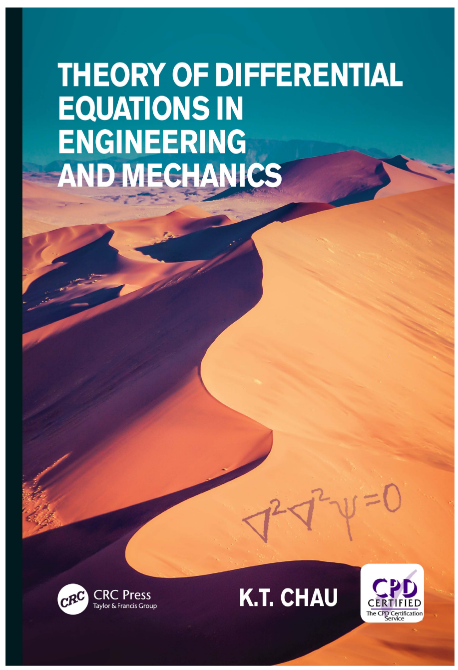
A Solution Manual For

THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018



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May 16, 2024

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1.1 problem Example 3.1

Internal problem ID [5834]
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Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - x^2(1 + y^2) = 0}$$

1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= x^2(y^2 + 1) \end{aligned}$$

Where $f(x) = x^2$ and $g(y) = y^2 + 1$. Integrating both sides gives

$$\frac{1}{y^2 + 1}\,dy = x^2\,dx$$

$$\int \frac{1}{y^2+1}\,dy = \int x^2\,dx$$

$$\arctan(y) = \frac{x^3}{3} + c_1$$

Which results in

$$y = \tan\left(\frac{x^3}{3} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{x^3}{3} + c_1\right) \tag{1}$$

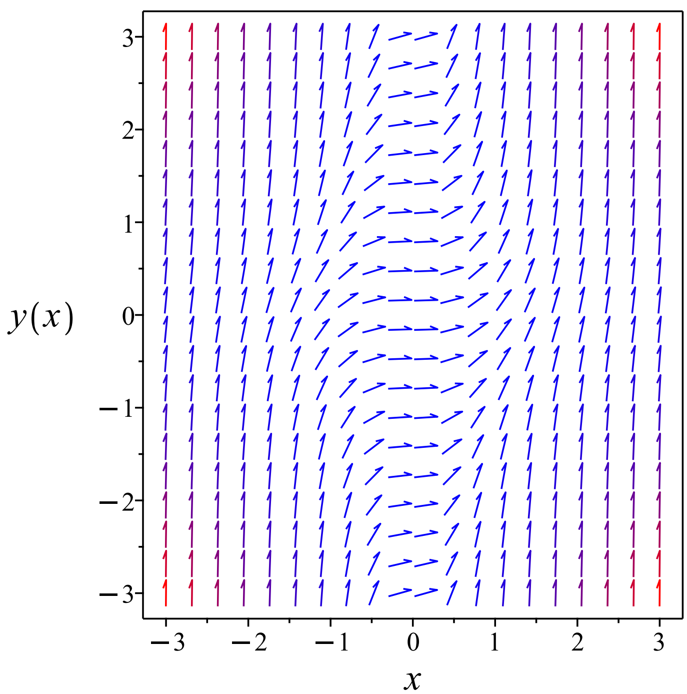


Figure 1: Slope field plot

Verification of solutions

$$y = \tan\left(\frac{x^3}{3} + c_1\right)$$

Verified OK.

1.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= x^2(y^2+1) \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{1}{x^2}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x^2}} dx \end{aligned}$$

Which results in

$$S = \frac{x^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x^2 (y^2 + 1)$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x^2 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^3}{3} = \arctan(y) + c_1$$

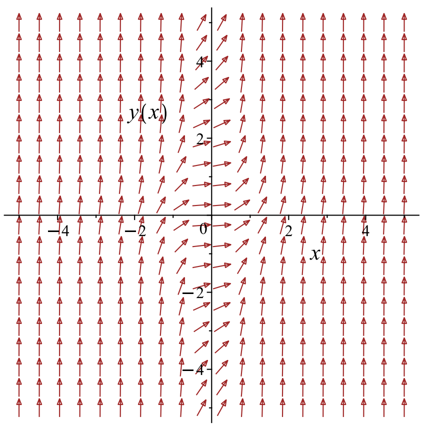
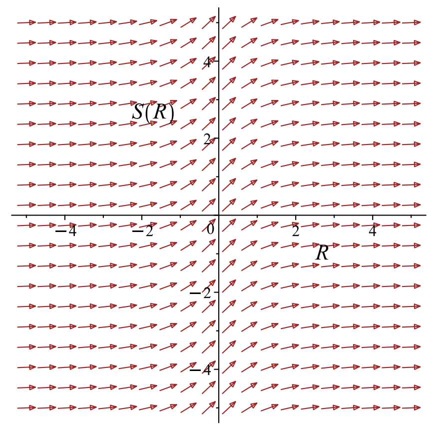
Which simplifies to

$$\frac{x^3}{3} = \arctan(y) + c_1$$

Which gives

$$y = -\tan\left(-\frac{x^3}{3} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2(y^2+1)$ 	$R = y$ $S = \frac{x^3}{3}$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Summary

The solution(s) found are the following

$$y = -\tan\left(-\frac{x^3}{3} + c_1\right) \tag{1}$$

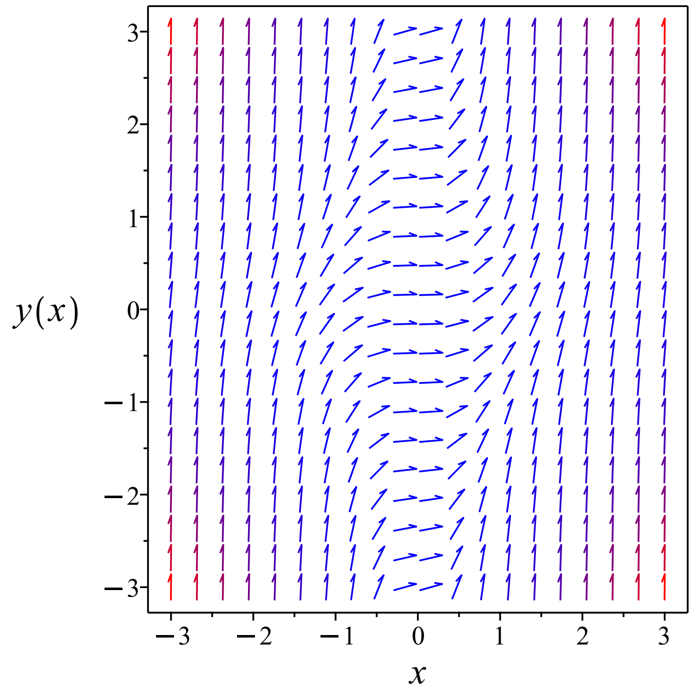


Figure 2: Slope field plot

Verification of solutions

$$y = -\tan\left(-\frac{x^3}{3} + c_1\right)$$

Verified OK.

1.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^2 + 1}\right) \mathrm{d}y = \left(x^2\right) \mathrm{d}x$$
$$\left(-x^2\right) \mathrm{d}x + \left(\frac{1}{y^2 + 1}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^2$$
$$N(x, y) = \frac{1}{y^2 + 1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-x^2\right)$$
$$= 0$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x^2\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{x^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$\frac{1}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2+1}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y^2+1}\right)\mathrm{d}y\\ f(y) &= \arctan(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} + \arctan(y)$$

The solution becomes

$$y = \tan\left(\frac{x^3}{3} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{x^3}{3} + c_1\right) \tag{1}$$

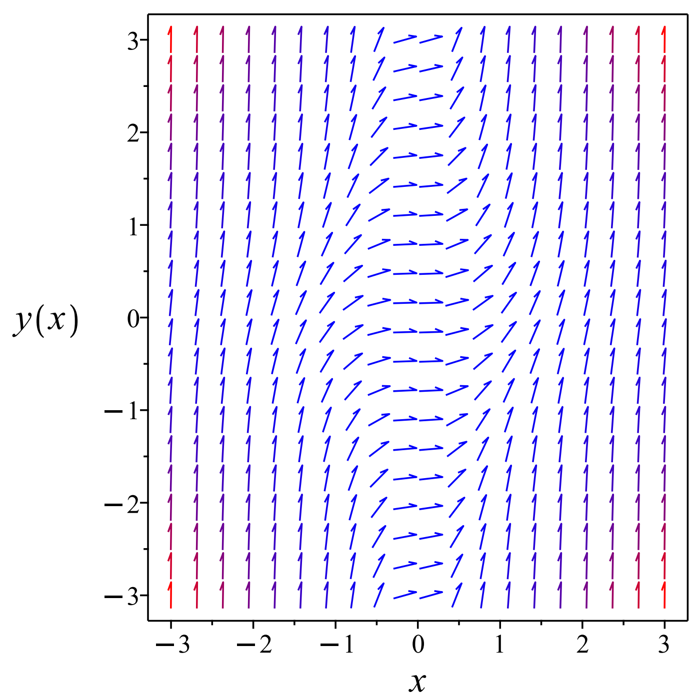


Figure 3: Slope field plot

Verification of solutions

$$y = \tan\left(\frac{x^3}{3} + c_1\right)$$

Verified OK.

1.1.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x^2(y^2 + 1) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2x^2 + x^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2$, $f_1(x) = 0$ and $f_2(x) = x^2$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^2 u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2x \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= x^6 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^2u''(x) - 2xu'(x) + x^6u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 \sin\left(\frac{x^3}{3}\right) + c_2 \cos\left(\frac{x^3}{3}\right)
$$

The above shows that

$$
u'(x) = x^2\left(c_1 \cos\left(\frac{x^3}{3}\right) - c_2 \sin\left(\frac{x^3}{3}\right)\right)
$$

Using the above in (1) gives the solution

$$
y = -\frac{c_1 \cos\left(\frac{x^3}{3}\right) - c_2 \sin\left(\frac{x^3}{3}\right)}{c_1 \sin\left(\frac{x^3}{3}\right) + c_2 \cos\left(\frac{x^3}{3}\right)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{-c_3 \cos\left(\frac{x^3}{3}\right) + \sin\left(\frac{x^3}{3}\right)}{c_3 \sin\left(\frac{x^3}{3}\right) + \cos\left(\frac{x^3}{3}\right)}
$$

Summary

The solution(s) found are the following

$$
y = \frac{-c_3 \cos\left(\frac{x^3}{3}\right) + \sin\left(\frac{x^3}{3}\right)}{c_3 \sin\left(\frac{x^3}{3}\right) + \cos\left(\frac{x^3}{3}\right)} \tag{1}
$$

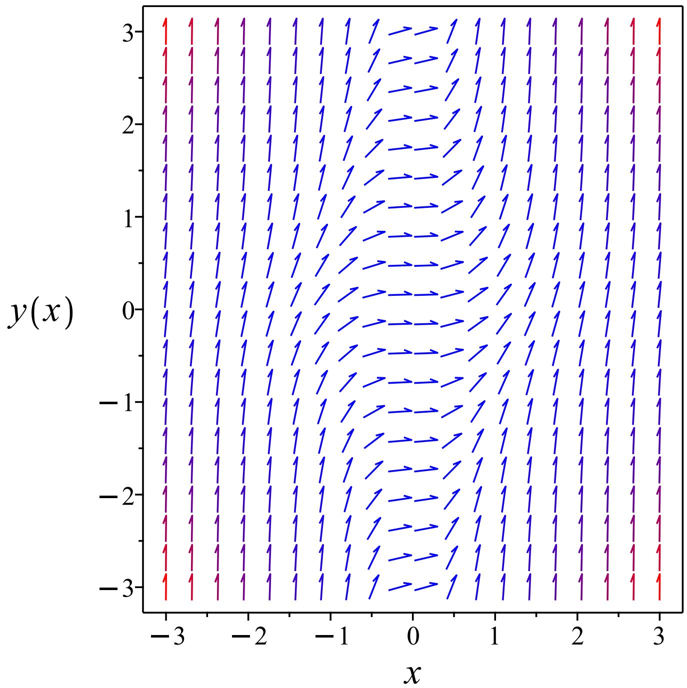


Figure 4: Slope field plot

Verification of solutions

$$y = \frac{-c_3 \cos\left(\frac{x^3}{3}\right) + \sin\left(\frac{x^3}{3}\right)}{c_3 \sin\left(\frac{x^3}{3}\right) + \cos\left(\frac{x^3}{3}\right)}$$

Verified OK.

1.1.5 Maple step by step solution

Let's solve

$y' - x^2(1 + y^2) = 0$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$\frac{y'}{1+y^2} = x^2$

- Integrate both sides with respect to x

$\int \frac{y'}{1+y^2} dx = \int x^2 dx + c_1$

- Evaluate integral

  $\arctan(y) = \frac{x^3}{3} + c_1$

- Solve for y

  $y = \tan\left(\frac{x^3}{3} + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=x^2*(y(x)^2+1),y(x), singsol=all)
```

$$y(x) = \tan\left(\frac{x^3}{3} + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.191 (sec). Leaf size: 30

```
DSolve[y'[x]==x^2*(y[x]^2+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan\left(\frac{x^3}{3} + c_1\right)$$
$$y(x) \to -i$$
$$y(x) \to i$$

1.2 problem Example 3.2

Internal problem ID [5835]
Internal file name [`OUTPUT/5083_Sunday_June_05_2022_03_23_41_PM_21933921/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x^2}{1 - y^2} = 0}$$

1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{x^2}{y^2 - 1} \end{aligned}$$

Where $f(x) = -x^2$ and $g(y) = \frac{1}{y^2-1}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y^2-1}}\,dy = -x^2\,dx$$

$$\int \frac{1}{\frac{1}{y^2-1}}\,dy = \int -x^2\,dx$$

$$\frac{1}{3}y^3 - y = -\frac{x^3}{3} + c_1$$

Which results in

$$y = \frac{\left(-4x^3 + 12c_1 + 4\sqrt{x^6 - 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(-4x^3 + 12c_1 + 4\sqrt{x^6 - 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}$$

$$y = -\frac{\left(-4x^3 + 12c_1 + 4\sqrt{x^6 - 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4x^3 + 12c_1 + 4\sqrt{x^6 - 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}\right)}{2}$$

$$y = -\frac{\left(-4x^3 + 12c_1 + 4\sqrt{x^6 - 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4x^3 + 12c_1 + 4\sqrt{x^6 - 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}} \tag{1}$$

$$y = -\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}\right)}{2} \tag{2}$$

$$y = -\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}\right)}{2} \tag{3}$$

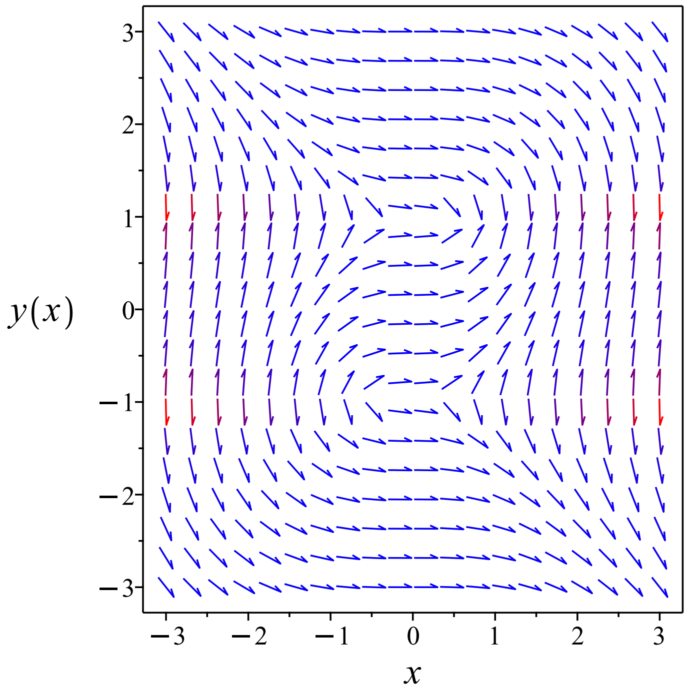


Figure 5: Slope field plot

Verification of solutions

$$y = \frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}$$

Verified OK.

$$y = -\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}\right)}{2}$$

Verified OK.

$$y = -\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}\right)}{2}$$

Verified OK.

