\cos(\ln(x))\ln(x)^2}{2} - \frac{\cos(\ln(x))}{2} - \ln(x)\sin(\ln(x)) + \ln(x)\cos(\ln(x)) - \sin(\ln(x))\right)(\ln(x)\sin(
\end{aligned}
$$

Therefore the particular solution is

$$y_p = \ln(x)^2 + 2\ln(x) - 3$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1\cos(\ln(x)) + c_2\sin(\ln(x)) + \ln(x)(c_3\cos(\ln(x)) + c_4\sin(\ln(x)))) \\
&\quad + \left(\ln(x)^2 + 2\ln(x) - 3\right)
\end{aligned}
$$

Which simplifies to

$$y = (\ln(x)c_3 + c_1)\cos(\ln(x)) + \sin(\ln(x))(c_4\ln(x) + c_2) + \ln(x)^2 + 2\ln(x) - 3$$

Summary

The solution(s) found are the following

$$y = (\ln(x) c_3 + c_1) \cos(\ln(x)) + \sin(\ln(x)) (c_4 \ln(x) + c_2) + \ln(x)^2 + 2\ln(x) - 3 \quad (1)$$

Verification of solutions

$$y = (\ln(x) c_3 + c_1) \cos(\ln(x)) + \sin(\ln(x)) (c_4 \ln(x) + c_2) + \ln(x)^2 + 2\ln(x) - 3$$

Verified OK.

29.6.1 Maple step by step solution

Let's solve

$x^4 y'''' + 6x^3 y''' + 9y'' x^2 + 3y' x + y = (\ln(x) + 1)^2$

- Highest derivative means the order of the ODE is 4

  y''''

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve(x^4*diff(y(x),x$4)+6*x^3*diff(y(x),x$3)+9*x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+y(x)=(1
```

$$y(x) = (c_3 \ln(x) + c_1) \cos(\ln(x)) + (c_4 \ln(x) + c_2) \sin(\ln(x)) + \ln(x)^2 + 2\ln(x) - 3$$

✓ Solution by Mathematica

Time used: 0.27 (sec). Leaf size: 39

```
DSolve[x^4*y''''[x]+6*x^3*y'''[x]+9*x^2*y''[x]+3*x*y'[x]+y[x]==(1+Log[x])^2,y[x],x,IncludeSi
```

$$y(x) \to \log^2(x) + 2\log(x) + (c_2 \log(x) + c_1)\cos(\log(x)) + (c_4 \log(x) + c_3)\sin(\log(x)) - 3$$

29.7 problem Ex 8

Internal problem ID [11282]
Internal file name [`OUTPUT/10267_Wednesday_December_21_2022_03_47_12_PM_79576549/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 52. Summary. Page 117
Problem number: Ex 8.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 2y'' + y' = x^2 - x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 2y'' + y' = 0$$

The characteristic equation is

$$\lambda^3 + 2\lambda^2 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + x\, e^{-x} c_2 + c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = x\,e^{-x}$$
$$y_3 = 1$$

Now the particular solution to the given ODE is found

$$y''' + 2y'' + y' = x^2 - x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\,e^{-x}, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3x^3 + A_2x^2 + A_1x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3x^2A_3 + 2xA_2 + 12xA_3 + A_1 + 4A_2 + 6A_3 = x^2 - x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 8, A_2 = -\frac{5}{2}, A_3 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 8x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + c_3) + \left(\frac{1}{3}x^3 - \frac{5}{2}x^2 + 8x\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + c_3 + \frac{x^3}{3} - \frac{5x^2}{2} + 8x$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + c_3 + \frac{x^3}{3} - \frac{5x^2}{2} + 8x \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + c_3 + \frac{x^3}{3} - \frac{5x^2}{2} + 8x$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _a^2-2*(diff(_b(_a), _a))-_b(
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$3)+2*diff(y(x),x$2)+diff(y(x),x)=x^2-x,y(x), singsol=all)
```

$$y(x) = (-c_1x - c_1 - c_2)\,\mathrm{e}^{-x} + \frac{x^3}{3} - \frac{5x^2}{2} + 8x + c_3$$

✓ Solution by Mathematica

Time used: 0.243 (sec). Leaf size: 39

```
DSolve[y'''[x]+2*y''[x]+y'[x]==x^2-x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}x(2x^2 - 15x + 48) - e^{-x}(c_2(x+1) + c_1) + c_3$$

29.8 problem Ex 9

Internal problem ID [11283]
Internal file name [`OUTPUT/10268_Wednesday_December_21_2022_03_47_13_PM_82388358/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 52. Summary. Page 117
Problem number: Ex 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \sin(x)^2}$$

29.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \sin(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x\cos(2x), \sin(2x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x \cos(2x) + A_3 \sin(2x)\, x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_2 \sin(2x) + 4A_3 \cos(2x) + 4A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = 0, A_3 = -\frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{8} - \frac{\sin(2x)\,x}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{1}{8} - \frac{\sin(2x)\,x}{8}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{8} - \frac{\sin(2x)\,x}{8} \tag{1}$$

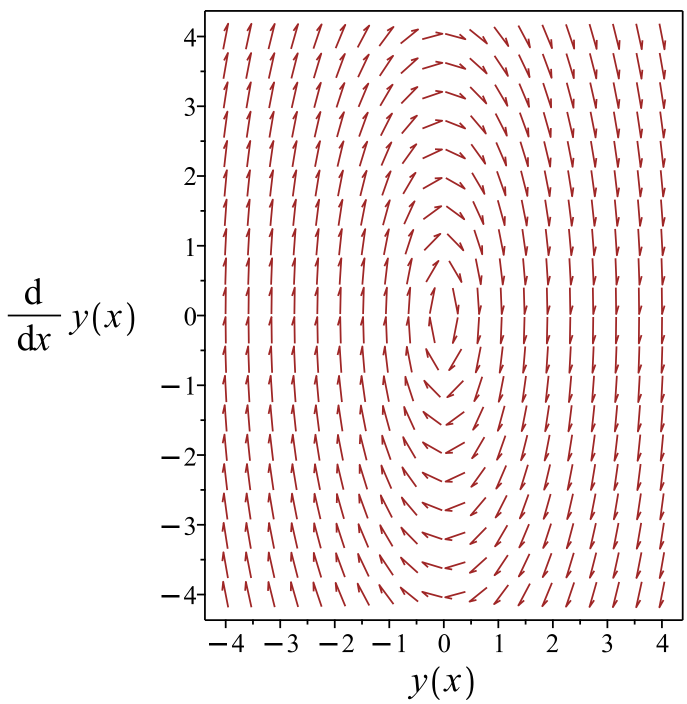

Figure 236: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{8} - \frac{\sin(2x)\,x}{8}$$

Verified OK.

29.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 168: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(2x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(2x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\
&= \cos(2x) \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x\cos(2x), \sin(2x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x \cos(2x) + A_3 \sin(2x)\, x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_2 \sin(2x) + 4A_3 \cos(2x) + 4A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = 0, A_3 = -\frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{8} - \frac{\sin(2x)\, x}{8}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\frac{1}{8} - \frac{\sin(2x)\, x}{8}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{1}{8} - \frac{\sin(2x)\, x}{8} \tag{1}
$$

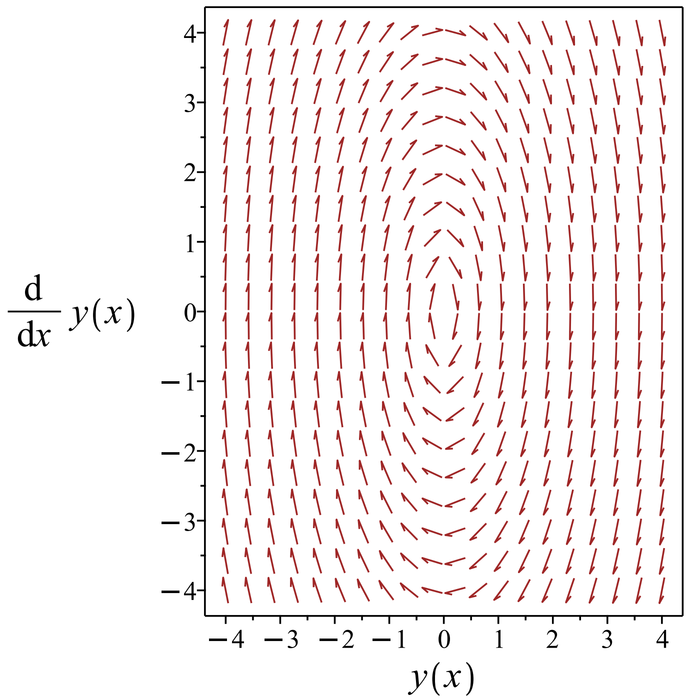

Figure 237: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{1}{8} - \frac{\sin(2x)\, x}{8}
$$

Verified OK.

29.8.3 Maple step by step solution

Let's solve

$y'' + 4y = \sin(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)^2\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- ∘ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(2x)\left(\int \sin(2x)\sin(x)^2 dx\right)}{2} + \frac{\sin(2x)\left(\int \cos(2x)\sin(x)^2 dx\right)}{2}$$

- Compute integrals

$$y_p(x) = -\frac{\sin(2x)x}{8} + \frac{1}{8} - \frac{\cos(2x)}{8}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2x) + c_2 \sin(2x) - \frac{\sin(2x)x}{8} + \frac{1}{8} - \frac{\cos(2x)}{8}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+4*y(x)=sin(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{(8c_1 - 1)\cos(2x)}{8} + \frac{1}{8} + \frac{(8c_2 - x)\sin(2x)}{8}$$

✓ Solution by Mathematica

Time used: 0.16 (sec). Leaf size: 34

```
DSolve[y''[x]+4*y[x]==Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}((-1 + 8c_1)\cos(2x) - (x - 8c_2)\sin(2x) + 1)$$

29.9 problem Ex 10

Internal problem ID [11284]
Internal file name [`OUTPUT/10269_Wednesday_December_21_2022_03_47_14_PM_98521348/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 52. Summary. Page 117
Problem number: Ex 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \sec(x)^2}$$

29.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \sec(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos{(2x)} + c_2 \sin{(2x)})$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(2x) \\ y_2 &= \sin(2x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}(\sin(2x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(2\cos(2x)) - (\sin(2x))(-2\sin(2x))$$

Which simplifies to

$$W = 2\cos(2x)^2 + 2\sin(2x)^2$$

Which simplifies to

$$W = 2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(2x)\sec(x)^2}{2}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\sec(x)^2}{2}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(2x)\sec(x)^2}{2}dx$$

Hence

$$u_2 = x - \frac{\tan(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(2x) + \left(x - \frac{\tan(x)}{2}\right)\sin(2x)$$

Which simplifies to

$$y_p(x) = \ln(\cos(x)) \left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(2x) + c_2\sin(2x)) + \left(\ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(2x) + c_2\sin(2x) + \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2 \tag{1}$$

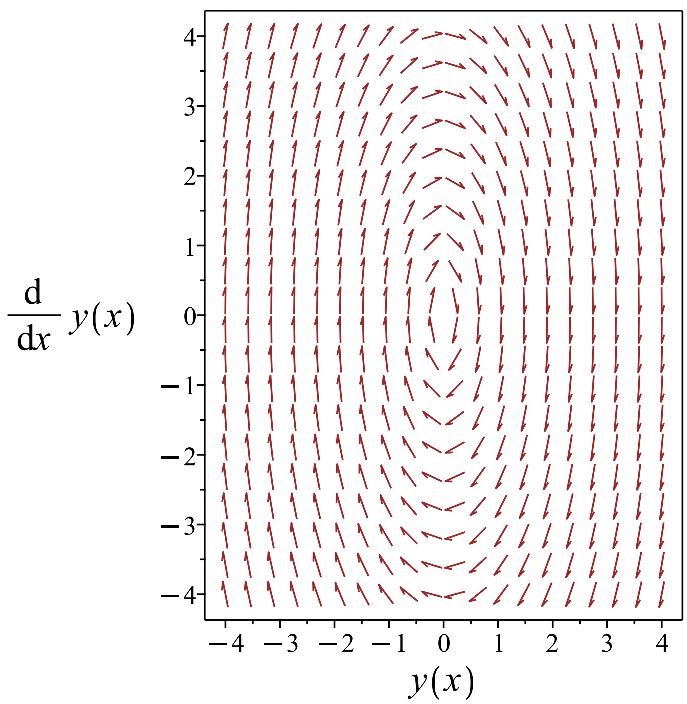

Figure 238: Slope field plot

Verification of solutions

$$y = c_1\cos(2x) + c_2\sin(2x) + \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2$$

Verified OK.

29.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 170: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)$$
$$y_2 = \frac{\sin(2x)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}\left(\frac{\sin(2x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ -2\sin(2x) & \cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(\cos(2x)) - \left(\frac{\sin(2x)}{2}\right)(-2\sin(2x))$$

Which simplifies to

$$W = \cos(2x)^2 + \sin(2x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(2x)\sec(x)^2}{2}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\sec(x)^2}{1}\,dx$$

Which simplifies to

$$u_2 = \int \cos(2x)\sec(x)^2\,dx$$

Hence

$$u_2 = 2x - \tan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(2x) + \frac{(2x - \tan(x))\sin(2x)}{2}$$

Which simplifies to

$$y_p(x) = \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)x - \sin(x)^2$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(c_1\cos(2x) + \frac{c_2\sin(2x)}{2}\right) + \left(\ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)x - \sin(x)^2\right)$$

Summary

The solution(s) found are the following

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)x - \sin(x)^2 \tag{1}$$

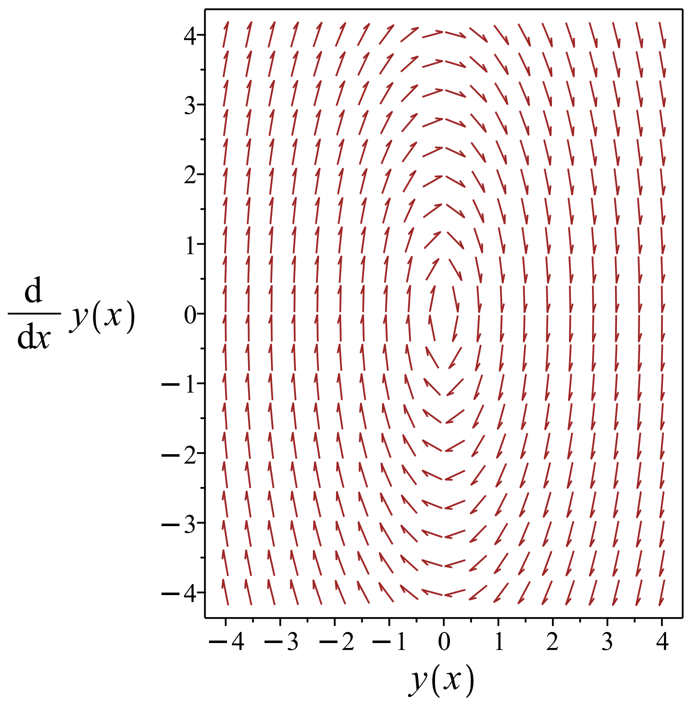

Figure 239: Slope field plot

Verification of solutions

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)x - \sin(x)^2$$

Verified OK.

29.9.3 Maple step by step solution

Let's solve

$y'' + 4y = \sec(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sec(x)^2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(2x)\left(\int \tan(x)\,dx\right) + \frac{\sin(2x)\left(\int \cos(2x)\sec(x)^2 dx\right)}{2}$$

- Compute integrals

$$y_p(x) = \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(diff(y(x),x$2)+4*y(x)=sec(x)^2,y(x), singsol=all)
```

$$y(x) = \left(-2\cos(x)^2 + 1\right)\ln(\sec(x)) + 2\cos(x)^2 c_1 + 2\sin(x)(c_2 + x)\cos(x) - \sin(x)^2 - c_1$$

✓ Solution by Mathematica

Time used: 0.168 (sec). Leaf size: 33

```
DSolve[y''[x]+4*y[x]==Sec[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(2x)(\log(\cos(x)) + c_1) + \sin(x)(-\sin(x) + 2(x + c_2)\cos(x))$$

29.10 problem Ex 12

Internal problem ID [11285]
Internal file name [`OUTPUT/10270_Wednesday_December_21_2022_03_47_16_PM_68194073/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 52. Summary. Page 117
Problem number: Ex 12.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _with_linear_symmetries]]`

$$\boxed{y'''' - y''' - 3y'' + 5y' - 2y = e^{3x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - y''' - 3y'' + 5y' - 2y = 0$$

The characteristic equation is

$$\lambda^4 - \lambda^3 - 3\lambda^2 + 5\lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = -2$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^x + x\, e^x c_3 + x^2 e^x c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = e^x$$
$$y_3 = x\, e^x$$
$$y_4 = x^2 e^x$$

Now the particular solution to the given ODE is found

$$y'''' - y''' - 3y'' + 5y' - 2y = e^{3x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^x, x^2 e^x, e^x, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$40A_1 e^{3x} = e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{40}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{3x}}{40}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{x} + x\, e^{x} c_3 + x^2 e^{x} c_4\right) + \left(\frac{e^{3x}}{40}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}\left(\left(c_4 x^2 + c_3 x + c_2\right) e^{3x} + c_1\right) + \frac{e^{3x}}{40}$$

Summary

The solution(s) found are the following

$$y = e^{-2x}\left(\left(c_4 x^2 + c_3 x + c_2\right) e^{3x} + c_1\right) + \frac{e^{3x}}{40} \tag{1}$$

Verification of solutions

$$y = e^{-2x}\left(\left(c_4 x^2 + c_3 x + c_2\right) e^{3x} + c_1\right) + \frac{e^{3x}}{40}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$4)-diff(y(x),x$3)-3*diff(y(x),x$2)+5*diff(y(x),x)-2*y(x)=exp(3*x),y(x), s
```

$$y(x) = \left(\left(c_3 x^2 + c_4 x + c_1\right) e^{3x} + c_2 + \frac{e^{5x}}{40}\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 39

```
DSolve[y''''[x]-y'''[x]-3*y''[x]+5*y'[x]-2*y[x]==Exp[3*x],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{e^{3x}}{40} + c_1 e^{-2x} + e^x(x(c_4 x + c_3) + c_2)$$

29.11 problem Ex 13

Internal problem ID [11286]
Internal file name [`OUTPUT/10271_Wednesday_December_21_2022_03_47_17_PM_72410501/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 52. Summary. Page 117
Problem number: Ex 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \cos(x) x}$$

29.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \cos(x) x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)\, x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \sin(x), \cos(x)\, x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \sin(x), \cos(x)\, x, \cos(x)\, x^2, \sin(x)\, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \sin(x) + A_2 \cos(x)\, x + A_3 \cos(x)\, x^2 + A_4 \sin(x)\, x^2$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \cos(x) - 2A_2 \sin(x) - 4A_3 \sin(x)\, x + 2A_3 \cos(x) + 4A_4 \cos(x)\, x + 2A_4 \sin(x) = \cos(x)\, x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{4}, A_3 = 0, A_4 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)\, x}{4} + \frac{\sin(x)\, x^2}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{\cos(x)\, x}{4} + \frac{\sin(x)\, x^2}{4}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) + \frac{\cos(x)\, x}{4} + \frac{\sin(x)\, x^2}{4} \tag{1}
$$

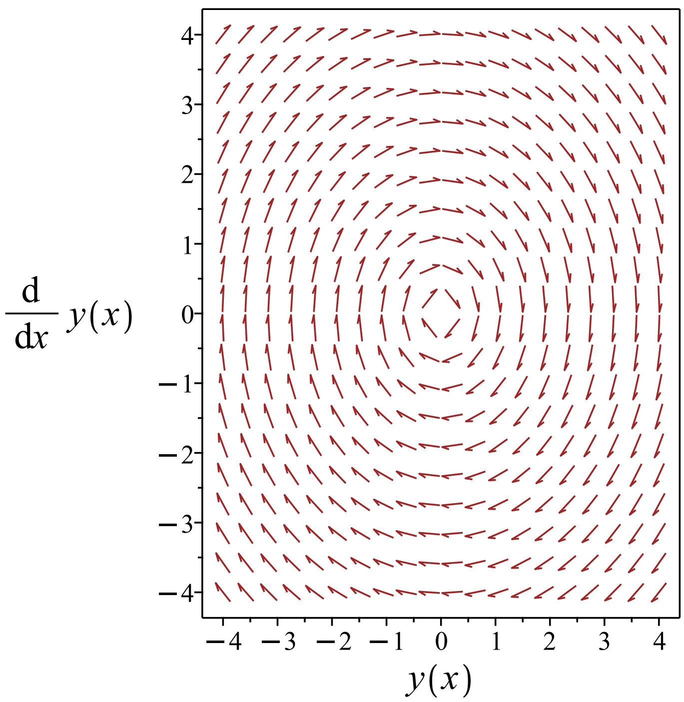

Figure 240: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) + \frac{\cos(x)\, x}{4} + \frac{\sin(x)\, x^2}{4}
$$

Verified OK.

29.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 172: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B=0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B=0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x) (\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x) (\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)\, x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \sin(x), \cos(x)\, x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \sin(x), \cos(x)\, x, \cos(x)\, x^2, \sin(x)\, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \sin(x) + A_2 \cos(x)\, x + A_3 \cos(x)\, x^2 + A_4 \sin(x)\, x^2$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_1 \cos(x) - 2A_2 \sin(x) - 4A_3 \sin(x)\, x + 2A_3 \cos(x) + 4A_4 \cos(x)\, x + 2A_4 \sin(x)\\ &= \cos(x)\, x\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{4}, A_3 = 0, A_4 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)\, x}{4} + \frac{\sin(x)\, x^2}{4}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{\cos(x)\, x}{4} + \frac{\sin(x)\, x^2}{4}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{\cos(x)\,x}{4} + \frac{\sin(x)\,x^2}{4} \tag{1}$$

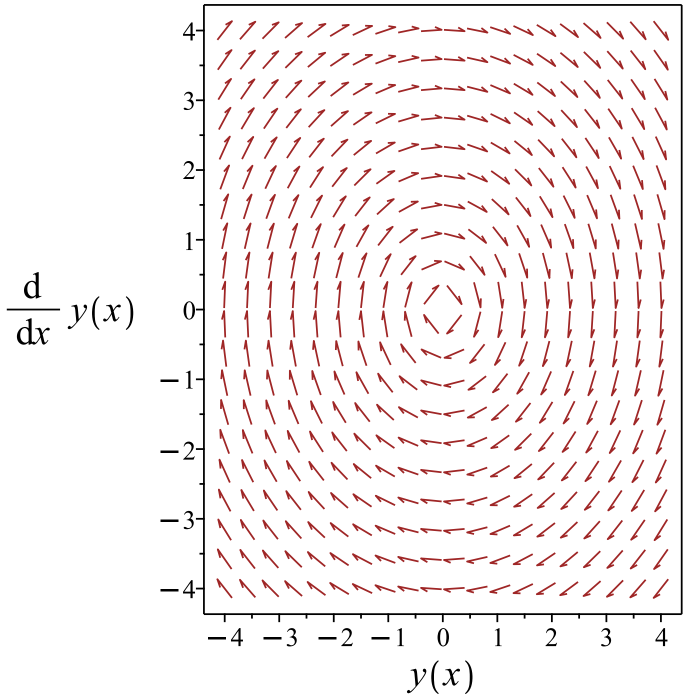

Figure 241: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{\cos(x)\,x}{4} + \frac{\sin(x)\,x^2}{4}$$

Verified OK.

29.11.3 Maple step by step solution

Let's solve

$y'' + y = \cos(x)\,x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cos(x)\, x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\cos(x) \left(\int \sin(2x) x dx\right)}{2} + \sin(x) \left(\int \cos(x)^2 x dx\right)$

  - Compute integrals

    $y_p(x) = \frac{\sin(x) x^2}{4} - \frac{\sin(x)}{4} + \frac{\cos(x) x}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x) x^2}{4} - \frac{\sin(x)}{4} + \frac{\cos(x) x}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+y(x)=x*cos(x),y(x), singsol=all)
```

$$y(x) = \frac{(x^2 + 4c_2 - 1)\sin(x)}{4} + \frac{\cos(x)(4c_1 + x)}{4}$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 34

```
DSolve[y''[x]+y[x]==x*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}\left((2x^2 - 1 + 8c_2)\sin(x) + 2(x + 4c_1)\cos(x)\right)$$

29.12 problem Ex 14

Internal problem ID [11287]
Internal file name [OUTPUT/10272_Wednesday_December_21_2022_03_47_18_PM_2332710/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 52. Summary. Page 117
Problem number: Ex 14.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^3y''' + 2x^2y'' - y'x + y = \frac{1}{x}}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' + 2x^2y'' - y'x + y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^3y''' + 2x^2y'' - y'x + y = \frac{1}{x}$$

gives

$$-x\lambda\, x^{\lambda-1} + 2x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + x^\lambda = 0$$

Which simplifies to

$$-\lambda\, x^\lambda + 2\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-\lambda + 2\lambda(\lambda-1) + \lambda(\lambda-1)\,(\lambda-2) + 1 = 0$$

Simplifying gives the characteristic equation as

$$(\lambda+1)\,(\lambda-1)^2 = 0$$

Solving the above gives the following roots

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
1	2	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + c_2 x + c_3 \ln(x)\, x$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \frac{1}{x} \\
y_2 &= x \\
y_3 &= x \ln(x)
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
x^3 y''' + 2x^2 y'' - y'x + y = \frac{1}{x}
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
\begin{aligned}
W &= \begin{bmatrix} \frac{1}{x} & x & x \ln(x) \\ -\frac{1}{x^2} & 1 & \ln(x) + 1 \\ \frac{2}{x^3} & 0 & \frac{1}{x} \end{bmatrix} \\
|W| &= \frac{4}{x^2}
\end{aligned}
$$

The determinant simplifies to

$$|W| = \frac{4}{x^2}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} x & x\ln(x) \\ 1 & \ln(x)+1 \end{bmatrix} \\ &= x \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} \frac{1}{x} & x\ln(x) \\ -\frac{1}{x^2} & \ln(x)+1 \end{bmatrix} \\ &= \frac{2\ln(x)+1}{x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} \frac{1}{x} & x \\ -\frac{1}{x^2} & 1 \end{bmatrix} \\ &= \frac{2}{x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2 \int \frac{\left(\frac{1}{x}\right)(x)}{(x^3)\left(\frac{4}{x^2}\right)}\,dx \\ &= \int \frac{1}{4x}\,dx \\ &= \int \left(\frac{1}{4x}\right) dx \\ &= \frac{\ln(x)}{4} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\
&= (-1)^1 \int \frac{\left(\frac{1}{x}\right)\left(\frac{2\ln(x)+1}{x}\right)}{(x^3)\left(\frac{4}{x^2}\right)}\, dx \\
&= -\int \frac{\frac{2\ln(x)+1}{x^2}}{4x}\, dx \\
&= -\int \left(\frac{2\ln(x)+1}{4x^3}\right) dx \\
&= \frac{\ln(x)}{4x^2} + \frac{1}{4x^2}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^0 \int \frac{\left(\frac{1}{x}\right)\left(\frac{2}{x}\right)}{(x^3)\left(\frac{4}{x^2}\right)}\, dx \\
&= \int \frac{\frac{2}{x^2}}{4x}\, dx \\
&= \int \left(\frac{1}{2x^3}\right) dx \\
&= -\frac{1}{4x^2}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p =& \left(\frac{\ln(x)}{4}\right)\left(\frac{1}{x}\right) \\
&+ \left(\frac{\ln(x)}{4x^2} + \frac{1}{4x^2}\right)(x) \\
&+ \left(-\frac{1}{4x^2}\right)(x\ln(x))
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\ln(x)+1}{4x}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x} + c_2 x + c_3 \ln (x)\, x\right) + \left(\frac{\ln (x) + 1}{4x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} + c_2 x + c_3 \ln (x)\, x + \frac{\ln (x) + 1}{4x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x} + c_2 x + c_3 \ln (x)\, x + \frac{\ln (x) + 1}{4x}
$$

Verified OK.

29.12.1 Maple step by step solution

Let's solve

$x^3 y''' + 2y'' x^2 - y' x + y = \frac{1}{x}$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = (c__1-_b(_a)*_a+_a^2*(diff(_b
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   <- high order exact linear fully integrable successful
<- high order exact_linear_nonhomogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(x^3*diff(y(x),x$3)+2*x^2*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=1/x,y(x), singsol=all)
```

$$y(x) = \frac{4\ln(x)\, c_2 x^2 + 4c_3 x^2 + \ln(x) + c_1 + 1}{4x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 33

```
DSolve[x^3*y'''[x]+2*x^2*y''[x]-x*y'[x]+y[x]==1/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\log(x) + 1}{4x} + \frac{c_1}{x} + c_2 x + c_3 x \log(x)$$

29.13 problem Ex 15

Internal problem ID [11288]
Internal file name [OUTPUT/10273_Wednesday_December_21_2022_03_47_20_PM_39957394/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VII, Linear differential equations with constant coefficients. Article 52. Summary. Page 117
Problem number: Ex 15.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - y = x\,\mathrm{e}^{x} + \cos(x)^{2}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y = 0$$

The characteristic equation is

$$\lambda^3 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$
$$\lambda_3 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$

$$y_2 = e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}$$

$$y_3 = e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}$$

Now the particular solution to the given ODE is found

$$y''' - y = x\,e^x + \cos(x)^2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,e^x + \cos(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{x\,e^x, e^x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^x, e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}, e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}\right\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x\,e^x, x^2e^x\}, \{\cos(2x), \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x\,e^x + A_3 x^2 e^x + A_4 \cos(2x) + A_5 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_2e^x + 6A_3e^x + 6A_3x\,e^x + 8A_4\sin(2x) - 8A_5\cos(2x) \\ - A_1 - A_4\cos(2x) - A_5\sin(2x) = x\,e^x + \cos(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = -\frac{1}{3}, A_3 = \frac{1}{6}, A_4 = -\frac{1}{130}, A_5 = -\frac{4}{65}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{2} - \frac{x\,\mathrm{e}^x}{3} + \frac{x^2\mathrm{e}^x}{6} - \frac{\cos(2x)}{130} - \frac{4\sin(2x)}{65}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^x c_1 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3\right) + \left(-\frac{1}{2} - \frac{x\,\mathrm{e}^x}{3} + \frac{x^2\mathrm{e}^x}{6} - \frac{\cos(2x)}{130} - \frac{4\sin(2x)}{65}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^x c_1 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3 - \frac{1}{2} - \frac{x\,\mathrm{e}^x}{3} + \frac{x^2\mathrm{e}^x}{6} - \frac{\cos(2x)}{130} - \frac{4\sin(2x)}{65} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^x c_1 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3 - \frac{1}{2} - \frac{x\,\mathrm{e}^x}{3} + \frac{x^2\mathrm{e}^x}{6} - \frac{\cos(2x)}{130} - \frac{4\sin(2x)}{65}$$

Verified OK.

29.13.1 Maple step by step solution

Let's solve

$y''' - y = x\,\mathrm{e}^x + \cos(x)^2$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $$y_3(x) = y''$$

- Isolate for $y_3'(x)$ using original ODE

  $$y_3'(x) = x\,\mathrm{e}^x + \cos(x)^2 + y_1(x)$$

  Convert linear ODE into a system of first order ODEs

  $$\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x\,\mathrm{e}^x + \cos(x)^2 + y_1(x)\right]$$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ x\,\mathrm{e}^x + \cos(x)^2 \end{array}\right]$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ x\,\mathrm{e}^x + \cos(x)^2 \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \cdot \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-\frac{x}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}\right)^2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{-\frac{1}{2} - \frac{I\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{-\frac{x}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}, \vec{y}_3(x) = e^{-\frac{x}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \left[\begin{array}{ccc} \frac{e^{x}}{3}+\frac{2 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{3} & \frac{e^{x}}{3}-\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{3}+\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{3} & \frac{e^{x}}{3}-\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{3} \cdots \\ \frac{e^{x}}{3}-\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{3}-\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{3} & \frac{e^{x}}{3}+\frac{2 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{3} & \frac{e^{x}}{3}-\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{3} \cdots \\ \frac{e^{x}}{3}-\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{3}+\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{3} & \frac{e^{x}}{3}-\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{3}-\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{3} & \frac{e^{x}}{3}+\cdots \end{array}\right.$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_{p}(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}_{p}^{\prime}(x) = \Phi^{\prime}(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}^{\prime}(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi^{\prime}(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}^{\prime}(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}^{\prime}(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}^{\prime}(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}^{\prime}(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_{0}^{x} \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_{p}(x) = \Phi(x) \cdot \left(\int_{0}^{x} \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_{p}(x) = \left[\begin{array}{c} -\frac{1}{2}+\frac{10 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{117}+\frac{2 e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{39}-\frac{\cos(2 x)}{130}-\frac{4 \sin(2 x)}{65}+\frac{(15 x^{2}-30 x+38) e^{x}}{90} \\ \frac{x^{2} e^{x}}{6}-\frac{8 e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{117}+\frac{4 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{117}+\frac{4 e^{x}}{45}-\frac{8 \cos(2 x)}{65}+\frac{\sin(2 x)}{65} \\ -\frac{14 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{117}+\frac{2 e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{117}+\frac{2 \cos(2 x)}{65}+\frac{16 \sin(2 x)}{65}+\frac{(15 x^{2}+30 x+8) e^{x}}{90} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \begin{bmatrix} -\frac{1}{2} + \frac{10\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{117} + \frac{2\,e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{39} - \frac{\cos(2x)}{130} - \frac{4\sin(2x)}{65} \\ \frac{x^2 e^x}{6} - \frac{8\,e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{117} + \frac{4\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{117} + \frac{4\,e^x}{45} - \frac{8\cos(}{65} \\ -\frac{14\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{117} + \frac{2\,e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{117} + \frac{2\cos(2x)}{65} + \frac{16\sin(2x)}{65} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{1}{2} - \frac{e^{-\frac{x}{2}}\left(\sqrt{3}\,c_3 + c_2 - \frac{20}{117}\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\left(\left(c_2 - \frac{4}{39}\right)\sqrt{3} - c_3\right)e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\cos(2x)}{130} - \frac{4\sin(2x)}{65} + \frac{(15x^2 + 90c_1 - 3}{90}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 61

```
dsolve(diff(y(x),x$3)-y(x)=x*exp(x)+cos(x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{1}{2} + c_2 e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_3 e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) - \frac{\cos(2x)}{130} - \frac{4\sin(2x)}{65} + \frac{(3x^2 + 18c_1 - 6x + 4)\,e^x}{18}$$

✓ Solution by Mathematica

Time used: 7.274 (sec). Leaf size: 98

```
DSolve[y'''[x]-y[x]==x*Exp[x]+Cos[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x x^2}{6} - \frac{e^x x}{3} + \frac{2e^x}{9} - \frac{4}{65}\sin(2x) - \frac{1}{130}\cos(2x) + c_1 e^x + c_2 e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right) + c_3 e^{-x/2}\sin\left(\frac{\sqrt{3}x}{2}\right) - \frac{1}{2}$$

30 Chapter VIII, Linear differential equations of the second order. Article 53. Change of dependent variable. Page 125

30.1 problem Ex 1

Internal problem ID [11289]
Internal file name [`OUTPUT/10274_Wednesday_December_21_2022_03_47_22_PM_26854596/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 53. Change of dependent variable. Page 125
Problem number: Ex 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - x^2 y' + yx = x}$$

30.1.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -x^2, C = x, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$y'' - x^2 y' + yx = 0$$

In normal form the ode

$$y'' - x^2 y' + yx = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -x^2 \\ q(x) &= x \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - nx + x = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} - x^2\right) v'(x) &= 0 \\ v''(x) + \frac{(-x^3 + 2)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(-x^3+2)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(x^3-2)}{x} \end{aligned}$$

Where $f(x) = \frac{x^3-2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{x^3-2}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{x^3-2}{x}\, dx \\ \ln(u) &= \frac{x^3}{3} - 2\ln(x) + c_1 \\ u &= \mathrm{e}^{\frac{x^3}{3} - 2\ln(x) + c_1} \\ &= c_1 \mathrm{e}^{\frac{x^3}{3} - 2\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 \mathrm{e}^{\frac{x^3}{3}}}{x^2}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= \frac{c_1 3^{\frac{2}{3}} (-1)^{\frac{1}{3}} \left(-\frac{3x^2(-1)^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)}{(-x^3)^{\frac{2}{3}}} + \frac{3\, 3^{\frac{1}{3}} (-1)^{\frac{2}{3}} \mathrm{e}^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{\frac{2}{3}} \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{2}{3}}} \right)}{9} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(\frac{c_1 3^{\frac{2}{3}} (-1)^{\frac{1}{3}} \left(-\frac{3x^2(-1)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{\frac{2}{3}}} + \frac{3\,3^{\frac{1}{3}}(-1)^{\frac{2}{3}}\mathrm{e}^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{2}{3}}}\right)}{9} + c_2\right) x \\
&= \frac{\left(3c_2 x - 3c_1 \mathrm{e}^{\frac{x^3}{3}}\right)(-x^3)^{\frac{2}{3}} + x^3 c_1 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3\,(-x^3)^{\frac{2}{3}}}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
y'' - x^2 y' + yx = x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}\right)}{3\,(-x^3)^{\frac{2}{3}}} - \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3\,(-x^3)^{\frac{2}{3}}} - \mathrm{e}^{\frac{x^3}{3}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{\frac{2}{3}}} - \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3(-x^3)^{\frac{2}{3}}} - \mathrm{e}^{\frac{x^3}{3}} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{\frac{2}{3}}} - \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3(-x^3)^{\frac{2}{3}}} - \mathrm{e}^{\frac{x^3}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{\frac{2}{3}}} - \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3(-x^3)^{\frac{2}{3}}} - \mathrm{e}^{\frac{x^3}{3}} \\ 1 & \frac{3^{\frac{2}{3}} x^2 \Gamma\left(\frac{2}{3}\right)}{(-x^3)^{\frac{2}{3}}} + \frac{2\,3^{\frac{2}{3}} x^5 \Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{\frac{5}{3}}} - \frac{3^{\frac{2}{3}} x^2 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{2}{3}}} - \frac{2\,3^{\frac{2}{3}} x^5 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3(-x^3)^{\frac{5}{3}}} - \frac{3^{\frac{2}{3}} x^5 \mathrm{e}^{\frac{x^3}{3}}}{3(-x^3)^{\frac{2}{3}} \left(-\frac{x^3}{3}\right)^{\frac{1}{3}}} - x^2 \mathrm{e}^{\frac{x^3}{3}} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{3^{\frac{2}{3}} x^2 \Gamma\left(\frac{2}{3}\right)}{(-x^3)^{\frac{2}{3}}} + \frac{2\,3^{\frac{2}{3}} x^5 \Gamma\left(\frac{2}{3}\right)}{3\left(-x^3\right)^{\frac{5}{3}}} - \frac{3^{\frac{2}{3}} x^2 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{2}{3}}} - \frac{2\,3^{\frac{2}{3}} x^5 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3\left(-x^3\right)^{\frac{5}{3}}} - \frac{3^{\frac{2}{3}} x^5 \mathrm{e}^{\frac{x^3}{3}}}{3\left(-x^3\right)^{\frac{2}{3}} \left(-\frac{x^3}{3}\right)^{\frac{1}{3}}} - x^2 \mathrm{e}^{\frac{x^3}{3}}\right) - \left(\frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}\right)}{3\left(-x^3\right)^{\frac{2}{3}}} - \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3\left(-x^3\right)^{\frac{2}{3}}} - \mathrm{e}^{\frac{x^3}{3}}\right) (1)$$

Which simplifies to

$$W = \frac{\mathrm{e}^{\frac{x^3}{3}}\left(3^{\frac{2}{3}} x^9 - 3(-x^3)^{\frac{5}{3}}\left(-\frac{x^3}{3}\right)^{\frac{1}{3}} x^3 + 3(-x^3)^{\frac{5}{3}}\left(-\frac{x^3}{3}\right)^{\frac{1}{3}}\right)}{3\left(-x^3\right)^{\frac{5}{3}}\left(-\frac{x^3}{3}\right)^{\frac{1}{3}}}$$

Which simplifies to

$$W = \mathrm{e}^{\frac{x^3}{3}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}\right)}{3(-x^3)^{\frac{2}{3}}} - \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3(-x^3)^{\frac{2}{3}}} - \mathrm{e}^{\frac{x^3}{3}}\right) x}{\mathrm{e}^{\frac{x^3}{3}}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x\left(-3(-x^3)^{\frac{2}{3}} + x^3 3^{\frac{2}{3}} \mathrm{e}^{-\frac{x^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} \mathrm{e}^{-\frac{\alpha^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{3\left(-\alpha^3\right)^{\frac{2}{3}}} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{\mathrm{e}^{\frac{x^3}{3}}} dx$$

Which simplifies to

$$u_2 = \int x^2 \mathrm{e}^{-\frac{x^3}{3}} dx$$

Hence

$$u_2 = -\mathrm{e}^{-\frac{x^3}{3}}$$

Which simplifies to

$$u_1 = -\frac{\left(\int_0^x \frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} \mathrm{e}^{-\frac{\alpha^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{\left(-\alpha^3\right)^{\frac{2}{3}}} d\alpha\right)}{3}$$

$$u_2 = -\mathrm{e}^{-\frac{x^3}{3}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x \frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} \mathrm{e}^{-\frac{\alpha^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{\left(-\alpha^3\right)^{\frac{2}{3}}} d\alpha\right) x}{3} - \mathrm{e}^{-\frac{x^3}{3}} \left(\frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}\right)}{3\left(-x^3\right)^{\frac{2}{3}}} - \frac{3^{\frac{2}{3}} x^3 \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{3\left(-x^3\right)^{\frac{2}{3}}} - \mathrm{e}^{\frac{x^3}{3}}\right)$$

Which simplifies to

$$y_p(x) = \frac{-3^{\frac{2}{3}} x^3 e^{-\frac{x^3}{3}} \Gamma\left(\frac{2}{3}\right) + 3^{\frac{2}{3}} x^3 e^{-\frac{x^3}{3}} \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) - \left(\int_0^x \frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha\right) x(-x^3)^{\frac{2}{3}} + 3}{3(-x^3)^{\frac{2}{3}}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(\frac{c_1 3^{\frac{2}{3}} (-1)^{\frac{1}{3}} \left(-\frac{3x^2(-1)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{\frac{2}{3}}} + \frac{3\, 3^{\frac{1}{3}} (-1)^{\frac{2}{3}} e^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{2}{3}}}\right)}{9} + c_2\right) x\right) \\ &\quad + \left(\frac{-3^{\frac{2}{3}} x^3 e^{-\frac{x^3}{3}} \Gamma\left(\frac{2}{3}\right) + 3^{\frac{2}{3}} x^3 e^{-\frac{x^3}{3}} \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) - \left(\int_0^x \frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha\right) x(-x^3)}{3(-x^3)^{\frac{2}{3}}}\right. \end{aligned}$$

$$\begin{aligned} &= \frac{-3^{\frac{2}{3}} x^3 e^{-\frac{x^3}{3}} \Gamma\left(\frac{2}{3}\right) + 3^{\frac{2}{3}} x^3 e^{-\frac{x^3}{3}} \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) - \left(\int_0^x \frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha\right) x(-x^3)^{\frac{2}{3}} +}{3(-x^3)^{\frac{2}{3}}} \\ &\quad + \left(\frac{c_1 3^{\frac{2}{3}} (-1)^{\frac{1}{3}} \left(-\frac{3x^2(-1)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{\frac{2}{3}}} + \frac{3\, 3^{\frac{1}{3}} (-1)^{\frac{2}{3}} e^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{2}{3}}}\right)}{9} + c_2\right) x \end{aligned}$$

Which simplifies to

$$y = -\frac{\left(\left(\int_0^x \frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha - 3c_2\right) x + 3c_1 e^{\frac{x^3}{3}} - 3\right)(-x^3)^{\frac{2}{3}} - x^3 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right.}{3(-x^3)^{\frac{2}{3}}}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(\left(\int_0^x \frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}}+\alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha - 3c_2\right) x + 3c_1 \mathrm{e}^{\frac{x^3}{3}} - 3\right) (-x^3)^{\frac{2}{3}} - x^3 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right.}{3\left(-x^3\right)^{\frac{2}{3}}} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\left(\int_0^x \frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}}+\alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha - 3c_2\right) x + 3c_1 \mathrm{e}^{\frac{x^3}{3}} - 3\right) (-x^3)^{\frac{2}{3}} - x^3 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right.}{3\left(-x^3\right)^{\frac{2}{3}}}$$

Verified OK.

30.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - x^2 y' + yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3 - 8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x(x^3 - 8) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3 - 8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 176: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} - \frac{2}{x} - \frac{4}{x^4} - \frac{16}{x^7} - \frac{80}{x^{10}} - \frac{448}{x^{13}} - \frac{2688}{x^{16}} - \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x(x^3 - 8)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^4 - 2x\right) + (0) \\ &= \frac{1}{4}x^4 - 2x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -2. Now b can be found.

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^2}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 2\right) &&= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 2\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3 - 8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	-3	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-) \left(\frac{x^2}{2} \right) \\ &= -\frac{x^2}{2} \\ &= -\frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x^2}{2}\right)(1) + \left((-x) + \left(-\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3 - 8)}{4}\right)\right) = 0$$
$$x a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, e^{\int -\frac{x^2}{2} dx} \\ &= (x) \, e^{-\frac{x^3}{6}} \\ &= x \, e^{-\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x^2}{1} \, dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1 \left(e^{\frac{x^3}{6}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) \right)}{3 \, (-x^3)^{\frac{2}{3}} \, x} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2 \left(x \left(\frac{-3(-x^3)^{\frac{2}{3}} \mathrm{e}^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) \right)}{3\left(-x^3\right)^{\frac{2}{3}} x} \right) \right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - x^2 y' + yx = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 x + \frac{c_2 \left(-3(-x^3)^{\frac{2}{3}} \mathrm{e}^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) \right) \right)}{3\left(-x^3\right)^{\frac{2}{3}}}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \frac{-3(-x^3)^{\frac{2}{3}} \mathrm{e}^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) \right)}{3\left(-x^3\right)^{\frac{2}{3}}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3(-x^3)^{\frac{2}{3}}} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3(-x^3)^{\frac{2}{3}}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3(-x^3)^{\frac{2}{3}}} \\ 1 & \frac{\frac{6 e^{\frac{x^3}{3}} x^2}{(-x^3)^{\frac{1}{3}}} - 3(-x^3)^{\frac{2}{3}} x^2 e^{\frac{x^3}{3}} + 3x^2 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right) - \frac{x^5 3^{\frac{2}{3}} e^{\frac{x^3}{3}}}{\left(-\frac{x^3}{3}\right)^{\frac{1}{3}}}}{3(-x^3)^{\frac{2}{3}}} + \frac{2\left(-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right) x^2}{3(-x^3)^{\frac{5}{3}}} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{\frac{6 e^{\frac{x^3}{3}} x^2}{(-x^3)^{\frac{1}{3}}} - 3(-x^3)^{\frac{2}{3}} x^2 e^{\frac{x^3}{3}} + 3x^2 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right) - \frac{x^5 3^{\frac{2}{3}} e^{\frac{x^3}{3}}}{\left(-\frac{x^3}{3}\right)^{\frac{1}{3}}}}{3\,(-x^3)^{\frac{2}{3}}} + \frac{2\left(-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right) x^2}{3\,(-x^3)^{\frac{5}{3}}}\right) - \left(\frac{-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3\,(-x^3)^{\frac{2}{3}}}\right)(1)$$

Which simplifies to

$$W = \frac{e^{\frac{x^3}{3}} \left(3^{\frac{2}{3}} x^9 - 3(-x^3)^{\frac{5}{3}} \left(-\frac{x^3}{3}\right)^{\frac{1}{3}} x^3 + 3(-x^3)^{\frac{5}{3}} \left(-\frac{x^3}{3}\right)^{\frac{1}{3}}\right)}{3 \left(-x^3\right)^{\frac{5}{3}} \left(-\frac{x^3}{3}\right)^{\frac{1}{3}}}$$

Which simplifies to

$$W = e^{\frac{x^3}{3}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\left(-3(-x^3)^{\frac{2}{3}} e^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right) x}{3(-x^3)^{\frac{2}{3}}}}{e^{\frac{x^3}{3}}} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{x\left(-3(-x^3)^{\frac{2}{3}} + x^3 3^{\frac{2}{3}} e^{-\frac{x^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3 \left(-x^3\right)^{\frac{2}{3}}} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} e^{-\frac{\alpha^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{3 \left(-\alpha^3\right)^{\frac{2}{3}}} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{e^{\frac{x^3}{3}}} \, dx$$

Which simplifies to

$$u_2 = \int x^2 e^{-\frac{x^3}{3}} dx$$

Hence

$$u_2 = -e^{-\frac{x^3}{3}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\left(\int_0^x \frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} \mathrm{e}^{-\frac{\alpha^3}{3}} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{3\left(-\alpha^3\right)^{\frac{2}{3}}} d\alpha\right) x - \frac{\mathrm{e}^{-\frac{x^3}{3}}\left(-3(-x^3)^{\frac{2}{3}} \mathrm{e}^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}}$$

Which simplifies to

$$y_p(x) = \frac{-3^{\frac{2}{3}} x^3 \mathrm{e}^{-\frac{x^3}{3}} \Gamma\left(\frac{2}{3}\right) + 3^{\frac{2}{3}} x^3 \mathrm{e}^{-\frac{x^3}{3}} \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) + \left(\int_0^x -\frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} \mathrm{e}^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha\right) x(-x^3)^{\frac{2}{3}} + }{3\left(-x^3\right)^{\frac{2}{3}}}$$

Therefore the general solution is

$$y = y_h + y_p = \left(c_1 x + \frac{c_2\left(-3(-x^3)^{\frac{2}{3}} \mathrm{e}^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}}\right) + \left(\frac{-3^{\frac{2}{3}} x^3 \mathrm{e}^{-\frac{x^3}{3}} \Gamma\left(\frac{2}{3}\right) + 3^{\frac{2}{3}} x^3 \mathrm{e}^{-\frac{x^3}{3}} \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) + \left(\int_0^x -\frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} \mathrm{e}^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha\right) x(-x}{3\left(-x^3\right)^{\frac{2}{3}}}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x + \frac{c_2\left(-3(-x^3)^{\frac{2}{3}} \mathrm{e}^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}} + \frac{-3^{\frac{2}{3}} x^3 \mathrm{e}^{-\frac{x^3}{3}} \Gamma\left(\frac{2}{3}\right) + 3^{\frac{2}{3}} x^3 \mathrm{e}^{-\frac{x^3}{3}} \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) + \left(\int_0^x -\frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}} \mathrm{e}^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha\right) x(-x^3)^{\frac{2}{3}}}{3\left(-x^3\right)^{\frac{2}{3}}} \tag{1}$$

Verification of solutions

$$y = c_1 x + \frac{c_2\left(-3(-x^3)^{\frac{2}{3}}\,\mathrm{e}^{\frac{x^3}{3}} + x^3 3^{\frac{2}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}}$$
$$+ \frac{-3^{\frac{2}{3}} x^3 \mathrm{e}^{-\frac{x^3}{3}}\Gamma\left(\frac{2}{3}\right) + 3^{\frac{2}{3}} x^3 \mathrm{e}^{-\frac{x^3}{3}}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) + \left(\int_0^x -\frac{\alpha\left(-3(-\alpha^3)^{\frac{2}{3}} + \alpha^3 3^{\frac{2}{3}}\mathrm{e}^{-\frac{\alpha^3}{3}}\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{\alpha^3}{3}\right)\right)\right)}{(-\alpha^3)^{\frac{2}{3}}} d\alpha\right) x(-x^3)^{\frac{2}{3}}}{3\left(-x^3\right)^{\frac{2}{3}}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 57

```
dsolve(diff(y(x),x$2)-x^2*diff(y(x),x)+x*y(x)=x,y(x), singsol=all)
```

$$y(x) = -\frac{\left(-3^{\frac{1}{3}}\mathrm{e}^{\frac{x^3}{3}} c_1 - c_2 x - 1\right)\left(-x^3\right)^{\frac{2}{3}} + x^3 c_1\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{\left(-x^3\right)^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 0.286 (sec). Leaf size: 42

```
DSolve[y''[x]-x^2*y'[x]+x*y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{c_2\sqrt[3]{-x^3}\Gamma\left(-\frac{1}{3},-\frac{x^3}{3}\right)}{3\sqrt[3]{3}}+c_1x+1$$

30.2 problem Ex 2

Internal problem ID [11290]
Internal file name [OUTPUT/10275_Wednesday_December_21_2022_03_47_24_PM_65438398/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 53. Change of dependent variable. Page 125
Problem number: Ex 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' - (2x+1)\,y' + y(1+x) = x^2 - x - 1}$$

30.2.1 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + (-2x-1)\,y' + y(1+x) = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-1 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 177: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\,(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x-1}{x} \, dx} \\ &= z_1 e^{x + \frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\, e^x\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-1}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
xy'' + (-2x - 1)\, y' + y(1 + x) = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x c_1 + \frac{c_2 x^2 \mathrm{e}^x}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^x$$

$$y_2 = \frac{x^2 \mathrm{e}^x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^x & \frac{x^2 \mathrm{e}^x}{2} \\ \frac{d}{dx}(\mathrm{e}^x) & \frac{d}{dx}\left(\frac{x^2 \mathrm{e}^x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^x & \frac{x^2 \mathrm{e}^x}{2} \\ \mathrm{e}^x & x\,\mathrm{e}^x + \frac{x^2 \mathrm{e}^x}{2} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^x)\left(x\,\mathrm{e}^x + \frac{x^2 \mathrm{e}^x}{2}\right) - \left(\frac{x^2 \mathrm{e}^x}{2}\right)(\mathrm{e}^x)$$

Which simplifies to

$$W = \mathrm{e}^{2x} x$$

Which simplifies to

$$W = \mathrm{e}^{2x} x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^2 e^x (x^2 - x - 1)}{2}}{x^2 e^{2x}} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{-x}(x^2 - x - 1)}{2} dx$$

Hence

$$u_1 = \frac{x(1 + x) e^{-x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^x (x^2 - x - 1)}{x^2 e^{2x}} \, dx$$

Which simplifies to

$$u_2 = \int \frac{e^{-x}(x^2 - x - 1)}{x^2} dx$$

Hence

$$u_2 = -\frac{(x - 1) e^{-x}}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x(1 + x) e^{-x} e^x}{2} - \frac{(x - 1) e^{-x} x e^x}{2}$$

Which simplifies to

$$y_p(x) = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + \frac{c_2 x^2 e^x}{2}\right) + (x) \end{aligned}$$

Which simplifies to

$$y = e^x \left(c_1 + \frac{c_2 x^2}{2}\right) + x$$

Summary
The solution(s) found are the following

$$y = e^x \left(c_1 + \frac{c_2 x^2}{2}\right) + x \tag{1}$$

Verification of solutions

$$y = e^x \left(c_1 + \frac{c_2 x^2}{2}\right) + x$$

Verified OK.
Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x$2)-(2*x+1)*diff(y(x),x)+(x+1)*y(x)=x^2-x-1,y(x), singsol=all)
```

$$y(x) = (c_1 x^2 + c_2) e^x + x$$

✓ Solution by Mathematica

Time used: 0.275 (sec). Leaf size: 25

```
DSolve[x*y''[x]-(2*x+1)*y'[x]+(x+1)*y[x]==x^2-x-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}c_2e^x x^2 + x + c_1e^x$$

30.3 problem Ex 3

Internal problem ID [11291]
Internal file name [`OUTPUT/10276_Wednesday_December_21_2022_03_47_25_PM_19071862/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 53. Change of dependent variable. Page 125
Problem number: Ex 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''+2y'x-2y=0}$$

30.3.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$(x^2+1)\,y''+2y'x-2y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2x}{x^2+1} \\ q(x) &= -\frac{2}{x^2+1} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{2n}{x^2+1} - \frac{2}{x^2+1} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} + \frac{2x}{x^2+1}\right) v'(x) &= 0 \\ v''(x) + \left(\frac{2}{x} + \frac{2x}{x^2+1}\right) v'(x) &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \left(\frac{2}{x} + \frac{2x}{x^2+1}\right) u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u(2x^2+1)}{x\,(x^2+1)} \end{aligned}$$

Where $f(x) = -\frac{2(2x^2+1)}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{2(2x^2+1)}{x\,(x^2+1)}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{2(2x^2+1)}{x\,(x^2+1)}\, dx \\
\ln(u) &= -2\ln(x) - \ln\left(x^2+1\right) + c_1 \\
u &= \mathrm{e}^{-2\ln(x)-\ln(x^2+1)+c_1} \\
&= c_1 \mathrm{e}^{-2\ln(x)-\ln(x^2+1)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_1}{x^2\,(x^2+1)}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= c_1\left(-\frac{1}{x} - \arctan(x)\right) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(c_1\left(-\frac{1}{x} - \arctan(x)\right) + c_2\right) x \\
&= -\arctan(x)\, c_1 x + c_2 x - c_1
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(c_1\left(-\frac{1}{x} - \arctan(x)\right) + c_2\right) x \tag{1}
$$

Verification of solutions

$$
y = \left(c_1\left(-\frac{1}{x} - \arctan(x)\right) + c_2\right) x
$$

Verified OK.

30.3.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 + 1 \\ B &= 2x \\ C &= -2 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2 + 1\right)(0) + (2x)(2) + (-2)(2x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$2x(x^2+1)v''+(8x^2+4)v'=0$$

Now by applying $v'=u$ the above becomes

$$2(x^2+1)xu'(x)+(8x^2+4)u(x)=0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{2u(2x^2+1)}{x(x^2+1)}\end{aligned}$$

Where $f(x)=-\frac{2(2x^2+1)}{x(x^2+1)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= -\frac{2(2x^2+1)}{x(x^2+1)}\,dx\\ \int\frac{1}{u}\,du &= \int -\frac{2(2x^2+1)}{x(x^2+1)}\,dx\\ \ln(u) &= -2\ln(x)-\ln(x^2+1)+c_1\\ u &= \mathrm{e}^{-2\ln(x)-\ln(x^2+1)+c_1}\\ &= c_1\mathrm{e}^{-2\ln(x)-\ln(x^2+1)}\end{aligned}$$

Which simplifies to

$$u(x)=\frac{c_1}{x^2(x^2+1)}$$

The ode for v now becomes

$$\begin{aligned}v' &= u\\ &= \frac{c_1}{x^2(x^2+1)}\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}v(x) &= \int\frac{c_1}{x^2(x^2+1)}\,\mathrm{d}x\\ &= c_1\left(-\frac{1}{x}-\arctan(x)\right)+c_2\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (2x)\left(c_1\left(-\frac{1}{x} - \arctan(x)\right) + c_2\right) \\
&= -2\arctan(x)\, c_1 x + 2c_2 x - 2c_1
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -2\arctan(x)\, c_1 x + 2c_2 x - 2c_1 \tag{1}
$$

Verification of solutions

$$
y = -2\arctan(x)\, c_1 x + 2c_2 x - 2c_1
$$

Verified OK.

30.3.3 Solving using Kovacic algorithm

Writing the ode as

$$
(x^2+1)\, y'' + 2y'x - 2y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 + 1 \\
B &= 2x \\
C &= -2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 + 3}{(x^2 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 + 3$$
$$t = (x^2 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 + 3}{(x^2 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 178: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{5i}{4(x-i)} + \frac{5i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+3}{(x^2+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2+3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} + (0) \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} \\
&= \frac{x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)(1) + \left(\left(-\frac{1}{2(x-i)^2} - \frac{1}{2(x+i)^2}\right) + \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)^2 - \left(\frac{2x^2+3}{(x^2+1)^2}\right.\right.$$
$$-\frac{2(x^2+1)\, a_0}{(-x+i)^2 (x+i)^2}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right) dx} \\
&= (x)\sqrt{x^2+1} \\
&= x\sqrt{x^2+1}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1}\, dx} \\
&= z_1 e^{-\frac{\ln(x^2+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x^2+1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x^2+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{1}{x} - \arctan(x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2 \left(x \left(-\frac{1}{x} - \arctan(x)\right)\right)
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x + c_2(-x \arctan(x) - 1) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 x + c_2(-x \arctan(x) - 1)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve((1+x^2)*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x + \arctan(x)\, xc_2 + c_2$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 48

