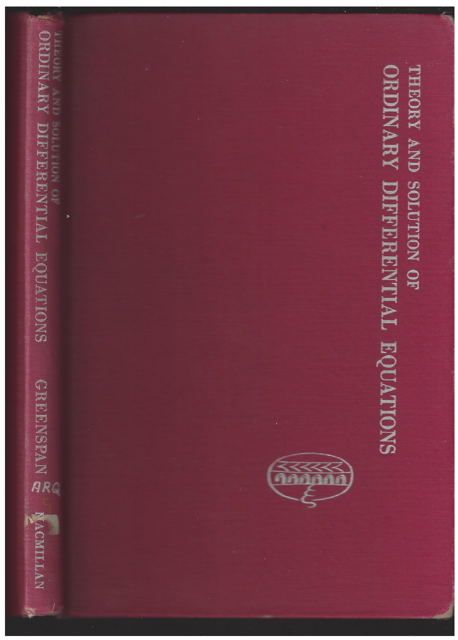
A Solution Manual For

Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960



Nasser M. Abbasi

May 16, 2024

Contents

1 Exercises, page 14

1.1 problem 1(a)

Internal problem ID [3002]
Internal file name [OUTPUT/2494_Sunday_June_05_2022_03_16_45_AM_98176719/index.tex]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 1(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = \mathrm{e}^{-x}}$$

1.1.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \mathrm{e}^{-x}\, \mathrm{d}x \\ &= -\mathrm{e}^{-x} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\mathrm{e}^{-x} + c_1 \tag{1}$$

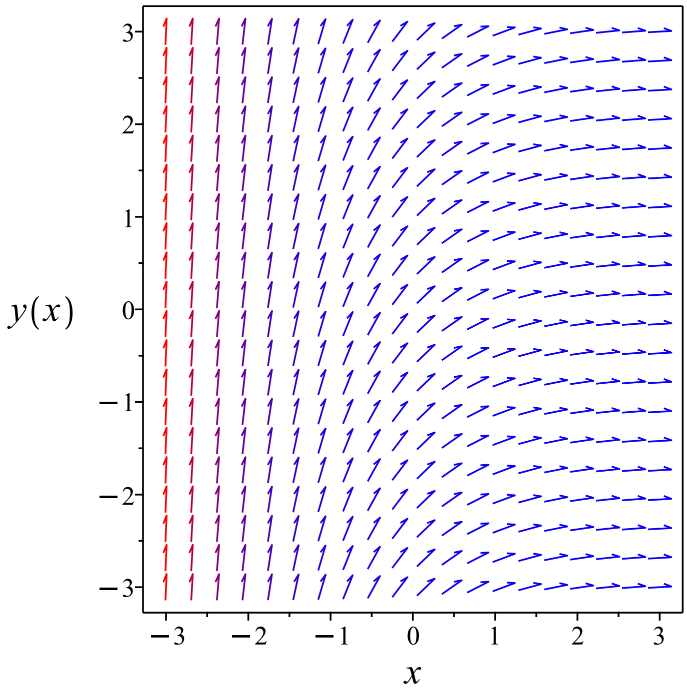


Figure 1: Slope field plot

<u>Verification of solutions</u>

$$y = -e^{-x} + c_1$$

Verified OK.

1.1.2 Maple step by step solution

Let's solve

$y' = e^{-x}$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int e^{-x} dx + c_1$

- Evaluate integral

  $y = -e^{-x} + c_1$

- Solve for y

  $y = -e^{-x} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=exp(-x),y(x), singsol=all)
```

$$y(x) = -\mathrm{e}^{-x} + c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 15

```
DSolve[y'[x]==Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{-x} + c_1$$

1.2 problem 1(b)

Internal problem ID [3003]
Internal file name [`OUTPUT/2495_Sunday_June_05_2022_03_16_47_AM_24614121/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 1(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = 1 - x^5 + \sqrt{x}}$$

1.2.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 1 - x^5 + \sqrt{x}\,\mathrm{d}x \\ &= x + \frac{2x^{\frac{3}{2}}}{3} - \frac{x^6}{6} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x + \frac{2x^{\frac{3}{2}}}{3} - \frac{x^6}{6} + c_1 \tag{1}$$

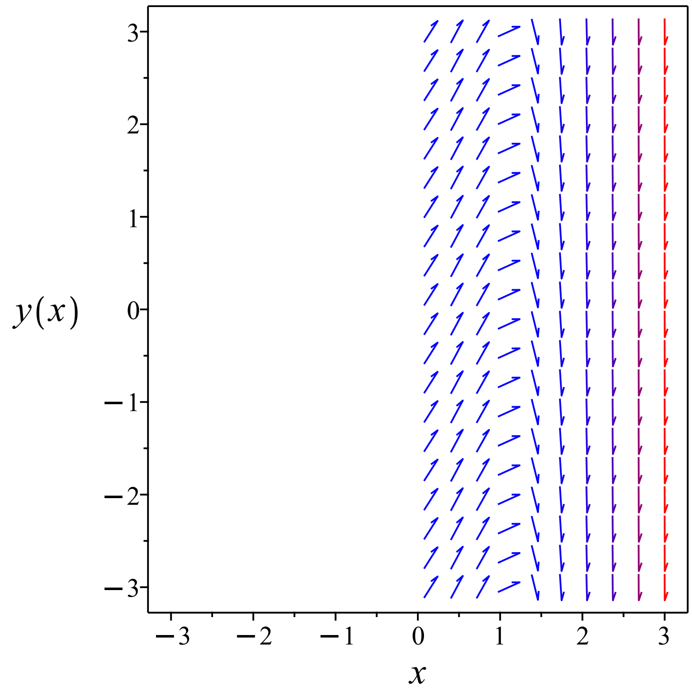


Figure 2: Slope field plot

Verification of solutions

$$y = x + \frac{2x^{\frac{3}{2}}}{3} - \frac{x^6}{6} + c_1$$

Verified OK.

1.2.2 Maple step by step solution

Let's solve

$y' = 1 - x^5 + \sqrt{x}$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \left(1 - x^5 + \sqrt{x}\right) dx + c_1$

- Evaluate integral

  $y = x + \frac{2x^{\frac{3}{2}}}{3} - \frac{x^6}{6} + c_1$

- Solve for y

$$y = x + \frac{2x^{\frac{3}{2}}}{3} - \frac{x^6}{6} + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)=1-x^5+sqrt(x),y(x), singsol=all)
```

$$y(x) = \frac{2x^{\frac{3}{2}}}{3} - \frac{x^6}{6} + x + c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 25

```
DSolve[y'[x]==1-x^5+Sqrt[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2x^{3/2}}{3} - \frac{x^6}{6} + x + c_1$$

1.3 problem 1(c)

Internal problem ID [3004]
Internal file name [`OUTPUT/2496_Sunday_June_05_2022_03_16_48_AM_49830060/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 1(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{3y + (3x - 2)\, y' = 2x}$$

1.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{3}{3x - 2}$$
$$q(x) = \frac{2x}{3x - 2}$$

Hence the ode is

$$y' + \frac{3y}{3x - 2} = \frac{2x}{3x - 2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{3}{3x-2}dx}$$
$$= 3x - 2$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{2x}{3x-2}\right)$$
$$\frac{d}{dx}((3x-2)\, y) = (3x-2)\left(\frac{2x}{3x-2}\right)$$
$$d((3x-2)\, y) = (2x)\ dx$$

Integrating gives

$$(3x-2)\, y = \int 2x\, dx$$
$$(3x-2)\, y = x^2 + c_1$$

Dividing both sides by the integrating factor $\mu = 3x - 2$ results in

$$y = \frac{x^2}{3x-2} + \frac{c_1}{3x-2}$$

which simplifies to

$$y = \frac{x^2 + c_1}{3x-2}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2 + c_1}{3x-2} \tag{1}$$

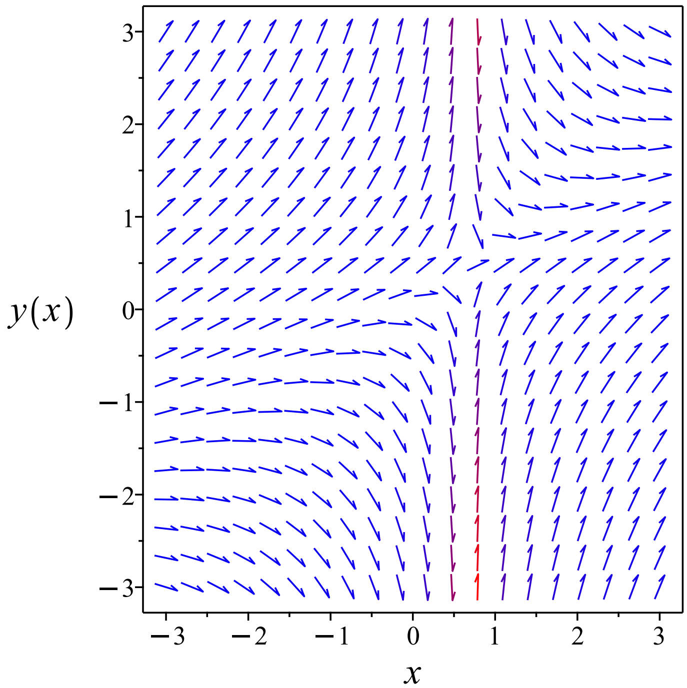


Figure 3: Slope field plot

Verification of solutions

$$y = \frac{x^2 + c_1}{3x - 2}$$

Verified OK.

1.3.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-3y + 2x}{3x - 2} \tag{1}$$

Which becomes

$$0 = (-3x + 2)\, dy + (-3y + 2x)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-3x + 2)\, dy + (-3y + 2x)\, dx = d\left(x^2 - 3xy + 2y\right)$$

Hence (2) becomes

$$0 = d\left(x^2 - 3xy + 2y\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{x^2 + c_1}{3x - 2} + c_1$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + c_1}{3x - 2} + c_1 \tag{1}$$

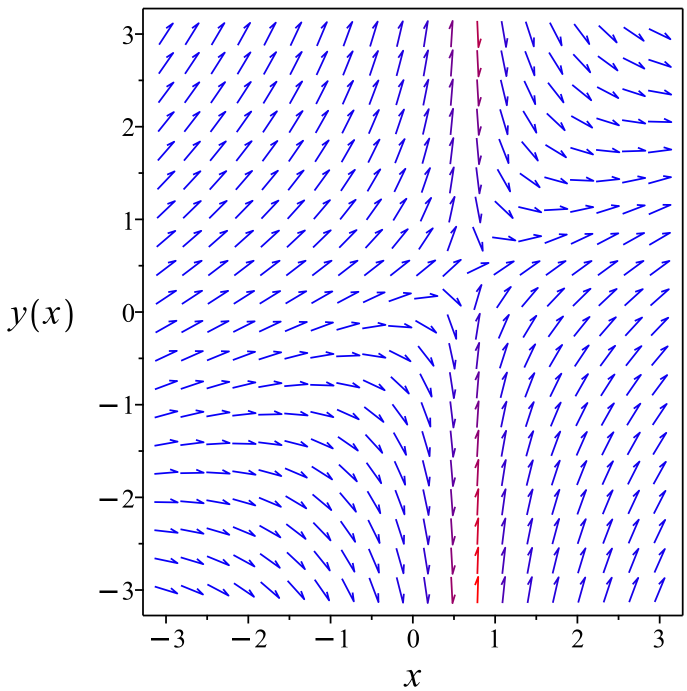


Figure 4: Slope field plot

Verification of solutions

$$y = \frac{x^2 + c_1}{3x - 2} + c_1$$

Verified OK.

1.3.3 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{-2X - 2x_0 + 3Y(X) + 3y_0}{3X + 3x_0 - 2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned} x_0 &= \frac{2}{3} \\ y_0 &= \frac{4}{9} \end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{-2X + 3Y(X)}{3X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{-2X + 3Y}{3X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = 2X - 3Y$ and $N = 3X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dX}X + u &= \frac{2}{3} - u \\ \frac{du}{dX} &= \frac{\frac{2}{3} - 2u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{2}{3} - 2u(X)}{X} = 0$$

Or

$$3\left(\frac{d}{dX}u(X)\right)X + 6u(X) - 2 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X)g(u) \\ &= \frac{-2u + \frac{2}{3}}{X} \end{aligned}$$

Where $f(X) = \frac{1}{X}$ and $g(u) = -2u + \frac{2}{3}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-2u + \frac{2}{3}}\, du &= \frac{1}{X}\, dX \\ \int \frac{1}{-2u + \frac{2}{3}}\, du &= \int \frac{1}{X}\, dX \\ -\frac{\ln(-3u + 1)}{2} &= \ln(X) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-3u + 1}} = e^{\ln(X) + c_2}$$

Which simplifies to

$$\frac{1}{\sqrt{-3u + 1}} = c_3 X$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \frac{(c_3^2 e^{2c_2} X^2 - 1)\, e^{-2c_2}}{3X\, c_3^2}$$

Using the solution for $Y(X)$

$$Y(X) = \frac{(c_3^2 e^{2c_2} X^2 - 1)\, e^{-2c_2}}{3X\, c_3^2}$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y + \frac{4}{9} \\
X &= x + \frac{2}{3}
\end{aligned}
$$

Then the solution in y becomes

$$
y - \frac{4}{9} = \frac{\left(c_3^2 e^{2c_2} \left(x - \frac{2}{3}\right)^2 - 1\right) e^{-2c_2}}{3 \left(x - \frac{2}{3}\right) c_3^2}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y - \frac{4}{9} = \frac{\left(c_3^2 e^{2c_2} \left(x - \frac{2}{3}\right)^2 - 1\right) e^{-2c_2}}{3 \left(x - \frac{2}{3}\right) c_3^2} \tag{1}
$$

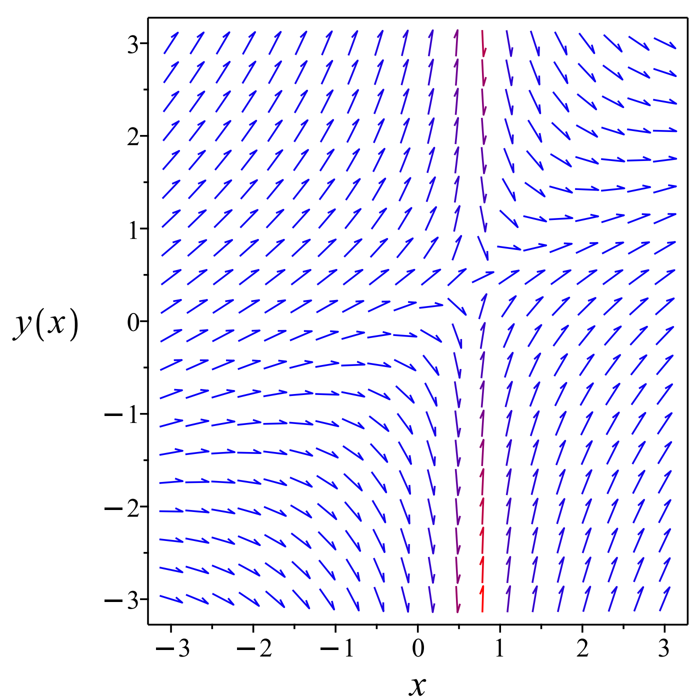


Figure 5: Slope field plot

Verification of solutions

$$y - \frac{4}{9} = \frac{\left(c_3^2 \mathrm{e}^{2c_2} \left(x - \frac{2}{3}\right)^2 - 1\right) \mathrm{e}^{-2c_2}}{3 \left(x - \frac{2}{3}\right) c_3^2}$$

Verified OK.

1.3.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3y - 2x}{3x - 2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 3: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{3x - 2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{3x-2}} dy \end{aligned}$$

Which results in

$$S = (3x - 2)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3y - 2x}{3x - 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3y \\ S_y &= 3x - 2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(3x - 2)\, y = x^2 + c_1$$

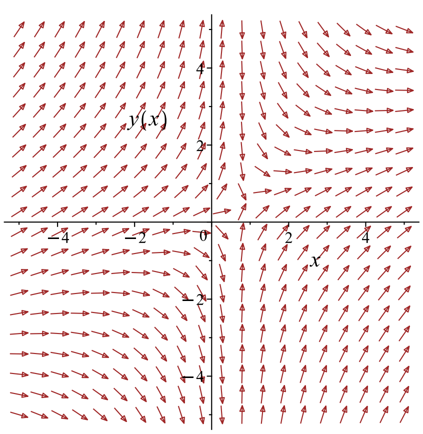
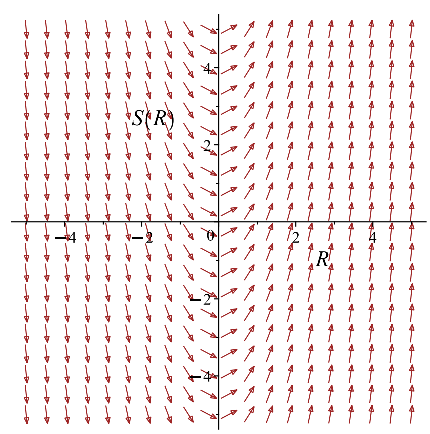
Which simplifies to

$$(3x - 2)\, y = x^2 + c_1$$

Which gives

$$y = \frac{x^2 + c_1}{3x - 2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y-2x}{3x-2}$ 	$R = x$ $S = (3x - 2)\, y$	$\frac{dS}{dR} = 2R$ 

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{x^2 + c_1}{3x - 2} \tag{1}$$

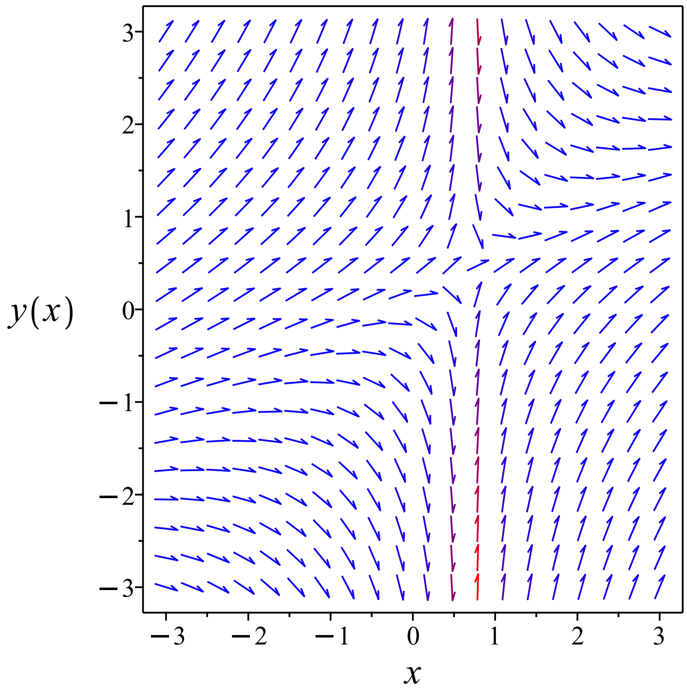


Figure 6: Slope field plot

Verification of solutions

$$y = \frac{x^2 + c_1}{3x - 2}$$

Verified OK.

1.3.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3x-2)\,\mathrm{d}y &= (-3y+2x)\,\mathrm{d}x\\ (3y-2x)\,\mathrm{d}x + (3x-2)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 3y - 2x\\ N(x, y) &= 3x - 2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3y - 2x)\\ &= 3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x - 2)\\ &= 3\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 3y - 2x \, \mathrm{d}x$$
$$\phi = -x(x - 3y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3x - 2$. Therefore equation (4) becomes

$$3x - 2 = 3x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-2) \, \mathrm{d}y$$
$$f(y) = -2y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x(x - 3y) - 2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x(x - 3y) - 2y$$

The solution becomes

$$y = \frac{x^2 + c_1}{3x - 2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + c_1}{3x - 2} \tag{1}$$

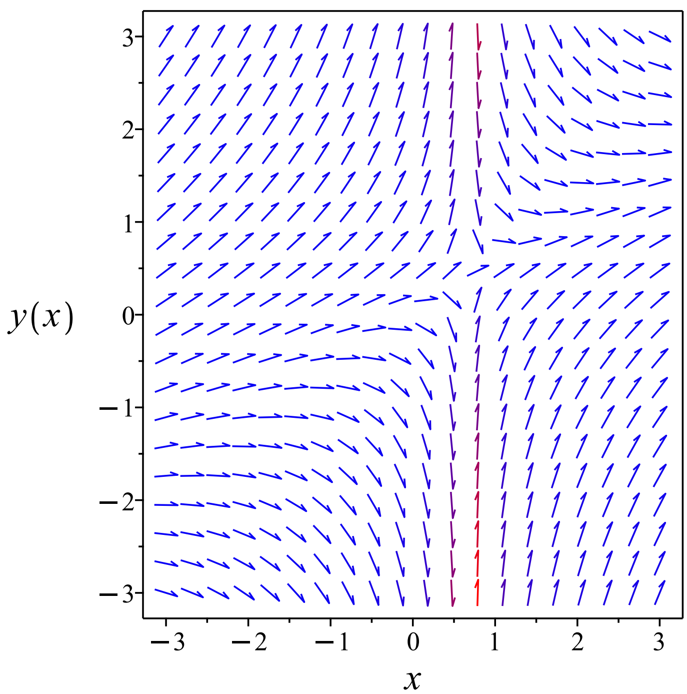


Figure 7: Slope field plot

Verification of solutions

$$y = \frac{x^2 + c_1}{3x - 2}$$

Verified OK.

1.3.6 Maple step by step solution

Let's solve

$3y + (3x-2)\, y' = 2x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{3y}{3x-2} + \frac{2x}{3x-2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{3y}{3x-2} = \frac{2x}{3x-2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' + \frac{3y}{3x-2}\right) = \frac{2\mu(x)x}{3x-2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' + \frac{3y}{3x-2}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)}{3x-2}$

- Solve to find the integrating factor

  $\mu(x) = 3x-2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{2\mu(x)x}{3x-2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{2\mu(x)x}{3x-2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{2\mu(x)x}{3x-2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = 3x-2$

  $y = \frac{\int 2x dx + c_1}{3x-2}$

- Evaluate the integrals on the rhs

  $y = \frac{x^2+c_1}{3x-2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve((3*y(x)-2*x)+(3*x-2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2 + c_1}{-2 + 3x}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 21

```
DSolve[(3*y[x]-2*x)+(3*x-2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2 - c_1}{3x - 2}$$

1.4 problem 1(d)

Internal problem ID [3005]
Internal file name [`OUTPUT/2497_Sunday_June_05_2022_03_16_50_AM_77083438/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 1(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(2yx + y)\, y' = -x^2 - x + 1}$$

1.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x^2 + x - 1}{y\,(1 + 2x)} \end{aligned}$$

Where $f(x) = -\frac{x^2+x-1}{1+2x}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\, dy = -\frac{x^2 + x - 1}{1 + 2x}\, dx$$

$$\int \frac{1}{\frac{1}{y}}\, dy = \int -\frac{x^2 + x - 1}{1 + 2x}\, dx$$

$$\frac{y^2}{2} = -\frac{x^2}{4} - \frac{x}{4} + \frac{5\ln(1 + 2x)}{8} + c_1$$

Which results in

$$y = \frac{\sqrt{-2x^2 + 5\ln(1+2x) + 8c_1} - 2x}{2}$$

$$y = -\frac{\sqrt{-2x^2 + 5\ln(1+2x) + 8c_1} - 2x}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-2x^2 + 5\ln(1+2x) + 8c_1} - 2x}{2} \tag{1}$$

$$y = -\frac{\sqrt{-2x^2 + 5\ln(1+2x) + 8c_1} - 2x}{2} \tag{2}$$

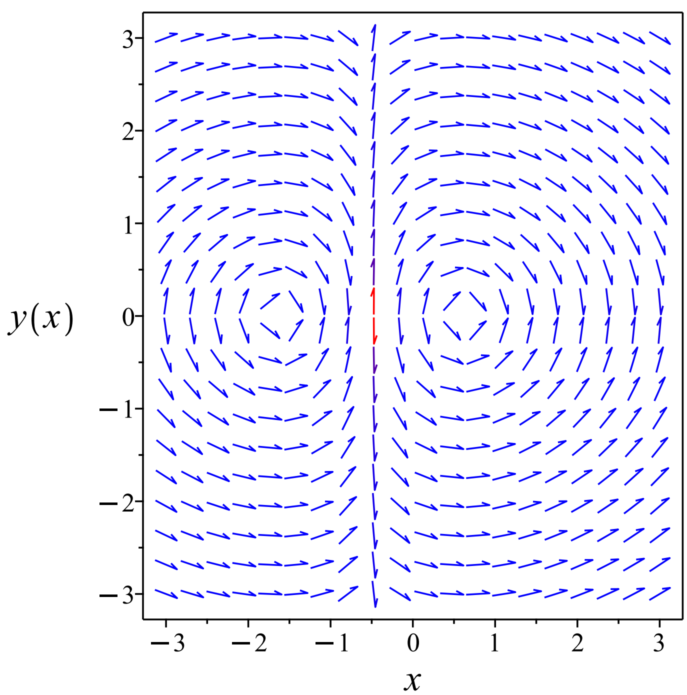


Figure 8: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-2x^2 + 5\ln(1+2x) + 8c_1} - 2x}{2}$$

Verified OK.

$$y = -\frac{\sqrt{-2x^2 + 5\ln(1+2x) + 8c_1} - 2x}{2}$$

Verified OK.

1.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2 + x - 1}{y(1+2x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 6: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{1 + 2x}{x^2 + x - 1} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{1+2x}{x^2+x-1}} dx
\end{aligned}$$

Which results in

$$S = -\frac{x^2}{4} - \frac{x}{4} + \frac{5\ln(1+2x)}{8}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x^2+x-1}{y(1+2x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{-x^2-x+1}{1+2x} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

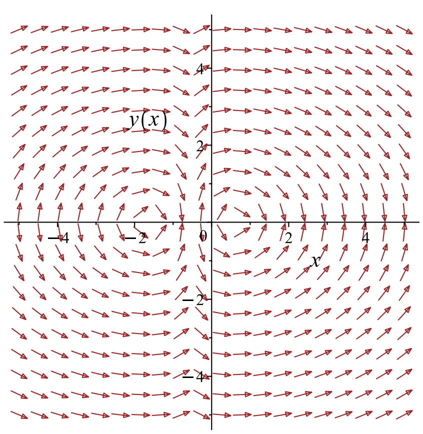
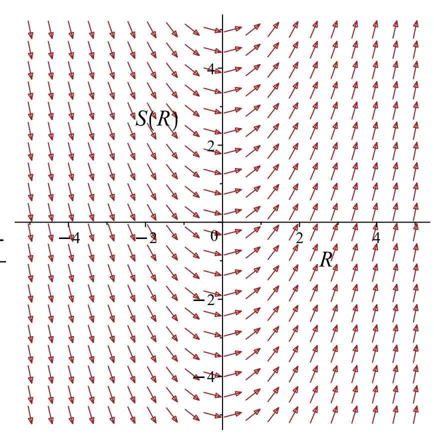
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{4} - \frac{x}{4} + \frac{5\ln(1+2x)}{8} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$-\frac{x^2}{4} - \frac{x}{4} + \frac{5\ln(1+2x)}{8} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{x^2+x-1}{y(1+2x)}$

</td><td>$R = y$
$S = -\frac{x^2}{4} - \frac{x}{4} + \frac{5\ln(1+}{8}$</td><td>$\frac{dS}{dR} = R$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$-\frac{x^2}{4} - \frac{x}{4} + \frac{5\ln(1+2x)}{8} = \frac{y^2}{2} + c_1 \tag{1}$$

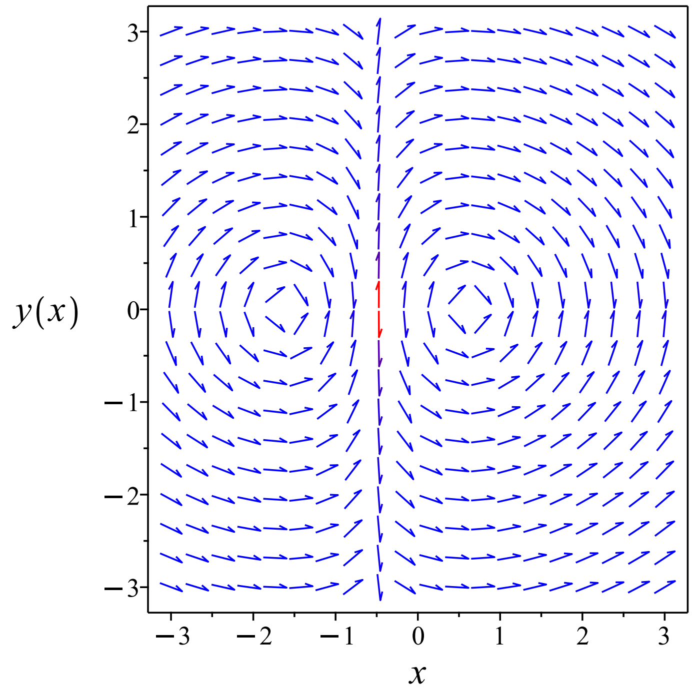


Figure 9: Slope field plot

Verification of solutions

$$-\frac{x^2}{4}-\frac{x}{4}+\frac{5\ln\left(1+2x\right)}{8}=\frac{y^2}{2}+c_1$$

Verified OK.

1.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-y)\,\mathrm{d}y &= \left(\frac{x^2 + x - 1}{1 + 2x}\right)\mathrm{d}x \\ \left(-\frac{x^2 + x - 1}{1 + 2x}\right)\mathrm{d}x + (-y)\,\mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{x^2 + x - 1}{1 + 2x} \\ N(x, y) &= -y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^2 + x - 1}{1 + 2x}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x^2+x-1}{1+2x}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{x^2}{4} - \frac{x}{4} + \frac{5\ln(1+2x)}{8} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y$. Therefore equation (4) becomes

$$-y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-y)\,\mathrm{d}y$$

$$f(y) = -\frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{4} - \frac{x}{4} + \frac{5\ln(1+2x)}{8} - \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{4} - \frac{x}{4} + \frac{5\ln(1+2x)}{8} - \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{4} + \frac{5\ln(1+2x)}{8} - \frac{x}{4} - \frac{y^2}{2} = c_1 \tag{1}$$

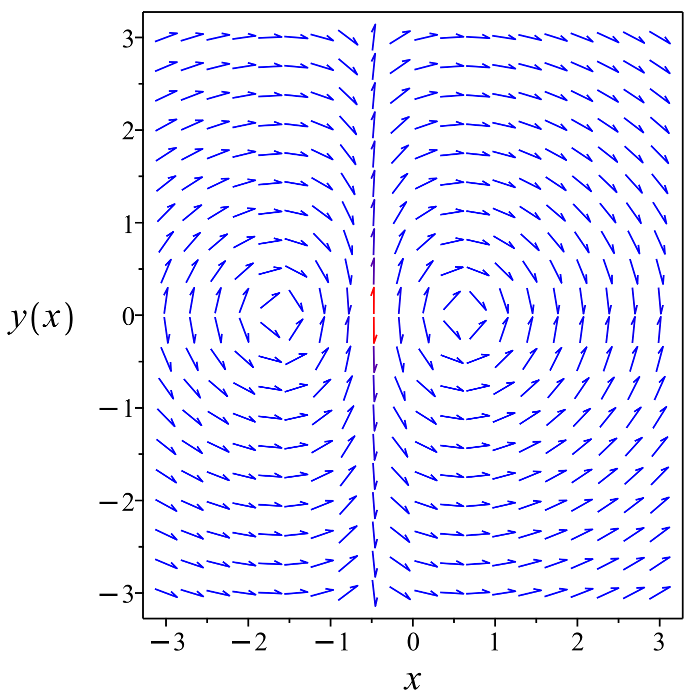


Figure 10: Slope field plot

Verification of solutions

$$-\frac{x^2}{4} + \frac{5\ln(1+2x)}{8} - \frac{x}{4} - \frac{y^2}{2} = c_1$$

Verified OK.

1.4.4 Maple step by step solution

Let's solve

$(2yx + y)\, y' = -x^2 - x + 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = -\frac{x^2+x-1}{1+2x}$

- Integrate both sides with respect to x

  $\int yy'dx = \int -\frac{x^2+x-1}{1+2x}dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = -\frac{x^2}{4} - \frac{x}{4} + \frac{5\ln(1+2x)}{8} + c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{-2x^2+5\ln(1+2x)+8c_1-2x}}{2}, y = \frac{\sqrt{-2x^2+5\ln(1+2x)+8c_1-2x}}{2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 55