1.2.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x^2}{1-y^2} \tag{1}$$

Which becomes

$$\left(y^2-1\right)dy = \left(-x^2\right)dx \tag{2}$$

But the RHS is complete differential because

$$
\left(-x^{2}\right) dx = d\left(-\frac{x^{3}}{3}\right)
$$

Hence (2) becomes

$$
\left(y^{2}-1\right) dy = d\left(-\frac{x^{3}}{3}\right)
$$

Integrating both sides gives gives these solutions

$$
y = \frac{\left(-4x^{3}+12c_{1}+4\sqrt{x^{6}-6c_{1}x^{3}+9c_{1}^{2}-4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(-4x^{3}+12c_{1}+4\sqrt{x^{6}-6c_{1}x^{3}+9c_{1}^{2}-4}\right)^{\frac{1}{3}}} + c_{1}
$$

$$
y = -\frac{\left(-4x^{3}+12c_{1}+4\sqrt{x^{6}-6c_{1}x^{3}+9c_{1}^{2}-4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4x^{3}+12c_{1}+4\sqrt{x^{6}-6c_{1}x^{3}+9c_{1}^{2}-4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}}{\;}
$$

$$
y = -\frac{\left(-4x^{3}+12c_{1}+4\sqrt{x^{6}-6c_{1}x^{3}+9c_{1}^{2}-4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4x^{3}+12c_{1}+4\sqrt{x^{6}-6c_{1}x^{3}+9c_{1}^{2}-4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}}{\;}
$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-4x^3 + 12c_1 + 4\sqrt{x^6 - 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(-4x^3 + 12c_1 + 4\sqrt{x^6 - 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}} + c_1 \tag{1}$$

$$y = -\frac{\left(-4x^3 + 12c_1 + 4\sqrt{x^6 - 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4x^3 + 12c_1 + 4\sqrt{x^6 - 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}\right)}{2} + c_1 \tag{2}$$

$$y = -\frac{\left(-4x^3 + 12c_1 + 4\sqrt{x^6 - 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4x^3 + 12c_1 + 4\sqrt{x^6 - 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}\right)}{2} + c_1 \tag{3}$$

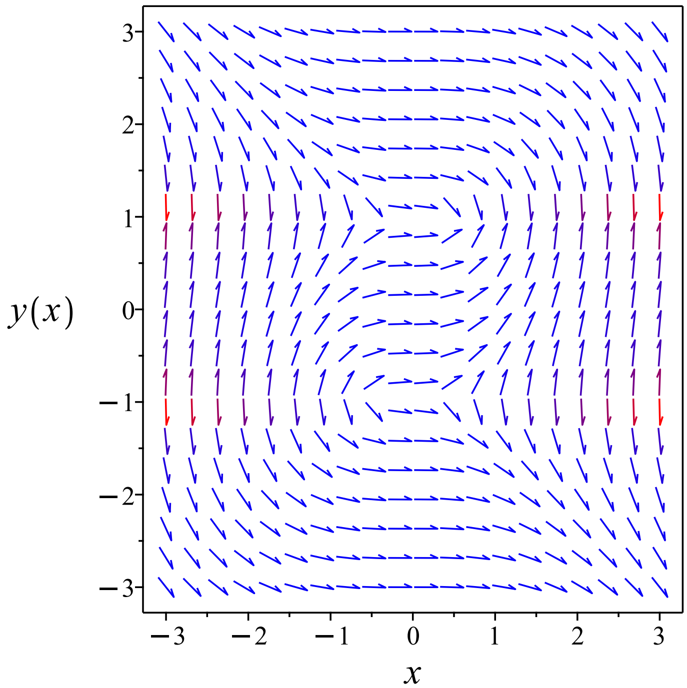


Figure 6: Slope field plot

Verification of solutions

$$y = \frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}} + c_1$$

Verified OK.

$$y = -\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}\right)}{2} + c_1$$

Verified OK.

$$y = -\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-4x^3+12c_1+4\sqrt{x^6-6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}\right)}{2} + c_1$$

Verified OK.

1.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2}{y^2-1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= -\frac{1}{x^2} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{x^2}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x^2}{y^2 - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -x^2 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y^2 - 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2 - 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{3}R^3 - R + c_1 \tag{4}$$

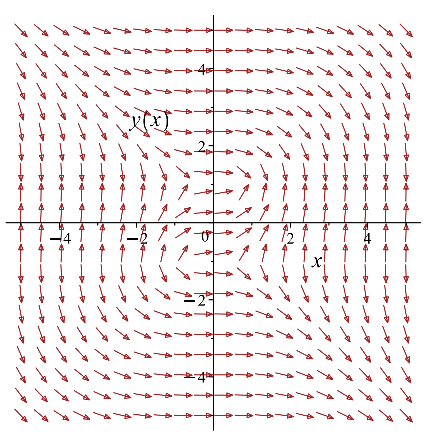
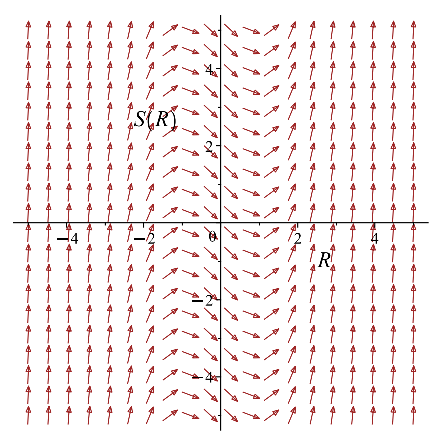
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^3}{3} = \frac{y^3}{3} - y + c_1$$

Which simplifies to

$$-\frac{x^3}{3} = \frac{y^3}{3} - y + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{x^2}{y^2-1}$

</td><td>$R = y$
$S = -\frac{x^3}{3}$</td><td>$\frac{dS}{dR} = R^2 - 1$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$-\frac{x^3}{3} = \frac{y^3}{3} - y + c_1 \tag{1}$$

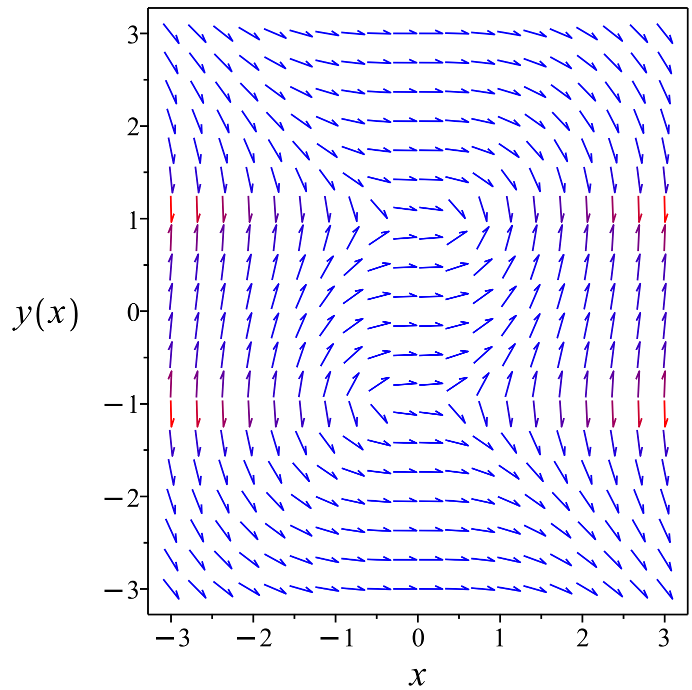


Figure 7: Slope field plot

Verification of solutions

$$-\frac{x^3}{3} = \frac{y^3}{3} - y + c_1$$

Verified OK.

1.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(-y^2+1\right) \mathrm{d} y &= \left(x^2\right) \mathrm{d} x \\
\left(-x^2\right) \mathrm{d} x+\left(-y^2+1\right) \mathrm{d} y &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -x^2 \\
N(x, y) &= -y^2+1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2\right) \\
&= 0
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y^2+1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x^2 \, \mathrm{d}x \\
\phi &= -\frac{x^3}{3} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y^2 + 1$. Therefore equation (4) becomes

$$
-y^2 + 1 = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -y^2 + 1
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int (-y^2 + 1) \, \mathrm{d}y \\
f(y) &= -\frac{1}{3}y^3 + y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{3}x^3 - \frac{1}{3}y^3 + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{3}x^3 - \frac{1}{3}y^3 + y$$

Summary

The solution(s) found are the following

$$-\frac{x^3}{3} - \frac{y^3}{3} + y = c_1 \tag{1}$$

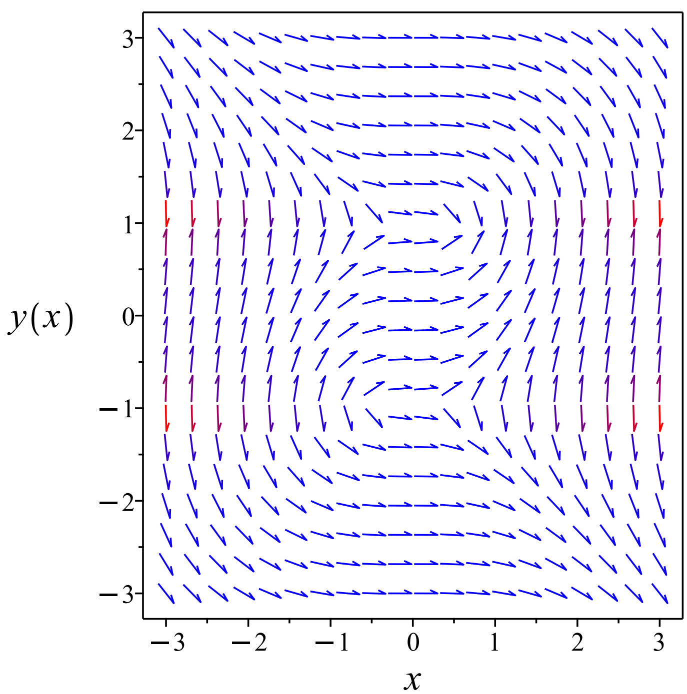


Figure 8: Slope field plot

Verification of solutions

$$-\frac{x^3}{3} - \frac{y^3}{3} + y = c_1$$

Verified OK.

1.2.5 Maple step by step solution

Let's solve

$y' - \frac{x^2}{1-y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $(1-y^2)\, y' = x^2$

- Integrate both sides with respect to x

  $\int (1-y^2)\, y' dx = \int x^2 dx + c_1$

- Evaluate integral

  $-\frac{y^3}{3} + y = \frac{x^3}{3} + c_1$

- Solve for y

  $y = \frac{\left(-4x^3-12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(-4x^3-12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2-4}\right)^{\frac{1}{3}}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 268

```
dsolve(diff(y(x),x)=x^2/(1-y(x)^2),y(x), singsol=all)
```