```
DSolve[(1+x^2)*y''[x]+2*x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}i(2c_1x - c_2x\log(1-ix) + c_2x\log(1+ix) + 2ic_2)$$

30.4 problem Ex 4

Internal problem ID [11292]
Internal file name [`OUTPUT/10277_Wednesday_December_21_2022_03_47_27_PM_14577643/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 53. Change of dependent variable. Page 125
Problem number: Ex 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1-x)\, y''+y'x-y=(1-x)^2}$$

30.4.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=1-x, B=x, C=-1, f(x)=(x-1)^2$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$(1-x)\, y''+y'x-y=0$$

In normal form the ode

$$(1-x)\,y''+y'x-y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=-\frac{x}{x-1}\\ q(x)&=\frac{1}{x-1}\end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right)v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right)v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}-\frac{n}{x-1}+\frac{1}{x-1}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(x)+\left(\frac{2}{x}-\frac{x}{x-1}\right)v'(x)&=0\\ v''(x)+\left(\frac{2}{x}-\frac{x}{x-1}\right)v'(x)&=0\end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right) u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(x^2-2x+2)}{x\,(x-1)} \end{aligned}$$

Where $f(x) = \frac{x^2-2x+2}{x(x-1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= \frac{x^2-2x+2}{x\,(x-1)}\,dx \\ \int \frac{1}{u}\,du &= \int \frac{x^2-2x+2}{x\,(x-1)}\,dx \\ \ln(u) &= x + \ln(x-1) - 2\ln(x) + c_1 \\ u &= e^{x+\ln(x-1)-2\ln(x)+c_1} \\ &= c_1 e^{x+\ln(x-1)-2\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = c_1\left(\frac{e^x}{x} - \frac{e^x}{x^2}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= \frac{e^x c_1}{x} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(\frac{e^x c_1}{x} + c_2\right) x \\ &= e^x c_1 + c_2 x \end{aligned}$$

Now the particular solution to this ODE is found

$$(1-x)\,y''+y'x-y=(x-1)^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1=x$$
$$y_2=\mathrm{e}^x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1=-\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2=\int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W=\begin{vmatrix} x & \mathrm{e}^x \\ \frac{d}{dx}(x) & \frac{d}{dx}(\mathrm{e}^x) \end{vmatrix}$$

Which gives

$$W=\begin{vmatrix} x & \mathrm{e}^x \\ 1 & \mathrm{e}^x \end{vmatrix}$$

Therefore

$$W=(x)\,(\mathrm{e}^x)-(\mathrm{e}^x)\,(1)$$

Which simplifies to

$$W = x\,\mathrm{e}^{x} - \mathrm{e}^{x}$$

Which simplifies to

$$W = \mathrm{e}^{x}(x - 1)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{x}(x-1)^2}{(1-x)\,\mathrm{e}^{x}\,(x-1)}\,dx$$

Which simplifies to

$$u_1 = -\int (-1)\,dx$$

Hence

$$u_1 = x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x-1)^2}{(1-x)\,\mathrm{e}^{x}\,(x-1)}\,dx$$

Which simplifies to

$$u_2 = \int -x\,\mathrm{e}^{-x}dx$$

Hence

$$u_2 = (1 + x)\,\mathrm{e}^{-x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x^2 + (1 + x)\,\mathrm{e}^{-x}\mathrm{e}^{x}$$

Which simplifies to

$$y_p(x) = x^2 + x + 1$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(\frac{e^x c_1}{x} + c_2\right) x\right) + (x^2 + x + 1) \\
&= x^2 + x + 1 + \left(\frac{e^x c_1}{x} + c_2\right) x
\end{aligned}
$$

Which simplifies to

$$
y = e^x c_1 + c_2 x + x^2 + x + 1
$$

Summary

The solution(s) found are the following

$$
y = e^x c_1 + c_2 x + x^2 + x + 1 \tag{1}
$$

Verification of solutions

$$
y = e^x c_1 + c_2 x + x^2 + x + 1
$$

Verified OK.

30.4.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + (2AB' + B^2)\, v' + (AB'' + BB' + CB)\, v &= 0
\end{aligned} \tag{1}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)\, v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)\, u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= 1 - x \\
B &= x \\
C &= -1 \\
F &= (x-1)^2
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (1-x)\,(0) + (x)\,(1) + (-1)\,(x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x(x-1)\, v'' + (x^2 - 2x + 2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$(-x^2 + x)\, u'(x) + (x^2 - 2x + 2)\, u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u(x^2 - 2x + 2)}{x\,(x-1)}
\end{aligned}$$

Where $f(x) = \frac{x^2-2x+2}{x(x-1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{x^2-2x+2}{x(x-1)}\,dx\\
\int \frac{1}{u}\,du &= \int \frac{x^2-2x+2}{x(x-1)}\,dx\\
\ln(u) &= x+\ln(x-1)-2\ln(x)+c_1\\
u &= \mathrm{e}^{x+\ln(x-1)-2\ln(x)+c_1}\\
&= c_1\mathrm{e}^{x+\ln(x-1)-2\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_1\left(\frac{\mathrm{e}^x}{x}-\frac{\mathrm{e}^x}{x^2}\right)
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u\\
&= c_1\left(\frac{\mathrm{e}^x}{x}-\frac{\mathrm{e}^x}{x^2}\right)
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1\mathrm{e}^x(x-1)}{x^2}\,\mathrm{d}x\\
&= \frac{\mathrm{e}^x c_1}{x}+c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv\\
&= (x)\left(\frac{\mathrm{e}^x c_1}{x}+c_2\right)\\
&= \mathrm{e}^x c_1+c_2 x
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1+u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = e^x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & e^x \\ \frac{d}{dx}(x) & \frac{d}{dx}(e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix}$$

Therefore

$$W = (x)(e^x) - (e^x)(1)$$

Which simplifies to

$$W = x e^x - e^x$$

Which simplifies to

$$W = e^x(x - 1)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^x (x-1)^2}{(1-x) e^x (x-1)} \, dx$$

Which simplifies to

$$u_1 = -\int (-1)\, dx$$

Hence

$$u_1 = x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x-1)^2}{(1-x)\,\mathrm{e}^x\,(x-1)}\, dx$$

Which simplifies to

$$u_2 = \int -x\,\mathrm{e}^{-x} dx$$

Hence

$$u_2 = (1+x)\,\mathrm{e}^{-x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x^2 + (1+x)\,\mathrm{e}^{-x}\mathrm{e}^x$$

Which simplifies to

$$y_p(x) = x^2 + x + 1$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (\mathrm{e}^x c_1 + c_2 x) + \left(x^2 + x + 1\right) \\ &= \mathrm{e}^x c_1 + c_2 x + x^2 + x + 1 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^x c_1 + c_2 x + x^2 + x + 1 \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^x c_1 + c_2 x + x^2 + x + 1$$

Verified OK.

30.4.3 Solving using Kovacic algorithm

Writing the ode as

$$(1-x)\,y''+y'x-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x+6}{4\,(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-4x+6 \\ t &= 4(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x+6}{4\,(x-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y=z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 179: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(x-1)} + \frac{3}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\,(x-1)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2\,(x-1)} + \frac{1}{2} \\ &= \frac{x-2}{2x-2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2\left(x-1\right)}+\frac{1}{2}\right)(0)+\left(\left(\frac{1}{2\left(x-1\right)^{2}}\right)+\left(-\frac{1}{2\left(x-1\right)}+\frac{1}{2}\right)^{2}-\left(\frac{x^{2}-4x+6}{4\left(x-1\right)^{2}}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1-x}\, dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x-1)}{2}} \\
&= z_1\left(\sqrt{x-1}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1-x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1\left(-x\,\mathrm{e}^{-x}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-x\,\mathrm{e}^{-x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
(1-x)\,y'' + y'x - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x c_1 - c_2 x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^x \\
y_2 &= -x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & -x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(-x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & -x \\ e^x & -1 \end{vmatrix}$$

Therefore

$$W = (e^x)(-1) - (-x)(e^x)$$

Which simplifies to

$$W = x\,e^x - e^x$$

Which simplifies to

$$W = e^x(x-1)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-x(x-1)^2}{(1-x)\,e^x\,(x-1)}\,dx$$

Which simplifies to

$$u_1 = -\int x\,e^{-x}dx$$

Hence

$$u_1 = (1+x)\,e^{-x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^x(x-1)^2}{(1-x)\,e^x\,(x-1)}\,dx$$

Which simplifies to

$$u_2 = \int (-1)\, dx$$

Hence

$$u_2 = -x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x^2 + (1 + x)\, \mathrm{e}^{-x} \mathrm{e}^{x}$$

Which simplifies to

$$y_p(x) = x^2 + x + 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\mathrm{e}^{x} c_1 - c_2 x) + \left(x^2 + x + 1\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{x} c_1 - c_2 x + x^2 + x + 1 \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{x} c_1 - c_2 x + x^2 + x + 1$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((1-x)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=(1-x)^2,y(x), singsol=all)
```

$$y(x) = c_2x + c_1\mathrm{e}^x + x^2 + 1$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 22

```
DSolve[(1-x)*y''[x]+x*y'[x]-y[x]==(1-x)^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + x - c_2x + c_1e^x + 1$$

30.5 problem Ex 5

Internal problem ID [11293]
Internal file name [OUTPUT/10278_Wednesday_December_21_2022_03_47_29_PM_61425486/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 53. Change of dependent variable. Page 125
Problem number: Ex 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{\sin(x)\, y'' + 2\cos(x)\, y' + 3\sin(x)\, y = \mathrm{e}^{x}}$$

30.5.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$\sin(x)\, y'' + 2\cos(x)\, y' + 3\sin(x)\, y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2\cos(x)}{\sin(x)} \\ q(x) &= 3 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= 3 - \frac{\left(\frac{2\cos(x)}{\sin(x)}\right)'}{2} - \frac{\left(\frac{2\cos(x)}{\sin(x)}\right)^2}{4} \\ &= 3 - \frac{\left(-2 - \frac{2\cos(x)^2}{\sin(x)^2}\right)}{2} - \frac{\left(\frac{4\cos(x)^2}{\sin(x)^2}\right)}{4} \\ &= 3 - \left(-1 - \frac{\cos(x)^2}{\sin(x)^2}\right) - \frac{\cos(x)^2}{\sin(x)^2} \\ &= 4 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{\frac{2\cos(x)}{\sin(x)}}{2}} \\ &= \csc(x) \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\csc(x) \tag{4}$$

Applying this change of variable to the original ode results in

$$4v(x) + v''(x) = \mathrm{e}^{x}$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \mathrm{e}^{x}$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$4v(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$v(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$v_p = A_1 \mathrm{e}^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 \mathrm{e}^x = \mathrm{e}^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = \frac{e^x}{5}$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{e^x}{5}\right) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 \cos(2x) + c_2 \sin(2x) + \frac{e^x}{5}\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \csc(x)$$

Hence (7) becomes

$$y = \left(c_1 \cos(2x) + c_2 \sin(2x) + \frac{e^x}{5}\right) \csc(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 \cos(2x) + c_2 \sin(2x) + \frac{e^x}{5}\right) \csc(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)\csc(x)$$
$$y_2 = \sin(2x)\csc(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x)\csc(x) & \sin(2x)\csc(x) \\ \frac{d}{dx}(\cos(2x)\csc(x)) & \frac{d}{dx}(\sin(2x)\csc(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x)\csc(x) & \sin(2x)\csc(x) \\ -2\sin(2x)\csc(x) - \cos(2x)\csc(x)\cot(x) & 2\cos(2x)\csc(x) - \sin(2x)\csc(x)\cot(x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x)\csc(x))(2\cos(2x)\csc(x) - \sin(2x)\csc(x)\cot(x)) - (\sin(2x)\csc(x))(-2\sin(2x)\csc(x) - \cos(2x)\csc(x)\cot(x))$$

Which simplifies to

$$W = 2\csc(x)^2\cos(2x)^2 + 2\csc(x)^2\sin(2x)^2$$

Which simplifies to

$$W = 2\csc(x)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(2x)\csc(x)\,e^x}{2\sin(x)\csc(x)^2}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^x \sin(2x)}{2}dx$$

Hence

$$u_1 = \frac{e^x \cos(2x)}{5} - \frac{e^x \sin(2x)}{10}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\csc(x)\,e^x}{2\sin(x)\csc(x)^2}\,dx$$

Which simplifies to

$$u_2 = \int \frac{e^x \cos(2x)}{2}dx$$

Hence

$$u_2 = \frac{e^x \cos(2x)}{10} + \frac{e^x \sin(2x)}{5}$$

Which simplifies to

$$u_1 = \frac{e^x(2\cos(2x) - \sin(2x))}{10}$$

$$u_2 = \frac{e^x(\cos(2x) + 2\sin(2x))}{10}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^x(2\cos(2x) - \sin(2x))\cos(2x)\csc(x)}{10} + \frac{e^x(\cos(2x) + 2\sin(2x))\sin(2x)\csc(x)}{10}$$

Which simplifies to

$$y_p(x) = \frac{\csc(x)\,e^x}{5}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(c_1 \cos(2x) + c_2 \sin(2x) + \frac{e^x}{5}\right) \csc(x)\right) + \left(\frac{\csc(x)\, e^x}{5}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(c_1 \cos(2x) + c_2 \sin(2x) + \frac{e^x}{5}\right) \csc(x) + \frac{\csc(x)\, e^x}{5} \tag{1}
$$

Verification of solutions

$$
y = \left(c_1 \cos(2x) + c_2 \sin(2x) + \frac{e^x}{5}\right) \csc(x) + \frac{\csc(x)\, e^x}{5}
$$

Verified OK.

30.5.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0
\end{aligned} \tag{1}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$AB u' + (2AB' + B^2)\, u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= \sin(x) \\
B &= 2\cos(x) \\
C &= 3\sin(x) \\
F &= e^{x}
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (\sin(x))\,(-2\cos(x)) + (2\cos(x))\,(-2\sin(x)) + (3\sin(x))\,(2\cos(x)) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$\sin(2x)\, v'' + (4\cos(2x))\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$4u(x)\cos(2x) + u'(x)\sin(2x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= -\frac{4u\cos(2x)}{\sin(2x)}
\end{aligned}$$

Where $f(x) = -\frac{4\cos(2x)}{\sin(2x)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{4\cos(2x)}{\sin(2x)}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{4\cos(2x)}{\sin(2x)}\, dx \\
\ln(u) &= -2\ln(\sin(2x)) + c_1 \\
u &= e^{-2\ln(\sin(2x)) + c_1} \\
&= \frac{c_1}{\sin(2x)^2}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{\sin(2x)^2} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{\sin(2x)^2}\, dx \\ &= -\frac{c_1 \cot(2x)}{2} + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (2\cos(x)) \left(-\frac{c_1 \cot(2x)}{2} + c_2\right) \\ &= (-\cot(x)\, c_1 + 2c_2) \cos(x) + \frac{\csc(x)\, c_1}{2} \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= -\cos(x)\cot(x) + \frac{\csc(x)}{2} \\ y_2 &= \cos(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -\cos(x)\cot(x) + \frac{\csc(x)}{2} & \cos(x) \\ \frac{d}{dx}\left(-\cos(x)\cot(x) + \frac{\csc(x)}{2}\right) & \frac{d}{dx}(\cos(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -\cos(x)\cot(x) + \frac{\csc(x)}{2} & \cos(x) \\ \sin(x)\cot(x) - \cos(x)\left(-1-\cot(x)^2\right) - \frac{\csc(x)\cot(x)}{2} & -\sin(x) \end{vmatrix}$$

Therefore

$$W = \left(-\cos(x)\cot(x) + \frac{\csc(x)}{2}\right)(-\sin(x)) - (\cos(x))\left(\sin(x)\cot(x) - \cos(x)\left(-1-\cot(x)^2\right) - \frac{\csc(x)\cot(x)}{2}\right)$$

Which simplifies to

$$W = -\frac{\csc(x)\sin(x)}{2} - \cos(x)^2\cot(x)^2 - \cos(x)^2 + \frac{\cos(x)\csc(x)\cot(x)}{2}$$

Which simplifies to

$$W = -\frac{\csc(x)^2}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^x\cos(x)}{-\frac{\sin(x)\csc(x)^2}{2}}\,dx$$

Which simplifies to

$$u_1 = -\int -e^x\sin(2x)\,dx$$

Hence

$$u_1 = -\frac{2\,e^x\cos(2x)}{5} + \frac{e^x\sin(2x)}{5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(-\cos\left(x\right)\cot\left(x\right)+\frac{\csc(x)}{2}\right)\mathrm{e}^{x}}{-\frac{\sin(x)\csc(x)^2}{2}}\,dx$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{x}\cos\left(2x\right)dx$$

Hence

$$u_2 = \frac{\mathrm{e}^{x}\cos\left(2x\right)}{5}+\frac{2\,\mathrm{e}^{x}\sin\left(2x\right)}{5}$$

Which simplifies to

$$u_1 = -\frac{\mathrm{e}^{x}\left(2\cos\left(2x\right)-\sin\left(2x\right)\right)}{5}$$
$$u_2 = \frac{\mathrm{e}^{x}\left(\cos\left(2x\right)+2\sin\left(2x\right)\right)}{5}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\mathrm{e}^{x}\left(2\cos\left(2x\right)-\sin\left(2x\right)\right)\left(-\cos\left(x\right)\cot\left(x\right)+\frac{\csc(x)}{2}\right)}{5} + \frac{\mathrm{e}^{x}\left(\cos\left(2x\right)+2\sin\left(2x\right)\right)\cos\left(x\right)}{5}$$

Which simplifies to

$$y_p(x) = \frac{\csc\left(x\right)\mathrm{e}^{x}}{5}$$

Hence the complete solution is

$$\begin{aligned}
y(x) &= y_h + y_p \\
&= \left(\left(-\cot\left(x\right)c_1+2c_2\right)\cos\left(x\right)+\frac{\csc\left(x\right)c_1}{2}\right)+\left(\frac{\csc\left(x\right)\mathrm{e}^{x}}{5}\right) \\
&= \frac{\csc\left(x\right)\left(20\cos\left(x\right)\sin\left(x\right)c_2-10c_1\cos\left(x\right)^2+2\,\mathrm{e}^{x}+5c_1\right)}{10}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\csc(x)\left(20\cos(x)\sin(x)\,c_2 - 10c_1\cos(x)^2 + 2\,\mathrm{e}^x + 5c_1\right)}{10} \tag{1}$$

Verification of solutions

$$y = \frac{\csc(x)\left(20\cos(x)\sin(x)\,c_2 - 10c_1\cos(x)^2 + 2\,\mathrm{e}^x + 5c_1\right)}{10}$$

Verified OK.

30.5.3 Solving using Kovacic algorithm

Writing the ode as

$$\sin(x)\,y'' + 2\cos(x)\,y' + 3\sin(x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= \sin(x) \\ B &= 2\cos(x) \\ C &= 3\sin(x) \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 180: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2\cos(x)}{\sin(x)}\,dx} \\
&= z_1 e^{-\ln(\sin(x))} \\
&= z_1(\csc(x))
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)\csc(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2\cos(x)}{\sin(x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(\sin(x))}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\tan(2x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2x)\csc(x)) + c_2\left(\cos(2x)\csc(x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$\sin(x)\, y'' + 2\cos(x)\, y' + 3\sin(x)\, y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \csc(x)\, c_1 \cos(2x) + \cos(x)\, c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)\csc(x)$$
$$y_2 = \cos(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x)\csc(x) & \cos(x) \\ \frac{d}{dx}(\cos(2x)\csc(x)) & \frac{d}{dx}(\cos(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x)\csc(x) & \cos(x) \\ -2\sin(2x)\csc(x) - \cos(2x)\csc(x)\cot(x) & -\sin(x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x)\csc(x))(-\sin(x)) - (\cos(x))(-2\sin(2x)\csc(x) - \cos(2x)\csc(x)\cot(x))$$

Which simplifies to

$$W = \cos(x)\csc(x)\cot(x)\cos(2x) - \cos(2x)\csc(x)\sin(x) + 2\cos(x)\csc(x)\sin(2x)$$

Which simplifies to

$$W = \csc(x)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^x \cos(x)}{\sin(x)\csc(x)^2}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^x \sin(2x)}{2}dx$$

Hence

$$u_1 = \frac{e^x \cos(2x)}{5} - \frac{e^x \sin(2x)}{10}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\csc(x)\,e^x}{\sin(x)\csc(x)^2}\,dx$$

Which simplifies to

$$u_2 = \int e^x \cos(2x)\,dx$$

Hence

$$u_2 = \frac{e^x \cos(2x)}{5} + \frac{2\,e^x \sin(2x)}{5}$$

Which simplifies to

$$u_1 = \frac{e^x(2\cos(2x) - \sin(2x))}{10}$$
$$u_2 = \frac{e^x(\cos(2x) + 2\sin(2x))}{5}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^x(2\cos(2x) - \sin(2x))\cos(2x)\csc(x)}{10} + \frac{e^x(\cos(2x) + 2\sin(2x))\cos(x)}{5}$$

Which simplifies to

$$y_p(x) = \frac{\csc(x)\,e^x}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\csc(x)\,c_1\cos(2x) + \cos(x)\,c_2) + \left(\frac{\csc(x)\,e^x}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \csc(x)\,c_1\cos(2x) + \cos(x)\,c_2 + \frac{\csc(x)\,e^x}{5} \tag{1}$$

Verification of solutions

$$y = \csc(x)\,c_1\cos(2x) + \cos(x)\,c_2 + \frac{\csc(x)\,e^x}{5}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve(sin(x)*diff(y(x),x$2)+2*cos(x)*diff(y(x),x)+3*sin(x)*y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = \frac{\csc(x)\left(10\cos(x)^2 c_1 + 10\sin(x)\cos(x)c_2 + e^x - 5c_1\right)}{5}$$

✓ Solution by Mathematica

Time used: 0.229 (sec). Leaf size: 56

```
DSolve[Sin[x]*y''[x]+2*Cos[x]*y'[x]+3*Sin[x]*y[x]==Exp[x],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{e^{-ix}\left(4ie^{(1+2i)x} + 5c_2e^{4ix} + 20ic_1\right)}{10\left(-1+e^{2ix}\right)}$$

30.6 problem Ex 6

Internal problem ID [11294]
Internal file name [`OUTPUT/10279_Wednesday_December_21_2022_03_47_31_PM_31216664/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 53. Change of dependent variable. Page 125
Problem number: Ex 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y' \tan(x) - (a^2 + 1)\, y = 0}$$

30.6.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -2\tan(x) \\ q(x) &= -a^2 - 1 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= -a^2 - 1 - \frac{(-2\tan(x))'}{2} - \frac{(-2\tan(x))^2}{4} \\
&= -a^2 - 1 - \frac{(-2\tan(x)^2 - 2)}{2} - \frac{(4\tan(x)^2)}{4} \\
&= -a^2 - 1 - (-\tan(x)^2 - 1) - \tan(x)^2 \\
&= -a^2
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-2\tan(x)}{2}} \\
&= \sec(x) \qquad (5)
\end{aligned}
$$

Hence (3) becomes

$$
y = v(x)\sec(x) \tag{4}
$$

Applying this change of variable to the original ode results in

$$
-\sec(x)\left(v(x)\, a^2 - v''(x)\right) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = -1, B = 0, C = a^2$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
-\lambda^2 \mathrm{e}^{\lambda x} + a^2 \mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
a^2 - \lambda^2 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = -1, B = 0, C = a^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(-1)} \pm \frac{1}{(2)\,(-1)}\sqrt{0^2 - (4)\,(-1)\,(a^2)}\\ &= \pm - \sqrt{a^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= + - \sqrt{a^2}\\ \lambda_2 &= - - \sqrt{a^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\sqrt{a^2}\\ \lambda_2 &= \sqrt{a^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ v(x) &= c_1 e^{\left(-\sqrt{a^2}\right)x} + c_2 e^{\left(\sqrt{a^2}\right)x}\end{aligned}$$

Or

$$v(x) = c_1 \mathrm{e}^{-\sqrt{a^2}\, x} + c_2 \mathrm{e}^{\sqrt{a^2}\, x}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x)\\ &= \left(c_1 \mathrm{e}^{-\sqrt{a^2}\, x} + c_2 \mathrm{e}^{\sqrt{a^2}\, x}\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \sec(x)$$

Hence (7) becomes

$$y = \left(c_1 \mathrm{e}^{-\sqrt{a^2}\, x} + c_2 \mathrm{e}^{\sqrt{a^2}\, x}\right) \sec(x)$$

Summary
The solution(s) found are the following

$$y = \left(c_1 \mathrm{e}^{-\sqrt{a^2}\, x} + c_2 \mathrm{e}^{\sqrt{a^2}\, x}\right) \sec(x) \tag{1}$$

Verification of solutions

$$y = \left(c_1 \mathrm{e}^{-\sqrt{a^2}\, x} + c_2 \mathrm{e}^{\sqrt{a^2}\, x}\right) \sec(x)$$

Verified OK.

30.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' \tan(x) + \left(-a^2 - 1\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2\tan(x) \\ C &= -a^2 - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (a^2)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 181: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{a^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2\tan(x)}{1}\,dx} \\
&= z_1 e^{-\ln(\cos(x))} \\
&= z_1(\sec(x))
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\operatorname{csgn}(a)ax}\sec(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2\tan(x)}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(\cos(x))}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\operatorname{csgn}(a)\,\mathrm{e}^{-2\operatorname{csgn}(a)ax}}{2a}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\operatorname{csgn}(a)ax}\sec(x)\right) + c_2\left(\mathrm{e}^{\operatorname{csgn}(a)ax}\sec(x)\left(-\frac{\operatorname{csgn}(a)\,\mathrm{e}^{-2\operatorname{csgn}(a)ax}}{2a}\right)\right)
\end{aligned}$$

Simplifying the solution $y = c_1\mathrm{e}^{\operatorname{csgn}(a)ax}\sec(x) - \frac{c_2\operatorname{csgn}(a)\sec(x)\mathrm{e}^{-\operatorname{csgn}(a)ax}}{2a}$ to $y = c_1\mathrm{e}^{ax}\sec(x) - \frac{c_2\sec(x)\mathrm{e}^{-ax}}{2a}$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{ax}\sec(x) - \frac{c_2\sec(x)\,\mathrm{e}^{-ax}}{2a} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{ax} \sec(x) - \frac{c_2 \sec(x) \mathrm{e}^{-ax}}{2a}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)-2*tan(x)*diff(y(x),x)-(a^2+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sec(x)\,(c_1 \sinh(ax) + c_2 \cosh(ax))$$

✓ Solution by Mathematica

Time used: 0.117 (sec). Leaf size: 32

```
DSolve[y''[x]-2*Tan[x]*y'[x]-(a^2+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sec(x)\left(c_1 e^{-ax} + \frac{c_2 e^{ax}}{2a}\right)$$

30.7 problem Ex 7

Internal problem ID [11295]
Internal file name [OUTPUT/10280_Wednesday_December_21_2022_03_47_33_PM_71027106/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 53. Change of dependent variable. Page 125
Problem number: Ex 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{4x^2y'' + 4y'x^3 + (x^2+1)\,y = 0}$$

30.7.1 Maple step by step solution

Let's solve

$4y''x^2 + 4y'x^3 + (x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y'x - \frac{(x^2+1)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x + \frac{(x^2+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = x, P_3(x) = \frac{x^2+1}{4x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4y''x^2 + 4y'x^3 + (x^2+1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^3 \cdot y'$ to series expansion

  $x^3 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+2}$

- Shift index using $k- >k-2$

  $x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2}(k-2+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+2r)^2 x^r + a_1(1+2r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-2}(4k-7+4r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$

- Each term must be 0

  $a_1(1+2r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r-1)^2 + a_{k-2}(4k-7+4r) = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(2k+3+2r)^2 + a_k(4k+4r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(4k+4r+1)}{(2k+3+2r)^2}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k(4k+3)}{(2k+4)^2}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k(4k+3)}{(2k+4)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 35

```
dsolve(4*x^2*diff(y(x),x$2)+4*x^3*diff(y(x),x)+(x^2+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{-\frac{x^2}{4}} \left(\text{WhittakerM}\left(-\frac{1}{8}, 0, \frac{x^2}{2}\right) c_1 + \text{WhittakerW}\left(-\frac{1}{8}, 0, \frac{x^2}{2}\right) c_2\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.239 (sec). Leaf size: 60

```
DSolve[4*x^2*y''[x]+4*x^3*y'[x]+(x^2+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 G_{1,2}^{2,0}\left(\frac{x^2}{16} \middle| \begin{array}{c} \frac{7}{8} \\ \frac{1}{4}, \frac{1}{4} \end{array}\right) + \frac{1}{2}\sqrt[4]{-1} c_1 \sqrt{x}\, \text{Hypergeometric1F1}\left(\frac{3}{8}, 1, -\frac{x^2}{16}\right)$$

30.8 problem Ex 8

Internal problem ID [11296]
Internal file name [`OUTPUT/10281_Wednesday_December_21_2022_03_47_34_PM_35017132/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 53. Change of dependent variable. Page 125
Problem number: Ex 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{xy'' + 2y' - yx = 2\,\mathrm{e}^{x}}$$

30.8.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$xy'' + 2y' - yx = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= -1
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= -1 - \frac{\left(\frac{2}{x}\right)'}{2} - \frac{\left(\frac{2}{x}\right)^2}{4} \\
&= -1 - \frac{\left(-\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\
&= -1 - \left(-\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= -1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{2}{x}}{2}} \\
&= \frac{1}{x}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = \frac{v(x)}{x} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(x) - v(x) = 2\,\mathrm{e}^{x}
$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = f(x)
$$

Where $A = 1, B = 0, C = -1, f(x) = 2\,\mathrm{e}^{x}$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$v''(x) - v(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$v(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$v(x) = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$v(x) = e^{x} c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution v_h is

$$v_h = e^{x} c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$v_p = A_1 x\,e^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{x} = 2\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = x\,e^{x}$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= \left(e^x c_1 + c_2 e^{-x}\right) + \left(x\, e^x\right) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(e^x c_1 + c_2 e^{-x} + x\, e^x\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x}$$

Hence (7) becomes

$$y = \frac{e^x c_1 + c_2 e^{-x} + x\, e^x}{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{e^x c_1 + c_2 e^{-x} + x\, e^x}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \frac{e^x}{x} \\ y_2 &= \frac{e^{-x}}{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{e^x}{x} & \frac{e^{-x}}{x} \\ \frac{d}{dx}\left(\frac{e^x}{x}\right) & \frac{d}{dx}\left(\frac{e^{-x}}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{e^x}{x} & \frac{e^{-x}}{x} \\ \frac{e^x}{x} - \frac{e^x}{x^2} & -\frac{e^{-x}}{x^2} - \frac{e^{-x}}{x} \end{vmatrix}$$

Therefore

$$W = \left(\frac{e^x}{x}\right)\left(-\frac{e^{-x}}{x^2} - \frac{e^{-x}}{x}\right) - \left(\frac{e^{-x}}{x}\right)\left(\frac{e^x}{x} - \frac{e^x}{x^2}\right)$$

Which simplifies to

$$W = -\frac{2\,e^x e^{-x}}{x^2}$$

Which simplifies to

$$W = -\frac{2}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2\,e^{-x} e^x}{x}}{-\frac{2}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int (-1)\,dx$$

Hence

$$u_1 = x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2\,\mathrm{e}^{2x}}{x}}{-\frac{2}{x}}\,dx$$

Which simplifies to

$$u_2 = \int -\mathrm{e}^{2x}\,dx$$

Hence

$$u_2 = -\frac{\mathrm{e}^{2x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \mathrm{e}^{x} - \frac{\mathrm{e}^{2x}\mathrm{e}^{-x}}{2x}$$

Which simplifies to

$$y_p(x) = \frac{\mathrm{e}^{x}(2x-1)}{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\mathrm{e}^{x}c_1 + c_2\mathrm{e}^{-x} + x\,\mathrm{e}^{x}}{x}\right) + \left(\frac{\mathrm{e}^{x}(2x-1)}{2x}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_2\mathrm{e}^{-x} + \mathrm{e}^{x}(x+c_1)}{x} + \frac{\mathrm{e}^{x}(2x-1)}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2\mathrm{e}^{-x} + \mathrm{e}^{x}(x+c_1)}{x} + \frac{\mathrm{e}^{x}(2x-1)}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{c_2 e^{-x} + e^x (x + c_1)}{x} + \frac{e^x (2x - 1)}{2x}$$

Verified OK.

30.8.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + 2y' x - x^2 y = 2x\, e^x \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + y' x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= -\frac{1}{2} \\ \beta &= i \\ n &= \frac{1}{2} \\ \gamma &= 1 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{i c_1 \sqrt{2}\, \sinh(x)}{\sqrt{x}\, \sqrt{\pi}\, \sqrt{ix}} - \frac{c_2 \sqrt{2}\, \cosh(x)}{\sqrt{x}\, \sqrt{\pi}\, \sqrt{ix}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{i c_1 \sqrt{2}\, \sinh(x)}{\sqrt{x}\, \sqrt{\pi}\, \sqrt{ix}} - \frac{c_2 \sqrt{2}\, \cosh(x)}{\sqrt{x}\, \sqrt{\pi}\, \sqrt{ix}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{e^x}{x}$$

$$y_2 = \frac{e^{-x}}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{e^x}{x} & \frac{e^{-x}}{x} \\ \frac{d}{dx}\left(\frac{e^x}{x}\right) & \frac{d}{dx}\left(\frac{e^{-x}}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{e^x}{x} & \frac{e^{-x}}{x} \\ \frac{e^x}{x} - \frac{e^x}{x^2} & -\frac{e^{-x}}{x^2} - \frac{e^{-x}}{x} \end{vmatrix}$$

Therefore

$$W = \left(\frac{e^x}{x}\right)\left(-\frac{e^{-x}}{x^2} - \frac{e^{-x}}{x}\right) - \left(\frac{e^{-x}}{x}\right)\left(\frac{e^x}{x} - \frac{e^x}{x^2}\right)$$

Which simplifies to

$$W = -\frac{2\,e^{x} e^{-x}}{x^2}$$

Which simplifies to

$$W = -\frac{2}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,e^{x} e^{-x}}{-2}\,dx$$

Which simplifies to

$$u_1 = -\int (-1)\,dx$$

Hence

$$u_1 = x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,e^{2x}}{-2}\,dx$$

Which simplifies to

$$u_2 = \int -e^{2x} dx$$

Hence

$$u_2 = -\frac{e^{2x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = e^{x} - \frac{e^{2x} e^{-x}}{2x}$$

Which simplifies to

$$y_p(x) = \frac{e^{x}(2x-1)}{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{i c_1 \sqrt{2}\, \sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}} - \frac{c_2 \sqrt{2}\, \cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}}\right) + \left(\frac{e^{x}(2x-1)}{2x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{i c_1 \sqrt{2}\, \sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}} - \frac{c_2 \sqrt{2}\, \cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}} + \frac{e^{x}(2x-1)}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{i c_1 \sqrt{2}\, \sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}} - \frac{c_2 \sqrt{2}\, \cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}} + \frac{e^{x}(2x-1)}{2x}$$

Verified OK.

30.8.3 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + 2y' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 183: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{e^{-x}}{x}\right) + c_2\left(\frac{e^{-x}}{x}\left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
xy'' + 2y' - yx = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1 e^{-x}}{x} + \frac{c_2 e^{x}}{2x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{e^{-x}}{x} \\
y_2 &= \frac{e^{x}}{2x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{e^{-x}}{x} & \frac{e^{x}}{2x} \\ \frac{d}{dx}\left(\frac{e^{-x}}{x}\right) & \frac{d}{dx}\left(\frac{e^{x}}{2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{e^{-x}}{x} & \frac{e^{x}}{2x} \\ -\frac{e^{-x}}{x^2} - \frac{e^{-x}}{x} & \frac{e^{x}}{2x} - \frac{e^{x}}{2x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{e^{-x}}{x}\right)\left(\frac{e^{x}}{2x} - \frac{e^{x}}{2x^2}\right) - \left(\frac{e^{x}}{2x}\right)\left(-\frac{e^{-x}}{x^2} - \frac{e^{-x}}{x}\right)$$

Which simplifies to

$$W = \frac{e^{x} e^{-x}}{x^2}$$

Which simplifies to

$$W = \frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{2x}}{x}}{\frac{1}{x}}\, dx$$

Which simplifies to

$$u_1 = -\int e^{2x} dx$$

Hence

$$u_1 = -\frac{e^{2x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2\, e^{-x} e^{x}}{x}}{\frac{1}{x}}\, dx$$

Which simplifies to

$$u_2 = \int 2dx$$

Hence

$$u_2 = 2x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \mathrm{e}^{x} - \frac{\mathrm{e}^{2x}\mathrm{e}^{-x}}{2x}$$

Which simplifies to

$$y_p(x) = \frac{\mathrm{e}^{x}(2x-1)}{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1\mathrm{e}^{-x}}{x} + \frac{c_2\mathrm{e}^{x}}{2x}\right) + \left(\frac{\mathrm{e}^{x}(2x-1)}{2x}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_2\mathrm{e}^{x} + 2c_1\mathrm{e}^{-x}}{2x} + \frac{\mathrm{e}^{x}(2x-1)}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2\mathrm{e}^{x} + 2c_1\mathrm{e}^{-x}}{2x} + \frac{\mathrm{e}^{x}(2x-1)}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{c_2\mathrm{e}^{x} + 2c_1\mathrm{e}^{-x}}{2x} + \frac{\mathrm{e}^{x}(2x-1)}{2x}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)-x*y(x)=2*exp(x),y(x), singsol=all)
```

$$y(x) = \frac{e^x x + \sinh(x) c_2 + \cosh(x) c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 35

```
DSolve[x*y''[x]+2*y'[x]-x*y[x]==2*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x}(e^{2x}(2x - 1 + c_2) + 2c_1)}{2x}$$

31 Chapter VIII, Linear differential equations of the second order. Article 54. Change of independent variable. Page 127

31.1 problem Ex 1

Internal problem ID [11297]
Internal file name [`OUTPUT/10282_Wednesday_December_21_2022_03_47_36_PM_84888588/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 54. Change of independent variable. Page 127
Problem number: Ex 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + (2\,\mathrm{e}^{x} - 1)\,y' + y\,\mathrm{e}^{2x} = \mathrm{e}^{4x}}$$

31.1.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + (2\,\mathrm{e}^{x} - 1)\,y' + y\,\mathrm{e}^{2x} = 0$$

In normal form the ode

$$y'' + (2\,\mathrm{e}^{x} - 1)\,y' + y\,\mathrm{e}^{2x} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= 2\,\mathrm{e}^{x} - 1 \\ q(x) &= \mathrm{e}^{2x} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int (2\,\mathrm{e}^{x}-1)dx\right)}dx \\ &= \int e^{x-2\,\mathrm{e}^{x}}\,dx \\ &= \int \mathrm{e}^{x-2\,\mathrm{e}^{x}}dx \\ &= -\frac{\mathrm{e}^{-2\,\mathrm{e}^{x}}}{2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'\,(x)^2} \\ &= \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x-4\,\mathrm{e}^{x}}} \\ &= \mathrm{e}^{4\,\mathrm{e}^{x}} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + e^{4\,e^x} y(\tau) = 0$$

But in terms of τ

$$e^{4\,e^x} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\left(c_1 - c_2\ln(2) + c_2\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^x}\right)\right)}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\left(c_1 - c_2\ln(2) + c_2\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^x}\right)\right)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}$$

$$y_2 = -\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^x}\right)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}} & -\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^x}\right)}{2} \\ \frac{d}{dx}\left(\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^x}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}} & -\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\ln(2)}{2}+\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^{x}}\right)}{2} \\ \frac{\mathrm{e}^{-2\,\mathrm{e}^{x}}\mathrm{e}^{x}}{\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}} & -\frac{\sqrt{2}\,\ln(2)\mathrm{e}^{-2\,\mathrm{e}^{x}}\mathrm{e}^{x}}{2\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}}+\frac{\sqrt{2}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^{x}}\right)\mathrm{e}^{-2\,\mathrm{e}^{x}}\mathrm{e}^{x}}{2\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}}-\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\mathrm{e}^{x} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\right)\left(-\frac{\sqrt{2}\,\ln\left(2\right)\mathrm{e}^{-2\,\mathrm{e}^{x}}\mathrm{e}^{x}}{2\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}}+\frac{\sqrt{2}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^{x}}\right)\mathrm{e}^{-2\,\mathrm{e}^{x}}\mathrm{e}^{x}}{2\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}}-\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\mathrm{e}^{x}\right) \\ & -\left(-\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\ln\left(2\right)}{2}+\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^{x}}\right)}{2}\right)\left(\frac{\mathrm{e}^{-2\,\mathrm{e}^{x}}\mathrm{e}^{x}}{\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}}\right) \end{aligned}$$

Which simplifies to

$$W = \sqrt{2}\,\mathrm{e}^{-2\,\mathrm{e}^{x}}\mathrm{e}^{x}$$

Which simplifies to

$$W = \sqrt{2}\,\mathrm{e}^{x-2\,\mathrm{e}^{x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\ln(2)}{2}+\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^{x}}\right)}{2}\right)\mathrm{e}^{4x}}{\sqrt{2}\,\mathrm{e}^{x-2\,\mathrm{e}^{x}}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\mathrm{e}^{3x+2\,\mathrm{e}^{x}}\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\left(-\ln\left(2\right)+\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^{x}}\right)\right)}{2}dx$$

Hence

$$u_1 = -\frac{\left(i\ln\left(2\right)-4i+\pi\right)\mathrm{e}\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}+2i\pi\left(\mathrm{e}^{x}-\frac{\mathrm{e}^{2x}}{2}-1\right)\mathrm{csgn}\left(i\mathrm{e}^{-2\,\mathrm{e}^{x}}\right)+\left(2\,\mathrm{e}^{x}-\mathrm{e}^{2x}-2\right)\ln\left(\mathrm{e}^{-2\,\mathrm{e}^{x}}\right)+\left(\ln\left(2\right)\right.}{2\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\mathrm{e}^{4x}}{\sqrt{2}\,\mathrm{e}^{x-2\,\mathrm{e}^{x}}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{2}\,\mathrm{e}^{3x+2\,\mathrm{e}^{x}}\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}}{2}dx$$

Hence

$$u_2 = \int_0^x \frac{\sqrt{2}\,\mathrm{e}^{3\alpha+2\,\mathrm{e}^{\alpha}}\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{\alpha}}}}{2}d\alpha$$

Which simplifies to

$$u_1 = -\frac{(i\ln(2)-4i+\pi)\,\mathrm{e}\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}} + 2i\pi\left(\mathrm{e}^{x}-\frac{\mathrm{e}^{2x}}{2}-1\right)\mathrm{csgn}\left(i\mathrm{e}^{-2\,\mathrm{e}^{x}}\right) + (2\,\mathrm{e}^{x}-\mathrm{e}^{2x}-2)\ln\left(\mathrm{e}^{-2\,\mathrm{e}^{x}}\right) + (\ln(2)}{2\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}}$$

$$u_2 = \frac{\sqrt{2}\left(\int_0^x \mathrm{e}^{3\alpha+2\,\mathrm{e}^{\alpha}}\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{\alpha}}}d\alpha\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &-\frac{(i\ln(2)-4i+\pi)\,\mathrm{e}\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}}{2} - i\pi\left(\mathrm{e}^{x}-\frac{\mathrm{e}^{2x}}{2}-1\right)\mathrm{csgn}\left(i\mathrm{e}^{-2\,\mathrm{e}^{x}}\right) \\ &- \frac{(2\,\mathrm{e}^{x}-\mathrm{e}^{2x}-2)\ln\left(\mathrm{e}^{-2\,\mathrm{e}^{x}}\right)}{2} - \frac{(\ln(2)-2)\,\mathrm{e}^{2x}}{2} - \frac{(-2\ln(2)+8)\,\mathrm{e}^{x}}{2} - \ln(2) \\ &+ 6 + \frac{\sqrt{2}\left(\int_0^x \mathrm{e}^{3\alpha+2\,\mathrm{e}^{\alpha}}\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{\alpha}}}d\alpha\right)\left(-\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^{x}}\right)}{2}\right)}{2} \end{aligned}$$

Which simplifies to

$$\begin{aligned} y_p(x) = &\frac{\left(2\,\mathrm{e}^{x-\mathrm{e}^{x}} + (-i\pi+\ln(2))\,\mathrm{e}^{-\mathrm{e}^{x}}\right)\left(\int_0^x \mathrm{e}^{3\alpha+\mathrm{e}^{\alpha}}d\alpha\right)}{2} + \frac{(\ln(2)-4-i\pi)\,\mathrm{e}^{1-\mathrm{e}^{x}}}{2} \\ &+ \frac{(6+i\pi-\ln(2))\,\mathrm{e}^{2x}}{2} - i\pi\,\mathrm{e}^{x} + i\pi + \mathrm{e}^{x}\ln(2) - \ln(2) - 6\,\mathrm{e}^{x} - \mathrm{e}^{3x} + 6 \end{aligned}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\left(c_1 - c_2\ln\left(2\right) + c_2\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^{x}}\right)\right)}{2}\right) \\
&\quad + \left(\frac{\left(2\,\mathrm{e}^{x-\mathrm{e}^{x}} + \left(-i\pi + \ln\left(2\right)\right)\mathrm{e}^{-\mathrm{e}^{x}}\right)\left(\int_0^x \mathrm{e}^{3\alpha+\mathrm{e}^{\alpha}}d\alpha\right)}{2} + \frac{\left(\ln\left(2\right) - 4 - i\pi\right)\mathrm{e}^{1-\mathrm{e}^{x}}}{2}\right. \\
&\qquad \left. + \frac{\left(6 + i\pi - \ln\left(2\right)\right)\mathrm{e}^{2x}}{2} - i\pi\,\mathrm{e}^{x} + i\pi + \mathrm{e}^{x}\ln\left(2\right) - \ln\left(2\right) - 6\,\mathrm{e}^{x} - \mathrm{e}^{3x} + 6\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\left(c_1 - c_2\ln\left(2\right) + c_2\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^{x}}\right)\right)}{2} \\
&\quad + \frac{\left(2\,\mathrm{e}^{x-\mathrm{e}^{x}} + \left(-i\pi + \ln\left(2\right)\right)\mathrm{e}^{-\mathrm{e}^{x}}\right)\left(\int_0^x \mathrm{e}^{3\alpha+\mathrm{e}^{\alpha}}d\alpha\right)}{2} + \frac{\left(\ln\left(2\right) - 4 - i\pi\right)\mathrm{e}^{1-\mathrm{e}^{x}}}{2} \\
&\quad + \frac{\left(6 + i\pi - \ln\left(2\right)\right)\mathrm{e}^{2x}}{2} - i\pi\,\mathrm{e}^{x} + i\pi + \mathrm{e}^{x}\ln\left(2\right) - \ln\left(2\right) - 6\,\mathrm{e}^{x} - \mathrm{e}^{3x} + 6
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^{x}}}\,\left(c_1 - c_2\ln\left(2\right) + c_2\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^{x}}\right)\right)}{2} \\
&\quad + \frac{\left(2\,\mathrm{e}^{x-\mathrm{e}^{x}} + \left(-i\pi + \ln\left(2\right)\right)\mathrm{e}^{-\mathrm{e}^{x}}\right)\left(\int_0^x \mathrm{e}^{3\alpha+\mathrm{e}^{\alpha}}d\alpha\right)}{2} + \frac{\left(\ln\left(2\right) - 4 - i\pi\right)\mathrm{e}^{1-\mathrm{e}^{x}}}{2} \\
&\quad + \frac{\left(6 + i\pi - \ln\left(2\right)\right)\mathrm{e}^{2x}}{2} - i\pi\,\mathrm{e}^{x} + i\pi + \mathrm{e}^{x}\ln\left(2\right) - \ln\left(2\right) - 6\,\mathrm{e}^{x} - \mathrm{e}^{3x} + 6
\end{aligned}
$$

Verified OK. `{0 < x}`

31.1.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = 1, B = 2\,\mathrm{e}^{x} - 1, C = \mathrm{e}^{2x}, f(x) = \mathrm{e}^{4x}$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$y'' + (2\,\mathrm{e}^{x} - 1)\,y' + y\,\mathrm{e}^{2x} = 0$$

In normal form the ode

$$y'' + (2\,\mathrm{e}^{x} - 1)\,y' + y\,\mathrm{e}^{2x} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= 2\,\mathrm{e}^{x} - 1 \\ q(x) &= \mathrm{e}^{2x} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\mathrm{e}^{2x}}}{c} \\ \tau'' &= \frac{\sqrt{\mathrm{e}^{2x}}}{c} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{\sqrt{\mathrm{e}^{2x}}}{c} + 2\,\mathrm{e}^{x} - 1\frac{\sqrt{\mathrm{e}^{2x}}}{c}}{\left(\frac{\sqrt{\mathrm{e}^{2x}}}{c}\right)^2} \\ &= 2c \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + 2c\left(\frac{d}{d\tau} y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = e^{-c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dx \\
&= \frac{\int \sqrt{e^{2x}} dx}{c} \\
&= \frac{\sqrt{e^{2x}}}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = e^{-e^x} c_1
$$

Now the particular solution to this ODE is found

$$
y'' + (2\,e^x - 1)\, y' + y\, e^{2x} = e^{4x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{-e^{-2\,e^x}} \\
y_2 &= -\frac{\sqrt{2}\,\sqrt{-e^{-2\,e^x}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-e^{-2\,e^x}}\,\ln\left(-e^{-2\,e^x}\right)}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}} & -\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^x}\right)}{2} \\ \frac{d}{dx}\left(\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^x}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}} & -\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^x}\right)}{2} \\ \frac{\mathrm{e}^{-2\,\mathrm{e}^x}\,\mathrm{e}^x}{\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}} & -\frac{\sqrt{2}\,\ln(2)\mathrm{e}^{-2\,\mathrm{e}^x}\,\mathrm{e}^x}{2\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}} + \frac{\sqrt{2}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^x}\right)\mathrm{e}^{-2\,\mathrm{e}^x}\,\mathrm{e}^x}{2\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}} - \sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\mathrm{e}^x \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\right)\left(-\frac{\sqrt{2}\,\ln(2)\,\mathrm{e}^{-2\,\mathrm{e}^x}\,\mathrm{e}^x}{2\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}} + \frac{\sqrt{2}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^x}\right)\mathrm{e}^{-2\,\mathrm{e}^x}\,\mathrm{e}^x}{2\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}} - \sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\mathrm{e}^x\right) - \left(-\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^x}\right)}{2}\right)\left(\frac{\mathrm{e}^{-2\,\mathrm{e}^x}\,\mathrm{e}^x}{\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}}\right)$$