```
dsolve((x^2+x-1)+(2*x*y(x)+y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2x^2 + 5\ln(2x+1) + 4c_1 - 2x}}{2}$$
$$y(x) = \frac{\sqrt{-2x^2 + 5\ln(2x+1) + 4c_1 - 2x}}{2}$$

✓ Solution by Mathematica

Time used: 0.119 (sec). Leaf size: 73

```
DSolve[(x^2+x-1)+(2*x*y[x]+y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}\sqrt{-2x^2 - 2x + 5\log(2x+1) - \frac{1}{2} + 8c_1}$$
$$y(x) \to \frac{1}{2}\sqrt{-2x^2 - 2x + 5\log(2x+1) - \frac{1}{2} + 8c_1}$$

1.5 problem 1(e)

Internal problem ID [3006]
Internal file name [`OUTPUT/2498_Sunday_June_05_2022_03_16_53_AM_28461493/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 1(e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{e^{2y} + (x+1)\, y' = 0}$$

1.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{e^{2y}}{x+1} \end{aligned}$$

Where $f(x) = -\frac{1}{x+1}$ and $g(y) = e^{2y}$. Integrating both sides gives

$$\frac{1}{e^{2y}}\, dy = -\frac{1}{x+1}\, dx$$

$$\int \frac{1}{e^{2y}}\, dy = \int -\frac{1}{x+1}\, dx$$

$$-\frac{e^{-2y}}{2} = -\ln(x+1) + c_1$$

Which results in

$$y = \frac{\ln\left(\frac{1}{2\ln(x+1)-2c_1}\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln\left(\frac{1}{2\ln(x+1)-2c_1}\right)}{2} \tag{1}$$

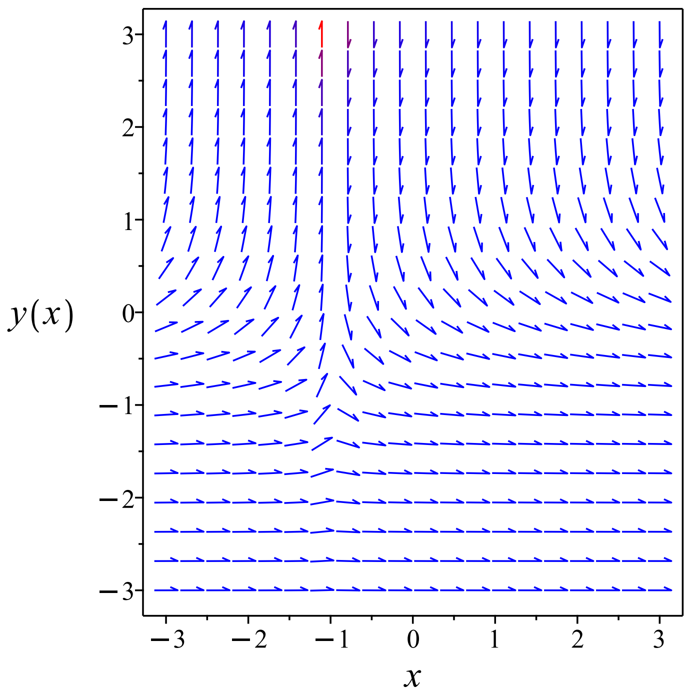


Figure 11: Slope field plot

Verification of solutions

$$y = \frac{\ln\left(\frac{1}{2\ln(x+1)-2c_1}\right)}{2}$$

Verified OK.

1.5.2 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = -\frac{e^{2y}}{x+1} \tag{1}$$

And using the substitution $u = e^{-2y}$ then

$$u' = -2y' e^{-2y}$$

The above shows that

$$\begin{aligned} y' &= -\frac{u'(x)\, e^{2y}}{2} \\ &= -\frac{u'(x)}{2u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{2u} = -\frac{1}{(x+1)\, u}$$

The above simplifies to

$$u'(x) = \frac{2}{x+1} \tag{2}$$

Now ode (2) is solved for $u(x)$ Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{2}{x+1}\, dx \\ &= 2\ln(x+1) + c_1 \end{aligned}$$

Substituting the solution found for $u(x)$ in $u = e^{-2y}$ gives

$$\begin{aligned} y &= -\frac{\ln(u(x))}{2} \\ &= -\frac{\ln(2\ln(x+1) + c_1)}{2} \\ &= -\frac{\ln(2\ln(x+1) + c_1)}{2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln(2\ln(x+1) + c_1)}{2} \tag{1}$$

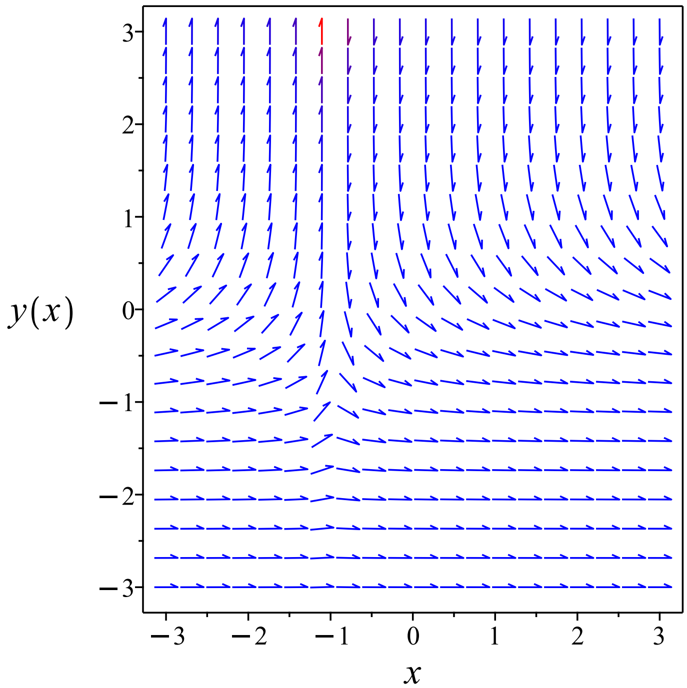


Figure 12: Slope field plot

Verification of solutions

$$y = -\frac{\ln (2\ln (x+1)+c_1)}{2}$$

Verified OK.

1.5.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{e^{2y}}{x+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 9: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -x - 1 \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-x-1} dx \end{aligned}$$

Which results in

$$S = -\ln(-x-1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{e^{2y}}{x+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{-x-1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{-2y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{-2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{e^{-2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(-x-1) = -\frac{e^{-2y}}{2} + c_1$$

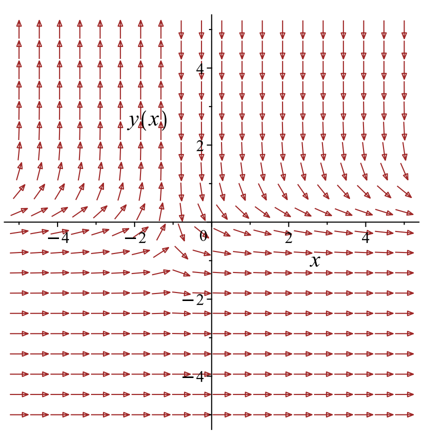
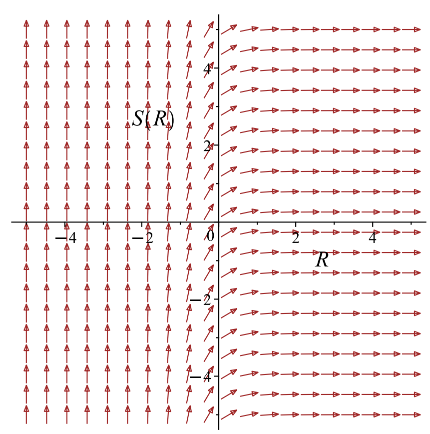
Which simplifies to

$$-\ln(-x-1) = -\frac{e^{-2y}}{2} + c_1$$

Which gives

$$y = -\frac{\ln(2\ln(-x-1) + 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{e^{2y}}{x+1}$ 	$R = y$ $S = -\ln(-x-1)$	$\frac{dS}{dR} = e^{-2R}$ 

Summary
The solution(s) found are the following

$$y = -\frac{\ln\left(2\ln\left(-x-1\right)+2c_1\right)}{2} \tag{1}$$

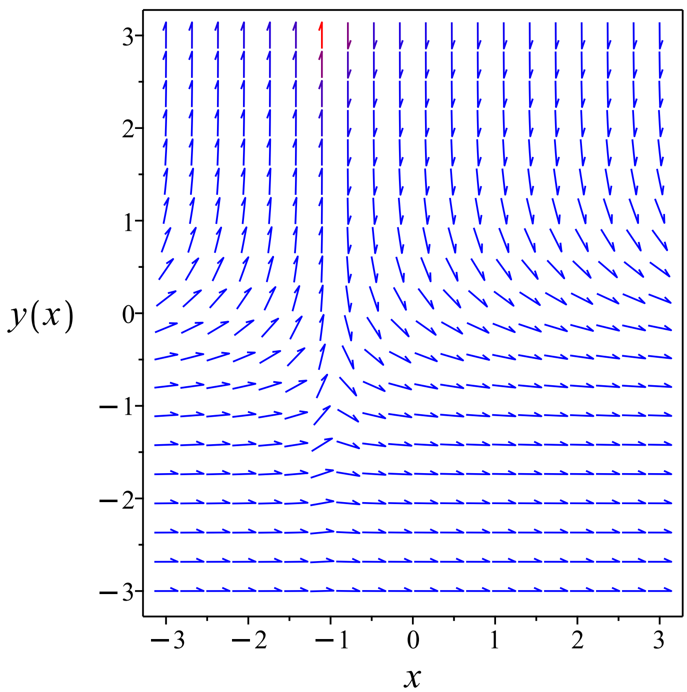


Figure 13: Slope field plot

Verification of solutions

$$y = -\frac{\ln\left(2\ln\left(-x-1\right)+2c_1\right)}{2}$$

Verified OK.

1.5.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\mathrm{e}^{-2y}\right)\mathrm{d}y &= \left(\frac{1}{x+1}\right)\mathrm{d}x \\ \left(-\frac{1}{x+1}\right)\mathrm{d}x + \left(-\mathrm{e}^{-2y}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x+1} \\ N(x, y) &= -\mathrm{e}^{-2y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x+1}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\mathrm{e}^{-2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{x+1} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x+1) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\mathrm{e}^{-2y}$. Therefore equation (4) becomes

$$
-\mathrm{e}^{-2y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= -\mathrm{e}^{-2y} \\
&= -\mathrm{e}^{-2y}
\end{aligned}
$$

Integrating the above w.r.t y results in

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\mathrm{e}^{-2y}\right) \mathrm{d}y \\
f(y) &= \frac{\mathrm{e}^{-2y}}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x+1) + \frac{e^{-2y}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x+1) + \frac{e^{-2y}}{2}$$

The solution becomes

$$y = -\frac{\ln(2\ln(x+1) + 2c_1)}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln(2\ln(x+1) + 2c_1)}{2} \tag{1}$$

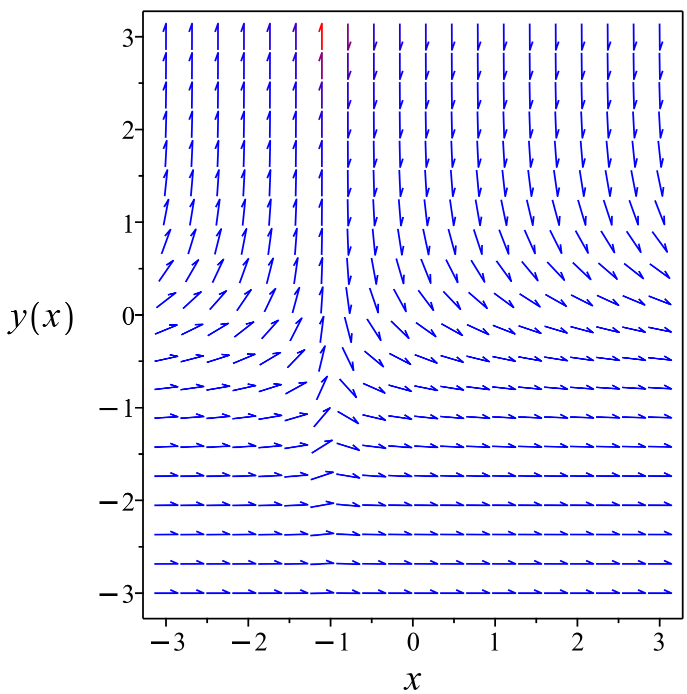


Figure 14: Slope field plot

Verification of solutions

$$y = -\frac{\ln(2\ln(x+1)+2c_1)}{2}$$

Verified OK.

1.5.5 Maple step by step solution

Let's solve

$e^{2y} + (x+1)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{e^{2y}} = -\frac{1}{x+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{e^{2y}}dx = \int -\frac{1}{x+1}dx + c_1$

- Evaluate integral

  $-\frac{1}{2\,e^{2y}} = -\ln(x+1) + c_1$

- Solve for y

  $y = \frac{\ln\left(\frac{1}{2(\ln(x+1)-c_1)}\right)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve(exp(2*y(x))+(1+x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\ln(2)}{2} - \frac{\ln(\ln(x+1) + c_1)}{2}$$

✓ Solution by Mathematica

Time used: 0.376 (sec). Leaf size: 21

```
DSolve[Exp[2*y[x]]+(1+x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}\log(2(\log(x+1) - c_1))$$

1.6 problem 1(f)

Internal problem ID [3007]
Internal file name [`OUTPUT/2499_Sunday_June_05_2022_03_16_55_AM_1089413/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 1(f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(x+1)\, y' - y^2x^2 = 0}$$

1.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{y^2x^2}{x+1} \end{aligned}$$

Where $f(x) = \frac{x^2}{x+1}$ and $g(y) = y^2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y^2}\, dy &= \frac{x^2}{x+1}\, dx\\ \int \frac{1}{y^2}\, dy &= \int \frac{x^2}{x+1}\, dx \end{aligned}$$

$$-\frac{1}{y} = -x + \frac{x^2}{2} + \ln(x+1) + c_1$$

Which results in

$$y = -\frac{2}{x^2 + 2\ln(x+1) + 2c_1 - 2x}$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{x^2 + 2\ln(x+1) + 2c_1 - 2x} \tag{1}$$

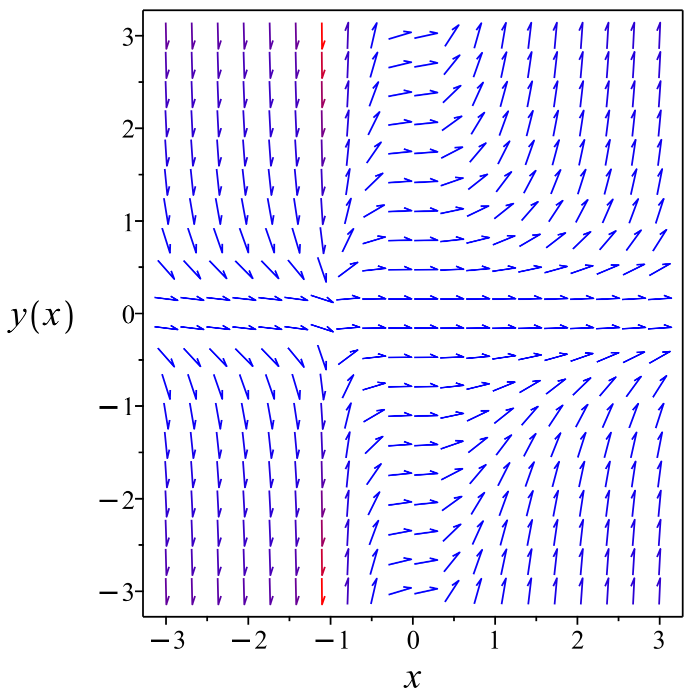


Figure 15: Slope field plot

Verification of solutions

$$y = -\frac{2}{x^2 + 2\ln(x+1) + 2c_1 - 2x}$$

Verified OK.

1.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y^2 x^2}{x+1} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 12: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{x + 1}{x^2} \\
\eta(x, y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{x+1}{x^2}} dx
\end{aligned}
$$

Which results in

$$
S = -x + \frac{x^2}{2} + \ln(x + 1)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \qquad (2)
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{y^2 x^2}{x + 1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{x^2}{x+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{1}{R} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-x + \frac{x^2}{2} + \ln(x+1) = -\frac{1}{y} + c_1
$$

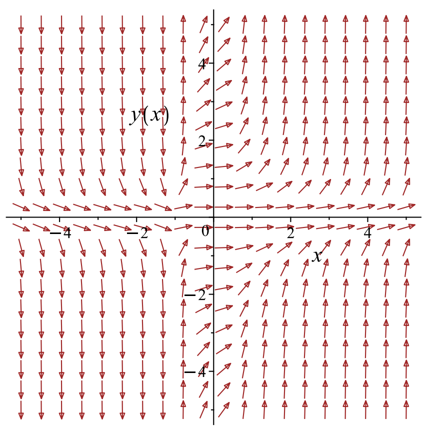
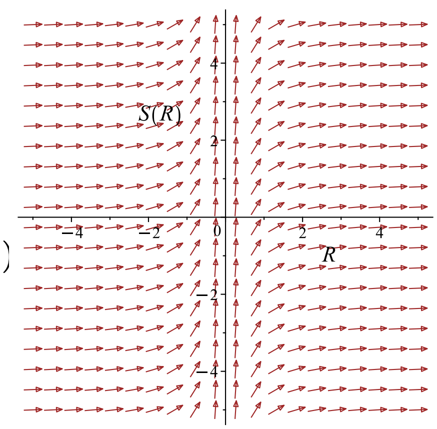
Which simplifies to

$$
-x + \frac{x^2}{2} + \ln(x+1) = -\frac{1}{y} + c_1
$$

Which gives

$$
y = -\frac{2}{x^2 + 2\ln(x+1) - 2c_1 - 2x}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2 x^2}{x+1}$ 	$R = y$ $S = -x + \frac{x^2}{2} + \ln(x+1)$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = -\frac{2}{x^2 + 2\ln(x+1) - 2c_1 - 2x} \tag{1}$$

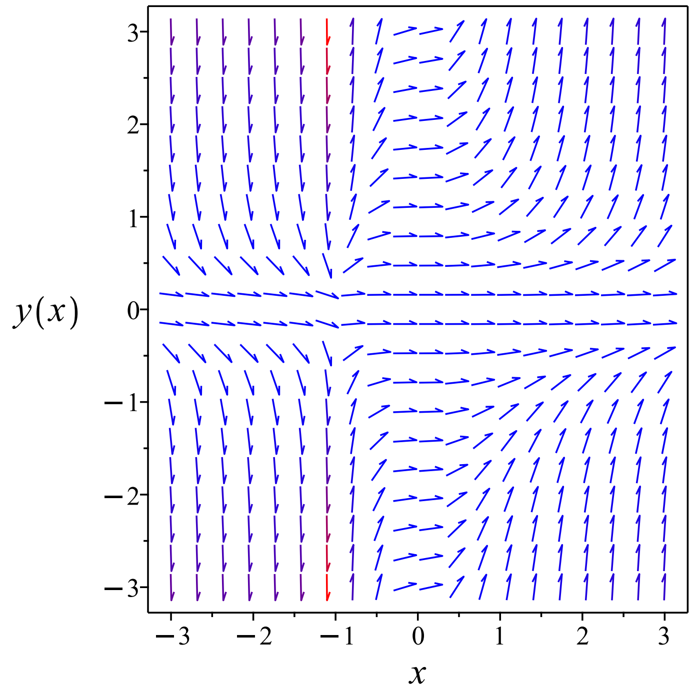


Figure 16: Slope field plot

Verification of solutions

$$y = -\frac{2}{x^2 + 2\ln(x+1) - 2c_1 - 2x}$$

Verified OK.

1.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^2}\right)\mathrm{d}y = \left(\frac{x^2}{x+1}\right)\mathrm{d}x$$
$$\left(-\frac{x^2}{x+1}\right)\mathrm{d}x + \left(\frac{1}{y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x^2}{x+1}$$
$$N(x,y) = \frac{1}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^2}{x+1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2}\right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x^2}{x+1} \, \mathrm{d}x\end{aligned}$$

$$\phi = -\frac{x^2}{2} + x - \ln(x+1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2}\right)\mathrm{d}y$$

$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + x - \ln(x+1) - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + x - \ln(x+1) - \frac{1}{y}$$

The solution becomes

$$y = -\frac{2}{x^2 + 2\ln(x+1) + 2c_1 - 2x}$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{x^2 + 2\ln(x+1) + 2c_1 - 2x} \tag{1}$$

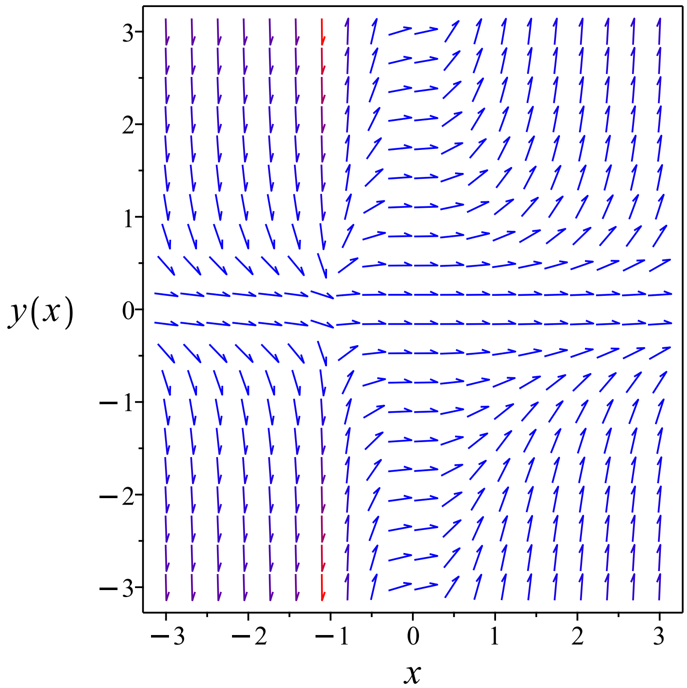


Figure 17: Slope field plot

Verification of solutions

$$y = -\frac{2}{x^2 + 2\ln(x+1) + 2c_1 - 2x}$$

Verified OK.

1.6.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y^2 x^2}{x+1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2 x^2}{x+1}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = \frac{x^2}{x+1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{x^2 u}{x+1}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2x}{x+1} - \frac{x^2}{(x+1)^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{x^2 u''(x)}{x+1} - \left(\frac{2x}{x+1} - \frac{x^2}{(x+1)^2}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 \ln(x+1) + \frac{(x^2 - 2x) c_2}{2} + c_1$$

The above shows that

$$u'(x) = \frac{c_2 x^2}{x+1}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{c_2 \ln(x+1) + \frac{(x^2-2x)c_2}{2} + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{2}{x^2 + 2\ln(x+1) + 2c_3 - 2x}$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{x^2 + 2\ln(x+1) + 2c_3 - 2x} \tag{1}$$

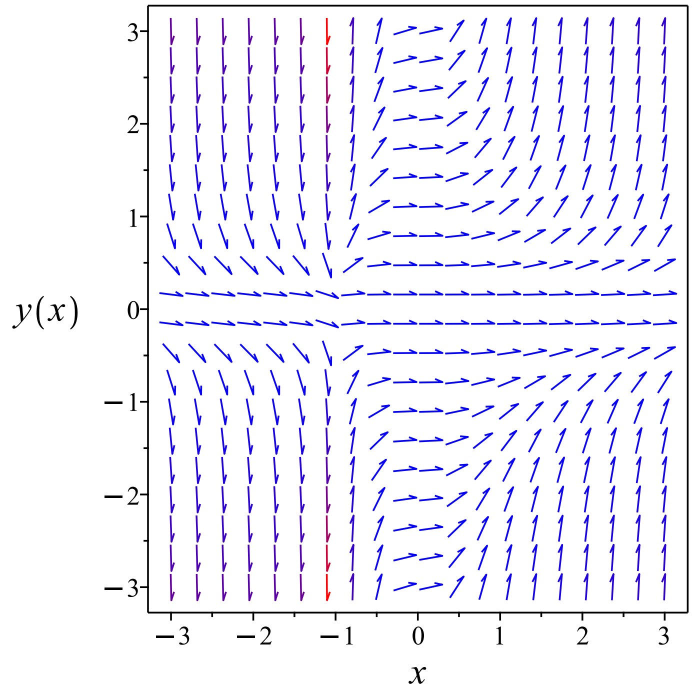


Figure 18: Slope field plot

Verification of solutions

$$y = -\frac{2}{x^2 + 2\ln(x+1) + 2c_3 - 2x}$$

Verified OK.

1.6.5 Maple step by step solution

Let's solve

$(x+1)\,y' - y^2x^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = \frac{x^2}{x+1}$

- Integrate both sides with respect to x

$\int \frac{y'}{y^2} dx = \int \frac{x^2}{x+1} dx + c_1$

- Evaluate integral

$-\frac{1}{y} = -x + \frac{x^2}{2} + \ln(x+1) + c_1$

- Solve for y

$y = -\frac{2}{x^2+2\ln(x+1)+2c_1-2x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve((x+1)*diff(y(x),x)-x^2*y(x)^2=0,y(x), singsol=all)
```

$$y(x) = -\frac{2}{x^2 + 2\ln(x+1) - 2c_1 - 2x}$$

✓ Solution by Mathematica

Time used: 0.162 (sec). Leaf size: 32

```
DSolve[(x+1)*y'[x]-x^2*y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2}{x^2 - 2x + 2\log(x+1) - 3 + 2c_1}$$
$$y(x) \to 0$$

1.7 problem 1(g)

Internal problem ID [3008]
Internal file name [`OUTPUT/2500_Sunday_June_05_2022_03_16_57_AM_75808387/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 1(g).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y - 2x}{x} = 0}$$

1.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = -2$$

Hence the ode is

$$y' - \frac{y}{x} = -2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(-2) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)(-2) \\ d\left(\frac{y}{x}\right) &= \left(-\frac{2}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int -\frac{2}{x} dx \\ \frac{y}{x} &= -2\ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -2\ln(x)\, x + c_1 x$$

which simplifies to

$$y = x(-2\ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = x(-2\ln(x) + c_1) \tag{1}$$

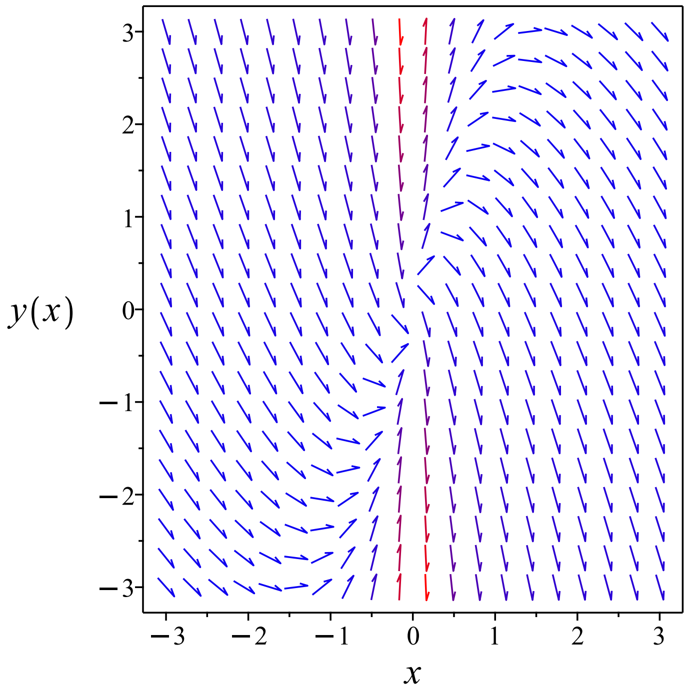


Figure 19: Slope field plot

Verification of solutions

$$y = x(-2\ln(x) + c_1)$$

Verified OK.

1.7.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)\,x - 2x}{x} = 0$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int -\frac{2}{x}\,\mathrm{d}x \\ &= -2\ln(x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x(-2\ln(x) + c_2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x(-2\ln(x) + c_2) \tag{1}$$

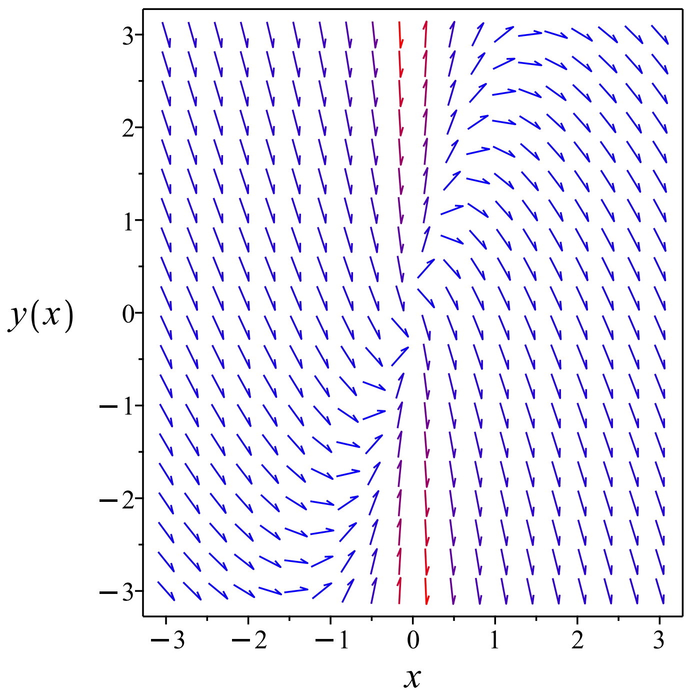


Figure 20: Slope field plot

Verification of solutions

$$y = x(-2\ln(x) + c_2)$$

Verified OK.

1.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-2x + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 15: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= x\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-2x + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = -2\ln(x) + c_1$$

Which simplifies to

$$\frac{y}{x} = -2\ln(x) + c_1$$

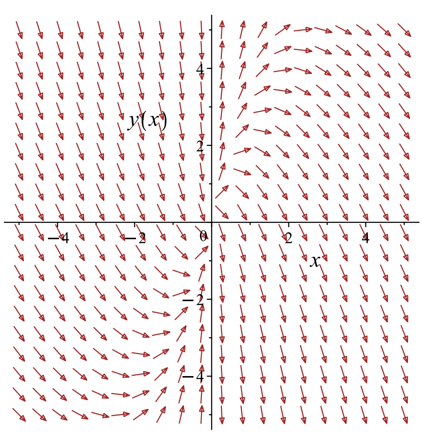
Which gives

$$y = -x(2\ln(x) - c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{-2x+y}{x}$



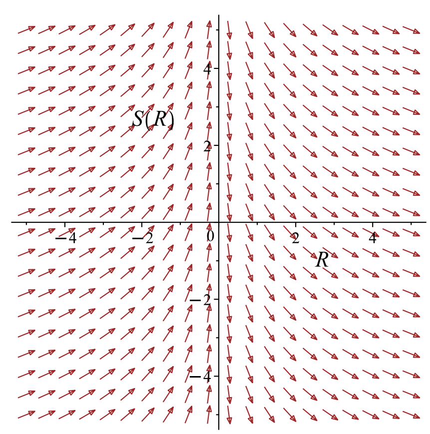
</td><td>

$R = x$

$S = \frac{y}{x}$

</td><td>

$\frac{dS}{dR} = -\frac{2}{R}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -x(2\ln(x) - c_1) \tag{1}$$

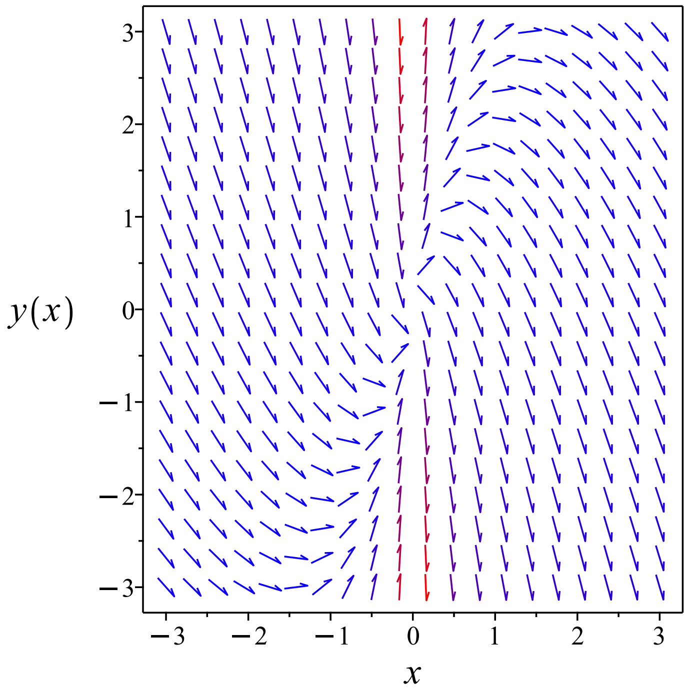


Figure 21: Slope field plot

Verification of solutions

$$y = -x(2\ln(x) - c_1)$$

Verified OK.