$$y(x) = \frac{\left(-4x^3 - 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{2}{3}} + 4}{2\left(-4x^3 - 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{(1 + i\sqrt{3})\left(-4x^3 - 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{2}{3}} - 4i\sqrt{3} + 4}{4\left(-4x^3 - 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{i\left(-4x^3 - 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{2}{3}}\sqrt{3} - 4i\sqrt{3} - \left(-4x^3 - 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{2}{3}}}{4\left(-4x^3 - 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 - 4}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 2.485 (sec). Leaf size: 320

```
DSolve[y'[x]==x^2/(1-y[x]^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{-x^3 + \sqrt{x^6 - 6c_1x^3 - 4 + 9c_1{}^2} + 3c_1}}{\sqrt[3]{2}} + \frac{\sqrt[3]{2}}{\sqrt[3]{-x^3 + \sqrt{x^6 - 6c_1x^3 - 4 + 9c_1{}^2} + 3c_1}}$$

$$y(x) \to \frac{i(\sqrt{3} + i)\sqrt[3]{-x^3 + \sqrt{x^6 - 6c_1x^3 - 4 + 9c_1{}^2} + 3c_1}}{2\sqrt[3]{2}} - \frac{1 + i\sqrt{3}}{2^{2/3}\sqrt[3]{-x^3 + \sqrt{x^6 - 6c_1x^3 - 4 + 9c_1{}^2} + 3c_1}}$$

$$y(x) \to \frac{i(\sqrt{3} + i)}{2^{2/3}\sqrt[3]{-x^3 + \sqrt{x^6 - 6c_1x^3 - 4 + 9c_1{}^2} + 3c_1}} - \frac{(1 + i\sqrt{3})\sqrt[3]{-x^3 + \sqrt{x^6 - 6c_1x^3 - 4 + 9c_1{}^2} + 3c_1}}{2\sqrt[3]{2}}$$

1.3 problem Example 3.3

Internal problem ID [5836]
Internal file name [`OUTPUT/5084_Sunday_June_05_2022_03_23_44_PM_9927366/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{3x^2 + 4x + 2}{2y - 2} = 0}$$

With initial conditions

$$[y(0) = -1]$$

1.3.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(x,y) \\
&= \frac{3x^2+4x+2}{2y-2}
\end{aligned}
$$

The x domain of $f(x,y)$ when $y=-1$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x=0$ is

$$
\{y < 1 \lor 1 < y\}
$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{3x^2+4x+2}{2y-2}\right) \\
&= -\frac{3x^2+4x+2}{2\left(y-1\right)^2}
\end{aligned}
$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=-1$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=0$ is

$$
\{y < 1 \lor 1 < y\}
$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

1.3.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{\frac{3}{2}x^2+2x+1}{y-1}
\end{aligned}
$$

Where $f(x) = \frac{3}{2}x^2 + 2x + 1$ and $g(y) = \frac{1}{y-1}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y-1}} dy = \frac{3}{2}x^2 + 2x + 1\, dx$$
$$\int \frac{1}{\frac{1}{y-1}} dy = \int \frac{3}{2}x^2 + 2x + 1\, dx$$
$$\frac{1}{2}y^2 - y = \frac{1}{2}x^3 + x^2 + x + c_1$$

Which results in

$$y = 1 + \sqrt{x^3 + 2x^2 + 2c_1 + 2x + 1}$$
$$y = 1 - \sqrt{x^3 + 2x^2 + 2c_1 + 2x + 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = 1 - \sqrt{2c_1 + 1}$$

$$c_1 = \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

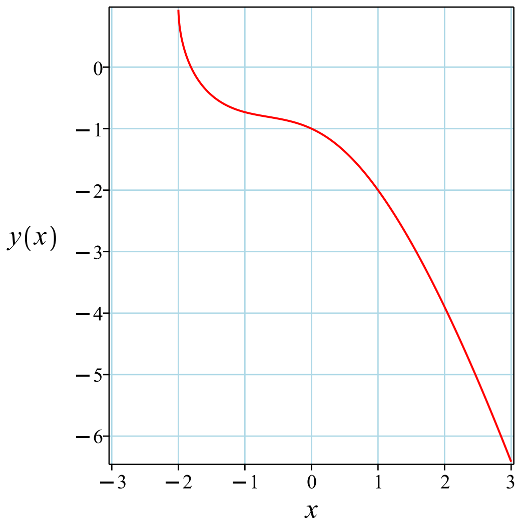
$$-1 = 1 + \sqrt{2c_1 + 1}$$

Warning: Unable to solve for constant of integration.

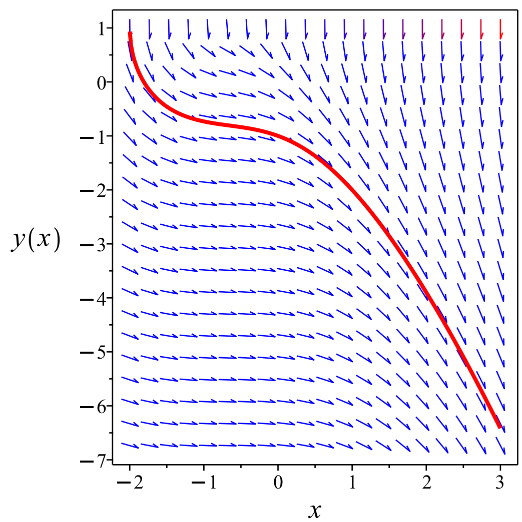
Summary

The solution(s) found are the following

$$y = 1 - \sqrt{x^3 + 2x^2 + 2$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

Verified OK.

1.3.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{3x^2 + 4x + 2}{2y - 2} \tag{1}$$

Which becomes

$$(2y - 2)\,dy = \left(3x^2 + 4x + 2\right)dx \tag{2}$$

But the RHS is complete differential because

$$\left(3x^2 + 4x + 2\right)dx = d\left(x^3 + 2x^2 + 2x\right)$$

Hence (2) becomes

$$(2y - 2)\,dy = d\left(x^3 + 2x^2 + 2x\right)$$

Integrating both sides gives gives these solutions

$$y = 1 + \sqrt{x^3 + 2x^2 + c_1 + 2x + 1} + c_1$$
$$y = 1 - \sqrt{x^3 + 2x^2 + c_1 + 2x + 1} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = 1 - \sqrt{c_1 + 1} + c_1$$

$$c_1 = -\frac{3}{2} - \frac{i\sqrt{3}}{2}$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{2} - \frac{\sqrt{4x^3 + 8x^2 - 2 - 2i\sqrt{3} + 8x}}{2} - \frac{i\sqrt{3}}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = 1 + \sqrt{c_1 + 1} + c_1$$

Warning: Unable to solve for constant of integration.

Summary

The solution(s) found are the following

$$y = -\frac{1}{2} - \frac{\sqrt{4x^3 + 8x^2 - 2 - 2i\sqrt{}}}{2}$$

Verification of solutions

$$y = -\frac{1}{2} - \frac{\sqrt{4x^3 + 8x^2 - 2 - 2i\sqrt{3} + 8x}}{2} - \frac{i\sqrt{3}}{2}$$

Verified OK.

1.3.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{3x^2 + 4x + 2}{2y - 2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 7: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= \frac{1}{\frac{3}{2}x^2 + 2x + 1} \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{\frac{3}{2}x^2+2x+1}} dx \end{aligned}$$

Which results in

$$S = \frac{1}{2}x^3 + x^2 + x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{3x^2 + 4x + 2}{2y - 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{3}{2}x^2 + 2x + 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y - 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R - 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{2}R^2 - R + c_1 \tag{4}$$

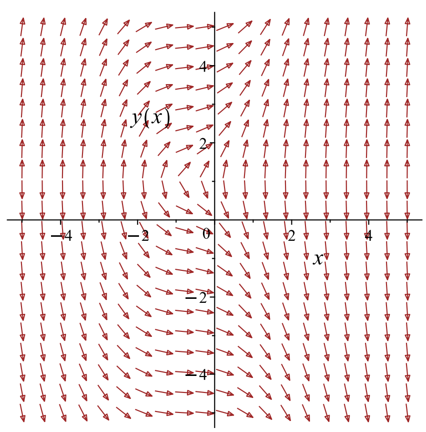
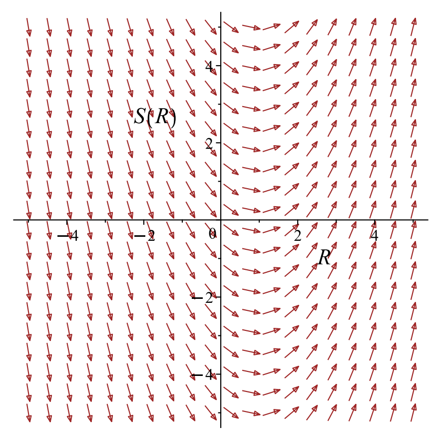
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{2}x^3 + x^2 + x = \frac{y^2}{2} - y + c_1$$

Which simplifies to

$$\frac{1}{2}x^3 + x^2 + x = \frac{y^2}{2} - y + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x^2+4x+2}{2y-2}$ 	$R = y$ $S = \frac{1}{2}x^3 + x^2 + x$	$\frac{dS}{dR} = R - 1$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{3}{2} + c_1$$

$$c_1 = -\frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{2}x^3 + x^2 + x = \frac{1}{2}y^2 - y - \frac{3}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$\frac{1}{2}x^3 + x^2 + x = \frac{y^2}{2} - y - \frac{3}{2} \tag{1}$$

<u>Verification of solutions</u>

$$\frac{1}{2}x^3 + x^2 + x = \frac{y^2}{2} - y - \frac{3}{2}$$

Verified OK.

1.3.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2y-2)\,\mathrm{d}y &= \left(3x^2+4x+2\right)\mathrm{d}x\\ \left(-3x^2-4x-2\right)\mathrm{d}x + (2y-2)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -3x^2-4x-2\\ N(x,y) &= 2y-2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3x^2-4x-2\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2y-2)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -3x^2 - 4x - 2 \, \mathrm{d}x \\
\phi &= -x^3 - 2x^2 - 2x + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y - 2$. Therefore equation (4) becomes

$$2y - 2 = 0 + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y - 2$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int (2y - 2) \, \mathrm{d}y \\
f(y) &= y^2 - 2y + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^3 - 2x^2 + y^2 - 2x - 2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^3 - 2x^2 + y^2 - 2x - 2y$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

$$-x^3 - 2x^2 + y^2 - 2x - 2y = 3$$

Summary
The solution(s) found are the following

$$-x^3 - 2x^2 + y^2 - 2x - 2y = 3 \tag{1}$$

Verification of solutions

$$-x^3 - 2x^2 + y^2 - 2x - 2y = 3$$

Verified OK.

1.3.6 Maple step by step solution

Let's solve

$\left[y' - \frac{3x^2+4x+2}{2y-2} = 0, y(0) = -1\right]$

- Highest derivative means the order of the ODE is 1

  y'
- Separate variables

  $y'(2y-2) = 3x^2 + 4x + 2$
- Integrate both sides with respect to x

  $\int y'(2y-2)\,dx = \int (3x^2 + 4x + 2)\,dx + c_1$
- Evaluate integral

  $y^2 - 2y = x^3 + 2x^2 + c_1 + 2x$
- Solve for y

  $\left\{y = 1 - \sqrt{x^3 + 2x^2 + c_1 + 2x + 1}, y = 1 + \sqrt{x^3 + 2x^2 + c_1 + 2x + 1}\right\}$
- Use initial condition $y(0) = -1$

$$-1 = 1 - \sqrt{c_1 + 1}$$

- Solve for c_1

  $c_1 = 3$

- Substitute $c_1 = 3$ into general solution and simplify

  $y = -\sqrt{(x+2)(x^2+2)} + 1$

- Use initial condition $y(0) = -1$

  $-1 = 1 + \sqrt{c_1 + 1}$

- Solution does not satisfy initial condition

- Solution to the IVP

  $y = -\sqrt{(x+2)(x^2+2)} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 19

```
dsolve([diff(y(x),x)=(3*x^2+4*x+2)/(2*(y(x)-1)),y(0) = -1],y(x), singsol=all)
```

$$y(x) = 1 - \sqrt{(x+2)(x^2+2)}$$

✓ Solution by Mathematica

Time used: 0.134 (sec). Leaf size: 26

```
DSolve[{y'[x]==(3*x^2+4*x+2)/(2*(y[x]-1)),{y[0]==-1}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

1.4 problem Example 3.4

Internal problem ID [5837]
Internal file name [`OUTPUT/5085_Sunday_June_05_2022_03_23_46_PM_57715086/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xy' - 2\sqrt{xy} - y = 0}$$

1.4.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{2\sqrt{xy} + y}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(2\sqrt{xy} + y)(b_3 - a_2)}{x} - \frac{(2\sqrt{xy} + y)^2 a_3}{x^2} - \left(\frac{y}{\sqrt{xy}\, x} - \frac{2\sqrt{xy} + y}{x^2}\right)(x a_2 + y a_3 + a_1) - \frac{\left(\frac{x}{\sqrt{xy}} + 1\right)(x b_2 + y b_3 + b_1)}{x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4(xy)^{\frac{3}{2}} a_3 - x^2 y b_3 + 3x\, y^2 a_3 + x^3 b_2 + x^2 y a_2 - x y a_1 + \sqrt{xy}\, x b_1 - \sqrt{xy}\, y a_1 + x^2 b_1}{\sqrt{xy}\, x^2} = 0$$

Setting the numerator to zero gives

$$-4(xy)^{\frac{3}{2}} a_3 - x^3 b_2 - x^2 y a_2 + x^2 y b_3 - 3x\, y^2 a_3 - \sqrt{xy}\, x b_1 + \sqrt{xy}\, y a_1 - x^2 b_1 + x y a_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-x^3 b_2 - x^2 y a_2 + x^2 y b_3 - 4xy\sqrt{xy}\, a_3 - 3x\, y^2 a_3 - x^2 b_1 - \sqrt{xy}\, x b_1 + x y a_1 + \sqrt{xy}\, y a_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{xy}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{xy} = v_3\}$$

The above PDE (6E) now becomes

$$-v_1^2 v_2 a_2 - 3v_1 v_2^2 a_3 - 4v_1 v_2 v_3 a_3 - v_1^3 b_2 + v_1^2 v_2 b_3 + v_1 v_2 a_1 + v_3 v_2 a_1 - v_1^2 b_1 - v_3 v_1 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$-v_1^3 b_2 + (b_3 - a_2) v_1^2 v_2 - v_1^2 b_1 - 3 v_1 v_2^2 a_3 - 4 v_1 v_2 v_3 a_3 + v_1 v_2 a_1 - v_3 v_1 b_1 + v_3 v_2 a_1 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -4a_3 &= 0 \\ -3a_3 &= 0 \\ -b_1 &= 0 \\ -b_2 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\,\xi \\ &= y - \left(\frac{2\sqrt{xy} + y}{x}\right)(x) \\ &= -2\sqrt{xy} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-2\sqrt{xy}} dy \end{aligned}$$

Which results in

$$S = -\frac{y}{\sqrt{xy}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2\sqrt{xy} + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\sqrt{y}}{2x^{\frac{3}{2}}} \\ S_y &= -\frac{1}{2\sqrt{x}\,\sqrt{y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\sqrt{xy}}{x^{\frac{3}{2}}\sqrt{y}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\sqrt{y}}{\sqrt{x}} = -\ln(x) + c_1$$

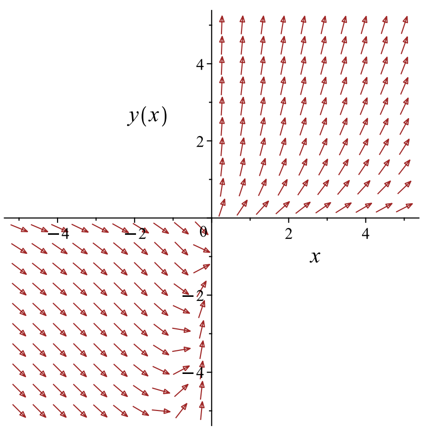
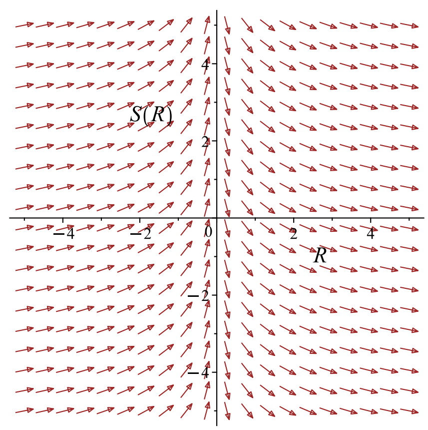
Which simplifies to

$$-\frac{\sqrt{y}}{\sqrt{x}} = -\ln(x) + c_1$$

Which gives

$$y = \ln(x)^2 x - 2\ln(x) x c_1 + c_1^2 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2\sqrt{xy}+y}{x}$ 	$R = x$ $S = -\frac{\sqrt{y}}{\sqrt{x}}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \ln(x)^2 x - 2\ln(x) x c_1 + c_1^2 x \tag{1}$$