Which simplifies to

$$W = \sqrt{2}\,\mathrm{e}^{-2\,\mathrm{e}^x}\,\mathrm{e}^x$$

Which simplifies to

$$W = \sqrt{2}\,\mathrm{e}^{x-2\,\mathrm{e}^x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^x}\right)}{2}\right)\mathrm{e}^{4x}}{\sqrt{2}\,\mathrm{e}^{x-2\,\mathrm{e}^x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{3x+2\,e^{x}} \sqrt{-e^{-2\,e^{x}}}\, \left(-\ln\left(2\right)+\ln\left(-e^{-2\,e^{x}}\right)\right)}{2} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{e^{3\alpha+2\,e^{\alpha}} \sqrt{-e^{-2\,e^{\alpha}}}\, \left(-\ln\left(2\right)+\ln\left(-e^{-2\,e^{\alpha}}\right)\right)}{2} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{-e^{-2\,e^{x}}}\, e^{4x}}{\sqrt{2}\, e^{x-2\,e^{x}}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{2}\, e^{3x+2\,e^{x}} \sqrt{-e^{-2\,e^{x}}}}{2} dx$$

Hence

$$u_2 = \int_0^x \frac{\sqrt{2}\, e^{3\alpha+2\,e^{\alpha}} \sqrt{-e^{-2\,e^{\alpha}}}}{2} d\alpha$$

Which simplifies to

$$u_1 = \frac{\left(\int_0^x \left(i\ln\left(2\right)+2i e^{\alpha}+\pi\right) e^{3\alpha+e^{\alpha}} d\alpha\right)}{2}$$

$$u_2 = \frac{i\sqrt{2}\left(\int_0^x e^{e^{\alpha}+3\alpha} d\alpha\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\int_0^x \left(i\ln\left(2\right)+2i e^{\alpha}+\pi\right) e^{3\alpha+e^{\alpha}} d\alpha\right) \sqrt{-e^{-2\,e^{x}}}}{2} + \frac{i\sqrt{2}\left(\int_0^x e^{e^{\alpha}+3\alpha} d\alpha\right)\left(-\frac{\sqrt{2}\,\sqrt{-e^{-2\,e^{x}}}\,\ln(2)}{2}+\frac{\sqrt{2}\,\sqrt{-e^{-2\,e^{x}}}\,\ln\left(-e^{-2\,e^{x}}\right)}{2}\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{i\left(\int_0^x (i\ln(2) + 2i\mathrm{e}^{\alpha} + \pi)\,\mathrm{e}^{3\alpha+\mathrm{e}^{\alpha}} d\alpha\right)\mathrm{e}^{-\mathrm{e}^x}}{2} - \frac{\left(-2\,\mathrm{e}^{x-\mathrm{e}^x} + (i\pi - \ln(2))\,\mathrm{e}^{-\mathrm{e}^x}\right)\left(\int_0^x \mathrm{e}^{\mathrm{e}^{\alpha}+3\alpha} d\alpha\right)}{2}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{-\mathrm{e}^x} c_1\right) + \left(\frac{i\left(\int_0^x (i\ln(2) + 2i\mathrm{e}^{\alpha} + \pi)\,\mathrm{e}^{3\alpha+\mathrm{e}^{\alpha}} d\alpha\right)\mathrm{e}^{-\mathrm{e}^x}}{2} - \frac{\left(-2\,\mathrm{e}^{x-\mathrm{e}^x} + (i\pi - \ln(2))\,\mathrm{e}^{-\mathrm{e}^x}\right)\left(\int_0^x \mathrm{e}^{\mathrm{e}^{\alpha}+3\alpha} d\alpha\right)}{2}\right) \\
&= \frac{i\left(\int_0^x (i\ln(2) + 2i\mathrm{e}^{\alpha} + \pi)\,\mathrm{e}^{3\alpha+\mathrm{e}^{\alpha}} d\alpha\right)\mathrm{e}^{-\mathrm{e}^x}}{2} - \frac{\left(-2\,\mathrm{e}^{x-\mathrm{e}^x} + (i\pi - \ln(2))\,\mathrm{e}^{-\mathrm{e}^x}\right)\left(\int_0^x \mathrm{e}^{\mathrm{e}^{\alpha}+3\alpha} d\alpha\right)}{2} + \mathrm{e}^{-\mathrm{e}^x} c_1
\end{aligned}$$

Which simplifies to

$$y = \frac{i\left(\int_0^x (i\ln(2) + 2i\mathrm{e}^{\alpha} + \pi)\,\mathrm{e}^{3\alpha+\mathrm{e}^{\alpha}} d\alpha\right)\mathrm{e}^{-\mathrm{e}^x}}{2} - \frac{\left(-2\,\mathrm{e}^{x-\mathrm{e}^x} + (i\pi - \ln(2))\,\mathrm{e}^{-\mathrm{e}^x}\right)\left(\int_0^x \mathrm{e}^{\mathrm{e}^{\alpha}+3\alpha} d\alpha\right)}{2} + \mathrm{e}^{-\mathrm{e}^x} c_1$$

Summary

The solution(s) found are the following

$$y = \frac{i\left(\int_0^x (i\ln(2) + 2i\mathrm{e}^{\alpha} + \pi)\,\mathrm{e}^{3\alpha+\mathrm{e}^{\alpha}} d\alpha\right)\mathrm{e}^{-\mathrm{e}^x}}{2} - \frac{\left(-2\,\mathrm{e}^{x-\mathrm{e}^x} + (i\pi - \ln(2))\,\mathrm{e}^{-\mathrm{e}^x}\right)\left(\int_0^x \mathrm{e}^{\mathrm{e}^{\alpha}+3\alpha} d\alpha\right)}{2} + \mathrm{e}^{-\mathrm{e}^x} c_1 \tag{1}$$

Verification of solutions

$$y = \frac{i\left(\int_0^x (i\ln(2) + 2i\mathrm{e}^{\alpha} + \pi)\,\mathrm{e}^{3\alpha+\mathrm{e}^{\alpha}} d\alpha\right)\mathrm{e}^{-\mathrm{e}^x}}{2} - \frac{\left(-2\,\mathrm{e}^{x-\mathrm{e}^x} + (i\pi - \ln(2))\,\mathrm{e}^{-\mathrm{e}^x}\right)\left(\int_0^x \mathrm{e}^{\mathrm{e}^{\alpha}+3\alpha} d\alpha\right)}{2} + \mathrm{e}^{-\mathrm{e}^x} c_1$$

Verified OK. `{0 < x}`

31.1.3 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + (2\,\mathrm{e}^{x} - 1)\,y' + y\,\mathrm{e}^{2x} = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= 2\,\mathrm{e}^{x} - 1\\ q(x) &= \mathrm{e}^{2x} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4}\\ &= \mathrm{e}^{2x} - \frac{(2\,\mathrm{e}^{x}-1)'}{2} - \frac{(2\,\mathrm{e}^{x}-1)^2}{4}\\ &= \mathrm{e}^{2x} - \frac{(2\,\mathrm{e}^{x})}{2} - \frac{\left((2\,\mathrm{e}^{x}-1)^2\right)}{4}\\ &= \mathrm{e}^{2x} - (\mathrm{e}^{x}) - \frac{(2\,\mathrm{e}^{x}-1)^2}{4}\\ &= -\frac{1}{4} \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2}dx\right)}\\ &= e^{-\int \frac{2\,\mathrm{e}^{x}-1}{2}}\\ &= \mathrm{e}^{\frac{x}{2}-\mathrm{e}^{x}} \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, e^{\frac{x}{2}-e^{x}} \tag{4}$$

Applying this change of variable to the original ode results in

$$e^{\frac{x}{2}-e^{x}}\left(4v''(x) - v(x)\right) = 4\, e^{4x}$$

Which is now solved for $v(x)$ Simplyfing the ode gives

$$v''(x) - \frac{v(x)}{4} = e^{\frac{7x}{2}+e^{x}}$$

This is second order non-homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = f(x)$$

Where $A = 1, B = 0, C = -\frac{1}{4}, f(x) = e^{\frac{7x}{2}+e^{x}}$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$v''(x) - \frac{v(x)}{4} = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -\frac{1}{4}$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \frac{e^{\lambda x}}{4} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \frac{1}{4} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -\frac{1}{4}$ into the above gives

$$\lambda_{1,2} = \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)\left(-\frac{1}{4}\right)}$$
$$= \pm \frac{1}{2}$$

Hence

$$\lambda_1 = +\frac{1}{2}$$
$$\lambda_2 = -\frac{1}{2}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{2}$$

Since roots are real and distinct, then the solution is

$$v(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$v(x) = c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{\left(-\frac{1}{2}\right)x}$$

Or

$$v(x) = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

The particular solution v_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$v_p(x) = u_1 v_1 + u_2 v_2 \tag{1}$$

Where u_1, u_2 to be determined, and v_1, v_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$v_1 = e^{\frac{x}{2}}$$

$$v_2 = e^{-\frac{x}{2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{v_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{v_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of v'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\frac{x}{2}} & e^{-\frac{x}{2}} \\ \frac{d}{dx}\left(e^{\frac{x}{2}}\right) & \frac{d}{dx}\left(e^{-\frac{x}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\frac{x}{2}} & e^{-\frac{x}{2}} \\ \frac{e^{\frac{x}{2}}}{2} & -\frac{e^{-\frac{x}{2}}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{\frac{x}{2}}\right)\left(-\frac{e^{-\frac{x}{2}}}{2}\right) - \left(e^{-\frac{x}{2}}\right)\left(\frac{e^{\frac{x}{2}}}{2}\right)$$

Which simplifies to

$$W = -e^{-\frac{x}{2}} e^{\frac{x}{2}}$$

Which simplifies to

$$W = -1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-\frac{x}{2}} e^{\frac{7x}{2} + e^x}}{-1}\, dx$$

Which simplifies to

$$u_1 = -\int -\mathrm{e}^{3x+\mathrm{e}^x} dx$$

Hence

$$u_1 = \mathrm{e}^{2x}\mathrm{e}^{\mathrm{e}^x} - 2\,\mathrm{e}^{x}\mathrm{e}^{\mathrm{e}^x} + 2\,\mathrm{e}^{\mathrm{e}^x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{\frac{x}{2}}\mathrm{e}^{\frac{7x}{2}+\mathrm{e}^x}}{-1}\, dx$$

Which simplifies to

$$u_2 = \int -\mathrm{e}^{4x+\mathrm{e}^x} dx$$

Hence

$$u_2 = -\mathrm{e}^{3x}\mathrm{e}^{\mathrm{e}^x} + 3\,\mathrm{e}^{2x}\mathrm{e}^{\mathrm{e}^x} - 6\,\mathrm{e}^{x}\mathrm{e}^{\mathrm{e}^x} + 6\,\mathrm{e}^{\mathrm{e}^x}$$

Which simplifies to

$$\begin{aligned} u_1 &= \mathrm{e}^{2x+\mathrm{e}^x} - 2\,\mathrm{e}^{x+\mathrm{e}^x} + 2\,\mathrm{e}^{\mathrm{e}^x} \\ u_2 &= -\mathrm{e}^{3x+\mathrm{e}^x} + 3\,\mathrm{e}^{2x+\mathrm{e}^x} - 6\,\mathrm{e}^{x+\mathrm{e}^x} + 6\,\mathrm{e}^{\mathrm{e}^x} \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$v_p(x) = \left(\mathrm{e}^{2x+\mathrm{e}^x} - 2\,\mathrm{e}^{x+\mathrm{e}^x} + 2\,\mathrm{e}^{\mathrm{e}^x}\right)\mathrm{e}^{\frac{x}{2}} + \left(-\mathrm{e}^{3x+\mathrm{e}^x} + 3\,\mathrm{e}^{2x+\mathrm{e}^x} - 6\,\mathrm{e}^{x+\mathrm{e}^x} + 6\,\mathrm{e}^{\mathrm{e}^x}\right)\mathrm{e}^{-\frac{x}{2}}$$

Which simplifies to

$$v_p(x) = \mathrm{e}^{\frac{3x}{2}+\mathrm{e}^x} - 4\,\mathrm{e}^{\frac{x}{2}+\mathrm{e}^x} + 6\,\mathrm{e}^{-\frac{x}{2}+\mathrm{e}^x}$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= \left(c_1\mathrm{e}^{\frac{x}{2}} + c_2\mathrm{e}^{-\frac{x}{2}}\right) + \left(\mathrm{e}^{\frac{3x}{2}+\mathrm{e}^x} - 4\,\mathrm{e}^{\frac{x}{2}+\mathrm{e}^x} + 6\,\mathrm{e}^{-\frac{x}{2}+\mathrm{e}^x}\right) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}} + \mathrm{e}^{\frac{3x}{2}+\mathrm{e}^x} - 4\,\mathrm{e}^{\frac{x}{2}+\mathrm{e}^x} + 6\,\mathrm{e}^{-\frac{x}{2}+\mathrm{e}^x}\right)(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \mathrm{e}^{\frac{x}{2}-\mathrm{e}^x}$$

Hence (7) becomes

$$y = \left(c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}} + \mathrm{e}^{\frac{3x}{2}+\mathrm{e}^x} - 4\,\mathrm{e}^{\frac{x}{2}+\mathrm{e}^x} + 6\,\mathrm{e}^{-\frac{x}{2}+\mathrm{e}^x}\right) \mathrm{e}^{\frac{x}{2}-\mathrm{e}^x}$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}} + \mathrm{e}^{\frac{3x}{2}+\mathrm{e}^x} - 4\,\mathrm{e}^{\frac{x}{2}+\mathrm{e}^x} + 6\,\mathrm{e}^{-\frac{x}{2}+\mathrm{e}^x}\right) \mathrm{e}^{\frac{x}{2}-\mathrm{e}^x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}} \\ y_2 &= -\frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\mathrm{e}^{-2\,\mathrm{e}^x}}\,\ln\left(-\mathrm{e}^{-2\,\mathrm{e}^x}\right)}{2} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-e^{-2e^x}} & -\frac{\sqrt{2}\sqrt{-e^{-2e^x}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-e^{-2e^x}}\ln\left(-e^{-2e^x}\right)}{2} \\ \frac{d}{dx}\left(\sqrt{-e^{-2e^x}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\sqrt{-e^{-2e^x}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-e^{-2e^x}}\ln\left(-e^{-2e^x}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-e^{-2e^x}} & -\frac{\sqrt{2}\sqrt{-e^{-2e^x}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-e^{-2e^x}}\ln\left(-e^{-2e^x}\right)}{2} \\ \frac{e^{-2e^x}e^x}{\sqrt{-e^{-2e^x}}} & -\frac{\sqrt{2}\ln(2)e^{-2e^x}e^x}{2\sqrt{-e^{-2e^x}}} + \frac{\sqrt{2}\ln\left(-e^{-2e^x}\right)e^{-2e^x}e^x}{2\sqrt{-e^{-2e^x}}} - \sqrt{2}\sqrt{-e^{-2e^x}}\,e^x \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-e^{-2e^x}}\right)\left(-\frac{\sqrt{2}\ln(2)e^{-2e^x}e^x}{2\sqrt{-e^{-2e^x}}} + \frac{\sqrt{2}\ln\left(-e^{-2e^x}\right)e^{-2e^x}e^x}{2\sqrt{-e^{-2e^x}}} - \sqrt{2}\sqrt{-e^{-2e^x}}\,e^x\right) - \left(-\frac{\sqrt{2}\sqrt{-e^{-2e^x}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-e^{-2e^x}}\ln\left(-e^{-2e^x}\right)}{2}\right)\left(\frac{e^{-2e^x}e^x}{\sqrt{-e^{-2e^x}}}\right)$$

Which simplifies to

$$W = \sqrt{2}\,e^{-2e^x}e^x$$

Which simplifies to

$$W = \sqrt{2}\,e^{x-2e^x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{2}\sqrt{-e^{-2e^x}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-e^{-2e^x}}\ln\left(-e^{-2e^x}\right)}{2}\right)e^{4x}}{\sqrt{2}\,e^{x-2e^x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{3x+2e^x}\sqrt{-e^{-2e^x}}\left(-\ln(2) + \ln\left(-e^{-2e^x}\right)\right)}{2}\,dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{e^{3\alpha+2 e^{\alpha}} \sqrt{-e^{-2 e^{\alpha}}}\, \left(-\ln(2)+\ln\left(-e^{-2 e^{\alpha}}\right)\right)}{2}d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{-e^{-2 e^{x}}}\, e^{4x}}{\sqrt{2}\, e^{x-2 e^{x}}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{2}\, e^{3x+2 e^{x}} \sqrt{-e^{-2 e^{x}}}}{2}dx$$

Hence

$$u_2 = \int_0^x \frac{\sqrt{2}\, e^{3\alpha+2 e^{\alpha}} \sqrt{-e^{-2 e^{\alpha}}}}{2}d\alpha$$

Which simplifies to

$$u_1 = \frac{\left(\int_0^x \left(i\ln(2)+2i e^{\alpha}+\pi\right) e^{3\alpha+e^{\alpha}}d\alpha\right)}{2}$$

$$u_2 = \frac{i\sqrt{2}\left(\int_0^x e^{e^{\alpha}+3\alpha}d\alpha\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\int_0^x \left(i\ln(2)+2i e^{\alpha}+\pi\right) e^{3\alpha+e^{\alpha}}d\alpha\right) \sqrt{-e^{-2 e^{x}}}}{2} + \frac{i\sqrt{2}\left(\int_0^x e^{e^{\alpha}+3\alpha}d\alpha\right)\left(-\frac{\sqrt{2}\,\sqrt{-e^{-2 e^{x}}}\, \ln(2)}{2}+\frac{\sqrt{2}\,\sqrt{-e^{-2 e^{x}}}\, \ln\left(-e^{-2 e^{x}}\right)}{2}\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{i\left(\int_0^x \left(i\ln(2)+2i e^{\alpha}+\pi\right) e^{3\alpha+e^{\alpha}}d\alpha\right) e^{-e^{x}}}{2} - \frac{\left(-2\, e^{x-e^{x}}+\left(i\pi-\ln(2)\right) e^{-e^{x}}\right)\left(\int_0^x e^{e^{\alpha}+3\alpha}d\alpha\right)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}} + \mathrm{e}^{\frac{3x}{2}+\mathrm{e}^x} - 4\,\mathrm{e}^{\frac{x}{2}+\mathrm{e}^x} + 6\,\mathrm{e}^{-\frac{x}{2}+\mathrm{e}^x}\right) \mathrm{e}^{\frac{x}{2}-\mathrm{e}^x}\right) \\ &\quad + \left(\frac{i\left(\int_0^x \left(i\ln\left(2\right) + 2i\mathrm{e}^{\alpha} + \pi\right) \mathrm{e}^{3\alpha+\mathrm{e}^{\alpha}} d\alpha\right) \mathrm{e}^{-\mathrm{e}^x}}{2} \right. \\ &\quad \left. - \frac{\left(-2\,\mathrm{e}^{x-\mathrm{e}^x} + \left(i\pi - \ln\left(2\right)\right) \mathrm{e}^{-\mathrm{e}^x}\right) \left(\int_0^x \mathrm{e}^{\mathrm{e}^{\alpha}+3\alpha} d\alpha\right)}{2}\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} y = c_1 \mathrm{e}^{x-\mathrm{e}^x} + \mathrm{e}^{2x} - 4\,\mathrm{e}^x + 6 + c_2 \mathrm{e}^{-\mathrm{e}^x} + \frac{i\left(\int_0^x \left(i\ln\left(2\right) + 2i\mathrm{e}^{\alpha} + \pi\right) \mathrm{e}^{3\alpha+\mathrm{e}^{\alpha}} d\alpha\right) \mathrm{e}^{-\mathrm{e}^x}}{2} \\ - \frac{\left(-2\,\mathrm{e}^{x-\mathrm{e}^x} + \left(i\pi - \ln\left(2\right)\right) \mathrm{e}^{-\mathrm{e}^x}\right) \left(\int_0^x \mathrm{e}^{\mathrm{e}^{\alpha}+3\alpha} d\alpha\right)}{2} \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1 \mathrm{e}^{x-\mathrm{e}^x} + \mathrm{e}^{2x} - 4\,\mathrm{e}^x + 6 + c_2 \mathrm{e}^{-\mathrm{e}^x} + \frac{i\left(\int_0^x \left(i\ln\left(2\right) + 2i\mathrm{e}^{\alpha} + \pi\right) \mathrm{e}^{3\alpha+\mathrm{e}^{\alpha}} d\alpha\right) \mathrm{e}^{-\mathrm{e}^x}}{2} \\ - \frac{\left(-2\,\mathrm{e}^{x-\mathrm{e}^x} + \left(i\pi - \ln\left(2\right)\right) \mathrm{e}^{-\mathrm{e}^x}\right) \left(\int_0^x \mathrm{e}^{\mathrm{e}^{\alpha}+3\alpha} d\alpha\right)}{2} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = c_1 \mathrm{e}^{x-\mathrm{e}^x} + \mathrm{e}^{2x} - 4\,\mathrm{e}^x + 6 + c_2 \mathrm{e}^{-\mathrm{e}^x} + \frac{i\left(\int_0^x \left(i\ln\left(2\right) + 2i\mathrm{e}^{\alpha} + \pi\right) \mathrm{e}^{3\alpha+\mathrm{e}^{\alpha}} d\alpha\right) \mathrm{e}^{-\mathrm{e}^x}}{2} \\ - \frac{\left(-2\,\mathrm{e}^{x-\mathrm{e}^x} + \left(i\pi - \ln\left(2\right)\right) \mathrm{e}^{-\mathrm{e}^x}\right) \left(\int_0^x \mathrm{e}^{\mathrm{e}^{\alpha}+3\alpha} d\alpha\right)}{2} \end{aligned}$$

Verified OK. `{0 < x}`

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
dsolve(diff(y(x),x$2)+(2*exp(x)-1)*diff(y(x),x)+exp(2*x)*y(x)=exp(4*x),y(x), singsol=all)
```

$$y(x) = e^{\frac{x}{2}-e^x} \sinh\left(\frac{x}{2}\right) c_2 + e^{\frac{x}{2}-e^x} \cosh\left(\frac{x}{2}\right) c_1 + e^{2x} - 4\,e^x + 6$$

✓ Solution by Mathematica

Time used: 0.121 (sec). Leaf size: 39

```
DSolve[y''[x]+(2*Exp[x]-1)*y'[x]+Exp[2*x]*y[x]==Exp[4*x],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -4e^x + e^{2x} + c_1 e^{-e^x} + c_2 e^{x-e^x} + 6$$

31.2 problem Ex 2

Internal problem ID [11298]
Internal file name [`OUTPUT/10283_Wednesday_December_21_2022_03_47_39_PM_28886466/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 54. Change of independent variable. Page 127
Problem number: Ex 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__change__of__variable__on__x__method__1", "second__order__change__of__variable__on__x__method__2"**

Maple gives the following as the ode type

```
[_Gegenbauer, [_2nd_order, _linear, `_with_symmetry_[0,F(x)]`]]
```

$$\boxed{\left(-x^2+1\right) y''-y'x+4y=0}$$

31.2.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$\left(-x^2+1\right) y''-y'x+4y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x}{x^2-1} \\ q(x) &= -\frac{4}{x^2-1} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{x}{x^2-1}dx\right)}dx \\
&= \int e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}}\,dx \\
&= \int \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}dx \\
&= \frac{\sqrt{(x-1)(1+x)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{1+x}}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{4}{x^2-1}}{\frac{1}{(x-1)(1+x)}} \\
&= -4
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 4y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y(\tau) = e^{\lambda \tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} - 4\, e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2\\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}\\ y(\tau) &= c_1 e^{(2)\tau} + c_2 e^{(-2)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{2\tau} + c_2 e^{-2\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(x+\sqrt{x^2-1}\right)^{\frac{2\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x+\sqrt{x^2-1}\right)^{-\frac{2\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(x+\sqrt{x^2-1}\right)^{\frac{2\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x+\sqrt{x^2-1}\right)^{-\frac{2\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} \tag{1}$$

Verification of solutions

$$y = c_1 \left(x+\sqrt{x^2-1}\right)^{\frac{2\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x+\sqrt{x^2-1}\right)^{-\frac{2\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}}$$

Verified OK.

31.2.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$\left(-x^2+1\right) y'' - y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x}{x^2-1} \\ q(x) &= -\frac{4}{x^2-1} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{-\frac{1}{x^2-1}}}{c} \\ \tau'' &= \frac{2x}{c\sqrt{-\frac{1}{x^2-1}}\,(x^2-1)^2}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{2x}{c\sqrt{-\frac{1}{x^2-1}}\,(x^2-1)^2} + \frac{x}{x^2-1}\frac{2\sqrt{-\frac{1}{x^2-1}}}{c}}{\left(\frac{2\sqrt{-\frac{1}{x^2-1}}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int 2\sqrt{-\frac{1}{x^2-1}}dx}{c} \\ &= \frac{2\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(2\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) + c_2 \sin\left(2\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(2\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) + c_2 \sin\left(2\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(2\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) + c_2 \sin\left(2\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)$$

Verified OK.

31.2.3 Solving using Kovacic algorithm

Writing the ode as

$$\left(-x^2+1\right)y'' - y'x + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 - 18}{4\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2 - 18 \\ t &= 4\left(x^2 - 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 - 18}{4\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 184: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{33}{16(x-1)} - \frac{3}{16(x-1)^2} - \frac{3}{16(1+x)^2} - \frac{33}{16(1+x)}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{15x^2 - 18}{4(x^2-1)^2}
$$

Since the gcd$(s, t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 - 18}{4\left(x^2-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2} \right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4\left(x-1\right)} + \frac{3}{4\left(1+x\right)} + (0) \\
&= \frac{3}{4\left(x-1\right)} + \frac{3}{4\left(1+x\right)} \\
&= \frac{3x}{2x^2 - 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4\left(x-1\right)} + \frac{3}{4\left(1+x\right)}\right)(1) + \left(\left(-\frac{3}{4\left(x-1\right)^2} - \frac{3}{4\left(1+x\right)^2}\right) + \left(\frac{3}{4\left(x-1\right)} + \frac{3}{4\left(1+x\right)}\right)^2 - \left(\frac{15}{4(\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(\frac{3}{4(x-1)} + \frac{3}{4(1+x)}\right) dx} \\
&= (x)\, \mathrm{e}^{\frac{3\ln(x-1)}{4} + \frac{3\ln(1+x)}{4}} \\
&= x(x-1)^{\frac{3}{4}} \left(1+x\right)^{\frac{3}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{-x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{4}-\frac{\ln(1+x)}{4}} \\
&= z_1 \left(\frac{1}{(x-1)^{\frac{1}{4}}\,(1+x)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\sqrt{x-1}\,\sqrt{1+x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-2x^2+1}{\sqrt{x-1}\,\sqrt{1+x}\,x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\sqrt{x-1}\,\sqrt{1+x}\right) + c_2\left(x\sqrt{x-1}\,\sqrt{1+x}\left(\frac{-2x^2+1}{\sqrt{x-1}\,\sqrt{1+x}\,x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\sqrt{x-1}\,\sqrt{1+x} + c_2\left(-2x^2+1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\sqrt{x-1}\,\sqrt{1+x} + c_2\left(-2x^2+1\right)
$$

Verified OK.

31.2.4 Maple step by step solution

Let's solve

$(-x^2+1)\,y''-y'x+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{xy'}{x^2-1}+\frac{4y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{xy'}{x^2-1}-\frac{4y}{x^2-1}=0$

- Multiply by denominators of ODE

  $(-x^2+1)\,y''-y'x+4y=0$

- Make a change of variables

  $\theta=\arccos(x)$

- Calculate y' with change of variables

  $y'=\left(\frac{d}{d\theta}y(\theta)\right)\theta'(x)$

- Compute 1st derivative y'

  $y'=-\frac{\frac{d}{d\theta}y(\theta)}{\sqrt{-x^2+1}}$

- Calculate y'' with change of variables

  $y''=\left(\frac{d^2}{d\theta^2}y(\theta)\right)\theta'(x)^2+\theta''(x)\left(\frac{d}{d\theta}y(\theta)\right)$

- Compute 2nd derivative y''

  $y''=\frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1}-\frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}$

- Apply the change of variables to the ODE

  $(-x^2+1)\left(\frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1}-\frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}\right)+\frac{\left(\frac{d}{d\theta}y(\theta)\right)x}{\sqrt{-x^2+1}}+4y=0$

- Multiply through

  $-\frac{\left(\frac{d^2}{d\theta^2}y(\theta)\right)x^2}{-x^2+1}+\frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1}+\frac{x^3\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}-\frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}+\frac{\left(\frac{d}{d\theta}y(\theta)\right)x}{\sqrt{-x^2+1}}+4y=0$

- Simplify ODE

$4y + \frac{d^2}{d\theta^2} y(\theta) = 0$

- ODE is that of a harmonic oscillator with given general solution

  $y(\theta) = c_1 \sin(2\theta) + c_2 \cos(2\theta)$

- Revert back to x

  $y = c_1 \sin(2\arccos(x)) + c_2 \cos(2\arccos(x))$

- Apply double angle identities to solution

  $y = c_1 \sin(\arccos(x)) \cos(\arccos(x)) + c_2\left(2\cos(\arccos(x))^2 - 1\right)$

- Use trig identity to simplify sin

  $\sin(\arccos(x)) = \sqrt{-x^2+1}$

- Simplify solution to the ODE

  $y = c_1 x\sqrt{-x^2+1} + c_2(2x^2-1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve((1-x^2)*diff(y(x),x$2)-x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(8x^3 - 4x)\, c_2\sqrt{x^2-1} + (8x^4 - 8x^2 + 1)\, c_2 + c_1}{\left(x + \sqrt{x^2-1}\right)^2}$$

✓ Solution by Mathematica

Time used: 0.316 (sec). Leaf size: 97

```
DSolve[(1-x^2)*y''[x]-x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cosh\left(\frac{4\sqrt{1-x^2}\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right)}{\sqrt{x^2-1}}\right) - ic_2 \sinh\left(\frac{4\sqrt{1-x^2}\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right)}{\sqrt{x^2-1}}\right)$$

31.3 problem Ex 3

Internal problem ID [11299]
Internal file name [`OUTPUT/10284_Wednesday_December_21_2022_03_47_40_PM_31667708/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 54. Change of independent variable. Page 127
Problem number: Ex 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' \tan(x) + \cos(x)^2 y = 0}$$

31.3.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$y'' + y' \tan(x) + \cos(x)^2 y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \tan(x)$$
$$q(x) = \cos(x)^2$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \tan(x)dx\right)} dx \\
&= \int e^{\ln(\cos(x))}\, dx \\
&= \int \cos(x)\, dx \\
&= \sin(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\cos(x)^2}{\cos(x)^2} \\
&= 1
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} + \mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\
&= \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +i \\
\lambda_2 &= -i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= i \\
\lambda_2 &= -i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(\sin(x)) + c_2 \sin(\sin(x))$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(\sin(x)) + c_2 \sin(\sin(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(\sin(x)) + c_2 \sin(\sin(x))$$

Verified OK.

31.3.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$y'' + y' \tan(x) + \cos(x)^2 y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= \cos(x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{\cos(2x)}{2} + \frac{1}{2}}}{c} \\ \tau'' &= -\frac{2\cos(x)\sin(x)}{c\sqrt{2\cos(2x) + 2}} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \\
&= \frac{-\frac{2\cos(x)\sin(x)}{c\sqrt{2\cos(2x)+2}} + \tan\left(x\right)\frac{\sqrt{\frac{\cos(2x)}{2}+\frac{1}{2}}}{c}}{\left(\frac{\sqrt{\frac{\cos(2x)}{2}+\frac{1}{2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos\left(c\tau\right) + c_2 \sin\left(c\tau\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{\cos(2x)}{2}+\frac{1}{2}}dx}{c} \\
&= \frac{\sqrt{\frac{\cos(2x)}{2}+\frac{1}{2}}\,\sin\left(x\right)}{c\cos\left(x\right)}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos\left(\sin\left(x\right)\right) + c_2 \sin\left(\tan\left(x\right)\left|\cos\left(x\right)\right|\right)
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos\left(\sin\left(x\right)\right) + c_2 \sin\left(\tan\left(x\right)\left|\cos\left(x\right)\right|\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 \cos\left(\sin\left(x\right)\right) + c_2 \sin\left(\tan\left(x\right)\left|\cos\left(x\right)\right|\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   <- constant coefficients successful
   Change of variables used:
      [x = arcsin(t)]
   Linear ODE actually solved:
      (-2*t^2+2)*u(t)+(-2*t^2+2)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)+tan(x)*diff(y(x),x)+cos(x)^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(\sin(x)) + c_2 \cos(\sin(x))$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 18

```
DSolve[y''[x]+Tan[x]*y'[x]+Cos[x]^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \sin(\sin(x)) + c_1 \cos(\sin(x))$$

31.4 problem Ex 4

Internal problem ID [11300]
Internal file name [`OUTPUT/10285_Wednesday_December_21_2022_03_47_42_PM_71014606/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 54. Change of independent variable. Page 127
Problem number: Ex 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^6y'' + 3x^5y' + y = \frac{1}{x^2}}$$

31.4.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^6y'' + 3x^5y' + y = 0$$

In normal form the ode

$$x^6 y'' + 3x^5 y' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{1}{x^6} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\ &= \int \mathrm{e}^{-\left(\int \frac{3}{x} dx\right)} dx \\ &= \int e^{-3\ln(x)}\, dx \\ &= \int \frac{1}{x^3} dx \\ &= -\frac{1}{2x^2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^6}}{\frac{1}{x^6}} \\
&= 1
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\
&= \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +i \\
\lambda_2 &= -i
\end{aligned}$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^{0}(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(\frac{1}{2x^2}\right) - c_2 \sin\left(\frac{1}{2x^2}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\frac{1}{2x^2}\right) - c_2 \sin\left(\frac{1}{2x^2}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{1}{2x^2}\right)$$
$$y_2 = \sin\left(\frac{1}{2x^2}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{2x^2}\right) & \sin\left(\frac{1}{2x^2}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{2x^2}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{2x^2}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{2x^2}\right) & \sin\left(\frac{1}{2x^2}\right) \\ \frac{\sin\left(\frac{1}{2x^2}\right)}{x^3} & -\frac{\cos\left(\frac{1}{2x^2}\right)}{x^3} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{1}{2x^2}\right)\right)\left(-\frac{\cos\left(\frac{1}{2x^2}\right)}{x^3}\right) - \left(\sin\left(\frac{1}{2x^2}\right)\right)\left(\frac{\sin\left(\frac{1}{2x^2}\right)}{x^3}\right)$$

Which simplifies to

$$W = -\frac{\cos\left(\frac{1}{2x^2}\right)^2 + \sin\left(\frac{1}{2x^2}\right)^2}{x^3}$$

Which simplifies to

$$W = -\frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin\left(\frac{1}{2x^2}\right)}{x^2}}{-x^3}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin\left(\frac{1}{2x^2}\right)}{x^5} dx$$

Hence

$$u_1 = \frac{\cos\left(\frac{1}{2x^2}\right)}{x^2} - 2\sin\left(\frac{1}{2x^2}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos\left(\frac{1}{2x^2}\right)}{x^2}}{-x^3}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{1}{2x^2}\right)}{x^5}dx$$

Hence

$$u_2 = \frac{\sin\left(\frac{1}{2x^2}\right)}{x^2} + 2\cos\left(\frac{1}{2x^2}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\cos\left(\frac{1}{2x^2}\right)}{x^2} - 2\sin\left(\frac{1}{2x^2}\right)\right)\cos\left(\frac{1}{2x^2}\right) + \left(\frac{\sin\left(\frac{1}{2x^2}\right)}{x^2} + 2\cos\left(\frac{1}{2x^2}\right)\right)\sin\left(\frac{1}{2x^2}\right)$$

Which simplifies to

$$y_p(x) = \frac{1}{x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos\left(\frac{1}{2x^2}\right) - c_2\sin\left(\frac{1}{2x^2}\right)\right) + \left(\frac{1}{x^2}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1\cos\left(\frac{1}{2x^2}\right) - c_2\sin\left(\frac{1}{2x^2}\right) + \frac{1}{x^2} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1\cos\left(\frac{1}{2x^2}\right) - c_2\sin\left(\frac{1}{2x^2}\right) + \frac{1}{x^2}$$

Verified OK.

31.4.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^6, B = 3x^5, C = 1, f(x) = \frac{1}{x^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^6y'' + 3x^5y' + y = 0$$

In normal form the ode

$$x^6y'' + 3x^5y' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{1}{x^6} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^6}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{3}{c\sqrt{\frac{1}{x^6}}\,x^7}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{3}{c\sqrt{\frac{1}{x^6}}\,x^7} + \frac{3}{x}\frac{\sqrt{\frac{1}{x^6}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^6}}}{c}\right)^2} \\
&= 0
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{1}{x^6}}dx}{c} \\
&= -\frac{x\sqrt{\frac{1}{x^6}}}{2c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\frac{1}{2x^2}\right) - c_2 \sin\left(\frac{1}{2x^2}\right)$$

Now the particular solution to this ODE is found

$$x^6 y'' + 3x^5 y' + y = \frac{1}{x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{1}{2x^2}\right)$$

$$y_2 = \sin\left(\frac{1}{2x^2}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{2x^2}\right) & \sin\left(\frac{1}{2x^2}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{2x^2}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{2x^2}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{2x^2}\right) & \sin\left(\frac{1}{2x^2}\right) \\ \frac{\sin\left(\frac{1}{2x^2}\right)}{x^3} & -\frac{\cos\left(\frac{1}{2x^2}\right)}{x^3} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{1}{2x^2}\right)\right)\left(-\frac{\cos\left(\frac{1}{2x^2}\right)}{x^3}\right) - \left(\sin\left(\frac{1}{2x^2}\right)\right)\left(\frac{\sin\left(\frac{1}{2x^2}\right)}{x^3}\right)$$

Which simplifies to

$$W = -\frac{\cos\left(\frac{1}{2x^2}\right)^2 + \sin\left(\frac{1}{2x^2}\right)^2}{x^3}$$

Which simplifies to

$$W = -\frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin\left(\frac{1}{2x^2}\right)}{x^2}}{-x^3}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin\left(\frac{1}{2x^2}\right)}{x^5} dx$$

Hence

$$u_1 = \frac{\cos\left(\frac{1}{2x^2}\right)}{x^2} - 2\sin\left(\frac{1}{2x^2}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos\left(\frac{1}{2x^2}\right)}{x^2}}{-x^3}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{1}{2x^2}\right)}{x^5} dx$$

Hence

$$u_2 = \frac{\sin\left(\frac{1}{2x^2}\right)}{x^2} + 2\cos\left(\frac{1}{2x^2}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\cos\left(\frac{1}{2x^2}\right)}{x^2} - 2\sin\left(\frac{1}{2x^2}\right)\right)\cos\left(\frac{1}{2x^2}\right) + \left(\frac{\sin\left(\frac{1}{2x^2}\right)}{x^2} + 2\cos\left(\frac{1}{2x^2}\right)\right)\sin\left(\frac{1}{2x^2}\right)$$

Which simplifies to

$$y_p(x) = \frac{1}{x^2}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos\left(\frac{1}{2x^2}\right) - c_2 \sin\left(\frac{1}{2x^2}\right)\right) + \left(\frac{1}{x^2}\right) \\
&= c_1 \cos\left(\frac{1}{2x^2}\right) - c_2 \sin\left(\frac{1}{2x^2}\right) + \frac{1}{x^2}
\end{aligned}$$

Which simplifies to

$$y = c_1 \cos\left(\frac{1}{2x^2}\right) - c_2 \sin\left(\frac{1}{2x^2}\right) + \frac{1}{x^2}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos\left(\frac{1}{2x^2}\right) - c_2 \sin\left(\frac{1}{2x^2}\right) + \frac{1}{x^2} \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(\frac{1}{2x^2}\right) - c_2 \sin\left(\frac{1}{2x^2}\right) + \frac{1}{x^2}$$

Verified OK.

31.4.3 Solving as second order bessel ode ode

Writing the ode as

$$y''x^2 + 3y'x + \frac{y}{x^4} = \frac{1}{x^6} \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$y''x^2 + y'x + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$y''x^2 + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= -1 \\
\beta &= \frac{1}{2} \\
n &= \frac{1}{2} \\
\gamma &= -2
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{2c_1 \sin\left(\frac{1}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{1}{x^2}}} - \frac{2c_2 \cos\left(\frac{1}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{1}{x^2}}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{2c_1 \sin\left(\frac{1}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{1}{x^2}}} - \frac{2c_2 \cos\left(\frac{1}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{1}{x^2}}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{1}{2x^2}\right)$$

$$y_2 = \sin\left(\frac{1}{2x^2}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{2x^2}\right) & \sin\left(\frac{1}{2x^2}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{2x^2}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{2x^2}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{2x^2}\right) & \sin\left(\frac{1}{2x^2}\right) \\ \frac{\sin\left(\frac{1}{2x^2}\right)}{x^3} & -\frac{\cos\left(\frac{1}{2x^2}\right)}{x^3} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{1}{2x^2}\right)\right)\left(-\frac{\cos\left(\frac{1}{2x^2}\right)}{x^3}\right) - \left(\sin\left(\frac{1}{2x^2}\right)\right)\left(\frac{\sin\left(\frac{1}{2x^2}\right)}{x^3}\right)$$

Which simplifies to

$$W = -\frac{\cos\left(\frac{1}{2x^2}\right)^2 + \sin\left(\frac{1}{2x^2}\right)^2}{x^3}$$

Which simplifies to

$$W = -\frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin\left(\frac{1}{2x^2}\right)}{x^6}}{-\frac{1}{x}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin\left(\frac{1}{2x^2}\right)}{x^5} dx$$

Hence

$$u_1 = \frac{\cos\left(\frac{1}{2x^2}\right)}{x^2} - 2\sin\left(\frac{1}{2x^2}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos\left(\frac{1}{2x^2}\right)}{x^6}}{-\frac{1}{x}}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{1}{2x^2}\right)}{x^5} dx$$

Hence

$$u_2 = \frac{\sin\left(\frac{1}{2x^2}\right)}{x^2} + 2\cos\left(\frac{1}{2x^2}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\cos\left(\frac{1}{2x^2}\right)}{x^2} - 2\sin\left(\frac{1}{2x^2}\right)\right)\cos\left(\frac{1}{2x^2}\right) + \left(\frac{\sin\left(\frac{1}{2x^2}\right)}{x^2} + 2\cos\left(\frac{1}{2x^2}\right)\right)\sin\left(\frac{1}{2x^2}\right)$$

Which simplifies to

$$y_p(x) = \frac{1}{x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{2c_1\sin\left(\frac{1}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{1}{x^2}}} - \frac{2c_2\cos\left(\frac{1}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{1}{x^2}}}\right) + \left(\frac{1}{x^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{2c_1\sin\left(\frac{1}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{1}{x^2}}} - \frac{2c_2\cos\left(\frac{1}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{1}{x^2}}} + \frac{1}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{2c_1 \sin\left(\frac{1}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{1}{x^2}}} - \frac{2c_2 \cos\left(\frac{1}{2x^2}\right)}{x\sqrt{\pi}\,\sqrt{\frac{1}{x^2}}} + \frac{1}{x^2}$$

Verified OK.

31.4.4 Solving using Kovacic algorithm

Writing the ode as

$$x^6 y'' + 3x^5 y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^6 \\ B &= 3x^5 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^4 - 4}{4x^6} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^4 - 4 \\ t &= 4x^6 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^4 - 4}{4x^6}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 186: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^6$. There is a pole at $x = 0$ of order 6. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned}\alpha_c^{+} &= \frac{1}{2}\left(\frac{b}{a} + v\right)\\ \alpha_c^{-} &= \frac{1}{2}\left(-\frac{b}{a} + v\right)\end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{3}{4x^2} - \frac{1}{x^6}$$

There is pole in r at $x = 0$ of order 6, hence $v = 3$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i}{x^3} - \frac{3ix}{8} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 3$ the above becomes

$$[\sqrt{r}]_c = \frac{i}{x^3} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^3}$ is

$$a = i$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^4}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned}b &= (0) - (0)\\ &= 0\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= \frac{i}{x^3} \\
\alpha_c^{+} &= \frac{1}{2}\left(\frac{b}{a}+v\right) &= \frac{1}{2}\left(\frac{0}{i}+3\right) &= \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2}\left(-\frac{b}{a}+v\right) &= \frac{1}{2}\left(-\frac{0}{i}+3\right) &= \frac{3}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^4 - 4}{4x^6}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^4 - 4}{4x^6}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	6	$\frac{i}{x^3}$	$\frac{3}{2}$	$\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{+} = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{+} - (\alpha_{c_1}^{-}) \\
&= \frac{3}{2} - \left(\frac{3}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{i}{x^3} + \frac{3}{2x} + (0) \\
&= -\frac{i}{x^3} + \frac{3}{2x} \\
&= -\frac{i}{x^3} + \frac{3}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{i}{x^3} + \frac{3}{2x}\right)(0) + \left(\left(\frac{3i}{x^4} - \frac{3}{2x^2}\right) + \left(-\frac{i}{x^3} + \frac{3}{2x}\right)^2 - \left(\frac{3x^4 - 4}{4x^6}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{i}{x^3} + \frac{3}{2x}\right) dx} \\
&= x^{\frac{3}{2}} e^{\frac{i}{2x^2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^5}{x^6}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{i}{2x^2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^5}{x^6}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i\mathrm{e}^{-\frac{i}{x^2}}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{i}{2x^2}}\right) + c_2\left(\mathrm{e}^{\frac{i}{2x^2}}\left(-\frac{i\mathrm{e}^{-\frac{i}{x^2}}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^6y'' + 3x^5y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{\frac{i}{2x^2}} - \frac{ic_2 e^{-\frac{i}{2x^2}}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{\frac{i}{2x^2}}$$

$$y_2 = -\frac{ie^{-\frac{i}{2x^2}}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\frac{i}{2x^2}} & -\frac{ie^{-\frac{i}{2x^2}}}{2} \\ \frac{d}{dx}\left(e^{\frac{i}{2x^2}}\right) & \frac{d}{dx}\left(-\frac{ie^{-\frac{i}{2x^2}}}{2}\right) \end{vmatrix}$$

Which gives

$$W=\begin{vmatrix} e^{\frac{i}{2x^2}} & -\frac{i e^{-\frac{i}{2x^2}}}{2} \\ -\frac{i e^{\frac{i}{2x^2}}}{x^3} & \frac{e^{-\frac{i}{2x^2}}}{2x^3} \end{vmatrix}$$

Therefore

$$W=\left(e^{\frac{i}{2x^2}}\right)\left(\frac{e^{-\frac{i}{2x^2}}}{2x^3}\right)-\left(-\frac{i e^{-\frac{i}{2x^2}}}{2}\right)\left(-\frac{i e^{\frac{i}{2x^2}}}{x^3}\right)$$

Which simplifies to

$$W=\frac{e^{\frac{i}{2x^2}} e^{-\frac{i}{2x^2}}}{x^3}$$

Which simplifies to

$$W=\frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1=-\int \frac{-\frac{i e^{-\frac{i}{2x^2}}}{2x^2}}{x^3}\, dx$$

Which simplifies to

$$u_1=-\int -\frac{i e^{-\frac{i}{2x^2}}}{2x^5} dx$$

Hence

$$u_1=undefined$$

And Eq. (3) becomes

$$u_2=\int \frac{\frac{e^{\frac{i}{2x^2}}}{x^2}}{x^3}\, dx$$

Which simplifies to

$$u_2=\int \frac{e^{\frac{i}{2x^2}}}{x^5} dx$$

Hence

$$u_2=undefined$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined } e^{\frac{i}{2x^2}} - i \text{ undefined } e^{-\frac{i}{2x^2}}$$

Which simplifies to

$$y_p(x) = \left(i e^{-\frac{i}{2x^2}} + e^{\frac{i}{2x^2}}\right) \text{undefined}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{i}{2x^2}} - \frac{i c_2 e^{-\frac{i}{2x^2}}}{2}\right) + \left(\left(i e^{-\frac{i}{2x^2}} + e^{\frac{i}{2x^2}}\right) \text{undefined}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{i}{2x^2}} - \frac{i c_2 e^{-\frac{i}{2x^2}}}{2} + \left(i e^{-\frac{i}{2x^2}} + e^{\frac{i}{2x^2}}\right) \text{undefined} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\frac{i}{2x^2}} - \frac{i c_2 e^{-\frac{i}{2x^2}}}{2} + \left(i e^{-\frac{i}{2x^2}} + e^{\frac{i}{2x^2}}\right) \text{undefined}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(x^6*diff(y(x),x$2)+3*x^5*diff(y(x),x)+y(x)=1/x^2,y(x), singsol=all)
```

$$y(x) = \sin\left(\frac{1}{2x^2}\right)c_2 + \cos\left(\frac{1}{2x^2}\right)c_1 + \frac{1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.099 (sec). Leaf size: 32

```
DSolve[x^6*y''[x]+3*x^5*y'[x]+y[x]==1/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x^2} + c_1\cos\left(\frac{1}{2x^2}\right) - c_2\sin\left(\frac{1}{2x^2}\right)$$

31.5 problem Ex 5

Internal problem ID [11301]
Internal file name [`OUTPUT/10286_Wednesday_December_21_2022_03_47_44_PM_51163665/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 54. Change of independent variable. Page 127
Problem number: Ex 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{xy'' - (2x^2 + 1)\,y' - 8yx^3 = 4x^3 \mathrm{e}^{-x^2}}$$

31.5.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$xy'' + (-2x^2 - 1)\,y' - 8yx^3 = 0$$

In normal form the ode

$$xy'' + (-2x^2 - 1)\,y' - 8yx^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{-2x^2 - 1}{x} \\ q(x) &= -8x^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x) dx\right)} dx \\ &= \int \mathrm{e}^{-\left(\int \frac{-2x^2-1}{x} dx\right)} dx \\ &= \int e^{x^2 + \ln(x)}\, dx \\ &= \int x\, \mathrm{e}^{x^2} dx \\ &= \frac{\mathrm{e}^{x^2}}{2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-8x^2}{x^2 \mathrm{e}^{2x^2}} \\ &= -8\, \mathrm{e}^{-2x^2} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - 8\,\mathrm{e}^{-2x^2} y(\tau) = 0$$

But in terms of τ

$$-8\,\mathrm{e}^{-2x^2} = -\frac{2}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{2y(\tau)}{\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 2\tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r - 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau} + c_2\tau^2$$

The above solution is now transformed back to y using (6) which results in

$$y = 2c_1 \mathrm{e}^{-x^2} + \frac{c_2 \mathrm{e}^{2x^2}}{4}$$

Therefore the homogeneous solution y_h is

$$y_h = 2c_1 \mathrm{e}^{-x^2} + \frac{c_2 \mathrm{e}^{2x^2}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x^2}$$
$$y_2 = \mathrm{e}^{2x^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x^2} & \mathrm{e}^{2x^2} \\ \frac{d}{dx}\left(\mathrm{e}^{-x^2}\right) & \frac{d}{dx}\left(\mathrm{e}^{2x^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x^2} & e^{2x^2} \\ -2x\,e^{-x^2} & 4\,e^{2x^2}x \end{vmatrix}$$

Therefore

$$W = \left(e^{-x^2}\right)\left(4\,e^{2x^2}x\right) - \left(e^{2x^2}\right)\left(-2x\,e^{-x^2}\right)$$

Which simplifies to

$$W = 6x\,e^{x^2}$$

Which simplifies to

$$W = 6x\,e^{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\,e^{2x^2}x^3e^{-x^2}}{6x^2e^{x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2x}{3}dx$$

Hence

$$u_1 = -\frac{x^2}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\left(e^{-x^2}\right)^2 x^3}{6x^2e^{x^2}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{2x\,e^{-3x^2}}{3}dx$$

Hence

$$u_2 = -\frac{e^{-3x^2}}{9}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2 e^{-x^2}}{3} - \frac{e^{-3x^2} e^{2x^2}}{9}$$

Which simplifies to

$$y_p(x) = -\frac{e^{-x^2}(3x^2+1)}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(2c_1 e^{-x^2} + \frac{c_2 e^{2x^2}}{4}\right) + \left(-\frac{e^{-x^2}(3x^2+1)}{9}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = 2c_1 e^{-x^2} + \frac{c_2 e^{2x^2}}{4} - \frac{e^{-x^2}(3x^2+1)}{9} \tag{1}$$

Verification of solutions

$$y = 2c_1 e^{-x^2} + \frac{c_2 e^{2x^2}}{4} - \frac{e^{-x^2}(3x^2+1)}{9}$$

Verified OK.

31.5.2 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + (-2x^2 - 1)\, y' - 8yx^3 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x^2 - 1 \\ C &= -8x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{36x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 36x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{36x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 187: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 9x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 3x + \frac{1}{8x^3} - \frac{1}{384x^7} + \frac{1}{9216x^{11}} - \frac{5}{884736x^{15}} + \frac{7}{21233664x^{19}} - \frac{7}{339738624x^{23}} + \frac{11}{8153726976x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 3$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 3x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 9x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{36x^4+3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= (9x^2) + \left(\frac{3}{4x^2}\right) \\
&= 9x^2 + \frac{3}{4x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$
\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 3x \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{3} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{3} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{36x^4+3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$3x$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha^{\pm}_{\infty}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha^{-}_{\infty} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha^{-}_{\infty} - \left(\alpha^{-}_{c_1}\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha^{-}_{c_1}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\left(3x\right) \\ &= -\frac{1}{2x} - 3x \\ &= -\frac{1}{2x} - 3x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x} - 3x\right)(0) + \left(\left(\frac{1}{2x^2} - 3\right) + \left(-\frac{1}{2x} - 3x\right)^2 - \left(\frac{36x^4 + 3}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2x} - 3x\right)dx} \\ &= \frac{e^{-\frac{3x^2}{2}}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-1}{x}\,dx} \\
&= z_1 e^{\frac{x^2}{2}+\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\,\mathrm{e}^{\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{3x^2}}{6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}\left(\frac{\mathrm{e}^{3x^2}}{6}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$xy''+\left(-2x^2-1\right)y'-8yx^3=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1\mathrm{e}^{-x^2}+\frac{c_2\mathrm{e}^{2x^2}}{6}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1,u_2 to be determined, and y_1,y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1=\mathrm{e}^{-x^2}$$
$$y_2=\frac{\mathrm{e}^{2x^2}}{6}$$

In the Variation of parameters u_1,u_2 are found using

$$u_1=-\int\frac{y_2f(x)}{aW(x)} \tag{2}$$
$$u_2=\int\frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}$. Hence

$$W=\begin{vmatrix}\mathrm{e}^{-x^2} & \frac{\mathrm{e}^{2x^2}}{6}\\ \frac{d}{dx}\left(\mathrm{e}^{-x^2}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{2x^2}}{6}\right)\end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x^2} & \frac{\mathrm{e}^{2x^2}}{6} \\ -2x\,\mathrm{e}^{-x^2} & \frac{2\,\mathrm{e}^{2x^2} x}{3} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x^2}\right)\left(\frac{2\,\mathrm{e}^{2x^2} x}{3}\right) - \left(\frac{\mathrm{e}^{2x^2}}{6}\right)\left(-2x\,\mathrm{e}^{-x^2}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-x^2}\mathrm{e}^{2x^2} x$$

Which simplifies to

$$W = x\,\mathrm{e}^{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2\,\mathrm{e}^{2x^2} x^3 \mathrm{e}^{-x^2}}{3}}{x^2 \mathrm{e}^{x^2}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{2x}{3} dx$$

Hence

$$u_1 = -\frac{x^2}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\,\mathrm{e}^{-2x^2} x^3}{x^2 \mathrm{e}^{x^2}}\, dx$$

Which simplifies to

$$u_2 = \int 4x\,\mathrm{e}^{-3x^2} dx$$

Hence

$$u_2 = -\frac{2\,\mathrm{e}^{-3x^2}}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2 e^{-x^2}}{3} - \frac{e^{-3x^2} e^{2x^2}}{9}$$

Which simplifies to

$$y_p(x) = -\frac{e^{-x^2}(3x^2+1)}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x^2} + \frac{c_2 e^{2x^2}}{6}\right) + \left(-\frac{e^{-x^2}(3x^2+1)}{9}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x^2} + \frac{c_2 e^{2x^2}}{6} - \frac{e^{-x^2}(3x^2+1)}{9} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x^2} + \frac{c_2 e^{2x^2}}{6} - \frac{e^{-x^2}(3x^2+1)}{9}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve(x*diff(y(x),x$2)-(2*x^2+1)*diff(y(x),x)-8*x^3*y(x)=4*x^3*exp(-x^2),y(x), singsol=all)
```

$$y(x) = \frac{(-x^2 + 3c_1)\,\mathrm{e}^{-x^2}}{3} + \mathrm{e}^{2x^2} c_2$$

✓ Solution by Mathematica

Time used: 0.105 (sec). Leaf size: 38

```
DSolve[x*y''[x]-(2*x^2+1)*y'[x]-8*x^3*y[x]==4*x^3*Exp[-x^2],y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{9} e^{-x^2} \left(-3x^2 + 9c_1 e^{3x^2} - 1 + 9c_2\right)$$

32 Chapter VIII, Linear differential equations of the second order. Article 55. Summary. Page 129

32.1 problem Ex 1

Internal problem ID [11302]
Internal file name [`OUTPUT/10287_Wednesday_December_21_2022_03_47_45_PM_74052029/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 55. Summary. Page 129
Problem number: Ex 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{xy'' - (x+3)\,y' + 3y = 0}$$

32.1.1 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + (-x-3)\,y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x-3 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6x + 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 6x + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 188: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4} + \frac{15}{4x^2} - \frac{3}{2x}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2x} + \frac{3}{2x^2} + \frac{9}{2x^3} + \frac{45}{4x^4} + \frac{81}{4x^5} + \frac{27}{4x^6} - \frac{567}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 6x + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6x + 15}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-6x + 15}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 6x + 15}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{3}{2x} \\
&= \frac{x-3}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{3}{2x}\right)(0) + \left(\left(\frac{3}{2x^2}\right) + \left(\frac{1}{2} - \frac{3}{2x}\right)^2 - \left(\frac{x^2-6x+15}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{2} - \frac{3}{2x}\right) dx} \\
&= \frac{e^{\frac{x}{2}}}{x^{\frac{3}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x-3}{x}\, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{3\ln(x)}{2}} \\
&= z_1 \left(x^{\frac{3}{2}} e^{\frac{x}{2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-3}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1\left(-\mathrm{e}^{-x}\left(x^3+3x^2+6x+6\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-\mathrm{e}^{-x}\left(x^3+3x^2+6x+6\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^x c_1 + c_2\left(-x^3-3x^2-6x-6\right) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^x c_1 + c_2\left(-x^3-3x^2-6x-6\right)$$

Verified OK.