1.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{-2x + y}{x}\right) \mathrm{d}x$$
$$\left(-\frac{-2x + y}{x}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{-2x + y}{x}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{-2x + y}{x}\right)\\ &= -\frac{1}{x}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{1}{x}\right) - (0)\right) \\
&= -\frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{1}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(x)} \\
&= \frac{1}{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x}\left(-\frac{-2x + y}{x}\right) \\
&= \frac{2x - y}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x}(1) \\
&= \frac{1}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{2x - y}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{2x - y}{x^2}\,\mathrm{d}x$$
$$\phi = \frac{y}{x} + 2\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y}{x} + 2\ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y}{x} + 2\ln(x)$$

The solution becomes

$$y = -x(2\ln(x) - c_1)$$

Summary

The solution(s) found are the following

$$y = -x(2\ln(x) - c_1) \tag{1}$$

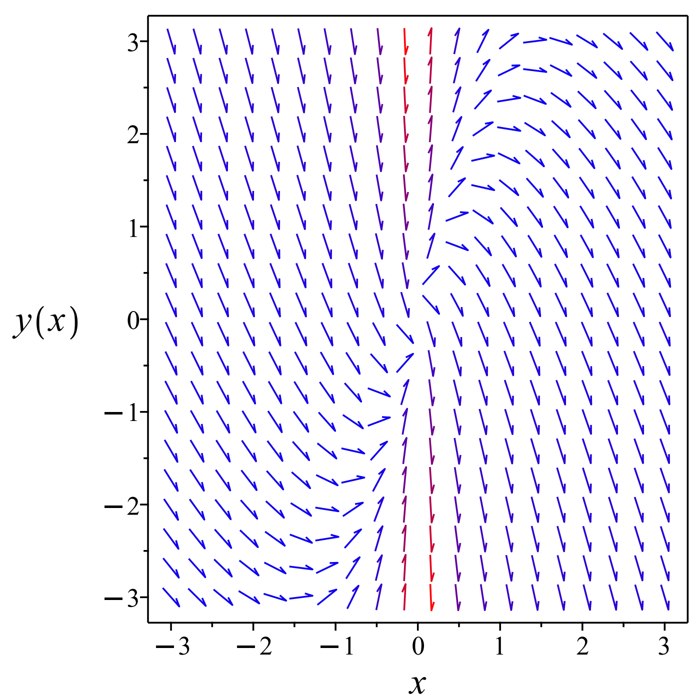


Figure 22: Slope field plot

Verification of solutions

$$y = -x(2\ln(x) - c_1)$$

Verified OK.

1.7.5 Maple step by step solution

Let's solve

$y' - \frac{y-2x}{x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2 + \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = -2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = -2\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -2\mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -2\mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int -2\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int -\frac{2}{x}dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(-2\ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=(y(x)-2*x)/x,y(x), singsol=all)
```

$$y(x) = (-2\ln(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 14

```
DSolve[y'[x]==(y[x]-2*x)/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(-2\log(x) + c_1)$$

1.8 problem 1(h)

Internal problem ID [3009]
Internal file name [`OUTPUT/2501_Sunday_June_05_2022_03_16_59_AM_35539033/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 1(h).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^3 - x y^2 y' = -x^3}$$

1.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^3 x^3 - x^3 u(x)^2 \left(u'(x)\,x + u(x)\right) = -x^3$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{1}{u^2 x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{1}{u^2}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{u^2}}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{\frac{1}{u^2}}\, du &= \int \frac{1}{x}\, dx \\
\frac{u^3}{3} &= \ln(x) + c_2
\end{aligned}
$$

The solution is

$$
\frac{u(x)^3}{3} - \ln(x) - c_2 = 0
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\frac{y^3}{3x^3} - \ln(x) - c_2 = 0 \\
\frac{y^3}{3x^3} - \ln(x) - c_2 = 0
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\frac{y^3}{3x^3} - \ln(x) - c_2 = 0 \tag{1}
$$

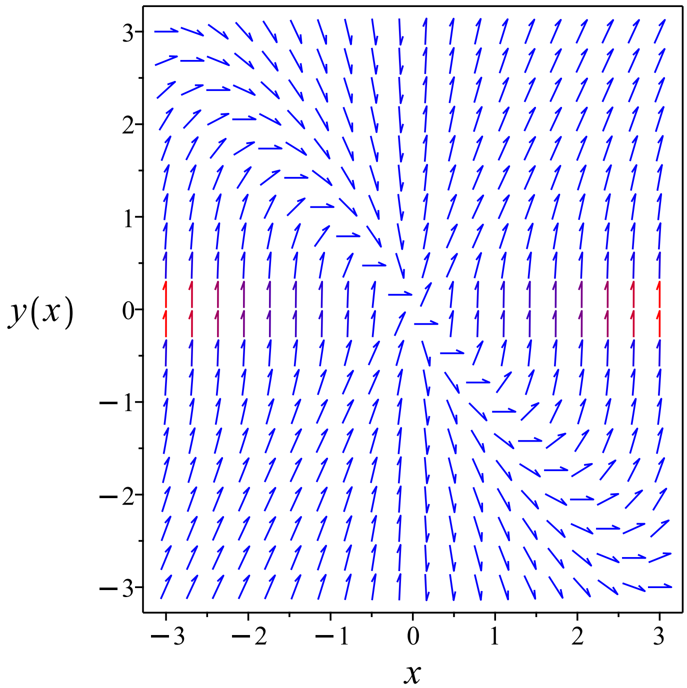


Figure 23: Slope field plot

Verification of solutions

$$\frac{y^3}{3x^3} - \ln(x) - c_2 = 0$$

Verified OK.

1.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^3 + y^3}{x\,y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 18: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \frac{x^3}{y^2} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^3}{y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3}{3x^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^3 + y^3}{x\,y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^3}{x^4} \\ S_y &= \frac{y^2}{x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

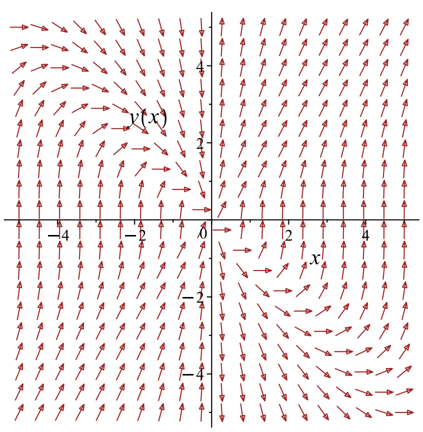
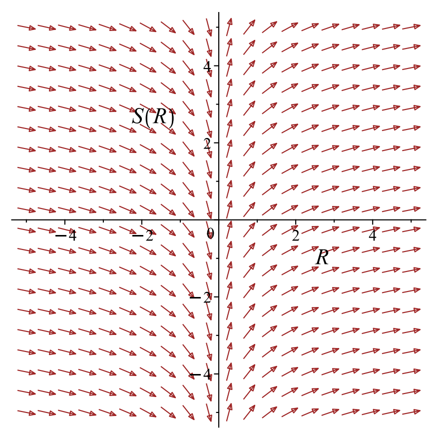
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3}{3x^3} = \ln(x) + c_1$$

Which simplifies to

$$\frac{y^3}{3x^3} = \ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3+y^3}{x\,y^2}$ 	$R = x$ $S = \frac{y^3}{3x^3}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$\frac{y^3}{3x^3} = \ln(x) + c_1 \tag{1}$$

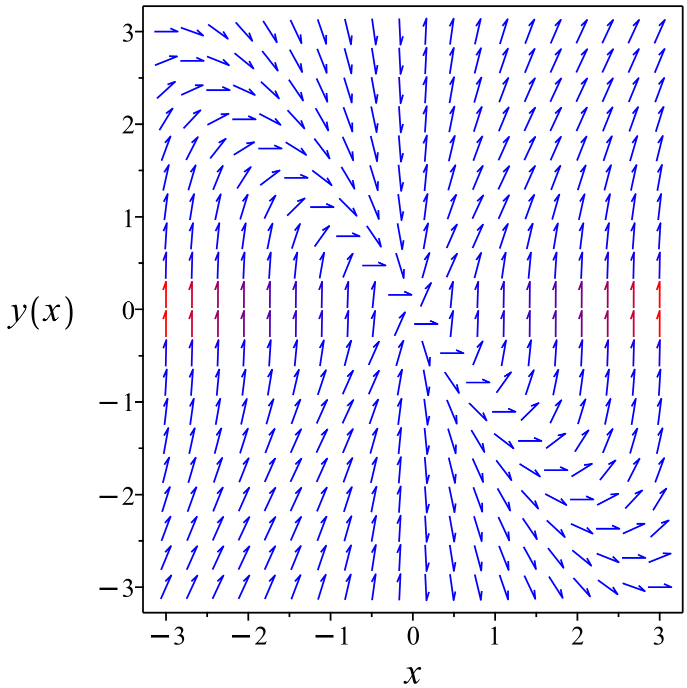


Figure 24: Slope field plot

<u>Verification of solutions</u>

$$\frac{y^3}{3x^3} = \ln(x) + c_1$$

Verified OK.

1.8.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^3 + y^3}{x\,y^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y + x^2\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= x^2 \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = \frac{y^3}{x} + x^2 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= \frac{w(x)}{x} + x^2 \\ w' &= \frac{3w}{x} + 3x^2 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= 3x^2 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{3w(x)}{x} = 3x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{3}{x}dx} \\ &= \frac{1}{x^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(3x^2\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(3x^2\right) \\ \mathrm{d}\left(\frac{w}{x^3}\right) &= \left(\frac{3}{x}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^3} &= \int \frac{3}{x}\,\mathrm{d}x \\ \frac{w}{x^3} &= 3\ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$w(x) = 3x^3 \ln(x) + c_1 x^3$$

which simplifies to

$$w(x) = x^3(3\ln(x) + c_1)$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = x^3(3\ln(x) + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= (3\ln(x) + c_1)^{\frac{1}{3}}\,x \\ y(x) &= \frac{(3\ln(x) + c_1)^{\frac{1}{3}}\left(-1 + i\sqrt{3}\right)x}{2} \\ y(x) &= -\frac{(3\ln(x) + c_1)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)x}{2}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = (3\ln(x)+c_1)^{\frac{1}{3}} x \tag{1}$$

$$y = \frac{(3\ln(x)+c_1)^{\frac{1}{3}} \left(-1+i\sqrt{3}\right) x}{2} \tag{2}$$

$$y = -\frac{(3\ln(x)+c_1)^{\frac{1}{3}} \left(1+i\sqrt{3}\right) x}{2} \tag{3}$$

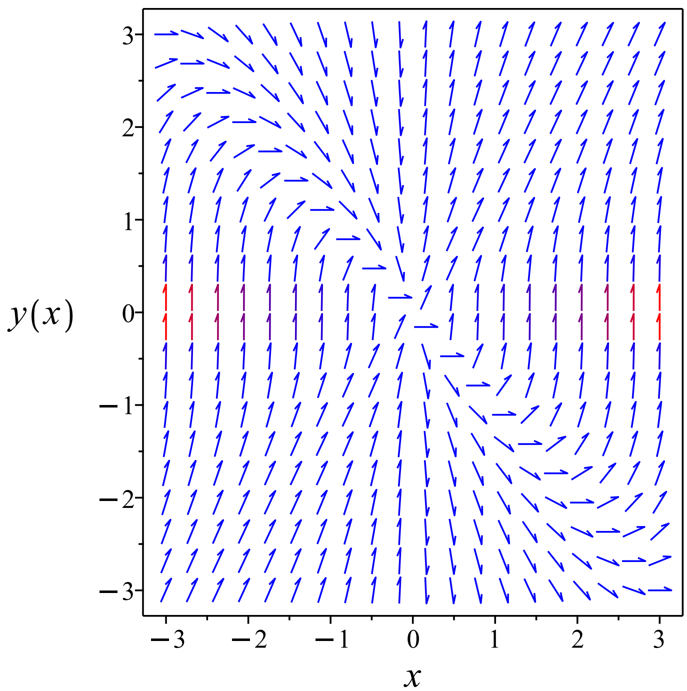


Figure 25: Slope field plot

Verification of solutions

$$y = (3\ln(x) + c_1)^{\frac{1}{3}} x$$

Verified OK.

$$y = \frac{(3\ln(x) + c_1)^{\frac{1}{3}} \left(-1 + i\sqrt{3}\right) x}{2}$$

Verified OK.

$$y = -\frac{(3\ln(x) + c_1)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right) x}{2}$$

Verified OK.

1.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-x\,y^2\right)\mathrm{d}y &= \left(-x^3 - y^3\right)\mathrm{d}x\\ \left(x^3 + y^3\right)\mathrm{d}x + \left(-x\,y^2\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x^3 + y^3\\ N(x,y) &= -x\,y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^3 + y^3\right)\\ &= 3y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x\,y^2\right)\\ &= -y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{x\,y^2}\left(\left(3y^2\right) - \left(-y^2\right)\right)\\ &= -\frac{4}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{4}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4\ln(x)}\\ &= \frac{1}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^4}\left(x^3+y^3\right)\\ &= \frac{x^3+y^3}{x^4}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^4}\left(-x\,y^2\right)\\ &= -\frac{y^2}{x^3}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{x^3+y^3}{x^4}\right)+\left(-\frac{y^2}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{x^3 + y^3}{x^4} \mathrm{d}x$$
$$\phi = -\frac{y^3}{3x^3} + \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{y^2}{x^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y^2}{x^3}$. Therefore equation (4) becomes

$$-\frac{y^2}{x^3} = -\frac{y^2}{x^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y^3}{3x^3} + \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y^3}{3x^3} + \ln(x)$$

Summary
The solution(s) found are the following

$$-\frac{y^3}{3x^3} + \ln(x) = c_1 \tag{1}$$

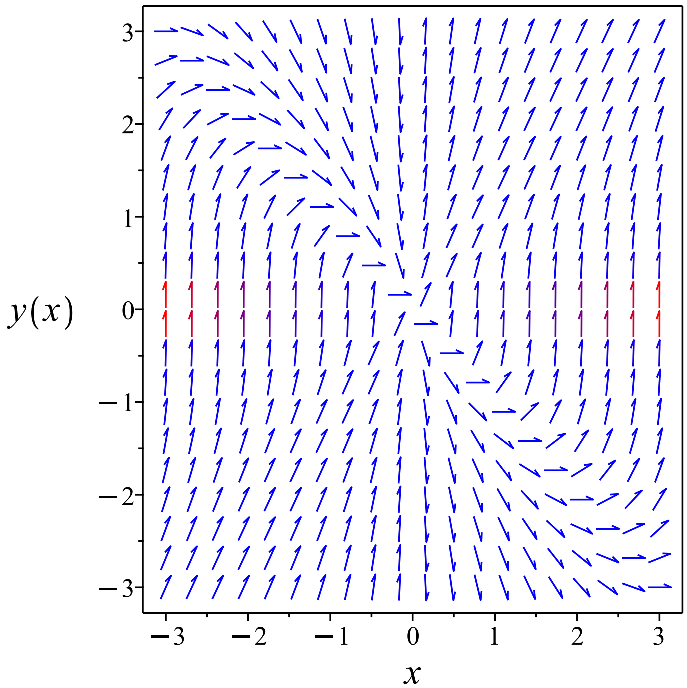


Figure 26: Slope field plot

Verification of solutions

$$-\frac{y^3}{3x^3}+\ln(x)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
dsolve((x^3+y(x)^3)-x*y(x)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (3\ln(x) + c_1)^{\frac{1}{3}} x$$
$$y(x) = -\frac{(3\ln(x) + c_1)^{\frac{1}{3}} (1 + i\sqrt{3}) x}{2}$$
$$y(x) = \frac{(3\ln(x) + c_1)^{\frac{1}{3}} (i\sqrt{3} - 1) x}{2}$$

✓ Solution by Mathematica

Time used: 0.193 (sec). Leaf size: 63

```
DSolve[(x^3+y[x]^3)-x*y[x]^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\sqrt[3]{3\log(x) + c_1}$$
$$y(x) \to -\sqrt[3]{-1}x\sqrt[3]{3\log(x) + c_1}$$
$$y(x) \to (-1)^{2/3}x\sqrt[3]{3\log(x) + c_1}$$

1.9 problem 1(i)

Internal problem ID [3010]
Internal file name [`OUTPUT/2502_Sunday_June_05_2022_03_17_03_AM_30374513/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 1(i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y = 0}$$

1.9.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{y} dy = \int dx$$
$$-\ln(y) = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{y} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{y} = c_2 e^x$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-x}}{c_2} \tag{1}$$

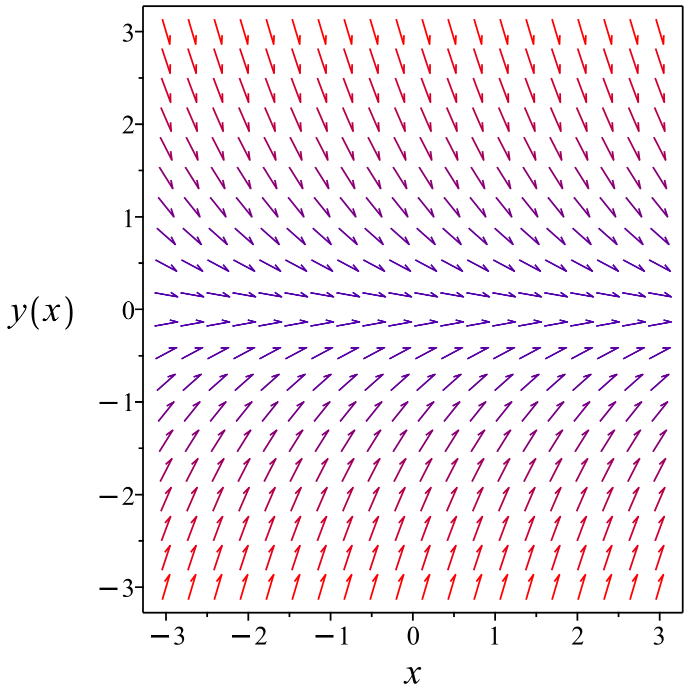


Figure 27: Slope field plot

Verification of solutions

$$y = \frac{e^{-x}}{c_2}$$

Verified OK.

1.9.2 Maple step by step solution

Let's solve

$y' + y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int (-1)\, dx + c_1$

- Evaluate integral

$$\ln(y) = -x + c_1$$

- Solve for y

$$y = e^{-x+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}c_1$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 18

```
DSolve[y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x}$$
$$y(x) \to 0$$

1.10 problem 1(j)

Internal problem ID [3011]
Internal file name [OUTPUT/2503_Sunday_June_05_2022_03_17_04_AM_34495165/index.tex]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 1(j).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = x^2 + 2}$$

1.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x^2 + 2 \end{aligned}$$

Hence the ode is

$$y' + y = x^2 + 2$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1 dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x^2+2\right)$$
$$\frac{d}{dx}(e^x y) = (e^x)\left(x^2+2\right)$$
$$d(e^x y) = \left(\left(x^2+2\right)e^x\right)\, dx$$

Integrating gives

$$e^x y = \int \left(x^2+2\right)e^x \, dx$$
$$e^x y = \left(x^2-2x+4\right)e^x + c_1$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(x^2-2x+4\right)e^x + c_1 e^{-x}$$

which simplifies to

$$y = x^2 - 2x + 4 + c_1 e^{-x}$$

Summary
The solution(s) found are the following

$$y = x^2 - 2x + 4 + c_1 e^{-x} \tag{1}$$

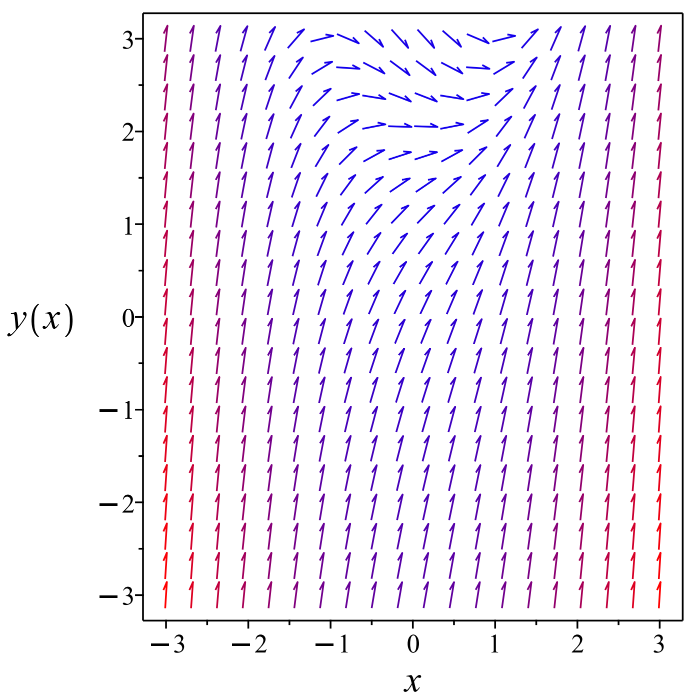


Figure 28: Slope field plot

Verification of solutions

$$y = x^2 - 2x + 4 + c_1 \mathrm{e}^{-x}$$

Verified OK.

1.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= x^2 - y + 2 \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 21: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy \end{aligned}$$

Which results in

$$S = e^x y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x^2 - y + 2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^x y \\ S_y &= e^x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (x^2 + 2) e^x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (R^2 + 2) e^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = (R^2 - 2R + 4)\, e^R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^x y = (x^2 - 2x + 4)\, e^x + c_1$$

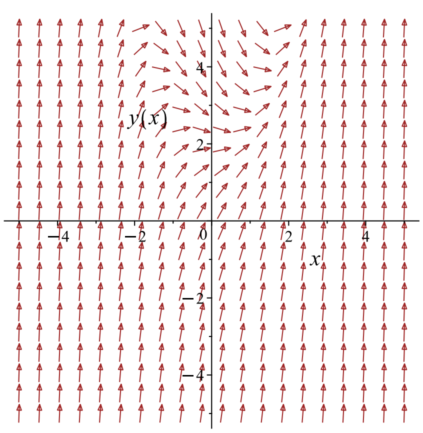
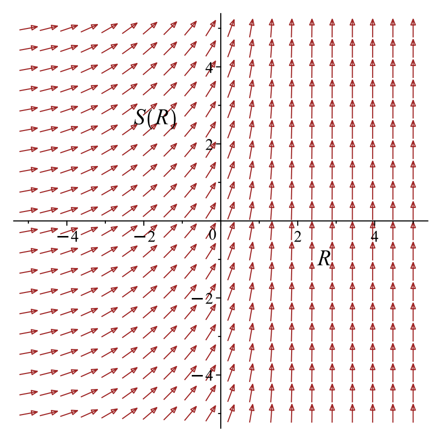
Which simplifies to

$$e^x y = (x^2 - 2x + 4)\, e^x + c_1$$

Which gives

$$y = (x^2 e^x - 2x\, e^x + 4\, e^x + c_1)\, e^{-x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2 - y + 2$ 	$R = x$ $S = e^x y$	$\frac{dS}{dR} = (R^2 + 2)\, e^R$ 

Summary
The solution(s) found are the following

$$y = (x^2 e^x - 2x\, e^x + 4\, e^x + c_1)\, e^{-x} \tag{1}$$

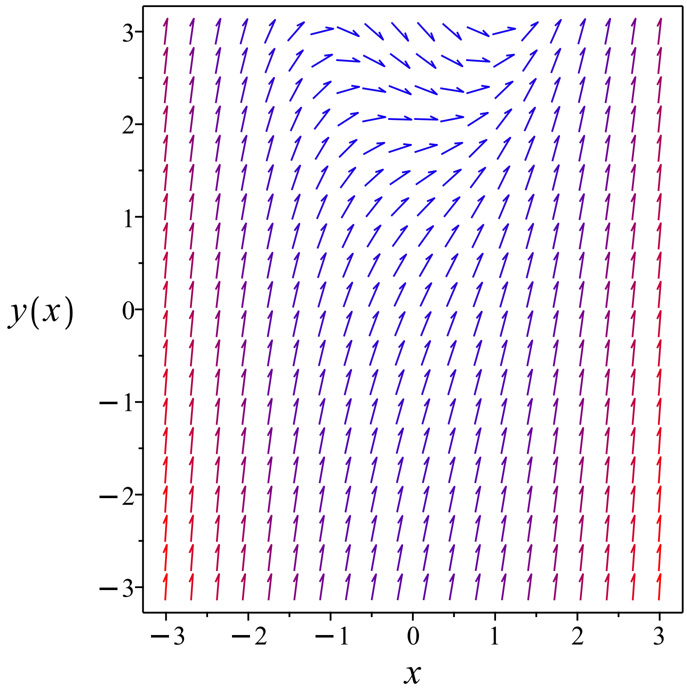


Figure 29: Slope field plot

<u>Verification of solutions</u>

$$y = \left(x^2 \mathrm{e}^x - 2x\,\mathrm{e}^x + 4\,\mathrm{e}^x + c_1\right)\mathrm{e}^{-x}$$

Verified OK.

1.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(x^2 - y + 2\right)\mathrm{d}x\\ \left(-x^2 + y - 2\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^2 + y - 2\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 + y - 2\right)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((1) - (0)) \\
&= 1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x}\left(-x^{2} + y - 2\right) \\
&= -\mathrm{e}^{x}\left(x^{2} - y + 2\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(1) \\
&= \mathrm{e}^{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{x}\left(x^{2} - y + 2\right)\right) + \left(\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\mathrm{e}^{x}\left(x^{2}-y+2\right)\mathrm{d}x$$
$$\phi = -\left(x^{2}-2x-y+4\right)\mathrm{e}^{x}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\left(x^{2}-2x-y+4\right)\mathrm{e}^{x}+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\left(x^{2}-2x-y+4\right)\mathrm{e}^{x}$$

The solution becomes

$$y = \left(x^{2}\mathrm{e}^{x}-2x\,\mathrm{e}^{x}+4\,\mathrm{e}^{x}+c_1\right)\mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = \left(x^{2}\mathrm{e}^{x}-2x\,\mathrm{e}^{x}+4\,\mathrm{e}^{x}+c_1\right)\mathrm{e}^{-x} \tag{1}$$

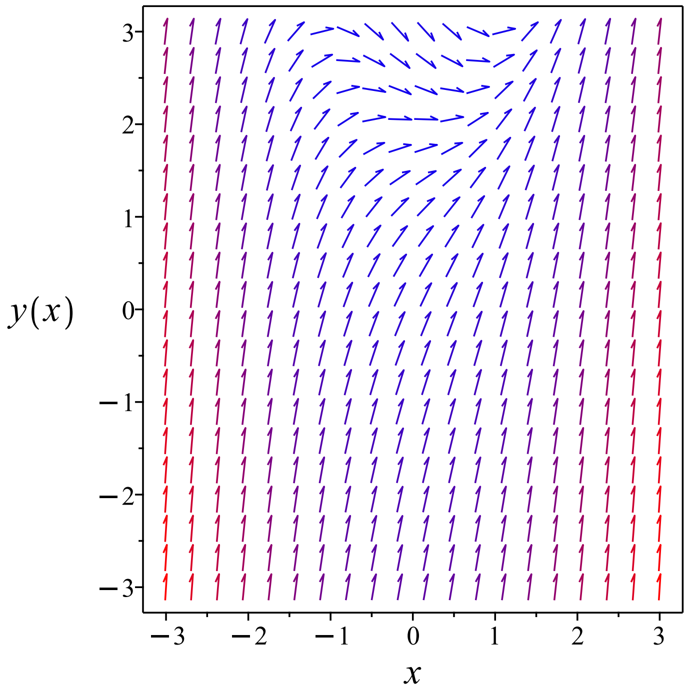


Figure 30: Slope field plot

Verification of solutions

$$y = \left(x^2 e^x - 2x\, e^x + 4\, e^x + c_1\right) e^{-x}$$

Verified OK.

1.10.4 Maple step by step solution

Let's solve

$y' + y = x^2 + 2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + x^2 + 2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = x^2 + 2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + y) = \mu(x)\,(x^2 + 2)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'+y) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,(x^2+2)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,(x^2+2)\,dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)(x^2+2)\,dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int (x^2+2)\mathrm{e}^{x}\,dx + c_1}{\mathrm{e}^{x}}$

- Evaluate the integrals on the rhs

  $y = \frac{(x^2-2x+4)\mathrm{e}^{x} + c_1}{\mathrm{e}^{x}}$

- Simplify

  $y = x^2 - 2x + 4 + c_1\mathrm{e}^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x)+y(x)=x^2+2,y(x), singsol=all)
```

$$y(x) = x^2 - 2x + 4 + e^{-x} c_1$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 21

```
DSolve[y'[x]+y[x]==x^2+2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow x^2 - 2x + c_1 e^{-x} + 4$$

1.11 problem 2(a)

Internal problem ID [3012]
Internal file name [`OUTPUT/2504_Sunday_June_05_2022_03_17_06_AM_75331586/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - y\tan(x) = x}$$

With initial conditions

$$[y(0) = 0]$$

1.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\tan(x)$$
$$q(x) = x$$

Hence the ode is

$$y' - y \tan(x) = x$$

The domain of $p(x) = -\tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z142 \lor \frac{1}{2}\pi + \pi\_Z142 < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.11.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\tan(x)dx} \\ &= \cos(x) \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)(x) \\ \frac{d}{dx}(\cos(x)\, y) &= (\cos(x))(x) \\ d(\cos(x)\, y) &= (\cos(x)\, x)\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \cos(x)\, y &= \int \cos(x)\, x\, dx \\ \cos(x)\, y &= x \sin(x) + \cos(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \cos(x)$ results in

$$y = \sec(x)(x \sin(x) + \cos(x)) + c_1 \sec(x)$$

which simplifies to

$$y = \tan(x)\, x + 1 + c_1 \sec(x)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + 1$$

$$c_1 = -1$$

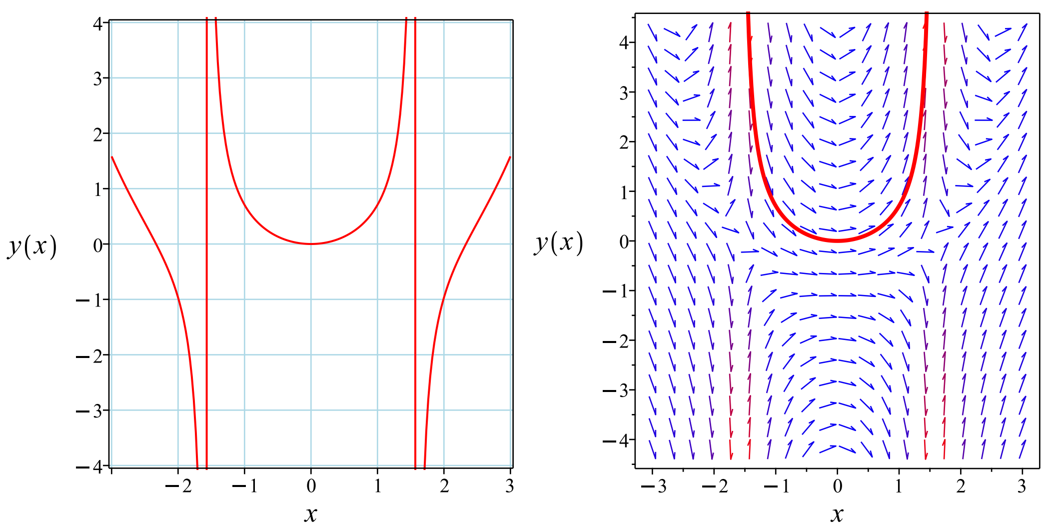
Substituting c_1 found above in the general solution gives

$$y = 1 + \tan(x)\, x - \sec(x)$$

Summary

The solution(s) found are the following

$$y = 1 + \tan(x)\, x - \sec(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 1 + \tan(x)\, x - \sec(x)$$

Verified OK.