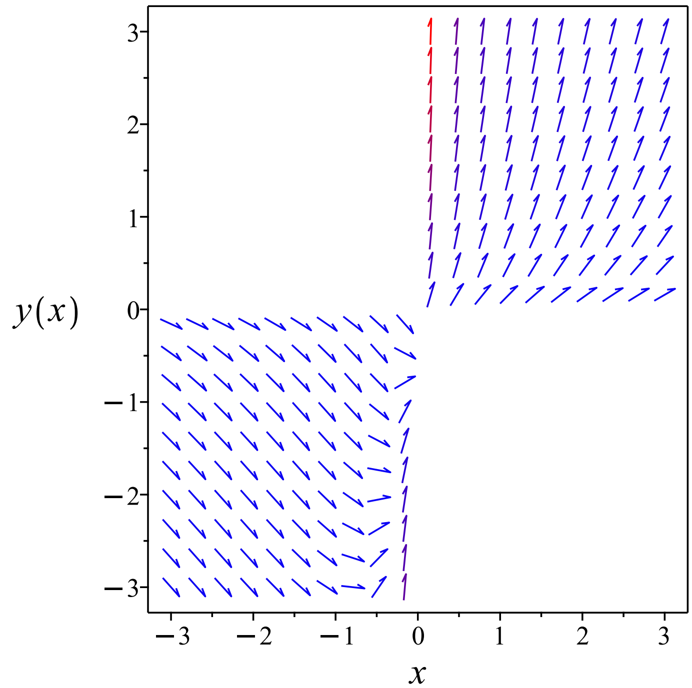


Figure 10: Slope field plot

Verification of solutions

$$y = \ln(x)^2 x - 2\ln(x)\, x c_1 + c_1^2 x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x)-2*sqrt(x*y(x))=y(x),y(x), singsol=all)
```

$$-\frac{y(x)}{\sqrt{xy(x)}}+\ln(x)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.182 (sec). Leaf size: 19

```
DSolve[x*y'[x]-2*Sqrt[x*y[x]]==y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{1}{4}x(2\log(x)+c_1)^2$$

1.5 problem Example 3.5

Internal problem ID [5838]
Internal file name [`OUTPUT/5086_Sunday_June_05_2022_03_23_49_PM_67090161/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{-1 + y + x}{x - y + 3} = 0}$$

1.5.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{-1 + Y(X) + y_0 + X + x_0}{-X - x_0 + Y(X) + y_0 - 3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -1$$
$$y_0 = 2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{Y(X) + X}{-X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= -\frac{Y+X}{-X+Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = Y+X$ and $N = X-Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-u-1}{u-1} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) X u(X) - \left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 1 = 0$$

Or

$$X(u(X)-1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u^2+1}{X(u-1)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\,du = -\frac{1}{X}\,dX$$
$$\int \frac{1}{\frac{u^2+1}{u-1}}\,du = \int -\frac{1}{X}\,dX$$
$$\frac{\ln\left(u^2+1\right)}{2} - \arctan\left(u\right) = -\ln\left(X\right) + c_2$$

The solution is

$$\frac{\ln\left(u(X)^2+1\right)}{2} - \arctan\left(u(X)\right) + \ln\left(X\right) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) + \ln\left(X\right) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) + \ln\left(X\right) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 2$$
$$X = x - 1$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{(y-2)^2}{(1+x)^2}+1\right)}{2} - \arctan\left(\frac{y-2}{1+x}\right) + \ln\left(1+x\right) - c_2 = 0$$

Summary
The solution(s) found are the following

$$\frac{\ln\left(\frac{(y-2)^2}{(1+x)^2}+1\right)}{2}-\arctan\left(\frac{y-2}{1+x}\right)+\ln(1+x)-c_2=0 \tag{1}$$

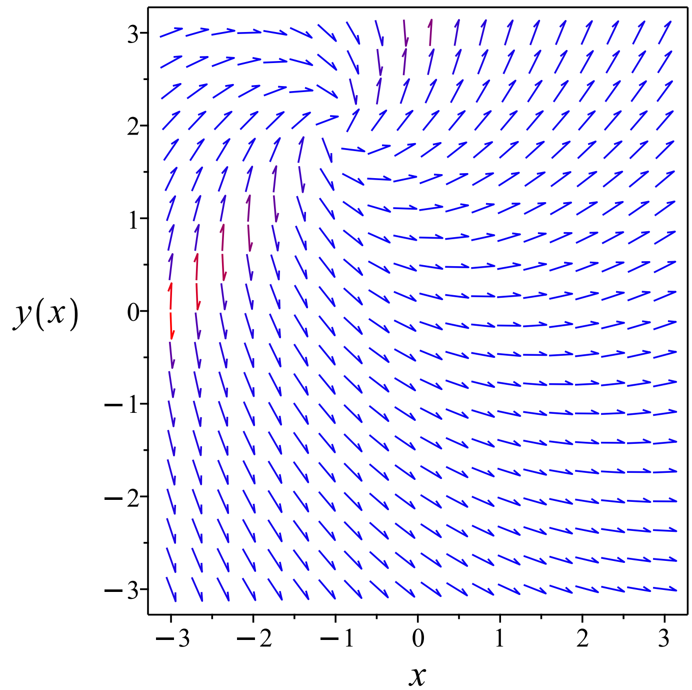


Figure 11: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{(y-2)^2}{(1+x)^2}+1\right)}{2}-\arctan\left(\frac{y-2}{1+x}\right)+\ln(1+x)-c_2=0$$

Verified OK.

1.5.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{y+x-1}{-x+y-3}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(y+x-1)(b_3-a_2)}{-x+y-3} - \frac{(y+x-1)^2 a_3}{(-x+y-3)^2} \\ &- \left(-\frac{1}{-x+y-3} - \frac{y+x-1}{(-x+y-3)^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(-\frac{1}{-x+y-3} + \frac{y+x-1}{(-x+y-3)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2 a_2 + x^2 a_3 + x^2 b_2 - x^2 b_3 - 2xy a_2 + 2xy a_3 + 2xy b_2 + 2xy b_3 - y^2 a_2 - y^2 a_3 - y^2 b_2 + y^2 b_3 + 6x a_2 - 2xa}{(x - y + 3}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^2 a_2 - x^2 a_3 - x^2 b_2 + x^2 b_3 + 2xy a_2 - 2xy a_3 - 2xy b_2 - 2xy b_3 + y^2 a_2 \\ &+ y^2 a_3 + y^2 b_2 - y^2 b_3 - 6x a_2 + 2x a_3 - 2x b_1 + 4x b_2 + 2x b_3 + 2y a_1 \\ &- 4y a_2 - 2y a_3 - 6y b_2 + 2y b_3 - 4a_1 + 3a_2 - a_3 - 2b_1 + 9b_2 - 3b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2 v_1^2 + 2a_2 v_1 v_2 + a_2 v_2^2 - a_3 v_1^2 - 2a_3 v_1 v_2 + a_3 v_2^2 - b_2 v_1^2 - 2b_2 v_1 v_2 + b_2 v_2^2 + b_3 v_1^2 - 2b_3 v_1 v_2 - b_3 v_2^2 + 2a_1 v_2 - 6a_2 v_1 - 4a_2 v_2 + 2a_3 v_1 - 2a_3 v_2 - 2b_1 v_1 + 4b_2 v_1 - 6b_2 v_2 + 2b_3 v_1 + 2b_3 v_2 - 4a_1 + 3a_2 - a_3 - 2b_1 + 9b_2 - 3b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 - b_2 + b_3) v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3) v_1 v_2 + (-6a_2 + 2a_3 - 2b_1 + 4b_2 + 2b_3) v_1 + (a_2 + a_3 + b_2 - b_3) v_2^2 + (2a_1 - 4a_2 - 2a_3 - 6b_2 + 2b_3) v_2 - 4a_1 + 3a_2 - a_3 - 2b_1 + 9b_2 - 3b_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-a_2 - a_3 - b_2 + b_3 &= 0 \\
a_2 + a_3 + b_2 - b_3 &= 0 \\
2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0 \\
2a_1 - 4a_2 - 2a_3 - 6b_2 + 2b_3 &= 0 \\
-6a_2 + 2a_3 - 2b_1 + 4b_2 + 2b_3 &= 0 \\
-4a_1 + 3a_2 - a_3 - 2b_1 + 9b_2 - 3b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 2b_2 + b_3 \\
a_2 &= b_3 \\
a_3 &= -b_2 \\
b_1 &= b_2 - 2b_3 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1+x \\
\eta &= y-2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y-2-\left(-\frac{y+x-1}{-x+y-3}\right)(1+x) \\
&= \frac{-x^2-y^2-2x+4y-5}{x-y+3} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2-y^2-2x+4y-5}{x-y+3}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(x^2+y^2+2x-4y+5)}{2} + \frac{2(-1-x)\arctan\left(\frac{2y-4}{2+2x}\right)}{2+2x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y + x - 1}{-x + y - 3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y + x - 1}{x^2 + y^2 + 2x - 4y + 5} \\
S_y &= \frac{-x + y - 3}{x^2 + y^2 + 2x - 4y + 5}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

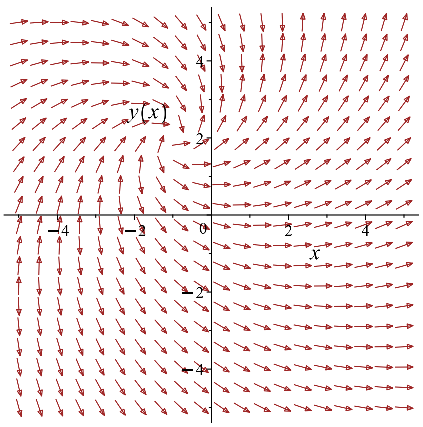
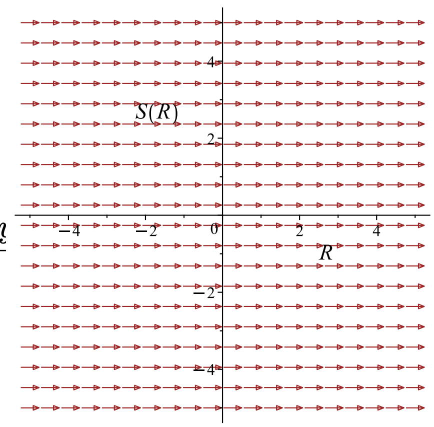
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2 + x^2 - 4y + 2x + 5)}{2} - \arctan\left(\frac{y - 2}{1 + x}\right) = c_1$$

Which simplifies to

$$\frac{\ln(y^2 + x^2 - 4y + 2x + 5)}{2} - \arctan\left(\frac{y - 2}{1 + x}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{y+x-1}{-x+y-3}$
</td><td>$R = x$
$S = \frac{\ln \left(x^{2}+y^{2}+2x-4y\right.}{2}$</td><td>$\frac{dS}{dR} = 0$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{\ln \left(y^{2}+x^{2}-4y+2x+5\right)}{2}-\arctan \left(\frac{y-2}{1+x}\right) = c_{1} \tag{1}$$

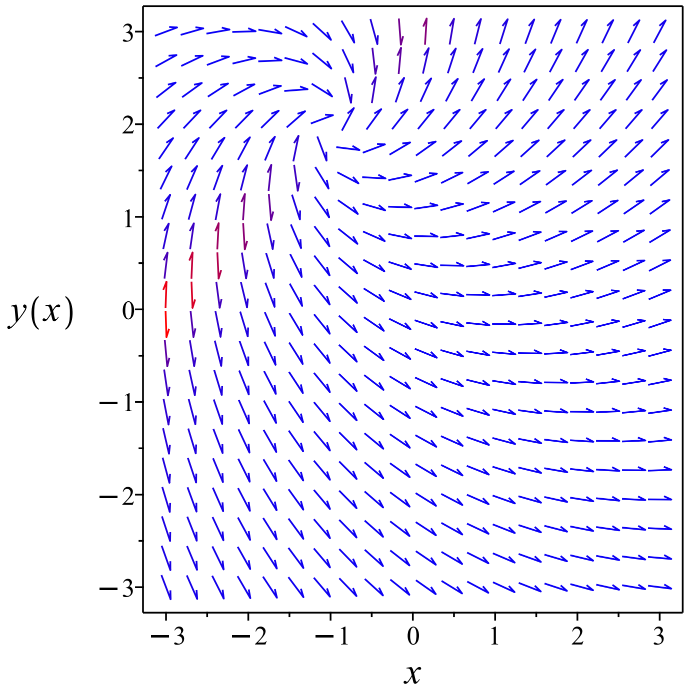


Figure 12: Slope field plot

Verification of solutions

$$\frac{\ln \left(y^{2}+x^{2}-4y+2x+5\right)}{2}-\arctan \left(\frac{y-2}{1+x}\right)=c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 32

```
dsolve(diff(y(x),x)=(x+y(x)-1)/(x-y(x)+3),y(x), singsol=all)
```