32.1.2 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$xy'' + (-x-3)\,y' + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$p(x) = \frac{-x-3}{x}$$
$$q(x) = \frac{3}{x}$$
$$r(x) = 0$$

The Lagrange adjoint ode is given by

$$\xi'' - (\xi\, p)' + \xi q = 0$$
$$\xi'' - \left(\frac{(-x-3)\,\xi(x)}{x}\right)' + \left(\frac{3\xi(x)}{x}\right) = 0$$
$$\xi''(x) - \frac{(-x-3)\,\xi'(x)}{x} + \left(\frac{4}{x} + \frac{-x-3}{x^2}\right)\xi(x) = 0$$

Which is solved for $\xi(x)$. In normal form the ode

$$\xi''(x)\, x^2 + \left(x^2 + 3x\right)\xi'(x) + (-3+3x)\,\xi(x) = 0 \tag{1}$$

Becomes

$$\xi''(x) + p(x)\,\xi'(x) + q(x)\,\xi(x) = 0 \tag{2}$$

Where

$$p(x) = \frac{x+3}{x}$$
$$q(x) = \frac{-3+3x}{x^2}$$

Applying change of variables on the depndent variable $\xi(x) = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not $\xi(x)$.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(x+3)}{x^2} + \frac{-3+3x}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(-\frac{6}{x} + \frac{x+3}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(x-3)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(x-3)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(x-3)\, u}{x} \end{aligned}$$

Where $f(x) = -\frac{x-3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{x-3}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{x-3}{x}\, dx \\ \ln(u) &= -x + 3\ln(x) + c_1 \\ u &= \mathrm{e}^{-x+3\ln(x)+c_1} \\ &= c_1 \mathrm{e}^{-x+3\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = c_1 x^3 \mathrm{e}^{-x}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\left(x^3+3x^2+6x+6\right) c_1 \mathrm{e}^{-x} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
\xi(x) &= v(x)\, x^n \\
&= \frac{-(x^3+3x^2+6x+6)\, c_1 \mathrm{e}^{-x} + c_2}{x^3} \\
&= \frac{-(x^3+3x^2+6x+6)\, c_1 \mathrm{e}^{-x} + c_2}{x^3}
\end{aligned}
$$

The original ode (2) now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}
\end{aligned}
$$

$$
y' + y\left(\frac{-x-3}{x} - \frac{\left(\frac{-(3x^2+6x+6)c_3\mathrm{e}^{-x}+(x^3+3x^2+6x+6)c_3\mathrm{e}^{-x}}{x^3} - \frac{3\left(-(x^3+3x^2+6x+6)c_3\mathrm{e}^{-x}+c_2\right)}{x^4}\right)x^3}{-\left(x^3+3x^2+6x+6\right)c_3\mathrm{e}^{-x}+c_2}\right) = 0
$$

Which is now a first order ode. This is now solved for y. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{y(-3c_3x^2 + c_2\mathrm{e}^x - 6c_3x - 6c_3)}{-c_3x^3 - 3c_3x^2 + c_2\mathrm{e}^x - 6c_3x - 6c_3}
\end{aligned}
$$

Where $f(x) = \frac{-3c_3x^2+c_2\mathrm{e}^x-6c_3x-6c_3}{-c_3x^3-3c_3x^2+c_2\mathrm{e}^x-6c_3x-6c_3}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= \frac{-3c_3x^2 + c_2\mathrm{e}^x - 6c_3x - 6c_3}{-c_3x^3 - 3c_3x^2 + c_2\mathrm{e}^x - 6c_3x - 6c_3}\, dx \\
\int \frac{1}{y}\, dy &= \int \frac{-3c_3x^2 + c_2\mathrm{e}^x - 6c_3x - 6c_3}{-c_3x^3 - 3c_3x^2 + c_2\mathrm{e}^x - 6c_3x - 6c_3}\, dx \\
\ln(y) &= \ln\left(-c_3x^3 - 3c_3x^2 + c_2\mathrm{e}^x - 6c_3x - 6c_3\right) + c_3 \\
y &= \mathrm{e}^{\ln\left(-c_3x^3-3c_3x^2+c_2\mathrm{e}^x-6c_3x-6c_3\right)+c_3} \\
&= c_3\left(-c_3x^3 - 3c_3x^2 + c_2\mathrm{e}^x - 6c_3x - 6c_3\right)
\end{aligned}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = e^{x} c_2 c_3 + c_3 \left(-c_3 x^3 - 3 c_3 x^2 - 6 c_3 x - 6 c_3\right)$$

Summary

The solution(s) found are the following

$$y = e^{x} c_2 c_3 + c_3 \left(-c_3 x^3 - 3 c_3 x^2 - 6 c_3 x - 6 c_3\right) \tag{1}$$

Verification of solutions

$$y = e^{x} c_2 c_3 + c_3 \left(-c_3 x^3 - 3 c_3 x^2 - 6 c_3 x - 6 c_3\right)$$

Verified OK.

32.1.3 Maple step by step solution

Let's solve

$y'' x + (-x - 3) y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y}{x} + \frac{(x+3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+3)y'}{x} + \frac{3y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x+3}{x}, P_3(x) = \frac{3}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$y''x + (-x-3)\,y' + 3y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$

  - Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-4+r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} (a_{k+1}(k+1+r)\,(k+r-3) - a_k(k+r-3))\,x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-4+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 4\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-3)\,(a_{k+1}(k+1+r) - a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 4$

  $$a_{k+1} = \frac{a_k}{k+5}$$

- Solution for $r = 4$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+1} = \frac{a_k}{k+5}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+4}\right), a_{1+k} = \frac{a_k}{1+k}, b_{1+k} = \frac{b_k}{k+5}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(x*diff(y(x),x$2)-(x+3)*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + c_2(x^3 + 3x^2 + 6x + 6)$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 29

```
DSolve[x*y''[x]-(x+3)*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2\left(x^3 + 3x^2 + 6x + 6\right)$$

32.2 problem Ex 2

Internal problem ID [11303]
Internal file name [`OUTPUT/10288_Wednesday_December_21_2022_03_47_47_PM_24759415/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 55. Summary. Page 129
Problem number: Ex 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-3)\,y''-(4x-9)\,y'+(3x-6)\,y=0}$$

32.2.1 Solving using Kovacic algorithm

Writing the ode as

$$(x-3)\,y''+(-4x+9)\,y'+(3x-6)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-3 \\ B &= -4x+9 \\ C &= 3x-6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 12x + 15}{4(x-3)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 - 12x + 15 \\ t &= 4(x-3)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 12x + 15}{4(x-3)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 190: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-3)^2$. There is a pole at $x = 3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{x-3} + \frac{15}{4\left(x-3\right)^2}$$

For the pole at $x = 3$ let b be the coefficient of $\frac{1}{(x-3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{3}{2x} + \frac{21}{4x^2} + \frac{135}{8x^3} + \frac{1665}{32x^4} + \frac{10071}{64x^5} + \frac{60453}{128x^6} + \frac{362151}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 12x + 15}{4x^2 - 24x + 36} \\ &= Q + \frac{R}{4x^2 - 24x + 36} \\ &= (1) + \left(\frac{12x - 21}{4x^2 - 24x + 36}\right) \\ &= 1 + \frac{12x - 21}{4x^2 - 24x + 36} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 12. Dividing this by leading coefficient in t which is 4 gives 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{3}{1} - 0\right) &&= \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{3}{1} - 0\right) &&= -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 - 12x + 15}{4(x-3)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
3	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{3}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{2(x-3)} + (-)(1) \\
&= -\frac{3}{2(x-3)} - 1 \\
&= -\frac{2x-3}{2(x-3)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{3}{2(x-3)} - 1\right)(0) + \left(\left(\frac{3}{2(x-3)^2}\right) + \left(-\frac{3}{2(x-3)} - 1\right)^2 - \left(\frac{4x^2 - 12x + 15}{4(x-3)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{3}{2(x-3)} - 1\right) dx} \\
&= \frac{\mathrm{e}^{-x}}{(x-3)^{\frac{3}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x+9}{x-3} \, dx} \\
&= z_1 e^{2x + \frac{3\ln(x-3)}{2}} \\
&= z_1 \left((x-3)^{\frac{3}{2}} \, \mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x+9}{x-3}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{4x+3\ln(x-3)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}(4x^3-42x^2+150x-183)}{8}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{\mathrm{e}^{2x}(4x^3-42x^2+150x-183)}{8}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^x c_1 + \frac{c_2 \mathrm{e}^{3x}(4x^3-42x^2+150x-183)}{8} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^x c_1 + \frac{c_2 \mathrm{e}^{3x}(4x^3-42x^2+150x-183)}{8}$$

Verified OK.

32.2.2 Maple step by step solution

Let's solve

$(x-3)\, y''+(-4x+9)\, y'+(3x-6)\, y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3(x-2)y}{x-3}+\frac{(4x-9)y'}{x-3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(4x-9)y'}{x-3}+\frac{3(x-2)y}{x-3}=0$

□ Check to see if $x_0=3$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{4x-9}{x-3}, P_3(x)=\frac{3(x-2)}{x-3}\right]$

  ◦ $(x-3)\cdot P_2(x)$ is analytic at $x=3$

  $\left((x-3)\cdot P_2(x)\right)\Big|_{x=3}=-3$

  ◦ $(x-3)^2\cdot P_3(x)$ is analytic at $x=3$

  $\left((x-3)^2\cdot P_3(x)\right)\Big|_{x=3}=0$

  ◦ $x=3$is a regular singular point

  Check to see if $x_0=3$ is a regular singular point

  $x_0=3$

- Multiply by denominators

  $(x-3)\, y''+(-4x+9)\, y'+(3x-6)\, y=0$

- Change variables using $x=u+3$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(-4u-3)\left(\frac{d}{du}y(u)\right)+(3u+3)\, y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^{m} \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^{m} \cdot y(u) = \sum_{k=0}^{\infty} a_{k} u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^{m} \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^{m} \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k->k+1$

  $$u \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}r(-4+r)\, u^{-1+r} + (a_{1}(1+r)\,(-3+r) - a_{0}(-3+4r))\, u^{r} + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-3+r)\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$r(-4+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 4\}$$

- Each term must be 0

  $$a_{1}(1+r)\,(-3+r) - a_{0}(-3+4r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)\,(k-3+r) - 4a_{k}k - 4a_{k}r + 3a_{k} + 3a_{k-1} = 0$$

- Shift index using $k->k+1$

  $$a_{k+2}(k+2+r)\,(k-2+r) - 4a_{k+1}(k+1) - 4ra_{k+1} + 3a_{k+1} + 3a_{k} = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{4ka_{k+1}+4ra_{k+1}-3a_k+a_{k+1}}{(k+2+r)(k-2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{4ka_{k+1}-3a_k+a_{k+1}}{(k+2)(k-2)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

$$a_{k+2} = \frac{4ka_{k+1}-3a_k+a_{k+1}}{(k+2)(k-2)}$$

- Recursion relation for $r = 4$

$$a_{k+2} = \frac{4ka_{k+1}-3a_k+17a_{k+1}}{(k+6)(k+2)}$$

- Solution for $r = 4$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+4}, a_{k+2} = \frac{4ka_{k+1}-3a_k+17a_{k+1}}{(k+6)(k+2)}, 5a_1 - 13a_0 = 0\right]$$

- Revert the change of variables $u = x - 3$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-3)^{k+4}, a_{k+2} = \frac{4ka_{k+1}-3a_k+17a_{k+1}}{(k+6)(k+2)}, 5a_1 - 13a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve((x-3)*diff(y(x),x$2)-(4*x-9)*diff(y(x),x)+(3*x-6)*y(x)=0,y(x), singsol=all)
```

$$y(x) = 4c_2\left(x^3 - \frac{21}{2}x^2 + \frac{75}{2}x - \frac{183}{4}\right)e^{3x} + c_1 e^x$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 42

```
DSolve[(x-3)*y''[x]-(4*x-9)*y'[x]+(3*x-6)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}c_2 e^{3x-9}(4x^3 - 42x^2 + 150x - 183) + c_1 e^{x-3}$$

32.3 problem Ex 3

Internal problem ID [11304]
Internal file name [`OUTPUT/10289_Wednesday_December_21_2022_03_47_49_PM_39648213/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 55. Summary. Page 129
Problem number: Ex 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y''x^2 + 4y'x + \left(-x^2 + 2\right) y = 0}$$

32.3.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{4}{x}$$
$$q(x) = \frac{-x^2 + 2}{x^2}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{-x^2+2}{x^2} - \frac{\left(\frac{4}{x}\right)'}{2} - \frac{\left(\frac{4}{x}\right)^2}{4} \\
&= \frac{-x^2+2}{x^2} - \frac{\left(-\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\
&= \frac{-x^2+2}{x^2} - \left(-\frac{2}{x^2}\right) - \frac{4}{x^2} \\
&= -1
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{4}{x}}{2}} \\
&= \frac{1}{x^2}
\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{x^2} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) - v(x) = 0$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ v(x) &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$v(x) = \mathrm{e}^x c_1 + c_2 \mathrm{e}^{-x}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x) \\ &= \left(\mathrm{e}^x c_1 + c_2 \mathrm{e}^{-x}\right) (z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x^2}$$

Hence (7) becomes

$$y = \frac{e^x c_1 + c_2 e^{-x}}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^x c_1 + c_2 e^{-x}}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{e^x c_1 + c_2 e^{-x}}{x^2}$$

Verified OK.

32.3.2 Solving as second order bessel ode ode

Writing the ode as

$$y''x^2 + 4y'x + (-x^2 + 2)\, y = 0 \tag{1}$$

Bessel ode has the form

$$y''x^2 + y'x + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$y''x^2 + (1 - 2\alpha)\, xy' + (\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= -\frac{3}{2} \\
\beta &= i \\
n &= -\frac{1}{2} \\
\gamma &= 1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \sqrt{2}\, \cosh(x)}{x^{\frac{3}{2}} \sqrt{\pi}\, \sqrt{ix}} + \frac{i c_2 \sqrt{2}\, \sinh(x)}{x^{\frac{3}{2}} \sqrt{\pi}\, \sqrt{ix}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1\sqrt{2}\,\cosh(x)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{ix}} + \frac{ic_2\sqrt{2}\,\sinh(x)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{ix}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{2}\,\cosh(x)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{ix}} + \frac{ic_2\sqrt{2}\,\sinh(x)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{ix}}$$

Verified OK.

32.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y''x^2 + 4y'x + (-x^2 + 2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= -x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 192: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1\left(\frac{1}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{-x}}{x^2}\right) + c_2\left(\frac{\mathrm{e}^{-x}}{x^2}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{-x}}{x^2} + \frac{c_2 e^{x}}{2x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{-x}}{x^2} + \frac{c_2 e^{x}}{2x^2}$$

Verified OK.

32.3.4 Maple step by step solution

Let's solve

$y''x^2 + 4y'x + (-x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2-2)y}{x^2} - \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{4y'}{x} - \frac{(x^2-2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = -\frac{x^2-2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$$y''x^2+4y'x+(-x^2+2)\,y=0$$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..2$

$$x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k->k-m$

$$x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x \cdot y'$ to series expansion

$$x \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$$

  - Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(2+r)\,(1+r)\,x^r+a_1(3+r)\,(2+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)\,(k+r+1)-a_{k-2}\right) x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)\,(1+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{-2,-1\}$$

- Each term must be 0

$$a_1(3+r)\,(2+r)=0$$

- Solve for the dependent coefficient(s)

$$a_1=0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+2)\,(k+r+1)-a_{k-2}=0$$

- Shift index using $k->k+2$

$$a_{k+2}(k+4+r)\,(k+3+r)-a_k=0$$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+4+r)(k+3+r)}$

- Recursion relation for $r = -2$

  $a_{k+2} = \frac{a_k}{(k+2)(k+1)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = -1$

  $a_{k+2} = \frac{a_k}{(k+3)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = \frac{a_k}{(k+2)(1+k)}, a_1 = 0, b_{k+2} = \frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+(2-x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sinh(x) + c_2 \cosh(x)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 28

```
DSolve[x^2*y''[x]+4*x*y'[x]+(2-x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-x} + c_2 e^x}{2x^2}$$

32.4 problem Ex 4

Internal problem ID [11305]
Internal file name [`OUTPUT/10290_Wednesday_December_21_2022_03_47_50_PM_93363174/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 55. Summary. Page 129
Problem number: Ex 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{\left(x^{2}+1\right) y''-2y'x+2y=0}$$

32.4.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$\left(x^{2}+1\right) y''-2y'x+2y=0 \tag{1}$$

Becomes

$$y''+p(x)\, y'+q(x)\, y=0 \tag{2}$$

Where

$$p(x) = -\frac{2x}{x^{2}+1}$$
$$q(x) = \frac{2}{x^{2}+1}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{2n}{x^2+1} + \frac{2}{x^2+1} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} - \frac{2x}{x^2+1}\right) v'(x) &= 0 \\ v''(x) + \frac{2v'(x)}{x\,(x^2+1)} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x\,(x^2+1)} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x\,(x^2+1)} \end{aligned}$$

Where $f(x) = -\frac{2}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x\,(x^2+1)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2}{x\,(x^2+1)}\,dx \\
\ln(u) &= -2\ln(x) + \ln\left(x^2+1\right) + c_1 \\
u &= \mathrm{e}^{-2\ln(x)+\ln(x^2+1)+c_1} \\
&= c_1 \mathrm{e}^{-2\ln(x)+\ln(x^2+1)}
\end{aligned}$$

Which simplifies to

$$u(x) = c_1\left(1 + \frac{1}{x^2}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1\left(x - \frac{1}{x}\right) + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(c_1\left(x - \frac{1}{x}\right) + c_2\right)x \\
&= c_1x^2 + c_2x - c_1
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(c_1\left(x - \frac{1}{x}\right) + c_2\right)x \tag{1}$$

Verification of solutions

$$y = \left(c_1\left(x - \frac{1}{x}\right) + c_2\right)x$$

Verified OK.

32.4.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 + 1 \\ B &= -2x \\ C &= 2 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2 + 1\right)(0) + (-2x)(-2) + (2)(-2x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x\left(x^2+1\right)v''+(-4)\,v'=0$$

Now by applying $v'=u$ the above becomes

$$\left(-2x^3-2x\right)u'(x)-4u(x)=0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{2u}{x\left(x^2+1\right)}
\end{aligned}$$

Where $f(x)=-\frac{2}{x(x^2+1)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x\left(x^2+1\right)}\,dx\\
\int \frac{1}{u}\,du &= \int -\frac{2}{x\left(x^2+1\right)}\,dx\\
\ln(u) &= -2\ln(x)+\ln\left(x^2+1\right)+c_1\\
u &= \mathrm{e}^{-2\ln(x)+\ln\left(x^2+1\right)+c_1}\\
&= c_1\mathrm{e}^{-2\ln(x)+\ln\left(x^2+1\right)}
\end{aligned}$$

Which simplifies to

$$u(x)=c_1\left(1+\frac{1}{x^2}\right)$$

The ode for v now becomes

$$\begin{aligned}
v' &= u\\
&= c_1\left(1+\frac{1}{x^2}\right)
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{\left(x^2+1\right)c_1}{x^2}\,\mathrm{d}x\\
&= c_1\left(x-\frac{1}{x}\right)+c_2
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (-2x)\left(c_1\left(x - \frac{1}{x}\right) + c_2\right) \\
&= -2c_1x^2 - 2c_2x + 2c_1
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = -2c_1x^2 - 2c_2x + 2c_1 \tag{1}
$$

Verification of solutions

$$
y = -2c_1x^2 - 2c_2x + 2c_1
$$

Verified OK.

32.4.3 Solving using Kovacic algorithm

Writing the ode as

$$
\left(x^2 + 1\right)y'' - 2y'x + 2y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 + 1 \\
B &= -2x \\
C &= 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = \left(x^2+1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 194: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (-)\,(0) \\ &= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} \\ &= \frac{x-2i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{\frac{3}{2}}}{(ix+1)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\
&= z_1 e^{\frac{\ln(x^2+1)}{2}} \\
&= z_1\left(\sqrt{x^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx$$

$$= y_1 \left(-\frac{x}{(x+i)^2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2 \left(\frac{(x^2+1)^2}{(ix+1)^2} \left(-\frac{x}{(x+i)^2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 (x^2+1)^2}{(ix+1)^2} + \frac{c_2 (x^2+1)^2 x}{(x-i)^2 (x+i)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 (x^2+1)^2}{(ix+1)^2} + \frac{c_2 (x^2+1)^2 x}{(x-i)^2 (x+i)^2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((x^2+1)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_2x^2 + c_1x - c_2$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 21

```
DSolve[(x^2+1)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2x - c_1(x - i)^2$$

32.5 problem Ex 5

Internal problem ID [11306]
Internal file name [`OUTPUT/10291_Wednesday_December_21_2022_03_47_52_PM_76349440/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 55. Summary. Page 129
Problem number: Ex 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' - (2x - 1)\,y' + y(x - 1) = 0}$$

32.5.1 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + (-2x + 1)\,y' + y(x - 1) = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x + 1 \\ C &= x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 195: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x+1}{x} \, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x+1}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\ln(x))) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \mathrm{e}^x c_1 + c_2 \mathrm{e}^x \ln(x) \tag{1}$$

<u>Verification of solutions</u>

$$y = \mathrm{e}^x c_1 + c_2 \mathrm{e}^x \ln(x)$$

Verified OK.

32.5.2 Maple step by step solution

Let's solve
$y''x + (-2x+1)\,y' + y(x-1) = 0$

- Highest derivative means the order of the ODE is 2
  y''
- Isolate 2nd derivative
  $y'' = -\frac{(x-1)y}{x} + \frac{(2x-1)y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $y'' - \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x + (-2x+1)\,y' + y(x-1) = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 - a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k+2r+1) + a_{k-1}\right) x^k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term must be 0

  $a_1(1+r)^2 - a_0(1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + (-2k-1)\, a_k + a_{k-1} = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2)^2 + (-2k-3)\, a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(x*diff(y(x),x$2)-(2*x-1)*diff(y(x),x)+(x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{x}(c_1 + c_2 \ln(x))$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 17

```
DSolve[x*y''[x]-(2*x-1)*y'[x]+(x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}(c_2 \log(x) + c_1)$$

32.6 problem Ex 6

Internal problem ID [11307]
Internal file name [`OUTPUT/10292_Wednesday_December_21_2022_03_47_53_PM_82182492/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 55. Summary. Page 129
Problem number: Ex 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y''x^2 - 4y'x + (x^2 + 6)\, y = 0}$$

32.6.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{x^2 + 6}{x^2}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{x^2+6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\
&= \frac{x^2+6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\
&= \frac{x^2+6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\
&= 1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{4}{x}}{2}} \\
&= x^2
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$y = v(x)\, x^2 \tag{4}$$

Applying this change of variable to the original ode results in

$$x^4(v''(x) + v(x)) = 0$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x) z(x) \\ &= (c_1 \cos(x) + c_2 \sin(x))(z(x)) \qquad (7)\end{aligned}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = (c_1 \cos(x) + c_2 \sin(x))\, x^2$$

Summary
The solution(s) found are the following

$$y = (c_1 \cos(x) + c_2 \sin(x))\, x^2 \tag{1}$$

Verification of solutions

$$y = (c_1 \cos(x) + c_2 \sin(x))\, x^2$$

Verified OK.

32.6.2 Solving as second order bessel ode ode

Writing the ode as

$$y''x^2 - 4y'x + (x^2 + 6)\, y = 0 \tag{1}$$

Bessel ode has the form

$$y''x^2 + y'x + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$y''x^2 + (1 - 2\alpha)\, xy' + (\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{5}{2}$$
$$\beta = 1$$
$$n = -\frac{1}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x^2 \sqrt{2}\, \cos(x)}{\sqrt{\pi}} + \frac{c_2 x^2 \sqrt{2}\, \sin(x)}{\sqrt{\pi}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2 \sqrt{2}\, \cos(x)}{\sqrt{\pi}} + \frac{c_2 x^2 \sqrt{2}\, \sin(x)}{\sqrt{\pi}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2 \sqrt{2}\, \cos(x)}{\sqrt{\pi}} + \frac{c_2 x^2 \sqrt{2}\, \sin(x)}{\sqrt{\pi}}$$

Verified OK.

32.6.3 Solving using Kovacic algorithm

Writing the ode as

$$y''x^2 - 4y'x + (x^2 + 6)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 197: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\ &= z_1 e^{2\ln(x)} \\ &= z_1\left(x^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\, x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\cos(x)\, x^2\right) + c_2\left(\cos(x)\, x^2(\tan(x))\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x)\, x^2 + c_2 \sin(x)\, x^2 \tag{1}$$

Verification of solutions

$$y = c_1 \cos(x)\, x^2 + c_2 \sin(x)\, x^2$$

Verified OK.

32.6.4 Maple step by step solution

Let's solve

$y'' x^2 - 4y' x + (x^2 + 6)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+6)y}{x^2} + \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x} + \frac{(x^2+6)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{4}{x}, P_3(x) = \frac{x^2+6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 6$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y'' x^2 - 4y' x + (x^2 + 6)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

  ◦ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-3+r)\, x^r + a_1(-1+r)\,(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-3) + a_{k-2})\, x^{k}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 3\}$

- Each term must be 0

  $a_1(-1+r)\,(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)\,(k+r-3) + a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+r)\,(k+r-1) + a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k}{(k+r)(k+r-1)}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{a_k}{(k+2)(1+k)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(6+x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^2(c_1 \sin(x) + c_2 \cos(x))$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]-4*x*y'[x]+(6+x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-ix}x^2\left(2c_1 - ic_2e^{2ix}\right)$$

32.7 problem Ex 7

Internal problem ID [11308]
Internal file name [`OUTPUT/10293_Wednesday_December_21_2022_03_47_55_PM_31561280/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 55. Summary. Page 129
Problem number: Ex 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^3-1)\,y''-6x^2y'+6yx=0}$$

32.7.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$(2x^3-1)\,y''-6x^2y'+6yx=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=-\frac{6x^2}{2x^3-1}\\ q(x)&=\frac{6x}{2x^3-1}\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{6nx}{2x^3 - 1} + \frac{6x}{2x^3 - 1} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} - \frac{6x^2}{2x^3 - 1}\right) v'(x) &= 0 \\ v''(x) + \frac{(-2x^3 - 2)\, v'(x)}{2x^4 - x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(-2x^3 - 2)\, u(x)}{2x^4 - x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= \frac{2u(x^3 + 1)}{x\,(2x^3 - 1)} \end{aligned}$$

Where $f(x) = \frac{2x^3+2}{x(2x^3-1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{2x^3+2}{x\left(2x^3-1\right)}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{2x^3+2}{x\left(2x^3-1\right)}\,dx \\
\ln\left(u\right) &= \ln\left(2x^3-1\right) - 2\ln\left(x\right) + c_1 \\
u &= \mathrm{e}^{\ln\left(2x^3-1\right) - 2\ln\left(x\right) + c_1} \\
&= c_1 \mathrm{e}^{\ln\left(2x^3-1\right) - 2\ln\left(x\right)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_1\left(2x - \frac{1}{x^2}\right)
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1\left(x^2 + \frac{1}{x}\right) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(c_1\left(x^2 + \frac{1}{x}\right) + c_2\right)x \\
&= c_1 x^3 + c_2 x + c_1
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(c_1\left(x^2 + \frac{1}{x}\right) + c_2\right)x \tag{1}
$$

Verification of solutions

$$
y = \left(c_1\left(x^2 + \frac{1}{x}\right) + c_2\right)x
$$

Verified OK.

32.7.2 Solving using Kovacic algorithm

Writing the ode as

$$(2x^3-1)\,y''-6x^2y'+6yx=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3-1\\B&=-6x^2\\C&=6x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x(x^3+4)}{(2x^3-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=3x(x^3+4)\\t&=(2x^3-1)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x(x^3+4)}{(2x^3-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y=z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 199: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 6 - 4\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^3-1)^2$. There is a pole at $x = \frac{2^{\frac{2}{3}}}{2}$ of order 2. There is a pole at $x = -\frac{2^{\frac{2}{3}}}{4} + \frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}$ of order 2. There is a pole at $x = -\frac{2^{\frac{2}{3}}}{4} - \frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1,2,4,6,12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(x-\frac{2^{\frac{2}{3}}}{2}\right)^2} + \frac{3}{4\left(x+\frac{2^{\frac{2}{3}}}{4}-\frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^2} + \frac{3}{4\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^2} - \frac{2^{\frac{1}{3}}}{2\left(x-\frac{2^{\frac{2}{3}}}{2}\right)} - \frac{\left(-\frac{2^{\frac{2}{3}}}{4}+\frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^2}{x+\frac{2^{\frac{2}{3}}}{4}-\frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}} - \frac{\left(-\frac{2^{\frac{2}{3}}}{4}-\frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^2}{x+\frac{2^{\frac{2}{3}}}{4}+\frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}}$$

For the pole at $x = \frac{2^{\frac{2}{3}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{\frac{2}{3}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -\frac{2^{\frac{2}{3}}}{4}+\frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2^{\frac{2}{3}}}{4}-\frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -\frac{2^{\frac{2}{3}}}{4}-\frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x(x^3+4)}{(2x^3-1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x(x^3+4)}{(2x^3-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{2^{\frac{2}{3}}}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-\frac{2^{\frac{2}{3}}}{4} + \frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-\frac{2^{\frac{2}{3}}}{4} - \frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x - \frac{2^{\frac{2}{3}}}{2}\right)} - \frac{1}{2\left(x + \frac{2^{\frac{2}{3}}}{4} - \frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)} - \frac{1}{2\left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)} + (-)(0) \\
&= -\frac{1}{2\left(x - \frac{2^{\frac{2}{3}}}{2}\right)} - \frac{1}{2\left(x + \frac{2^{\frac{2}{3}}}{4} - \frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)} - \frac{1}{2\left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)} \\
&= -\frac{3x^2}{2x^3-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x - \frac{2^{\frac{2}{3}}}{2}\right)} - \frac{1}{2\left(x + \frac{2^{\frac{2}{3}}}{4} - \frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)} - \frac{1}{2\left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)} \right)(1) + \left(\left(\frac{1}{2\left(x - \frac{2^{\frac{2}{3}}}{2}\right)^2} + \frac{}{2\left(x + \right.} \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{1}{2\left(x-\frac{2^{\frac{2}{3}}}{2}\right)}-\frac{1}{2\left(x+\frac{2^{\frac{2}{3}}}{4}-\frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)}-\frac{1}{2\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{i\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)}\right)dx} \\
&= (x)\, \mathrm{e}^{-\frac{\ln\left(-2^{\frac{2}{3}}+2x\right)}{2}-\frac{\ln\left(8\,2^{\frac{1}{3}}+8x2^{\frac{2}{3}}+16x^2\right)}{2}} \\
&= \frac{x\sqrt{2}}{4\sqrt{-2^{\frac{2}{3}}+2x}\,\sqrt{x2^{\frac{2}{3}}+2x^2+2^{\frac{1}{3}}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6x^2}{2x^3-1}\, dx} \\
&= z_1 e^{\frac{\ln\left(2x^3-1\right)}{2}} \\
&= z_1 \left(\sqrt{2x^3-1}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{2}\, x\sqrt{2x^3-1}}{4\sqrt{4x^3-2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x^2}{2x^3-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln\left(2x^3-1\right)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{16x^3+16}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1\left(\frac{\sqrt{2}\,x\sqrt{2x^3-1}}{4\sqrt{4x^3-2}}\right) + c_2\left(\frac{\sqrt{2}\,x\sqrt{2x^3-1}}{4\sqrt{4x^3-2}}\left(\frac{16x^3+16}{x}\right)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{2}\,x\sqrt{2x^3-1}}{4\sqrt{4x^3-2}} + c_2\left(4x^3+4\right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{2}\,x\sqrt{2x^3-1}}{4\sqrt{4x^3-2}} + c_2\left(4x^3+4\right)$$

Verified OK.

32.7.3 Maple step by step solution

Let's solve

$(2x^3-1)\,y''-6x^2y'+6yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{6x^2y'}{2x^3-1}-\frac{6xy}{2x^3-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{6x^2y'}{2x^3-1}+\frac{6xy}{2x^3-1}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{6x^2}{2x^3-1}, P_3(x)=\frac{6x}{2x^3-1}\right]$

- ◦ $\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)\cdot P_2(x)$ is analytic at $x=-\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

  $\left(\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}}=0$

- $\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^{2}\cdot P_{3}(x)$ is analytic at $x=-\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

  $\left(\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^{2}\cdot P_{3}(x)\right)\Bigg|_{x=-\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}}=0$

- $x=-\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

- Multiply by denominators

  $(2x^3-1)\,y''-6x^2y'+6yx=0$

- Change variables using $x=u-\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$ so that the regular singular point is at $u=0$

  $\left(2u^3-\frac{3u^2 2^{\frac{2}{3}}}{2}-\frac{3\,\mathrm{I}u^2\sqrt{3}\,2^{\frac{2}{3}}}{2}-\frac{3u2^{\frac{1}{3}}}{2}+\frac{3\,\mathrm{I}u2^{\frac{1}{3}}\sqrt{3}}{2}\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(-6u^2+3u2^{\frac{2}{3}}+3\,\mathrm{I}u\sqrt{3}\,2^{\frac{2}{3}}+\frac{3\,2^{\frac{1}{3}}}{2}-3\right.$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

$$u^{m} \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$\frac{3\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)r(r-2)a_{0}u^{r-1}}{2} + \left(\frac{3\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)(1+r)(r-1)a_{1}}{2} - \frac{3\,2^{\frac{2}{3}}\left(1+\mathrm{I}\sqrt{3}\right)(r^{2}-3r+1)a_{0}}{2}\right)u^{r} + \left(\sum_{k=1}^{\infty}\left(\frac{3\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\frac{3\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)r(r-2)}{2} = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$\frac{3\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)(1+r)(r-1)a_{1}}{2} - \frac{3\,2^{\frac{2}{3}}\left(1+\mathrm{I}\sqrt{3}\right)(r^{2}-3r+1)a_{0}}{2} = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-\frac{3a_{k}\left(1+\mathrm{I}\sqrt{3}\right)(k^{2}+(2r-3)k+r^{2}-3r+1)2^{\frac{2}{3}}}{2} + \frac{3\,\mathrm{I}(k+1+r)a_{k+1}2^{\frac{1}{3}}(k+r-1)\sqrt{3}}{2} - \frac{3a_{k+1}(k+1+r)(k+r-1)2^{\frac{1}{3}}}{2} + 2a_{k-1}(k-$$

- Shift index using $k->k+1$

$$-\frac{3a_{k+1}\left(1+\mathrm{I}\sqrt{3}\right)\left((k+1)^{2}+(2r-3)(k+1)+r^{2}-3r+1\right)2^{\frac{2}{3}}}{2} + \frac{3\,\mathrm{I}(k+2+r)a_{k+2}2^{\frac{1}{3}}(k+r)\sqrt{3}}{2} - \frac{3a_{k+2}(k+2+r)(k+r)2^{\frac{1}{3}}}{2} + 2a_{k}(k$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{\left(3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,r^{2}a_{k+1}+6\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,kra_{k+1}-3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,ka_{k+1}-3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,ra_{k+1}-3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,a_{k+1}+3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,k^{2}a_{k+1}+3\,2^{\frac{2}{3}}k^{2}a_{k+1}+\right.}{6\left(\mathrm{I}\sqrt{3}\,k^{2}+2\,\mathrm{I}\sqrt{3}\,kr+\mathrm{I}\sqrt{3}\,r^{2}+2\,\mathrm{I}\sqrt{3}\,k+2\right.}$$

- Recursion relation for $r=0$

$$a_{k+2} = \frac{\left(-3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,ka_{k+1}-3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,a_{k+1}+3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,k^{2}a_{k+1}+3\,2^{\frac{2}{3}}k^{2}a_{k+1}-3\,2^{\frac{2}{3}}ka_{k+1}-3a_{k+1}2^{\frac{2}{3}}-4k^{2}a_{k}+16ka_{k}-12a_{k}\right)2^{\frac{2}{3}}}{6\left(\mathrm{I}\sqrt{3}\,k^{2}+2\,\mathrm{I}\sqrt{3}\,k-k^{2}-2k\right)}$$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$

$$a_{k+2} = \frac{\left(-3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,ka_{k+1}-3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,a_{k+1}+3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,k^{2}a_{k+1}+3\,2^{\frac{2}{3}}k^{2}a_{k+1}-3\,2^{\frac{2}{3}}ka_{k+1}-3a_{k+1}2^{\frac{2}{3}}-4k^{2}a_{k}+16ka_{k}-12a_{k}\right)2^{\frac{2}{3}}}{6\left(\mathrm{I}\sqrt{3}\,k^{2}+2\,\mathrm{I}\sqrt{3}\,k-k^{2}-2k\right)}$$

- Recursion relation for $r=2$

$$a_{k+2} = \frac{\left(9\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,ka_{k+1}+3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,a_{k+1}+3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,k^{2}a_{k+1}+3\,2^{\frac{2}{3}}k^{2}a_{k+1}+9\,2^{\frac{2}{3}}ka_{k+1}+3a_{k+1}2^{\frac{2}{3}}-4k^{2}a_{k}+4a_{k}\right)2^{\frac{2}{3}}}{6\left(\mathrm{I}\sqrt{3}\,k^{2}+6\,\mathrm{I}\sqrt{3}\,k+8\,\mathrm{I}\sqrt{3}-k^{2}-6k-8\right)}$$

- Solution for $r=2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_{k}u^{k+2}, a_{k+2} = \frac{\left(9\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,ka_{k+1}+3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,a_{k+1}+3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,k^{2}a_{k+1}+3\,2^{\frac{2}{3}}k^{2}a_{k+1}+9\,2^{\frac{2}{3}}ka_{k+1}+3a_{k+1}2^{\frac{2}{3}}-4k^{2}\right.}{6\left(\mathrm{I}\sqrt{3}\,k^{2}+6\,\mathrm{I}\sqrt{3}\,k+8\,\mathrm{I}\sqrt{3}-k^{2}-6k-8\right)}\right.$$

- Revert the change of variables $u = x + \frac{2^{\frac{2}{3}}}{4} + \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^{k+2}, a_{k+2} = \frac{\left(9\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,k a_{k+1} + 3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,a_{k+1} + 3\,\mathrm{I}\,2^{\frac{2}{3}}\sqrt{3}\,k^2 a_{k+1} + 3\,2^{\frac{2}{3}}\,k^2 a_{k+1} + 9\,2^{\frac{2}{3}}\,ka\right.}{6\left(\mathrm{I}\sqrt{3}\,k^2 + 6\,\mathrm{I}\sqrt{3}\,k + 8\,\mathrm{I}\sqrt{3} - k^2 - 6k - 8\right)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve((2*x^3-1)*diff(y(x),x$2)-6*x^2*diff(y(x),x)+6*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_2 x^3 + c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 2.452 (sec). Leaf size: 19

```
DSolve[(2*x^3-1)*y''[x]-6*x^2*y'[x]+6*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - c_2(x^3 + 1)$$

32.8 problem Ex 8

Internal problem ID [11309]
Internal file name [`OUTPUT/10294_Wednesday_December_21_2022_03_47_56_PM_17563448/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 55. Summary. Page 129
Problem number: Ex 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y''x^2 - 2x(1+x)\,y' + 2y(1+x) = x^3}$$

32.8.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y''x^2 + \left(-2x^2 - 2x\right)y' + (2+2x)\,y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{-2x^2 - 2x}{x^2} \\
q(x) &= \frac{2 + 2x}{x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{2 + 2x}{x^2} - \frac{\left(\frac{-2x^2-2x}{x^2}\right)'}{2} - \frac{\left(\frac{-2x^2-2x}{x^2}\right)^2}{4} \\
&= \frac{2 + 2x}{x^2} - \frac{\left(\frac{-4x-2}{x^2} - \frac{2(-2x^2-2x)}{x^3}\right)}{2} - \frac{\left(\frac{(-2x^2-2x)^2}{x^4}\right)}{4} \\
&= \frac{2 + 2x}{x^2} - \left(\frac{-4x - 2}{2x^2} - \frac{-2x^2 - 2x}{x^3}\right) - \frac{(-2x^2 - 2x)^2}{4x^4} \\
&= -1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{-2x^2-2x}{x^2}}{2}} \\
&= x\,\mathrm{e}^{x}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, x\,\mathrm{e}^{x} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
\mathrm{e}^{x}(v''(x) - v(x)) = 1
$$

Which is now solved for $v(x)$ Simplyfing the ode gives

$$
v''(x) - v(x) = \mathrm{e}^{-x}
$$

This is second order non-homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = \mathrm{e}^{-x}$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$v''(x) - v(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
v(x) &= c_1 e^{(1)x} + c_2 e^{(-1)x}
\end{aligned}
$$

Or

$$
v(x) = \mathrm{e}^{x} c_1 + c_2 \mathrm{e}^{-x}
$$

Therefore the homogeneous solution v_h is

$$
v_h = \mathrm{e}^{x} c_1 + c_2 \mathrm{e}^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{-x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{x}, \mathrm{e}^{-x}\}
$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\,\mathrm{e}^{-x}\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
v_p = A_1 x\,\mathrm{e}^{-x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-2A_1 \mathrm{e}^{-x} = \mathrm{e}^{-x}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = -\frac{x\,\mathrm{e}^{-x}}{2}$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= \left(\mathrm{e}^x c_1 + c_2 \mathrm{e}^{-x}\right) + \left(-\frac{x\,\mathrm{e}^{-x}}{2}\right) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\,z(x) \\ &= \left(\mathrm{e}^x c_1 + c_2 \mathrm{e}^{-x} - \frac{x\,\mathrm{e}^{-x}}{2}\right)(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x\,\mathrm{e}^x$$

Hence (7) becomes

$$y = \left(\mathrm{e}^x c_1 + c_2 \mathrm{e}^{-x} - \frac{x\,\mathrm{e}^{-x}}{2}\right) x\,\mathrm{e}^x$$

Therefore the homogeneous solution y_h is

$$y_h = \left(\mathrm{e}^x c_1 + c_2 \mathrm{e}^{-x} - \frac{x\,\mathrm{e}^{-x}}{2}\right) x\,\mathrm{e}^x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{2x} x$$

$$y_2 = \mathrm{e}^{-x} x\, \mathrm{e}^{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{2x} x & \mathrm{e}^{-x} x\, \mathrm{e}^{x} \\ \frac{d}{dx}\left(\mathrm{e}^{2x} x\right) & \frac{d}{dx}\left(\mathrm{e}^{-x} x\, \mathrm{e}^{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{2x} x & \mathrm{e}^{-x} x\, \mathrm{e}^{x} \\ \mathrm{e}^{2x} + 2\,\mathrm{e}^{2x} x & \mathrm{e}^{x} \mathrm{e}^{-x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{2x} x\right)\left(\mathrm{e}^{x} \mathrm{e}^{-x}\right) - \left(\mathrm{e}^{-x} x\, \mathrm{e}^{x}\right)\left(\mathrm{e}^{2x} + 2\,\mathrm{e}^{2x} x\right)$$

Which simplifies to

$$W = -2\,\mathrm{e}^{-x} \mathrm{e}^{3x} x^2$$

Which simplifies to

$$W = -2x^2 \mathrm{e}^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{-x} x^4 \mathrm{e}^{x}}{-2x^4 \mathrm{e}^{2x}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-2x}}{2} dx$$

Hence

$$u_1 = -\frac{e^{-2x}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 e^{2x}}{-2x^4 e^{2x}} \, dx$$

Which simplifies to

$$u_2 = \int -\frac{1}{2} dx$$

Hence

$$u_2 = -\frac{x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^{-2x} x \, e^{2x}}{4} - \frac{x^2 e^{-x} e^x}{2}$$

Which simplifies to

$$y_p(x) = -\frac{1}{4}x - \frac{1}{2}x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(e^x c_1 + c_2 e^{-x} - \frac{x \, e^{-x}}{2}\right) x \, e^x\right) + \left(-\frac{1}{4}x - \frac{1}{2}x^2\right) \end{aligned}$$

Which simplifies to

$$y = -\frac{x(-2c_1 e^{2x} - 2c_2 + x)}{2} - \frac{x}{4} - \frac{x^2}{2}$$

Summary
The solution(s) found are the following

$$y = -\frac{x(-2c_1 e^{2x} - 2c_2 + x)}{2} - \frac{x}{4} - \frac{x^2}{2} \tag{1}$$

Verification of solutions

$$y = -\frac{x(-2c_1 e^{2x} - 2c_2 + x)}{2} - \frac{x}{4} - \frac{x^2}{2}$$

Verified OK.

32.8.2 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -2x^2 - 2x, C = 2 + 2x, f(x) = x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$y''x^2 + \left(-2x^2 - 2x\right) y' + (2 + 2x)\, y = 0$$

In normal form the ode

$$y''x^2 + \left(-2x^2 - 2x\right) y' + (2 + 2x)\, y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{-2x - 2}{x} \\ q(x) &= \frac{2 + 2x}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(-2x-2)}{x^2} + \frac{2+2x}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \left(\frac{2}{x} + \frac{-2x-2}{x}\right) v'(x) = 0$$
$$v''(x) - 2v'(x) = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) - 2u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. Integrating both sides gives

$$\int \frac{1}{2u} du = \int dx$$
$$\frac{\ln(u)}{2} = x + c_1$$

Raising both side to exponential gives

$$\sqrt{u} = e^{x+c_1}$$

Which simplifies to

$$\sqrt{u} = c_2 \mathrm{e}^{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_3 \\ &= \frac{\mathrm{e}^{2x} c_2^2}{2} + c_3 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(\frac{\mathrm{e}^{2x} c_2^2}{2} + c_3\right) x \\ &= \frac{\left(\mathrm{e}^{2x} c_2^2 + 2c_3\right) x}{2} \end{aligned}$$

Now the particular solution to this ODE is found

$$y'' x^2 + \left(-2x^2 - 2x\right) y' + (2 + 2x)\, y = x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \mathrm{e}^{2x} x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & e^{2x}x \\ \frac{d}{dx}(x) & \frac{d}{dx}(e^{2x}x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & e^{2x}x \\ 1 & e^{2x} + 2\,e^{2x}x \end{vmatrix}$$

Therefore

$$W = (x)\left(e^{2x} + 2\,e^{2x}x\right) - \left(e^{2x}x\right)(1)$$

Which simplifies to

$$W = 2x^2 e^{2x}$$

Which simplifies to

$$W = 2x^2 e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^4 e^{2x}}{2x^4 e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{2} dx$$

Hence

$$u_1 = -\frac{x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4}{2x^4 e^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{e^{-2x}}{2} dx$$

Hence

$$u_2 = -\frac{e^{-2x}}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2}{2} - \frac{e^{-2x} x\, e^{2x}}{4}$$

Which simplifies to

$$y_p(x) = -\frac{1}{4}x - \frac{1}{2}x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(\frac{e^{2x} c_2^2}{2} + c_3\right) x\right) + \left(-\frac{1}{4}x - \frac{1}{2}x^2\right) \\ &= -\frac{x}{4} - \frac{x^2}{2} + \left(\frac{e^{2x} c_2^2}{2} + c_3\right) x \end{aligned}$$

Which simplifies to

$$y = -\frac{\left(-e^{2x} c_2^2 + x - 2c_3 + \frac{1}{2}\right) x}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{\left(-e^{2x} c_2^2 + x - 2c_3 + \frac{1}{2}\right) x}{2} \tag{1}$$

<u>Verification of solutions</u>

$$y = -\frac{\left(-e^{2x} c_2^2 + x - 2c_3 + \frac{1}{2}\right) x}{2}$$

Verified OK.

32.8.3 Solving using Kovacic algorithm

Writing the ode as

$$y''x^2 + (-2x^2 - 2x)\, y' + (2 + 2x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2 - 2x \\ C &= 2 + 2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 201: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

$$
\begin{aligned}
&= z_1 e^{-\int \frac{1}{2} \frac{-2x^2-2x}{x^2} dx} \\
&= z_1 e^{x+\ln(x)} \\
&= z_1 (x\, \mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-2x}{x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2 \left(x\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y''x^2 + \left(-2x^2 - 2x\right) y' + (2 + 2x) y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x + \frac{c_2 \mathrm{e}^{2x} x}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \frac{\mathrm{e}^{2x} x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{\mathrm{e}^{2x} x}{2} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{\mathrm{e}^{2x} x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{\mathrm{e}^{2x} x}{2} \\ 1 & \frac{\mathrm{e}^{2x}}{2} + \mathrm{e}^{2x} x \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{e^{2x}}{2} + e^{2x}x\right) - \left(\frac{e^{2x}x}{2}\right)(1)$$

Which simplifies to

$$W = x^2 e^{2x}$$

Which simplifies to

$$W = x^2 e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^4 e^{2x}}{2}}{x^4 e^{2x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{2} dx$$

Hence

$$u_1 = -\frac{x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4}{x^4 e^{2x}}\, dx$$

Which simplifies to

$$u_2 = \int e^{-2x} dx$$

Hence

$$u_2 = -\frac{e^{-2x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2}{2} - \frac{e^{-2x} x\, e^{2x}}{4}$$

Which simplifies to

$$y_p(x) = -\frac{1}{4}x - \frac{1}{2}x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x + \frac{c_2 \mathrm{e}^{2x} x}{2}\right) + \left(-\frac{1}{4}x - \frac{1}{2}x^2\right) \end{aligned}$$

Which simplifies to

$$y = x\left(c_1 + \frac{c_2 \mathrm{e}^{2x}}{2}\right) - \frac{x}{4} - \frac{x^2}{2}$$

Summary

The solution(s) found are the following

$$y = x\left(c_1 + \frac{c_2 \mathrm{e}^{2x}}{2}\right) - \frac{x}{4} - \frac{x^2}{2} \tag{1}$$

Verification of solutions

$$y = x\left(c_1 + \frac{c_2 \mathrm{e}^{2x}}{2}\right) - \frac{x}{4} - \frac{x^2}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)-2*x*(1+x)*diff(y(x),x)+2*(1+x)*y(x)=x^3,y(x), singsol=all)
```

$$y(x) = -\frac{x(-2\,e^{2x}c_1 - 2c_2 + x)}{2}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 28

```
DSolve[x^2*y''[x]-2*x*(1+x)*y'[x]+2*(1+x)*y[x]==x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{4}x\left(2x - 2c_2e^{2x} + 1 - 4c_1\right)$$

32.9 problem Ex 9

Internal problem ID [11310]
Internal file name [`OUTPUT/10295_Wednesday_December_21_2022_03_47_59_PM_73161115/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 55. Summary. Page 129
Problem number: Ex 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y''x^2 - 2nx(1+x)\,y' + \left(a^2x^2 + n^2 + n\right)y = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 89

```
dsolve(x^2*diff(y(x),x$2)-2*n*x*(1+x)*diff(y(x),x)+(n^2+n+a^2*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{xn} x^n \left(\mathrm{WhittakerM}\left(\frac{i n^2}{\sqrt{a-n}\,\sqrt{a+n}}, \frac{1}{2}, 2i\sqrt{a-n}\,\sqrt{a+n}\,x\right) c_1 + \mathrm{WhittakerW}\left(\frac{i n^2}{\sqrt{a-n}\,\sqrt{a+n}}, \frac{1}{2}, 2i\sqrt{a-n}\,\sqrt{a+n}\,x\right) c_2\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*y''[x]-2*n*x*(1+x)*y'[x]+(n^2+n+a^2*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions
```

Not solved

32.10 problem Ex 10

Internal problem ID [11311]
Internal file name [`OUTPUT/10296_Wednesday_December_21_2022_03_48_01_PM_43515805/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter VIII, Linear differential equations of the second order. Article 55. Summary. Page 129
Problem number: Ex 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x^4 y'' + 2x^3(1+x)\, y' + y n^2 = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunD  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.422 (sec). Leaf size: 297

```
dsolve(x^4*diff(y(x),x$2)+2*x^3*(1+x)*diff(y(x),x)+n^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \operatorname{HeunD}\left(8(-n^2)^{\frac{1}{4}}, \frac{-8i(-n^2)^{\frac{3}{4}}-n+8\sqrt{-n^2}\,n}{n}, -\frac{16i(-n^2)^{\frac{3}{4}}}{n}, \frac{n-8i(-n^2)^{\frac{3}{4}}-8\sqrt{-n^2}\,n}{n}, \frac{(-n^2)^{\frac{1}{4}}x-in}{(-n^2)^{\frac{1}{4}}x+in}\right) e^{\frac{i\sqrt{-n^2}\,x^2+in^2-n\,x^2}{xn}} \cdots}{\cdots}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^4*y''[x]+2*x^3*(1+x)*y'[x]+n^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

33 Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 57. Dependent variable absent. Page 132

33.1 problem Ex 1

Internal problem ID [11312]
Internal file name [`OUTPUT/10297_Tuesday_December_27_2022_04_05_54_AM_9550685/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 57. Dependent variable absent. Page 132
Problem number: Ex 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{(x^2+1)\,y''+y'^2=-1}$$

33.1.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x)=y'$$

Then

$$p'(x)=y''$$

Hence the ode becomes

$$(x^2+1)\,p'(x)+1+p(x)^2=0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned}
p' &= F(x,p)\\
&= f(x)g(p)\\
&= \frac{-p^2-1}{x^2+1}
\end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(p) = -p^2 - 1$. Integrating both sides gives

$$\frac{1}{-p^2-1}\,dp = \frac{1}{x^2+1}\,dx$$
$$\int \frac{1}{-p^2-1}\,dp = \int \frac{1}{x^2+1}\,dx$$
$$-\arctan(p) = \arctan(x) + c_1$$

The solution is

$$-\arctan(p(x)) - \arctan(x) - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\arctan(y') - \arctan(x) - c_1 = 0$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\tan(\arctan(x) + c_1)\,\mathrm{d}x \\ &= \frac{i\mathrm{e}^{4ic_1}x}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{ix}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{4\,\mathrm{e}^{2ic_1}\ln\left((-\mathrm{e}^{2ic_1}+1)\,x + i\mathrm{e}^{2ic_1} + i\right)}{(\mathrm{e}^{2ic_1}-1)^2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{i\mathrm{e}^{4ic_1}x}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{ix}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{4\,\mathrm{e}^{2ic_1}\ln\left((-\mathrm{e}^{2ic_1}+1)\,x + i\mathrm{e}^{2ic_1} + i\right)}{(\mathrm{e}^{2ic_1}-1)^2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{i\mathrm{e}^{4ic_1}x}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{ix}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{4\,\mathrm{e}^{2ic_1}\ln\left((-\mathrm{e}^{2ic_1}+1)\,x + i\mathrm{e}^{2ic_1} + i\right)}{(\mathrm{e}^{2ic_1}-1)^2} + c_2$$

Verified OK.