1.11.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \tan(x)\, y + x \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 24: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{\cos\left(x\right)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\cos(x)}} dy \end{aligned}$$

Which results in

$$S = \cos\left(x\right) y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \tan\left(x\right) y + x$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= -\sin(x)\, y\\
S_y &= \cos(x)
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \cos(x)\, x \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \cos(R)\, R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \cos(R) + \sin(R)\, R + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\cos(x)\, y = x \sin(x) + \cos(x) + c_1
$$

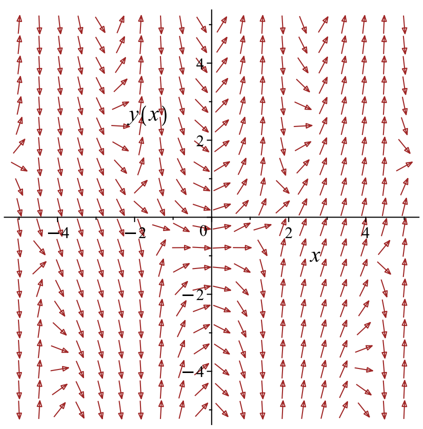
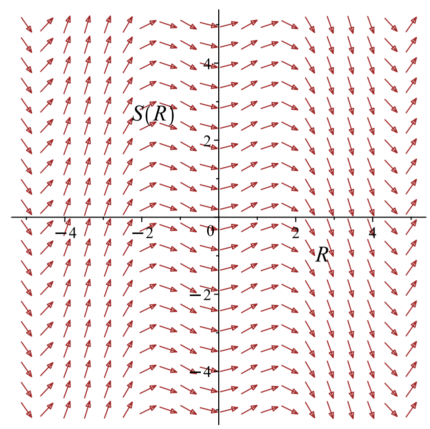
Which simplifies to

$$
\cos(x)\, y = x \sin(x) + \cos(x) + c_1
$$

Which gives

$$
y = \frac{x \sin(x) + \cos(x) + c_1}{\cos(x)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \tan(x) y + x$
</td><td>$R = x$
$S = \cos(x) y$</td><td>$\frac{dS}{dR} = \cos(R) R$
</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + 1$$

$$c_1 = -1$$

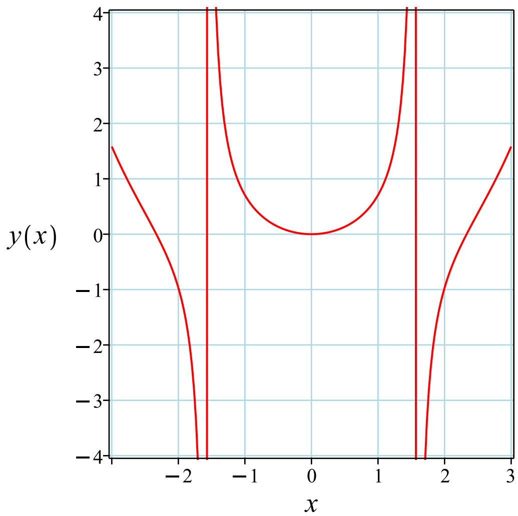
Substituting c_1 found above in the general solution gives

$$y = 1 + \tan(x) x - \sec(x)$$

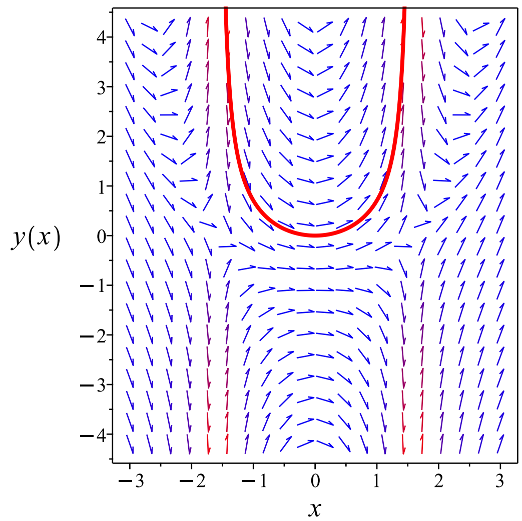
<u>Summary</u>

The solution(s) found are the following

$$y = 1 + \tan(x) x - \sec(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 + \tan(x)\, x - \sec(x)$$

Verified OK.

1.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= (\tan(x)\,y + x)\,\mathrm{d}x \\ (-\tan(x)\,y - x)\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\tan(x)\,y - x \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\tan(x)\,y - x) \\ &= -\tan(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-\tan(x)) - (0)) \\ &= -\tan(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int -\tan(x) \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(\cos(x))} \\
&= \cos(x)
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \cos(x)\,(-\tan(x)\,y - x) \\
&= -\cos(x)\,x - \sin(x)\,y
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \cos(x)\,(1) \\
&= \cos(x)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(-\cos(x)\,x - \sin(x)\,y) + (\cos(x))\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\cos(x)\,x - \sin(x)\,y \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = (y-1)\cos(x) - x\sin(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(x)$. Therefore equation (4) becomes

$$\cos(x) = \cos(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (y - 1)\cos(x) - x\sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y - 1)\cos(x) - x\sin(x)$$

The solution becomes

$$y = \frac{x\sin(x) + \cos(x) + c_1}{\cos(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + 1$$

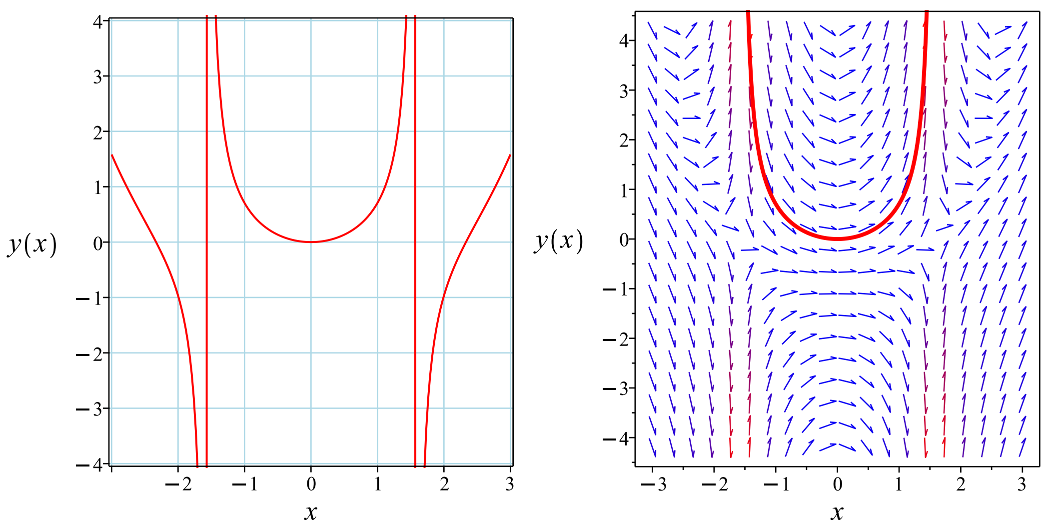
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$y = 1 + \tan(x)\,x - \sec(x)$$

Summary
The solution(s) found are the following

$$y = 1 + \tan(x)\, x - \sec(x) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 1 + \tan(x)\, x - \sec(x)$$

Verified OK.

1.11.5 Maple step by step solution

Let's solve

$[y' - y\tan(x) = x, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y\tan(x) + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y\tan(x) = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y'-y\tan\left(x\right)\right)=\mu(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y'-y\tan\left(x\right)\right)=\mu'(x)\, y+\mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x)=-\mu(x)\tan\left(x\right)$

- Solve to find the integrating factor

  $\mu(x)=\cos\left(x\right)$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\, y)\right)dx=\int\mu(x)\, xdx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y=\int\mu(x)\, xdx+c_1$

- Solve for y

  $y=\frac{\int\mu(x)xdx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\cos\left(x\right)$

  $y=\frac{\int\cos(x)xdx+c_1}{\cos(x)}$

- Evaluate the integrals on the rhs

  $y=\frac{x\sin(x)+\cos(x)+c_1}{\cos(x)}$

- Simplify

  $y=\tan\left(x\right)x+1+c_1\sec\left(x\right)$

- Use initial condition $y(0)=0$

  $0=c_1+1$

- Solve for c_1

  $c_1=-1$

- Substitute $c_1=-1$ into general solution and simplify

  $y=1+\tan\left(x\right)x-\sec\left(x\right)$

- Solution to the IVP

  $y=1+\tan\left(x\right)x-\sec\left(x\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)-y(x)*tan(x)=x,y(0) = 0],y(x), singsol=all)
```

$$y(x) = 1 + \tan(x) x - \sec(x)$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 15

```
DSolve[{y'[x]-y[x]*Tan[x]==x,y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \tan(x) - \sec(x) + 1$$

1.12 problem 2(b)

Internal problem ID [3013]
Internal file name [`OUTPUT/2505_Sunday_June_05_2022_03_17_09_AM_46567828/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - e^{x-2y} = 0}$$

With initial conditions

$$[y(0) = 0]$$

1.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= e^{x-2y} \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{x-2y}\right) \\ &= -2\,\mathrm{e}^{x-2y}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

1.12.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \mathrm{e}^{x}\mathrm{e}^{-2y}\end{aligned}$$

Where $f(x) = \mathrm{e}^{x}$ and $g(y) = \mathrm{e}^{-2y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\mathrm{e}^{-2y}}\,dy &= \mathrm{e}^{x}\,dx \\ \int \frac{1}{\mathrm{e}^{-2y}}\,dy &= \int \mathrm{e}^{x}\,dx \\ \frac{\mathrm{e}^{2y}}{2} &= \mathrm{e}^{x} + c_1\end{aligned}$$

Which results in

$$y = \frac{\ln\left(2\,\mathrm{e}^{x} + 2c_1\right)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\ln(2)}{2} + \frac{\ln(c_1 + 1)}{2}$$

$$c_1 = -\frac{1}{2}$$

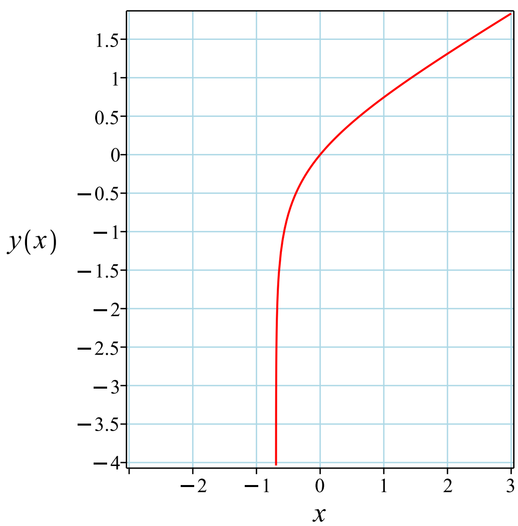
Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(2\,\mathrm{e}^x - 1)}{2}$$

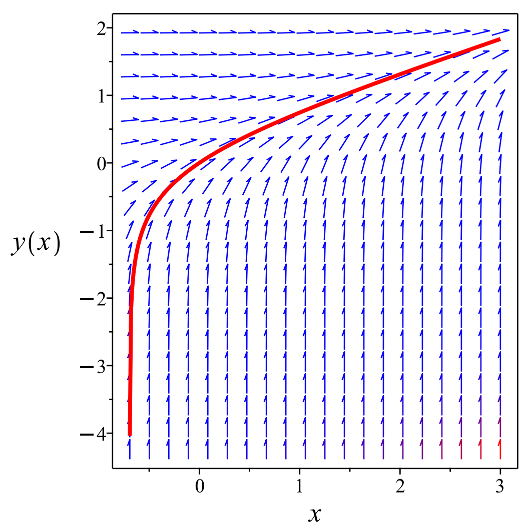
Summary

The solution(s) found are the following

$$y = \frac{\ln(2\,\mathrm{e}^x - 1)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\ln(2\,\mathrm{e}^x - 1)}{2}$$

Verified OK.

1.12.3 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = e^{x-2y} \tag{1}$$

And using the substitution $u = e^{2y}$ then

$$u' = 2y' e^{2y}$$

The above shows that

$$\begin{aligned} y' &= \frac{u'(x)\,e^{-2y}}{2} \\ &= \frac{u'(x)}{2u} \end{aligned}$$

Substituting this in (1) gives

$$\frac{u'(x)}{2u} = \frac{e^{x}}{u}$$

The above simplifies to

$$u'(x) = 2\,e^{x} \tag{2}$$

Now ode (2) is solved for $u(x)$ Integrating both sides gives

$$\begin{aligned} u(x) &= \int 2\,e^{x}\,dx \\ &= 2\,e^{x} + c_1 \end{aligned}$$

Substituting the solution found for $u(x)$ in $u = e^{2y}$ gives

$$\begin{aligned} y &= \frac{\ln (u(x))}{2} \\ &= \frac{\ln (2\,e^{x} + c_1)}{2} \\ &= \frac{\ln (2\,e^{x} + c_1)}{2} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\ln (2 + c_1)}{2}$$

$$c_1 = -1$$

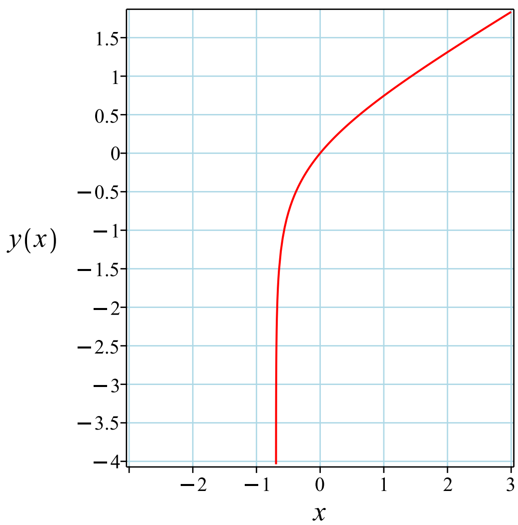
Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(2\,e^x - 1)}{2}$$

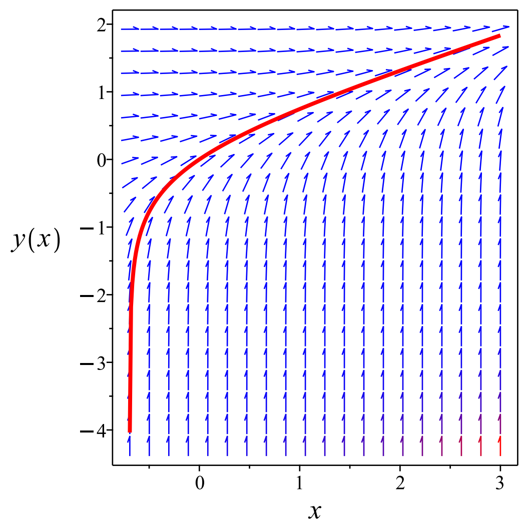
Summary

The solution(s) found are the following

$$y = \frac{\ln(2\,e^x - 1)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\ln(2\,e^x - 1)}{2}$$

Verified OK.

1.12.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= e^{x-2y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 27: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \mathrm{e}^{-x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\mathrm{e}^{-x}} dx
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \mathrm{e}^{x-2y}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \mathrm{e}^{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{2y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x} = \frac{e^{2y}}{2} + c_1$$

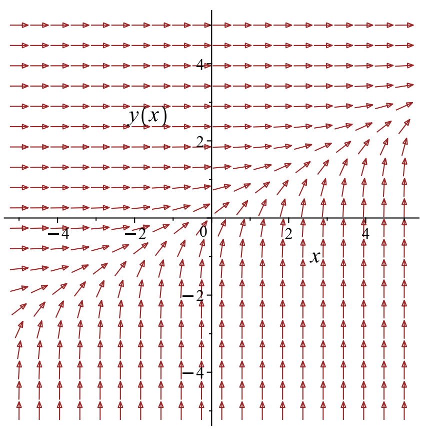
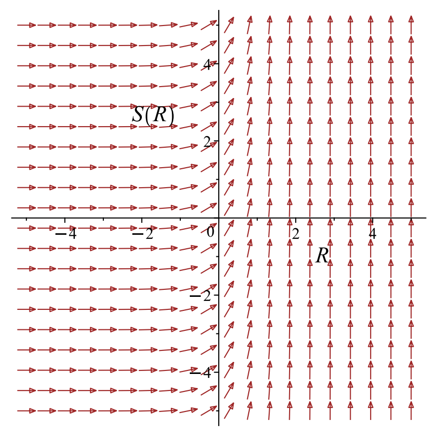
Which simplifies to

$$e^{x} = \frac{e^{2y}}{2} + c_1$$

Which gives

$$y = \frac{\ln(2\,e^{x} - 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = e^{x-2y}$ 	$R = y$ $S = e^{x}$	$\frac{dS}{dR} = e^{2R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\ln(2)}{2} + \frac{\ln(1 - c_1)}{2}$$

$$c_1 = \frac{1}{2}$$

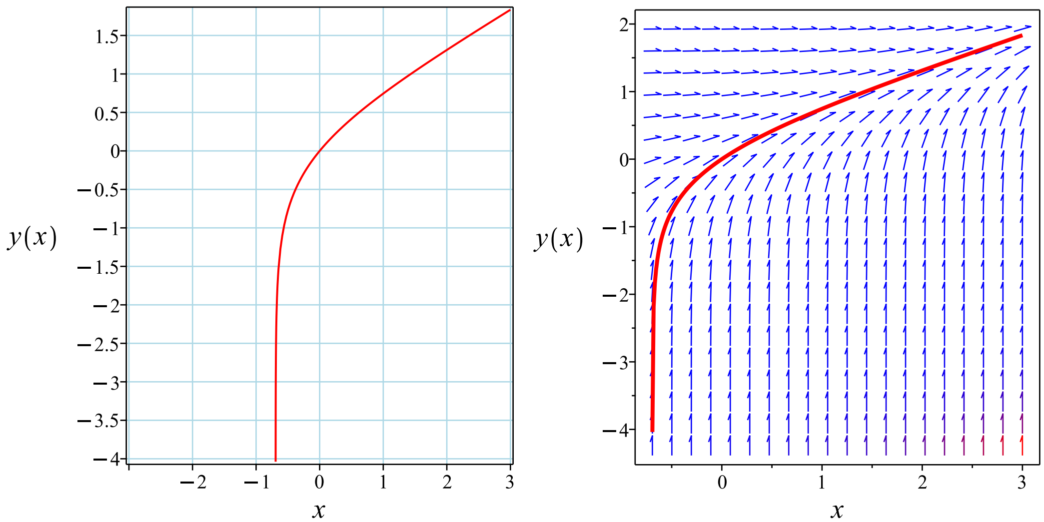
Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(2 e^{x} - 1)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(2 e^{x} - 1)}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\ln(2\,e^{x} - 1)}{2}$$

Verified OK.

1.12.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^{2y})\,\mathrm{d}y &= (\mathrm{e}^{x})\,\mathrm{d}x\\ (-\mathrm{e}^{x})\,\mathrm{d}x + (\mathrm{e}^{2y})\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\mathrm{e}^{x}\\ N(x,y) &= \mathrm{e}^{2y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\mathrm{e}^{x})\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^{2y})\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\mathrm{e}^{x} \mathrm{d}x \\
\phi &= -\mathrm{e}^{x} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2y}$. Therefore equation (4) becomes

$$
\mathrm{e}^{2y} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \mathrm{e}^{2y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(\mathrm{e}^{2y}\right) \mathrm{d}y \\
f(y) &= \frac{\mathrm{e}^{2y}}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\mathrm{e}^{x} + \frac{\mathrm{e}^{2y}}{2} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\mathrm{e}^{x} + \frac{\mathrm{e}^{2y}}{2}
$$

The solution becomes

$$y = \frac{\ln(2\,e^{x} + 2c_1)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\ln(2)}{2} + \frac{\ln(c_1 + 1)}{2}$$

$$c_1 = -\frac{1}{2}$$

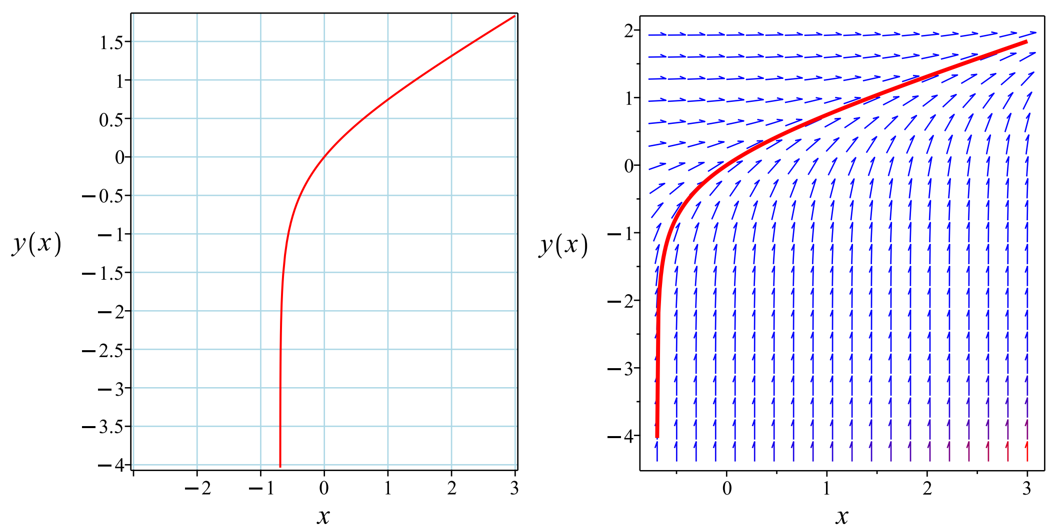
Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(2\,e^{x} - 1)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(2\,e^{x} - 1)}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\ln(2\,e^{x} - 1)}{2}$$

Verified OK.

1.12.6 Maple step by step solution

Let's solve

$[y' - e^{x-2y} = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(e^{y})^{2} = e^{x}$

- Integrate both sides with respect to x

  $\int y'(e^{y})^{2} dx = \int e^{x} dx + c_1$

- Evaluate integral

  $\frac{(e^{y})^{2}}{2} = e^{x} + c_1$

- Solve for y

  $y = \frac{\ln(2\,e^{x}+2c_1)}{2}$

- Use initial condition $y(0) = 0$

  $0 = \frac{\ln(2+2c_1)}{2}$

- Solve for c_1

  $c_1 = -\frac{1}{2}$

- Substitute $c_1 = -\frac{1}{2}$ into general solution and simplify

  $y = \frac{\ln(2\,e^{x}-1)}{2}$

- Solution to the IVP

  $y = \frac{\ln(2\,e^{x}-1)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)=exp(x-2*y(x)),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\ln(2\,e^{x} - 1)}{2}$$

✓ Solution by Mathematica

Time used: 0.824 (sec). Leaf size: 17

```
DSolve[{y'[x]==Exp[x-2*y[x]],y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\log(2e^{x} - 1)$$

1.13 problem 2(c)

Internal problem ID [3014]
Internal file name [`OUTPUT/2506_Sunday_June_05_2022_03_17_11_AM_66234823/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y' - \frac{y^2 + x^2}{2x^2} = 0}$$

1.13.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)^2\,x^2 + x^2}{2x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-u + \frac{1}{2}u^2 + \frac{1}{2}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -u + \frac{1}{2}u^2 + \frac{1}{2}$. Integrating both sides gives

$$\frac{1}{-u + \frac{1}{2}u^2 + \frac{1}{2}}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{-u + \frac{1}{2}u^2 + \frac{1}{2}}\,du = \int \frac{1}{x}\,dx$$
$$-\frac{2}{u-1} = \ln(x) + c_2$$

The solution is

$$-\frac{2}{u(x)-1} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{2}{\frac{y}{x}-1} - \ln(x) - c_2 = 0$$
$$\frac{(\ln(x) + c_2)\,y - x(c_2 + \ln(x) - 2)}{-y + x} = 0$$

Summary

The solution(s) found are the following

$$\frac{(\ln(x) + c_2)\,y - x(c_2 + \ln(x) - 2)}{-y + x} = 0 \tag{1}$$

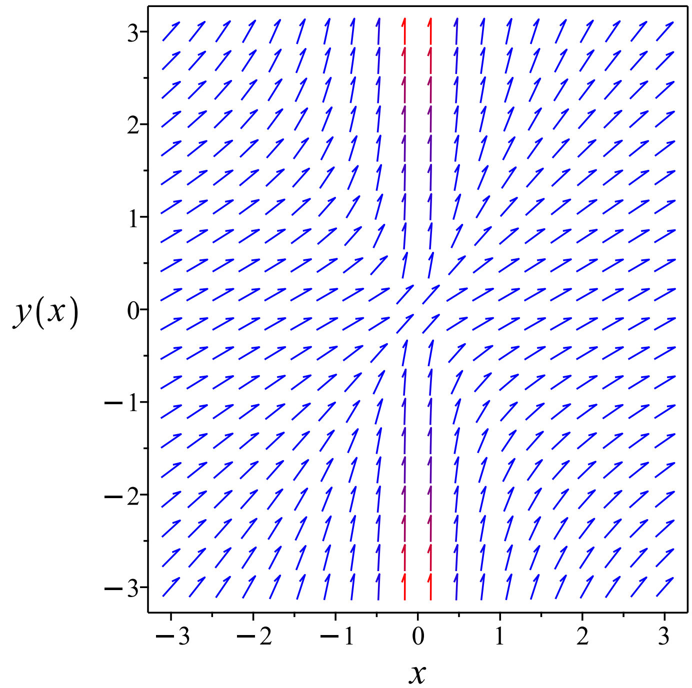


Figure 38: Slope field plot

Verification of solutions

$$\frac{(\ln(x) + c_2)\, y - x(c_2 + \ln(x) - 2)}{-y + x} = 0$$

Verified OK.

1.13.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x^2 + y^2}{2x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(x^2+y^2)(b_3-a_2)}{2x^2} - \frac{(x^2+y^2)^2 a_3}{4x^4} \\ & - \left(\frac{1}{x} - \frac{x^2+y^2}{x^3}\right)(xa_2+ya_3+a_1) - \frac{y(xb_2+yb_3+b_1)}{x^2} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^4a_2 + x^4a_3 - 4b_2x^4 - 2x^4b_3 + 4x^3yb_2 - 2x^2y^2a_2 + 2x^2y^2a_3 + 2x^2y^2b_3 - 4x\,y^3a_3 + y^4a_3 + 4x^2yb_1 - 4x}{4x^4}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -2x^4a_2 - x^4a_3 + 4b_2x^4 + 2x^4b_3 - 4x^3yb_2 + 2x^2y^2a_2 - 2x^2y^2a_3 \\ & - 2x^2y^2b_3 + 4x\,y^3a_3 - y^4a_3 - 4x^2yb_1 + 4x\,y^2a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -2a_2v_1^4 + 2a_2v_1^2v_2^2 - a_3v_1^4 - 2a_3v_1^2v_2^2 + 4a_3v_1v_2^3 - a_3v_2^4 + 4b_2v_1^4 \\ & - 4b_2v_1^3v_2 + 2b_3v_1^4 - 2b_3v_1^2v_2^2 + 4a_1v_1v_2^2 - 4b_1v_1^2v_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-2a_2 - a_3 + 4b_2 + 2b_3) v_1^4 - 4b_2 v_1^3 v_2 + (2a_2 - 2a_3 - 2b_3) v_1^2 v_2^2 \\
&- 4b_1 v_1^2 v_2 + 4a_3 v_1 v_2^3 + 4a_1 v_1 v_2^2 - a_3 v_2^4 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
4a_1 &= 0 \\
-a_3 &= 0 \\
4a_3 &= 0 \\
-4b_1 &= 0 \\
-4b_2 &= 0 \\
2a_2 - 2a_3 - 2b_3 &= 0 \\
-2a_2 - a_3 + 4b_2 + 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{x^2 + y^2}{2x^2}\right)(x) \\
&= \frac{-x^2 + 2xy - y^2}{2x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2+2xy-y^2}{2x}} dy \end{aligned}$$

Which results in

$$S = \frac{2x}{y - x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 + y^2}{2x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2y}{(-y + x)^2} \\ S_y &= -\frac{2x}{(-y + x)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2x}{-y+x} = -\ln(x) + c_1$$

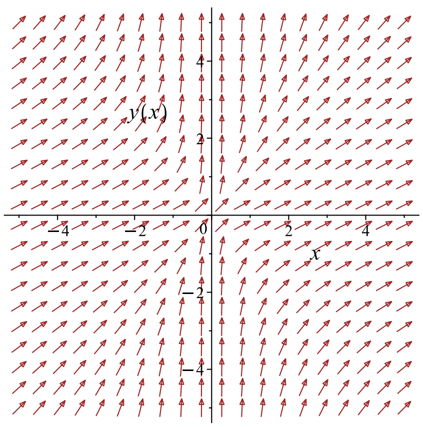
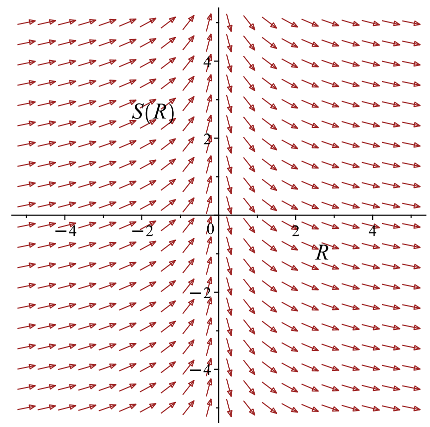
Which simplifies to

$$-\frac{2x}{-y+x} = -\ln(x) + c_1$$

Which gives

$$y = \frac{x(\ln(x) - c_1 - 2)}{\ln(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y^2}{2x^2}$ 	$R = x$ $S = -\frac{2x}{-y+x}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{x(\ln(x) - c_1 - 2)}{\ln(x) - c_1} \tag{1}$$

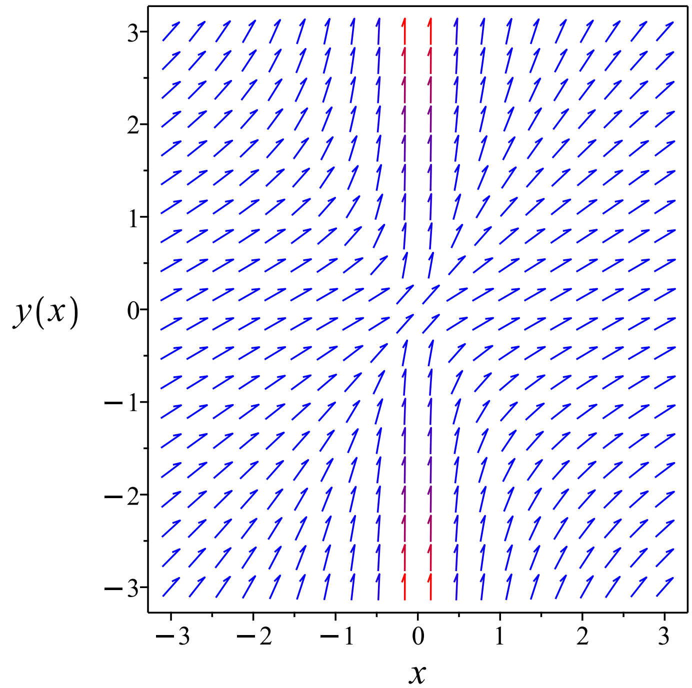


Figure 39: Slope field plot

Verification of solutions

$$y = \frac{x(\ln(x) - c_1 - 2)}{\ln(x) - c_1}$$

Verified OK.

1.13.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + y^2}{2x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{1}{2} + \frac{y^2}{2x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{1}{2}$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{2x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{2x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{x^3} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{1}{8x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{2x^2} + \frac{u'(x)}{x^3} + \frac{u(x)}{8x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_1 + c_2 \ln(x)}{\sqrt{x}}$$

The above shows that

$$u'(x) = -\frac{c_2 \ln(x) + c_1 - 2c_2}{2x^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$y = \frac{(c_2 \ln(x) + c_1 - 2c_2)\, x}{c_1 + c_2 \ln(x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(\ln(x) + c_3 - 2)\, x}{c_3 + \ln(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{(\ln(x) + c_3 - 2)\, x}{c_3 + \ln(x)} \tag{1}$$

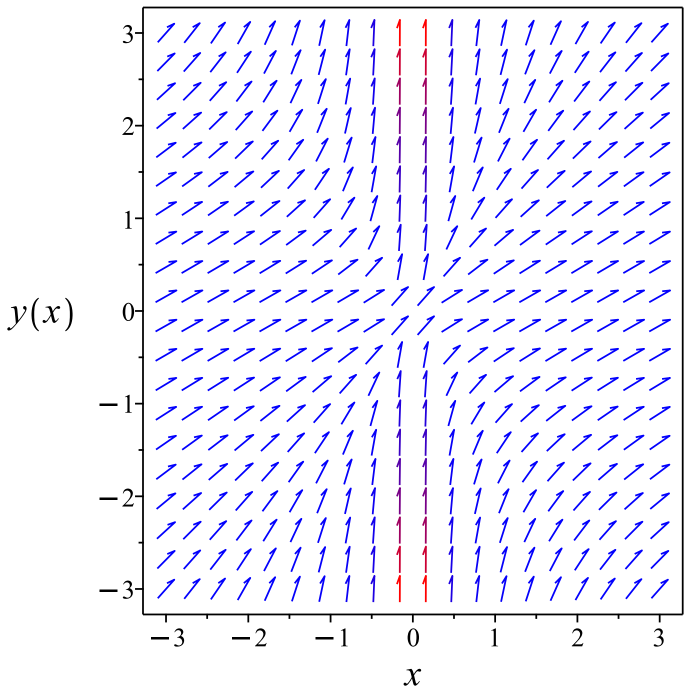


Figure 40: Slope field plot

Verification of solutions

$$y = \frac{(\ln(x) + c_3 - 2)\, x}{c_3 + \ln(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)=(x^2+y(x)^2)/(2*x^2),y(x), singsol=all)
```