$$y(x) = 2 + \tan\left(\operatorname{RootOf}\left(2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x+1\right) + 2c_1\right)\right)(-x-1)$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 59

```
DSolve[y'[x]==(x+y[x]-1)/(x-y[x]+3),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2\arctan\left(1 - \frac{2(x+1)}{-y(x)+x+3}\right) + \log\left(\frac{x^2+y(x)^2-4y(x)+2x+5}{2(x+1)^2}\right) + 2\log(x+1) + c_1 = 0, y(x)\right]$$

1.6 problem Example 3.6

Internal problem ID [5839]
Internal file name [`OUTPUT/5087_Sunday_June_05_2022_03_23_52_PM_37258556/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{y + (x - 2\sin(y))\, y' = -\mathrm{e}^{x}}$$

1.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x - 2\sin(y))\,\mathrm{d}y &= (-y - \mathrm{e}^x)\,\mathrm{d}x \\ (y + \mathrm{e}^x)\,\mathrm{d}x + (x - 2\sin(y))\,\mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y + \mathrm{e}^x \\ N(x, y) &= x - 2\sin(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y + \mathrm{e}^x) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x - 2\sin(y)) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int y+\mathrm{e}^{x}\,\mathrm{d}x$$
$$\phi=xy+\mathrm{e}^{x}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x-2\sin(y)$. Therefore equation (4) becomes

$$x-2\sin(y)=x+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=-2\sin(y)$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int(-2\sin(y))\,\mathrm{d}y$$
$$f(y)=2\cos(y)+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=xy+\mathrm{e}^{x}+2\cos(y)+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy + e^x + 2\cos(y)$$

Summary

The solution(s) found are the following

$$xy + e^x + 2\cos(y) = c_1 \tag{1}$$

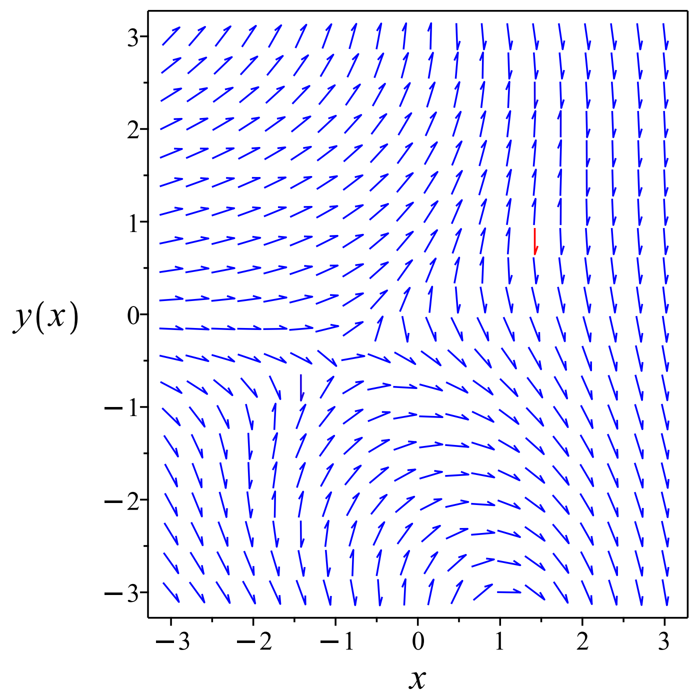


Figure 13: Slope field plot

Verification of solutions

$$xy + e^x + 2\cos(y) = c_1$$

Verified OK.

1.6.2 Maple step by step solution

Let's solve

$y + (x - 2\sin(y))\, y' = -e^{x}$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $1 = 1$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (y + e^{x})\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = xy + e^{x} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $x - 2\sin(y) = x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = -2\sin(y)$

- Solve for $f_1(y)$

  $f_1(y) = 2\cos(y)$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = xy + e^x + 2\cos(y)$$

- Substitute $F(x, y)$ into the solution of the ODE

$$xy + e^x + 2\cos(y) = c_1$$

- Solve for y

$$y = RootOf(-x\_Z - e^x - 2\cos(\_Z) + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((exp(x)+y(x))+(x-2*sin(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$e^x + xy(x) + 2\cos(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.233 (sec). Leaf size: 19

```
DSolve[(Exp[x]+y[x])+(x-2*Sin[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}[xy(x) + 2\cos(y(x)) + e^x = c_1, y(x)]$$

1.7 problem Example 3.7

Internal problem ID [5840]
Internal file name [`OUTPUT/5088_Sunday_June_05_2022_03_23_55_PM_49293309/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{\frac{6}{y}+\left(\frac{x^2}{y}+\frac{3y}{x}\right)y' = -3x}$$

1.7.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-3x-\frac{6}{y}}{\frac{x^2}{y}+\frac{3y}{x}} \tag{1}$$

Which becomes

$$\left(3y^2\right)dy = \left(-x^3\right)dy + \left(-3x(xy+2)\right)dx \tag{2}$$

But the RHS is complete differential because

$$\left(-x^3\right)dy + \left(-3x(xy+2)\right)dx = d\left(-y\,x^3-3x^2\right)$$

Hence (2) becomes

$$\left(3y^2\right)dy = d\left(-y\,x^3-3x^2\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{6} - \frac{2x^3}{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4}\right.}$$

$$y = -\frac{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{12} + \frac{x^3}{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729}\right.}$$

$$y = -\frac{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{12} + \frac{x^3}{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729}\right.}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{6} - \frac{2x^3}{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}} + c_1 \tag{1}$$

$$y = -\frac{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{12} + \frac{x^3}{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}{6} + \frac{2x^3}{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1 \tag{2}$$

$$y = -\frac{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{12} + \frac{x^3}{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}{6} + \frac{2x^3}{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1 \tag{3}$$

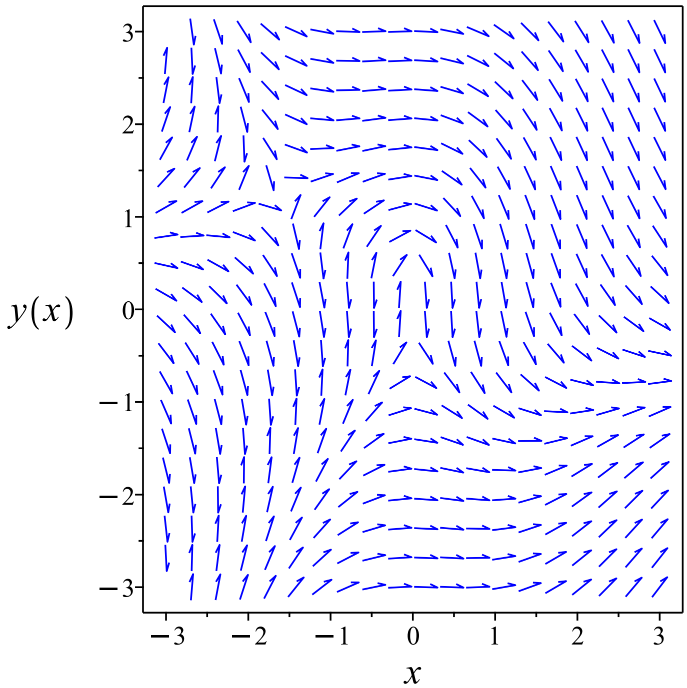


Figure 14: Slope field plot

Verification of solutions

$$y = \frac{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{6} - \frac{2x^3}{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}} + c_1$$

Verified OK.

$$y = -\frac{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{12} + \frac{x^3}{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}{6} + \frac{2x^3}{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1$$

Verified OK.

$$y = -\frac{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{12} + \frac{x^3}{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}{6} + \frac{2x^3}{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1$$

Verified OK.

1.7.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^3 + 3y^2)\,\mathrm{d}y &= (-3x(xy+2))\,\mathrm{d}x\\ (3x(xy+2))\,\mathrm{d}x + (x^3 + 3y^2)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 3x(xy+2)\\ N(x,y) &= x^3 + 3y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x(xy+2)) \\ &= 3x^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^3+3y^2) \\ &= 3x^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 3x(xy+2) \, \mathrm{d}x \\ \phi &= y\,x^3 + 3x^2 + f(y) \end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3 + 3y^2$. Therefore equation (4) becomes

$$x^3 + 3y^2 = x^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y^2$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int (3y^2)\, \mathrm{d}y$$
$$f(y) = y^3 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y\, x^3 + y^3 + 3x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\, x^3 + y^3 + 3x^2$$

<u>Summary</u>
The solution(s) found are the following

$$yx^3 + y^3 + 3x^2 = c_1 \tag{1}$$

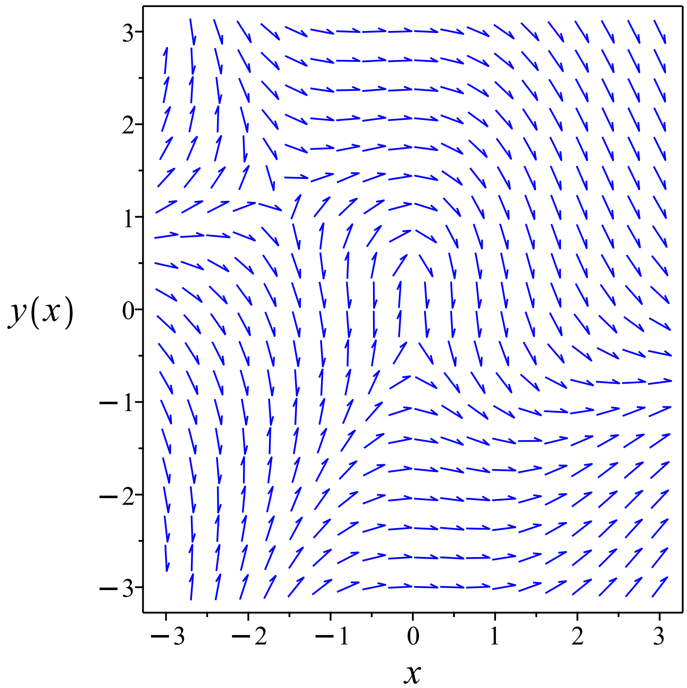


Figure 15: Slope field plot

Verification of solutions

$$yx^3 + y^3 + 3x^2 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 326

```
dsolve((3*x+6/y(x))+(x^2/y(x)+3*y(x)/x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-12x^3 + \left(-324x^2 - 108c_1 + 12\sqrt{12x^9 + 729x^4 + 486c_1x^2 + 81c_1^2}\right)^{\frac{2}{3}}}{6\left(-324x^2 - 108c_1 + 12\sqrt{12x^9 + 729x^4 + 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{\left(1 + i\sqrt{3}\right)\left(-324x^2 - 108c_1 + 12\sqrt{12x^9 + 729x^4 + 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{12} - \frac{\left(i\sqrt{3} - 1\right)x^3}{\left(-324x^2 - 108c_1 + 12\sqrt{12x^9 + 729x^4 + 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{12i\sqrt{3}\,x^3 + i\left(-324x^2 - 108c_1 + 12\sqrt{12x^9 + 729x^4 + 486c_1x^2 + 81c_1^2}\right)^{\frac{2}{3}}\sqrt{3} + 12x^3 - \left(-324x^2 - 108c_1\right.}{12\left(-324x^2 - 108c_1 + 12\sqrt{12x^9 + 729x^4 + 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 4.542 (sec). Leaf size: 331

```
DSolve[(3*x+6/y[x])+(x^2/y[x]+3*y[x]/x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{-81x^2+\sqrt{108x^9+729\left(-3x^2+c_1\right)^2}+27c_1}}{3\sqrt[3]{2}} - \frac{\sqrt[3]{2}x^3}{\sqrt[3]{-81x^2+\sqrt{108x^9+729\left(-3x^2+c_1\right)^2}+27c_1}}$$

$$y(x) \to \frac{\left(-1+i\sqrt{3}\right)\sqrt[3]{-81x^2+\sqrt{108x^9+729\left(-3x^2+c_1\right)^2}+27c_1}}{6\sqrt[3]{2}} + \frac{\left(1+i\sqrt{3}\right)x^3}{2^{2/3}\sqrt[3]{-81x^2+\sqrt{108x^9+729\left(-3x^2+c_1\right)^2}+27c_1}}$$

$$y(x) \to \frac{\left(1-i\sqrt{3}\right)x^3}{2^{2/3}\sqrt[3]{-81x^2+\sqrt{108x^9+729\left(-3x^2+c_1\right)^2}+27c_1}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{-81x^2+\sqrt{108x^9+729\left(-3x^2+c_1\right)^2}+27c_1}}{6\sqrt[3]{2}}$$

1.8 problem Example 3.8

Internal problem ID [5841]
Internal file name [`OUTPUT/5089_Sunday_June_05_2022_03_23_58_PM_10799289/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 - xy + x^2 y' = 0}$$

1.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\, x^2 - x^2 u(x) + x^2(u'(x)\, x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{u^2}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u^2$. Integrating both sides gives

$$\frac{1}{u^2}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{u^2}\,du = \int -\frac{1}{x}\,dx$$
$$-\frac{1}{u} = -\ln(x) + c_2$$

The solution is

$$-\frac{1}{u(x)} + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x}{y} + \ln(x) - c_2 = 0$$
$$-\frac{x}{y} + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\frac{x}{y} + \ln(x) - c_2 = 0 \tag{1}$$

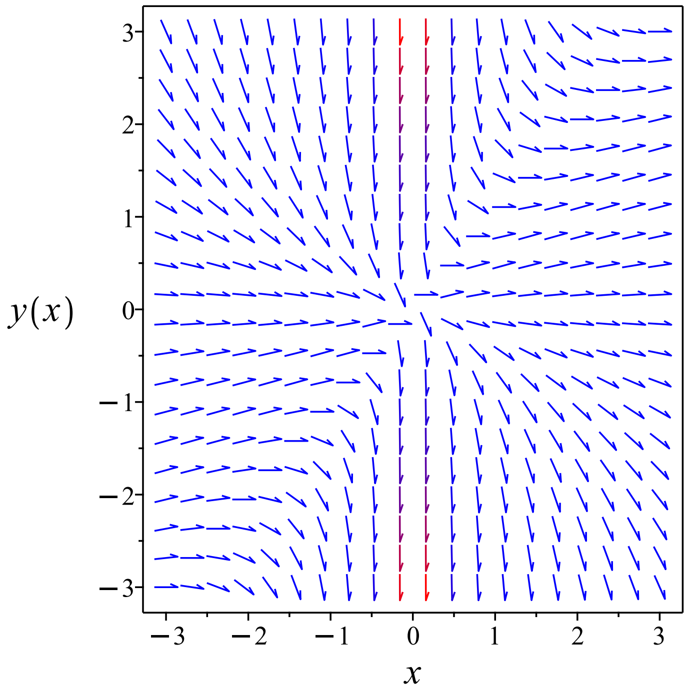


Figure 16: Slope field plot

Verification of solutions

$$-\frac{x}{y}+\ln(x)-c_2=0$$

Verified OK.