33.1.2 Maple step by step solution

Let's solve

$(x^2+1)\,y''+y'^2=-1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $(x^2+1)\,u'(x)+u(x)^2=-1$

- Separate variables

  $\frac{u'(x)}{-u(x)^2-1}=\frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{-u(x)^2-1}dx=\int \frac{1}{x^2+1}dx+c_1$

- Evaluate integral

  $-\arctan(u(x))=\arctan(x)+c_1$

- Solve for $u(x)$

  $u(x)=-\tan(\arctan(x)+c_1)$

- Solve 1st ODE for $u(x)$

  $u(x)=-\tan(\arctan(x)+c_1)$

- Make substitution $u=y'$

  $y'=-\tan(\arctan(x)+c_1)$

- Integrate both sides to solve for y

  $\int y'dx=\int -\tan(\arctan(x)+c_1)\,dx+c_2$

- Compute integrals

  $y=\frac{\mathrm{I}\,\mathrm{e}^{4\,\mathrm{I}c_1}x}{(\mathrm{e}^{2\,\mathrm{I}c_1}-1)^2}-\frac{\mathrm{I}x}{(\mathrm{e}^{2\,\mathrm{I}c_1}-1)^2}-\frac{4\,\mathrm{e}^{2\,\mathrm{I}c_1}\ln\left((-\mathrm{e}^{2\,\mathrm{I}c_1}+1)x+\mathrm{I}\,\mathrm{e}^{2\,\mathrm{I}c_1}+\mathrm{I}\right)}{(\mathrm{e}^{2\,\mathrm{I}c_1}-1)^2}+c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 33

```
dsolve((1+x^2)*diff(y(x),x$2)+1+diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = \frac{\ln (c_1 x - 1) c_1^2 + c_2 c_1^2 + c_1 x + \ln (c_1 x - 1)}{c_1^2}$$

✓ Solution by Mathematica

Time used: 12.052 (sec). Leaf size: 33

```
DSolve[(1+x^2)*y''[x]+1+(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x \cot(c_1) + \csc^2(c_1) \log(-x \sin(c_1) - \cos(c_1)) + c_2$$

33.2 problem Ex 2

Internal problem ID [11313]
Internal file name [`OUTPUT/10298_Tuesday_December_27_2022_04_05_55_AM_57654390/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 57. Dependent variable absent. Page 132
Problem number: Ex 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y], [_3rd_order, _with_linear_symmetries
  ]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
   *** Sublevel 2 ***
   Methods for third order ODEs:
   Successful isolation of d^3y/dx^3: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for third order ODEs:
      --- Trying classification methods ---
      trying 3rd order ODE linearizable_by_differentiation
      -> Calling odsolve with the ODE`, diff(diff(diff(diff(y(x), x), x), x), x), y(x)`
         Methods for high order ODEs:
         --- Trying classification methods ---
         trying a quadrature
         <- quadrature successful
      <- 3rd order ODE linearizable_by_differentiation successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for third order ODEs:
      --- Trying classification methods ---
      trying 3rd order ODE linearizable_by_differentiation
      <- 3rd order ODE linearizable_by_differentiation successful
-> Calling odsolve with the ODE`, (diff(diff(y(x), x), x))^2 = -x^2+1, y(x), singsol = none`
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      <- quadrature successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      <- quadrature successful`
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 94

```
dsolve((x*diff(y(x),x$3)-diff(y(x),x$2))^2=diff(y(x),x$3)^2+1,y(x), singsol=all)
```

$$y(x) = \frac{(x^2+2)\sqrt{-x^2+1}}{6} + c_1 x + \frac{x \arcsin(x)}{2} + c_2$$
$$y(x) = -\frac{x^2\sqrt{-x^2+1}}{6} - \frac{\sqrt{-x^2+1}}{3} - \frac{x \arcsin(x)}{2} + c_1 x + c_2$$
$$y(x) = \frac{\sqrt{c_1^2-1}\, x^3}{6} + \frac{c_1 x^2}{2} + c_2 x + c_3$$

✓ Solution by Mathematica

Time used: 0.241 (sec). Leaf size: 75

```
DSolve[(x*y'''[x]-y''[x])^2==(y'''[x])^2+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 x^3}{6} - \frac{1}{2}\sqrt{1+c_1{}^2}x^2 + c_3 x + c_2$$
$$y(x) \to \frac{c_1 x^3}{6} + \frac{1}{2}\sqrt{1+c_1{}^2}x^2 + c_3 x + c_2$$

33.3 problem Ex 3

Internal problem ID [11314]
Internal file name [`OUTPUT/10299_Tuesday_December_27_2022_04_05_56_AM_53925476/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 57. Dependent variable absent. Page 132
Problem number: Ex 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y'x = x}$$

33.3.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x)\, x - x = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= x(-p+1) \end{aligned}$$

Where $f(x) = x$ and $g(p) = -p + 1$. Integrating both sides gives

$$\frac{1}{-p+1}\,dp = x\,dx$$

$$\int \frac{1}{-p+1}\,dp = \int x\,dx$$

$$-\ln(p-1) = \frac{x^2}{2} + c_1$$

Raising both side to exponential gives

$$\frac{1}{p-1} = e^{\frac{x^2}{2}+c_1}$$

Which simplifies to

$$\frac{1}{p-1} = c_2 e^{\frac{x^2}{2}}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{\left(c_2 e^{\frac{x^2}{2}+c_1} + 1\right) e^{-\frac{x^2}{2}-c_1}}{c_2}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\left(c_2 e^{\frac{x^2}{2}+c_1} + 1\right) e^{-\frac{x^2}{2}-c_1}}{c_2}\, dx \\ &= \frac{c_2 x + \frac{e^{-c_1}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2}}{c_2} + c_3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2 x + \frac{e^{-c_1}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2}}{c_2} + c_3 \tag{1}$$

Verification of solutions

$$y = \frac{c_2 x + \frac{e^{-c_1}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2}}{c_2} + c_3$$

Verified OK.

33.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y'x = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 203: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞.

Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned}
\tag{8}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{x}{2} + \frac{1}{2x} - \frac{1}{4x^3} + \frac{1}{4x^5} - \frac{5}{16x^7} + \frac{7}{16x^9} - \frac{21}{32x^{11}} + \frac{33}{32x^{13}} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2}
\end{aligned}
\tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} + \frac{1}{2}\right) + (0) \\
&= \frac{x^2}{4} + \frac{1}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 1\right) &= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 1\right) &= -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	0	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 0$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^{+} \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= (+)[\sqrt{r}]_\infty \\
&= 0 + \left(\frac{x}{2}\right) \\
&= \frac{x}{2} \\
&= \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{x}{2}\right)(0) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{1}{2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \frac{x}{2} dx} \\
&= \mathrm{e}^{\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2 \left(1 \left(\frac{\sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right)}{2} \right) \right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y'x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + \frac{c_2 \sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 1 \\ y_2 &= \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} \\ 0 & \mathrm{e}^{-\frac{x^2}{2}} \end{vmatrix}$$

Therefore

$$W = (1)\left(\mathrm{e}^{-\frac{x^2}{2}}\right) - \left(\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2}\right)(0)$$

Which simplifies to

$$W = \mathrm{e}^{-\frac{x^2}{2}}$$

Which simplifies to

$$W = \mathrm{e}^{-\frac{x^2}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)x}{2}}{\mathrm{e}^{-\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)x\,\mathrm{e}^{\frac{x^2}{2}}}{2}dx$$

Hence

$$u_1 = -\frac{\sqrt{2}\left(-\sqrt{2}\,\mathrm{e}^{\frac{x^2}{2}}\,x\,\mathrm{e}^{-\frac{x^2}{2}} + \mathrm{e}^{\frac{x^2}{2}}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{\pi}\right)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x}{\mathrm{e}^{-\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int x\,\mathrm{e}^{\frac{x^2}{2}}\,dx$$

Hence

$$u_2 = \mathrm{e}^{\frac{x^2}{2}}$$

Which simplifies to

$$u_1 = \frac{\sqrt{2}\left(-\mathrm{e}^{\frac{x^2}{2}}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{\pi} + \sqrt{2}\,x\right)}{2}$$

$$u_2 = \mathrm{e}^{\frac{x^2}{2}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sqrt{2}\left(-\mathrm{e}^{\frac{x^2}{2}} \operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{\pi} + \sqrt{2}\,x\right)}{2} + \frac{\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right)\mathrm{e}^{\frac{x^2}{2}}}{2}$$

Which simplifies to

$$y_p(x) = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2}\right) + (x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \frac{c_2\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} + x \tag{1}$$

Verification of solutions

$$y = c_1 + \frac{c_2\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} + x$$

Verified OK.

33.3.3 Maple step by step solution

Let's solve

$y'' + y'x = x$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + u(x)\,x = x$

- Separate variables

$\frac{u'(x)}{u(x)-1} = -x$

- Integrate both sides with respect to x

$\int \frac{u'(x)}{u(x)-1} dx = \int -x dx + c_1$

- Evaluate integral

$\ln(u(x)-1) = -\frac{x^2}{2} + c_1$

- Solve for $u(x)$

$u(x) = e^{-\frac{x^2}{2}+c_1} + 1$

- Solve 1st ODE for $u(x)$

$u(x) = e^{-\frac{x^2}{2}+c_1} + 1$

- Make substitution $u = y'$

$y' = e^{-\frac{x^2}{2}+c_1} + 1$

- Integrate both sides to solve for y

$\int y' dx = \int \left(e^{-\frac{x^2}{2}+c_1} + 1\right) dx + c_2$

- Compute integrals

$y = x + \frac{e^{c_1}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{2} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)*_a+_a, _b(_a)`   *** Sublevel 2
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)=x,y(x), singsol=all)
```

$$y(x) = \frac{c_1\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{x\sqrt{2}}{2}\right)}{2} + x + c_2$$

✓ Solution by Mathematica

Time used: 0.137 (sec). Leaf size: 29

```
DSolve[y''[x]+x*y'[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{\frac{\pi}{2}}c_1\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) + x + c_2$$

33.4 problem Ex 4

Internal problem ID [11315]
Internal file name [`OUTPUT/10300_Tuesday_December_27_2022_04_05_58_AM_2399981/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 57. Dependent variable absent. Page 132
Problem number: Ex 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = x\,\mathrm{e}^{x}}$$

33.4.1 Solving as second order ode quadrature ode

Integrating once gives

$$y' = \mathrm{e}^{x}(x-1)+c_1$$

Integrating again gives

$$y = (x-2)\,\mathrm{e}^{x}+c_1x+c_2$$

Summary
The solution(s) found are the following

$$y = (x-2)\,\mathrm{e}^{x}+c_1x+c_2 \tag{1}$$

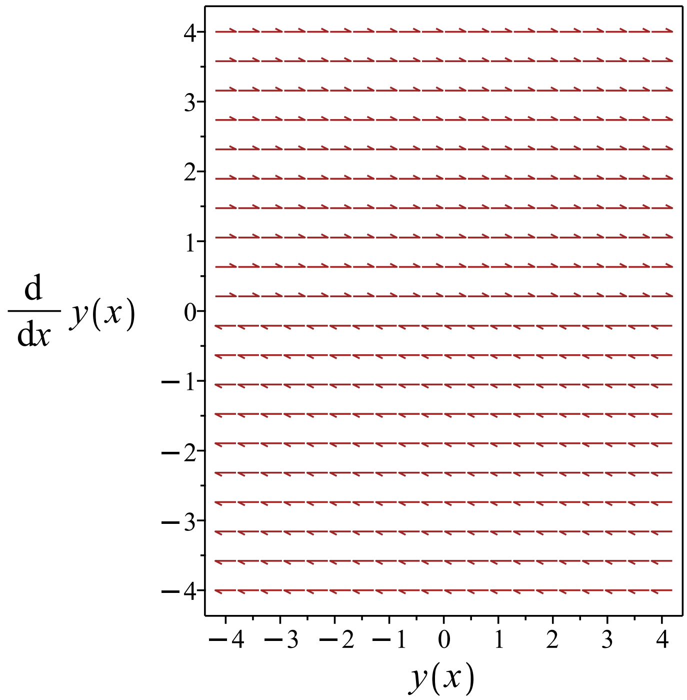

Figure 242: Slope field plot

Verification of solutions

$$y = (x-2)\,e^{x} + c_1 x + c_2$$

Verified OK.

33.4.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = x\,e^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^x + A_2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x + A_1 x e^x + A_2 e^x = x e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x e^x - 2 e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + (x e^x - 2 e^x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x + c_1 + x e^x - 2 e^x \tag{1}$$

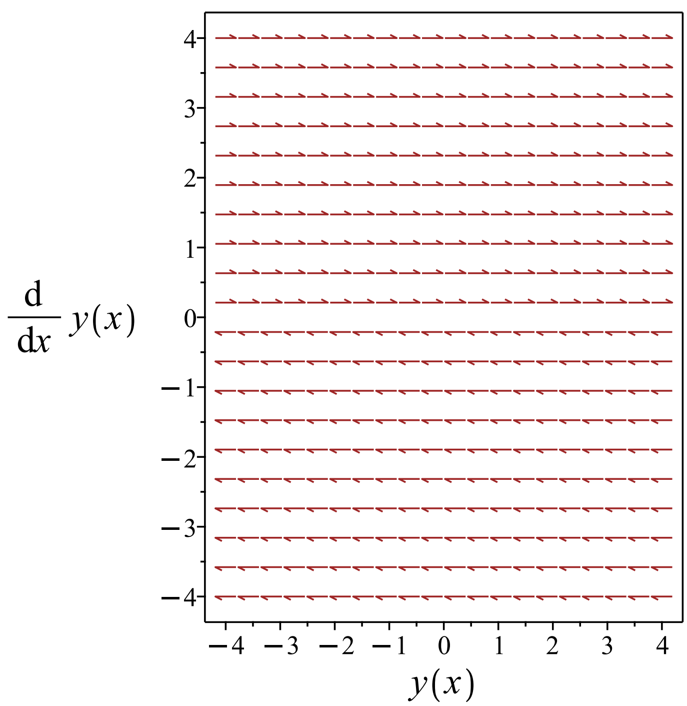

Figure 243: Slope field plot

Verification of solutions

$$y = c_2 x + c_1 + x\,\mathrm{e}^x - 2\,\mathrm{e}^x$$

Verified OK.

33.4.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int x\,\mathrm{e}^x dx$$
$$y' = \mathrm{e}^x(x-1) + c_1$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int x\,\mathrm{e}^x - \mathrm{e}^x + c_1 \;\mathrm{d}x \\ &= x\,\mathrm{e}^x + c_1 x - 2\,\mathrm{e}^x + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x\,\mathrm{e}^x + c_1 x - 2\,\mathrm{e}^x + c_2 \tag{1}$$

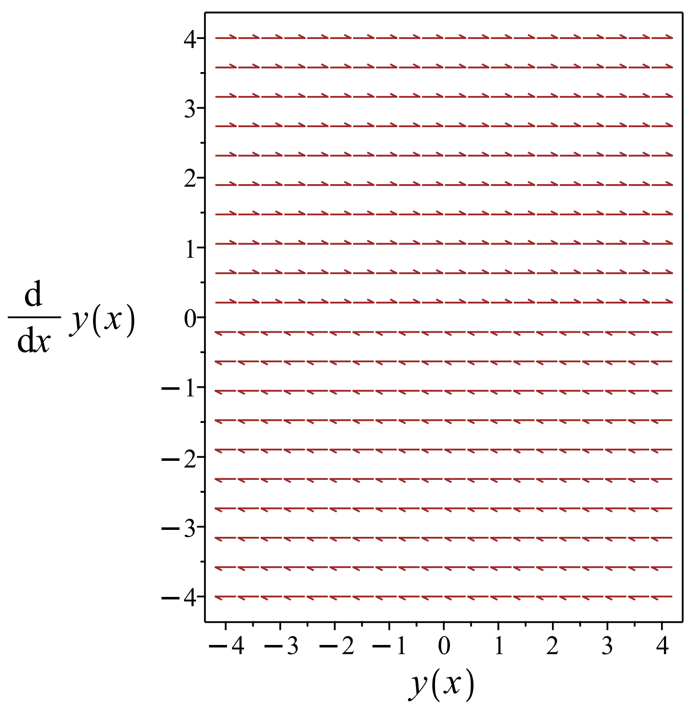

Figure 244: Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2$$

Verified OK.

33.4.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - x\,\mathrm{e}^{x} = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int x\,\mathrm{e}^{x}\,\mathrm{d}x \\ &= \mathrm{e}^{x}(x-1) + c_1 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \mathrm{e}^{x}(x-1) + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int x\,\mathrm{e}^{x} - \mathrm{e}^{x} + c_1\,\mathrm{d}x \\ &= x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2 \tag{1}$$

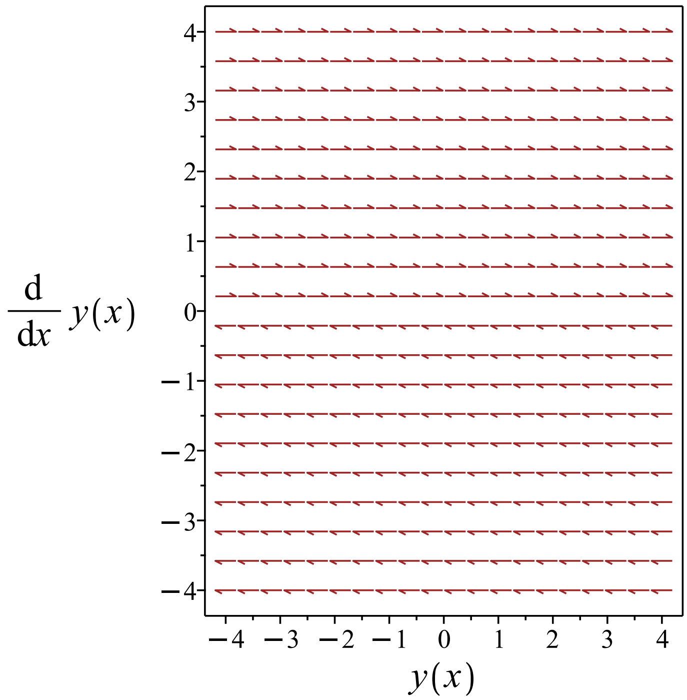

Figure 245: Slope field plot

Verification of solutions

$$y = x\,e^{x} + c_1 x - 2\,e^{x} + c_2$$

Verified OK.

33.4.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 205: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(1) + c_2(1(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_2x + c_1
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x\,\mathrm{e}^{x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, x\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1x\,\mathrm{e}^{x} + A_2\mathrm{e}^{x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_1\mathrm{e}^{x} + A_1x\,\mathrm{e}^{x} + A_2\mathrm{e}^{x} = x\,\mathrm{e}^{x}
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,e^x - 2\,e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + (x\,e^x - 2\,e^x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 x + c_1 + x\,e^x - 2\,e^x \tag{1}$$

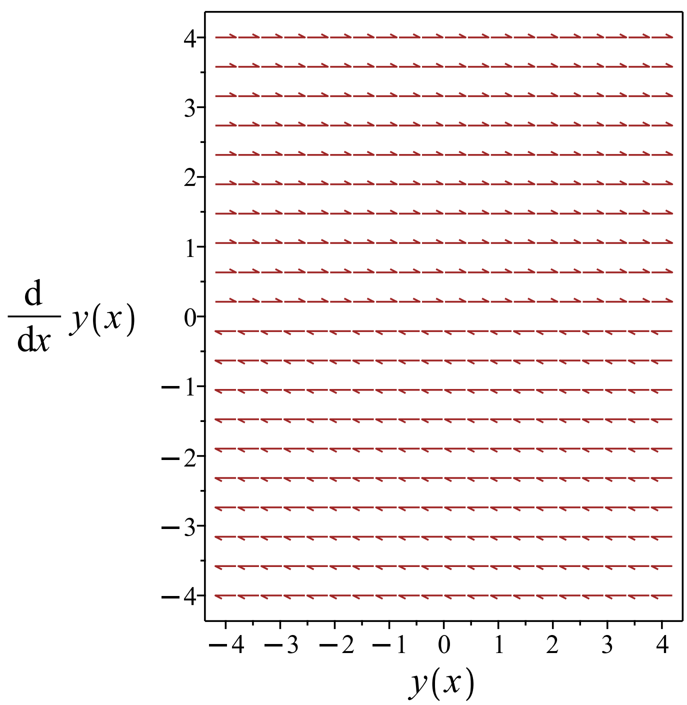

Figure 246: Slope field plot

Verification of solutions

$$y = c_2 x + c_1 + x\,e^x - 2\,e^x$$

Verified OK.

33.4.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= x\,\mathrm{e}^{x} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\, y' + (q(x) - p'(x))\, y\right)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y' = \int x\,\mathrm{e}^{x}\, dx$$

We now have a first order ode to solve which is

$$y' = \mathrm{e}^{x}(x - 1) + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int x\,\mathrm{e}^{x} - \mathrm{e}^{x} + c_1 \, \mathrm{d}x \\ &= x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2 \tag{1}$$

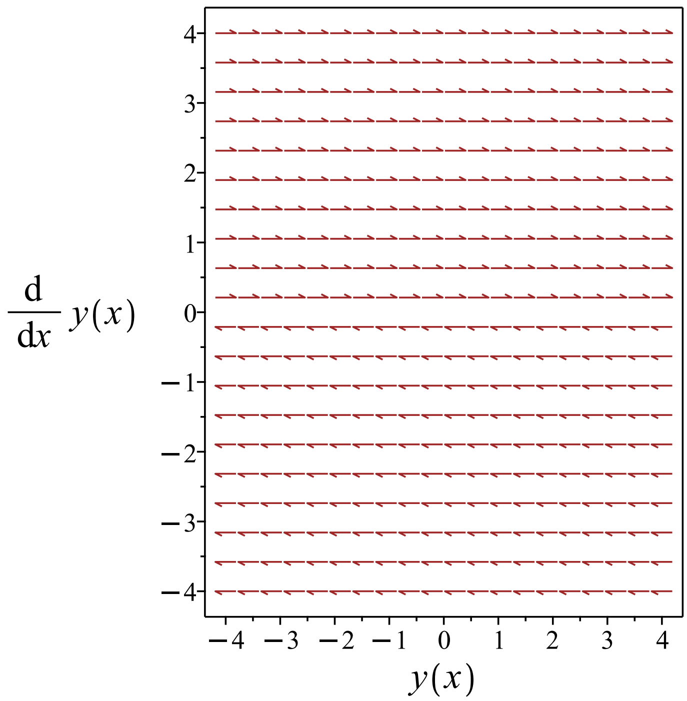

Figure 247: Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2$$

Verified OK.

33.4.7 Maple step by step solution

Let's solve

$y'' = x\,e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^{2} = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\,e^{x}\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\left(\int x^2 e^x dx\right) + x\left(\int x\, e^x dx\right)$$

- Compute integrals

$$y_p(x) = (x-2)\, e^x$$

- Substitute particular solution into general solution to ODE

$$y = (x-2)\, e^x + c_2 x + c_1$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)=x*exp(x),y(x), singsol=all)
```

$$y(x) = (x-2)\, e^x + c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 19

```
DSolve[y''[x]==x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(x-2) + c_2 x + c_1$$

33.5 problem Ex 5

Internal problem ID [11316]
Internal file name [`OUTPUT/10301_Tuesday_December_27_2022_04_05_59_AM_47236562/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 57. Dependent variable absent. Page 132
Problem number: Ex 5.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{(y' - xy'')^2 - y''^2 = 1}$$

33.5.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$\left(p'(x)\, x^2 - 2p(x)\, x - p'(x)\right) p'(x) + p(x)^2 - 1 = 0$$

Which is now solve for $p(x)$ as first order ode. This is Clairaut ODE. It has the form

$$p = xp'(x) + g(p'(x))$$

Where g is function of $p'(x)$. Let $p = p'(x)$ the ode becomes

$$\left(p\, x^2 - 2px - p\right) p + p^2 = 1$$

Solving for $p(x)$ from the above results in

$$p(x) = px + \sqrt{p^2 + 1} \tag{1A}$$
$$p(x) = px - \sqrt{p^2 + 1} \tag{2A}$$

Each of the above ode's is a Clairaut ode which is now solved. Solving ode 1A We start by replacing $p'(x)$ by p which gives

$$\begin{aligned} p(x) &= px + \sqrt{p^2 + 1} \\ &= px + \sqrt{p^2 + 1} \end{aligned}$$

Writing the ode as

$$p(x) = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$p = px + g \tag{1}$$

Then we see that

$$g = \sqrt{p^2 + 1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x + g')\frac{dp}{dx} \\ 0 &= (x + g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$p(x) = c_1 x + \sqrt{c_1^2 + 1}$$

The singular solution is found from solving for p from

$$x+g'(p)=0$$

And substituting the result back in (1). Since we found above that $g=\sqrt{p^2+1}$, then the above equation becomes

$$\begin{aligned}x+g'(p)&=x+\frac{p}{\sqrt{p^2+1}}\\&=0\end{aligned}$$

Solving the above for p results in

$$p_1=-x\sqrt{-\frac{1}{x^2-1}}$$

Substituting the above back in (1) results in

$$p(x)_1=\sqrt{-\frac{1}{x^2-1}}\,(-x^2+1)$$

Solving ode 2A We start by replacing $p'(x)$ by p which gives

$$\begin{aligned}p(x)&=px-\sqrt{p^2+1}\\&=px-\sqrt{p^2+1}\end{aligned}$$

Writing the ode as

$$p(x)=px+g(p)$$

We now write $g\equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$p=px+g \tag{1}$$

Then we see that

$$g=-\sqrt{p^2+1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}p&=\frac{d}{dx}(xp+g)\\p&=\left(p+x\frac{dp}{dx}\right)+\left(g'\frac{dp}{dx}\right)\\p&=p+(x+g')\frac{dp}{dx}\\0&=(x+g')\frac{dp}{dx}\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$
\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}
$$

Substituting this in (1) gives the general solution as

$$
p(x) = c_2 x - \sqrt{c_2^2 + 1}
$$

The singular solution is found from solving for p from

$$
x + g'(p) = 0
$$

And substituting the result back in (1). Since we found above that $g = -\sqrt{p^2 + 1}$, then the above equation becomes

$$
\begin{aligned}
x + g'(p) &= x - \frac{p}{\sqrt{p^2 + 1}} \\
&= 0
\end{aligned}
$$

Solving the above for p results in

$$
p_1 = x\sqrt{-\frac{1}{x^2 - 1}}
$$

Substituting the above back in (1) results in

$$
p(x)_1 = \sqrt{-\frac{1}{x^2 - 1}}\,(x^2 - 1)
$$

For solution (1) found earlier, since $p = y'$ then the new first order ode to solve is

$$
y' = c_1 x + \sqrt{c_1^2 + 1}
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int c_1 x + \sqrt{c_1^2 + 1}\, \mathrm{d}x \\
&= \frac{c_1 x^2}{2} + \sqrt{c_1^2 + 1}\, x + c_3
\end{aligned}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = \sqrt{-\frac{1}{x^2 - 1}}\,(-x^2 + 1)
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int -\sqrt{-\frac{1}{x^2-1}}\,(x^2-1)\, \mathrm{d}x \\
&= -\frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\left(x\sqrt{x^2-1}-\ln\left(x+\sqrt{x^2-1}\right)\right)}{2} + c_4
\end{aligned}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = c_2 x - \sqrt{c_2^2+1}
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int c_2 x - \sqrt{c_2^2+1}\, \mathrm{d}x \\
&= \frac{c_2 x^2}{2} - \sqrt{c_2^2+1}\,x + c_5
\end{aligned}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = \sqrt{-\frac{1}{x^2-1}}\,(x^2-1)
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \sqrt{-\frac{1}{x^2-1}}\,(x^2-1)\, \mathrm{d}x \\
&= \frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\left(x\sqrt{x^2-1}-\ln\left(x+\sqrt{x^2-1}\right)\right)}{2} + c_6
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^2}{2} + \sqrt{c_1^2+1}\,x + c_3 \tag{1}
$$

$$
y = -\frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\left(x\sqrt{x^2-1}-\ln\left(x+\sqrt{x^2-1}\right)\right)}{2} + c_4 \tag{2}
$$

$$
y = \frac{c_2 x^2}{2} - \sqrt{c_2^2+1}\,x + c_5 \tag{3}
$$

$$
y = \frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\left(x\sqrt{x^2-1}-\ln\left(x+\sqrt{x^2-1}\right)\right)}{2} + c_6 \tag{4}
$$

Verification of solutions

$$y = \frac{c_1 x^2}{2} + \sqrt{c_1^2+1}\, x + c_3$$

Verified OK.

$$y = -\frac{\sqrt{-\frac{1}{x^2-1}}\, \sqrt{x^2-1}\, \left(x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)\right)}{2} + c_4$$

Verified OK.

$$y = \frac{c_2 x^2}{2} - \sqrt{c_2^2+1}\, x + c_5$$

Verified OK.

$$y = \frac{\sqrt{-\frac{1}{x^2-1}}\, \sqrt{x^2-1}\, \left(x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)\right)}{2} + c_6$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      <- 2nd order ODE linearizable_by_differentiation successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      <- 2nd order ODE linearizable_by_differentiation successful
-> Calling odsolve with the ODE`, (diff(y(x), x))^2 = -x^2+1, y(x), singsol = none`   *** Su
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   <- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 63

```
dsolve((diff(y(x),x)-x*diff(y(x),x$2))^2=1+diff(y(x),x$2)^2,y(x), singsol=all)
```

$$y(x) = \frac{x\sqrt{-x^2+1}}{2} + \frac{\arcsin(x)}{2} + c_1$$
$$y(x) = -\frac{x\sqrt{-x^2+1}}{2} - \frac{\arcsin(x)}{2} + c_1$$
$$y(x) = \frac{\sqrt{c_1^2-1}\,x^2}{2} + c_1x + c_2$$

✓ Solution by Mathematica

Time used: 0.215 (sec). Leaf size: 58

```
DSolve[(y'[x]-x*y''[x])^2==1+(y''[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1x^2}{2} - \sqrt{1+c_1{}^2}x + c_2$$
$$y(x) \to \frac{c_1x^2}{2} + \sqrt{1+c_1{}^2}x + c_2$$

34 Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 58. Independent variable absent. Page 135

34.1 problem Ex 1

Internal problem ID [11317]
Internal file name [OUTPUT/10302_Tuesday_December_27_2022_04_06_01_AM_77622999/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 58. Independent variable absent. Page 135
Problem number: Ex 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order,
   _with_potential_symmetries], [_2nd_order, _reducible, _mu_xy
   ]]
```

$$\boxed{yy'' - y'^2 - y^2y' = 0}$$

34.1.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right)+\left(-p(y)-y^{2}\right)p(y)=0$$

Which is now solved as first order ode for $p(y)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dy}p(y)+p(y)p(y)=q(y)$$

Where here

$$\begin{aligned}p(y)&=-\frac{1}{y}\\ q(y)&=y\end{aligned}$$

Hence the ode is

$$\frac{d}{dy}p(y)-\frac{p(y)}{y}=y$$

The integrating factor μ is

$$\begin{aligned}\mu&=\mathrm{e}^{\int -\frac{1}{y}dy}\\ &=\frac{1}{y}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}(\mu p)&=(\mu)(y)\\ \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{p}{y}\right)&=\left(\frac{1}{y}\right)(y)\\ \mathrm{d}\left(\frac{p}{y}\right)&=\mathrm{d}y\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{p}{y}&=\int \mathrm{d}y\\ \frac{p}{y}&=y+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu=\frac{1}{y}$ results in

$$p(y)=c_1y+y^{2}$$

which simplifies to

$$p(y) = y(y + c_1)$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = y(y + c_1)$$

Integrating both sides gives

$$\int \frac{1}{y\,(y + c_1)} dy = \int dx$$

$$\frac{\ln (y)}{c_1} - \frac{\ln (y + c_1)}{c_1} = x + c_2$$

The above can be written as

$$\left(\frac{1}{c_1}\right)(\ln (y) - \ln (y + c_1)) = x + c_2$$

$$\ln (y) - \ln (y + c_1) = (c_1)(x + c_2)$$

$$= c_1(x + c_2)$$

Raising both side to exponential gives

$$e^{\ln(y) - \ln(y + c_1)} = c_1 c_2 e^{c_1 x}$$

Which simplifies to

$$\frac{y}{y + c_1} = c_3 e^{c_1 x}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_3 e^{c_1 x} c_1}{-1 + c_3 e^{c_1 x}} \tag{1}$$

Verification of solutions

$$y = -\frac{c_3 e^{c_1 x} c_1}{-1 + c_3 e^{c_1 x}}$$

Verified OK.

34.1.2 Maple step by step solution

Let's solve

$yy'' + (-y' - y^2)\, y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $yu(y)\left(\frac{d}{dy}u(y)\right) + (-u(y) - y^2)\, u(y) = 0$

- Isolate the derivative

  $\frac{d}{dy}u(y) = \frac{u(y)}{y} + y$

- Group terms with $u(y)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{d}{dy}u(y) - \frac{u(y)}{y} = y$

- The ODE is linear; multiply by an integrating factor $\mu(y)$

  $\mu(y)\left(\frac{d}{dy}u(y) - \frac{u(y)}{y}\right) = \mu(y)\, y$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dy}(\mu(y)\, u(y))$

  $\mu(y)\left(\frac{d}{dy}u(y) - \frac{u(y)}{y}\right) = \left(\frac{d}{dy}\mu(y)\right) u(y) + \mu(y)\left(\frac{d}{dy}u(y)\right)$

- Isolate $\frac{d}{dy}\mu(y)$

  $\frac{d}{dy}\mu(y) = -\frac{\mu(y)}{y}$

- Solve to find the integrating factor

  $\mu(y) = \frac{1}{y}$

- Integrate both sides with respect to y

  $\int \left(\frac{d}{dy}(\mu(y)\, u(y))\right) dy = \int \mu(y)\, y dy + c_1$

- Evaluate the integral on the lhs

  $\mu(y)\, u(y) = \int \mu(y)\, y dy + c_1$

- Solve for $u(y)$

  $u(y) = \frac{\int \mu(y) y dy + c_1}{\mu(y)}$

- Substitute $\mu(y) = \frac{1}{y}$

  $u(y) = y\left(\int 1 dy + c_1\right)$

- Evaluate the integrals on the rhs

  $u(y) = y(y + c_1)$

- Solve 1st ODE for $u(y)$

  $u(y) = y(y + c_1)$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = y(y + c_1)$

- Separate variables

  $\frac{y'}{y(y+c_1)} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(y+c_1)} dx = \int 1 dx + c_2$

- Evaluate integral

  $\frac{\ln(y)}{c_1} - \frac{\ln(y+c_1)}{c_1} = x + c_2$

- Solve for y

  $y = -\frac{c_1 \mathrm{e}^{c_2 c_1 + c_1 x}}{-1+\mathrm{e}^{c_2 c_1 + c_1 x}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x$2)-diff(y(x),x)^2-y(x)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = -\frac{c_1 \mathrm{e}^{c_1(c_2+x)}}{-1+\mathrm{e}^{c_1(c_2+x)}}$$

✓ Solution by Mathematica

Time used: 2.444 (sec). Leaf size: 43

```
DSolve[y[x]*y''[x]-y'[x]^2-y[x]^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{c_1 e^{c_1(x+c_2)}}{-1+e^{c_1(x+c_2)}}$$
$$y(x) \to -\frac{1}{x+c_2}$$

34.2 problem Ex 2

Internal problem ID [11318]
Internal file name [`OUTPUT/10303_Tuesday_December_27_2022_04_06_03_AM_59945777/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 58. Independent variable absent. Page 135
Problem number: Ex 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]`

$$\boxed{yy'' - y'^2 = -1}$$

34.2.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) - p(y)^2 = -1$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{p^2-1}{yp} \end{aligned}$$

Where $f(y)=\frac{1}{y}$ and $g(p)=\frac{p^2-1}{p}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{p^2-1}{p}}\,dp &= \frac{1}{y}\,dy \\ \int \frac{1}{\frac{p^2-1}{p}}\,dp &= \int \frac{1}{y}\,dy \\ \frac{\ln(p-1)}{2}+\frac{\ln(p+1)}{2} &= \ln(y)+c_1 \end{aligned}$$

The above can be written as

$$\begin{aligned} \left(\frac{1}{2}\right)(\ln(p-1)+\ln(p+1)) &= \ln(y)+2c_1 \\ \ln(p-1)+\ln(p+1) &= (2)(\ln(y)+2c_1) \\ &= 2\ln(y)+4c_1 \end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(p-1)+\ln(p+1)} = e^{2\ln(y)+2c_1}$$

Which simplifies to

$$\begin{aligned} p^2-1 &= 2c_1y^2 \\ &= c_2y^2 \end{aligned}$$

The solution is

$$p(y)^2-1 = c_2y^2$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$y'^2-1 = c_2y^2$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{c_2 y^2 + 1} \tag{1}$$
$$y' = -\sqrt{c_2 y^2 + 1} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\int \frac{1}{\sqrt{c_2 y^2 + 1}} dy = \int dx$$
$$\frac{\ln\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)}{\sqrt{c_2}} = c_3 + x$$

Raising both side to exponential gives

$$e^{\frac{\ln\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)}{\sqrt{c_2}}} = e^{c_3 + x}$$

Which simplifies to

$$\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)^{\frac{1}{\sqrt{c_2}}} = e^x c_4$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\int -\frac{1}{\sqrt{c_2 y^2 + 1}} dy = \int dx$$
$$-\frac{\ln\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)}{\sqrt{c_2}} = x + c_5$$

Raising both side to exponential gives

$$e^{-\frac{\ln\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)}{\sqrt{c_2}}} = e^{x + c_5}$$

Which simplifies to

$$\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)^{-\frac{1}{\sqrt{c_2}}} = c_6 e^x$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\left(\mathrm{e}^{x} c_4\right)^{2\sqrt{c_2}}-1\right)\left(\mathrm{e}^{x} c_4\right)^{-\sqrt{c_2}}}{2\sqrt{c_2}} \tag{1}$$

$$y = \frac{\left(\left(c_6\,\mathrm{e}^{x}\right)^{-2\sqrt{c_2}}-1\right)\left(c_6\,\mathrm{e}^{x}\right)^{\sqrt{c_2}}}{2\sqrt{c_2}} \tag{2}$$

Verification of solutions

$$y = \frac{\left(\left(\mathrm{e}^{x} c_4\right)^{2\sqrt{c_2}}-1\right)\left(\mathrm{e}^{x} c_4\right)^{-\sqrt{c_2}}}{2\sqrt{c_2}}$$

Verified OK.

$$y = \frac{\left(\left(c_6\,\mathrm{e}^{x}\right)^{-2\sqrt{c_2}}-1\right)\left(c_6\,\mathrm{e}^{x}\right)^{\sqrt{c_2}}}{2\sqrt{c_2}}$$

Verified OK.

34.2.2 Maple step by step solution

Let's solve

$y y'' - y'^2 = -1$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$$y u(y) \left(\frac{d}{dy}u(y)\right)-u(y)^{2}=-1$$

- Separate variables

$$\frac{\left(\frac{d}{dy}u(y)\right) u(y)}{u(y)^{2}-1}=\frac{1}{y}$$

- Integrate both sides with respect to y

$$\int \frac{\left(\frac{d}{dy}u(y)\right) u(y)}{u(y)^{2}-1}d y =\int \frac{1}{y}d y +c_{1}$$

- Evaluate integral

$$\frac{\ln \left(u(y)-1\right)}{2}+\frac{\ln \left(u(y)+1\right)}{2}=\ln \left(y\right)+c_{1}$$

- Solve for $u(y)$

$$\left\{u(y)=\frac{{\mathrm e}^{2 c_{1}} \left(-1+\sqrt{\left({\mathrm e}^{c_{1}}\right)^{2} y^{2}+1}\right)}{\left({\mathrm e}^{c_{1}}\right)^{2}}+1, u(y)=-\frac{{\mathrm e}^{2 c_{1}} \left(1+\sqrt{\left({\mathrm e}^{c_{1}}\right)^{2} y^{2}+1}\right)}{\left({\mathrm e}^{c_{1}}\right)^{2}}+1\right\}$$

- Solve 1st ODE for $u(y)$

$$u(y)=\frac{{\mathrm e}^{2 c_{1}} \left(-1+\sqrt{\left({\mathrm e}^{c_{1}}\right)^{2} y^{2}+1}\right)}{\left({\mathrm e}^{c_{1}}\right)^{2}}+1$$

- Revert to original variables with substitution $u(y)=y', y=y$

$$y'=\frac{{\mathrm e}^{2 c_{1}} \left(-1+\sqrt{\left({\mathrm e}^{c_{1}}\right)^{2} y^{2}+1}\right)}{\left({\mathrm e}^{c_{1}}\right)^{2}}+1$$

- Separate variables

$$\frac{y'}{\frac{{\mathrm e}^{2 c_{1}} \left(-1+\sqrt{\left({\mathrm e}^{c_{1}}\right)^{2} y^{2}+1}\right)}{\left({\mathrm e}^{c_{1}}\right)^{2}}+1}=1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\frac{{\mathrm e}^{2 c_{1}} \left(-1+\sqrt{\left({\mathrm e}^{c_{1}}\right)^{2} y^{2}+1}\right)}{\left({\mathrm e}^{c_{1}}\right)^{2}}+1}d x =\int 1d x +c_{2}$$

- Evaluate integral

$$(\mathrm{e}^{c_1})^2 \left(\frac{\mathrm{e}^{2c_1} \arctan\left(\frac{\mathrm{e}^{4c_1} y}{\sqrt{\mathrm{e}^{2c_1} \mathrm{e}^{4c_1}}}\right)}{(\mathrm{e}^{c_1})^2 \sqrt{\mathrm{e}^{2c_1} \mathrm{e}^{4c_1}}} + \frac{\mathrm{e}^{2c_1} \left(-\frac{-\sqrt{(\mathrm{e}^{c_1})^2 \left(y-\sqrt{-\mathrm{e}^{6c_1}}\, \mathrm{e}^{-4c_1}\right)^2 + 2(\mathrm{e}^{c_1})^2 \sqrt{-\mathrm{e}^{6c_1}}\, \mathrm{e}^{-4c_1} \left(y-\sqrt{-\mathrm{e}^{6c_1}}\, \mathrm{e}^{-4c_1}\right) - (\mathrm{e}^{c_1})^2 \mathrm{e}^{6c_1} \left(\mathrm{e}^{\cdots}\right.}}{\cdots} \right.}{\cdots} \right.$$

- Solve for y

$$y = -\frac{\left((\mathrm{e}^{c_1})^2 - \left(\mathrm{e}^{\mathrm{e}^{c_1} c_2 + \mathrm{e}^{c_1} x + c_1}\right)^2\right) \mathrm{e}^{-3c_1 - \mathrm{e}^{c_1} c_2 - \mathrm{e}^{c_1} x}}{2}$$

- Solve 2nd ODE for $u(y)$

$$u(y) = -\frac{\mathrm{e}^{2c_1} \left(1+\sqrt{(\mathrm{e}^{c_1})^2 y^2 + 1}\right)}{(\mathrm{e}^{c_1})^2} + 1$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = -\frac{\mathrm{e}^{2c_1} \left(1+\sqrt{(\mathrm{e}^{c_1})^2 y^2 + 1}\right)}{(\mathrm{e}^{c_1})^2} + 1$$

- Separate variables

$$\frac{y'}{-\frac{\mathrm{e}^{2c_1} \left(1+\sqrt{(\mathrm{e}^{c_1})^2 y^2 + 1}\right)}{(\mathrm{e}^{c_1})^2} + 1} = 1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{-\frac{\mathrm{e}^{2c_1} \left(1+\sqrt{(\mathrm{e}^{c_1})^2 y^2 + 1}\right)}{(\mathrm{e}^{c_1})^2} + 1} dx = \int 1 dx + c_2$$

- Evaluate integral

$$-\left(e^{c_1}\right)^2 \left(\frac{\arctan\left(\frac{e^{4c_1} y}{\sqrt{e^{2c_1} e^{4c_1}}}\right)}{\sqrt{e^{2c_1} e^{4c_1}}} + \frac{e^{2c_1} \left(-\frac{-\sqrt{\left(e^{c_1}\right)^2 \left(y-\sqrt{-e^{6c_1}}\, e^{-4c_1}\right)^2 + 2\left(e^{c_1}\right)^2 \sqrt{-e^{6c_1}}\, e^{-4c_1} \left(y-\sqrt{-e^{6c_1}}\, e^{-4c_1}\right) - \left(e^{c_1}\right)^2 e^{6c_1} \left(e^{-4 \cdots}\right.}}{\cdots} \right.}{\cdots} \right.$$

- Solve for y

$$y = \frac{\left(\left(e^{c_1} - e^{c_1} c_2 - e^{c_1} x\right)^2 - \left(e^{c_1}\right)^2\right) e^{-3c_1 + e^{c_1} c_2 + e^{c_1} x}}{2}$$

<u>Maple trace</u>

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 59

```
dsolve(y(x)*diff(y(x),x$2)-diff(y(x),x)^2+1=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1\left(-\mathrm{e}^{\frac{c_2+x}{c_1}}+\mathrm{e}^{\frac{-c_2-x}{c_1}}\right)}{2}$$
$$y(x) = -\frac{c_1\left(-\mathrm{e}^{\frac{c_2+x}{c_1}}+\mathrm{e}^{\frac{-c_2-x}{c_1}}\right)}{2}$$

✓ Solution by Mathematica

Time used: 60.222 (sec). Leaf size: 85

```
DSolve[y[x]*y''[x]-y'[x]^2+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{ie^{-c_1}\tanh\left(e^{c_1}(x+c_2)\right)}{\sqrt{-\operatorname{sech}^2\left(e^{c_1}(x+c_2)\right)}}$$
$$y(x) \to \frac{ie^{-c_1}\tanh\left(e^{c_1}(x+c_2)\right)}{\sqrt{-\operatorname{sech}^2\left(e^{c_1}(x+c_2)\right)}}$$

34.3 problem Ex 3

Internal problem ID [11319]
Internal file name [`OUTPUT/10304_Tuesday_December_27_2022_04_06_07_AM_10455936/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 58. Independent variable absent. Page 135
Problem number: Ex 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{2y'' - e^{y} = 0}$$

34.3.1 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$2y'y'' - y'e^{y} = 0$$

Integrating the above w.r.t x gives

$$\int (2y'y'' - y'e^{y})\, dx = 0$$
$$y'^2 - e^{y} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{e^{y} + c_1} \tag{1}$$
$$y' = -\sqrt{e^{y} + c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{e^{y}+c_1}}dy = \int dx$$

$$-\frac{2\,\operatorname{arctanh}\left(\frac{\sqrt{e^{y}+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{e^{y}+c_1}}dy = \int dx$$

$$\frac{2\,\operatorname{arctanh}\left(\frac{\sqrt{e^{y}+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_3 + x$$

Summary

The solution(s) found are the following

$$-\frac{2\,\operatorname{arctanh}\left(\frac{\sqrt{e^{y}+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2 \tag{1}$$

$$\frac{2\,\operatorname{arctanh}\left(\frac{\sqrt{e^{y}+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_3 + x \tag{2}$$

Verification of solutions

$$-\frac{2\,\operatorname{arctanh}\left(\frac{\sqrt{e^{y}+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$$

Verified OK.

$$\frac{2\,\operatorname{arctanh}\left(\frac{\sqrt{e^{y}+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_3 + x$$

Verified OK.

34.3.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}$$

Hence the ode becomes

$$2p(y)\left(\frac{d}{dy}p(y)\right) = e^{y}$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= \frac{e^{y}}{2p}
\end{aligned}$$

Where $f(y) = \frac{e^{y}}{2}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{1}{p}}\,dp &= \frac{e^{y}}{2}\,dy \\
\int \frac{1}{\frac{1}{p}}\,dp &= \int \frac{e^{y}}{2}\,dy \\
\frac{p^{2}}{2} &= \frac{e^{y}}{2} + c_1
\end{aligned}$$

The solution is

$$\frac{p(y)^{2}}{2} - \frac{e^{y}}{2} - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{y'^2}{2} - \frac{e^y}{2} - c_1 = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{e^y + 2c_1} \tag{1}$$
$$y' = -\sqrt{e^y + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{e^y + 2c_1}} dy = \int dx$$
$$-\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{e^y+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{e^y + 2c_1}} dy = \int dx$$
$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{e^y+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_3 + x$$

Summary

The solution(s) found are the following

$$y = \ln\left(2\tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)^2 c_1 - 2c_1\right) \tag{1}$$
$$y = \ln\left(2\tanh\left(\frac{\sqrt{c_1}\,(c_3 + x)\,\sqrt{2}}{2}\right)^2 c_1 - 2c_1\right) \tag{2}$$

Verification of solutions

$$y = \ln\left(2\tanh\left(\frac{\sqrt{c_1}\,(x+c_2)\,\sqrt{2}}{2}\right)^2 c_1 - 2c_1\right)$$

Verified OK.

$$y = \ln\left(2\tanh\left(\frac{\sqrt{c_1}\,(c_3+x)\,\sqrt{2}}{2}\right)^2 c_1 - 2c_1\right)$$

Verified OK.

34.3.3 Maple step by step solution

Let's solve

$2y'' = e^y$

- Highest derivative means the order of the ODE is 2

  y''
- Define new dependent variable u

  $u(x) = y'$
- Compute y''

  $u'(x) = y''$
- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$
- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$
- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $2u(y)\left(\frac{d}{dy}u(y)\right) = e^y$
- Integrate both sides with respect to y

  $\int 2u(y)\left(\frac{d}{dy}u(y)\right)dy = \int e^y dy + c_1$
- Evaluate integral

  $u(y)^2 = e^y + c_1$

- Solve for $u(y)$

  $\left\{u(y)=\sqrt{\mathrm{e}^{y}+c_1}, u(y)=-\sqrt{\mathrm{e}^{y}+c_1}\right\}$

- Solve 1st ODE for $u(y)$

  $u(y)=\sqrt{\mathrm{e}^{y}+c_1}$

- Revert to original variables with substitution $u(y)=y', y=y$

  $y'=\sqrt{\mathrm{e}^{y}+c_1}$

- Separate variables

  $\frac{y'}{\sqrt{\mathrm{e}^{y}+c_1}}=1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{\mathrm{e}^{y}+c_1}}dx=\int 1dx+c_2$

- Evaluate integral

  $-\frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{\mathrm{e}^{y}+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}}=x+c_2$

- Solve for y

  $y=\ln\left(\tanh\left(\frac{c_2\sqrt{c_1}}{2}+\frac{x\sqrt{c_1}}{2}\right)^{2}c_1-c_1\right)$

- Solve 2nd ODE for $u(y)$

  $u(y)=-\sqrt{\mathrm{e}^{y}+c_1}$

- Revert to original variables with substitution $u(y)=y', y=y$

  $y'=-\sqrt{\mathrm{e}^{y}+c_1}$

- Separate variables

  $\frac{y'}{\sqrt{\mathrm{e}^{y}+c_1}}=-1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{\mathrm{e}^{y}+c_1}}dx=\int (-1)\,dx+c_2$

- Evaluate integral

  $-\frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{\mathrm{e}^{y}+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}}=-x+c_2$

- Solve for y

  $y=\ln\left(\tanh\left(\frac{c_2\sqrt{c_1}}{2}-\frac{x\sqrt{c_1}}{2}\right)^{2}c_1-c_1\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(1/2)*exp(_a) = 0, _b(_a), HINT
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 1/2*_b]
```

✓ Solution by Maple

Time used: 0.469 (sec). Leaf size: 20

```
dsolve(2*diff(y(x),x$2)=exp(y(x)),y(x), singsol=all)
```

$$y(x) = \ln\left(\frac{\sec\left(\frac{c_2+x}{2c_1}\right)^2}{c_1^2}\right)$$

✓ Solution by Mathematica

Time used: 60.049 (sec). Leaf size: 30

```
DSolve[2*y''[x]==Exp[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log\left(-c_1\operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_1(x+c_2)^2}\right)\right)$$

34.4 problem Ex 4

Internal problem ID [11320]
Internal file name [`OUTPUT/10305_Tuesday_December_27_2022_04_06_08_AM_66861636/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 58. Independent variable absent. Page 135
Problem number: Ex 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1],
  [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{yy'' + 2y' - y'^2 = 0}$$

34.4.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right)+(2-p(y))\,p(y)=0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}p' &= F(y,p)\\ &= f(y)g(p)\\ &= \frac{-2+p}{y}\end{aligned}$$

Where $f(y)=\frac{1}{y}$ and $g(p)=-2+p$. Integrating both sides gives

$$\begin{aligned}\frac{1}{-2+p}\,dp &= \frac{1}{y}\,dy\\ \int\frac{1}{-2+p}\,dp &= \int\frac{1}{y}\,dy\\ \ln(-2+p) &= \ln(y)+c_1\end{aligned}$$

Raising both side to exponential gives

$$-2+p=\mathrm{e}^{\ln(y)+c_1}$$

Which simplifies to

$$-2+p=c_2y$$

Which simplifies to

$$p(y)=c_2y\,\mathrm{e}^{c_1}+2$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$y'=c_2y\,\mathrm{e}^{c_1}+2$$

Integrating both sides gives

$$\begin{aligned}\int\frac{1}{c_2y\,\mathrm{e}^{c_1}+2}dy &= \int dx\\ \frac{\ln(c_2y\,\mathrm{e}^{c_1}+2)\,\mathrm{e}^{-c_1}}{c_2} &= c_3+x\end{aligned}$$

Raising both side to exponential gives

$$e^{\frac{\ln(c_2 y\, e^{c_1}+2)e^{-c_1}}{c_2}} = e^{c_3+x}$$

Which simplifies to

$$(c_2 y\, e^{c_1}+2)^{\frac{e^{-c_1}}{c_2}} = e^x c_4$$

Summary
The solution(s) found are the following

$$y = \frac{\left((e^x c_4)^{e^{c_1} c_2} - 2\right) e^{-c_1}}{c_2} \tag{1}$$

Verification of solutions

$$y = \frac{\left((e^x c_4)^{e^{c_1} c_2} - 2\right) e^{-c_1}}{c_2}$$

Verified OK.