$$y(x) = \frac{x(\ln(x) + c_1 - 2)}{\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.149 (sec). Leaf size: 29

```
DSolve[y'[x]==(x^2+y[x]^2)/(2*x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x(\log(x) - 2 + 2c_1)}{\log(x) + 2c_1}$$
$$y(x) \to x$$

1.14 problem 2(d)

Internal problem ID [3015]
Internal file name [`OUTPUT/2507_Sunday_June_05_2022_03_17_14_AM_17946256/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' - y = x}$$

With initial conditions

$$[y(-1) = -1]$$

1.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 1$$

Hence the ode is

$$y' - \frac{y}{x} = 1$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

1.14.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \frac{1}{x} \\ \mathrm{d}\left(\frac{y}{x}\right) &= \frac{1}{x}\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{1}{x}\,\mathrm{d}x \\ \frac{y}{x} &= \ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x + \ln(x)\, x$$

which simplifies to

$$y = x(\ln(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -i\pi - c_1$$

$$c_1 = -i\pi + 1$$

Substituting c_1 found above in the general solution gives

$$y = -i\pi x + \ln(x) x + x$$

Summary
The solution(s) found are the following

$$y = -i\pi x + \ln(x) x + x \tag{1}$$

Verification of solutions

$$y = -i\pi x + \ln(x) x + x$$

Verified OK.

1.14.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x) x$ on the above ode results in new ode in $u(x)$

$$x(u'(x) x + u(x)) - u(x) x = x$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{1}{x} \, dx \\ &= \ln(x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x(\ln(x) + c_2) \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -i\pi - c_2$$

$$c_2 = -i\pi + 1$$

Substituting c_2 found above in the general solution gives

$$y = -i\pi x + \ln(x)\,x + x$$

Summary

The solution(s) found are the following

$$y = -i\pi x + \ln(x)\,x + x \tag{1}$$

Verification of solutions

$$y = -i\pi x + \ln(x)\,x + x$$

Verified OK.

1.14.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y+x}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 30: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + x}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \ln(x) + c_1$$

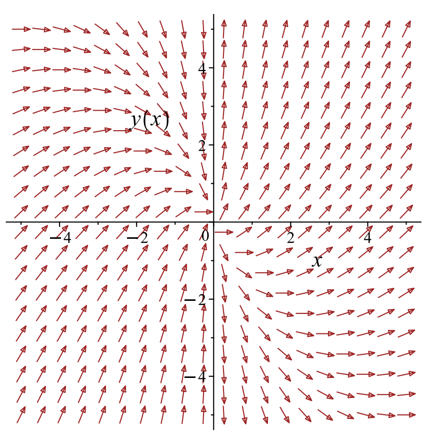
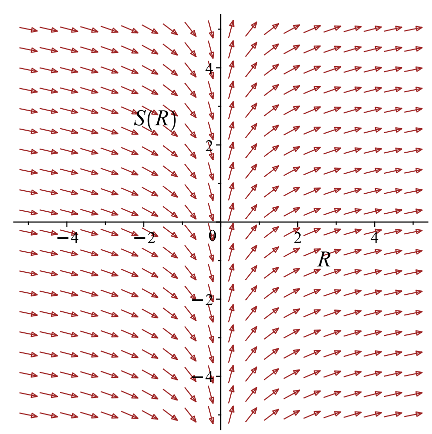
Which simplifies to

$$\frac{y}{x} = \ln(x) + c_1$$

Which gives

$$y = x(\ln(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+x}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -i\pi - c_1$$

$$c_1 = -i\pi + 1$$

Substituting c_1 found above in the general solution gives

$$y = -i\pi x + \ln(x)\, x + x$$

Summary

The solution(s) found are the following

$$y = -i\pi x + \ln(x)\, x + x \tag{1}$$

Verification of solutions

$$y = -i\pi x + \ln(x)\, x + x$$

Verified OK.

1.14.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= (y+x)\,\mathrm{d}x \\ (-y-x)\,\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -y-x \\ N(x,y) &= x \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y-x) \\ &= -1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((-1)-(1)) \\ &= -\frac{2}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{2}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^2}(-y-x) \\ &= \frac{-y-x}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^2}(x) \\ &= \frac{1}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-y-x}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \frac{-y-x}{x^2} \,\mathrm{d}x \\
\phi &= \frac{y}{x} - \ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y}{x} - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y}{x} - \ln(x)$$

The solution becomes

$$y = x(\ln(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -i\pi - c_1$$

$$c_1 = -i\pi + 1$$

Substituting c_1 found above in the general solution gives

$$y = -i\pi x + \ln(x)\, x + x$$

Summary

The solution(s) found are the following

$$y = -i\pi x + \ln(x)\, x + x \tag{1}$$

Verification of solutions

$$y = -i\pi x + \ln(x)\, x + x$$

Verified OK.

1.14.6 Maple step by step solution

Let's solve

$[xy' - y = x, y(-1) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

$\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \frac{1}{x}dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(\ln(x) + c_1)$

- Use initial condition $y(-1) = -1$

  $-1 = -\mathrm{I}\pi - c_1$

- Solve for c_1

  $c_1 = -\mathrm{I}\pi + 1$

- Substitute $c_1 = -\mathrm{I}\pi + 1$ into general solution and simplify

  $y = (\ln(x) - \mathrm{I}\pi + 1)\, x$

- Solution to the IVP

  $y = (\ln(x) - \mathrm{I}\pi + 1)\, x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([x*diff(y(x),x)=x+y(x),y(-1) = -1],y(x), singsol=all)
```

$$y(x) = (\ln (x) + 1 - i\pi)\, x$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 16

```
DSolve[{x*y'[x]==x+y[x],y[-1]==-1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(\log(x) - i\pi + 1)$$

1.15 problem 2(e)

Internal problem ID [3016]
Internal file name [`OUTPUT/2508_Sunday_June_05_2022_03_17_16_AM_91803760/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{e^{-y} + \left(x^2+1\right) y' = 0}$$

With initial conditions

$$[y(0) = 0]$$

1.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{e^{-y}}{x^2+1} \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{e^{-y}}{x^2+1}\right) \\ &= \frac{e^{-y}}{x^2+1}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

1.15.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{e^{-y}}{x^2+1}\end{aligned}$$

Where $f(x) = -\frac{1}{x^2+1}$ and $g(y) = e^{-y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{e^{-y}}\,dy &= -\frac{1}{x^2+1}\,dx \\ \int \frac{1}{e^{-y}}\,dy &= \int -\frac{1}{x^2+1}\,dx \\ e^{y} &= -\arctan(x) + c_1\end{aligned}$$

Which results in

$$y = -\ln\left(-\frac{1}{\arctan(x) - c_1}\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\ln\left(\frac{1}{c_1}\right)$$

$$c_1 = 1$$

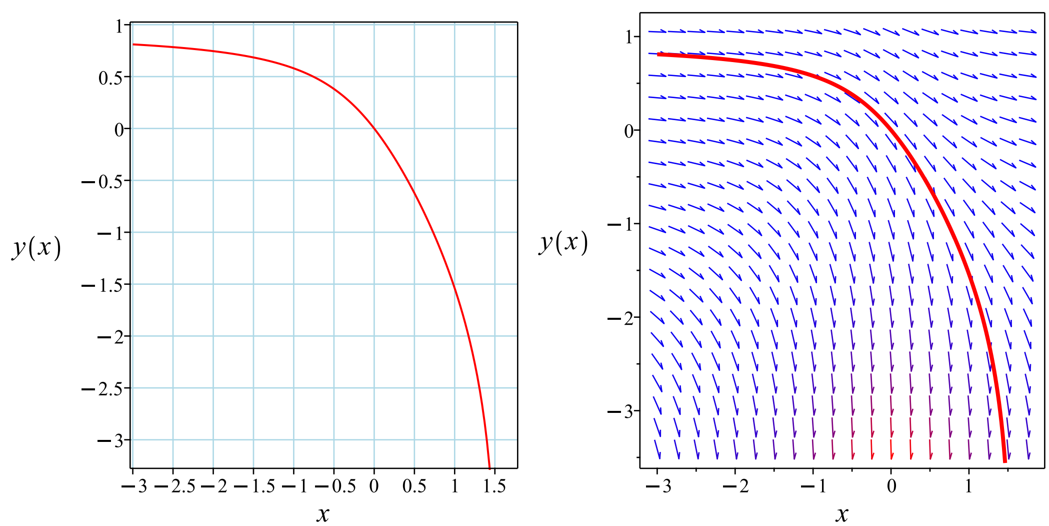
Substituting c_1 found above in the general solution gives

$$y = -\ln\left(-\frac{1}{\arctan(x) - 1}\right)$$

Summary

The solution(s) found are the following

$$y = -\ln\left(-\frac{1}{\arctan(x) - 1}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\ln\left(-\frac{1}{\arctan(x) - 1}\right)$$

Verified OK.

1.15.3 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = -\frac{e^{-y}}{x^2+1} \tag{1}$$

And using the substitution $u = e^y$ then

$$u' = y' e^y$$

The above shows that

$$\begin{aligned} y' &= u'(x)\, e^{-y} \\ &= \frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$\frac{u'(x)}{u} = -\frac{1}{(x^2+1)\, u}$$

The above simplifies to

$$u'(x) = -\frac{1}{x^2+1} \tag{2}$$

Now ode (2) is solved for $u(x)$ Integrating both sides gives

$$\begin{aligned} u(x) &= \int -\frac{1}{x^2+1}\, dx \\ &= -\arctan(x) + c_1 \end{aligned}$$

Substituting the solution found for $u(x)$ in $u = e^y$ gives

$$\begin{aligned} y &= \ln(u(x)) \\ &= \ln(-\arctan(x) + c_1) \\ &= \ln(-\arctan(x) + c_1) \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln(c_1)$$

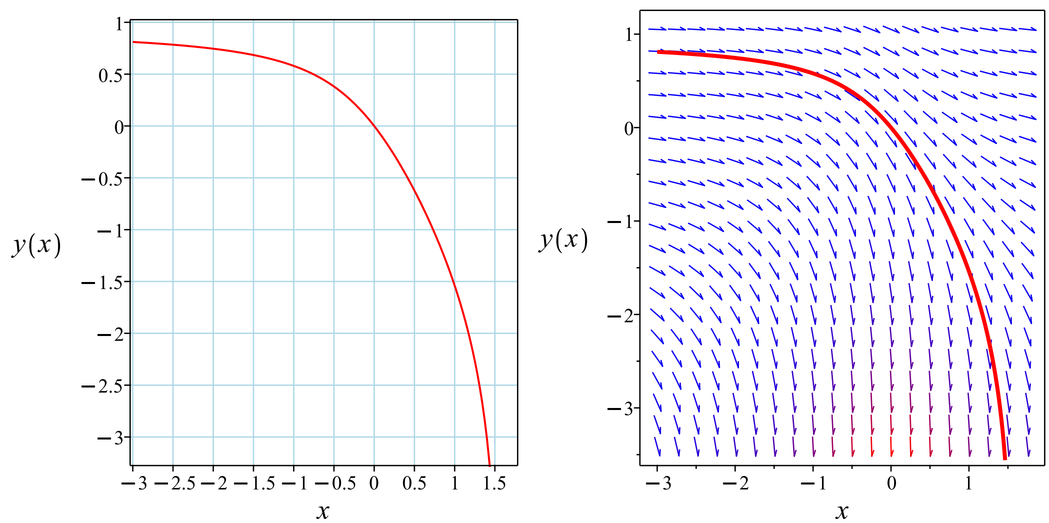
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = \ln(-\arctan(x) + 1)$$

Summary
The solution(s) found are the following

$$y = \ln(-\arctan(x) + 1) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \ln(-\arctan(x) + 1)$$

Verified OK.

1.15.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{e^{-y}}{x^2 + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 33: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -x^2 - 1 \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-x^2 - 1} dx \end{aligned}$$

Which results in

$$S = -\arctan(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{e^{-y}}{x^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{1}{x^2 + 1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \mathrm{e}^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\arctan(x) = \mathrm{e}^{y} + c_1$$

Which simplifies to

$$-\arctan(x) = \mathrm{e}^{y} + c_1$$

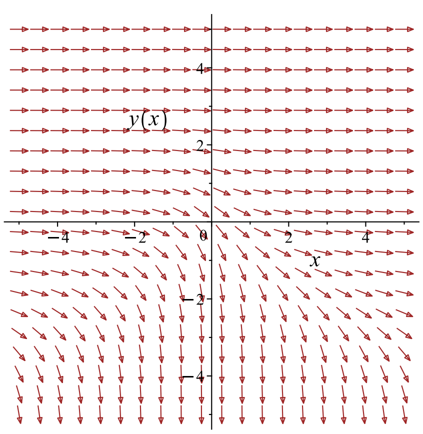
Which gives

$$y = \ln(-\arctan(x) - c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{\mathrm{e}^{-y}}{x^2+1}$



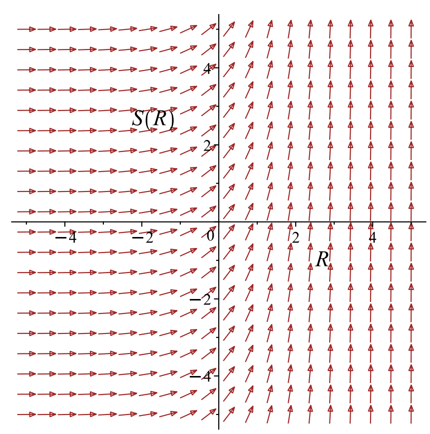
</td><td>

$R = y$

$S = -\arctan(x)$

</td><td>

$\frac{dS}{dR} = \mathrm{e}^{R}$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln(-c_1)$$

$$c_1 = -1$$

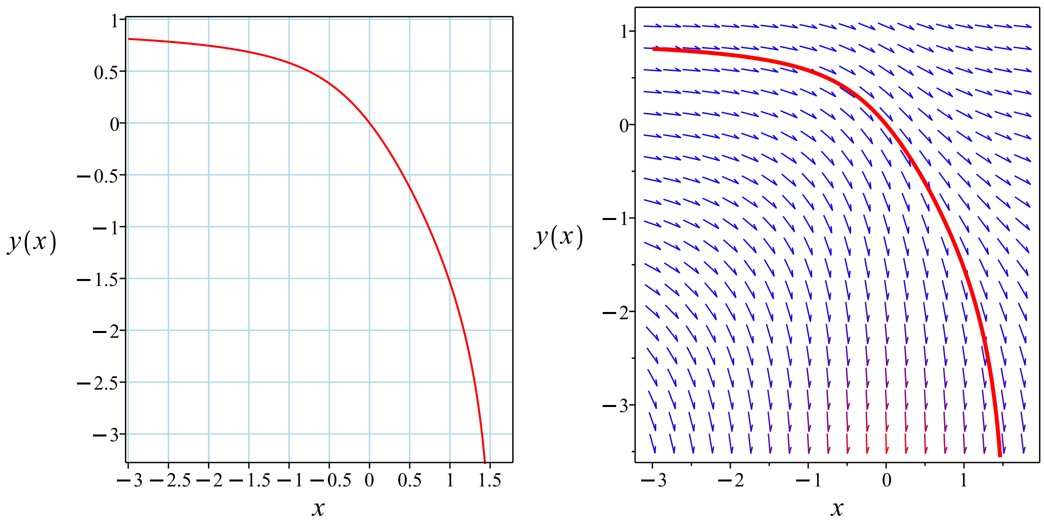
Substituting c_1 found above in the general solution gives

$$y = \ln(-\arctan(x) + 1)$$

Summary

The solution(s) found are the following

$$y = \ln(-\arctan(x) + 1) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \ln(-\arctan(x) + 1)$$

Verified OK.

1.15.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-\mathrm{e}^{y})\,\mathrm{d}y&=\left(\frac{1}{x^2+1}\right)\mathrm{d}x\\ \left(-\frac{1}{x^2+1}\right)\mathrm{d}x+(-\mathrm{e}^{y})\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{x^2+1}\\ N(x,y)&=-\mathrm{e}^{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{x^2+1}\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-\mathrm{e}^{y}) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x^2+1} \, \mathrm{d}x \\ \phi &= -\arctan(x) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\mathrm{e}^{y}$. Therefore equation (4) becomes

$$-\mathrm{e}^{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y) &= -\mathrm{e}^{y} \\ &= -\mathrm{e}^{y}\end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned}\int f'(y) \, \mathrm{d}y &= \int (-\mathrm{e}^{y}) \, \mathrm{d}y \\ f(y) &= -\mathrm{e}^{y} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arctan(x) - e^{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arctan(x) - e^{y}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

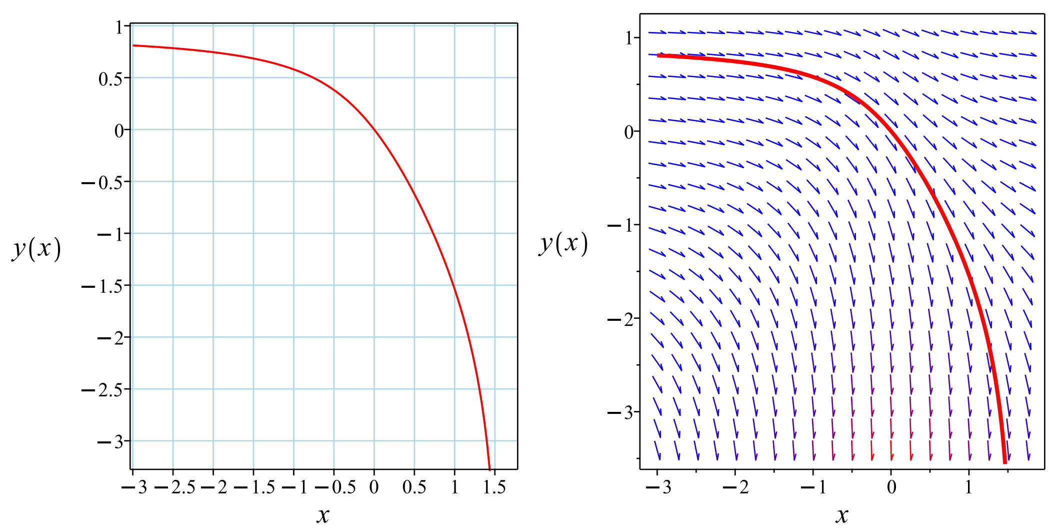
$$-\arctan(x) - e^{y} = -1$$

Solving for y from the above gives

$$y = \ln(-\arctan(x) + 1)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \ln(-\arctan(x) + 1) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \ln(-\arctan(x) + 1)$$

Verified OK.

1.15.6 Maple step by step solution

Let's solve

$\left[e^{-y} + (x^2+1)\,y' = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{e^{-y}} = -\frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{e^{-y}}dx = \int -\frac{1}{x^2+1}dx + c_1$

- Evaluate integral

  $\frac{1}{e^{-y}} = -\arctan(x) + c_1$

- Solve for y

  $y = -\ln\left(-\frac{1}{\arctan(x)-c_1}\right)$

- Use initial condition $y(0) = 0$

  $0 = -\ln\left(\frac{1}{c_1}\right)$

- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

  $y = -\ln\left(-\frac{1}{\arctan(x)-1}\right)$

- Solution to the IVP

  $y = -\ln\left(-\frac{1}{\arctan(x)-1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 11

```
dsolve([exp(-y(x))+(1+x^2)*diff(y(x),x)=0,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \ln(-\arctan(x) + 1)$$

✓ Solution by Mathematica

Time used: 0.391 (sec). Leaf size: 12

```
DSolve[{Exp[-y[x]]+(1+x^2)*y'[x]==0,y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(1 - \arctan(x))$$

1.16 problem 2(f)

Internal problem ID [3017]
Internal file name [`OUTPUT/2509_Sunday_June_05_2022_03_17_18_AM_43139761/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = e^{x} \sin(x)}$$

With initial conditions

$$[y(0) = 0]$$

1.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= e^{x} \sin(x) \end{aligned}$$

Hence the ode is

$$y' = e^{x} \sin(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = e^x \sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.16.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int e^x \sin(x)\, dx \\ &= -\frac{\cos(x)\, e^x}{2} + \frac{e^x \sin(x)}{2} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 - \frac{1}{2}$$

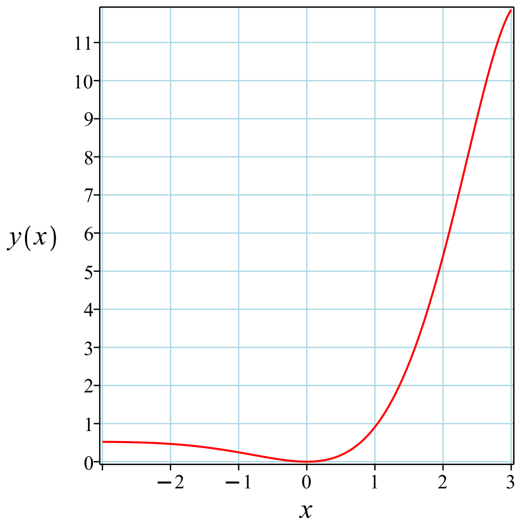
$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

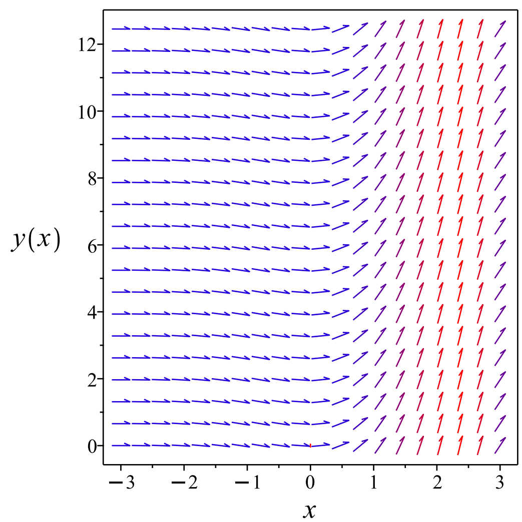
$$y = -\frac{\cos(x)\, e^x}{2} + \frac{e^x \sin(x)}{2} + \frac{1}{2}$$

Summary
The solution(s) found are the following

$$y = -\frac{\cos(x)\, e^x}{2} + \frac{e^x \sin(x)}{2} + \frac{1}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\cos(x)\,e^{x}}{2} + \frac{e^{x}\sin(x)}{2} + \frac{1}{2}$$

Verified OK.

1.16.3 Maple step by step solution

Let's solve

$[y' = e^{x}\sin(x), y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'dx = \int e^{x}\sin(x)\,dx + c_1$

- Evaluate integral

  $y = -\frac{\cos(x)e^{x}}{2} + \frac{e^{x}\sin(x)}{2} + c_1$

- Solve for y

  $y = -\frac{\cos(x)e^{x}}{2} + \frac{e^{x}\sin(x)}{2} + c_1$

- Use initial condition $y(0) = 0$

  $0 = c_1 - \frac{1}{2}$

- Solve for c_1

  $c_1 = \frac{1}{2}$
- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

  $y = \frac{1}{2} + \frac{(-\cos(x)+\sin(x))\mathrm{e}^x}{2}$
- Solution to the IVP

  $y = \frac{1}{2} + \frac{(-\cos(x)+\sin(x))\mathrm{e}^x}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)=exp(x)*sin(x),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{1}{2} + \frac{\mathrm{e}^x(\sin(x) - \cos(x))}{2}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 24

```
DSolve[{y'[x]==Exp[x]*Sin[x],y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(e^x \sin(x) - e^x \cos(x) + 1)$$

1.17 problem 2(g)

Internal problem ID [3018]
Internal file name [`OUTPUT/2510_Sunday_June_05_2022_03_17_20_AM_75488550/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(g).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 3y = e^{3x} + e^{-3x}}$$

With initial conditions

$$[y(5) = 5]$$

1.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= e^{3x} + e^{-3x} \end{aligned}$$

Hence the ode is

$$y' - 3y = e^{3x} + e^{-3x}$$

The domain of $p(x) = -3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 5$ is inside this domain. The domain of $q(x) = e^{3x} + e^{-3x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 5$ is also inside this domain. Hence solution exists and is unique.

1.17.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-3)dx} \\ &= e^{-3x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(e^{3x} + e^{-3x}\right) \\ \frac{d}{dx}\left(e^{-3x}y\right) &= \left(e^{-3x}\right)\left(e^{3x} + e^{-3x}\right) \\ d\left(e^{-3x}y\right) &= \left(e^{-6x} + 1\right)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-3x}y &= \int e^{-6x} + 1\, dx \\ e^{-3x}y &= x - \frac{e^{-6x}}{6} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-3x}$ results in

$$y = e^{3x}\left(x - \frac{e^{-6x}}{6}\right) + c_1 e^{3x}$$

which simplifies to

$$y = (x + c_1)\, e^{3x} - \frac{e^{-3x}}{6}$$

Initial conditions are used to solve for c_1. Substituting $x = 5$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = e^{15} c_1 + 5\,e^{15} - \frac{e^{-15}}{6}$$

$$c_1 = -\frac{\left(30\,e^{15} - e^{-15} - 30\right) e^{-15}}{6}$$

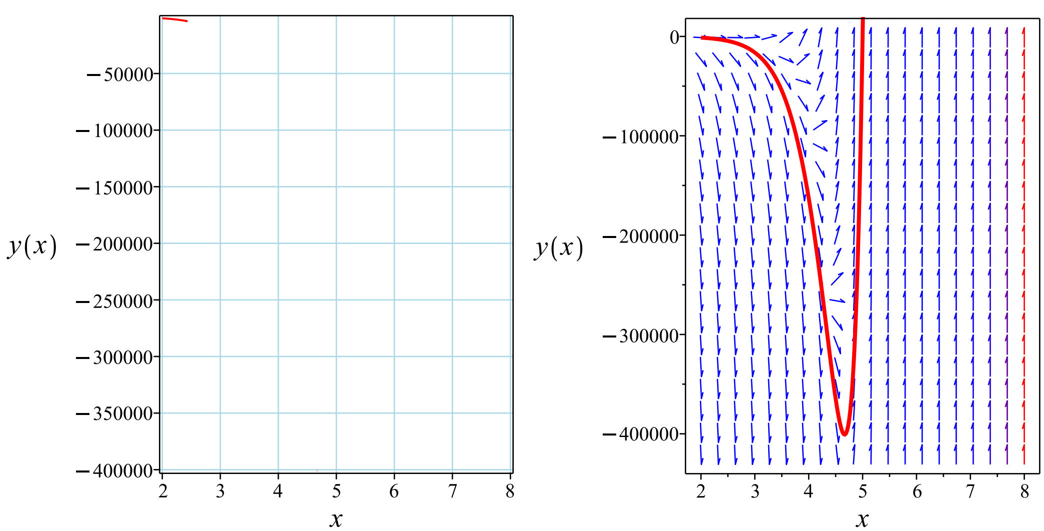
Substituting c_1 found above in the general solution gives

$$y = x\,e^{3x} - 5\,e^{3x} + \frac{e^{3x-30}}{6} + 5\,e^{3x-15} - \frac{e^{-3x}}{6}$$

Summary

The solution(s) found are the following

$$y = x\,e^{3x} - 5\,e^{3x} + \frac{e^{3x-30}}{6} + 5\,e^{3x-15} - \frac{e^{-3x}}{6} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x\,e^{3x} - 5\,e^{3x} + \frac{e^{3x-30}}{6} + 5\,e^{3x-15} - \frac{e^{-3x}}{6}$$

Verified OK.

1.17.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 3y + e^{3x} + e^{-3x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 37: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \mathrm{e}^{3x}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta}dy\\ &= \int \frac{1}{\mathrm{e}^{3x}}dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{-3x}y$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 3y + \mathrm{e}^{3x} + \mathrm{e}^{-3x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= -3\,\mathrm{e}^{-3x}y\\ S_y &= \mathrm{e}^{-3x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{-6x} + 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{-6R} + 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{e^{-6R}}{6} + R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-3x}y = -\frac{e^{-6x}}{6} + x + c_1$$

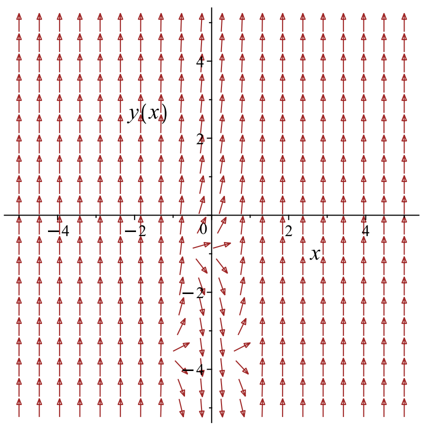
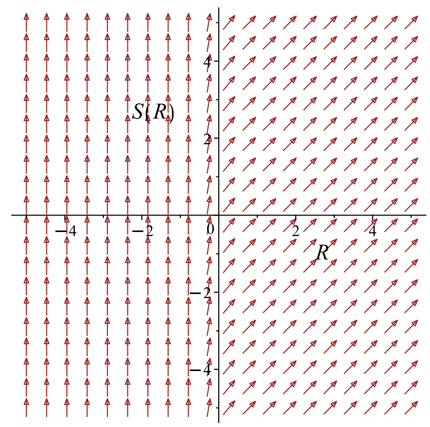
Which simplifies to

$$e^{-3x}y = -\frac{e^{-6x}}{6} + x + c_1$$

Which gives

$$y = -\frac{(e^{-6x} - 6c_1 - 6x)\, e^{3x}}{6}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3y + \mathrm{e}^{3x} + \mathrm{e}^{-3x}$ 	$R = x$ $S = \mathrm{e}^{-3x} y$	$\frac{dS}{dR} = \mathrm{e}^{-6R} + 1$ 

Initial conditions are used to solve for c_1. Substituting $x = 5$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = \mathrm{e}^{15} c_1 + 5\,\mathrm{e}^{15} - \frac{\mathrm{e}^{-15}}{6}$$

$$c_1 = -\frac{(30\,\mathrm{e}^{15} - \mathrm{e}^{-15} - 30)\,\mathrm{e}^{-15}}{6}$$

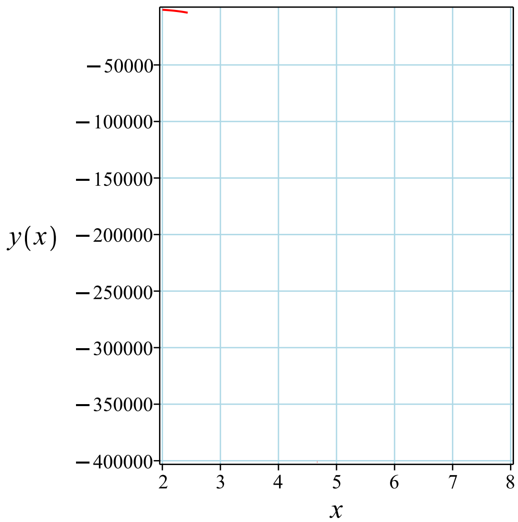
Substituting c_1 found above in the general solution gives

$$y = x\,\mathrm{e}^{3x} - 5\,\mathrm{e}^{3x} + \frac{\mathrm{e}^{3x-30}}{6} + 5\,\mathrm{e}^{3x-15} - \frac{\mathrm{e}^{-3x}}{6}$$

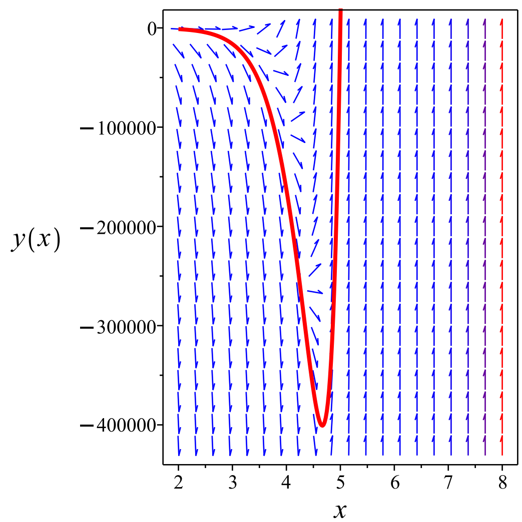
<u>Summary</u>

The solution(s) found are the following

$$y = x\,\mathrm{e}^{3x} - 5\,\mathrm{e}^{3x} + \frac{\mathrm{e}^{3x-30}}{6} + 5\,\mathrm{e}^{3x-15} - \frac{\mathrm{e}^{-3x}}{6} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{3x} - 5\,\mathrm{e}^{3x} + \frac{\mathrm{e}^{3x-30}}{6} + 5\,\mathrm{e}^{3x-15} - \frac{\mathrm{e}^{-3x}}{6}$$

Verified OK.