1.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(-x+y)}{x^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 11: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{y^2}{x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(-x+y)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{y} \\ S_y &= \frac{x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = -\ln(x) + c_1$$

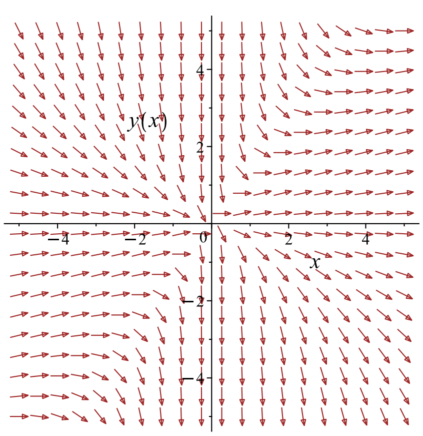
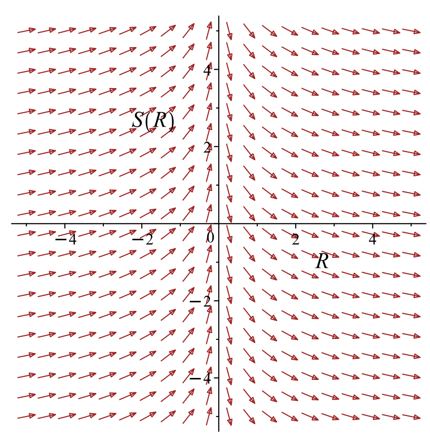
Which simplifies to

$$-\frac{x}{y} = -\ln(x) + c_1$$

Which gives

$$y = \frac{x}{\ln(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-x+y)}{x^2}$ 	$R = x$ $S = -\frac{x}{y}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary
The solution(s) found are the following

$$y = \frac{x}{\ln(x) - c_1} \tag{1}$$

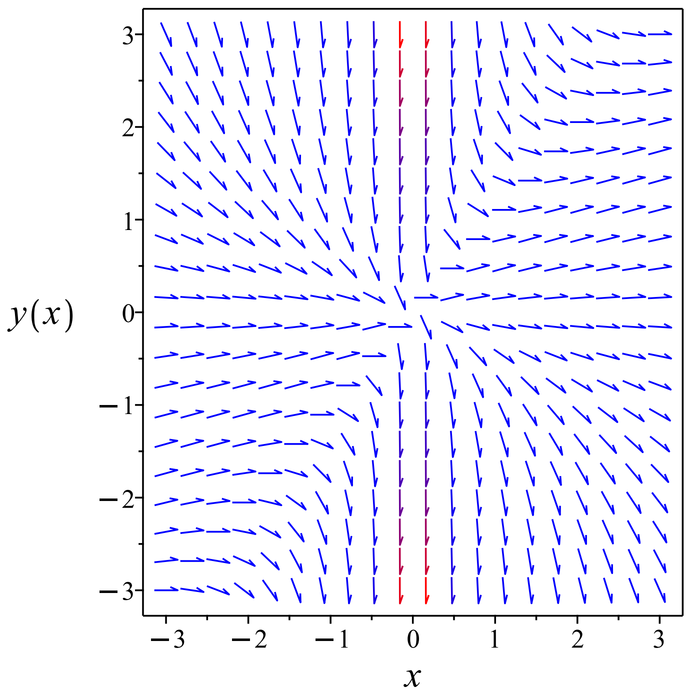


Figure 17: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) - c_1}$$

Verified OK.

1.8.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(-x + y)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y - \frac{1}{x^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= -\frac{1}{x^2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{yx} - \frac{1}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} - \frac{1}{x^2} \\ w' &= -\frac{w}{x} + \frac{1}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = \frac{1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x} dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu) \left(\frac{1}{x^2}\right) \\ \frac{d}{dx}(xw) &= (x) \left(\frac{1}{x^2}\right) \\ d(xw) &= \frac{1}{x}\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} xw &= \int \frac{1}{x}\, dx \\ xw &= \ln(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = \frac{\ln(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$w(x) = \frac{\ln(x) + c_1}{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{\ln(x) + c_1}{x}$$

Or

$$y = \frac{x}{\ln(x) + c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{\ln(x) + c_1} \tag{1}$$

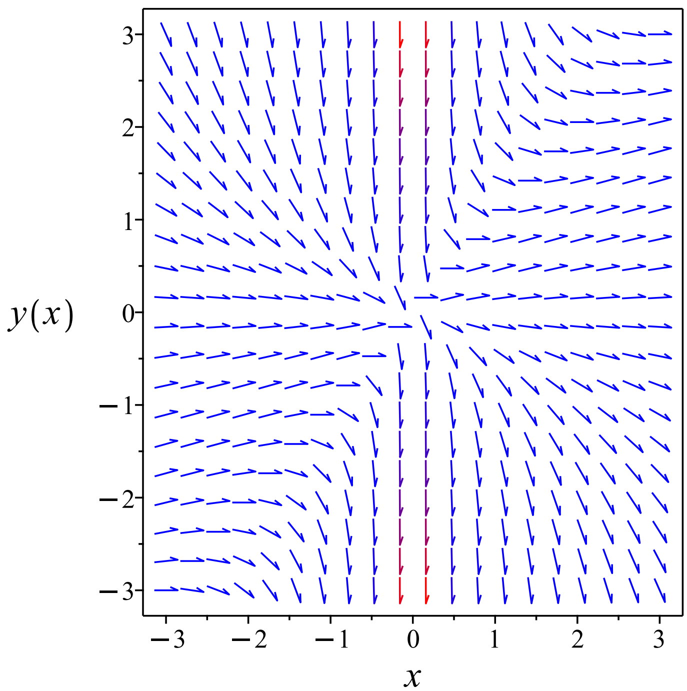


Figure 18: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) + c_1}$$

Verified OK.

1.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2)\,\mathrm{d}y &= (xy - y^2)\,\mathrm{d}x\\ (-xy + y^2)\,\mathrm{d}x + (x^2)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -xy + y^2\\ N(x,y) &= x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-xy + y^2\right)\\ &= -x + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2\right)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = y^2 - xy$ and $N = x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{y^2 - xy}{xy^2}\\ N &= \frac{x}{y^2}\end{aligned}$$

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{x}{y^2}\right)\mathrm{d}y &= \left(-\frac{-xy+y^2}{x\,y^2}\right)\mathrm{d}x \\
\left(\frac{-xy+y^2}{x\,y^2}\right)\mathrm{d}x + \left(\frac{x}{y^2}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= \frac{-xy+y^2}{x\,y^2} \\
N(x,y) &= \frac{x}{y^2}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-xy+y^2}{x\,y^2}\right) \\
&= \frac{1}{y^2}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right) \\
&= \frac{1}{y^2}
\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$

$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int \frac{-xy+y^2}{x\,y^2}\,\mathrm{d}x$$

$$\phi=\ln (x)-\frac{x}{y}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=\frac{x}{y^2}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{x}{y^2}$. Therefore equation (4) becomes

$$\frac{x}{y^2}=\frac{x}{y^2}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=\ln (x)-\frac{x}{y}+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) - \frac{x}{y}$$

The solution becomes

$$y = \frac{x}{\ln(x) - c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{\ln(x) - c_1} \tag{1}$$

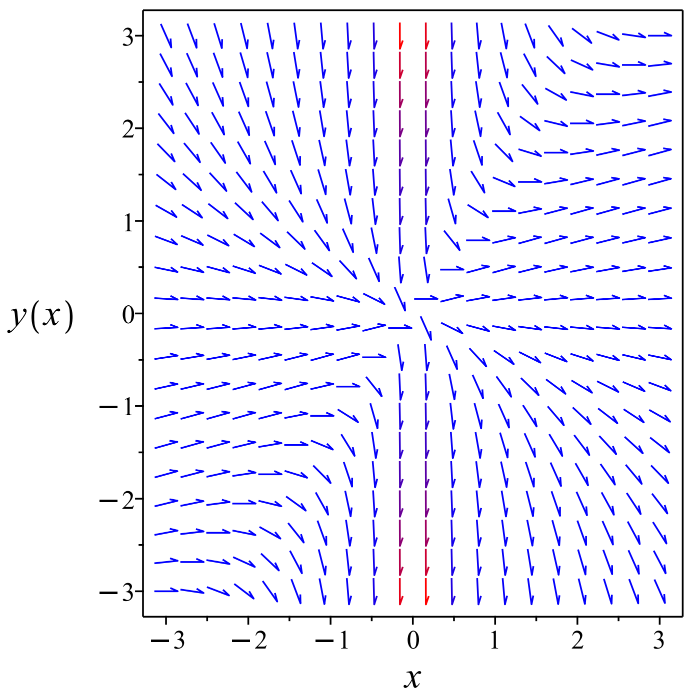


Figure 19: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) - c_1}$$

Verified OK.

1.8.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(-x+y)}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x} - \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = -\frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2}{x^3} \\ f_1 f_2 &= -\frac{1}{x^3} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x^2} - \frac{u'(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + c_2 \ln(x)$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$y = \frac{c_2 x}{c_1 + c_2 \ln(x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x}{c_3 + \ln(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{c_3 + \ln(x)} \tag{1}$$