34.4.2 Maple step by step solution

Let's solve

$yy'' + (2-y')\,y' = 0$

- Highest derivative means the order of the ODE is 2

  y''
- Define new dependent variable u

  $u(x) = y'$
- Compute y''

  $u'(x) = y''$
- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$
- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$
- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$$yu(y)\left(\frac{d}{dy}u(y)\right)+(2-u(y))\,u(y)=0$$

- Separate variables

$$\frac{\frac{d}{dy}u(y)}{2-u(y)}=-\frac{1}{y}$$

- Integrate both sides with respect to y

$$\int \frac{\frac{d}{dy}u(y)}{2-u(y)}dy=\int -\frac{1}{y}dy+c_1$$

- Evaluate integral

$$-\ln\left(2-u(y)\right)=-\ln\left(y\right)+c_1$$

- Solve for $u(y)$

$$u(y)=\frac{2\,\mathrm{e}^{c_1}-y}{\mathrm{e}^{c_1}}$$

- Solve 1st ODE for $u(y)$

$$u(y)=\frac{2\,\mathrm{e}^{c_1}-y}{\mathrm{e}^{c_1}}$$

- Revert to original variables with substitution $u(y)=y', y=y$

$$y'=\frac{2\,\mathrm{e}^{c_1}-y}{\mathrm{e}^{c_1}}$$

- Separate variables

$$\frac{y'}{2\,\mathrm{e}^{c_1}-y}=\frac{1}{\mathrm{e}^{c_1}}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{2\,\mathrm{e}^{c_1}-y}dx=\int \frac{1}{\mathrm{e}^{c_1}}dx+c_2$$

- Evaluate integral

$$-\ln\left(2\,\mathrm{e}^{c_1}-y\right)=\frac{x}{\mathrm{e}^{c_1}}+c_2$$

- Solve for y

$$y=-\mathrm{e}^{-\frac{\mathrm{e}^{c_1}c_2+x}{\mathrm{e}^{c_1}}}+2\,\mathrm{e}^{c_1}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-_b(_a)*(_b(_a)-2)/_a = 0, _b(_a)
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 20

```
dsolve(y(x)*diff(y(x),x$2)+2*diff(y(x),x)-diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \frac{e^{c_1(c_2+x)} - 2}{c_1}$$

✓ Solution by Mathematica

Time used: 2.726 (sec). Leaf size: 26

```
DSolve[y[x]*y''[x]+2*y'[x]-y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-2 + e^{c_1(x+c_2)}}{c_1}$$
$$y(x) \to \text{Indeterminate}$$

35 Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 59. Linear equations with particular integral known. Page 136

35.1 problem Ex 1

Internal problem ID [11321]
Internal file name [`OUTPUT/10306_Tuesday_December_27_2022_04_06_09_AM_65226813/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 59. Linear equations with particular integral known. Page 136
Problem number: Ex 1.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_3rd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{\left(x^{2}-2x+2\right) y'''-y''x^{2}+2y'x-2y=0}$$

Unable to solve this ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
trying to convert to a linear ODE with constant coefficients
trying differential order: 3; missing the dependent variable
Equation is the LCLM of -1/x*y(x)+diff(y(x),x), -2/x*y(x)+diff(y(x),x), -y(x)+diff(y(x),x)
trying differential order: 1; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful
trying differential order: 1; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful
trying differential order: 1; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful
<- solving the LCLM ode successful `
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve((x^2-2*x+2)*diff(y(x),x$3)-x^2*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=0,y(x), singsol
```

$$y(x) = c_1x + c_2x^2 + c_3e^x$$

✓ Solution by Mathematica

Time used: 0.124 (sec). Leaf size: 27

```
DSolve[(x^2-2*x+2)*y'''[x]-x^2*y''[x]+2*x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{2}\left(c_2x^2 + 2c_1x + c_3e^x\right)$$

35.2 problem Ex 2

Internal problem ID [11322]
Internal file name [`OUTPUT/10307_Tuesday_December_27_2022_04_06_10_AM_81835214/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 59. Linear equations with particular integral known. Page 136
Problem number: Ex 2.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_3rd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{xy''' - y'' - y'x + y = -x^2 + 1}$$

Unable to solve this ODE.

35.2.1 Maple step by step solution

Let's solve

$y'''x - y'' - y'x + y = -x^2 + 1$

- Highest derivative means the order of the ODE is 3

y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
Equation is the LCLM of -1/x*y(x)+diff(y(x),x), y(x)+diff(y(x),x), -y(x)+diff(y(x),x)
trying differential order: 1; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful
trying differential order: 1; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful
trying differential order: 1; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful
<- solving the LCLM ode successful `
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 22

```
dsolve(x*diff(y(x),x$3)-diff(y(x),x$2)-x*diff(y(x),x)+y(x)=1-x^2,y(x), singsol=all)
```

$$y(x) = x^2 + 3 + c_1 x + c_2 e^x + c_3 e^{-x}$$

✓ Solution by Mathematica
Time used: 0.242 (sec). Leaf size: 28

```
DSolve[x*y'''[x]-y''[x]-x*y'[x]+y[x]==1-x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + c_1 x - c_2 \cosh(x) + i c_3 \sinh(x) + 3$$

36 Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 60. Exact equation. Integrating factor. Page 139

36.1 problem Ex 1

Internal problem ID [11323]
Internal file name [`OUTPUT/10308_Tuesday_December_27_2022_04_06_11_AM_15567907/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 60. Exact equation. Integrating factor. Page 139
Problem number: Ex 1.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{(x+2)^2 y''' + (x+2) y'' + y' = 1}$$

Since y is missing from the ode then we can use the substitution $y' = v(x)$ to reduce the order by one. The ODE becomes

$$v(x) + (x+2) v'(x) + \left(x^2 + 4x + 4\right) v''(x) = 0$$

In normal form the ode

$$v''(x) (x+2)^2 + (x+2) v'(x) + v(x) = 0 \tag{1}$$

Becomes

$$v''(x) + p(x) v'(x) + q(x) v(x) = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x+2} \\ q(x) &= \frac{1}{(x+2)^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}v(\tau) + p_1\left(\frac{d}{d\tau}v(\tau)\right) + q_1 v(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x+2}dx\right)}dx \\
&= \int e^{-\ln(x+2)}\,dx \\
&= \int \frac{1}{x+2}dx \\
&= \ln(x+2) \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{(x+2)^2}}{\frac{1}{(x+2)^2}} \\
&= 1 \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}v(\tau) + q_1 v(\tau) &= 0 \\
\frac{d^2}{d\tau^2}v(\tau) + v(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $v(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(\tau) + Bv'(\tau) + Cv(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(\tau) = e^{\lambda \tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda \tau} + \mathrm{e}^{\lambda \tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda \tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(\tau) = e^{\alpha \tau}(c_1 \cos(\beta \tau) + c_2 \sin(\beta \tau))$$

Which becomes

$$v(\tau) = e^0(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$v(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to $v(x)$ using (6) which results in

$$v(x) = c_1 \cos(\ln(x+2)) + c_2 \sin(\ln(x+2))$$

But since $y' = v(x)$ then we now need to solve the ode $y' = c_1 \cos(\ln(x+2)) + c_2 \sin(\ln(x+2))$. Integrating both sides gives

$$\begin{aligned} y &= \int c_1 \cos(\ln(x+2)) + c_2 \sin(\ln(x+2))\, \mathrm{d}x \\ &= c_1 \left(\frac{\cos(\ln(x+2))(x+2)}{2} + \frac{(x+2)\sin(\ln(x+2))}{2}\right) + c_2 \left(-\frac{\cos(\ln(x+2))(x+2)}{2} + \frac{(x+2)\sin(\mathrm{l}}{2}\right. \end{aligned}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y' + (x+2)\, y'' + (x^2 + 4x + 4)\, y''' = 0$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} 1 & -\frac{(x+2)(\cos(\ln(x+2))-\sin(\ln(x+2)))}{2} & \frac{(x+2)(\cos(\ln(x+2))+\sin(\ln(x+2)))}{2} \\ 0 & \sin(\ln(x+2)) & \cos(\ln(x+2)) \\ 0 & \frac{\cos(\ln(x+2))}{x+2} & -\frac{\sin(\ln(x+2))}{x+2} \end{bmatrix}$$

$$|W| = -\frac{\cos(\ln(x+2))^2 + \sin(\ln(x+2))^2}{x+2}$$

The determinant simplifies to

$$|W| = -\frac{1}{x+2}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} -\frac{(x+2)(\cos(\ln(x+2))-\sin(\ln(x+2)))}{2} & \frac{(x+2)(\cos(\ln(x+2))+\sin(\ln(x+2)))}{2} \\ \sin(\ln(x+2)) & \cos(\ln(x+2)) \end{bmatrix}$$
$$= -\frac{x}{2} - 1$$

$$W_2(x) = \det \begin{bmatrix} 1 & \frac{(x+2)(\cos(\ln(x+2))+\sin(\ln(x+2)))}{2} \\ 0 & \cos(\ln(x+2)) \end{bmatrix}$$
$$= \cos(\ln(x+2))$$

$$W_3(x) = \det \begin{bmatrix} 1 & -\frac{(x+2)(\cos(\ln(x+2))-\sin(\ln(x+2)))}{2} \\ 0 & \sin(\ln(x+2)) \end{bmatrix}$$
$$= \sin(\ln(x+2))$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\ &= (-1)^2 \int \frac{\left(-(x+2)^2 y''' + 1 + (x^2+4x+4)\, y'''\right)\left(-\frac{x}{2} - 1\right)}{(x^2+4x+4)\left(-\frac{1}{x+2}\right)}\, dx \\ &= \int \frac{\left(-(x+2)^2 y''' + 1 + (x^2+4x+4)\, y'''\right)\left(-\frac{x}{2} - 1\right)}{-\frac{x^2+4x+4}{x+2}}\, dx \\ &= \int \left(\frac{1}{2}\right) dx \\ &= \frac{x}{2} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)}\, dx \\
&= (-1)^1 \int \frac{\left(-(x+2)^2 y''' + 1 + (x^2+4x+4)\, y'''\right)(\cos(\ln(x+2)))}{(x^2+4x+4)\left(-\frac{1}{x+2}\right)}\, dx \\
&= -\int \frac{\left(-(x+2)^2 y''' + 1 + (x^2+4x+4)\, y'''\right)\cos(\ln(x+2))}{-\frac{x^2+4x+4}{x+2}}\, dx \\
&= -\int \left(-\frac{\cos(\ln(x+2))}{x+2}\right) dx \\
&= \sin(\ln(x+2))
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{\left(-(x+2)^2 y''' + 1 + (x^2+4x+4)\, y'''\right)(\sin(\ln(x+2)))}{(x^2+4x+4)\left(-\frac{1}{x+2}\right)}\, dx \\
&= \int \frac{\left(-(x+2)^2 y''' + 1 + (x^2+4x+4)\, y'''\right)\sin(\ln(x+2))}{-\frac{x^2+4x+4}{x+2}}\, dx \\
&= \int \left(-\frac{\sin(\ln(x+2))}{x+2}\right) dx \\
&= \cos(\ln(x+2))
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = {} & \left(\frac{x}{2}\right) \\
& + (\sin(\ln(x+2)))\left(-\frac{\cos(\ln(x+2))\, x}{2} - \cos(\ln(x+2)) + \frac{\sin(\ln(x+2))\, x}{2} + \sin(\ln(x+2))\right) \\
& + (\cos(\ln(x+2)))\left(\frac{\cos(\ln(x+2))\, x}{2} + \cos(\ln(x+2)) + \frac{\sin(\ln(x+2))\, x}{2} + \sin(\ln(x+2))\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = 1 + x
$$

Therefore the general solution is

$$
y = y_h + y_p
$$

$$
\begin{aligned}
&= \Bigg(y \\
&= c_1\left(\frac{\cos(\ln(x+2))(x+2)}{2}+\frac{(x+2)\sin(\ln(x+2))}{2}\right) \\
&\quad + c_2\left(-\frac{\cos(\ln(x+2))(x+2)}{2}+\frac{(x+2)\sin(\ln(x+2))}{2}\right)+c_3\Bigg)+(1+x)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(\frac{\cos(\ln(x+2))(x+2)}{2}+\frac{(x+2)\sin(\ln(x+2))}{2}\right) \\
&\quad + c_2\left(-\frac{\cos(\ln(x+2))(x+2)}{2}+\frac{(x+2)\sin(\ln(x+2))}{2}\right)+c_3+1+x
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\left(\frac{\cos(\ln(x+2))(x+2)}{2}+\frac{(x+2)\sin(\ln(x+2))}{2}\right) \\
&\quad + c_2\left(-\frac{\cos(\ln(x+2))(x+2)}{2}+\frac{(x+2)\sin(\ln(x+2))}{2}\right)+c_3+1+x
\end{aligned}
$$

Verified OK.

36.1.1 Maple step by step solution

Let's solve

$(x+2)^2 y''' + (x+2) y'' + y' = 1$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -((diff(_b(_a), _a))*_a+2*(di
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      <- LODE of Euler type successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve((x+2)^2*diff(y(x),x$3)+(x+2)*diff(y(x),x$2)+diff(y(x),x)=1,y(x), singsol=all)
```

$$y(x) = \frac{(c_1 - c_2)(x+2)\cos(\ln(x+2))}{2} + \frac{(c_2 + c_1)(x+2)\sin(\ln(x+2))}{2} + x + c_3$$

✓ Solution by Mathematica

Time used: 0.202 (sec). Leaf size: 45

```
DSolve[(x+2)^2*y'''[x]+(x+2)*y''[x]+y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + \frac{1}{2}(c_1 - c_2)(x+2)\cos(\log(x+2)) + \frac{1}{2}(c_1 + c_2)(x+2)\sin(\log(x+2)) + c_3$$

36.2 problem Ex 2

Internal problem ID [11324]
Internal file name [`OUTPUT/10309_Tuesday_December_27_2022_04_06_12_AM_11809622/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 60. Exact equation. Integrating factor. Page 139
Problem number: Ex 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{y''x^2 + 3y'x + y = x}$$

36.2.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 3x, C = 1, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$y''x^2 + 3y'x + y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 3xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 3r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 3r + 1 = 0$$

Or

$$r^2 + 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = -1$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = \frac{c_1}{x} + \frac{c_2\ln(x)}{x}$$

Next, we find the particular solution to the ODE

$$y''x^2 + 3y'x + y = x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{\ln(x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ -\frac{1}{x^2} & -\frac{\ln(x)}{x^2} + \frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(-\frac{\ln(x)}{x^2} + \frac{1}{x^2}\right) - \left(\frac{\ln(x)}{x}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int x\ln(x)\,dx$$

Hence

$$u_1 = -\frac{x^2\ln(x)}{2} + \frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{1}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_2 = \int x dx$$

Hence

$$u_2 = \frac{x^2}{2}$$

Which simplifies to

$$u_1 = -\frac{x^2(-1 + 2\ln(x))}{4}$$

$$u_2 = \frac{x^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x(-1+2\ln(x))}{4} + \frac{x\ln(x)}{2}$$

Which simplifies to

$$y_p(x) = \frac{x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{x}{4} + \frac{c_1}{x} + \frac{c_2\ln(x)}{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{4} + \frac{c_1}{x} + \frac{c_2\ln(x)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{x}{4} + \frac{c_1}{x} + \frac{c_2\ln(x)}{x}$$

Verified OK.

36.2.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''x^2 + 3y'x + y = 0$$

In normal form the ode

$$y''x^2 + 3y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\ &= \int \mathrm{e}^{-\left(\int \frac{3}{x}dx\right)} dx \\ &= \int e^{-3\ln(x)}\, dx \\ &= \int \frac{1}{x^3} dx \\ &= -\frac{1}{2x^2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{x^2}}{\frac{1}{x^6}} \\ &= x^4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + x^4 y(\tau) = 0$$

But in terms of τ

$$x^4 = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{x^2}\right)\right)}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{x^2}\right)\right)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{-\frac{1}{x^2}}$$

$$y_2 = -\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\frac{1}{x^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2} \\ \frac{d}{dx}\left(\sqrt{-\frac{1}{x^2}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\frac{1}{x^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2} \\ \frac{1}{\sqrt{-\frac{1}{x^2}}\,x^3} & -\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{x^2}\right)}{2\sqrt{-\frac{1}{x^2}}\,x^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}}{x} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-\frac{1}{x^2}}\right)\left(-\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{x^2}\right)}{2\sqrt{-\frac{1}{x^2}}\,x^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}}{x}\right) - \left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2}\right)\left(\frac{1}{\sqrt{-\frac{1}{x^2}}\,x^3}\right)$$

Which simplifies to

$$W = \frac{\sqrt{2}}{x^3}$$

Which simplifies to

$$W = \frac{\sqrt{2}}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2}\right)x}{\frac{\sqrt{2}}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{-\frac{1}{x^2}}\,\left(-\ln\left(2\right)+\ln\left(-\frac{1}{x^2}\right)\right)x^2}{2}dx$$

Hence

$$u_1 = -\frac{x^3\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{4}+\frac{x^3\sqrt{-\frac{1}{x^2}}\,\left(\ln\left(2\right)-1\right)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{-\frac{1}{x^2}}\,x}{\frac{\sqrt{2}}{x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{-\frac{1}{x^2}}\,x^2\sqrt{2}}{2}dx$$

Hence

$$u_2 = \frac{x^3\sqrt{-\frac{1}{x^2}}\,\sqrt{2}}{4}$$

Which simplifies to

$$u_1 = -\frac{\sqrt{-\frac{1}{x^2}}\,x^3\left(\ln\left(-\frac{1}{x^2}\right)-\ln\left(2\right)+1\right)}{4}$$

$$u_2 = \frac{x^3\sqrt{-\frac{1}{x^2}}\,\sqrt{2}}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x\left(\ln\left(-\frac{1}{x^2}\right)-\ln\left(2\right)+1\right)}{4}+\frac{x^3\sqrt{-\frac{1}{x^2}}\,\sqrt{2}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2}+\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}\right)}{4}$$

Which simplifies to

$$y_p(x) = \frac{x}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\left(c_1 - c_2\ln\left(2\right) + c_2\ln\left(-\frac{1}{x^2}\right)\right)}{2}\right) + \left(\frac{x}{4}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\left(c_1 - c_2\ln\left(2\right) + c_2\ln\left(-\frac{1}{x^2}\right)\right)}{2} + \frac{x}{4} \tag{1}
$$

Verification of solutions

$$
y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\left(c_1 - c_2\ln\left(2\right) + c_2\ln\left(-\frac{1}{x^2}\right)\right)}{2} + \frac{x}{4}
$$

Verified OK.

36.2.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^2, B = 3x, C = 1, f(x) = x$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
y''x^2 + 3y'x + y = 0
$$

In normal form the ode

$$
y''x^2 + 3y'x + y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{3}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{3}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= 2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{-c\tau} c_1$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{c_1}{x}$$

Now the particular solution to this ODE is found

$$y'' x^2 + 3y' x + y = x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \sqrt{-\frac{1}{x^2}} \\
y_2 &= -\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\, \ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\, \ln\left(-\frac{1}{x^2}\right)}{2}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\frac{1}{x^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2} \\ \frac{d}{dx}\left(\sqrt{-\frac{1}{x^2}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\frac{1}{x^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2} \\ \frac{1}{\sqrt{-\frac{1}{x^2}}\,x^3} & -\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{x^2}\right)}{2\sqrt{-\frac{1}{x^2}}\,x^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}}{x} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-\frac{1}{x^2}}\right)\left(-\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{x^2}\right)}{2\sqrt{-\frac{1}{x^2}}\,x^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}}{x}\right) - \left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2}\right)\left(\frac{1}{\sqrt{-\frac{1}{x^2}}\,x^3}\right)$$

Which simplifies to

$$W = \frac{\sqrt{2}}{x^3}$$

Which simplifies to

$$W = \frac{\sqrt{2}}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}\right) x}{\frac{\sqrt{2}}{x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{-\frac{1}{x^2}}\,\left(-\ln(2) + \ln\left(-\frac{1}{x^2}\right)\right) x^2}{2} dx$$

Hence

$$u_1 = -\frac{x^3 \sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{4} + \frac{x^3 \sqrt{-\frac{1}{x^2}}\,(\ln(2) - 1)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{-\frac{1}{x^2}}\,x}{\frac{\sqrt{2}}{x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{-\frac{1}{x^2}}\,x^2 \sqrt{2}}{2} dx$$

Hence

$$u_2 = \frac{x^3 \sqrt{-\frac{1}{x^2}}\,\sqrt{2}}{4}$$

Which simplifies to

$$u_1 = -\frac{\sqrt{-\frac{1}{x^2}}\,x^3 \left(\ln\left(-\frac{1}{x^2}\right) - \ln(2) + 1\right)}{4}$$

$$u_2 = \frac{x^3 \sqrt{-\frac{1}{x^2}}\,\sqrt{2}}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x\left(\ln\left(-\frac{1}{x^2}\right) - \ln(2) + 1\right)}{4} + \frac{x^3\sqrt{-\frac{1}{x^2}}\,\sqrt{2}\left(-\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}\right)}{4}$$

Which simplifies to

$$y_p(x) = \frac{x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x}\right) + \left(\frac{x}{4}\right) \\ &= \frac{x}{4} + \frac{c_1}{x} \end{aligned}$$

Which simplifies to

$$y = \frac{x}{4} + \frac{c_1}{x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{x}{4} + \frac{c_1}{x} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{x}{4} + \frac{c_1}{x}$$

Verified OK.

36.2.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 3x, C = 1, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$y''x^2 + 3y'x + y = 0$$

In normal form the ode

$$y''x^2 + 3y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{3n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \frac{v'(x)}{x} = 0$$

$$v''(x) + \frac{v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \frac{c_1 \ln(x) + c_2}{x} \\ &= \frac{c_1 \ln(x) + c_2}{x} \end{aligned}$$

Now the particular solution to this ODE is found

$$y''x^2 + 3y'x + y = x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$

$$y_2 = \frac{\ln(x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ -\frac{1}{x^2} & -\frac{\ln(x)}{x^2} + \frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W=\left(\frac{1}{x}\right)\left(-\frac{\ln(x)}{x^2}+\frac{1}{x^2}\right)-\left(\frac{\ln(x)}{x}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W=\frac{1}{x^3}$$

Which simplifies to

$$W=\frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1=-\int\frac{\ln(x)}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_1=-\int x\ln(x)\,dx$$

Hence

$$u_1=-\frac{x^2\ln(x)}{2}+\frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2=\int\frac{1}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_2=\int x dx$$

Hence

$$u_2=\frac{x^2}{2}$$

Which simplifies to

$$u_1=-\frac{x^2(-1+2\ln(x))}{4}$$

$$u_2=\frac{x^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x(-1 + 2\ln(x))}{4} + \frac{x\ln(x)}{2}$$

Which simplifies to

$$y_p(x) = \frac{x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \ln(x) + c_2}{x}\right) + \left(\frac{x}{4}\right) \\ &= \frac{x}{4} + \frac{c_1 \ln(x) + c_2}{x} \end{aligned}$$

Which simplifies to

$$y = \frac{x}{4} + \frac{c_1 \ln(x) + c_2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{4} + \frac{c_1 \ln(x) + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{x}{4} + \frac{c_1 \ln(x) + c_2}{x}$$

Verified OK.

36.2.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y''x^2 + 3y'x + y)\, dx = \int x dx$$

$$x^2 y' + yx = \frac{x^2}{2} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$

$$q(x) = \frac{x^2 + 2c_1}{2x^2}$$

Hence the ode is

$$y' + \frac{y}{x} = \frac{x^2 + 2c_1}{2x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{x} dx} = x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^2 + 2c_1}{2x^2}\right)$$

$$\frac{d}{dx}(xy) = (x)\left(\frac{x^2 + 2c_1}{2x^2}\right)$$

$$d(xy) = \left(\frac{x^2 + 2c_1}{2x}\right) dx$$

Integrating gives

$$xy = \int \frac{x^2 + 2c_1}{2x}\, dx$$

$$xy = \frac{x^2}{4} + c_1 \ln(x) + c_2$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{\frac{x^2}{4} + c_1 \ln(x)}{x} + \frac{c_2}{x}$$

which simplifies to

$$y = \frac{c_1 \ln(x) + \frac{x^2}{4} + c_2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(x) + \frac{x^2}{4} + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(x) + \frac{x^2}{4} + c_2}{x}$$

Verified OK.

36.2.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y''x^2 + 3y'x + y = x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y''x^2 + 3y'x + y)\, dx = \int x dx$$
$$x^2y' + yx = \frac{x^2}{2} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x^2 + 2c_1}{2x^2}$$

Hence the ode is

$$y' + \frac{y}{x} = \frac{x^2 + 2c_1}{2x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x} dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^2 + 2c_1}{2x^2}\right) \\ \frac{d}{dx}(xy) &= (x)\left(\frac{x^2 + 2c_1}{2x^2}\right) \\ d(xy) &= \left(\frac{x^2 + 2c_1}{2x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xy &= \int \frac{x^2 + 2c_1}{2x} dx \\ xy &= \frac{x^2}{4} + c_1 \ln(x) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{\frac{x^2}{4} + c_1 \ln(x)}{x} + \frac{c_2}{x}$$

which simplifies to

$$y = \frac{c_1 \ln(x) + \frac{x^2}{4} + c_2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(x) + \frac{x^2}{4} + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(x) + \frac{x^2}{4} + c_2}{x}$$

Verified OK.

36.2.7 Solving using Kovacic algorithm

Writing the ode as

$$y''x^2 + 3y'x + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 213: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}(\ln(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y''x^2 + 3y'x + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{\ln(x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ -\frac{1}{x^2} & -\frac{\ln(x)}{x^2} + \frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(-\frac{\ln(x)}{x^2} + \frac{1}{x^2}\right) - \left(\frac{\ln(x)}{x}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)}{\frac{1}{x}} dx$$

Which simplifies to

$$u_1 = -\int x \ln(x) dx$$

Hence

$$u_1 = -\frac{x^2 \ln(x)}{2} + \frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{1}{\frac{1}{x}} dx$$

Which simplifies to

$$u_2 = \int x dx$$

Hence

$$u_2 = \frac{x^2}{2}$$

Which simplifies to

$$u_1 = -\frac{x^2(-1 + 2\ln(x))}{4}$$

$$u_2 = \frac{x^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x(-1+2\ln(x))}{4} + \frac{x\ln(x)}{2}$$

Which simplifies to

$$y_p(x) = \frac{x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2\ln(x)}{x}\right) + \left(\frac{x}{4}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_1 + \ln(x)\, c_2}{x} + \frac{x}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 + \ln(x)\, c_2}{x} + \frac{x}{4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 + \ln(x)\, c_2}{x} + \frac{x}{4}$$

Verified OK.

36.2.8 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x^2 \\
q(x) &= 3x \\
r(x) &= 1 \\
s(x) &= x
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 3
\end{aligned}
$$

Therefore (1) becomes

$$
2-(3)+(1)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)
$$

Integrating gives

$$
p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx
$$

Substituting the above values for p, q, r, s gives

$$
x^2y'+yx=\int x\,dx
$$

We now have a first order ode to solve which is

$$
x^2y'+yx=\frac{x^2}{2}+c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y'+p(x)y=q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{x^2+2c_1}{2x^2}
\end{aligned}
$$

Hence the ode is

$$y' + \frac{y}{x} = \frac{x^2 + 2c_1}{2x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x} dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu) \left(\frac{x^2 + 2c_1}{2x^2}\right) \\ \frac{d}{dx}(xy) &= (x) \left(\frac{x^2 + 2c_1}{2x^2}\right) \\ d(xy) &= \left(\frac{x^2 + 2c_1}{2x}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} xy &= \int \frac{x^2 + 2c_1}{2x} dx \\ xy &= \frac{x^2}{4} + c_1 \ln(x) + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{\frac{x^2}{4} + c_1 \ln(x)}{x} + \frac{c_2}{x}$$

which simplifies to

$$y = \frac{c_1 \ln(x) + \frac{x^2}{4} + c_2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(x) + \frac{x^2}{4} + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(x) + \frac{x^2}{4} + c_2}{x}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+y(x)=x,y(x), singsol=all)
```

$$y(x) = \frac{c_2}{x} + \frac{x}{4} + \frac{\ln(x)\, c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 26

```
DSolve[x^2*y''[x]+3*x*y'[x]+y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2 + 4c_2 \log(x) + 4c_1}{4x}$$

36.3 problem Ex 3

Internal problem ID [11325]
Internal file name [`OUTPUT/10310_Tuesday_December_27_2022_04_06_14_AM_40293410/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 60. Exact equation. Integrating factor. Page 139
Problem number: Ex 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$
\boxed{(x-1)^2 y'' + 4(x-1)\, y' + 2y = \cos(x)}
$$

36.3.1 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = \frac{4}{x-1}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int \frac{4}{x-1}\, dx} \\ &= (x-1)^2 \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \cos(x) \\ \left((x-1)^2 y\right)'' &= \cos(x) \end{aligned}$$

Integrating once gives

$$\left((x-1)^2 y\right)' = \sin(x) + c_1$$

Integrating again gives

$$\left((x-1)^2 y\right) = c_1 x - \cos(x) + c_2$$

Hence the solution is

$$y = \frac{c_1 x - \cos(x) + c_2}{(x-1)^2}$$

Or

$$y = \frac{c_1 x}{(x-1)^2} + \frac{c_2}{(x-1)^2} - \frac{\cos(x)}{(x-1)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x}{(x-1)^2} + \frac{c_2}{(x-1)^2} - \frac{\cos(x)}{(x-1)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x}{(x-1)^2} + \frac{c_2}{(x-1)^2} - \frac{\cos(x)}{(x-1)^2}$$

Verified OK.

36.3.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$(x-1)^2 y'' + (4x-4) y' + 2y = 0$$

In normal form the ode

$$(x-1)^2 y'' + (4x-4) y' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x-1} \\ q(x) &= \frac{2}{(x-1)^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\ &= \int e^{-\left(\int \frac{4}{x-1}dx\right)} dx \\ &= \int e^{-4\ln(x-1)} dx \\ &= \int \frac{1}{(x-1)^4} dx \\ &= -\frac{1}{3(x-1)^3}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{2}{(x-1)^2}}{\frac{1}{(x-1)^8}} \\ &= 2(x-1)^6\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) + 2(x-1)^6 y(\tau) = 0$$

But in terms of τ

$$2(x-1)^6 = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{3}} + c_2\tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}$$

$$y_2 = \left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}} \\ \frac{1}{\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}(x-1)^4} & \frac{2}{\left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}(x-1)^4} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}\right)\left(\frac{2}{\left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}(x-1)^4}\right) - \left(\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}\right)\left(\frac{1}{\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}(x-1)^4}\right)$$

Which simplifies to

$$W = \frac{1}{(x-1)^4}$$

Which simplifies to

$$W = \frac{1}{(x-1)^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}\cos(x)}{\frac{1}{(x-1)^2}}\,dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}\cos(x)(x-1)^2\,dx$$

Hence

$$u_1 = -\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}\sin(x)(x-1)^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}\cos(x)}{\frac{1}{(x-1)^2}}\,dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}} \cos(x)(x-1)^2 \, dx$$

Hence

$$u_2 = \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}} \cos(x)(x-1) + \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}} \sin(x)(x-1)^2$$

Which simplifies to

$$u_1 = -\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}} \sin(x)(x-1)^2$$

$$u_2 = \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}} ((x-1)\sin(x) + \cos(x))(x-1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sin(x)}{x-1} - \frac{(x-1)\sin(x) + \cos(x)}{(x-1)^2}$$

Which simplifies to

$$y_p(x) = -\frac{\cos(x)}{(x-1)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}}{3}\right) + \left(-\frac{\cos(x)}{(x-1)^2}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}}{3} - \frac{\cos(x)}{(x-1)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}}{3} - \frac{\cos(x)}{(x-1)^2}$$

Verified OK.

36.3.3 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$(x-1)^2 y'' + (4x-4)y' + 2y = 0$$

In normal form the given ode is written as

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4x-4}{x^2-2x+1} \\ q(x) &= \frac{2}{x^2-2x+1} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{2}{x^2-2x+1} - \frac{\left(\frac{4x-4}{x^2-2x+1}\right)'}{2} - \frac{\left(\frac{4x-4}{x^2-2x+1}\right)^2}{4} \\ &= \frac{2}{x^2-2x+1} - \frac{\left(\frac{4}{x^2-2x+1} - \frac{(4x-4)(2x-2)}{(x^2-2x+1)^2}\right)}{2} - \frac{\left(\frac{(4x-4)^2}{(x^2-2x+1)^2}\right)}{4} \\ &= \frac{2}{x^2-2x+1} - \left(\frac{2}{x^2-2x+1} - \frac{(4x-4)(2x-2)}{2(x^2-2x+1)^2}\right) - \frac{(4x-4)^2}{4(x^2-2x+1)^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{4x-4}{x^2-2x+1}}{2}} \\
&= \frac{1}{(x-1)^2}
\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{(x-1)^2} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) = \cos(x)$$

Which is now solved for $v(x)$ Integrating once gives

$$v'(x) = \sin(x) + c_1$$

Integrating again gives

$$v(x) = -\cos(x) + c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned}
y &= v(x)\, z(x) \\
&= (c_1 x - \cos(x) + c_2)\,(z(x))
\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{(x-1)^2}$$

Hence (7) becomes

$$y = \frac{c_1 x - \cos(x) + c_2}{(x-1)^2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x - \cos(x) + c_2}{(x-1)^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}$$
$$y_2 = \left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W=\begin{vmatrix}\left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}\\ \frac{1}{\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}(x-1)^4} & \frac{2}{\left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}(x-1)^4}\end{vmatrix}$$

Therefore

$$W=\left(\left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}\right)\left(\frac{2}{\left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}}(x-1)^4}\right)-\left(\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}\right)\left(\frac{1}{\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}(x-1)^4}\right)$$

Which simplifies to

$$W=\frac{1}{(x-1)^4}$$

Which simplifies to

$$W=\frac{1}{(x-1)^4}$$

Therefore Eq. (2) becomes

$$u_1=-\int\frac{\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}\cos(x)}{\frac{1}{(x-1)^2}}\,dx$$

Which simplifies to

$$u_1=-\int\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}\cos(x)(x-1)^2\,dx$$

Hence

$$u_1=-\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}}\sin(x)(x-1)^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}} \cos(x)}{\frac{1}{(x-1)^2}} dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}} \cos(x) (x-1)^2 dx$$

Hence

$$u_2 = \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}} \cos(x) (x-1) + \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}} \sin(x) (x-1)^2$$

Which simplifies to

$$u_1 = -\left(-\frac{1}{(x-1)^3}\right)^{\frac{2}{3}} \sin(x) (x-1)^2$$

$$u_2 = \left(-\frac{1}{(x-1)^3}\right)^{\frac{1}{3}} ((x-1)\sin(x) + \cos(x)) (x-1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sin(x)}{x-1} - \frac{(x-1)\sin(x) + \cos(x)}{(x-1)^2}$$

Which simplifies to

$$y_p(x) = -\frac{\cos(x)}{(x-1)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x - \cos(x) + c_2}{(x-1)^2}\right) + \left(-\frac{\cos(x)}{(x-1)^2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x - \cos(x) + c_2}{(x-1)^2} - \frac{\cos(x)}{(x-1)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x - \cos(x) + c_2}{(x-1)^2} - \frac{\cos(x)}{(x-1)^2}$$

Verified OK.

36.3.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left((x-1)^2 y'' + (4x-4)\, y' + 2y\right) dx = \int \cos(x)\, dx$$
$$(2x-2)\, y + \left(x^2 - 2x + 1\right) y' = \sin(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x-1}$$
$$q(x) = \frac{\sin(x) + c_1}{(x-1)^2}$$

Hence the ode is

$$y' + \frac{2y}{x-1} = \frac{\sin(x) + c_1}{(x-1)^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{x-1} dx}$$
$$= (x-1)^2$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\sin(x) + c_1}{(x-1)^2}\right)$$
$$\frac{d}{dx}\left((x-1)^2 y\right) = \left((x-1)^2\right)\left(\frac{\sin(x) + c_1}{(x-1)^2}\right)$$
$$d\left((x-1)^2 y\right) = (\sin(x) + c_1)\, dx$$

Integrating gives

$$(x-1)^2 y = \int \sin(x) + c_1\, dx$$
$$(x-1)^2 y = c_1 x - \cos(x) + c_2$$

Dividing both sides by the integrating factor $\mu = (x-1)^2$ results in

$$y = \frac{c_1 x - \cos(x)}{(x-1)^2} + \frac{c_2}{(x-1)^2}$$

which simplifies to

$$y = \frac{c_1 x - \cos(x) + c_2}{(x-1)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x - \cos(x) + c_2}{(x-1)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x - \cos(x) + c_2}{(x-1)^2}$$

Verified OK.

36.3.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(x-1)^2 y'' + (4x-4)y' + 2y = \cos(x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left((x-1)^2 y'' + (4x-4)y' + 2y\right) dx = \int \cos(x)\, dx$$
$$(2x-2)y + \left(x^2 - 2x + 1\right)y' = \sin(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x-1} \\ q(x) &= \frac{\sin(x) + c_1}{(x-1)^2} \end{aligned}$$

Hence the ode is

$$y' + \frac{2y}{x-1} = \frac{\sin(x) + c_1}{(x-1)^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{2}{x-1} dx} \\ &= (x-1)^2 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu) \left(\frac{\sin(x) + c_1}{(x-1)^2}\right) \\ \frac{d}{dx}\left((x-1)^2 y\right) &= \left((x-1)^2\right) \left(\frac{\sin(x) + c_1}{(x-1)^2}\right) \\ d\left((x-1)^2 y\right) &= (\sin(x) + c_1)\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} (x-1)^2 y &= \int \sin(x) + c_1\, dx \\ (x-1)^2 y &= c_1 x - \cos(x) + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = (x-1)^2$ results in

$$y = \frac{c_1 x - \cos(x)}{(x-1)^2} + \frac{c_2}{(x-1)^2}$$

which simplifies to

$$y = \frac{c_1 x - \cos(x) + c_2}{(x-1)^2}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x - \cos(x) + c_2}{(x-1)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x - \cos(x) + c_2}{(x-1)^2}$$

Verified OK.

36.3.6 Solving using Kovacic algorithm

Writing the ode as

$$(x-1)^2 y'' + (4x-4) y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x-1)^2 \\ B &= 4x-4 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 214: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x-4}{(x-1)^2}\,dx} \\
&= z_1 e^{-2\ln(x-1)} \\
&= z_1 \left(\frac{1}{(x-1)^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(x-1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x-4}{(x-1)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{(x-1)^2}\right) + c_2 \left(\frac{1}{(x-1)^2}(x)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(x-1)^2 y'' + (4x-4)\, y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{(x-1)^2} + \frac{c_2 x}{(x-1)^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{(x-1)^2}$$
$$y_2 = \frac{x}{(x-1)^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{(x-1)^2} & \frac{x}{(x-1)^2} \\ \frac{d}{dx}\left(\frac{1}{(x-1)^2}\right) & \frac{d}{dx}\left(\frac{x}{(x-1)^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{(x-1)^2} & \frac{x}{(x-1)^2} \\ -\frac{2}{(x-1)^3} & \frac{1}{(x-1)^2} - \frac{2x}{(x-1)^3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{(x-1)^2}\right)\left(\frac{1}{(x-1)^2} - \frac{2x}{(x-1)^3}\right) - \left(\frac{x}{(x-1)^2}\right)\left(-\frac{2}{(x-1)^3}\right)$$

Which simplifies to

$$W = \frac{1}{(x-1)^4}$$

Which simplifies to

$$W = \frac{1}{(x-1)^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\cos(x)}{(x-1)^2}}{\frac{1}{(x-1)^2}}\, dx$$

Which simplifies to

$$u_1 = -\int \cos(x)\, x dx$$

Hence

$$u_1 = -\cos(x) - x\sin(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(x)}{(x-1)^2}}{\frac{1}{(x-1)^2}} \, dx$$

Which simplifies to

$$u_2 = \int \cos(x) \, dx$$

Hence

$$u_2 = \sin(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\cos(x) - x\sin(x)}{(x-1)^2} + \frac{x\sin(x)}{(x-1)^2}$$

Which simplifies to

$$y_p(x) = -\frac{\cos(x)}{(x-1)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{(x-1)^2} + \frac{c_2 x}{(x-1)^2}\right) + \left(-\frac{\cos(x)}{(x-1)^2}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_2 x + c_1}{(x-1)^2} - \frac{\cos(x)}{(x-1)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2 x + c_1}{(x-1)^2} - \frac{\cos(x)}{(x-1)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_2 x + c_1}{(x-1)^2} - \frac{\cos(x)}{(x-1)^2}$$

Verified OK.

36.3.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)$$

is exact if

$$p''(x)-q'(x)+r(x)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x)&=(x-1)^2\\q(x)&=4x-4\\r(x)&=2\\s(x)&=\cos\left(x\right)\end{aligned}$$

Hence

$$\begin{aligned}p''(x)&=2\\q'(x)&=4\end{aligned}$$

Therefore (1) becomes

$$2-(4)+(2)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+(q(x)-p'(x))\,y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$(x-1)^2\,y'+(2x-2)\,y=\int\cos\left(x\right)\,dx$$

We now have a first order ode to solve which is

$$(x-1)^2\,y'+(2x-2)\,y=\sin\left(x\right)+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{2}{x-1}\\ q(x) &= \frac{\sin(x)+c_1}{(x-1)^2}\end{aligned}$$

Hence the ode is

$$y' + \frac{2y}{x-1} = \frac{\sin(x)+c_1}{(x-1)^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x-1}dx}\\ &= (x-1)^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\sin(x)+c_1}{(x-1)^2}\right)\\ \frac{d}{dx}\left((x-1)^2 y\right) &= \left((x-1)^2\right)\left(\frac{\sin(x)+c_1}{(x-1)^2}\right)\\ d\left((x-1)^2 y\right) &= (\sin(x)+c_1)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}(x-1)^2 y &= \int \sin(x)+c_1\, dx\\ (x-1)^2 y &= c_1 x - \cos(x) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = (x-1)^2$ results in

$$y = \frac{c_1 x - \cos(x)}{(x-1)^2} + \frac{c_2}{(x-1)^2}$$

which simplifies to

$$y = \frac{c_1 x - \cos(x) + c_2}{(x-1)^2}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x - \cos(x) + c_2}{(x-1)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x - \cos(x) + c_2}{(x-1)^2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 19

```
dsolve((x-1)^2*diff(y(x),x$2)+4*(x-1)*diff(y(x),x)+2*y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = \frac{c_2 + c_1 x - \cos(x)}{(-1+x)^2}$$

✓ Solution by Mathematica
Time used: 0.134 (sec). Leaf size: 24

```
DSolve[(x-1)^2*y''[x]+4*(x-1)*y'[x]+2*y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-\cos(x) + c_1(x-1) + c_2}{(x-1)^2}$$

36.4 problem Ex 4

Internal problem ID [11326]
Internal file name [`OUTPUT/10311_Tuesday_December_27_2022_04_06_15_AM_43338447/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 60. Exact equation. Integrating factor. Page 139
Problem number: Ex 4.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_3rd_order, _fully, _exact, _linear]]`

Unable to solve or complete the solution.

$$\boxed{(x^3 - x)\, y''' + (8x^2 - 3)\, y'' + 14y'x + 4y = 0}$$

Unable to solve this ODE.

36.4.1 Maple step by step solution

Let's solve

$(x^3 - x)\, y''' + (8x^2 - 3)\, y'' + 14y'x + 4y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{8x^2-3}{(x^2-1)x}, P_3(x) = \frac{14}{x^2-1}, P_4(x) = \frac{4}{(x^2-1)x}\right]$
  - $(1 + x) \cdot P_2(x)$ is analytic at $x = -1$

$\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1} = \frac{5}{2}$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $(1+x)^3 \cdot P_4(x)$ is analytic at $x=-1$

  $\left((1+x)^3 \cdot P_4(x)\right)\bigg|_{x=-1} = 0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\, x y''' + (8x^2-3)\, y'' + 14y'x + 4y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^3-3u^2+2u)\left(\frac{d^3}{du^3}y(u)\right) + (8u^2-16u+5)\left(\frac{d^2}{du^2}y(u)\right) + (14u-14)\left(\frac{d}{du}y(u)\right) + 4y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^3}{du^3}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^3}{du^3}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)(k+r-2)\,u^{k+r-3+m}$$

- Shift index using $k- >k+3-m$

  $$u^m \cdot \left(\frac{d^3}{du^3}y(u)\right) = \sum_{k=-3+m}^{\infty} a_{k+3-m}(k+3-m+r)(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0r(1+2r)(-1+r)\,u^{-2+r} + (a_1(r+1)(2r+3)\,r - a_0r(3r+4)(r+1))\,u^{-1+r} + \left(\sum_{k=0}^{\infty} (a_{k+2}(2+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(1+2r)(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, 1, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r+1)((a_k - 3a_{k+1} + 2a_{k+2})\,k + (a_k - 3a_{k+1} + 2a_{k+2})\,r + 2a_k - 7a_{k+1} + 5a_{k+2})(2+k+r)$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{ka_k - 3ka_{k+1} + ra_k - 3ra_{k+1} + 2a_k - 7a_{k+1}}{2k+5+2r}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{ka_k - 3ka_{k+1} + 2a_k - 7a_{k+1}}{2k+5}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{ka_k - 3ka_{k+1} + 2a_k - 7a_{k+1}}{2k+5}, 0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+2} = -\frac{ka_k - 3ka_{k+1} + 2a_k - 7a_{k+1}}{2k+5}, 0 = 0\right]$$

- Recursion relation for $r = 1$

  $$a_{k+2} = -\frac{ka_k - 3ka_{k+1} + 3a_k - 10a_{k+1}}{2k+7}$$

- Solution for $r = 1$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = -\frac{ka_k - 3ka_{k+1} + 3a_k - 10a_{k+1}}{2k+7}, 10a_1 - 14a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y=\sum_{k=0}^{\infty} a_k(1+x)^{k+1}, a_{k+2}=-\frac{k a_k-3k a_{k+1}+3a_k-10a_{k+1}}{2k+7}, 10a_1-14a_0=0\right]$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+2}=-\frac{k a_k-3k a_{k+1}+\frac{3}{2}a_k-\frac{11}{2}a_{k+1}}{2k+4}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+2}=-\frac{k a_k-3k a_{k+1}+\frac{3}{2}a_k-\frac{11}{2}a_{k+1}}{2k+4}, -\frac{a_1}{2}+\frac{5a_0}{8}=0\right]$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty} a_k(1+x)^{k-\frac{1}{2}}, a_{k+2}=-\frac{k a_k-3k a_{k+1}+\frac{3}{2}a_k-\frac{11}{2}a_{k+1}}{2k+4}, -\frac{a_1}{2}+\frac{5a_0}{8}=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k(1+x)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k(1+x)^{1+k}\right)+\left(\sum_{k=0}^{\infty} c_k(1+x)^{k-\frac{1}{2}}\right), a_{k+2}=-\frac{k a_k-3k a_{1+k}+2a_k-7a_{1}}{2k+5}\right.$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 41

```
dsolve((x^3-x)*diff(y(x),x$3)+(8*x^2-3)*diff(y(x),x$2)+14*x*diff(y(x),x)+4*y(x)=0,y(x), sing
```

$$y(x)=\frac{\frac{c_3}{\sqrt{1+x}\,\sqrt{-1+x}}+c_1+\frac{c_2\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}}}{x}$$

✓ Solution by Mathematica

Time used: 0.135 (sec). Leaf size: 51

```
DSolve[(x^3-x)*y'''[x]+(8*x^2-3)*y''[x]+14*x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{-\frac{c_2}{\sqrt{x^2-1}} + \frac{c_3 \log\left(\sqrt{x^2-1}-x\right)}{\sqrt{x^2-1}} + c_1}{x}$$

36.5 problem Ex 5

Internal problem ID [11327]
Internal file name [`OUTPUT/10312_Tuesday_December_27_2022_04_06_16_AM_28478648/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 60. Exact equation. Integrating factor. Page 139
Problem number: Ex 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _nonlinear], [_3rd_order,
   _with_linear_symmetries]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying 3rd order ODE linearizable_by_differentiation
differential order: 3; trying a linearization to 4th order
trying differential order: 3; missing variables
trying differential order: 3; exact nonlinear
-> Calling odsolve with the ODE`, _b(_a)*(diff(diff(_b(_a), _a), _a))*_a^3+_a^3*(diff(_b(_a)
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying 2nd order Liouville
   trying 2nd order WeierstrassP
   trying 2nd order JacobiSN
   differential order: 2; trying a linearization to 3rd order
   trying 2nd order ODE linearizable_by_differentiation
   trying 2nd order, 2 integrating factors of the form mu(x,y)
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
   <- 2nd order, 2 integrating factors of the form mu(x,y) successful
<- differential order: 3; exact nonlinear successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 56

```
dsolve(2*x^3*y(x)*diff(y(x),x$3)+6*x^3*diff(y(x),x)*diff(y(x),x$2)+18*x^2*y(x)*diff(y(x),x$2
```

$$y(x) = 0$$
$$y(x) = \frac{\sqrt{-x\,(c_1x^2 + 2c_2x - 2c_3)}}{x^2}$$
$$y(x) = -\frac{\sqrt{-x\,(c_1x^2 + 2c_2x - 2c_3)}}{x^2}$$

✓ Solution by Mathematica

Time used: 0.389 (sec). Leaf size: 60

```
DSolve[2*x^3*y[x]*y'''[x]+6*x^3*y'[x]*y''[x]+18*x^2*y[x]*y''[x]+18*x^2*y'[x]^2+36*x*y[x]*y'[
```

$$y(x) \to -\frac{\sqrt{c_1x^2 + c_3x + 2c_2}}{x^{3/2}}$$

$$y(x) \to \frac{\sqrt{c_1x^2 + c_3x + 2c_2}}{x^{3/2}}$$

36.6 problem Ex 6

Internal problem ID [11328]
Internal file name [`OUTPUT/10313_Tuesday_December_27_2022_04_06_17_AM_93956182/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 60. Exact equation. Integrating factor. Page 139
Problem number: Ex 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__change__of__variable__on__y__method__2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^5y'' + (2x^4 - x)\, y' - (2x^3 - 1)\, y = 0}$$

36.6.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^5y'' + (2x^4 - x)\, y' + (-2x^3 + 1)\, y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2x^3 - 1}{x^4}$$
$$q(x) = \frac{-2x^3 + 1}{x^5}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(2x^3-1)}{x^5} + \frac{-2x^3+1}{x^5} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} + \frac{2x^3-1}{x^4}\right) v'(x) &= 0 \\ v''(x) + \frac{(4x^3-1)\, v'(x)}{x^4} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(4x^3-1)\, u(x)}{x^4} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(4x^3-1)\, u}{x^4} \end{aligned}$$

Where $f(x) = -\frac{4x^3-1}{x^4}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{4x^3-1}{x^4}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{4x^3-1}{x^4}\,dx \\
\ln(u) &= -\frac{1}{3x^3} - 4\ln(x) + c_1 \\
u &= \mathrm{e}^{-\frac{1}{3x^3}-4\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{-\frac{1}{3x^3}-4\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 \mathrm{e}^{-\frac{1}{3x^3}}}{x^4}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1 \mathrm{e}^{-\frac{1}{3x^3}} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(c_1 \mathrm{e}^{-\frac{1}{3x^3}} + c_2\right) x \\
&= \left(c_1 \mathrm{e}^{-\frac{1}{3x^3}} + c_2\right) x
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(c_1 \mathrm{e}^{-\frac{1}{3x^3}} + c_2\right) x \tag{1}$$

<u>Verification of solutions</u>

$$y = \left(c_1 \mathrm{e}^{-\frac{1}{3x^3}} + c_2\right) x$$

Verified OK.