1.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= \left(3y + \mathrm{e}^{3x} + \mathrm{e}^{-3x}\right) \mathrm{d}x \\ \left(-3y - \mathrm{e}^{3x} - \mathrm{e}^{-3x}\right) \mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -3y - \mathrm{e}^{3x} - \mathrm{e}^{-3x} \\ N(x, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3y - \mathrm{e}^{3x} - \mathrm{e}^{-3x}\right) \\ &= -3 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-3) - (0)) \\ &= -3 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -3 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3x} \\ &= \mathrm{e}^{-3x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-3x}\left(-3y - \mathrm{e}^{3x} - \mathrm{e}^{-3x}\right) \\ &= \left(-\mathrm{e}^{6x} - 3y\,\mathrm{e}^{3x} - 1\right)\mathrm{e}^{-6x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-3x}(1) \\ &= \mathrm{e}^{-3x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\left(-\mathrm{e}^{6x} - 3y\,\mathrm{e}^{3x} - 1\right)\mathrm{e}^{-6x}\right) + \left(\mathrm{e}^{-3x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \left(-\mathrm{e}^{6x} - 3y\,\mathrm{e}^{3x} - 1\right)\mathrm{e}^{-6x}\,\mathrm{d}x\end{aligned}$$

$$\phi = -x + \frac{\mathrm{e}^{-6x}}{6} + \mathrm{e}^{-3x}y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-3x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-3x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-3x} = \mathrm{e}^{-3x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \frac{\mathrm{e}^{-6x}}{6} + \mathrm{e}^{-3x} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \frac{\mathrm{e}^{-6x}}{6} + \mathrm{e}^{-3x} y$$

The solution becomes

$$y = -\frac{(\mathrm{e}^{-6x} - 6c_1 - 6x)\,\mathrm{e}^{3x}}{6}$$

Initial conditions are used to solve for c_1. Substituting $x = 5$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = \mathrm{e}^{15} c_1 + 5\,\mathrm{e}^{15} - \frac{\mathrm{e}^{-15}}{6}$$

$$c_1 = -\frac{(30\,\mathrm{e}^{15} - \mathrm{e}^{-15} - 30)\,\mathrm{e}^{-15}}{6}$$

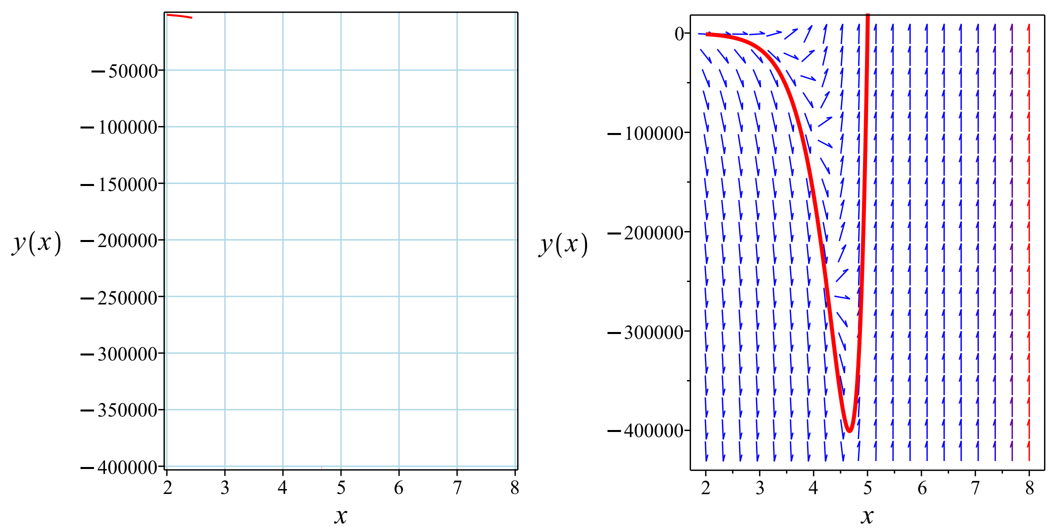
Substituting c_1 found above in the general solution gives

$$y = x\,e^{3x} - 5\,e^{3x} + \frac{e^{3x-30}}{6} + 5\,e^{3x-15} - \frac{e^{-3x}}{6}$$

Summary

The solution(s) found are the following

$$y = x\,e^{3x} - 5\,e^{3x} + \frac{e^{3x-30}}{6} + 5\,e^{3x-15} - \frac{e^{-3x}}{6} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = x\,e^{3x} - 5\,e^{3x} + \frac{e^{3x-30}}{6} + 5\,e^{3x-15} - \frac{e^{-3x}}{6}$$

Verified OK.

1.17.5 Maple step by step solution

Let's solve

$[y' - 3y = e^{3x} + e^{-3x}, y(5) = 5]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = 3y + e^{3x} + e^{-3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 3y = e^{3x} + e^{-3x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - 3y) = \mu(x)\,(e^{3x} + e^{-3x})$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' - 3y) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -3\mu(x)$

- Solve to find the integrating factor

  $\mu(x) = e^{-3x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,(e^{3x} + e^{-3x})\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,(e^{3x} + e^{-3x})\,dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)(e^{3x}+e^{-3x})dx+c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{-3x}$

  $y = \frac{\int (e^{3x}+e^{-3x})e^{-3x}dx+c_1}{e^{-3x}}$

- Evaluate the integrals on the rhs

  $y = \frac{x-\frac{1}{6(e^x)^6}+c_1}{e^{-3x}}$

- Simplify

  $y = (x + c_1)\,e^{3x} - \frac{e^{-3x}}{6}$

- Use initial condition $y(5) = 5$

  $5 = (c_1 + 5)\,e^{15} - \frac{e^{-15}}{6}$

- Solve for c_1

  $c_1 = -\frac{30\,e^{15}-e^{-15}-30}{6\,e^{15}}$

- Substitute $c_1 = -\frac{30e^{15}-e^{-15}-30}{6e^{15}}$ into general solution and simplify

  $y = \frac{e^{3x-30}}{6} + 5\,e^{3x-15} + (x - 5)\,e^{3x} - \frac{e^{-3x}}{6}$

- Solution to the IVP

  $y = \frac{e^{3x-30}}{6} + 5\,e^{3x-15} + (x-5)\,e^{3x} - \frac{e^{-3x}}{6}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 31

```
dsolve([diff(y(x),x)-3*y(x)=exp(3*x)+exp(-3*x),y(5) = 5],y(x), singsol=all)
```

$$y(x) = \frac{e^{3x-30}}{6} + 5\,e^{3x-15} + (x-5)\,e^{3x} - \frac{e^{-3x}}{6}$$

✓ Solution by Mathematica

Time used: 0.077 (sec). Leaf size: 48

```
DSolve[{y'[x]-3*y[x]==Exp[3*x]+Exp[-3*x],y[5]==5},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}e^{-3(x+10)}\left(6e^{6(x+5)}(x-5) + e^{6x} + 30e^{6x+15} - e^{30}\right)$$

1.18 problem 2(h)

Internal problem ID [3019]
Internal file name [`OUTPUT/2511_Sunday_June_05_2022_03_17_23_AM_72583526/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(h).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = x + \frac{1}{x}}$$

With initial conditions

$$[y(-2) = 5]$$

1.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \frac{x^2 + 1}{x}$$

Hence the ode is

$$y' = \frac{x^2 + 1}{x}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -2$ is inside this domain. The domain of $q(x) = \frac{x^2+1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -2$ is also inside this domain. Hence solution exists and is unique.

1.18.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{x^2+1}{x} \, \mathrm{d}x \\ &= \ln(x) + \frac{x^2}{2} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = -2$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = \ln(2) + i\pi + 2 + c_1$$

$$c_1 = -i\pi - \ln(2) + 3$$

Substituting c_1 found above in the general solution gives

$$y = \ln(x) + \frac{x^2}{2} - i\pi - \ln(2) + 3$$

Summary
The solution(s) found are the following

$$y = \ln(x) + \frac{x^2}{2} - i\pi - \ln(2) + 3 \tag{1}$$

Verification of solutions

$$y = \ln(x) + \frac{x^2}{2} - i\pi - \ln(2) + 3$$

Verified OK.

1.18.3 Maple step by step solution

Let's solve

$\left[y' = x + \frac{1}{x}, y(-2) = 5\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \left(x + \frac{1}{x}\right) dx + c_1$

- Evaluate integral

  $y = \ln(x) + \frac{x^2}{2} + c_1$

- Solve for y

  $y = \ln(x) + \frac{x^2}{2} + c_1$

- Use initial condition $y(-2) = 5$

  $5 = \ln(2) + \mathrm{I}\pi + 2 + c_1$

- Solve for c_1

  $c_1 = -\mathrm{I}\pi - \ln(2) + 3$

- Substitute $c_1 = -\mathrm{I}\pi - \ln(2) + 3$ into general solution and simplify

  $y = \ln(x) + \frac{x^2}{2} - \mathrm{I}\pi - \ln(2) + 3$

- Solution to the IVP

  $y = \ln(x) + \frac{x^2}{2} - \mathrm{I}\pi - \ln(2) + 3$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 21

```
dsolve([diff(y(x),x)=x+1/x,y(-2) = 5],y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} + \ln(x) + 3 - \ln(2) - i\pi$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 25

```
DSolve[{y'[x]==x+1/x,y[-2]==5},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} + \log\left(\frac{x}{2}\right) - i\pi + 3$$

1.19 problem 2(i)

Internal problem ID [3020]
Internal file name [`OUTPUT/2512_Sunday_June_05_2022_03_17_25_AM_82627847/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{xy' + 2y = (2 + 3x)\,\mathrm{e}^{3x}}$$

With initial conditions

$$[y(1) = 1]$$

1.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= \frac{\mathrm{e}^{3x}(2 + 3x)}{x} \end{aligned}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{e^{3x}(2+3x)}{x}$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{e^{3x}(2+3x)}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

1.19.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x}dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{e^{3x}(2+3x)}{x}\right) \\ \frac{d}{dx}\left(x^2 y\right) &= \left(x^2\right)\left(\frac{e^{3x}(2+3x)}{x}\right) \\ d\left(x^2 y\right) &= \left(e^{3x}(2+3x)\,x\right)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^2 y &= \int e^{3x}(2+3x)\,x\,dx \\ x^2 y &= x^2 e^{3x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = e^{3x} + \frac{c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = e^3 + c_1$$

$$c_1 = 1 - e^3$$

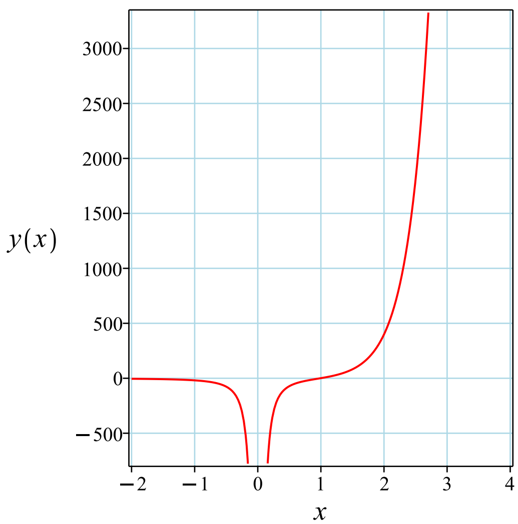
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 e^{3x} - e^3 + 1}{x^2}$$

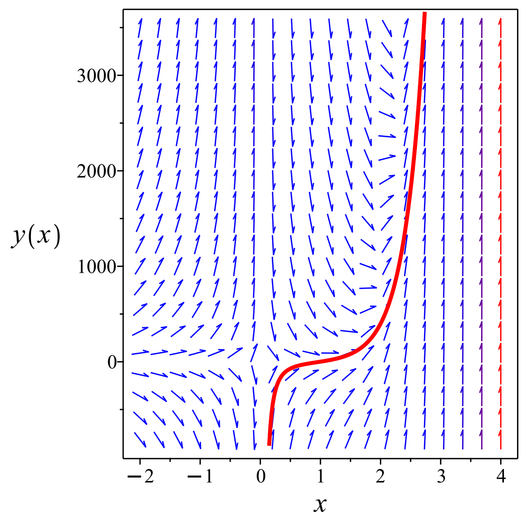
Summary

The solution(s) found are the following

$$y = \frac{x^2 e^{3x} - e^3 + 1}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2 e^{3x} - e^3 + 1}{x^2}$$

Verified OK.

1.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{3x\,\mathrm{e}^{3x} + 2\,\mathrm{e}^{3x} - 2y}{x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 41: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{x^2}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy\end{aligned}$$

Which results in

$$S = x^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{3x\,\mathrm{e}^{3x} + 2\,\mathrm{e}^{3x} - 2y}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 2xy \\
S_y &= x^2
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \mathrm{e}^{3x}(2+3x)\,x \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \mathrm{e}^{3R}(2+3R)\,R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \mathrm{e}^{3R}R^2 + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
x^2y = x^2\mathrm{e}^{3x} + c_1
$$

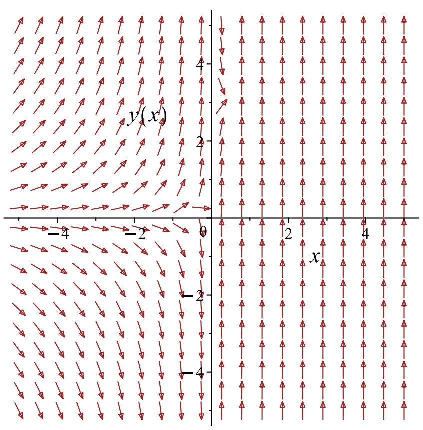
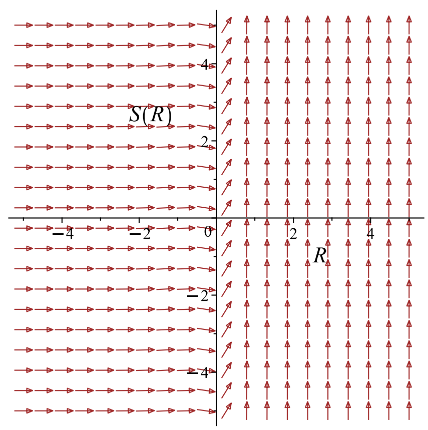
Which simplifies to

$$
x^2y = x^2\mathrm{e}^{3x} + c_1
$$

Which gives

$$
y = \frac{x^2\mathrm{e}^{3x} + c_1}{x^2}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x\,e^{3x}+2\,e^{3x}-2y}{x}$ 	$R = x$ $S = x^2 y$	$\frac{dS}{dR} = e^{3R}(2+3R)\,R$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = e^3 + c_1$$

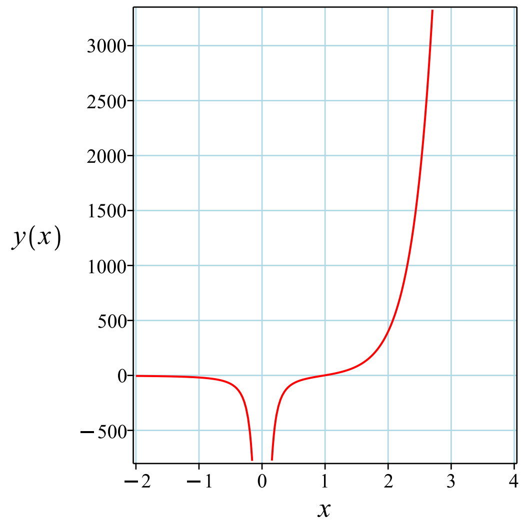
$$c_1 = 1 - e^3$$

Substituting c_1 found above in the general solution gives

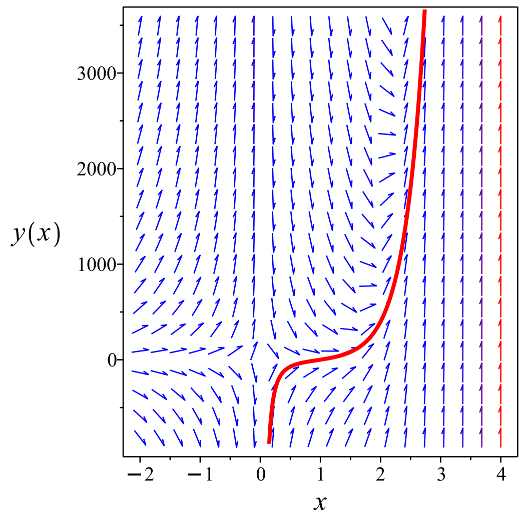
$$y = \frac{x^2 e^{3x} - e^3 + 1}{x^2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{x^2 e^{3x} - e^3 + 1}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2 e^{3x} - e^3 + 1}{x^2}$$

Verified OK.

1.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x) \, \mathrm{d}y &= \left(-2y + (2 + 3x) \, \mathrm{e}^{3x}\right) \mathrm{d}x \\ \left(2y - (2 + 3x) \, \mathrm{e}^{3x}\right) \mathrm{d}x + (x) \, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 2y - (2 + 3x) \, \mathrm{e}^{3x} \\ N(x, y) &= x \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y - (2 + 3x) \, \mathrm{e}^{3x}\right) \\ &= 2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((2) - (1)) \\ &= \frac{1}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int \frac{1}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)} \\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x\left(2y - (2+3x)\,\mathrm{e}^{3x}\right) \\ &= \left(-3x^2 - 2x\right)\mathrm{e}^{3x} + 2xy\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x(x) \\ &= x^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\left(-3x^2 - 2x\right)\mathrm{e}^{3x} + 2xy\right) + \left(x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \left(-3x^2 - 2x\right)\mathrm{e}^{3x} + 2xy\,\mathrm{d}x\end{aligned}$$

$$\phi = x^2\left(-\mathrm{e}^{3x} + y\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2(-e^{3x} + y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2(-e^{3x} + y)$$

The solution becomes

$$y = \frac{x^2 e^{3x} + c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = e^3 + c_1$$

$$c_1 = 1 - e^3$$

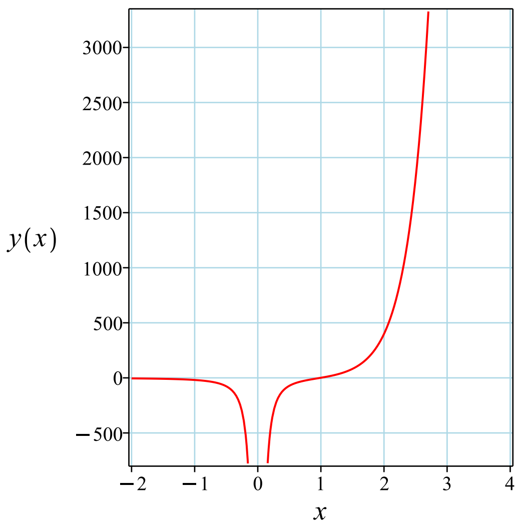
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 e^{3x} - e^3 + 1}{x^2}$$

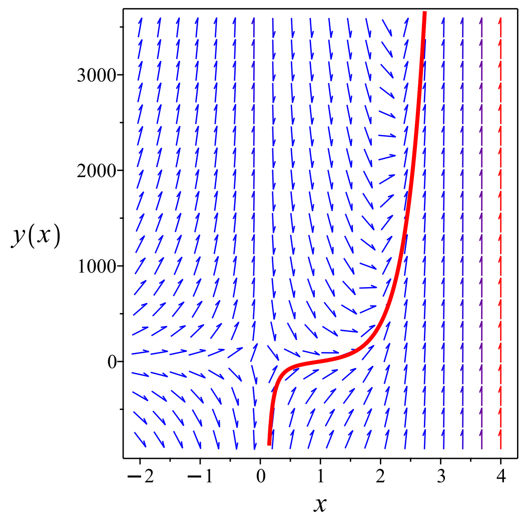
Summary

The solution(s) found are the following

$$y = \frac{x^2 e^{3x} - e^3 + 1}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2 e^{3x} - e^3 + 1}{x^2}$$

Verified OK.

1.19.5 Maple step by step solution

Let's solve

$[xy' + 2y = (2 + 3x) e^{3x}, y(1) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + \frac{e^{3x}(2+3x)}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = \frac{e^{3x}(2+3x)}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{2y}{x}\right)=\frac{\mu(x)\mathrm{e}^{3x}(2+3x)}{x}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{2y}{x}\right)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=\frac{2\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x)=x^2$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)\mathrm{e}^{3x}(2+3x)}{x}dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{\mu(x)\mathrm{e}^{3x}(2+3x)}{x}dx+c_1$
- Solve for y

  $y=\frac{\int\frac{\mu(x)\mathrm{e}^{3x}(2+3x)}{x}dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=x^2$

  $y=\frac{\int\mathrm{e}^{3x}(2+3x)x\,dx+c_1}{x^2}$
- Evaluate the integrals on the rhs

  $y=\frac{x^2\mathrm{e}^{3x}+c_1}{x^2}$
- Use initial condition $y(1)=1$

  $1=\mathrm{e}^3+c_1$
- Solve for c_1

  $c_1=1-\mathrm{e}^3$
- Substitute $c_1=1-\mathrm{e}^3$ into general solution and simplify

  $y=\frac{x^2\mathrm{e}^{3x}-\mathrm{e}^3+1}{x^2}$
- Solution to the IVP

  $y=\frac{x^2\mathrm{e}^{3x}-\mathrm{e}^3+1}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 19

```
dsolve([x*diff(y(x),x)+2*y(x)=(3*x+2)*exp(3*x),y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{x^2 e^{3x} - e^3 + 1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 22

```
DSolve[{x*y'[x]+2*y[x]==(3*x+2)*Exp[3*x],y[1]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^3}{x^2} + \frac{1}{x^2} + e^{3x}$$

1.20 problem 2(j)

Internal problem ID [3021]
Internal file name [`OUTPUT/2513_Sunday_June_05_2022_03_17_27_AM_79562150/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(j).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{2\sin(3x)\sin(2y)\,y' - 3\cos(3x)\cos(2y) = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{12}\right) = \frac{\pi}{8}\right]$$

1.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{3\cos(3x)\cos(2y)}{2\sin(3x)\sin(2y)} \end{aligned}$$

The x domain of $f(x,y)$ when $y = \frac{\pi}{8}$ is

$$\left\{x < \frac{\pi\_Z155}{3} \vee \frac{\pi\_Z155}{3} < x\right\}$$

And the point $x_0 = \frac{\pi}{12}$ is inside this domain. The y domain of $f(x,y)$ when $x = \frac{\pi}{12}$ is

$$\left\{y < \frac{\pi\_Z156}{2} \vee \frac{\pi\_Z156}{2} < y\right\}$$

And the point $y_0 = \frac{\pi}{8}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{3\cos(3x)\cos(2y)}{2\sin(3x)\sin(2y)}\right)\\ &= -\frac{3\cos(3x)}{\sin(3x)} - \frac{3\cos(3x)\cos(2y)^2}{\sin(3x)\sin(2y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{\pi}{8}$ is

$$\left\{x < \frac{\pi\_Z155}{3} \vee \frac{\pi\_Z155}{3} < x\right\}$$

And the point $x_0 = \frac{\pi}{12}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \frac{\pi}{12}$ is

$$\left\{y < \frac{\pi\_Z156}{2} \vee \frac{\pi\_Z156}{2} < y\right\}$$

And the point $y_0 = \frac{\pi}{8}$ is inside this domain. Therefore solution exists and is unique.

1.20.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{3\cos(3x)\cot(2y)}{2\sin(3x)}\end{aligned}$$

Where $f(x) = \frac{3\cos(3x)}{2\sin(3x)}$ and $g(y) = \cot(2y)$. Integrating both sides gives

$$\frac{1}{\cot(2y)}\,dy = \frac{3\cos(3x)}{2\sin(3x)}\,dx$$

$$\int \frac{1}{\cot(2y)}\,dy = \int \frac{3\cos(3x)}{2\sin(3x)}\,dx$$

$$-\frac{\ln(\cos(2y))}{2} = \frac{\ln(\sin(3x))}{2} + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{\cos(2y)}} = e^{\frac{\ln(\sin(3x))}{2}+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{\cos(2y)}} = c_2\sqrt{\sin(3x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{12}$ and $y = \frac{\pi}{8}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{8} = \frac{\pi}{4} - \frac{\arcsin\left(\frac{\sqrt{2}\,e^{-2c_1}}{c_2^2}\right)}{2}$$

$$c_1 = -\frac{\ln\left(\frac{c_2^2}{2}\right)}{2}$$

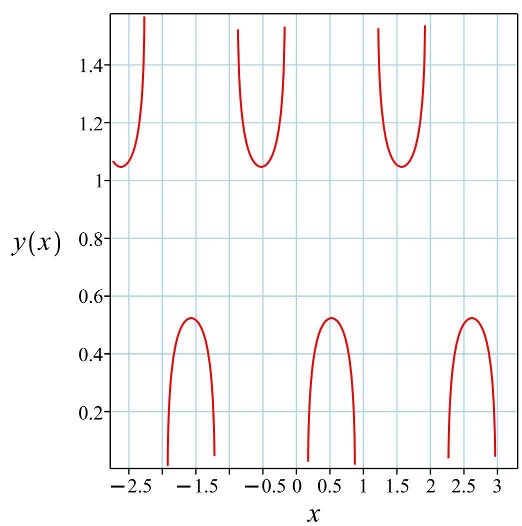
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{4} - \frac{\arcsin\left(\frac{1}{2\sin(3x)}\right)}{2}$$

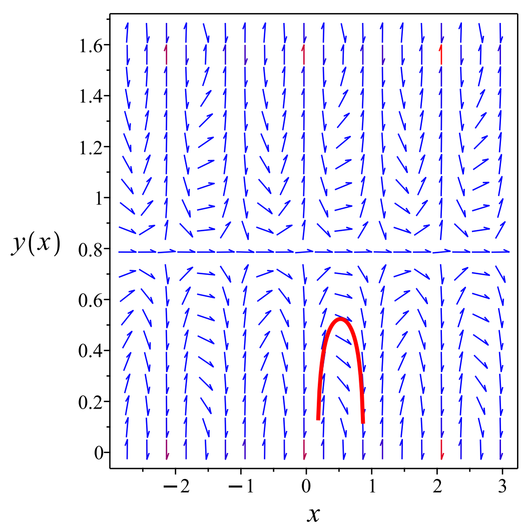
Summary

The solution(s) found are the following

$$y = \frac{\pi}{4} - \frac{\arcsin\left(\frac{1}{2\sin(3x)}\right)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{4} - \frac{\arcsin\left(\frac{1}{2\sin(3x)}\right)}{2}$$

Verified OK.

1.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{3\cos(3x)\cos(2y)}{2\sin(3x)\sin(2y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 44: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{2 \sin(3x)}{3 \cos(3x)} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{2\sin(3x)}{3\cos(3x)}} dx \end{aligned}$$

Which results in

$$S = \frac{\ln (\sin (3x))}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{3\cos (3x) \cos (2y)}{2\sin (3x) \sin (2y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{3\cot (3x)}{2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \tan (2y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \tan (2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln\left(1+\tan\left(2R\right)^{2}\right)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(\sin\left(3x\right)\right)}{2} = \frac{\ln\left(1+\tan\left(2y\right)^{2}\right)}{4} + c_1$$

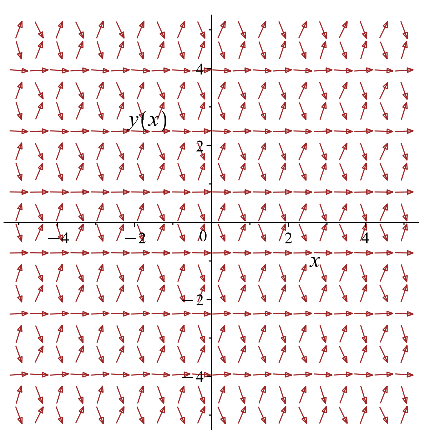
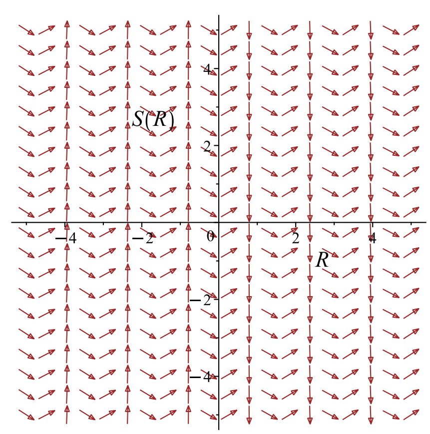
Which simplifies to

$$\frac{\ln\left(\sin\left(3x\right)\right)}{2} - \frac{\ln\left(\sec\left(2y\right)\right)}{2} - c_1 = 0$$

Which gives

$$y = \frac{\operatorname{arcsec}\left(\sin\left(3x\right) \mathrm{e}^{-2c_1}\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3\cos(3x)\cos(2y)}{2\sin(3x)\sin(2y)}$ 	$R = y$ $S = \frac{\ln(\sin(3x))}{2}$	$\frac{dS}{dR} = \tan(2R)$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{12}$ and $y = \frac{\pi}{8}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{8} = \frac{\pi}{4} - \frac{\arcsin\left(\sqrt{2}\,\mathrm{e}^{2c_1}\right)}{2}$$

$$c_1 = -\frac{\ln(2)}{2}$$

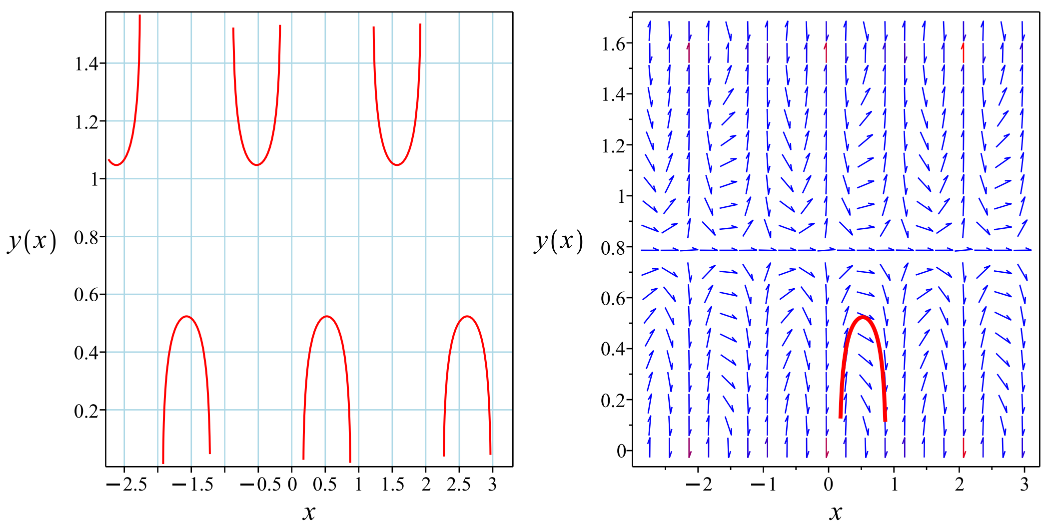
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{4} - \frac{\arcsin\left(\frac{1}{2\sin(3x)}\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\pi}{4} - \frac{\arcsin\left(\frac{1}{2\sin(3x)}\right)}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{4} - \frac{\arcsin\left(\frac{1}{2\sin(3x)}\right)}{2}$$

Verified OK.

1.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{2\sin(2y)}{3\cos(2y)}\right)\mathrm{d}y&=\left(\frac{\cos(3x)}{\sin(3x)}\right)\mathrm{d}x\\ \left(-\frac{\cos(3x)}{\sin(3x)}\right)\mathrm{d}x+\left(\frac{2\sin(2y)}{3\cos(2y)}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{\cos(3x)}{\sin(3x)} \\
N(x,y) &= \frac{2\sin(2y)}{3\cos(2y)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\cos(3x)}{\sin(3x)}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2\sin(2y)}{3\cos(2y)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{\cos(3x)}{\sin(3x)} \, \mathrm{d}x \\
\phi &= -\frac{\ln(\sin(3x))}{3} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2\sin(2y)}{3\cos(2y)}$. Therefore equation (4) becomes

$$\frac{2\sin(2y)}{3\cos(2y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= \frac{2\sin(2y)}{3\cos(2y)} \\ &= \frac{2\tan(2y)}{3} \end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned} \int f'(y)\,\mathrm{d}y &= \int \left(\frac{2\tan(2y)}{3}\right) \mathrm{d}y \\ f(y) &= \frac{\ln\left(1 + \tan(2y)^2\right)}{6} + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(\sin(3x))}{3} + \frac{\ln\left(1 + \tan(2y)^2\right)}{6} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(\sin(3x))}{3} + \frac{\ln\left(1 + \tan(2y)^2\right)}{6}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{12}$ and $y = \frac{\pi}{8}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\ln(2)}{3} = c_1$$

$$c_1 = \frac{\ln(2)}{3}$$

Substituting c_1 found above in the general solution gives

$$-\frac{\ln(\sin(3x))}{3}+\frac{\ln\left(1+\tan(2y)^2\right)}{6}=\frac{\ln(2)}{3}$$

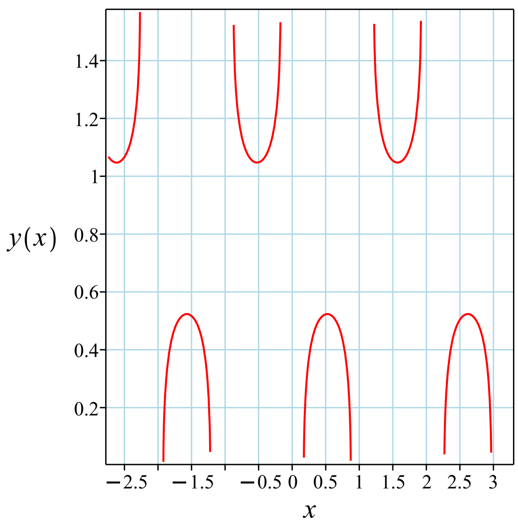
Solving for y from the above gives

$$y=\frac{\operatorname{arcsec}(2\sin(3x))}{2}$$

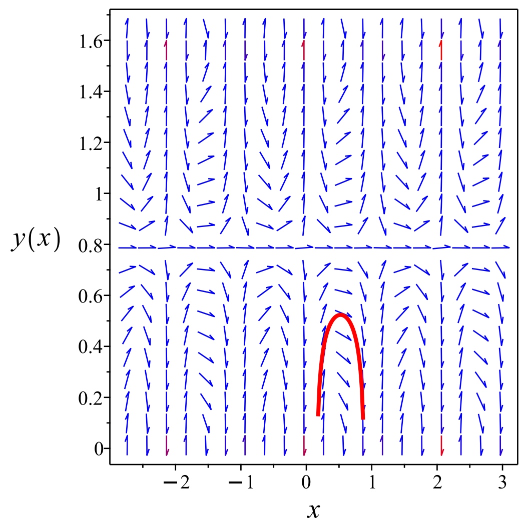
Summary

The solution(s) found are the following

$$y=\frac{\operatorname{arcsec}(2\sin(3x))}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y=\frac{\operatorname{arcsec}(2\sin(3x))}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.359 (sec). Leaf size: 17