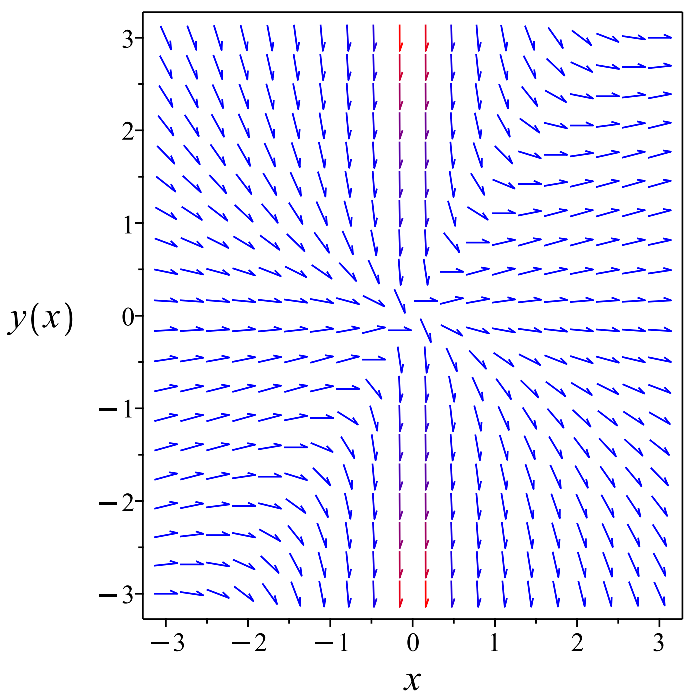


Figure 20: Slope field plot

Verification of solutions

$$y = \frac{x}{c_3 + \ln(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((y(x)^2-x*y(x))+x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 19

```
DSolve[(y[x]^2-x*y[x])+x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{\log(x) + c_1}$$
$$y(x) \to 0$$

1.9 problem Example 3.9

Internal problem ID [5842]
Internal file name [`OUTPUT/5090_Sunday_June_05_2022_03_24_01_PM_50793666/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y-(x-y)\,y'=-x}$$

1.9.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x-(x-u(x)\,x)\,(u'(x)\,x+u(x))=-x$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{u^2+1}{x\,(u-1)}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2+1}{u-1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u^2+1)}{2} - \arctan(u) = -\ln(x) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2+1\right)}{2} - \arctan(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

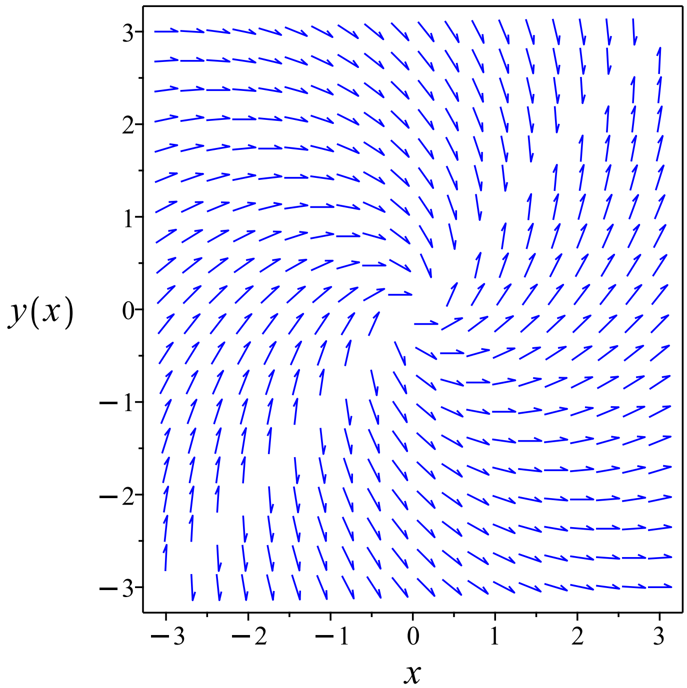


Figure 21: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2}-\arctan\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Verified OK.

1.9.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{x+y}{-x+y}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 - \frac{(x+y)(b_3-a_2)}{-x+y} - \frac{(x+y)^2 a_3}{(-x+y)^2} \\ - \left(-\frac{1}{-x+y} - \frac{x+y}{(-x+y)^2}\right)(xa_2+ya_3+a_1) \\ - \left(-\frac{1}{-x+y} + \frac{x+y}{(-x+y)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 + 2xyb_2 + 2xyb_3 - y^2a_2 - y^2a_3 - y^2b_2 + y^2b_3 + 2xb_1 - 2ya}{(x-y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} -x^2a_2 - x^2a_3 - x^2b_2 + x^2b_3 + 2xya_2 - 2xya_3 - 2xyb_2 \\ - 2xyb_3 + y^2a_2 + y^2a_3 + y^2b_2 - y^2b_3 - 2xb_1 + 2ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -a_2v_1^2 + 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 \\ - 2b_2v_1v_2 + b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 - b_3v_2^2 + 2a_1v_2 - 2b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 - b_2 + b_3) v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3) v_1 v_2 \\ - 2b_1 v_1 + (a_2 + a_3 + b_2 - b_3) v_2^2 + 2a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 &= 0 \\ -2b_1 &= 0 \\ -a_2 - a_3 - b_2 + b_3 &= 0 \\ a_2 + a_3 + b_2 - b_3 &= 0 \\ 2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= -b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(-\frac{x + y}{-x + y}\right)(x) \\ &= \frac{-x^2 - y^2}{x - y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2-y^2}{x-y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right)$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x+y}{-x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x+y}{x^2+y^2} \\ S_y &= \frac{-x+y}{x^2+y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

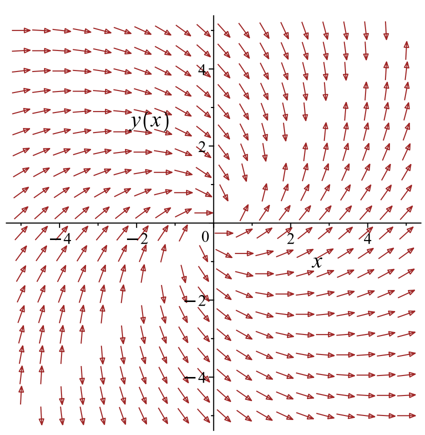
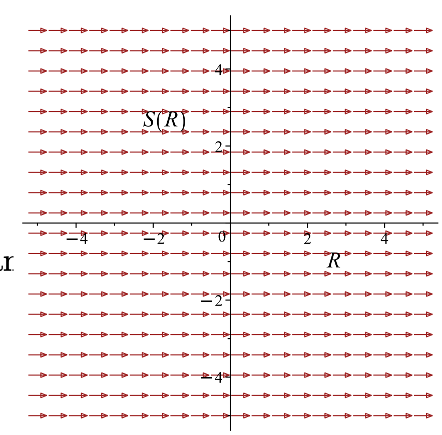
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Which simplifies to

$$\frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y}{-x+y}$ 	$R = x$ $S = \frac{\ln(x^2+y^2)}{2} - \arctan$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln\left(x^2+y^2\right)}{2}-\arctan\left(\frac{y}{x}\right)=c_1 \tag{1}$$

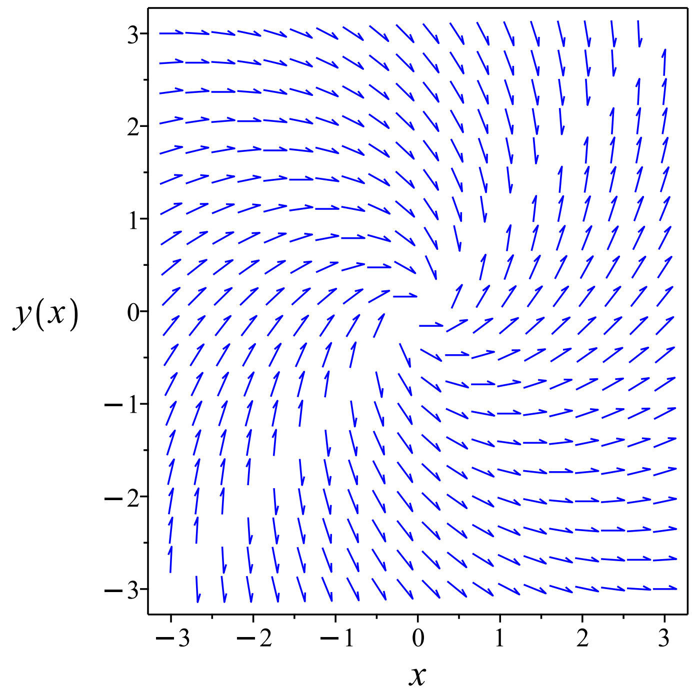


Figure 22: Slope field plot

Verification of solutions

$$\frac{\ln\left(x^2+y^2\right)}{2}-\arctan\left(\frac{y}{x}\right)=c_1$$

Verified OK.

1.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x+y)\,\mathrm{d}y&=(-y-x)\,\mathrm{d}x\\ (x+y)\,\mathrm{d}x+(-x+y)\,\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x+y\\ N(x,y)&=-x+y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(x+y)\\ &=1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x+y)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = x+y$ and $N = -x+y$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{x+y}{x^2+y^2}\\ N &= \frac{-x+y}{x^2+y^2}\end{aligned}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-x+y}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{x+y}{x^2+y^2}\right)\mathrm{d}x$$
$$\left(\frac{x+y}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{-x+y}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{x+y}{x^2+y^2}$$
$$N(x,y) = \frac{-x+y}{x^2+y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x+y}{x^2+y^2}\right)\\ &= \frac{x^2-2xy-y^2}{(x^2+y^2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x+y}{x^2+y^2}\right)\\ &= \frac{x^2-2xy-y^2}{(x^2+y^2)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x+y}{x^2+y^2}\,\mathrm{d}x\\ \phi &= \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right) + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= \frac{y}{x^2+y^2} - \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \qquad (4)\\ &= \frac{-x+y}{x^2+y^2} + f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+y}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{-x+y}{x^2+y^2} = \frac{-x+y}{x^2+y^2} + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right)$$

Summary

The solution(s) found are the following

$$\frac{\ln \left(x^2+y^2\right)}{2}+\arctan \left(\frac{x}{y}\right)=c_1 \tag{1}$$

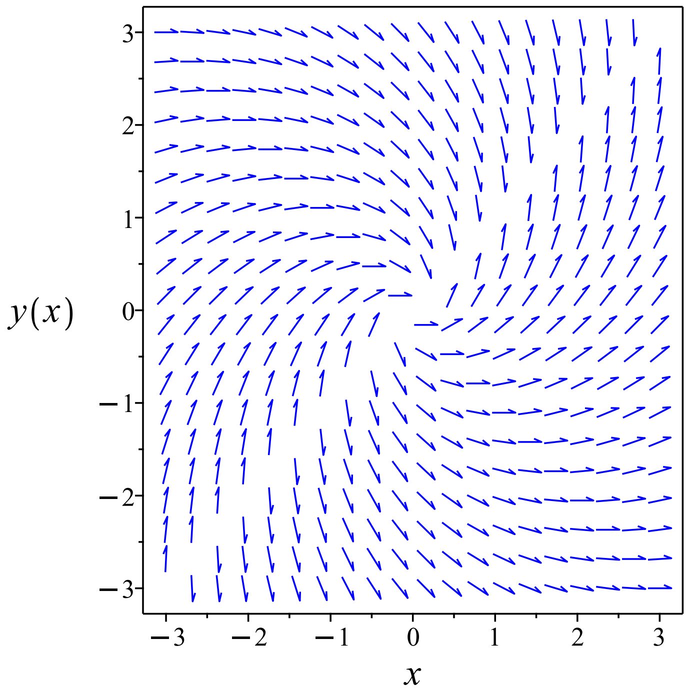


Figure 23: Slope field plot

Verification of solutions

$$\frac{\ln \left(x^2+y^2\right)}{2}+\arctan \left(\frac{x}{y}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve((x+y(x))-(x-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan\left(\operatorname{RootOf}\left(-2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x\right) + 2c_1\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 36

```
DSolve[(x+y[x])-(x-y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right)-\arctan\left(\frac{y(x)}{x}\right)=-\log(x)+c_1,y(x)\right]$$

1.10 problem Example 3.10

Internal problem ID [5843]
Internal file name [`OUTPUT/5091_Sunday_June_05_2022_03_24_03_PM_18778660/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y' - \frac{y}{2x} - \frac{x^2}{2y} = 0}$$

1.10.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^3 + y^2}{2xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 13: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^3 + y^2}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2}{2x^2} \\ S_y &= \frac{y}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{4} + c_1 \tag{4}$$

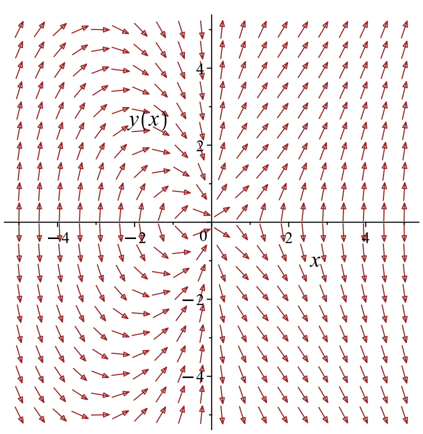
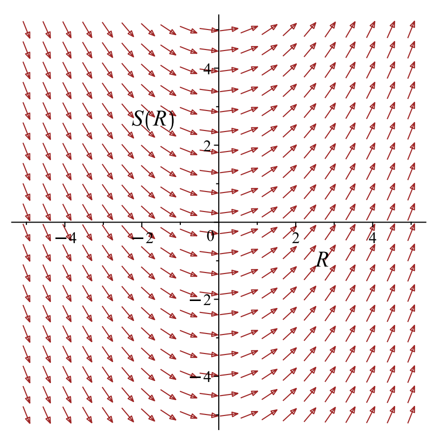
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x} = \frac{x^2}{4} + c_1$$

Which simplifies to

$$\frac{y^2}{2x} = \frac{x^2}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3+y^2}{2xy}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = \frac{R}{2}$ 

Summary
The solution(s) found are the following

$$\frac{y^2}{2x} = \frac{x^2}{4} + c_1 \tag{1}$$

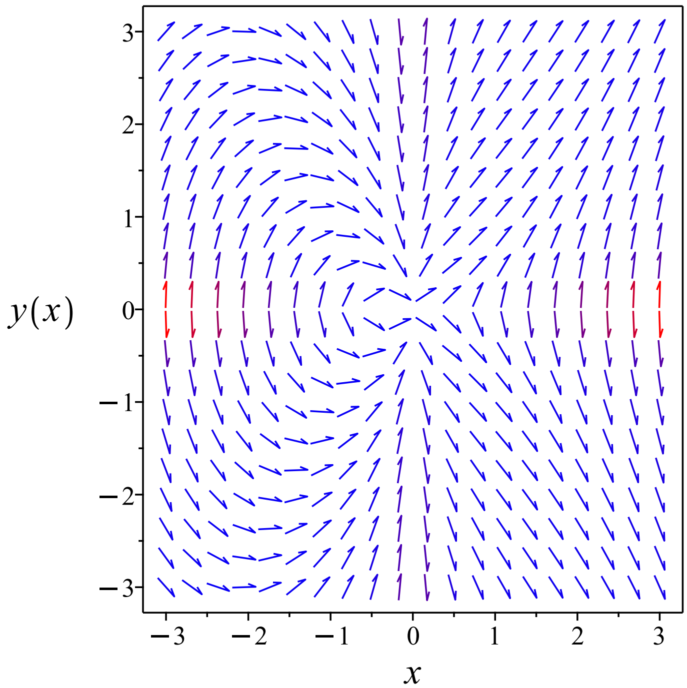


Figure 24: Slope field plot

Verification of solutions

$$\frac{y^2}{2x} = \frac{x^2}{4} + c_1$$

Verified OK.

1.10.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^3 + y^2}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y + \frac{x^2}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{2x} \\
f_1(x) &= \frac{x^2}{2} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = \frac{y^2}{2x} + \frac{x^2}{2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{w(x)}{2x} + \frac{x^2}{2} \\
w' &= \frac{w}{x} + x^2
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= x^2
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(x^2) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(x^2) \\ d\left(\frac{w}{x}\right) &= x\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int x\,dx \\ \frac{w}{x} &= \frac{x^2}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = \frac{1}{2}x^3 + c_1 x$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{1}{2}x^3 + c_1 x$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{2}\,\sqrt{x\,(x^2 + 2c_1)}}{2} \\ y(x) &= -\frac{\sqrt{2}\,\sqrt{x\,(x^2 + 2c_1)}}{2}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\,\sqrt{x\,(x^2 + 2c_1)}}{2} \tag{1}$$

$$y = -\frac{\sqrt{2}\,\sqrt{x\,(x^2 + 2c_1)}}{2} \tag{2}$$

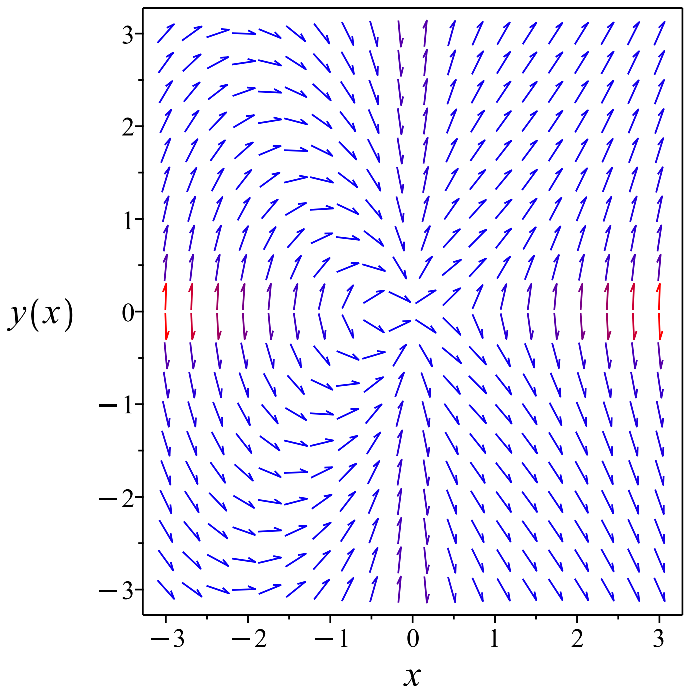


Figure 25: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{2}\,\sqrt{x\,(x^2 + 2c_1)}}{2}$$

Verified OK.

$$y = -\frac{\sqrt{2}\,\sqrt{x\,(x^2 + 2c_1)}}{2}$$

Verified OK.

1.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy)\,\mathrm{d}y&=\left(x^3+y^2\right)\mathrm{d}x\\ \left(-x^3-y^2\right)\mathrm{d}x+(2xy)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^3-y^2\\ N(x,y)&=2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3 - y^2\right)\\ &= -2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy)\\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2yx}((-2y) - (2y))\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int -\frac{2}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}\left(-x^3 - y^2\right)\\ &= \frac{-x^3 - y^2}{x^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(2xy) \\
&= \frac{2y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x^3 - y^2}{x^2}\right) + \left(\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-x^3 - y^2}{x^2}\,\mathrm{d}x \\
\phi &= -\frac{x^2}{2} + \frac{y^2}{x} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{2y}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y}{x}$. Therefore equation (4) becomes

$$
\frac{2y}{x} = \frac{2y}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^2}{x}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} + \frac{y^2}{x} = c_1 \tag{1}$$

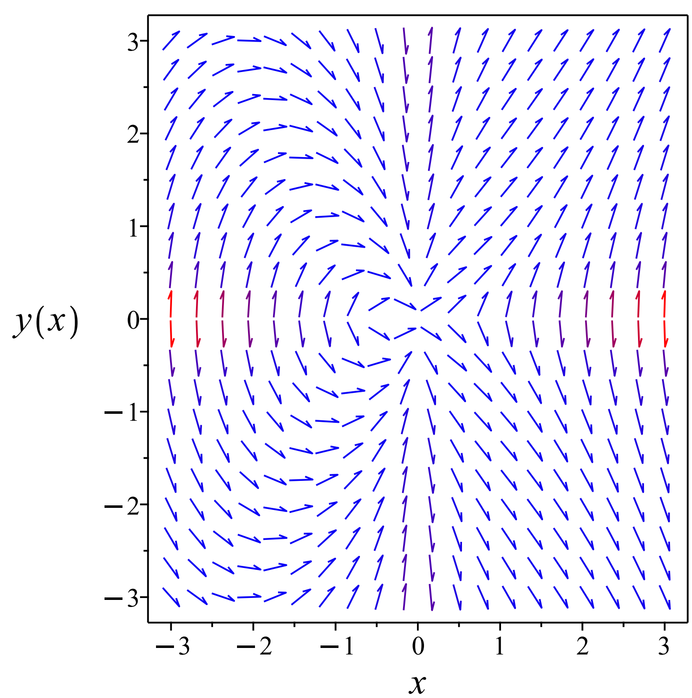


Figure 26: Slope field plot

Verification of solutions

$$-\frac{x^2}{2}+\frac{y^2}{x}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve(diff(y(x),x)=y(x)/(2*x)+x^2/(2*y(x)),y(x), singsol=all)
```

$$y(x)=-\frac{\sqrt{2}\,\sqrt{x\,(x^2+2c_1)}}{2}$$
$$y(x)=\frac{\sqrt{2}\,\sqrt{x\,(x^2+2c_1)}}{2}$$

✓ Solution by Mathematica

Time used: 0.194 (sec). Leaf size: 56