36.6.2 Solving using Kovacic algorithm

Writing the ode as

$$x^5y'' + (2x^4 - x)\,y' + (-2x^3 + 1)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^5 \\ B &= 2x^4 - x \\ C &= -2x^3 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x^6 + 1}{4x^8} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8x^6 + 1 \\ t &= 4x^8 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x^6 + 1}{4x^8}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 216: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 8 - 6 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^8$. There is a pole at $x = 0$ of order 8. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\alpha_c^+ = \frac{1}{2}\left(\frac{b}{a} + v\right)$$
$$\alpha_c^- = \frac{1}{2}\left(-\frac{b}{a} + v\right)$$

The partial fraction decomposition of r is

$$r = \frac{2}{x^2} + \frac{1}{4x^8}$$

There is pole in r at $x = 0$ of order 8, hence $v = 4$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^4} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 4$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^4} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^4}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^5}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^4} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) = \frac{1}{2}\left(\frac{0}{\frac{1}{2}} + 4\right) = 2 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) = \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} + 4\right) = 2 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{8x^6 + 1}{4x^8}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8x^6 + 1}{4x^8}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	8	$\frac{1}{2x^4}$	2	2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= 2 - (2) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x^4} + \frac{2}{x} + (0) \\
&= -\frac{1}{2x^4} + \frac{2}{x} \\
&= -\frac{1}{2x^4} + \frac{2}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x^4} + \frac{2}{x}\right)(0) + \left(\left(\frac{2}{x^5} - \frac{2}{x^2}\right) + \left(-\frac{1}{2x^4} + \frac{2}{x}\right)^2 - \left(\frac{8x^6 + 1}{4x^8}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x^4} + \frac{2}{x}\right) dx} \\
&= x^2 \mathrm{e}^{\frac{1}{6x^3}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^4 - x}{x^5} \, dx} \\
&= z_1 e^{-\frac{1}{6x^3} - \ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{1}{6x^3}}}{x} \right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^4-x}{x^5}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{1}{3x^3}-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\mathrm{e}^{-\frac{1}{3x^3}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2\left(x\left(\mathrm{e}^{-\frac{1}{3x^3}}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2 \mathrm{e}^{-\frac{1}{3x^3}} x \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2 \mathrm{e}^{-\frac{1}{3x^3}} x$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x^5*diff(y(x),x$2)+(2*x^4-x)*diff(y(x),x)-(2*x^3-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x\left(c_1 + c_2 e^{-\frac{1}{3x^3}}\right)$$

✓ Solution by Mathematica
Time used: 0.152 (sec). Leaf size: 22

```
DSolve[x^5*y''[x]+(2*x^4-x)*y'[x]-(2*x^3-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\left(c_2 e^{-\frac{1}{3x^3}} + c_1\right)$$

36.7 problem Ex 7

Internal problem ID [11329]
Internal file name [OUTPUT/10314_Tuesday_December_27_2022_04_06_19_AM_27016616/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 60. Exact equation. Integrating factor. Page 139
Problem number: Ex 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^2(-x^3+1)\, y'' - y' x^3 - 2y = 0}$$

36.7.1 Maple step by step solution

Let's solve

$(-x^5+x^2)\, y'' - y' x^3 - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x^2(x^3-1)} - \frac{y' x}{x^3-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y' x}{x^3-1} + \frac{2y}{x^2(x^3-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{x}{x^3-1}, P_3(x) = \frac{2}{x^2(x^3-1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -2$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2(x^3 - 1) + y'x^3 + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^3 \cdot y'$ to series expansion

  $x^3 \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r+2}$

- Shift index using $k- >k - 2$

  $x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2}(k - 2 + r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..5$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(1 + r)\,(-2 + r)\, x^r - a_1(2 + r)\,(-1 + r)\, x^{1+r} + (-a_2(3 + r)\, r + a_0 r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (-a_k(k + r\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-(1+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$

- The coefficients of each power of x must be 0

  $[-a_1(2+r)(-1+r) = 0, -a_2(3+r)r + a_0 r = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = \frac{a_0}{3+r}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r+1)(k-2+r) + a_{k-2}(k-2+r) + a_{k-3}(k-3+r)(k-4+r) = 0$

- Shift index using $k- >k+3$

  $-a_{k+3}(k+4+r)(k+r+1) + a_{k+1}(k+r+1) + a_k(k+r)(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+3} = \frac{k^2 a_k + 2kra_k + r^2 a_k - ka_k + ka_{k+1} - ra_k + ra_{k+1} + a_{k+1}}{(k+4+r)(k+r+1)}$

- Recursion relation for $r = -1$

  $a_{k+3} = \frac{k^2 a_k - 3ka_k + ka_{k+1} + 2a_k}{(k+3)k}$

- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 0$

  $a_{k+3} = \frac{k^2 a_k - 3ka_k + ka_{k+1} + 2a_k}{(k+3)k}$

- Recursion relation for $r = 2$

  $a_{k+3} = \frac{k^2 a_k + 3ka_k + ka_{k+1} + 2a_k + 3a_{k+1}}{(k+6)(k+3)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+3} = \frac{k^2 a_k + 3ka_k + ka_{k+1} + 2a_k + 3a_{k+1}}{(k+6)(k+3)}, a_1 = 0, a_2 = \frac{a_0}{5}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
```

✗ Solution by Maple

```
dsolve(x^2*(1-x^3)*diff(y(x),x$2)-x^3*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*(1-x^3)*y''[x]-x^3*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

36.8 problem Ex 8

Internal problem ID [11330]
Internal file name [OUTPUT/10315_Tuesday_December_27_2022_04_07_29_AM_43734921/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 60. Exact equation. Integrating factor. Page 139
Problem number: Ex 8.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_3rd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x^2 y''' - 5xy'' + \left(4x^4 + 5\right) y' - 8y x^3 = 0}$$

Unable to solve this ODE.

36.8.1 Maple step by step solution

Let's solve

$y''' x^2 - 5y'' x + (4x^4 + 5) y' - 8y x^3 = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{5}{x}, P_3(x) = \frac{4x^4+5}{x^2}, P_4(x) = -8x\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\Big|_{x=0} = -5$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 5$$

- $x^3 \cdot P_4(x)$ is analytic at $x = 0$

  $$\left(x^3 \cdot P_4(x)\right)\Big|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

  $$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$

- Shift index using $k- >k-3$

  $$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..4$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y'''$ to series expansion

  $x^2 \cdot y''' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)(k+r-2)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x^2 \cdot y''' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)(-6+r)\, x^{-1+r} + a_1(1+r)(-1+r)(-5+r)\, x^r + a_2(2+r)\, r(-4+r)\, x^{1+r} + a_3(3$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r)(-6+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2, 6\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)(-1+r)(-5+r) = 0, a_2(2+r)\, r(-4+r) = 0, a_3(3+r)(1+r)(-3+r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k-5+r)(a_{k+1}(k+1+r)(k+r-1) + 4a_{k-3}) = 0$

- Shift index using $k- >k+3$

  $(k+r-2)(a_{k+4}(k+4+r)(k+2+r) + 4a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+2+r)}$

- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{4a_k}{(k+4)(k+2)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4a_k}{(k+4)(k+2)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

- Recursion relation for $r = 2$

  $a_{k+4} = -\frac{4a_k}{(k+6)(k+4)}$

- Solution for $r = 2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+4}=-\frac{4a_k}{(k+6)(k+4)}, a_1=0, a_2=0, a_3=0\right]$$

- Recursion relation for $r=6$

$$a_{k+4}=-\frac{4a_k}{(k+10)(k+8)}$$

- Solution for $r=6$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+4}=-\frac{4a_k}{(k+10)(k+8)}, a_1=0, a_2=0, a_3=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right)+\left(\sum_{k=0}^{\infty} c_k x^{k+6}\right), a_{k+4}=-\frac{4a_k}{(k+4)(k+2)}, a_1=0, a_2=0, a_3=0, b_{k\cdot}\right.$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
trying to convert to a linear ODE with constant coefficients
trying differential order: 3; missing the dependent variable
Equation is the LCLM of -2/x*y(x)+diff(y(x),x), 4*x^2*y(x)-1/x*diff(y(x),x)+diff(diff(y(x),x
trying differential order: 1; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving the LCLM ode successful `
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(x^2*diff(y(x),x$3)-5*x*diff(y(x),x$2)+(4*x^4+5)*diff(y(x),x)-8*x^3*y(x)=0,y(x), sings
```

$$y(x) = c_1 x^2 + c_2 \cos\left(x^2\right) + c_3 \sin\left(x^2\right)$$

✓ Solution by Mathematica

Time used: 0.507 (sec). Leaf size: 44

```
DSolve[x^2*y'''[x]-5*x*y''[x]+(4*x^4+5)*y'[x]-8*x^3*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to c_1 x^2 + \frac{1}{2} i c_2 e^{-ix^2} - \frac{1}{8} c_3 e^{ix^2}$$

36.9 problem Ex 10

Internal problem ID [11331]
Internal file name [OUTPUT/10316_Tuesday_December_27_2022_04_07_30_AM_93030666/index.tex]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 60. Exact equation. Integrating factor. Page 139
Problem number: Ex 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + 2\cot(x) y' + 2\tan(x) {y'}^2 = 0}$$

36.9.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + (2\cot(x) + 2p(x)\tan(x))\, p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Writing the ode as

$$\begin{aligned} p'(x) &= -2p(p\tan(x) + \cot(x)) \\ p'(x) &= \omega(x, p) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_p - \xi_x) - \omega^2 \xi_p - \omega_x \xi - \omega_p \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 219: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, p) &= 0 \\ \eta(x, p) &= p^2 \sin(x)^2 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, p) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dp}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial p}\right) S(x,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{p^2 \sin(x)^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{\sin(x)^2 p}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,p) S_p}{R_x + \omega(x,p) R_p} \tag{2}$$

Where in the above R_x, R_p, S_x, S_p are all partial derivatives and $\omega(x,p)$ is the right hand side of the original ode given by

$$\omega(x,p) = -2p(p\tan(x) + \cot(x))$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_p &= 0 \\ S_x &= \frac{2\csc(x)^2 \cot(x)}{p} \\ S_p &= \frac{\csc(x)^2}{p^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -2\sec(x)\csc(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -2\sec(R)\csc(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(\tan(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, p coordinates. This results in

$$-\frac{\csc(x)^2}{p(x)} = -2\ln(\tan(x)) + c_1$$

Which simplifies to

$$-\frac{\csc(x)^2}{p(x)} = -2\ln(\tan(x)) + c_1$$

Which gives

$$p(x) = \frac{\csc(x)^2}{2\ln(\tan(x)) - c_1}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{\csc(x)^2}{2\ln(\tan(x)) - c_1}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\csc(x)^2}{2\ln(\tan(x)) - c_1}\, dx \\ &= -\frac{e^{-\frac{c_1}{2}} \operatorname{expIntegral}_1\left(\ln(\tan(x)) - \frac{c_1}{2}\right)}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-\frac{c_1}{2}} \operatorname{expIntegral}_1\left(\ln(\tan(x)) - \frac{c_1}{2}\right)}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{e^{-\frac{c_1}{2}} \operatorname{expIntegral}_1\left(\ln\left(\tan\left(x\right)\right) - \frac{c_1}{2}\right)}{2} + c_2$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -2*tan(_a)*_b(_a)^2-2*cot(_a)*_b(_a), _
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+2*cot(x)*diff(y(x),x)+2*tan(x)*diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = -\frac{e^{\frac{c_1}{2}} \operatorname{expIntegral}_1\left(\ln\left(\tan\left(x\right)\right) + \frac{c_1}{2}\right)}{2} + c_2$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+2*Cot[x]*y'[x]+2*Tan[x]*y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

37 Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 61. Transformation of variables. Page 143

37.1 problem Ex 1

Internal problem ID [11332]
Internal file name [`OUTPUT/10317_Tuesday_December_27_2022_04_07_32_AM_86152449/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 61. Transformation of variables. Page 143
Problem number: Ex 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _reducible
   , _mu_y_y1], [_2nd_order, _reducible, _mu_xy]]
```

Unable to solve or complete the solution.

$$x^2yy'' + (y'x - y)^2 = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 44

```
dsolve(x^2*y(x)*diff(y(x),x$2)+(x*diff(y(x),x)-y(x))^2=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \sqrt{2}\,\sqrt{-x\,(c_1x - c_2)}$$
$$y(x) = -\sqrt{2}\,\sqrt{-x\,(c_1x - c_2)}$$

✓ Solution by Mathematica

Time used: 0.388 (sec). Leaf size: 23

```
DSolve[x^2*y[x]*y''[x]+(x*y'[x]-y[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2\sqrt{x}\sqrt{2x + c_1}$$

37.2 problem Ex 2

Internal problem ID [11333]
Internal file name [`OUTPUT/10318_Tuesday_December_27_2022_04_07_33_AM_17358228/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 61. Transformation of variables. Page 143
Problem number: Ex 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _reducible
   , _mu_xy]]
```

Unable to solve or complete the solution.

$$x^3y'' - (y'x - y)^2 = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
<- quadrature successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve(x^3*diff(y(x),x$2)-(x*diff(y(x),x)-y(x))^2=0,y(x), singsol=all)
```

$$y(x) = -x \ln \left(\frac{c_1 x - c_2}{x}\right)$$

✓ Solution by Mathematica

Time used: 1.65 (sec). Leaf size: 21

```
DSolve[x^3*y''[x]-(x*y'[x]-y[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x \log \left(-\frac{c_2 x + c_1}{x}\right)$$

37.3 problem Ex 3

Internal problem ID [11334]
Internal file name [`OUTPUT/10319_Tuesday_December_27_2022_04_07_34_AM_59523079/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 61. Transformation of variables. Page 143
Problem number: Ex 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _reducible, _mu_xy]]`

Unable to solve or complete the solution.

$$\boxed{yy'' - y'^2 - y^2 \ln(y) + x^2 y^2 = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 22

```
dsolve(y(x)*diff(y(x),x$2)-diff(y(x),x)^2=y(x)^2*ln(y(x))-x^2*y(x)^2,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{x^2+2-\frac{c_2 \mathrm{e}^x}{2}+\frac{c_1 \mathrm{e}^{-x}}{2}}$$

✓ Solution by Mathematica

Time used: 1.156 (sec). Leaf size: 30

```
DSolve[y[x]*y''[x]-y'[x]^2==y[x]^2*Log[y[x]]-x^2*y[x]^2,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{x^2-\frac{c_1 e^x}{2}-c_2 e^{-x}+2}$$

37.4 problem Ex 4

Internal problem ID [11335]
Internal file name [`OUTPUT/10320_Tuesday_December_27_2022_04_07_35_AM_56107345/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 61. Transformation of variables. Page 143
Problem number: Ex 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\sin(x)^2 y'' - 2y = 0$$

37.4.1 Maple step by step solution

Let's solve
$\sin(x)^2 y'' - 2y = 0$

- Highest derivative means the order of the ODE is 2
  y''
- Isolate 2nd derivative
  $y'' = \frac{2y}{\sin(x)^2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $y'' - \frac{2y}{\sin(x)^2} = 0$
- Multiply by denominators of the ODE
  $\sin(x)^2 y'' - 2y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\sin(x)^2 \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2y(t) = 0$

- Simplify

  $\frac{\sin(x)^2\left(\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t)\right)}{x^2} - 2y(t) = 0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{2x^2y(t)}{\sin(x)^2} + \frac{d}{dt}y(t)$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - \frac{2x^2y(t)}{\sin(x)^2} - \frac{d}{dt}y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{2x^2}{\sin(x)^2} - r = 0$

- Factor the characteristic polynomial

  $\frac{r^2\sin(x)^2 - r\sin(x)^2 - 2x^2}{\sin(x)^2} = 0$

- Roots of the characteristic polynomial

  $r = \left(\frac{\frac{\sin(x)}{2} + \frac{\sqrt{\sin(x)^2+8x^2}}{2}}{\sin(x)}, \frac{\frac{\sin(x)}{2} - \frac{\sqrt{\sin(x)^2+8x^2}}{2}}{\sin(x)}\right)$

- 1st solution of the ODE

$$y_1(t) = e^{\frac{\left(\frac{\sin(x)}{2}+\frac{\sqrt{\sin(x)^2+8x^2}}{2}\right)t}{\sin(x)}}$$

- 2nd solution of the ODE

$$y_2(t) = e^{\frac{\left(\frac{\sin(x)}{2}-\frac{\sqrt{\sin(x)^2+8x^2}}{2}\right)t}{\sin(x)}}$$

- General solution of the ODE

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

- Substitute in solutions

$$y(t) = c_1 e^{\frac{\left(\frac{\sin(x)}{2}+\frac{\sqrt{\sin(x)^2+8x^2}}{2}\right)t}{\sin(x)}} + c_2 e^{\frac{\left(\frac{\sin(x)}{2}-\frac{\sqrt{\sin(x)^2+8x^2}}{2}\right)t}{\sin(x)}}$$

- Change variables back using $t = \ln(x)$

$$y = c_1 e^{\frac{\left(\frac{\sin(x)}{2}+\frac{\sqrt{\sin(x)^2+8x^2}}{2}\right)\ln(x)}{\sin(x)}} + c_2 e^{\frac{\left(\frac{\sin(x)}{2}-\frac{\sqrt{\sin(x)^2+8x^2}}{2}\right)\ln(x)}{\sin(x)}}$$

- Simplify

$$y = \sqrt{x}\left(c_1 x^{\frac{\csc(x)\sqrt{\sin(x)^2+8x^2}}{2}} + c_2 x^{-\frac{\csc(x)\sqrt{\sin(x)^2+8x^2}}{2}}\right)$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
   Change of variables used:
      [x = 1/2*arccos(t)]
   Linear ODE actually solved:
      -u(t)+(t^2-t)*diff(u(t),t)+(t^3-t^2-t+1)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 31

```
dsolve(sin(x)^2*diff(y(x),x$2)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = -i\cot(x)\ln(\cos(2x) + i\sin(2x))\,c_2 - 2c_2 + c_1\cot(x)$$

✓ Solution by Mathematica

Time used: 0.339 (sec). Leaf size: 46

```
DSolve[Sin[x]^2*y''[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\cos(x)\left(c_1 - c_2\log\left(\sqrt{-\sin^2(x)} - \cos(x)\right)\right)}{\sqrt{-\sin^2(x)}} - c_2$$

38 Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 62. Summary. Page 144

38.1 problem Ex 1

Internal problem ID [11336]
Internal file name [`OUTPUT/10322_Friday_January_27_2023_02_36_51_AM_75656588/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 62. Summary. Page 144
Problem number: Ex 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y'' - y'^2 = 1}$$

38.1.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - p(x)^2 - 1 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{p^2 + 1} dp = x + c_1$$
$$\arctan(p) = x + c_1$$

Solving for p gives these solutions

$$p_1 = \tan(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \tan(x + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \tan(x + c_1)\, \mathrm{d}x \\ &= \frac{\ln(1 + \tan(x + c_1)^2)}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(1 + \tan(x + c_1)^2)}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{\ln(1 + \tan(x + c_1)^2)}{2} + c_2$$

Verified OK.

38.1.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) - p(y)^2 = 1$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int \frac{p}{p^2+1}dp = \int dy$$
$$\frac{\ln(p^2+1)}{2} = y + c_1$$

Raising both side to exponential gives

$$\sqrt{p^2+1} = e^{y+c_1}$$

Which simplifies to

$$\sqrt{p^2+1} = c_2 e^{y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \mathrm{RootOf}\left(\_Z^2 - c_2^2 e^{2y} + 1\right)$$

Integrating both sides gives

$$\int \frac{1}{\mathrm{RootOf}\left(\_Z^2 - c_2^2 e^{2y} + 1\right)}dy = \int dx$$
$$\int^y \frac{1}{\mathrm{RootOf}\left(\_Z^2 - c_2^2 e^{2\_a} + 1\right)}d\_a = c_3 + x$$

Summary

The solution(s) found are the following

$$\int^y \frac{1}{\mathrm{RootOf}\left(\_Z^2 - c_2^2 e^{2\_a} + 1\right)}d\_a = c_3 + x \tag{1}$$

Verification of solutions

$$\int^y \frac{1}{\mathrm{RootOf}\left(\_Z^2 - c_2^2 e^{2\_a} + 1\right)}d\_a = c_3 + x$$

Verified OK.

38.1.3 Maple step by step solution

Let's solve

$y'' - y'^2 = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) - u(x)^2 = 1$

- Separate variables

  $\frac{u'(x)}{u(x)^2+1} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^2+1} dx = \int 1 dx + c_1$

- Evaluate integral

  $\arctan(u(x)) = x + c_1$

- Solve for $u(x)$

  $u(x) = \tan(x + c_1)$

- Solve 1st ODE for $u(x)$

  $u(x) = \tan(x + c_1)$

- Make substitution $u = y'$

  $y' = \tan(x + c_1)$

- Integrate both sides to solve for y

  $\int y' dx = \int \tan(x + c_1) dx + c_2$

- Compute integrals

  $y = \frac{\ln\left(1+\tan(x+c_1)^2\right)}{2} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)=diff(y(x),x)^2+1,y(x), singsol=all)
```

$$y(x) = -\ln\left(c_1 \sin\left(x\right) - c_2 \cos\left(x\right)\right)$$

✓ Solution by Mathematica

Time used: 3.079 (sec). Leaf size: 16

```
DSolve[y''[x]==y'[x]^2+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \log(\cos(x + c_1))$$

38.2 problem Ex 2

Internal problem ID [11337]
Internal file name [`OUTPUT/10323_Friday_January_27_2023_02_36_54_AM_64126668/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 62. Summary. Page 144
Problem number: Ex 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{\left(-x^{2}+1\right) y''-y'x=2}$$

38.2.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x)=y'$$

Then

$$p'(x)=y''$$

Hence the ode becomes

$$\left(-x^{2}+1\right) p'(x)-p(x)\, x-2=0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{x}{x^2-1}\\ q(x) &= -\frac{2}{x^2-1}\end{aligned}$$

Hence the ode is

$$p'(x) + \frac{xp(x)}{x^2-1} = -\frac{2}{x^2-1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{x}{x^2-1}dx}\\ &= \mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(1+x)}{2}}\end{aligned}$$

Which simplifies to

$$\mu = \sqrt{x-1}\,\sqrt{1+x}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(-\frac{2}{x^2-1}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x-1}\,\sqrt{1+x}\,p\right) &= \left(\sqrt{x-1}\,\sqrt{1+x}\right)\left(-\frac{2}{x^2-1}\right)\\ \mathrm{d}\left(\sqrt{x-1}\,\sqrt{1+x}\,p\right) &= \left(-\frac{2\sqrt{x-1}\,\sqrt{1+x}}{x^2-1}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\sqrt{x-1}\,\sqrt{1+x}\,p &= \int -\frac{2\sqrt{x-1}\,\sqrt{1+x}}{x^2-1}\,\mathrm{d}x\\ \sqrt{x-1}\,\sqrt{1+x}\,p &= -\frac{2\sqrt{x-1}\,\sqrt{1+x}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{x-1}\,\sqrt{1+x}$ results in

$$p(x) = -\frac{2\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}} + \frac{c_1}{\sqrt{x-1}\,\sqrt{1+x}}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{2\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}} + \frac{c_1}{\sqrt{x-1}\,\sqrt{1+x}}$$

Integrating both sides gives

$$\begin{aligned}
y &= \int \frac{-2\sqrt{x-1}\,\sqrt{1+x}\,\ln\left(x+\sqrt{x^2-1}\right)+c_1\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{1+x}}\,\mathrm{d}x \\
&= \frac{c_1\sqrt{(x-1)\,(1+x)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{1+x}} - \ln\left(x+\sqrt{x^2-1}\right)^2 + c_2
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1\sqrt{(x-1)\,(1+x)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{1+x}} - \ln\left(x+\sqrt{x^2-1}\right)^2 + c_2 \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1\sqrt{(x-1)\,(1+x)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{1+x}} - \ln\left(x+\sqrt{x^2-1}\right)^2 + c_2$$

Verified OK.

38.2.2 Solving using Kovacic algorithm

Writing the ode as

$$\left(-x^2+1\right)y'' - y'x = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= -x^2+1 \\
B &= -x \\
C &= 0
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 2}{4(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 2$$
$$t = 4(x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 2}{4(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 223: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16x - 16} - \frac{3}{16(x-1)^2} - \frac{3}{16(1+x)^2} - \frac{1}{16(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 2}{4(x^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^2-2}{4\left(x^2-1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4x-4} + \frac{1}{4+4x} + (-)(0) \\
&= \frac{1}{4x-4} + \frac{1}{4+4x} \\
&= \frac{x}{2x^2-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{4x-4} + \frac{1}{4+4x}\right)(0) + \left(\left(-\frac{1}{4(x-1)^2} - \frac{1}{4(1+x)^2}\right) + \left(\frac{1}{4x-4} + \frac{1}{4+4x}\right)^2 - \left(\frac{-x^2-2}{4(x^2-1)^2}\right.\right.
$$

$$
0 =
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{4x-4} + \frac{1}{4+4x}\right) dx} \\
&= (x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{-x^2+1} \, dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{4} - \frac{\ln(1+x)}{4}} \\
&= z_1 \left(\frac{1}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right) + c_2\left(\frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\left(\ln\left(x+\sqrt{x^2-1}\right)\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$\left(-x^2+1\right)y'' - y'x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} + \frac{c_2(x^2-1)^{\frac{1}{4}}\ln\left(x+\sqrt{x^2-1}\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}$$

$$y_2 = \frac{(x^2-1)^{\frac{1}{4}} \ln\left(x+\sqrt{x^2-1}\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} & \frac{(x^2-1)^{\frac{1}{4}} \ln\left(x+\sqrt{x^2-1}\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} \\ \frac{d}{dx}\left(\frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right) & \frac{d}{dx}\left(\frac{(x^2-1)^{\frac{1}{4}} \ln\left(x+\sqrt{x^2-1}\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} & \frac{(x^2-1)^{\frac{1}{4}} \ln\left(x+ \right.}{(x-1)^{\frac{1}{4}}(1+} \\ \frac{x}{2(x^2-1)^{\frac{3}{4}}(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1)^{\frac{1}{4}}}{4(x-1)^{\frac{5}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1)^{\frac{1}{4}}}{4(x-1)^{\frac{1}{4}}(1+x)^{\frac{5}{4}}} & \frac{\ln\left(x+\sqrt{x^2-1}\right)x}{2(x^2-1)^{\frac{3}{4}}(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1)^{\frac{1}{4}}\ln\left(x+\sqrt{x^2-1}\right)}{4(x-1)^{\frac{5}{4}}(1+x)^{\frac{1}{4}}} - (x \end{vmatrix}$$

Therefore

$$W = \left(\frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right)\left(\frac{\ln\left(x+\sqrt{x^2-1}\right)x}{2(x^2-1)^{\frac{3}{4}}(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1)^{\frac{1}{4}}\ln\left(x+\sqrt{x^2-1}\right)}{4(x-1)^{\frac{5}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1)^{\frac{1}{4}}\ln\left(x+\sqrt{x^2-1}\right)}{4(x-1)^{\frac{1}{4}}(1+x)^{\frac{5}{4}}} + \frac{(x^2-1)^{\frac{1}{4}}\left(1+\frac{x}{\sqrt{x^2-1}}\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}\left(x+\sqrt{x^2-1}\right)}\right) - \left(\frac{(x^2-1)^{\frac{1}{4}}\ln\left(x+\sqrt{x^2-1}\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right)\left(\frac{x}{2(x^2-1)^{\frac{3}{4}}(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1)^{\frac{1}{4}}}{4(x-1)^{\frac{5}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1)^{\frac{1}{4}}}{4(x-1)^{\frac{1}{4}}(1+x)^{\frac{5}{4}}}\right)$$

Which simplifies to

$$W = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$$

Which simplifies to

$$W = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2(x^2-1)^{\frac{1}{4}}\ln\left(x+\sqrt{x^2-1}\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}}{\frac{-x^2+1}{\sqrt{x-1}\,\sqrt{1+x}}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{2(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}\ln\left(x+\sqrt{x^2-1}\right)}{(x^2-1)^{\frac{3}{4}}}dx$$

Hence

$$u_1 = -\left(\int_0^x -\frac{2(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha^2-1)^{\frac{3}{4}}}d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}}{\frac{-x^2+1}{\sqrt{x-1}\,\sqrt{1+x}}}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{2(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}{(x^2-1)^{\frac{3}{4}}} dx$$

Hence

$$u_2 = \int_0^x -\frac{2(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}} d\alpha$$

Which simplifies to

$$u_1 = 2\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha^2-1)^{\frac{3}{4}}} d\alpha\right)$$

$$u_2 = -2\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}} d\alpha\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha^2-1)^{\frac{3}{4}}} d\alpha\right)(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} - \frac{2\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}} d\alpha\right)(x^2-1)^{\frac{1}{4}}\ln\left(x+\sqrt{x^2-1}\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}$$

Which simplifies to

$$y_p(x) = -\frac{2(x^2-1)^{\frac{1}{4}}\left(\ln\left(x+\sqrt{x^2-1}\right)\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}} d\alpha\right) - \left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha^2-1)^{\frac{3}{4}}} d\alpha\right)\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(\frac{c_1(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} + \frac{c_2(x^2-1)^{\frac{1}{4}}\ln\left(x+\sqrt{x^2-1}\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right)$$
$$+ \left(-\frac{2(x^2-1)^{\frac{1}{4}}\left(\ln\left(x+\sqrt{x^2-1}\right)\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}}d\alpha\right) - \left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha^2-1)^{\frac{3}{4}}}d\alpha\right)\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right)$$

Which simplifies to

$$y = \frac{(x^2-1)^{\frac{1}{4}}\left(\ln\left(x+\sqrt{x^2-1}\right)c_2 + c_1\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}$$
$$-\frac{2(x^2-1)^{\frac{1}{4}}\left(\ln\left(x+\sqrt{x^2-1}\right)\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}}d\alpha\right) - \left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha^2-1)^{\frac{3}{4}}}d\alpha\right)\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}$$

Summary

The solution(s) found are the following

$$y = \frac{(x^2-1)^{\frac{1}{4}}\left(\ln\left(x+\sqrt{x^2-1}\right)c_2 + c_1\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} \tag{1}$$
$$-\frac{2(x^2-1)^{\frac{1}{4}}\left(\ln\left(x+\sqrt{x^2-1}\right)\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}}d\alpha\right) - \left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha^2-1)^{\frac{3}{4}}}d\alpha\right)\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}$$

Verification of solutions

$$y = \frac{(x^2-1)^{\frac{1}{4}}\left(\ln\left(x+\sqrt{x^2-1}\right)c_2 + c_1\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}$$
$$-\frac{2(x^2-1)^{\frac{1}{4}}\left(\ln\left(x+\sqrt{x^2-1}\right)\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}}d\alpha\right) - \left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha^2-1)^{\frac{3}{4}}}d\alpha\right)\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}$$

Verified OK.

38.2.3 Maple step by step solution

Let's solve

$(-x^2+1)\,y''-y'x=2$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $(-x^2+1)\,u'(x)-u(x)\,x=2$

- Isolate the derivative

  $u'(x)=-\frac{xu(x)}{x^2-1}-\frac{2}{x^2-1}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x)+\frac{xu(x)}{x^2-1}=-\frac{2}{x^2-1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x)+\frac{xu(x)}{x^2-1}\right)=-\frac{2\mu(x)}{x^2-1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x)+\frac{xu(x)}{x^2-1}\right)=\mu'(x)\,u(x)+\mu(x)\,u'(x)$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)x}{x^2-1}$

- Solve to find the integrating factor

  $\mu(x)=\sqrt{x-1}\,\sqrt{1+x}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,u(x))\right)dx=\int-\frac{2\mu(x)}{x^2-1}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,u(x)=\int-\frac{2\mu(x)}{x^2-1}dx+c_1$

- Solve for $u(x)$

  $u(x)=\frac{\int-\frac{2\mu(x)}{x^2-1}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\sqrt{x-1}\,\sqrt{1+x}$

  $u(x)=\frac{\int-\frac{2\sqrt{x-1}\,\sqrt{1+x}}{x^2-1}dx+c_1}{\sqrt{x-1}\,\sqrt{1+x}}$

- Evaluate the integrals on the rhs

$$u(x) = \frac{-\frac{2\sqrt{x-1}\,\sqrt{1+x}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}}+c_1}{\sqrt{x-1}\,\sqrt{1+x}}$$

- Simplify

$$u(x) = \frac{-2\sqrt{x-1}\,\sqrt{1+x}\,\ln\left(x+\sqrt{x^2-1}\right)+c_1\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{1+x}}$$

- Solve 1st ODE for $u(x)$

$$u(x) = \frac{-2\sqrt{x-1}\,\sqrt{1+x}\,\ln\left(x+\sqrt{x^2-1}\right)+c_1\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{1+x}}$$

- Make substitution $u = y'$

$$y' = \frac{-2\sqrt{x-1}\,\sqrt{1+x}\,\ln\left(x+\sqrt{x^2-1}\right)+c_1\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{1+x}}$$

- Integrate both sides to solve for y

$$\int y'dx = \int \frac{-2\sqrt{x-1}\,\sqrt{1+x}\,\ln\left(x+\sqrt{x^2-1}\right)+c_1\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{1+x}}dx + c_2$$

- Compute integrals

$$y = \frac{c_1\sqrt{(x-1)(1+x)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{1+x}} - \ln\left(x+\sqrt{x^2-1}\right)^2 + c_2$$

✓ Solution by Maple

Time used: 0.235 (sec). Leaf size: 59

```
dsolve((1-x^2)*diff(y(x),x$2)-x*diff(y(x),x)=2,y(x), singsol=all)
```

$$y(x) = -\left(\int -\frac{-2\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\sqrt{-1+x}\,\sqrt{1+x}+c_1(x^2-1)}{(-1+x)^{\frac{3}{2}}\,(1+x)^{\frac{3}{2}}}dx\right) + c_2$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 48

```
DSolve[(1-x^2)*y''[x]-x*y'[x]==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{1}{4}\left(\log\left(1-\frac{x}{\sqrt{x^2-1}}\right) - \log\left(\frac{x}{\sqrt{x^2-1}}+1\right) + c_1\right)^2$$

38.3 problem Ex 3

Internal problem ID [11338]
Internal file name [`OUTPUT/10324_Friday_January_27_2023_02_36_57_AM_26690807/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 62. Summary. Page 144
Problem number: Ex 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
  _Lagerstrom, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

$$\boxed{y'' + yy' = 0}$$

38.3.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + yy')\, dx = 0$$
$$\frac{y^2}{2} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2}+c_1}dy = x + c_2$$
$$\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(x+c_2)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Summary
The solution(s) found are the following

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x+c_2)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Verification of solutions

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x+c_2)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Verified OK.

38.3.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + yp(y) = 0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned} p(y) &= \int -y \,\mathrm{d}y \\ &= -\frac{y^2}{2} + c_1 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{y^2}{2} + c_1$$

Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2} + c_1} dy = x + c_2$$

$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Summary

The solution(s) found are the following

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Verification of solutions

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Verified OK.

38.3.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + yy' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + yy')\, dx = 0$$
$$\frac{y^2}{2} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2} + c_1} dy = x + c_2$$
$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Summary

The solution(s) found are the following

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Verification of solutions

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Verified OK.

38.3.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x,y,y')\,y''+a_1(x,y,y')\,y'+a_0(x,y,y')=0$$

Where the following conditions are satisfied

$$\begin{aligned}\frac{\partial a_2}{\partial y}&=\frac{\partial a_1}{\partial y'}\\ \frac{\partial a_2}{\partial x}&=\frac{\partial a_0}{\partial y'}\\ \frac{\partial a_1}{\partial x}&=\frac{\partial a_0}{\partial y}\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}a_2&=1\\ a_1&=y\\ a_0&=0\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}\int a_2\,dy'+\int a_1\,dy+\int a_0\,dx&=c_1\\ \int 1\,dy'+\int y\,dy+\int 0\,dx&=c_1\end{aligned}$$

Which results in

$$\frac{y^2}{2}+y'=c_1$$

Which is now solved Integrating both sides gives

$$\begin{aligned}\int \frac{1}{-\frac{y^2}{2}+c_1}dy&=x+c_2\\ \frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}}&=x+c_2\end{aligned}$$

Solving for y gives these solutions

$$y_1=\tanh\left(\frac{\sqrt{c_1}\,(x+c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Summary

The solution(s) found are the following

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x+c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Verification of solutions

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x+c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Verified OK.

38.3.5 Maple step by step solution

Let's solve

$y'' + yy' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $u(y)\left(\frac{d}{dy}u(y)\right) + yu(y) = 0$

- Separate variables

  $\frac{d}{dy}u(y) = -y$

- Integrate both sides with respect to y

  $\int \left(\frac{d}{dy}u(y)\right) dy = \int -y dy + c_1$

- Evaluate integral

  $u(y) = -\frac{y^2}{2} + c_1$
- Solve for $u(y)$

  $u(y) = -\frac{y^2}{2} + c_1$
- Solve 1st ODE for $u(y)$

  $u(y) = -\frac{y^2}{2} + c_1$
- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = -\frac{y^2}{2} + c_1$
- Separate variables

  $\frac{y'}{-\frac{y^2}{2}+c_1} = 1$
- Integrate both sides with respect to x

  $\int \frac{y'}{-\frac{y^2}{2}+c_1} dx = \int 1 dx + c_2$
- Evaluate integral

  $\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$
- Solve for y

  $y = \tanh\left(\frac{\sqrt{c_1}\,(x+c_2)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)*_a = 0, _b(_a), HINT = [[
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 2*_b]
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\tanh\left(\frac{(c_2+x)\sqrt{2}}{2c_1}\right)\sqrt{2}}{c_1}$$

✓ Solution by Mathematica

Time used: 20.03 (sec). Leaf size: 34

```
DSolve[y''[x]+y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{2}\sqrt{c_1}\tanh\left(\frac{\sqrt{c_1}(x+c_2)}{\sqrt{2}}\right)$$

38.4 problem Ex 4

Internal problem ID [11339]
Internal file name [`OUTPUT/10325_Friday_January_27_2023_02_36_58_AM_31623902/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 62. Summary. Page 144
Problem number: Ex 4.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_3rd_order, _fully, _exact, _linear]]`

Unable to solve or complete the solution.

$$\boxed{(x^3+1)\,y''' + 9y''x^2 + 18y'x + 6y = 0}$$

Unable to solve this ODE.

38.4.1 Maple step by step solution

Let's solve

$(x^3+1)\,y''' + 9y''x^2 + 18y'x + 6y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{9x^2}{x^3+1}, P_3(x) = \frac{18x}{x^3+1}, P_4(x) = \frac{6}{x^3+1}\right]$

  ◦ $\left(x - \frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right) \cdot P_2(x)$ is analytic at $x = \frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}$

$$\left(\left(x-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{1}{2}-\frac{I\sqrt{3}}{2}}=0$$

- $\left(x-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{1}{2}-\frac{I\sqrt{3}}{2}$

  $$\left(\left(x-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{1}{2}-\frac{I\sqrt{3}}{2}}=0$$

- $\left(x-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)^3\cdot P_4(x)$ is analytic at $x=\frac{1}{2}-\frac{I\sqrt{3}}{2}$

  $$\left(\left(x-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)^3\cdot P_4(x)\right)\bigg|_{x=\frac{1}{2}-\frac{I\sqrt{3}}{2}}=0$$

- $x=\frac{1}{2}-\frac{I\sqrt{3}}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=\frac{1}{2}-\frac{I\sqrt{3}}{2}$

- Change variables using $x=u+\frac{1}{2}-\frac{I\sqrt{3}}{2}$ so that the regular singular point is at $u=0$

  $$\left(u^3+\frac{3u^2}{2}-\frac{3\,Iu^2\sqrt{3}}{2}-\frac{3u}{2}-\frac{3\,Iu\sqrt{3}}{2}\right)\left(\frac{d^3}{du^3}y(u)\right)+\left(9u^2+9u-9\,Iu\sqrt{3}-\frac{9}{2}-\frac{9\,I\sqrt{3}}{2}\right)\left(\frac{d^2}{du^2}y(u)\right)+(1$$

- Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=0..2$

  $$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^3}{du^3}y(u)\right)$ to series expansion for $m=1..3$

$$u^m \cdot \left(\frac{d^3}{du^3}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)(k+r-2)\,u^{k+r-3+m}$$

- Shift index using $k- >k+3-m$

$$u^m \cdot \left(\frac{d^3}{du^3}y(u)\right) = \sum_{k=-3+m}^{\infty} a_{k+3-m}(k+3-m+r)(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-\frac{3\left(1+\mathrm{I}\sqrt{3}\right)(r-1)(r+1)r a_0 u^{-2+r}}{2} + \left(-\frac{3\left(1+\mathrm{I}\sqrt{3}\right)r(r+2)(r+1)a_1}{2} - \frac{3\left(\mathrm{I}\sqrt{3}-1\right)r(r+1)(r+2)a_0}{2}\right)u^{r-1} + \left(\sum_{k=0}^{\infty}\left(-\frac{3(1\cdot\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-\frac{3\left(1+\mathrm{I}\sqrt{3}\right)(r-1)(r+1)r}{2} = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0, 1\}$$

- Each term in the series must be 0, giving the recursion relation

$$\frac{3(r+1+k)(r+3+k)(r+2+k)\left(-\mathrm{I}(a_{k+1}+a_{k+2})\sqrt{3}+\frac{2a_k}{3}+a_{k+1}-a_{k+2}\right)}{2} = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{3\,\mathrm{I}\sqrt{3}\,a_{k+1}-2a_k-3a_{k+1}}{3\left(1+\mathrm{I}\sqrt{3}\right)}$$

- Recursion relation for $r=-1$

$$a_{k+2} = -\frac{3\,\mathrm{I}\sqrt{3}\,a_{k+1}-2a_k-3a_{k+1}}{3\left(1+\mathrm{I}\sqrt{3}\right)}$$

- Solution for $r=-1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{3\,\mathrm{I}\sqrt{3}\,a_{k+1}-2a_k-3a_{k+1}}{3\left(1+\mathrm{I}\sqrt{3}\right)}, 0=0\right]$$

- Revert the change of variables $u = x-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k\left(x-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^{k-1}, a_{k+2} = -\frac{3\,\mathrm{I}\sqrt{3}\,a_{k+1}-2a_k-3a_{k+1}}{3\left(1+\mathrm{I}\sqrt{3}\right)}, 0=0\right]$$

- Recursion relation for $r=0$

$$a_{k+2} = -\frac{3\,\mathrm{I}\sqrt{3}\,a_{k+1}-2a_k-3a_{k+1}}{3\left(1+\mathrm{I}\sqrt{3}\right)}$$

- Solution for $r=0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k}, a_{k+2} = -\frac{3\,\mathrm{I}\sqrt{3}\,a_{k+1}-2a_k-3a_{k+1}}{3\left(1+\mathrm{I}\sqrt{3}\right)}, 0=0\right]$$

- Revert the change of variables $u = x - \frac{1}{2} + \frac{I\sqrt{3}}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)^k, a_{k+2} = -\frac{3 I\sqrt{3}\, a_{k+1} - 2a_k - 3a_{k+1}}{3\left(1 + I\sqrt{3}\right)}, 0 = 0\right]$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{3 I\sqrt{3}\, a_{k+1} - 2a_k - 3a_{k+1}}{3\left(1 + I\sqrt{3}\right)}$$

- Solution for $r = 1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = -\frac{3 I\sqrt{3}\, a_{k+1} - 2a_k - 3a_{k+1}}{3\left(1 + I\sqrt{3}\right)}, -9\left(1 + I\sqrt{3}\right) a_1 - 9\left(I\sqrt{3} - 1\right) a_0 = 0\right]$$

- Revert the change of variables $u = x - \frac{1}{2} + \frac{I\sqrt{3}}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)^{k+1}, a_{k+2} = -\frac{3 I\sqrt{3}\, a_{k+1} - 2a_k - 3a_{k+1}}{3\left(1 + I\sqrt{3}\right)}, -9\left(1 + I\sqrt{3}\right) a_1 - 9\left(I\sqrt{3} - 1\right) a_0 = 0\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x - \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x - \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)^{k}\right) + \left(\sum_{k=0}^{\infty} c_k \left(x - \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)^{1+k}\right), a_{k+2}\right.$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve((1+x^3)*diff(y(x),x$3)+9*x^2*diff(y(x),x$2)+18*x*diff(y(x),x)+6*y(x)=0,y(x), singsol=
```

$$y(x) = \frac{c_1 x^2 + c_2 x + c_3}{(1 + x)(x^2 - x + 1)}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 31

```
DSolve[(1+x^3)*y'''[x]+9*x^2*y''[x]+18*x*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{c_3x^2 + 2c_2x + 2c_1}{2x^3 + 2}$$

38.5 problem Ex 5

Internal problem ID [11340]
Internal file name [`OUTPUT/10326_Friday_January_27_2023_02_36_59_AM_16834199/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 62. Summary. Page 144
Problem number: Ex 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2 - x)\, y'' + (4x + 2)\, y' + 2y = 0}$$

38.5.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(x^2 - x\right) y'' + (4x + 2)\, y' + 2y\right) dx = 0$$
$$(3 + 2x)\, y + \left(x^2 - x\right) y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{-2x-3}{x\left(x-1\right)} \\
q(x) &= \frac{c_1}{x\left(x-1\right)}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{\left(-2x-3\right)y}{x\left(x-1\right)} = \frac{c_1}{x\left(x-1\right)}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{-2x-3}{x(x-1)}dx} \\
&= \mathrm{e}^{5\ln(x-1)-3\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{\left(x-1\right)^5}{x^3}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x\left(x-1\right)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\left(x-1\right)^5 y}{x^3}\right) &= \left(\frac{\left(x-1\right)^5}{x^3}\right)\left(\frac{c_1}{x\left(x-1\right)}\right) \\
\mathrm{d}\left(\frac{\left(x-1\right)^5 y}{x^3}\right) &= \left(\frac{\left(x-1\right)^4 c_1}{x^4}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\left(x-1\right)^5 y}{x^3} &= \int \frac{\left(x-1\right)^4 c_1}{x^4}\,\mathrm{d}x \\
\frac{\left(x-1\right)^5 y}{x^3} &= c_1\left(x + \frac{2}{x^2} - \frac{1}{3x^3} - \frac{6}{x} - 4\ln\left(x\right)\right) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{(x-1)^5}{x^3}$ results in

$$
y = \frac{x^3 c_1\left(x + \frac{2}{x^2} - \frac{1}{3x^3} - \frac{6}{x} - 4\ln\left(x\right)\right)}{\left(x-1\right)^5} + \frac{c_2 x^3}{\left(x-1\right)^5}
$$

which simplifies to

$$y = \frac{-12 \ln(x) c_1 x^3 + (3x^4 - 18x^2 + 6x - 1) c_1 + 3c_2 x^3}{3(x-1)^5}$$

Summary

The solution(s) found are the following

$$y = \frac{-12 \ln(x) c_1 x^3 + (3x^4 - 18x^2 + 6x - 1) c_1 + 3c_2 x^3}{3(x-1)^5} \tag{1}$$

Verification of solutions

$$y = \frac{-12 \ln(x) c_1 x^3 + (3x^4 - 18x^2 + 6x - 1) c_1 + 3c_2 x^3}{3(x-1)^5}$$

Verified OK.

38.5.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(x^2 - x) y'' + (4x + 2) y' + 2y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left((x^2 - x) y'' + (4x + 2) y' + 2y\right) dx = 0$$
$$(3 + 2x) y + (x^2 - x) y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-2x - 3}{x(x-1)}$$
$$q(x) = \frac{c_1}{x(x-1)}$$

Hence the ode is

$$y' - \frac{(-2x - 3) y}{x(x-1)} = \frac{c_1}{x(x-1)}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-2x-3}{x(x-1)}dx}$$
$$= e^{5\ln(x-1)-3\ln(x)}$$

Which simplifies to

$$\mu = \frac{(x-1)^5}{x^3}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x(x-1)}\right)$$
$$\frac{d}{dx}\left(\frac{(x-1)^5 y}{x^3}\right) = \left(\frac{(x-1)^5}{x^3}\right)\left(\frac{c_1}{x(x-1)}\right)$$
$$d\left(\frac{(x-1)^5 y}{x^3}\right) = \left(\frac{(x-1)^4 c_1}{x^4}\right) dx$$

Integrating gives

$$\frac{(x-1)^5 y}{x^3} = \int \frac{(x-1)^4 c_1}{x^4} dx$$
$$\frac{(x-1)^5 y}{x^3} = c_1\left(x + \frac{2}{x^2} - \frac{1}{3x^3} - \frac{6}{x} - 4\ln(x)\right) + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{(x-1)^5}{x^3}$ results in

$$y = \frac{x^3 c_1\left(x + \frac{2}{x^2} - \frac{1}{3x^3} - \frac{6}{x} - 4\ln(x)\right)}{(x-1)^5} + \frac{c_2 x^3}{(x-1)^5}$$

which simplifies to

$$y = \frac{-12\ln(x)c_1x^3 + (3x^4 - 18x^2 + 6x - 1)c_1 + 3c_2x^3}{3(x-1)^5}$$

Summary

The solution(s) found are the following

$$y = \frac{-12\ln(x)c_1x^3 + (3x^4 - 18x^2 + 6x - 1)c_1 + 3c_2x^3}{3(x-1)^5} \tag{1}$$

Verification of solutions

$$y = \frac{-12\ln(x)c_1x^3 + (3x^4 - 18x^2 + 6x - 1)c_1 + 3c_2x^3}{3(x-1)^5}$$

Verified OK.

38.5.3 Solving using Kovacic algorithm

Writing the ode as

$$(x^2 - x) y'' + (4x + 2) y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 - x \\ B &= 4x + 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x + 2}{(x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x + 2 \\ t &= (x^2 - x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x + 2}{(x^2 - x)^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 227: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} - \frac{8}{x - 1} + \frac{6}{(x - 1)^2} + \frac{8}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x+2}{(x^2-x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1
1	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\ &= 0 - (0) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{2}{x} - \frac{2}{x-1} + (0) \\ &= \frac{2}{x} - \frac{2}{x-1} \\ &= -\frac{2}{x(x-1)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{2}{x} - \frac{2}{x-1}\right)(0) + \left(\left(-\frac{2}{x^2} + \frac{2}{(x-1)^2}\right) + \left(\frac{2}{x} - \frac{2}{x-1}\right)^2 - \left(\frac{4x+2}{(x^2-x)^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{2}{x} - \frac{2}{x-1}\right) dx} \\ &= \frac{x^2}{(x-1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x+2}{x^2-x}\,dx} \\
&= z_1 e^{-3\ln(x-1)+\ln(x)} \\
&= z_1 \left(\frac{x}{(x-1)^3}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^3}{(x-1)^5}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x+2}{x^2-x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-6\ln(x-1)+2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(x + \frac{2}{x^2} - \frac{1}{3x^3} - \frac{6}{x} - 4\ln(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^3}{(x-1)^5}\right) + c_2 \left(\frac{x^3}{(x-1)^5}\left(x + \frac{2}{x^2} - \frac{1}{3x^3} - \frac{6}{x} - 4\ln(x)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^3}{(x-1)^5} + \frac{c_2(-12\ln(x)\,x^3 + 3x^4 - 18x^2 + 6x - 1)}{3\,(x-1)^5} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 x^3}{(x-1)^5} + \frac{c_2(-12\ln(x)\,x^3 + 3x^4 - 18x^2 + 6x - 1)}{3(x-1)^5}$$

Verified OK.

38.5.4 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 - x \\ q(x) &= 4x + 2 \\ r(x) &= 2 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 4 \end{aligned}$$

Therefore (1) becomes

$$2 - (4) + (2) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$(3 + 2x)\,y + (x^2 - x)\,y' = c_1$$

We now have a first order ode to solve which is

$$(3+2x)\,y+\left(x^2-x\right)y'=c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{-2x-3}{x\,(x-1)}\\
q(x) &= \frac{c_1}{x\,(x-1)}
\end{aligned}$$

Hence the ode is

$$y'-\frac{(-2x-3)\,y}{x\,(x-1)}=\frac{c_1}{x\,(x-1)}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{-2x-3}{x(x-1)}dx}\\
&= \mathrm{e}^{5\ln(x-1)-3\ln(x)}
\end{aligned}$$

Which simplifies to

$$\mu=\frac{(x-1)^5}{x^3}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x\,(x-1)}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(x-1)^5\,y}{x^3}\right) &= \left(\frac{(x-1)^5}{x^3}\right)\left(\frac{c_1}{x\,(x-1)}\right)\\
\mathrm{d}\left(\frac{(x-1)^5\,y}{x^3}\right) &= \left(\frac{(x-1)^4\,c_1}{x^4}\right)\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{(x-1)^5\,y}{x^3} &= \int \frac{(x-1)^4\,c_1}{x^4}\,\mathrm{d}x\\
\frac{(x-1)^5\,y}{x^3} &= c_1\left(x+\frac{2}{x^2}-\frac{1}{3x^3}-\frac{6}{x}-4\ln(x)\right)+c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{(x-1)^5}{x^3}$ results in

$$y = \frac{x^3 c_1 \left(x + \frac{2}{x^2} - \frac{1}{3x^3} - \frac{6}{x} - 4\ln\left(x\right)\right)}{\left(x-1\right)^5} + \frac{c_2 x^3}{\left(x-1\right)^5}$$

which simplifies to

$$y = \frac{-12\ln\left(x\right) c_1 x^3 + \left(3x^4 - 18x^2 + 6x - 1\right) c_1 + 3c_2 x^3}{3\left(x-1\right)^5}$$

Summary

The solution(s) found are the following

$$y = \frac{-12\ln\left(x\right) c_1 x^3 + \left(3x^4 - 18x^2 + 6x - 1\right) c_1 + 3c_2 x^3}{3\left(x-1\right)^5} \tag{1}$$

Verification of solutions

$$y = \frac{-12\ln\left(x\right) c_1 x^3 + \left(3x^4 - 18x^2 + 6x - 1\right) c_1 + 3c_2 x^3}{3\left(x-1\right)^5}$$

Verified OK.

38.5.5 Maple step by step solution

Let's solve

$(x^2 - x) y'' + (4x + 2) y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x(x-1)} - \frac{2(2x+1)y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(2x+1)y'}{x(x-1)} + \frac{2y}{x(x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2(2x+1)}{x(x-1)}, P_3(x) = \frac{2}{x(x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -2$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$y''x(x-1) + (4x+2)\, y' + 2y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-3+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+r+1)\,(k-2+r) + a_k(k+r+2)\,(k+r+1)\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+1)\left((-k-r+2)\, a_{k+1} + a_k(k+r+2)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+2)}{k-2+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+2)}{k-2}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k(k+2)}{k-2}$

- Recursion relation for $r = 3$

  $a_{k+1} = \frac{a_k(k+5)}{k+1}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{a_k(k+5)}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve((x^2-x)*diff(y(x),x$2)+(4*x+2)*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{12\ln(x)\, c_1 x^3 + (-3x^4 + 18x^2 - 6x + 1)\, c_1 + c_2 x^3}{(-1+x)^5}$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 52

```
DSolve[(x^2-x)*y''[x]+(4*x+2)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-3c_2x^4 - 3c_1x^3 + 12c_2x^3\log(x) + 18c_2x^2 - 6c_2x + c_2}{3(x-1)^5}$$

38.6 problem Ex 6

Internal problem ID [11341]
Internal file name [`OUTPUT/10327_Friday_January_27_2023_02_37_01_AM_40293410/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 62. Summary. Page 144
Problem number: Ex 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y(1-\ln(y))\,y''+(1+\ln(y))\,{y'}^2=0}$$

38.6.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y'=p(y)$$

Then

$$\begin{aligned}y''&=\frac{dp}{dx}\\&=\frac{dy}{dx}\frac{dp}{dy}\\&=p\frac{dp}{dy}\end{aligned}$$

Hence the ode becomes

$$(-\ln(y)\,y+y)\,p(y)\left(\frac{d}{dy}p(y)\right)+(\ln(y)\,p(y)+p(y))\,p(y)=0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p)\\
&= f(y)g(p)\\
&= \frac{p(1+\ln(y))}{y(-1+\ln(y))}
\end{aligned}$$

Where $f(y)=\frac{1+\ln(y)}{y(-1+\ln(y))}$ and $g(p)=p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= \frac{1+\ln(y)}{y(-1+\ln(y))}\,dy\\
\int\frac{1}{p}\,dp &= \int\frac{1+\ln(y)}{y(-1+\ln(y))}\,dy\\
\ln(p) &= \ln(y)+2\ln(-1+\ln(y))+c_1\\
p &= e^{\ln(y)+2\ln(-1+\ln(y))+c_1}\\
&= c_1 e^{\ln(y)+2\ln(-1+\ln(y))}
\end{aligned}$$

Which simplifies to

$$p(y)=c_1\left(y-2\ln(y)\,y+y\ln(y)^2\right)$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$y'=c_1\left(y-2\ln(y)\,y+y\ln(y)^2\right)$$

Integrating both sides gives

$$\begin{aligned}
\int\frac{1}{c_1 y\left(1-2\ln(y)+\ln(y)^2\right)}dy &= \int dx\\
-\frac{1}{c_1(-1+\ln(y))} &= x+c_2
\end{aligned}$$

Raising both side to exponential gives

$$e^{-\frac{1}{c_1(-1+\ln(y))}}=e^{x+c_2}$$

Which simplifies to

$$e^{-\frac{1}{c_1(-1+\ln(y))}}=c_3e^x$$

Summary
The solution(s) found are the following

$$y = e^{\frac{c_1 \ln(c_3)+c_1 x-1}{(\ln(c_3)+x)c_1}} \tag{1}$$

Verification of solutions

$$y = e^{\frac{c_1 \ln(c_3)+c_1 x-1}{(\ln(c_3)+x)c_1}}$$

Verified OK.