```
dsolve([2*sin(3*x)*sin(2*y(x))*diff(y(x),x)-3*cos(3*x)*cos(2*y(x))=0,y(1/12*Pi) = 1/8*Pi],y(
```

$$y(x) = \frac{\pi}{4} - \frac{\arctan\left(\frac{1}{\sqrt{1-2\cos(6x)}}\right)}{2}$$

✓ Solution by Mathematica

Time used: 6.727 (sec). Leaf size: 18

```
DSolve[{2*Sin[3*x]*Sin[2*y[x]]*y'[x]-3*Cos[3*x]*Cos[2*y[x]]==0,y[Pi/12]==Pi/8},y[x],x,Includ
```

$$y(x) \to \frac{1}{2}\arccos\left(\frac{1}{2}\csc(3x)\right)$$

1.21 problem 2(k)

Internal problem ID [3022]
Internal file name [`OUTPUT/2514_Sunday_June_05_2022_03_17_31_AM_74195310/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(k).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{xyy' - (x+1)(y+1) = 0}$$

With initial conditions

$$[y(1) = 1]$$

1.21.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{xy + x + y + 1}{xy} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{xy + x + y + 1}{xy}\right) \\ &= \frac{x + 1}{xy} - \frac{xy + x + y + 1}{x\, y^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.21.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{(x + 1)\,(y + 1)}{xy}\end{aligned}$$

Where $f(x) = \frac{x+1}{x}$ and $g(y) = \frac{y+1}{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{y+1}{y}}\, dy &= \frac{x + 1}{x}\, dx \\ \int \frac{1}{\frac{y+1}{y}}\, dy &= \int \frac{x + 1}{x}\, dx \\ y - \ln (y + 1) &= x + \ln (x) + c_1\end{aligned}$$

Which results in

$$y = -\operatorname{LambertW}\left(-\frac{e^{-1-x-c_1}}{x}\right) - 1$$

Since c_1 is constant, then exponential powers of this constant are constants also, and these can be simplified to just c_1 in the above solution. The solution becomes

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x-1}}{c_1 x}\right) - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\operatorname{LambertW}\left(-\frac{e^{-2}}{c_1}\right) - 1$$

$$c_1 = \frac{1}{2}$$

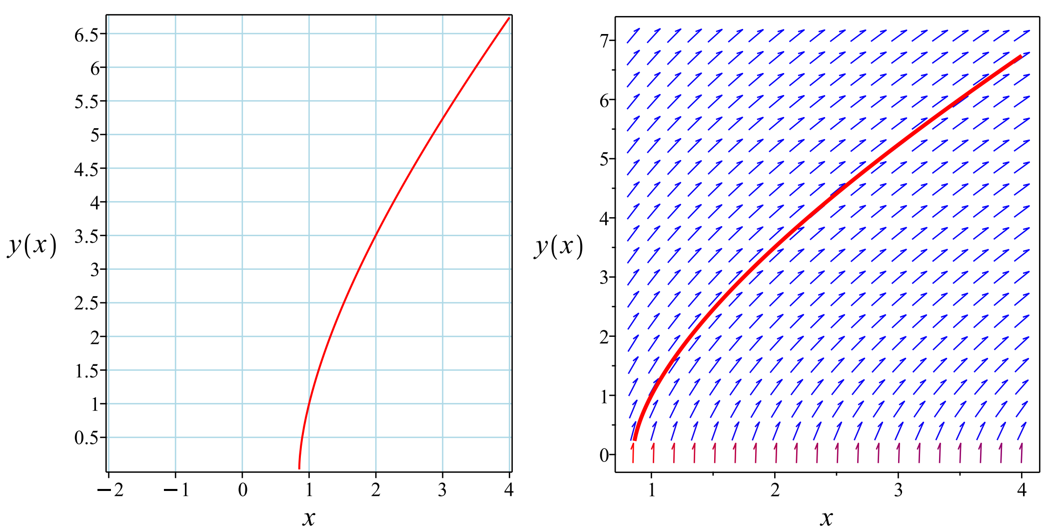
Substituting c_1 found above in the general solution gives

$$y = -\operatorname{LambertW}\left(-\frac{2\,e^{-x-1}}{x}\right) - 1$$

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-1, -\frac{2\,e^{-x-1}}{x}\right) - 1 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = -\operatorname{LambertW}\left(-1, -\frac{2\,\mathrm{e}^{-x-1}}{x}\right) - 1
$$

Verified OK.

1.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{xy + x + y + 1}{xy} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 46: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= \frac{x}{x+1} \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x}{x+1}} dx \end{aligned}$$

Which results in

$$S = x + \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{xy + x + y + 1}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 1 + \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R - \ln(R+1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x + \ln(x) = y - \ln(y+1) + c_1$$

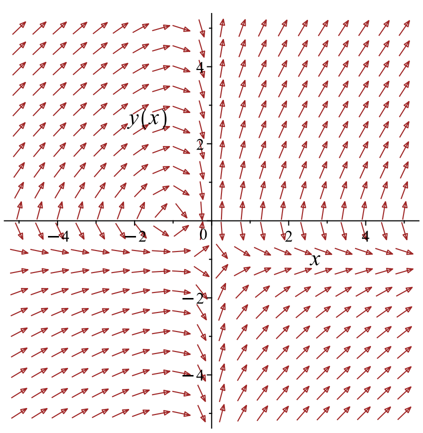
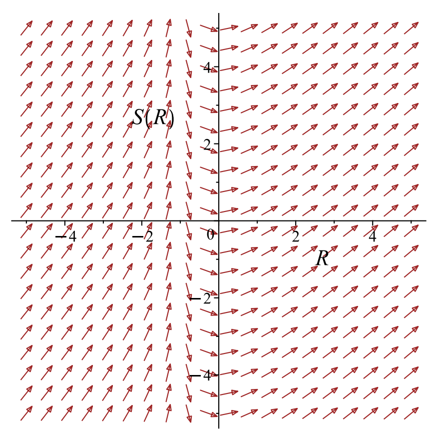
Which simplifies to

$$x + \ln(x) = y - \ln(y+1) + c_1$$

Which gives

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x-1+c_1}}{x}\right) - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{xy+x+y+1}{xy}$ 	$R = y$ $S = x + \ln(x)$	$\frac{dS}{dR} = \frac{R}{R+1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\operatorname{LambertW}\left(-e^{-2+c_1}\right) - 1$$

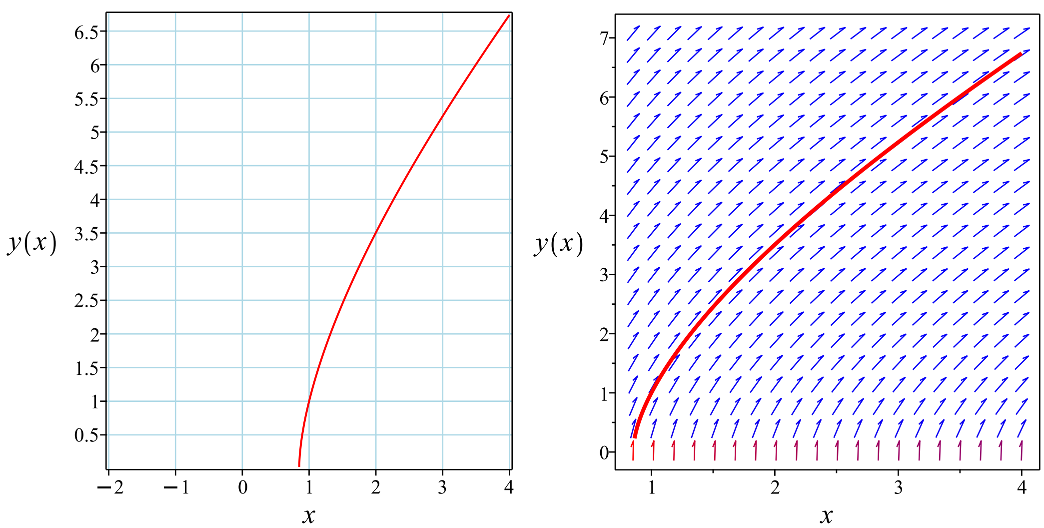
$$c_1 = \ln(2)$$

Substituting c_1 found above in the general solution gives

$$y = -\operatorname{LambertW}\left(-\frac{2\,e^{-x-1}}{x}\right) - 1$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-1, -\frac{2\,e^{-x-1}}{x}\right) - 1 \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\operatorname{LambertW}\left(-1, -\frac{2\,e^{-x-1}}{x}\right) - 1$$

Verified OK.

1.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{y+1}\right)\mathrm{d}y &= \left(\frac{x+1}{x}\right)\mathrm{d}x\\ \left(-\frac{x+1}{x}\right)\mathrm{d}x + \left(\frac{y}{y+1}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{x+1}{x} \\
N(x,y) &= \frac{y}{y+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x+1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{y+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{x+1}{x} \,\mathrm{d}x \\
\phi &= -x - \ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{y+1}$. Therefore equation (4) becomes

$$\frac{y}{y+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{y+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y}{y+1}\right) \mathrm{d}y$$
$$f(y) = y - \ln(y+1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \ln(x) + y - \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \ln(x) + y - \ln(y+1)$$

The solution becomes

$$y = -\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-1-x-c_1}}{x}\right) - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\operatorname{LambertW}\left(-\mathrm{e}^{-c_1-2}\right) - 1$$

$$c_1 = -\ln(2)$$

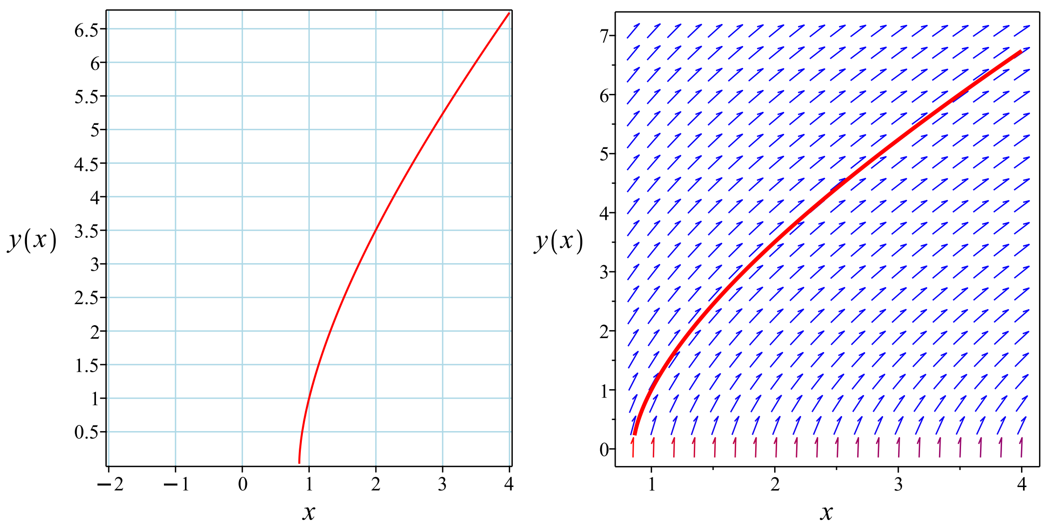
Substituting c_1 found above in the general solution gives

$$y = -\operatorname{LambertW}\left(-\frac{2\,e^{-x-1}}{x}\right) - 1$$

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-1, -\frac{2\,e^{-x-1}}{x}\right) - 1 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-1, -\frac{2\,e^{-x-1}}{x}\right) - 1$$

Verified OK.

1.21.5 Maple step by step solution

Let's solve

$[xyy' - (x+1)(y+1) = 0, y(1) = 1]$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

  $\frac{y'y}{y+1} = \frac{x+1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{y+1}dx = \int \frac{x+1}{x}dx + c_1$

- Evaluate integral

  $y - \ln(y+1) = x + \ln(x) + c_1$

- Solve for y

  $y = -\mathit{LambertW}\left(-\frac{e^{-1-x-c_1}}{x}\right) - 1$

- Use initial condition $y(1) = 1$

  $1 = -\mathit{LambertW}(-e^{-c_1-2}) - 1$

- Solve for c_1

  $c_1 = -\ln(2)$

- Substitute $c_1 = -\ln(2)$ into general solution and simplify

  $y = -\mathit{LambertW}\left(-\frac{2\,e^{-x-1}}{x}\right) - 1$

- Solution to the IVP

  $y = -\mathit{LambertW}\left(-\frac{2\,e^{-x-1}}{x}\right) - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 21

```
dsolve([x*y(x)*diff(y(x),x)=(x+1)*(y(x)+1),y(1) = 1],y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}\left(-1, -\frac{2\,\mathrm{e}^{-x-1}}{x}\right) - 1$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{x*y[x]*y'[x]==(x+1)*(y[x]+1),y[1]==1},y[x],x,IncludeSingularSolutions -> True]
```

{}

1.22 problem 2(L)

Internal problem ID [3023]
Internal file name [`OUTPUT/2515_Sunday_June_05_2022_03_17_33_AM_12604559/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(L).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{2x - y}{y + 2x} = 0}$$

With initial conditions

$$[y(2) = 2]$$

1.22.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{-2x + y}{2x + y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 2$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x, y)$ when $x = 2$ is

$$\{y < -4 \lor -4 < y\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{-2x+y}{2x+y}\right)\\ &= -\frac{1}{2x+y} + \frac{-2x+y}{(2x+y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 2$ is

$$\{y < -4 \lor -4 < y\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

1.22.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - \frac{2x - u(x)\, x}{u(x)\, x + 2x} = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u)\\ &= f(x)g(u)\\ &= -\frac{u^2 + 3u - 2}{x(u+2)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+3u-2}{u+2}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+3u-2}{u+2}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u^2+3u-2}{u+2}}\, du = \int -\frac{1}{x}\, dx$$

$$\frac{\ln(u^2 + 3u - 2)}{2} - \frac{\sqrt{17}\, \operatorname{arctanh}\left(\frac{(2u+3)\sqrt{17}}{17}\right)}{17} = -\ln(x) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2+3u(x)-2\right)}{2}-\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{(2u(x)+3)\sqrt{17}}{17}\right)}{17}+\ln(x)-c_2=0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y^2}{x^2}+\frac{3y}{x}-2\right)}{2}-\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{\left(\frac{2y}{x}+3\right)\sqrt{17}}{17}\right)}{17}+\ln(x)-c_2=0$$
$$\frac{\ln\left(\frac{y^2}{x^2}+\frac{3y}{x}-2\right)}{2}-\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{(3x+2y)\sqrt{17}}{17x}\right)}{17}+\ln(x)-c_2=0$$

Substituting initial conditions and solving for c_2 gives $c_2=\frac{3\ln(2)}{2}-\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{5\sqrt{17}}{17}\right)}{17}$.

Summary

The solution(s) found are the following

Hence the solution becomes
$$\frac{\ln\left(\frac{y^2}{x^2}+\frac{3y}{x}-2\right)}{2}-\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{(3x+2y)\sqrt{17}}{17x}\right)}{17}+\ln(x)-\frac{3\ln(2)}{2}+\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{5\sqrt{17}}{17}\right)}{17}=0$$

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+\frac{3y}{x}-2\right)}{2}-\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{(3x+2y)\sqrt{17}}{17x}\right)}{17}+\ln(x)-\frac{3\ln(2)}{2}+\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{5\sqrt{17}}{17}\right)}{17}=0$$

Verified OK.

1.22.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{-2x+y}{2x+y}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(-2x+y)(b_3 - a_2)}{2x+y} - \frac{(-2x+y)^2 a_3}{(2x+y)^2} - \left(\frac{2}{2x+y} + \frac{-4x+2y}{(2x+y)^2}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{1}{2x+y} + \frac{-2x+y}{(2x+y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4x^2a_2 + 4x^2a_3 - 8x^2b_2 - 4x^2b_3 + 4xya_2 - 4xya_3 - 4xyb_2 - 4xyb_3 - y^2a_2 + 5y^2a_3 - y^2b_2 + y^2b_3 - 4xb_1}{(2x+y)^2} = 0$$

Setting the numerator to zero gives

$$-4x^2a_2 - 4x^2a_3 + 8x^2b_2 + 4x^2b_3 - 4xya_2 + 4xya_3 + 4xyb_2 + 4xyb_3 + y^2a_2 - 5y^2a_3 + y^2b_2 - y^2b_3 + 4xb_1 - 4ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-4a_2v_1^2 - 4a_2v_1v_2 + a_2v_2^2 - 4a_3v_1^2 + 4a_3v_1v_2 - 5a_3v_2^2 + 8b_2v_1^2 \\
&+ 4b_2v_1v_2 + b_2v_2^2 + 4b_3v_1^2 + 4b_3v_1v_2 - b_3v_2^2 - 4a_1v_2 + 4b_1v_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-4a_2 - 4a_3 + 8b_2 + 4b_3)\, v_1^2 + (-4a_2 + 4a_3 + 4b_2 + 4b_3)\, v_1v_2 \\
&+ 4b_1v_1 + (a_2 - 5a_3 + b_2 - b_3)\, v_2^2 - 4a_1v_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-4a_1 &= 0 \\
4b_1 &= 0 \\
-4a_2 - 4a_3 + 8b_2 + 4b_3 &= 0 \\
-4a_2 + 4a_3 + 4b_2 + 4b_3 &= 0 \\
a_2 - 5a_3 + b_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 3a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= 2a_3 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{-2x+y}{2x+y}\right)(x) \\
&= \frac{-2x^2+3xy+y^2}{2x+y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2x^2+3xy+y^2}{2x+y}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln\left(-2x^2+3xy+y^2\right)}{2} - \frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{(3x+2y)\sqrt{17}}{17x}\right)}{17}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{-2x+y}{2x+y}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2x-y}{2x^2-3xy-y^2} \\
S_y &= \frac{-2x-y}{2x^2-3xy-y^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

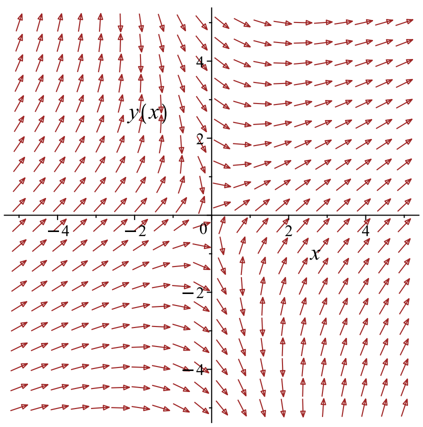
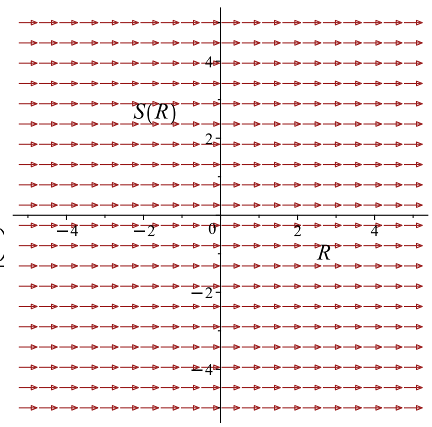
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(y^2+3yx-2x^2)}{2} - \frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{(3x+2y)\sqrt{17}}{17x}\right)}{17} = c_1
$$

Which simplifies to

$$
\frac{\ln(y^2+3yx-2x^2)}{2} - \frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{(3x+2y)\sqrt{17}}{17x}\right)}{17} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-2x+y}{2x+y}$ 	$R = x$ $S = \frac{\ln(-2x^2+3xy+y^2)}{2}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3\ln(2)}{2} - \frac{\sqrt{17}\,\operatorname{arccoth}\left(\frac{5\sqrt{17}}{17}\right)}{17} + \frac{i\sqrt{17}\,\pi}{34} = c_1$$

$$c_1 = \frac{3\ln(2)}{2} - \frac{\sqrt{17}\,\operatorname{arccoth}\left(\frac{5\sqrt{17}}{17}\right)}{17} + \frac{i\sqrt{17}\,\pi}{34}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(-2x^2+3xy+y^2)}{2} - \frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{(3x+2y)\sqrt{17}}{17x}\right)}{17} = \frac{3\ln(2)}{2} - \frac{\sqrt{17}\,\operatorname{arccoth}\left(\frac{5\sqrt{17}}{17}\right)}{17} + \frac{i\sqrt{17}\,\pi}{34}$$

Summary

The solution(s) found are the following

$$\frac{\ln(y^2+3yx-2x^2)}{2} - \frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{(3x+2y)\sqrt{17}}{17x}\right)}{17} = \frac{3\ln(2)}{2} - \frac{\sqrt{17}\,\operatorname{arccoth}\left(\frac{5\sqrt{17}}{17}\right)}{17} + \frac{i\sqrt{17}\,\pi}{34} \quad (1)$$

Verification of solutions

$$\frac{\ln(y^2+3yx-2x^2)}{2}-\frac{\sqrt{17}\,\operatorname{arctanh}\left(\frac{(3x+2y)\sqrt{17}}{17x}\right)}{17} = \frac{3\ln(2)}{2}-\frac{\sqrt{17}\,\operatorname{arccoth}\left(\frac{5\sqrt{17}}{17}\right)}{17}+\frac{i\sqrt{17}\,\pi}{34}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 2.297 (sec). Leaf size: 66

```
dsolve([diff(y(x),x)=(2*x-y(x))/(2*x+y(x)),y(2) = 2],y(x), singsol=all)
```

$$y(x) = \operatorname{RootOf}\left(-2\sqrt{17}\,\operatorname{arctanh}\left(\frac{5\sqrt{17}}{17}\right)+2\sqrt{17}\,\operatorname{arctanh}\left(\frac{(3x+2\_Z)\sqrt{17}}{17x}\right) + 51\ln(2)-34\ln(x)-17\ln\left(\frac{\_Z^2+3x\_Z-2x^2}{x^2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 137

```
DSolve[{y'[x]==(2*x-y[x])/(2*x+y[x]),y[2]==2},y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{34}\left(\left(17+\sqrt{17}\right)\log\left(-\frac{2y(x)}{x}+\sqrt{17}-3\right)-\left(\sqrt{17}-17\right)\log\left(\frac{2y(x)}{x}+\sqrt{17}+3\right)\right)=-\log(x)+\frac{1}{34}i\left(17+\sqrt{17}\right)\pi+\frac{1}{34}\left(34\log(2)+17\log\left(5-\sqrt{17}\right)+\sqrt{17}\log\left(5-\sqrt{17}\right)+17\log\left(5+\sqrt{17}\right)-\sqrt{17}\log\left(5+\sqrt{17}\right)\right),y(x)\right]$$

1.23 problem 2(m)

Internal problem ID [3024]
Internal file name [`OUTPUT/2516_Sunday_June_05_2022_03_17_42_AM_65814307/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(m).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{3x - y + 1}{3y - x + 5} = 0}$$

With initial conditions

$$[y(0) = 0]$$

1.23.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{-3x + y - 1}{3y - x + 5} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{x < 5 \lor 5 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\left\{y < -\frac{5}{3} \lor -\frac{5}{3} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{-3x+y-1}{3y-x+5}\right)\\ &= -\frac{1}{3y-x+5} + \frac{-9x+3y-3}{(3y-x+5)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < 5 \lor 5 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\left\{y < -\frac{5}{3} \lor -\frac{5}{3} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

1.23.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{-3X - 3x_0 + Y(X) + y_0 - 1}{3Y(X) + 3y_0 - X - x_0 + 5}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= -1\\ y_0 &= -2\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{-3X + Y(X)}{3Y(X) - X}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X,Y) \\
&= -\frac{-3X+Y}{3Y-X}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X,Y)
$$

In this case, it can be seen that both $M = -3X + Y$ and $N = -3Y + X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-u+3}{3u-1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-u(X)+3}{3u(X)-1} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{-u(X)+3}{3u(X)-1} - u(X)}{X} = 0
$$

Or

$$
3\left(\frac{d}{dX}u(X)\right)Xu(X) - \left(\frac{d}{dX}u(X)\right)X + 3u(X)^2 - 3 = 0
$$

Or

$$
-3 + X(3u(X) - 1)\left(\frac{d}{dX}u(X)\right) + 3u(X)^2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= -\frac{3(u^2-1)}{X(3u-1)}
\end{aligned}
$$

Where $f(X) = -\frac{3}{X}$ and $g(u) = \frac{u^2-1}{3u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{3u-1}}\,du = -\frac{3}{X}\,dX$$

$$\int \frac{1}{\frac{u^2-1}{3u-1}}\,du = \int -\frac{3}{X}\,dX$$

$$\ln(u-1) + 2\ln(u+1) = -3\ln(X) + c_2$$

Raising both side to exponential gives

$$e^{\ln(u-1)+2\ln(u+1)} = e^{-3\ln(X)+c_2}$$

Which simplifies to

$$(u-1)(u+1)^2 = \frac{c_3}{X^3}$$

The solution is

$$(u(X)-1)(u(X)+1)^2 = \frac{c_3}{X^3}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\left(\frac{Y(X)}{X}-1\right)\left(\frac{Y(X)}{X}+1\right)^2 = \frac{c_3}{X^3}$$

Which simplifies to

$$-(-Y(X)+X)(Y(X)+X)^2 = c_3$$

Using the solution for $Y(X)$

$$-(-Y(X)+X)(Y(X)+X)^2 = c_3$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 2$$
$$X = x - 1$$

Then the solution in y becomes

$$-(-y-1+x)(y+3+x)^2 = c_3$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$9 = c_3$$

$$c_3 = 9$$

Substituting c_3 found above in the general solution gives

$$-(x-1-y)(x+3+y)^2 = 9$$

Summary

The solution(s) found are the following

$$-(-y-1+x)(y+3+x)^2 = 9 \tag{1}$$

Verification of solutions

$$-(-y-1+x)(y+3+x)^2 = 9$$

Verified OK.