38.6.2 Maple step by step solution

Let's solve

$(-\ln(y)\, y + y)\, y'' + (\ln(y)\, y' + y')\, y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $(-\ln(y)\, y + y)\, u(y)\left(\frac{d}{dy}u(y)\right) + (\ln(y)\, u(y) + u(y))\, u(y) = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)} = \frac{1+\ln(y)}{y(-1+\ln(y))}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)} dy = \int \frac{1+\ln(y)}{y(-1+\ln(y))} dy + c_1$

- Evaluate integral

$$\ln (u(y)) = \ln (y) + 2 \ln (-1 + \ln (y)) + c_1$$

- Solve for $u(y)$

$$u(y) = e^{c_1} y(-1 + \ln (y))^2$$

- Solve 1st ODE for $u(y)$

$$u(y) = e^{c_1} y(-1 + \ln (y))^2$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = e^{c_1} y(\ln (y) - 1)^2$$

- Separate variables

$$\frac{y'}{y(\ln(y)-1)^2} = e^{c_1}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{y(\ln(y)-1)^2} dx = \int e^{c_1} dx + c_2$$

- Evaluate integral

$$-\frac{1}{\ln(y)-1} = e^{c_1} x + c_2$$

- Solve for y

$$y = e^{\frac{e^{c_1} x + c_2 - 1}{e^{c_1} x + c_2}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(y(x)*(1-ln(y(x)))*diff(y(x),x$2)+(1+ln(y(x)))*diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = e^{\frac{c_1 x + c_2 - 1}{c_1 x + c_2}}$$

✓ Solution by Mathematica

Time used: 1.021 (sec). Leaf size: 34

```
DSolve[y[x]*(1-Log[y[x]])*y''[x]+(1+Log[y[x]])*y'[x]^2==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to e^{\frac{c_1x-1+c_2c_1}{c_1(x+c_2)}}$$
$$y(x) \to e$$

38.7 problem Ex 7

Internal problem ID [11342]
Internal file name [`OUTPUT/10328_Friday_January_27_2023_02_37_03_AM_6094788/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 62. Summary. Page 144
Problem number: Ex 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + \frac{y'}{x} = 0}$$

38.7.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' + y')\, dx = 0$$
$$y'x = c_1$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_1}{x}\, \mathrm{d}x \\ &= c_1 \ln (x) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \ln (x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln (x) + c_2$$

Verified OK.

38.7.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)\, x + p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x, p) \\ &= f(x) g(p) \\ &= -\frac{p}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= -\frac{1}{x}\,dx \\
\int \frac{1}{p}\,dp &= \int -\frac{1}{x}\,dx \\
\ln(p) &= -\ln(x) + c_1 \\
p &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c_1}{x}$$

Integrating both sides gives

$$\begin{aligned}
y &= \int \frac{c_1}{x}\,\mathrm{d}x \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x) + c_2$$

Verified OK.

38.7.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0
\end{aligned} \tag{1}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x \\
B &= 1 \\
C &= 0 \\
F &= 0
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (x)(0) + (1)(0) + (0)(1) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
xv'' + (1) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
xu'(x) + u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{u}{x}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= \frac{c_1}{x}
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{c_1}{x}\,\mathrm{d}x \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y(x) &= Bv \\
&= (1)\,(c_1 \ln(x) + c_2) \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x) + c_2$$

Verified OK.

38.7.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' + y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' + y')\, dx = 0$$
$$y'x = c_1$$

Which is now solved for y. Integrating both sides gives

$$y = \int \frac{c_1}{x}\, \mathrm{d}x$$
$$= c_1 \ln (x) + c_2$$

Summary
The solution(s) found are the following

$$y = c_1 \ln (x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln (x) + c_2$$

Verified OK.

38.7.5 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 230: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{1}{x} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(\ln(x))) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \ln(x)\, c_2 \tag{1}$$

Verification of solutions

$$y = c_1 + \ln(x)\, c_2$$

Verified OK.

38.7.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx
$$

Substituting the above values for p, q, r, s gives

$$
y'x = c_1
$$

We now have a first order ode to solve which is

$$
y'x = c_1
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{c_1}{x}\, \mathrm{d}x \\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \ln(x) + c_2 \tag{1}
$$

Verification of solutions

$$
y = c_1 \ln(x) + c_2
$$

Verified OK.

38.7.7 Maple step by step solution

Let's solve

$y''x + y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} = 0$

- Multiply by denominators of the ODE

  $y''x + y' = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  ◦ Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x + \frac{\frac{d}{dt}y(t)}{x} = 0$

- Simplify

  $\frac{\frac{d^2}{dt^2}y(t)}{x} = 0$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

  $y_1(t) = 1$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_2 t + c_1$

- Change variables back using $t = \ln(x)$

  $y = c_1 + \ln(x) c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x$2)+1/x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 \ln(x)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 13

```
DSolve[y''[x]+1/x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \log(x) + c_2$$

38.8 problem Ex 8

Internal problem ID [11343]
Internal file name [`OUTPUT/10329_Friday_January_27_2023_02_37_04_AM_83539008/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 62. Summary. Page 144
Problem number: Ex 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _nonlinear], [_2nd_order, _reducible,
  _mu_xy]]
```

$$\boxed{x(x+2y)\,y''+2xy'^2+4(y+x)\,y'+2y=-x^2}$$

38.8.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(2yx+x^2\right)y''+\left(\left(2y'+4\right)x+4y\right)y'+2y\right)dx=\int -x^2dx$$

$$2yx+y^2+\left(2yx+x^2\right)y'=-\frac{x^3}{3}+c_1$$

Which is now solved for y.

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x^2 + 2xy\right)\mathrm{d}y = \left(-2xy - y^2 - \frac{1}{3}x^3 + c_1\right)\mathrm{d}x$$

$$\left(2xy + y^2 + \frac{1}{3}x^3 - c_1\right)\mathrm{d}x + \left(x^2 + 2xy\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2xy + y^2 + \frac{1}{3}x^3 - c_1\\ N(x, y) &= x^2 + 2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2xy + y^2 + \frac{1}{3}x^3 - c_1\right) \\
&= 2y + 2x
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + 2xy) \\
&= 2y + 2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int 2xy + y^2 + \frac{1}{3}x^3 - c_1 \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x(x^3 + 12xy + 12y^2 - 12c_1)}{12} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x(12x + 24y)}{12} + f'(y) \tag{4}
$$

$$
= x(x + 2y) + f'(y)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + 2xy$. Therefore equation (4) becomes

$$
x^2 + 2xy = x(x + 2y) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_2$$

Where c_2 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x^3 + 12xy + 12y^2 - 12c_1)}{12} + c_2$$

But since ϕ itself is a constant function, then let $\phi = c_3$ where c_3 is new constant and combining c_2 and c_3 constants into new constant c_2 gives the solution as

$$c_2 = \frac{x(x^3 + 12xy + 12y^2 - 12c_1)}{12}$$

Summary

The solution(s) found are the following

$$\frac{x(x^3 + 12yx + 12y^2 - 12c_1)}{12} = c_2 \tag{1}$$

Verification of solutions

$$\frac{x(x^3 + 12yx + 12y^2 - 12c_1)}{12} = c_2$$

Verified OK.

38.8.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(2yx + x^2)\, y'' + ((2y' + 4)\, x + 4y)\, y' + 2y = -x^2$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left((2yx + x^2)\, y'' + ((2y' + 4)\, x + 4y)\, y' + 2y\right) dx = \int -x^2 dx$$

$$2yx + y^2 + (2yx + x^2)\, y' = -\frac{x^3}{3} + c_1$$

Which is now solved for y.

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$(x^2+2xy)\,\mathrm{d}y=\left(-2xy-y^2-\frac{1}{3}x^3+c_1\right)\mathrm{d}x$$

$$\left(2xy+y^2+\frac{1}{3}x^3-c_1\right)\mathrm{d}x+(x^2+2xy)\,\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= 2xy + y^2 + \frac{1}{3}x^3 - c_1 \\
N(x,y) &= x^2 + 2xy
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2xy + y^2 + \frac{1}{3}x^3 - c_1\right) \\
&= 2y + 2x
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + 2xy) \\
&= 2y + 2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int 2xy + y^2 + \frac{1}{3}x^3 - c_1 \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x(x^3 + 12xy + 12y^2 - 12c_1)}{12} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{x(12x + 24y)}{12} + f'(y) \qquad (4) \\
&= x(x + 2y) + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = x^2 + 2xy$. Therefore equation (4) becomes

$$x^2 + 2xy = x(x + 2y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_2$$

Where c_2 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x^3 + 12xy + 12y^2 - 12c_1)}{12} + c_2$$

But since ϕ itself is a constant function, then let $\phi = c_3$ where c_3 is new constant and combining c_2 and c_3 constants into new constant c_2 gives the solution as

$$c_2 = \frac{x(x^3 + 12xy + 12y^2 - 12c_1)}{12}$$

Summary

The solution(s) found are the following

$$\frac{x(x^3 + 12yx + 12y^2 - 12c_1)}{12} = c_2 \tag{1}$$

Verification of solutions

$$\frac{x(x^3 + 12yx + 12y^2 - 12c_1)}{12} = c_2$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 80

```
dsolve(x*(x+2*y(x))*diff(y(x),x$2)+2*x*(diff(y(x),x))^2+4*(x+y(x))*diff(y(x),x)+2*y(x)+x^2=0
```

$$y(x) = \frac{-3x^2 + \sqrt{3}\,\sqrt{-x\,(x^4 - 3x^3 + 12c_2x - 12c_1)}}{6x}$$
$$y(x) = \frac{-3x^2 - \sqrt{3}\,\sqrt{-x\,(x^4 - 3x^3 + 12c_2x - 12c_1)}}{6x}$$

✓ Solution by Mathematica

Time used: 2.35 (sec). Leaf size: 104

```
DSolve[x*(x+2*y[x])*y''[x]+2*x*(y'[x])^2+4*(x+y[x])*y'[x]+2*y[x]+x^2==0,y[x],x,IncludeSingul
```

$$y(x) \to \frac{1}{6}\left(-3x - \sqrt{3}\sqrt{\frac{1}{x^2}}\sqrt{x\,(-x^4 + 3x^3 + 12c_2x + 12c_1)}\right)$$
$$y(x) \to \frac{1}{6}\left(-3x + \sqrt{3}\sqrt{\frac{1}{x^2}}\sqrt{x\,(-x^4 + 3x^3 + 12c_2x + 12c_1)}\right)$$

38.9 problem Ex 9

Internal problem ID [11344]
Internal file name [`OUTPUT/10330_Friday_January_27_2023_02_37_06_AM_1081674/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 62. Summary. Page 144
Problem number: Ex 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__ode__missing__x", "second__order__ode__missing__y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y'' + {y'}^2 = -1}$$

38.9.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x)^2 + 1 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{-p^2 - 1} dp = x + c_1$$
$$-\arctan(p) = x + c_1$$

Solving for p gives these solutions

$$p_1 = -\tan(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\tan(x + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\tan(x + c_1)\,\mathrm{d}x \\ &= -\frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2$$

Verified OK.

38.9.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + p(y)^2 = -1$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int -\frac{p}{p^2+1}dp = \int dy$$
$$-\frac{\ln(p^2+1)}{2} = y + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{p^2+1}} = e^{y+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{p^2+1}} = c_2 e^y$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \operatorname{RootOf}\left(\_Z^2 c_2^2 e^{2y} + c_2^2 e^{2y} - 1\right)$$

Integrating both sides gives

$$\int \frac{1}{\operatorname{RootOf}\left(\_Z^2 c_2^2 e^{2y} + c_2^2 e^{2y} - 1\right)} dy = \int dx$$
$$\int^y \frac{1}{\operatorname{RootOf}\left(\_Z^2 c_2^2 e^{2\_a} + c_2^2 e^{2\_a} - 1\right)} d\_a = c_3 + x$$

Summary

The solution(s) found are the following

$$\int^y \frac{1}{\operatorname{RootOf}\left(\_Z^2 c_2^2 e^{2\_a} + c_2^2 e^{2\_a} - 1\right)} d\_a = c_3 + x \tag{1}$$

Verification of solutions

$$\int^y \frac{1}{\operatorname{RootOf}\left(\_Z^2 c_2^2 e^{2\_a} + c_2^2 e^{2\_a} - 1\right)} d\_a = c_3 + x$$

Verified OK.

38.9.3 Maple step by step solution

Let's solve

$y'' + y'^2 = -1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + u(x)^2 = -1$

- Separate variables

  $\frac{u'(x)}{-u(x)^2-1} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{-u(x)^2-1} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\arctan(u(x)) = x + c_1$

- Solve for $u(x)$

  $u(x) = -\tan(x + c_1)$

- Solve 1st ODE for $u(x)$

  $u(x) = -\tan(x + c_1)$

- Make substitution $u = y'$

  $y' = -\tan(x + c_1)$

- Integrate both sides to solve for y

  $\int y' dx = \int -\tan(x + c_1) dx + c_2$

- Compute integrals

  $y = -\frac{\ln\left(1+\tan(x+c_1)^2\right)}{2} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)+diff(y(x),x)^2+1=0,y(x), singsol=all)
```

$$y(x) = \ln\left(-c_1 \sin\left(x\right) + c_2 \cos\left(x\right)\right)$$

✓ Solution by Mathematica

Time used: 3.113 (sec). Leaf size: 16

```
DSolve[y''[x]+y'[x]^2+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(\cos(x - c_1)) + c_2$$

38.10 problem Ex 10

Internal problem ID [11345]
Internal file name [`OUTPUT/10331_Friday_January_27_2023_02_37_09_AM_12373053/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 62. Summary. Page 144
Problem number: Ex 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{(-x^2+1)\,y''-\frac{y'}{x}=-x^2}$$

38.10.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x)=y'$$

Then

$$p'(x)=y''$$

Hence the ode becomes

$$(-x^2+1)\,p'(x)-\frac{p(x)}{x}+x^2=0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = \frac{1}{x^3 - x}$$
$$q(x) = \frac{x^2}{x^2 - 1}$$

Hence the ode is

$$p'(x) + \frac{p(x)}{x^3 - x} = \frac{x^2}{x^2 - 1}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{x^3 - x} dx}$$
$$= e^{\frac{\ln(x-1)}{2} + \frac{\ln(1+x)}{2} - \ln(x)}$$

Which simplifies to

$$\mu = \frac{\sqrt{x-1}\,\sqrt{1+x}}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu p) = (\mu)\left(\frac{x^2}{x^2 - 1}\right)$$
$$\frac{d}{dx}\left(\frac{\sqrt{x-1}\,\sqrt{1+x}\,p}{x}\right) = \left(\frac{\sqrt{x-1}\,\sqrt{1+x}}{x}\right)\left(\frac{x^2}{x^2 - 1}\right)$$
$$d\left(\frac{\sqrt{x-1}\,\sqrt{1+x}\,p}{x}\right) = \left(\frac{x\sqrt{x-1}\,\sqrt{1+x}}{x^2 - 1}\right) dx$$

Integrating gives

$$\frac{\sqrt{x-1}\,\sqrt{1+x}\,p}{x} = \int \frac{x\sqrt{x-1}\,\sqrt{1+x}}{x^2 - 1} dx$$
$$\frac{\sqrt{x-1}\,\sqrt{1+x}\,p}{x} = \frac{(x-1)^{\frac{3}{2}}(1+x)^{\frac{3}{2}}}{x^2 - 1} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{x-1}\,\sqrt{1+x}}{x}$ results in

$$p(x) = \frac{(x-1)(1+x)x}{x^2 - 1} + \frac{c_1 x}{\sqrt{x-1}\,\sqrt{1+x}}$$

which simplifies to

$$p(x) = x\left(1+\frac{c_1}{\sqrt{x-1}\,\sqrt{1+x}}\right)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x\left(1+\frac{c_1}{\sqrt{x-1}\,\sqrt{1+x}}\right)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{x\left(\sqrt{x-1}\,\sqrt{1+x}+c_1\right)}{\sqrt{x-1}\,\sqrt{1+x}}\,\mathrm{d}x \\ &= \frac{x^2}{2}+c_1\sqrt{x-1}\,\sqrt{1+x}+c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2}+c_1\sqrt{x-1}\,\sqrt{1+x}+c_2 \tag{1}$$

Verification of solutions

$$y = \frac{x^2}{2}+c_1\sqrt{x-1}\,\sqrt{1+x}+c_2$$

Verified OK.

38.10.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The ODE is now normalized to

$$
x\left(-x^2 + 1\right) y'' - y' = -x^2
$$

Where now

$$
\begin{aligned}
A &= x\left(-x^2 + 1\right) \\
B &= -1 \\
C &= 0 \\
F &= -x^2
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(x\left(-x^2 + 1\right)\right)(0) + (-1)(0) + (0)(-1) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
x^3 - xv'' + (1)\, v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
\left(x^3 - x\right) u'(x) + u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{(x^2-1)\,x} \end{aligned}$$

Where $f(x) = -\frac{1}{(x^2-1)x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{(x^2-1)\,x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{(x^2-1)\,x}\,dx \\ \ln(u) &= -\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2} + \ln(x) + c_1 \\ u &= \mathrm{e}^{-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2} + \ln(x) + c_1} \\ &= c_1 \mathrm{e}^{-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2} + \ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x}{\sqrt{x-1}\,\sqrt{1+x}}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1 x}{\sqrt{x-1}\,\sqrt{1+x}} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1 x}{\sqrt{x-1}\,\sqrt{1+x}}\,\mathrm{d}x \\ &= c_1\sqrt{x-1}\,\sqrt{1+x} + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (-1)\left(c_1\sqrt{x-1}\,\sqrt{1+x} + c_2\right) \\ &= -c_1\sqrt{x-1}\,\sqrt{1+x} - c_2 \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= -1 \\ y_2 &= \sqrt{x-1}\,\sqrt{1+x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -1 & \sqrt{x-1}\,\sqrt{1+x} \\ \frac{d}{dx}(-1) & \frac{d}{dx}\left(\sqrt{x-1}\,\sqrt{1+x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -1 & \sqrt{x-1}\,\sqrt{1+x} \\ 0 & \frac{\sqrt{1+x}}{2\sqrt{x-1}} + \frac{\sqrt{x-1}}{2\sqrt{1+x}} \end{vmatrix}$$

Therefore

$$W = (-1)\left(\frac{\sqrt{1+x}}{2\sqrt{x-1}} + \frac{\sqrt{x-1}}{2\sqrt{1+x}}\right) - \left(\sqrt{x-1}\,\sqrt{1+x}\right)(0)$$

Which simplifies to

$$W = -\frac{x}{\sqrt{x-1}\,\sqrt{1+x}}$$

Which simplifies to

$$W = -\frac{x}{\sqrt{x-1}\,\sqrt{1+x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\sqrt{x-1}\,\sqrt{1+x}\,x^2}{-\frac{x^2(-x^2+1)}{\sqrt{x-1}\,\sqrt{1+x}}}\,dx$$

Which simplifies to

$$u_1 = -\int (-1)\,dx$$

Hence

$$u_1 = x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{-\frac{x^2(-x^2+1)}{\sqrt{x-1}\,\sqrt{1+x}}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{x-1}\,\sqrt{1+x}}{x^2-1}dx$$

Hence

$$u_2 = \frac{\sqrt{x-1}\,\sqrt{1+x}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x + \frac{(x-1)\,(1+x)\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}}$$

Which simplifies to

$$y_p(x) = \frac{\ln\left(x+\sqrt{x^2-1}\right)x^2 - x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}}$$

Hence the complete solution is

$$
\begin{aligned}
y(x) &= y_h + y_p \\
&= \left(-c_1\sqrt{x-1}\,\sqrt{1+x} - c_2\right) + \left(\frac{\ln\left(x+\sqrt{x^2-1}\right)x^2 - x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}}\right) \\
&= \frac{(x^2-1)\ln\left(x+\sqrt{x^2-1}\right) - \sqrt{x^2-1}\left(c_1\sqrt{x-1}\,\sqrt{1+x}+x+c_2\right)}{\sqrt{x^2-1}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(x^2-1)\ln\left(x+\sqrt{x^2-1}\right) - \sqrt{x^2-1}\left(c_1\sqrt{x-1}\,\sqrt{1+x}+x+c_2\right)}{\sqrt{x^2-1}} \tag{1}
$$

Verification of solutions

$$
y = \frac{(x^2-1)\ln\left(x+\sqrt{x^2-1}\right) - \sqrt{x^2-1}\left(c_1\sqrt{x-1}\,\sqrt{1+x}+x+c_2\right)}{\sqrt{x^2-1}}
$$

Verified OK.

38.10.3 Solving using Kovacic algorithm

Writing the ode as

$$
\left(-x^3+x\right)y'' - y' = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= -x^3 + x \\
B &= -1 \\
C &= 0
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-6x^2 + 3}{4\left(x^3 - x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -6x^2 + 3 \\
t &= 4\left(x^3 - x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-6x^2 + 3}{4\left(x^3 - x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 233: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 2 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 - x)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4x^2} - \frac{3}{16(x-1)} - \frac{3}{16(x-1)^2} - \frac{3}{16(1+x)^2} + \frac{3}{16(1+x)}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

For the pole at $x=-1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-6x^2+3}{4\left(x^3-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=0$ then

$$\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}+\alpha_{c_3}^{-}\right) \\
&= 0-(0) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + \frac{1}{4x - 4} + \frac{1}{4 + 4x} + (0) \\
&= -\frac{1}{2x} + \frac{1}{4x - 4} + \frac{1}{4 + 4x} \\
&= \frac{1}{2x^3 - 2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} + \frac{1}{4x - 4} + \frac{1}{4 + 4x}\right)(0) + \left(\left(\frac{1}{2x^2} - \frac{1}{4(x-1)^2} - \frac{1}{4(1+x)^2}\right) + \left(-\frac{1}{2x} + \frac{1}{4x - 4} + \frac{1}{4 + 4x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2x} + \frac{1}{4x-4} + \frac{1}{4+4x}\right) dx} \\
&= \frac{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{-x^3+x}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{4}-\frac{\ln(1+x)}{4}+\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\sqrt{x}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{-x^3+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\sqrt{x^2-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right) + c_2\left(\frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\left(\sqrt{x^2-1}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$\left(-x^3+x\right) y''-y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1 (x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}+\frac{c_2 (x^2-1)^{\frac{3}{4}}}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}$$

$$y_2 = \frac{(x^2-1)^{\frac{3}{4}}}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} & \frac{(x^2-1)^{\frac{3}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} \\ \frac{d}{dx}\left(\frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right) & \frac{d}{dx}\left(\frac{(x^2-1)^{\frac{3}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} & \frac{(x^2-1)^{\frac{3}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} \\ \frac{x}{2(x^2-1)^{\frac{3}{4}}(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1)^{\frac{1}{4}}}{4(x-1)^{\frac{5}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1)^{\frac{1}{4}}}{4(x-1)^{\frac{1}{4}}(1+x)^{\frac{5}{4}}} & \frac{3x}{2(x^2-1)^{\frac{1}{4}}(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1)^{\frac{3}{4}}}{4(x-1)^{\frac{5}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1}{4(x-1)^{\frac{1}{4}}(} \end{vmatrix}$$

Therefore

$$W = \left(\frac{(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right)\left(\frac{3x}{2(x^2-1)^{\frac{1}{4}}(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1)^{\frac{3}{4}}}{4(x-1)^{\frac{5}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1)^{\frac{3}{4}}}{4(x-1)^{\frac{1}{4}}(1+x)^{\frac{5}{4}}}\right) - \left(\frac{(x^2-1)^{\frac{3}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right)\left(\frac{x}{2(x^2-1)^{\frac{3}{4}}(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1)^{\frac{1}{4}}}{4(x-1)^{\frac{5}{4}}(1+x)^{\frac{1}{4}}} - \frac{(x^2-1)^{\frac{1}{4}}}{4(x-1)^{\frac{1}{4}}(1+x)^{\frac{5}{4}}}\right)$$

Which simplifies to

$$W = \frac{x}{\sqrt{x-1}\,\sqrt{1+x}}$$

Which simplifies to

$$W = \frac{x}{\sqrt{x-1}\,\sqrt{1+x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{(x^2-1)^{\frac{3}{4}}x^2}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}}{\frac{(-x^3+x)x}{\sqrt{x-1}\,\sqrt{1+x}}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}{(x^2-1)^{\frac{1}{4}}} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}} (1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{1}{4}}} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\frac{(x^2-1)^{\frac{1}{4}} x^2}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}}{\frac{(-x^3+x)x}{\sqrt{x-1}\,\sqrt{1+x}}} dx$$

Which simplifies to

$$u_2 = \int \frac{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}{(x^2-1)^{\frac{3}{4}}} dx$$

Hence

$$u_2 = \int_0^x \frac{(\alpha-1)^{\frac{1}{4}} (1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{1}{4}}} d\alpha\right) (x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}} + \frac{\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}} d\alpha\right) (x^2-1)^{\frac{3}{4}}}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}$$

Which simplifies to

$$y_p(x) = \frac{(x^2-1)^{\frac{1}{4}} \left(\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}} d\alpha\right) \sqrt{x^2-1} - \left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{1}{4}}} d\alpha\right)\right)}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1(x^2-1)^{\frac{1}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} + \frac{c_2(x^2-1)^{\frac{3}{4}}}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right) \\
&\quad + \left(\frac{(x^2-1)^{\frac{1}{4}}\left(\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}}d\alpha\right)\sqrt{x^2-1} - \left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{1}{4}}}d\alpha\right)\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y &= \frac{(x^2-1)^{\frac{1}{4}}\left(c_2\sqrt{x^2-1}+c_1\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} \\
&\quad + \frac{(x^2-1)^{\frac{1}{4}}\left(\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}}d\alpha\right)\sqrt{x^2-1} - \left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{1}{4}}}d\alpha\right)\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{(x^2-1)^{\frac{1}{4}}\left(c_2\sqrt{x^2-1}+c_1\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} \\
&\quad + \frac{(x^2-1)^{\frac{1}{4}}\left(\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}}d\alpha\right)\sqrt{x^2-1} - \left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{1}{4}}}d\alpha\right)\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{(x^2-1)^{\frac{1}{4}}\left(c_2\sqrt{x^2-1}+c_1\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} \\
&\quad + \frac{(x^2-1)^{\frac{1}{4}}\left(\left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{3}{4}}}d\alpha\right)\sqrt{x^2-1} - \left(\int_0^x \frac{(\alpha-1)^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}}{(\alpha^2-1)^{\frac{1}{4}}}d\alpha\right)\right)}{(x-1)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}
\end{aligned}
$$

Verified OK.

38.10.4 Maple step by step solution

Let's solve

$x(-x^2+1)\,y''-y'=-x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $x(-x^2+1)\,u'(x)-u(x)=-x^2$

- Isolate the derivative

  $u'(x)=-\frac{u(x)}{(x^2-1)x}+\frac{x}{x^2-1}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x)+\frac{u(x)}{(x^2-1)x}=\frac{x}{x^2-1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x)+\frac{u(x)}{(x^2-1)x}\right)=\frac{\mu(x)x}{x^2-1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x)+\frac{u(x)}{(x^2-1)x}\right)=\mu'(x)\,u(x)+\mu(x)\,u'(x)$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)}{(x^2-1)x}$

- Solve to find the integrating factor

  $\mu(x)=\frac{\sqrt{x-1}\,\sqrt{1+x}}{x}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,u(x))\right)dx=\int\frac{\mu(x)x}{x^2-1}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,u(x)=\int\frac{\mu(x)x}{x^2-1}dx+c_1$

- Solve for $u(x)$

  $u(x)=\frac{\int\frac{\mu(x)x}{x^2-1}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\frac{\sqrt{x-1}\,\sqrt{1+x}}{x}$

  $u(x)=\frac{x\left(\int\frac{\sqrt{x-1}\,\sqrt{1+x}}{x^2-1}dx+c_1\right)}{\sqrt{x-1}\,\sqrt{1+x}}$

- Evaluate the integrals on the rhs

$$u(x)=\frac{x\left(\frac{\sqrt{x-1}\,\sqrt{1+x}\,\ln\left(x+\sqrt{x^{2}-1}\right)}{\sqrt{x^{2}-1}}+c_{1}\right)}{\sqrt{x-1}\,\sqrt{1+x}}$$

- Simplify

$$u(x)=\frac{x\left(\sqrt{x-1}\,\sqrt{1+x}\,\ln\left(x+\sqrt{x^{2}-1}\right)+c_{1}\sqrt{x^{2}-1}\right)}{\sqrt{x^{2}-1}\,\sqrt{x-1}\,\sqrt{1+x}}$$

- Solve 1st ODE for $u(x)$

$$u(x)=\frac{x\left(\sqrt{x-1}\,\sqrt{1+x}\,\ln\left(x+\sqrt{x^{2}-1}\right)+c_{1}\sqrt{x^{2}-1}\right)}{\sqrt{x^{2}-1}\,\sqrt{x-1}\,\sqrt{1+x}}$$

- Make substitution $u=y'$

$$y'=\frac{x\left(\sqrt{x-1}\,\sqrt{1+x}\,\ln\left(x+\sqrt{x^{2}-1}\right)+c_{1}\sqrt{x^{2}-1}\right)}{\sqrt{x^{2}-1}\,\sqrt{x-1}\,\sqrt{1+x}}$$

- Integrate both sides to solve for y

$$\int y'dx=\int\frac{x\left(\sqrt{x-1}\,\sqrt{1+x}\,\ln\left(x+\sqrt{x^{2}-1}\right)+c_{1}\sqrt{x^{2}-1}\right)}{\sqrt{x^{2}-1}\,\sqrt{x-1}\,\sqrt{1+x}}dx+c_{2}$$

- Compute integrals

$$y=\int\frac{x\left(\sqrt{x-1}\,\sqrt{1+x}\,\ln\left(x+\sqrt{x^{2}-1}\right)+c_{1}\sqrt{x^{2}-1}\right)}{\sqrt{x^{2}-1}\,\sqrt{x-1}\,\sqrt{1+x}}dx+c_{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(-_a^3+_b(_a))/(_a*(_a-1)*(_a+1)), _b(
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve((1-x^2)*diff(y(x),x$2)-1/x*diff(y(x),x)+x^2=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} + \sqrt{-1+x}\,\sqrt{1+x}\,c_1 + c_2$$

✓ Solution by Mathematica

Time used: 0.084 (sec). Leaf size: 30

```
DSolve[(1-x^2)*y''[x]-1/x*y'[x]+x^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} - c_1\sqrt{1-x^2} + c_2$$

38.11 problem Ex 11

Internal problem ID [11346]
Internal file name [`OUTPUT/10332_Friday_January_27_2023_02_37_11_AM_81805463/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 62. Summary. Page 144
Problem number: Ex 11.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{4x^2y''' + 8xy'' + y' = 0}$$

Since y is missing from the ode then we can use the substitution $y' = v(x)$ to reduce the order by one. The ODE becomes

$$4v''(x)\,x^2 + 8v'(x)\,x + v(x) = 0$$

This is Euler second order ODE. Let the solution be $v(x) = x^r$, then $v' = rx^{r-1}$ and $v'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$4x^2(r(r-1))x^{r-2} + 8xrx^{r-1} + x^r = 0$$

Simplifying gives

$$4r(r-1)\,x^r + 8r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$4r(r-1) + 8r + 1 = 0$$

Or

$$4r^2 + 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since the roots are equal, then the general solution is

$$v(x) = c_1 v_1 + c_2 v_2$$

Where $v_1 = x^r$ and $v_2 = x^r \ln(x)$. Hence

$$v(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2 \ln(x)}{\sqrt{x}}$$

But since $y' = v(x)$ then we now need to solve the ode $y' = \frac{c_1}{\sqrt{x}} + \frac{c_2 \ln(x)}{\sqrt{x}}$. Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_1 + \ln(x) c_2}{\sqrt{x}} dx \\ &= 2\sqrt{x}\, c_1 + 2\ln(x)\sqrt{x}\, c_2 - 4\sqrt{x}\, c_2 + c_3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = 2\sqrt{x}\, c_1 + 2\ln(x)\sqrt{x}\, c_2 - 4\sqrt{x}\, c_2 + c_3 \tag{1}$$

Verification of solutions

$$y = 2\sqrt{x}\, c_1 + 2\ln(x)\sqrt{x}\, c_2 - 4\sqrt{x}\, c_2 + c_3$$

Verified OK.

38.11.1 Maple step by step solution

Let's solve

$4y'''x^2 + 8y''x + y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{y' + 8y''x}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{2y''}{x} + \frac{y'}{4x^2} = 0$

- Multiply by denominators of the ODE

  $4y'''x^2 + 8y''x + y' = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  - Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\, t''(x) \left(\frac{d^2}{dt^2}y(t)\right) + t'''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $4\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) x^2 + 8\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x + \frac{\frac{d}{dt}y(t)}{x} = 0$

- Simplify

  $\frac{4\frac{d^3}{dt^3}y(t) - 4\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t)}{x} = 0$

- Isolate 3rd derivative

  $\frac{d^3}{dt^3}y(t) = \frac{d^2}{dt^2}y(t) - \frac{\frac{d}{dt}y(t)}{4}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^3}{dt^3}y(t) - \frac{d^2}{dt^2}y(t) + \frac{\frac{d}{dt}y(t)}{4} = 0$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = y_3(t) - \frac{y_2(t)}{4}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = y_3(t) - \frac{y_2(t)}{4}\right]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{4} & 1 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{4} & 1 \end{bmatrix}$$

- Rewrite the system as

  $\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue $\frac{1}{2}$

$$\vec{y}_2(t) = \mathrm{e}^{\frac{t}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = \frac{1}{2}$ is the eigenvalue, an

$$\vec{y}_3(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obta
- Substitute $\vec{y}_3(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$

- Make use of the identity matrix I

$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$

- Condition $\vec{p}$ must meet for $\vec{y}_3(t)$ to be a solution to the homogeneous system

$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue $\frac{1}{2}$

$$\left(\left[\begin{array}{ccc}0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{4} & 1\end{array}\right]-\frac{1}{2}\cdot\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]\right)\cdot\vec{p}=\left[\begin{array}{c}4 \\ 2 \\ 1\end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p}=\left[\begin{array}{c}-8 \\ 0 \\ 0\end{array}\right]$$

- Second solution from eigenvalue $\frac{1}{2}$

$$\vec{y}_3(t)=\mathrm{e}^{\frac{t}{2}}\cdot\left(t\cdot\left[\begin{array}{c}4 \\ 2 \\ 1\end{array}\right]+\left[\begin{array}{c}-8 \\ 0 \\ 0\end{array}\right]\right)$$

- General solution to the system of ODEs

$\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2(t)+c_3\vec{y}_3(t)$

- Substitute solutions into the general solution

$$\vec{y}=c_2\mathrm{e}^{\frac{t}{2}}\cdot\left[\begin{array}{c}4 \\ 2 \\ 1\end{array}\right]+c_3\mathrm{e}^{\frac{t}{2}}\cdot\left(t\cdot\left[\begin{array}{c}4 \\ 2 \\ 1\end{array}\right]+\left[\begin{array}{c}-8 \\ 0 \\ 0\end{array}\right]\right)+\left[\begin{array}{c}c_1 \\ 0 \\ 0\end{array}\right]$$

- First component of the vector is the solution to the ODE

$y(t)=((4t-8)\,c_3+4c_2)\,\mathrm{e}^{\frac{t}{2}}+c_1$

- Change variables back using $t=\ln(x)$

$y=((4\ln(x)-8)\,c_3+4c_2)\sqrt{x}+c_1$

- Simplify

$$y = (4c_3 \ln(x) + 4c_2 - 8c_3)\sqrt{x} + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(4*x^2*diff(y(x),x$3)+8*x*diff(y(x),x$2)+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (c_3 \ln(x) + c_2)\sqrt{x} + c_1$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 28

```
DSolve[4*x^2*y'''[x]+8*x*y''[x]+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x}(c_2 \log(x) + 2c_1 - 2c_2) + c_3$$

38.12 problem Ex 12

Internal problem ID [11347]
Internal file name [`OUTPUT/10333_Friday_January_27_2023_02_37_12_AM_61649637/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter IX, Miscellaneous methods for solving equations of higher order than first. Article 62. Summary. Page 144
Problem number: Ex 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{\sin(x) y'' - \cos(x) y' + 2\sin(x) y = 0}$$

38.12.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (\sin(x) y'' - \cos(x) y' + 2\sin(x) y) \, dx = 0$$
$$-2y\cos(x) + y'\sin(x) = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -2\cot(x)$$
$$q(x) = \csc(x)\, c_1$$

Hence the ode is

$$y' - 2y\cot(x) = \csc(x)\, c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -2\cot(x)dx} \\ &= \frac{1}{\sin(x)^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\csc(x)\, c_1) \\ \frac{d}{dx}\left(\frac{y}{\sin(x)^2}\right) &= \left(\frac{1}{\sin(x)^2}\right)(\csc(x)\, c_1) \\ d\left(\frac{y}{\sin(x)^2}\right) &= \left(\csc(x)^3 c_1\right)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{\sin(x)^2} &= \int \csc(x)^3 c_1\, dx \\ \frac{y}{\sin(x)^2} &= c_1\left(-\frac{\csc(x)\cot(x)}{2} + \frac{\ln(\csc(x) - \cot(x))}{2}\right) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sin(x)^2}$ results in

$$y = \sin(x)^2 c_1\left(-\frac{\csc(x)\cot(x)}{2} + \frac{\ln(\csc(x) - \cot(x))}{2}\right) + c_2\sin(x)^2$$

which simplifies to

$$y = \sin(x)^2\left(\frac{c_1(-\csc(x)\cot(x) + \ln(\csc(x) - \cot(x)))}{2} + c_2\right)$$

Summary

The solution(s) found are the following

$$y = \sin(x)^2\left(\frac{c_1(-\csc(x)\cot(x) + \ln(\csc(x) - \cot(x)))}{2} + c_2\right) \tag{1}$$

Verification of solutions

$$y = \sin(x)^2 \left(\frac{c_1(-\csc(x)\cot(x) + \ln(\csc(x) - \cot(x)))}{2} + c_2\right)$$

Verified OK.

38.12.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$\sin(x)\, y'' - \cos(x)\, y' + 2\sin(x)\, y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (\sin(x)\, y'' - \cos(x)\, y' + 2\sin(x)\, y)\, dx = 0$$
$$-2y\cos(x) + y'\sin(x) = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -2\cot(x)$$
$$q(x) = \csc(x)\, c_1$$

Hence the ode is

$$y' - 2y\cot(x) = \csc(x)\, c_1$$

The integrating factor μ is

$$\mu = e^{\int -2\cot(x)dx}$$
$$= \frac{1}{\sin(x)^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(\csc(x)\, c_1)$$
$$\frac{d}{dx}\left(\frac{y}{\sin(x)^2}\right) = \left(\frac{1}{\sin(x)^2}\right)(\csc(x)\, c_1)$$
$$d\left(\frac{y}{\sin(x)^2}\right) = \left(\csc(x)^3 c_1\right)\, dx$$

Integrating gives

$$\frac{y}{\sin(x)^2} = \int \csc(x)^3 c_1 \, dx$$

$$\frac{y}{\sin(x)^2} = c_1 \left(-\frac{\csc(x)\cot(x)}{2} + \frac{\ln(\csc(x) - \cot(x))}{2}\right) + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sin(x)^2}$ results in

$$y = \sin(x)^2 c_1 \left(-\frac{\csc(x)\cot(x)}{2} + \frac{\ln(\csc(x) - \cot(x))}{2}\right) + c_2 \sin(x)^2$$

which simplifies to

$$y = \sin(x)^2 \left(\frac{c_1(-\csc(x)\cot(x) + \ln(\csc(x) - \cot(x)))}{2} + c_2\right)$$

Summary

The solution(s) found are the following

$$y = \sin(x)^2 \left(\frac{c_1(-\csc(x)\cot(x) + \ln(\csc(x) - \cot(x)))}{2} + c_2\right) \tag{1}$$

Verification of solutions

$$y = \sin(x)^2 \left(\frac{c_1(-\csc(x)\cot(x) + \ln(\csc(x) - \cot(x)))}{2} + c_2\right)$$

Verified OK.

38.12.3 Solving as exact linear second order ode ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$p(x) = \sin(x)$$
$$q(x) = -\cos(x)$$
$$r(x) = 2\sin(x)$$
$$s(x) = 0$$

Hence

$$
\begin{aligned}
p''(x) &= -\sin(x) \\
q'(x) &= \sin(x)
\end{aligned}
$$

Therefore (1) becomes

$$
-\sin(x) - (\sin(x)) + (2\sin(x)) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
-2y\cos(x) + y'\sin(x) = c_1
$$

We now have a first order ode to solve which is

$$
-2y\cos(x) + y'\sin(x) = c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -2\cot(x) \\
q(x) &= \csc(x)\, c_1
\end{aligned}
$$

Hence the ode is

$$
y' - 2y\cot(x) = \csc(x)\, c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -2\cot(x)dx} \\
&= \frac{1}{\sin(x)^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\csc(x) c_1\right) \\
\frac{d}{dx}\left(\frac{y}{\sin(x)^2}\right) &= \left(\frac{1}{\sin(x)^2}\right)\left(\csc(x) c_1\right) \\
d\left(\frac{y}{\sin(x)^2}\right) &= \left(\csc(x)^3 c_1\right)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{\sin(x)^2} &= \int \csc(x)^3 c_1\, dx \\
\frac{y}{\sin(x)^2} &= c_1\left(-\frac{\csc(x)\cot(x)}{2} + \frac{\ln(\csc(x) - \cot(x))}{2}\right) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sin(x)^2}$ results in

$$
y = \sin(x)^2 c_1\left(-\frac{\csc(x)\cot(x)}{2} + \frac{\ln(\csc(x) - \cot(x))}{2}\right) + c_2 \sin(x)^2
$$

which simplifies to

$$
y = \sin(x)^2\left(\frac{c_1(-\csc(x)\cot(x) + \ln(\csc(x) - \cot(x)))}{2} + c_2\right)
$$

Summary

The solution(s) found are the following

$$
y = \sin(x)^2\left(\frac{c_1(-\csc(x)\cot(x) + \ln(\csc(x) - \cot(x)))}{2} + c_2\right) \tag{1}
$$

Verification of solutions

$$
y = \sin(x)^2\left(\frac{c_1(-\csc(x)\cot(x) + \ln(\csc(x) - \cot(x)))}{2} + c_2\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
   Change of variables used:
      [x = arccos(t)]
   Linear ODE actually solved:
      2*u(t)+(-t^2+1)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 39

```
dsolve(sin(x)*diff(y(x),x$2)-cos(x)*diff(y(x),x)+2*sin(x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \ln(\cos(x) - 1) c_2 \sin(x)^2 - \ln(\cos(x) + 1) c_2 \sin(x)^2 + c_1 \sin(x)^2 - 2c_2 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.235 (sec). Leaf size: 45

```
DSolve[Sin[x]*y''[x]-Cos[x]*y'[x]+2*Sin[x]*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -c_1 \sin^2(x) - \frac{1}{4} c_2 \left(2\cos(x) + \sin^2(x)(\log(\cos(x)+1) - \log(1-\cos(x)))\right)$$

39 Chapter X, System of simulataneous equations. Article 64. Systems of linear equations with constant coefficients. Page 150

39.1 problem Ex 1

Internal problem ID [11348]
Internal file name [`OUTPUT/10334_Friday_January_27_2023_02_37_14_AM_38590850/index.tex`]

Book: An elementary treatise on differential equations by Abraham Cohen. DC heath publishers. 1906
Section: Chapter X, System of simulataneous equations. Article 64. Systems of linear equations with constant coefficients. Page 150
Problem number: Ex 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -x(t) - \frac{2y(t)}{3} + \frac{e^t}{3} \\
y'(t) &= \frac{4x(t)}{3} + y(t) - t
\end{aligned}
$$

39.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & -\frac{2}{3} \\ \frac{4}{3} & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \frac{e^t}{3} \\ -t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 2\,\mathrm{e}^{-\frac{t}{3}} - \mathrm{e}^{\frac{t}{3}} & -\mathrm{e}^{\frac{t}{3}} + \mathrm{e}^{-\frac{t}{3}} \\ 2\,\mathrm{e}^{\frac{t}{3}} - 2\,\mathrm{e}^{-\frac{t}{3}} & -\mathrm{e}^{-\frac{t}{3}} + 2\,\mathrm{e}^{\frac{t}{3}} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 2\,\mathrm{e}^{-\frac{t}{3}} - \mathrm{e}^{\frac{t}{3}} & -\mathrm{e}^{\frac{t}{3}} + \mathrm{e}^{-\frac{t}{3}} \\ 2\,\mathrm{e}^{\frac{t}{3}} - 2\,\mathrm{e}^{-\frac{t}{3}} & -\mathrm{e}^{-\frac{t}{3}} + 2\,\mathrm{e}^{\frac{t}{3}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(2\,\mathrm{e}^{-\frac{t}{3}} - \mathrm{e}^{\frac{t}{3}}\right) c_1 + \left(-\mathrm{e}^{\frac{t}{3}} + \mathrm{e}^{-\frac{t}{3}}\right) c_2 \\ \left(2\,\mathrm{e}^{\frac{t}{3}} - 2\,\mathrm{e}^{-\frac{t}{3}}\right) c_1 + \left(-\mathrm{e}^{-\frac{t}{3}} + 2\,\mathrm{e}^{\frac{t}{3}}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} (2c_1 + c_2)\,\mathrm{e}^{-\frac{t}{3}} - \mathrm{e}^{\frac{t}{3}}(c_1 + c_2) \\ (-2c_1 - c_2)\,\mathrm{e}^{-\frac{t}{3}} + 2\,\mathrm{e}^{\frac{t}{3}}(c_1 + c_2) \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} -\mathrm{e}^{-\frac{t}{3}} + 2\,\mathrm{e}^{\frac{t}{3}} & \mathrm{e}^{\frac{t}{3}} - \mathrm{e}^{-\frac{t}{3}} \\ -2\,\mathrm{e}^{\frac{t}{3}} + 2\,\mathrm{e}^{-\frac{t}{3}} & 2\,\mathrm{e}^{-\frac{t}{3}} - \mathrm{e}^{\frac{t}{3}} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 2\,\mathrm{e}^{-\frac{t}{3}} - \mathrm{e}^{\frac{t}{3}} & -\mathrm{e}^{\frac{t}{3}} + \mathrm{e}^{-\frac{t}{3}} \\ 2\,\mathrm{e}^{\frac{t}{3}} - 2\,\mathrm{e}^{-\frac{t}{3}} & -\mathrm{e}^{-\frac{t}{3}} + 2\,\mathrm{e}^{\frac{t}{3}} \end{bmatrix} \int \begin{bmatrix} -\mathrm{e}^{-\frac{t}{3}} + 2\,\mathrm{e}^{\frac{t}{3}} & \mathrm{e}^{\frac{t}{3}} - \mathrm{e}^{-\frac{t}{3}} \\ -2\,\mathrm{e}^{\frac{t}{3}} + 2\,\mathrm{e}^{-\frac{t}{3}} & 2\,\mathrm{e}^{-\frac{t}{3}} - \mathrm{e}^{\frac{t}{3}} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{e}^{t}}{3} \\ -t \end{bmatrix} dt \\
&= \begin{bmatrix} 2\,\mathrm{e}^{-\frac{t}{3}} - \mathrm{e}^{\frac{t}{3}} & -\mathrm{e}^{\frac{t}{3}} + \mathrm{e}^{-\frac{t}{3}} \\ 2\,\mathrm{e}^{\frac{t}{3}} - 2\,\mathrm{e}^{-\frac{t}{3}} & -\mathrm{e}^{-\frac{t}{3}} + 2\,\mathrm{e}^{\frac{t}{3}} \end{bmatrix} \begin{bmatrix} 3(-3-t)\,\mathrm{e}^{-\frac{t}{3}} + 3(3-t)\,\mathrm{e}^{\frac{t}{3}} - \frac{\mathrm{e}^{\frac{2t}{3}}}{2} + \frac{\mathrm{e}^{\frac{4t}{3}}}{2} \\ 6(3+t)\,\mathrm{e}^{-\frac{t}{3}} + 3(-3+t)\,\mathrm{e}^{\frac{t}{3}} + \mathrm{e}^{\frac{2t}{3}} - \frac{\mathrm{e}^{\frac{4t}{3}}}{2} \end{bmatrix} \\
&= \begin{bmatrix} -6t \\ 9 + \frac{\mathrm{e}^{t}}{2} + 9t \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} (2c_1+c_2)\,\mathrm{e}^{-\frac{t}{3}} + (-c_1-c_2)\,\mathrm{e}^{\frac{t}{3}} - 6t \\ (-2c_1-c_2)\,\mathrm{e}^{-\frac{t}{3}} + 2\,\mathrm{e}^{\frac{t}{3}}(c_1+c_2) + 9 + \frac{\mathrm{e}^t}{2} + 9t \end{bmatrix}$$

39.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & -\frac{2}{3} \\ \frac{4}{3} & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \frac{\mathrm{e}^t}{3} \\ -t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -\frac{2}{3} \\ \frac{4}{3} & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & -\frac{2}{3} \\ \frac{4}{3} & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - \frac{1}{9} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{1}{3}$$
$$\lambda_2 = -\frac{1}{3}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{3}$	1	real eigenvalue
$\frac{1}{3}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -\frac{2}{3} \\ \frac{4}{3} & 1 \end{bmatrix} - \left(-\frac{1}{3}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} \\ \frac{4}{3} & \frac{4}{3} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{2}{3} & -\frac{2}{3} & 0 \\ \frac{4}{3} & \frac{4}{3} & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} -\frac{2}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{1}{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -\frac{2}{3} \\ \frac{4}{3} & 1 \end{bmatrix} - \left(\frac{1}{3}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{4}{3} & -\frac{2}{3} \\ \frac{4}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{4}{3} & -\frac{2}{3} & 0 \\ \frac{4}{3} & \frac{2}{3} & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -\frac{4}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{4}{3} & -\frac{2}{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{1}{3}$	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
$-\frac{1}{3}$	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{1}{3}$ is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{\frac{t}{3}}$$
$$= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{\frac{t}{3}}$$

Since eigenvalue $-\frac{1}{3}$ is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-\frac{t}{3}}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-\frac{t}{3}}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{e^{\frac{t}{3}}}{2} \\ e^{\frac{t}{3}} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-\frac{t}{3}} \\ e^{-\frac{t}{3}} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\frac{e^{\frac{t}{3}}}{2} & -e^{-\frac{t}{3}} \\ e^{\frac{t}{3}} & e^{-\frac{t}{3}} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} 2\,e^{-\frac{t}{3}} & 2\,e^{-\frac{t}{3}} \\ -2\,e^{\frac{t}{3}} & -e^{\frac{t}{3}} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -\frac{e^{\frac{t}{3}}}{2} & -e^{-\frac{t}{3}} \\ e^{\frac{t}{3}} & e^{-\frac{t}{3}} \end{bmatrix} \int \begin{bmatrix} 2\,e^{-\frac{t}{3}} & 2\,e^{-\frac{t}{3}} \\ -2\,e^{\frac{t}{3}} & -e^{\frac{t}{3}} \end{bmatrix} \begin{bmatrix} \frac{e^t}{3} \\ -t \end{bmatrix} dt$$

$$= \begin{bmatrix} -\frac{e^{\frac{t}{3}}}{2} & -e^{-\frac{t}{3}} \\ e^{\frac{t}{3}} & e^{-\frac{t}{3}} \end{bmatrix} \int \begin{bmatrix} \frac{2\,e^{\frac{2t}{3}}}{3} - 2t\,e^{-\frac{t}{3}} \\ -\frac{2\,e^{\frac{4t}{3}}}{3} + t\,e^{\frac{t}{3}} \end{bmatrix} dt$$

$$= \begin{bmatrix} -\frac{e^{\frac{t}{3}}}{2} & -e^{-\frac{t}{3}} \\ e^{\frac{t}{3}} & e^{-\frac{t}{3}} \end{bmatrix} \begin{bmatrix} 6(3+t)\,e^{-\frac{t}{3}} + e^{\frac{2t}{3}} \\ 3(-3+t)\,e^{\frac{t}{3}} - \frac{e^{\frac{4t}{3}}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -6t \\ 9 + \frac{e^t}{2} + 9t \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{\frac{t}{3}}}{2} \\ c_1 e^{\frac{t}{3}} \end{bmatrix} + \begin{bmatrix} -c_2 e^{-\frac{t}{3}} \\ c_2 e^{-\frac{t}{3}} \end{bmatrix} + \begin{bmatrix} -6t \\ 9 + \frac{e^t}{2} + 9t \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{\frac{t}{3}}}{2} - c_2 e^{-\frac{t}{3}} - 6t \\ c_1 e^{\frac{t}{3}} + c_2 e^{-\frac{t}{3}} + 9 + \frac{e^t}{2} + 9t \end{bmatrix}$$

39.1.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -x(t) - \frac{2y(t)}{3} + \frac{e^t}{3}, y'(t) = \frac{4x(t)}{3} + y(t) - t\right]$$

- Define vector

$$\overrightarrow{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & -\frac{2}{3} \\ \frac{4}{3} & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{e^t}{3} \\ -t \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & -\frac{2}{3} \\ \frac{4}{3} & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{e^t}{3} \\ -t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} \frac{e^t}{3} \\ -t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & -\frac{2}{3} \\ \frac{4}{3} & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{3}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[\frac{1}{3}, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{1}{3}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-\frac{t}{3}} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{1}{3}, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{\frac{t}{3}} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{x}_p(t)$$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -e^{-\frac{t}{3}} & -\frac{e^{\frac{t}{3}}}{2} \\ e^{-\frac{t}{3}} & e^{\frac{t}{3}} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -e^{-\frac{t}{3}} & -\frac{e^{\frac{t}{3}}}{2} \\ e^{-\frac{t}{3}} & e^{\frac{t}{3}} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} 2\,e^{-\frac{t}{3}} - e^{\frac{t}{3}} & -e^{\frac{t}{3}} + e^{-\frac{t}{3}} \\ 2\,e^{\frac{t}{3}} - 2\,e^{-\frac{t}{3}} & -e^{-\frac{t}{3}} + 2\,e^{\frac{t}{3}} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}_p^{\prime}(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}^{\prime}(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}^{\prime}(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}^{\prime}(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- Cancel like terms

$\Phi(t)\cdot \vec{v}'(t)=\vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t)=\int_0^t \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t)=\Phi(t)\cdot\left(\int_0^t \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t)=\left[\begin{array}{c} -6t+\frac{19\,e^{\frac{t}{3}}}{2}-\frac{19\,e^{-\frac{t}{3}}}{2} \\ 9+\frac{e^t}{2}+\frac{19\,e^{-\frac{t}{3}}}{2}+9t-19\,e^{\frac{t}{3}} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{x}(t)=c_1\vec{x}_1+c_2\vec{x}_2+\left[\begin{array}{c} -6t+\frac{19\,e^{\frac{t}{3}}}{2}-\frac{19\,e^{-\frac{t}{3}}}{2} \\ 9+\frac{e^t}{2}+\frac{19\,e^{-\frac{t}{3}}}{2}+9t-19\,e^{\frac{t}{3}} \end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right]=\left[\begin{array}{c} \frac{(-2c_1-19)e^{-\frac{t}{3}}}{2}+\frac{(-c_2+19)e^{\frac{t}{3}}}{2}-6t \\ 9+\frac{(2c_1+19)e^{-\frac{t}{3}}}{2}+(c_2-19)\,e^{\frac{t}{3}}+9t+\frac{e^t}{2} \end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x(t)=\frac{(-2c_1-19)e^{-\frac{t}{3}}}{2}+\frac{(-c_2+19)e^{\frac{t}{3}}}{2}-6t, y(t)=9+\frac{(2c_1+19)e^{-\frac{t}{3}}}{2}+(c_2-19)\,e^{\frac{t}{3}}+9t+\frac{e^t}{2}\right\}$$

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 47

```
dsolve([3*diff(x(t),t)+3*x(t)+2*y(t)=exp(t),4*x(t)-3*diff(y(t),t)+3*y(t)=3*t],singsol=all)
```

$$x(t)=-\frac{e^{\frac{t}{3}}c_2}{2}-e^{-\frac{t}{3}}c_1-6t$$
$$y(t)=e^{\frac{t}{3}}c_2+e^{-\frac{t}{3}}c_1+9t+9+\frac{e^t}{2}$$

✓ Solution by Mathematica

Time used: 1.125 (sec). Leaf size: 90

```
DSolve[{3*x'[t]+3*x[t]+2*y[t]==Exp[t],4*x[t]-3*y'[t]+3*y[t]==3*t},{x[t],y[t]},t,IncludeSingu
```

$$x(t) \to e^{-t/3}\left(-6e^{t/3}t - (c_1 + c_2)e^{2t/3} + 2c_1 + c_2\right)$$
$$y(t) \to 9(t+1) + \frac{e^t}{2} + 2(c_1 + c_2)e^{t/3} - (2c_1 + c_2)e^{-t/3}$$