1.23.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{-3x+y-1}{3y-x+5}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(-3x+y-1)(b_3-a_2)}{3y-x+5} - \frac{(-3x+y-1)^2 a_3}{(3y-x+5)^2} \\ & - \left(\frac{3}{3y-x+5} - \frac{-3x+y-1}{(3y-x+5)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{1}{3y-x+5} + \frac{-9x+3y-3}{(3y-x+5)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^2a_2 - 9x^2a_3 + 9x^2b_2 - 3x^2b_3 - 18xya_2 + 6xya_3 - 6xyb_2 + 18xyb_3 + 3y^2a_2 - 9y^2a_3 + 9y^2b_2 - 3y^2b_3 - 3}{(-3} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 3x^2a_2 - 9x^2a_3 + 9x^2b_2 - 3x^2b_3 - 18xya_2 + 6xya_3 - 6xyb_2 + 18xyb_3 + 3y^2a_2 \\ & - 9y^2a_3 + 9y^2b_2 - 3y^2b_3 - 30xa_2 - 6xa_3 + 8xb_1 - 2xb_2 + 14xb_3 - 8ya_1 \\ & + 2ya_2 - 14ya_3 + 30yb_2 + 6yb_3 - 16a_1 - 5a_2 - a_3 + 8b_1 + 25b_2 + 5b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 3a_2v_1^2 - 18a_2v_1v_2 + 3a_2v_2^2 - 9a_3v_1^2 + 6a_3v_1v_2 - 9a_3v_2^2 + 9b_2v_1^2 - 6b_2v_1v_2 + 9b_2v_2^2 \\ & - 3b_3v_1^2 + 18b_3v_1v_2 - 3b_3v_2^2 - 8a_1v_2 - 30a_2v_1 + 2a_2v_2 - 6a_3v_1 - 14a_3v_2 + 8b_1v_1 \\ & - 2b_2v_1 + 30b_2v_2 + 14b_3v_1 + 6b_3v_2 - 16a_1 - 5a_2 - a_3 + 8b_1 + 25b_2 + 5b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(3a_2 - 9a_3 + 9b_2 - 3b_3) v_1^2 + (-18a_2 + 6a_3 - 6b_2 + 18b_3) v_1 v_2 \\ + (-30a_2 - 6a_3 + 8b_1 - 2b_2 + 14b_3) v_1 + (3a_2 - 9a_3 + 9b_2 - 3b_3) v_2^2 \\ + (-8a_1 + 2a_2 - 14a_3 + 30b_2 + 6b_3) v_2 - 16a_1 - 5a_2 - a_3 + 8b_1 + 25b_2 + 5b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-18a_2 + 6a_3 - 6b_2 + 18b_3 &= 0\\ 3a_2 - 9a_3 + 9b_2 - 3b_3 &= 0\\ -8a_1 + 2a_2 - 14a_3 + 30b_2 + 6b_3 &= 0\\ -30a_2 - 6a_3 + 8b_1 - 2b_2 + 14b_3 &= 0\\ -16a_1 - 5a_2 - a_3 + 8b_1 + 25b_2 + 5b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 2b_2 + b_3\\ a_2 &= b_3\\ a_3 &= b_2\\ b_1 &= b_2 + 2b_3\\ b_2 &= b_2\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x + 1\\ \eta &= y + 2\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi\\ &= y + 2 - \left(-\frac{-3x + y - 1}{3y - x + 5}\right)(x + 1)\\ &= \frac{3x^2 - 3y^2 + 6x - 12y - 9}{-3y + x - 5}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{3x^2 - 3y^2 + 6x - 12y - 9}{-3y + x - 5}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln (y + 1 - x)}{3} + \frac{2 \ln (x + 3 + y)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-3x + y - 1}{3y - x + 5}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{-3 + 3x - 3y} + \frac{2}{3x + 9 + 3y} \\ S_y &= \frac{-3y + x - 5}{3 (x + 3 + y) (x - 1 - y)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

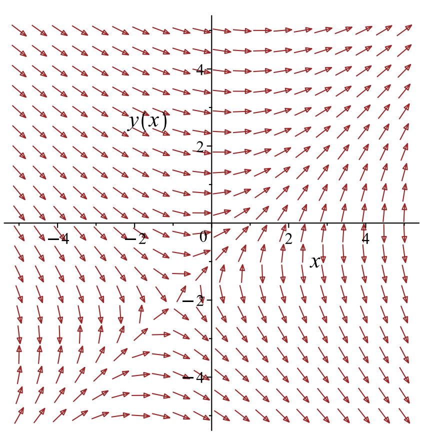
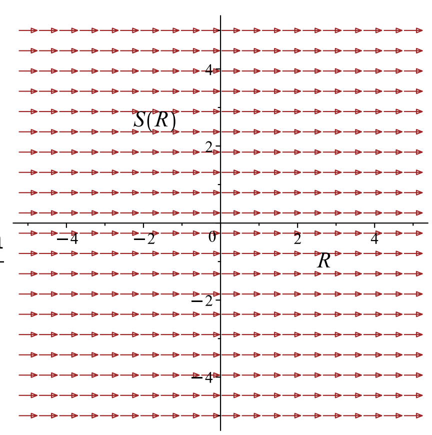
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y+1-x)}{3} + \frac{2\ln(y+3+x)}{3} = c_1$$

Which simplifies to

$$\frac{\ln(y+1-x)}{3} + \frac{2\ln(y+3+x)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-3x+y-1}{3y-x+5}$ 	$R = x$ $S = \frac{\ln(y+1-x)}{3} + \frac{2\ln}{}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{2\ln(3)}{3} = c_1$$

$$c_1 = \frac{2\ln(3)}{3}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(y+1-x)}{3} + \frac{2\ln(x+3+y)}{3} = \frac{2\ln(3)}{3}$$

Summary

The solution(s) found are the following

$$\frac{\ln(y+1-x)}{3} + \frac{2\ln(y+3+x)}{3} = \frac{2\ln(3)}{3} \tag{1}$$

Verification of solutions

$$\frac{\ln(y+1-x)}{3} + \frac{2\ln(y+3+x)}{3} = \frac{2\ln(3)}{3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 2.937 (sec). Leaf size: 84

```
dsolve([diff(y(x),x)=(3*x-y(x)+1)/(3*y(x)-x+5),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\left(-324+12\sqrt{96x^3+288x^2+288x+825}\right)^{\frac{4}{3}}-12\left(-324+12\sqrt{96x^3+288x^2+288x+825}\right)^{\frac{2}{3}}x-84\left(-3\right.}{36\left(-324+12\sqrt{96x^3+288x^2+288x+82\right.}}$$

✓ Solution by Mathematica

Time used: 60.775 (sec). Leaf size: 341

```
DSolve[{y'[x]==(3*x-y[x]+1)/(3*y[x]-x+5),y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x\text{Root}\left[\#1^6(1024x^6+6144x^5+15360x^4+20480x^3+15360x^2+6144x-58025)+\#1^4(-384x^4-153\right.}{}$$

1.24 problem 2(n)

Internal problem ID [3025]
Internal file name [`OUTPUT/2517_Sunday_June_05_2022_03_17_49_AM_46988216/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(n).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{3y+(7y-3x+3)\,y'=7x-7}$$

With initial conditions

$$[y(0)=0]$$

1.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned}y' &= f(x,y)\\ &= -\frac{3y-7x+7}{7y-3x+3}\end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{x<1\vee 1<x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\left\{y < -\frac{3}{7} \vee -\frac{3}{7} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{3y - 7x + 7}{7y - 3x + 3}\right) \\ &= -\frac{3}{7y - 3x + 3} + \frac{21y - 49x + 49}{(7y - 3x + 3)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < 1 \vee 1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\left\{y < -\frac{3}{7} \vee -\frac{3}{7} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

1.24.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{3Y(X) + 3y_0 - 7X - 7x_0 + 7}{7Y(X) + 7y_0 - 3X - 3x_0 + 3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= 1 \\ y_0 &= 0\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{3Y(X) - 7X}{7Y(X) - 3X}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X,Y) \\
&= -\frac{3Y-7X}{7Y-3X}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X,Y)
$$

In this case, it can be seen that both $M = 3Y - 7X$ and $N = -7Y + 3X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-3u+7}{7u-3} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-3u(X)+7}{7u(X)-3} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{-3u(X)+7}{7u(X)-3} - u(X)}{X} = 0
$$

Or

$$
7\left(\frac{d}{dX}u(X)\right)Xu(X) - 3\left(\frac{d}{dX}u(X)\right)X + 7u(X)^2 - 7 = 0
$$

Or

$$
-7 + X(7u(X) - 3)\left(\frac{d}{dX}u(X)\right) + 7u(X)^2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= -\frac{7(u^2-1)}{X\,(7u-3)}
\end{aligned}
$$

Where $f(X) = -\frac{7}{X}$ and $g(u) = \frac{u^2-1}{7u-3}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{7u-3}}\, du = -\frac{7}{X}\, dX$$

$$\int \frac{1}{\frac{u^2-1}{7u-3}}\, du = \int -\frac{7}{X}\, dX$$

$$2\ln(u-1) + 5\ln(u+1) = -7\ln(X) + c_2$$

Raising both side to exponential gives

$$e^{2\ln(u-1)+5\ln(u+1)} = e^{-7\ln(X)+c_2}$$

Which simplifies to

$$(u-1)^2 (u+1)^5 = \frac{c_3}{X^7}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \text{RootOf}\left(X^7 + 3X^6\_Z + X^5\_Z^2 - 5X^4\_Z^3 - 5X^3\_Z^4 + X^2\_Z^5 + 3X\_Z^6 + \_Z^7 - c_3\right)$$

Using the solution for $Y(X)$

$$Y(X) = \text{RootOf}\left(X^7 + 3X^6\_Z + X^5\_Z^2 - 5X^4\_Z^3 - 5X^3\_Z^4 + X^2\_Z^5 + 3X\_Z^6 + \_Z^7 - c_3\right)$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x + 1$$

Then the solution in y becomes

$$y = \text{RootOf}\left(\_Z^7 + (3x-3)\_Z^6 + \left(x^2 - 2x + 1\right)\_Z^5 + \left(-5x^3 + 15x^2 - 15x + 5\right)\_Z^4 + \left(-5x^4 + 20x^3\right.$$

Unable to solve for constant of integration due to RootOf in solution.

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = \operatorname{RootOf}\Big(&\_Z^7 + (3x-3)\_Z^6 + (x^2-2x+1)\_Z^5 + (-5x^3+15x^2-15x+5)\_Z^4 \\
&+ (-5x^4+20x^3-30x^2+20x-5)\_Z^3 + (x^5-5x^4+10x^3-10x^2+5x-1)\_Z^2 \\
&+ (3x^6-18x^5+45x^4-60x^3+45x^2-18x+3)\_Z + x^7-7x^6+21x^5-35x^4 \\
&+ 35x^3-21x^2-c_3+7x-1\Big)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = \operatorname{RootOf}\Big(&\_Z^7 + (3x-3)\_Z^6 + (x^2-2x+1)\_Z^5 + (-5x^3+15x^2-15x+5)\_Z^4 \\
&+ (-5x^4+20x^3-30x^2+20x-5)\_Z^3 + (x^5-5x^4+10x^3-10x^2+5x-1)\_Z^2 \\
&+ (3x^6-18x^5+45x^4-60x^3+45x^2-18x+3)\_Z + x^7-7x^6+21x^5-35x^4 \\
&+ 35x^3-21x^2-c_3+7x-1\Big)
\end{aligned}
$$

Verified OK.

1.24.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{3y-7x+7}{7y-3x+3} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(3y-7x+7)(b_3-a_2)}{7y-3x+3} - \frac{(3y-7x+7)^2 a_3}{(7y-3x+3)^2} \\
&- \left(\frac{7}{7y-3x+3} - \frac{3(3y-7x+7)}{(7y-3x+3)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{3}{7y-3x+3} + \frac{21y-49x+49}{(7y-3x+3)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
\frac{21x^2a_2 - 49x^2a_3 + 49x^2b_2 - 21x^2b_3 - 98xya_2 + 42xya_3 - 42xyb_2 + 98xyb_3 + 21y^2a_2 - 49y^2a_3 + 49y^2b_2 -}{}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
&21x^2a_2 - 49x^2a_3 + 49x^2b_2 - 21x^2b_3 - 98xya_2 + 42xya_3 - 42xyb_2 + 98xyb_3 \\
&+ 21y^2a_2 - 49y^2a_3 + 49y^2b_2 - 21y^2b_3 - 42xa_2 + 98xa_3 + 40xb_1 - 58xb_2 + 42xb_3 \\
&- 40ya_1 + 58ya_2 - 42ya_3 + 42yb_2 - 98yb_3 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&21a_2v_1^2 - 98a_2v_1v_2 + 21a_2v_2^2 - 49a_3v_1^2 + 42a_3v_1v_2 - 49a_3v_2^2 + 49b_2v_1^2 \\
&- 42b_2v_1v_2 + 49b_2v_2^2 - 21b_3v_1^2 + 98b_3v_1v_2 - 21b_3v_2^2 - 40a_1v_2 \\
&- 42a_2v_1 + 58a_2v_2 + 98a_3v_1 - 42a_3v_2 + 40b_1v_1 - 58b_2v_1 + 42b_2v_2 \\
&+ 42b_3v_1 - 98b_3v_2 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(21a_2-49a_3+49b_2-21b_3)\,v_1^2+(-98a_2+42a_3-42b_2+98b_3)\,v_1v_2 \\
&+(-42a_2+98a_3+40b_1-58b_2+42b_3)\,v_1+(21a_2-49a_3+49b_2-21b_3)\,v_2^2 \\
&+(-40a_1+58a_2-42a_3+42b_2-98b_3)\,v_2+21a_2-49a_3-40b_1+9b_2-21b_3=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-98a_2+42a_3-42b_2+98b_3&=0\\
21a_2-49a_3+49b_2-21b_3&=0\\
-40a_1+58a_2-42a_3+42b_2-98b_3&=0\\
-42a_2+98a_3+40b_1-58b_2+42b_3&=0\\
21a_2-49a_3-40b_1+9b_2-21b_3&=0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1&=-b_3\\
a_2&=b_3\\
a_3&=b_2\\
b_1&=-b_2\\
b_2&=b_2\\
b_3&=b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi&=y\\
\eta&=x-1
\end{aligned}
$$

Shifting is now applied to make $\xi=0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta&=\eta-\omega(x,y)\,\xi\\
&=x-1-\left(-\frac{3y-7x+7}{7y-3x+3}\right)(y)\\
&=\frac{3x^2-3y^2-6x+3}{-7y+3x-3}\\
\xi&=0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{3x^2-3y^2-6x+3}{-7y+3x-3}} dy \end{aligned}$$

Which results in

$$S = \frac{5 \ln(x - 1 + y)}{3} + \frac{2 \ln(y + 1 - x)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3y - 7x + 7}{7y - 3x + 3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{5}{3x - 3 + 3y} + \frac{2}{-3 + 3x - 3y} \\ S_y &= \frac{5}{3x - 3 + 3y} - \frac{2}{-3 + 3x - 3y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

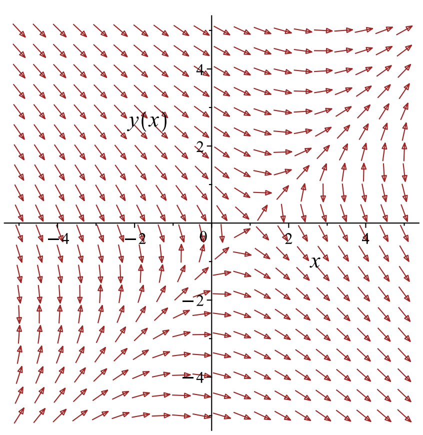
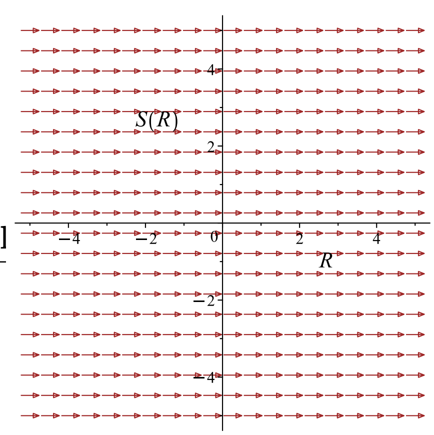
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{5 \ln (y + x - 1)}{3} + \frac{2 \ln (y + 1 - x)}{3} = c_1$$

Which simplifies to

$$\frac{5 \ln (y + x - 1)}{3} + \frac{2 \ln (y + 1 - x)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y-7x+7}{7y-3x+3}$ 	$R = x$ $S = \frac{5 \ln (x - 1 + y)}{3} + 2l$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{5i\pi}{3} = c_1$$

$$c_1 = \frac{5i\pi}{3}$$

Substituting c_1 found above in the general solution gives

$$\frac{5\ln(x-1+y)}{3} + \frac{2\ln(y+1-x)}{3} = \frac{5i\pi}{3}$$

Summary

The solution(s) found are the following

$$\frac{5\ln(y+x-1)}{3} + \frac{2\ln(y+1-x)}{3} = \frac{5i\pi}{3} \tag{1}$$

Verification of solutions

$$\frac{5\ln(y+x-1)}{3} + \frac{2\ln(y+1-x)}{3} = \frac{5i\pi}{3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.843 (sec). Leaf size: 5735

```
dsolve([(3*y(x)-7*x+7)+(7*y(x)-3*x+3)*diff(y(x),x)=0,y(0) = 0],y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 88.015 (sec). Leaf size: 1602

```
DSolve[{(3*y[x]-7*x+7)+(7*y[x]-3*x+3)*y'[x]==0,y[0]==0},y[x],x,IncludeSingularSolutions -> T
```

Too large to display

1.25 problem 2(o)

Internal problem ID [3026]
Internal file name [`OUTPUT/2518_Sunday_June_05_2022_03_17_55_AM_70956231/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(o).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{(2-x+2y)\,y'-xy(y'-1)=-x}$$

1.25.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}y &= \int \frac{x}{x-2}\,\mathrm{d}x \\ &= x+2\ln(x-2)+c_1\end{aligned}$$

Summary
The solution(s) found are the following

$$y = x+2\ln(x-2)+c_1 \tag{1}$$

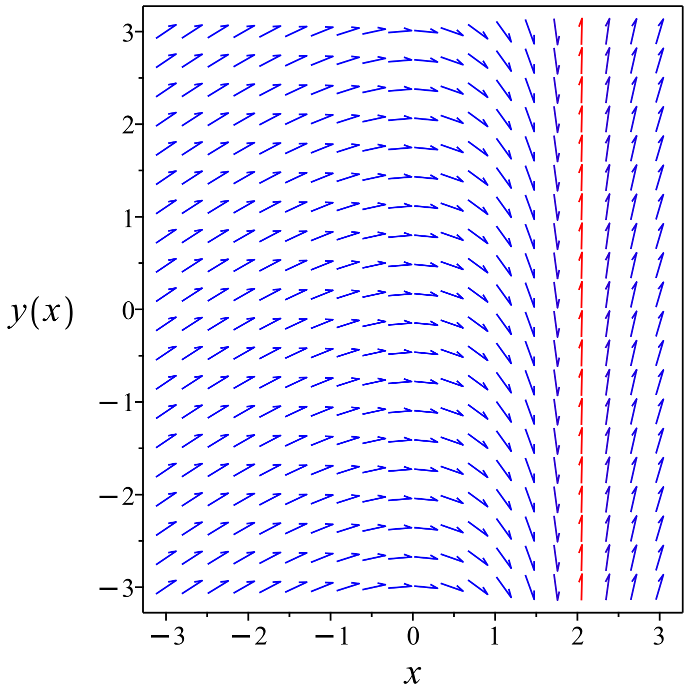


Figure 58: Slope field plot

Verification of solutions

$$y = x + 2\ln(x-2) + c_1$$

Verified OK.

1.25.2 Maple step by step solution

Let's solve

$(2 - x + 2y)\, y' - xy(y' - 1) = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' = \frac{x}{x-2}$

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{x}{x-2} dx + c_1$

- Evaluate integral

$$y = x + 2\ln(x-2) + c_1$$

- Solve for y

$$y = x + 2\ln(x-2) + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(x+(2-x+2*y(x))*diff(y(x),x)=x*y(x)*(diff(y(x),x)-1),y(x), singsol=all)
```

$$y(x) = -1$$
$$y(x) = x + 2\ln(-2+x) + c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 20

```
DSolve[x+(2-x+2*y[x])*y'[x]==x*y[x]*(y'[x]-1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -1$$
$$y(x) \to x + 2\log(x-2) + c_1$$

1.26 problem 2(p)

Internal problem ID [3027]
Internal file name [`OUTPUT/2519_Sunday_June_05_2022_03_17_56_AM_91197894/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(p).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' \cos(x) + y \sin(x) = 1}$$

With initial conditions

$$[y(0) = 0]$$

1.26.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \tan(x)$$
$$q(x) = \sec(x)$$

Hence the ode is

$$y' + y\tan(x) = \sec(x)$$

The domain of $p(x) = \tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z142 \lor \frac{1}{2}\pi + \pi\_Z142 < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \sec(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z142 \lor \frac{1}{2}\pi + \pi\_Z142 < x\right\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.26.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \tan(x)dx} \\ &= \frac{1}{\cos(x)}\end{aligned}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\sec(x)) \\ \frac{d}{dx}(\sec(x)\, y) &= (\sec(x))(\sec(x)) \\ d(\sec(x)\, y) &= \sec(x)^2\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\sec(x)\, y &= \int \sec(x)^2\, dx \\ \sec(x)\, y &= \tan(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$y = \cos(x)\tan(x) + c_1 \cos(x)$$

which simplifies to

$$y = c_1 \cos(x) + \sin(x)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

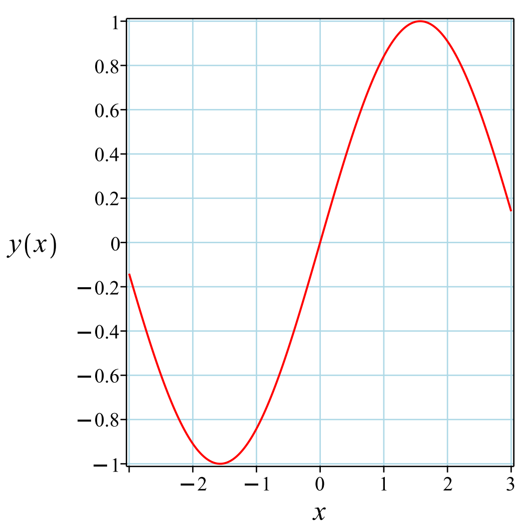
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

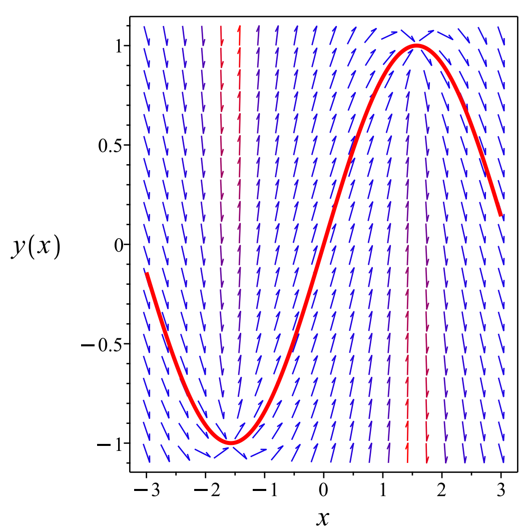
$$y = \sin(x)$$

Summary
The solution(s) found are the following

$$y = \sin(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sin(x)$$

Verified OK.

1.26.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-1+\sin(x)\,y}{\cos(x)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 50: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \cos(x)\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\cos(x)} dy\end{aligned}$$

Which results in

$$S = \frac{y}{\cos(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-1 + \sin(x)\, y}{\cos(x)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \sec(x)\tan(x)\,y \\
S_y &= \sec(x)
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sec(x)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sec(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \tan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\sec(x) = \tan(x) + c_1$$

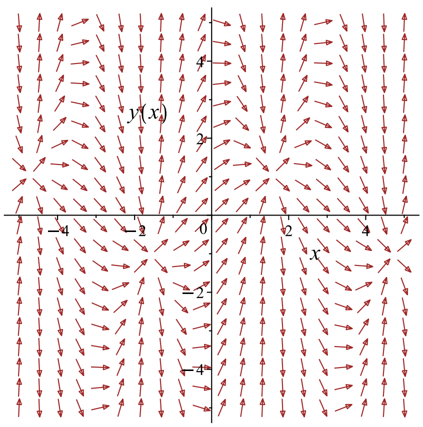
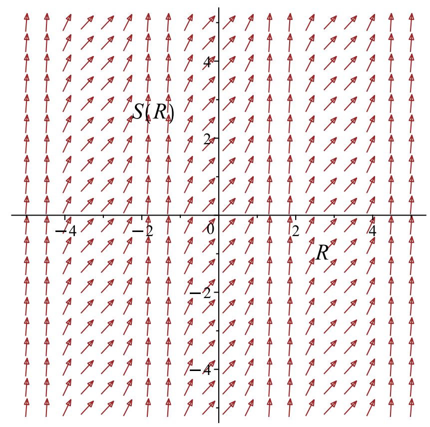
Which simplifies to

$$y\sec(x) = \tan(x) + c_1$$

Which gives

$$y = \frac{\tan(x) + c_1}{\sec(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-1+\sin(x)y}{\cos(x)}$ 	$R = x$ $S = \sec(x)\, y$	$\frac{dS}{dR} = \sec(R)^2$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

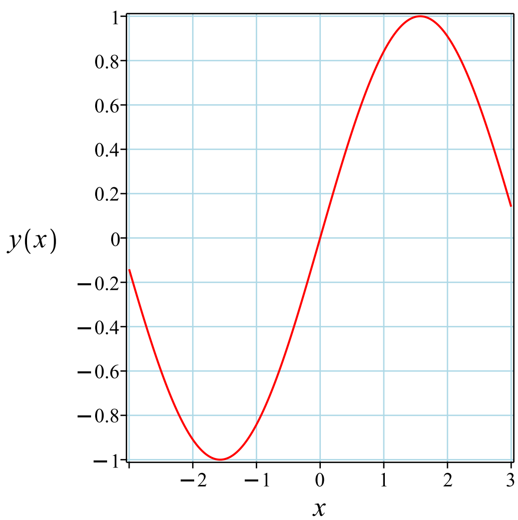
Substituting c_1 found above in the general solution gives

$$y = \sin(x)$$

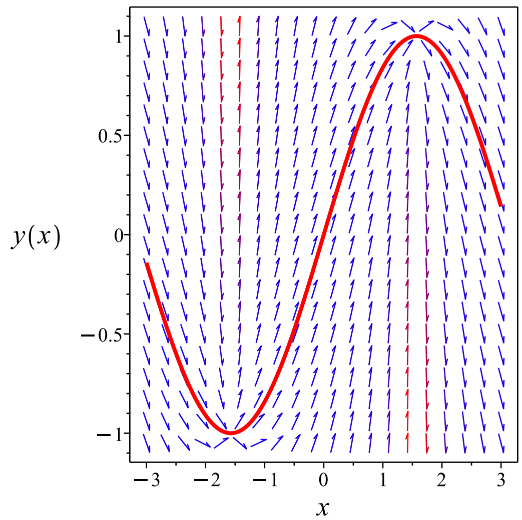
Summary

The solution(s) found are the following

$$y = \sin(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sin(x)$$

Verified OK.

1.26.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(x))\,\mathrm{d}y &= (-\sin(x)\,y+1)\,\mathrm{d}x\\ (-1+\sin(x)\,y)\,\mathrm{d}x + (\cos(x))\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -1+\sin(x)\,y\\ N(x,y) &= \cos(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1+\sin(x)\,y)\\ &= \sin(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(x))\\ &= -\sin(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \sec(x)\,((\sin(x)) - (-\sin(x)))\\ &= 2\tan(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int 2\tan(x)\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(\cos(x))}\\ &= \sec(x)^2\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \sec(x)^2(-1+\sin(x)\,y)\\ &= (-1+\sin(x)\,y)\sec(x)^2\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \sec(x)^2(\cos(x))\\ &= \sec(x)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left((-1+\sin(x)\,y)\sec(x)^2\right)+(\sec(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x\\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int(-1+\sin(x)\,y)\sec(x)^2\,\mathrm{d}x\end{aligned}$$

$$\phi = \sec(x)\,y-\tan(x)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sec(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sec(x)$. Therefore equation (4) becomes

$$\sec(x) = \sec(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sec(x)\,y - \tan(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sec(x)\,y - \tan(x)$$

The solution becomes

$$y = \frac{\tan(x) + c_1}{\sec(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

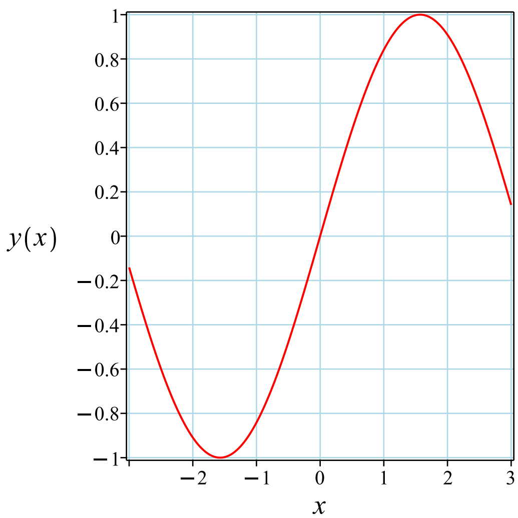
Substituting c_1 found above in the general solution gives

$$y = \sin(x)$$

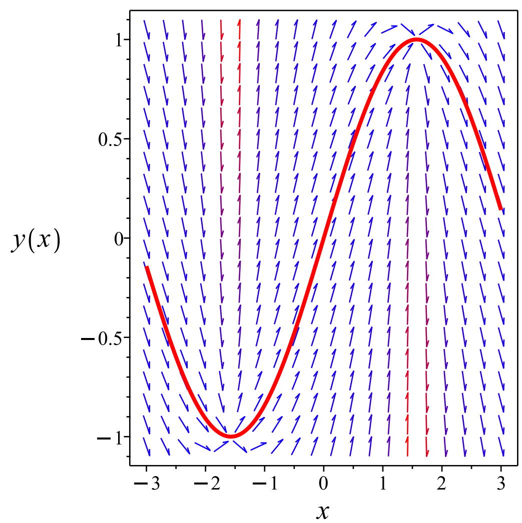
Summary

The solution(s) found are the following

$$y = \sin(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sin(x)$$

Verified OK.

1.26.5 Maple step by step solution

Let's solve

$[y' \cos(x) + y \sin(x) = 1, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{\sin(x) y}{\cos(x)} + \frac{1}{\cos(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{\sin(x) y}{\cos(x)} = \frac{1}{\cos(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x) \left(y' + \frac{\sin(x) y}{\cos(x)}\right) = \frac{\mu(x)}{\cos(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' + \frac{\sin(x) y}{\cos(x)}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x) \sin(x)}{\cos(x)}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{\cos(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{\cos(x)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{\cos(x)} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{\cos(x)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\cos(x)}$

  $y = \cos(x) \left(\int \frac{1}{\cos(x)^2} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = \cos(x) (\tan(x) + c_1)$

- Simplify

  $y = c_1 \cos(x) + \sin(x)$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = \sin(x)$

- Solution to the IVP

  $y = \sin(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 6

```
dsolve([diff(y(x),x)*cos(x)+y(x)*sin(x)=1,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \sin(x)$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 7

```
DSolve[{y'[x]*Cos[x]+y[x]*Sin[x]==1,y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x)$$

1.27 problem 2(q)

Internal problem ID [3028]
Internal file name [`OUTPUT/2520_Sunday_June_05_2022_03_17_59_AM_84763816/index.tex`]

Book: Theory and solutions of Ordinary Differential equations, Donald Greenspan, 1960
Section: Exercises, page 14
Problem number: 2(q).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{(x+y^2)\,y'+y=x^2}$$

With initial conditions

$$[y(1)=1]$$

1.27.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{-x^2+y}{y^2+x} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{x<-1 \lor -1<x\}$$

And the point $x_0=1$ is inside this domain. The y domain of $f(x,y)$ when $x=1$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{-x^2+y}{y^2+x}\right) \\
&= -\frac{1}{y^2+x} + \frac{2(-x^2+y)\,y}{(y^2+x)^2}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.27.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x^2 - y}{x + y^2} \tag{1}$$

Which becomes

$$\left(y^2\right) dy = (-x)\, dy + \left(x^2 - y\right) dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + \left(x^2 - y\right) dx = d\left(\frac{1}{3}x^3 - xy\right)$$

Hence (2) becomes

$$\left(y^2\right) dy = d\left(\frac{1}{3}x^3 - xy\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+4x^3+9c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{2x}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+4x^3+9c_1^2}\right)^{\frac{1}{3}}} + c_1$$

$$y = -\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+4x^3+9c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+4x^3+9c_1^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}}{}$$

$$y = -\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+4x^3+9c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+4x^3+9c_1^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}}{}$$

Initial conditions are used to solve for c_1. Substituting $x=1$ and $y=1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-i\sqrt{3}\left(4+12c_1+4\sqrt{9c_1^2+6c_1+5}\right)^{\frac{2}{3}} - 4i\sqrt{3} - \left(4+12c_1+4\sqrt{9c_1^2+6c_1+5}\right)^{\frac{2}{3}} + 4c_1\left(4+12c_1+4\right.}{4\left(4+12c_1+4\sqrt{9c_1^2+6c_1+5}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x=1$ and $y=1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{i\sqrt{3}\left(4+12c_1+4\sqrt{9c_1^2+6c_1+5}\right)^{\frac{2}{3}} + 4i\sqrt{3} - \left(4+12c_1+4\sqrt{9c_1^2+6c_1+5}\right)^{\frac{2}{3}} + 4c_1\left(4+12c_1+4\sqrt{}\right.}{4\left(4+12c_1+4\sqrt{9c_1^2+6c_1+5}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x=1$ and $y=1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\left(4+12c_1+4\sqrt{9c_1^2+6c_1+5}\right)^{\frac{2}{3}} + 2c_1\left(4+12c_1+4\sqrt{9c_1^2+6c_1+5}\right)^{\frac{1}{3}} - 4}{2\left(4+12c_1+4\sqrt{9c_1^2+6c_1+5}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration.

<u>Verification of solutions</u> N/A

1.27.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(y^2+x\right)\mathrm{d}y&=\left(x^2-y\right)\mathrm{d}x\\ \left(-x^2+y\right)\mathrm{d}x+\left(y^2+x\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^2+y\\ N(x,y)&=y^2+x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^2 + y) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^2 + x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x^2 + y \, \mathrm{d}x\end{aligned}$$

$$\phi = -\frac{1}{3}x^3 + xy + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^2 + x$. Therefore equation (4) becomes

$$y^2 + x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y^2)\,\mathrm{d}y$$

$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{3}x^3 + xy + \frac{1}{3}y^3$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$-\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 = 1$$

<u>Summary</u>
The solution(s) found are the following

$$-\frac{x^3}{3} + yx + \frac{y^3}{3} = 1 \tag{1}$$

<u>Verification of solutions</u>

$$-\frac{x^3}{3} + yx + \frac{y^3}{3} = 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 56

```
dsolve([(x+y(x)^2)*diff(y(x),x)+(y(x)-x^2)=0,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\left(12+4x^3+4\sqrt{x^6+10x^3+9}\right)^{\frac{2}{3}} - 4x}{2\left(12+4x^3+4\sqrt{x^6+10x^3+9}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 3.931 (sec). Leaf size: 66

```
DSolve[{(x+y[x]^2)*y'[x]+(y[x]-x^2)==0,y[1]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{x^3+\sqrt{x^6+10x^3+9}+3}}{\sqrt[3]{2}} - \frac{\sqrt[3]{2}x}{\sqrt[3]{x^3+\sqrt{x^6+10x^3+9}+3}}$$