```
DSolve[y'[x]==y[x]/(2*x)+x^2/(2*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\frac{\sqrt{x}\sqrt{x^2+2c_1}}{\sqrt{2}}$$
$$y(x)\to \frac{\sqrt{x}\sqrt{x^2+2c_1}}{\sqrt{2}}$$

1.11 problem Example 3.11

Internal problem ID [5844]
Internal file name [`OUTPUT/5092_Sunday_June_05_2022_03_24_07_PM_77953227/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y}{t} - \frac{y^2}{t} = -\frac{2}{t}}$$

1.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= f(t)g(y) \\ &= \frac{y^2 + y - 2}{t} \end{aligned}$$

Where $f(t) = \frac{1}{t}$ and $g(y) = y^2 + y - 2$. Integrating both sides gives

$$\frac{1}{y^2 + y - 2}\, dy = \frac{1}{t}\, dt$$
$$\int \frac{1}{y^2 + y - 2}\, dy = \int \frac{1}{t}\, dt$$
$$\frac{\ln(y-1)}{3} - \frac{\ln(y+2)}{3} = c_1 + \ln(t)$$

The above can be written as

$$\left(\frac{1}{3}\right)(\ln(y-1) - \ln(y+2)) = 2c_1 + \ln(t)$$
$$\ln(y-1) - \ln(y+2) = (3)(2c_1 + \ln(t))$$
$$= 6c_1 + 3\ln(t)$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(y-1) - \ln(y+2)} = \mathrm{e}^{3c_1 + 3\ln(t)}$$

Which simplifies to

$$\frac{y-1}{y+2} = 3c_1 t^3$$
$$= c_2 t^3$$

Summary

The solution(s) found are the following

$$y = -\frac{2c_2 t^3 + 1}{c_2 t^3 - 1} \tag{1}$$

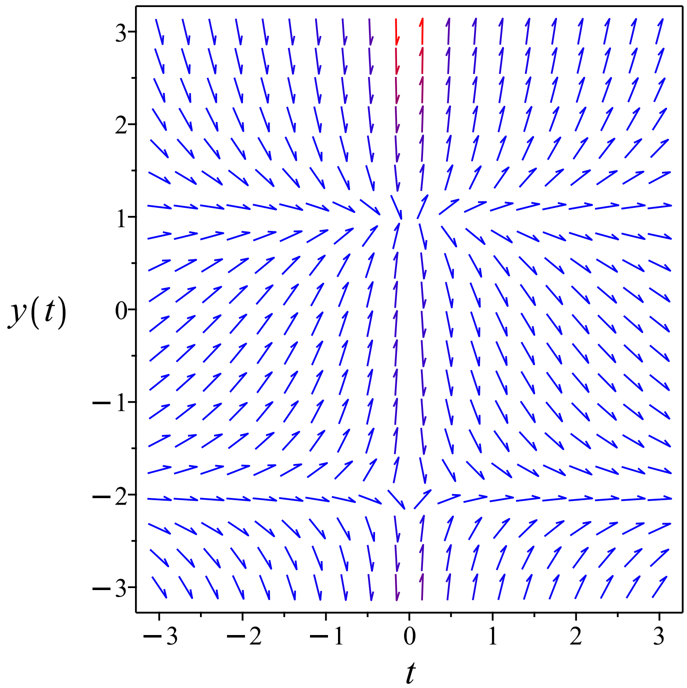


Figure 27: Slope field plot

Verification of solutions

$$y = -\frac{2c_2t^3 + 1}{c_2t^3 - 1}$$

Verified OK.

1.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2 + y - 2}{t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 15: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= t \\
\eta(t, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{t} dt \end{aligned}$$

Which results in

$$S = \ln(t)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{y^2 + y - 2}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= \frac{1}{t} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2 + y - 2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 + R - 2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln (R-1)}{3} - \frac{\ln (R+2)}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\ln (t) = \frac{\ln (y-1)}{3} - \frac{\ln (y+2)}{3} + c_1$$

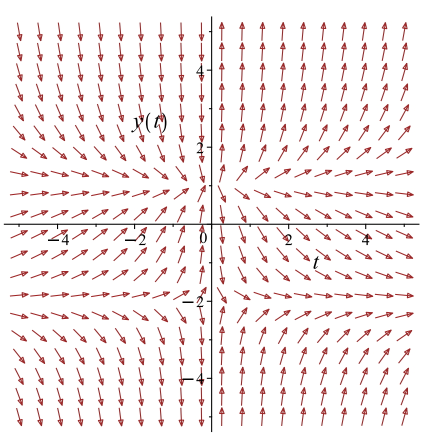
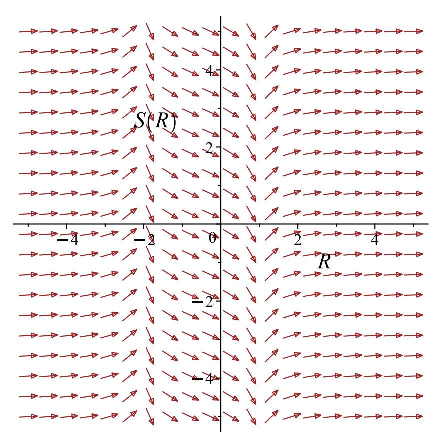
Which simplifies to

$$\ln (t) = \frac{\ln (y-1)}{3} - \frac{\ln (y+2)}{3} + c_1$$

Which gives

$$y = \frac{e^{3c_1} + 2t^3}{e^{3c_1} - t^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{y^2+y-2}{t}$ 	$R = y$ $S = \ln (t)$	$\frac{dS}{dR} = \frac{1}{R^2+R-2}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{3c_1} + 2t^3}{e^{3c_1} - t^3} \tag{1}$$

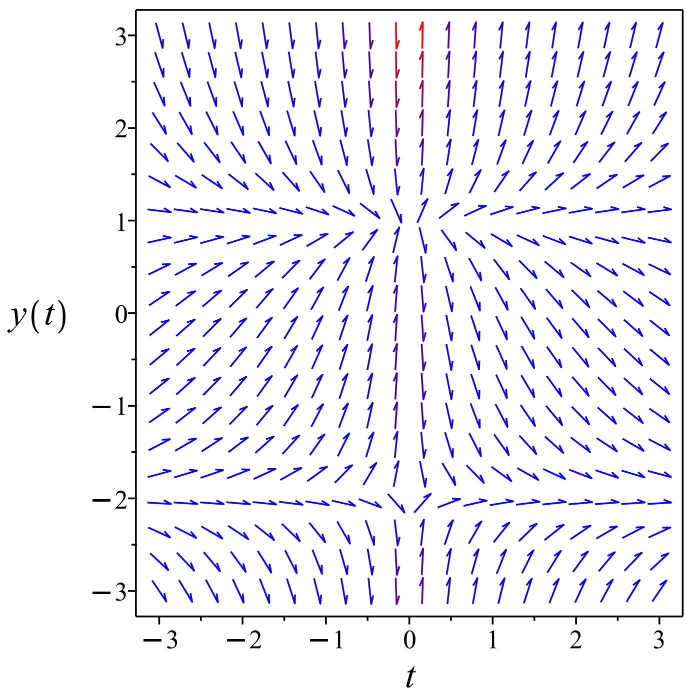


Figure 28: Slope field plot

Verification of solutions

$$y = \frac{e^{3c_1} + 2t^3}{e^{3c_1} - t^3}$$

Verified OK.

1.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2+y-2}\right)\mathrm{d}y&=\left(\frac{1}{t}\right)\mathrm{d}t\\ \left(-\frac{1}{t}\right)\mathrm{d}t+\left(\frac{1}{y^2+y-2}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-\frac{1}{t}\\ N(t,y)&=\frac{1}{y^2+y-2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{t}\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y^2+y-2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{1}{t}\,\mathrm{d}t\end{aligned}$$

$$\phi = -\ln(t) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+y-2}$. Therefore equation (4) becomes

$$\frac{1}{y^2+y-2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2+y-2}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y^2+y-2}\right)\mathrm{d}y\\ f(y) &= \frac{\ln(y-1)}{3} - \frac{\ln(y+2)}{3} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(t) + \frac{\ln(y-1)}{3} - \frac{\ln(y+2)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(t) + \frac{\ln(y-1)}{3} - \frac{\ln(y+2)}{3}$$

The solution becomes

$$y = \frac{-2\,e^{3c_1}t^3 - 1}{e^{3c_1}t^3 - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-2\,e^{3c_1}t^3 - 1}{e^{3c_1}t^3 - 1} \tag{1}$$

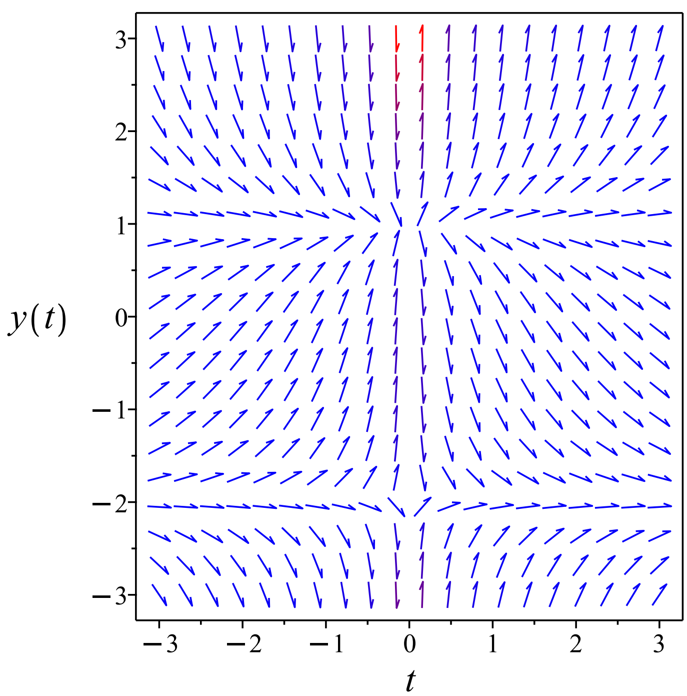


Figure 29: Slope field plot

Verification of solutions

$$y = \frac{-2\,\mathrm{e}^{3c_1}t^3 - 1}{\mathrm{e}^{3c_1}t^3 - 1}$$

Verified OK.

1.11.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= \frac{y^2 + y - 2}{t} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{2}{t} + \frac{y}{t} + \frac{y^2}{t}$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = -\frac{2}{t}$, $f_1(t) = \frac{1}{t}$ and $f_2(t) = \frac{1}{t}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{t}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{t^2} \\ f_1 f_2 &= \frac{1}{t^2} \\ f_2^2 f_0 &= -\frac{2}{t^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(t)}{t} - \frac{2u(t)}{t^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = \frac{c_1 t^3 + c_2}{t}$$

The above shows that

$$u'(t) = \frac{2c_1 t^3 - c_2}{t^2}$$

Using the above in (1) gives the solution

$$y = -\frac{2c_1 t^3 - c_2}{c_1 t^3 + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-2c_3 t^3 + 1}{c_3 t^3 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-2c_3 t^3 + 1}{c_3 t^3 + 1} \tag{1}$$

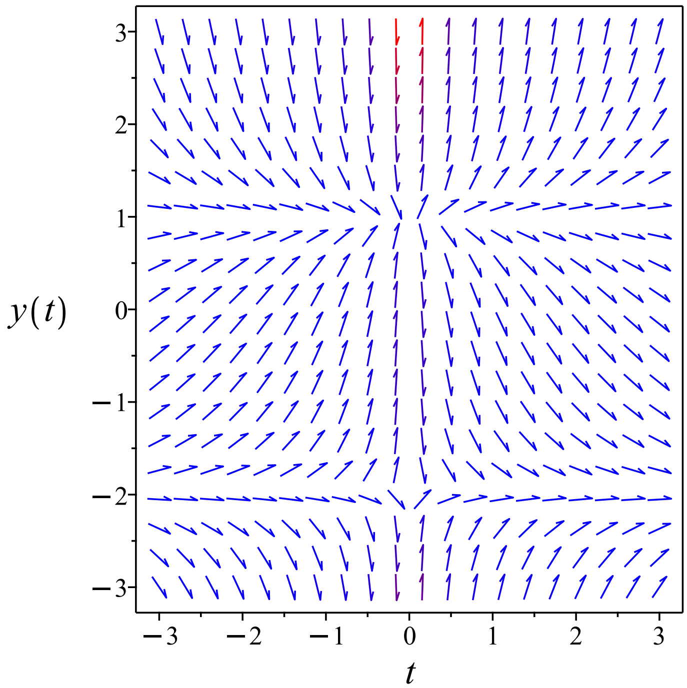


Figure 30: Slope field plot

Verification of solutions

$$y = \frac{-2c_3t^3 + 1}{c_3t^3 + 1}$$

Verified OK.

1.11.5 Maple step by step solution

Let's solve

$y' - \frac{y}{t} - \frac{y^2}{t} = -\frac{2}{t}$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(y+2)(y-1)} = \frac{1}{t}$

- Integrate both sides with respect to t

  $\int \frac{y'}{(y+2)(y-1)} dt = \int \frac{1}{t} dt + c_1$

- Evaluate integral

$$\frac{\ln(y-1)}{3} - \frac{\ln(y+2)}{3} = c_1 + \ln(t)$$

- Solve for y

$$y = \frac{t^3 (e^{c_1})^3 - 3\,e^{3c_1} t^3 - 1}{t^3 (e^{c_1})^3 - 1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.438 (sec). Leaf size: 22

```
dsolve(diff(y(t),t)=-2/t+1/t*y(t)+1/t*y(t)^2,y(t), singsol=all)
```

$$y(t) = \frac{-2c_1 t^3 - 1}{c_1 t^3 - 1}$$

✓ Solution by Mathematica

Time used: 1.263 (sec). Leaf size: 43

```
DSolve[y'[t]==-2/t+1/t*y[t]+1/t*y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1 - 2e^{3c_1} t^3}{1 + e^{3c_1} t^3}$$
$$y(t) \to -2$$
$$y(t) \to 1$$

1.12 problem Example 3.12

Internal problem ID [5845]
Internal file name [OUTPUT/5093_Sunday_June_05_2022_03_24_10_PM_796261/index.tex]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{y' + \frac{y}{t} + y^2 = -1}$$

1.12.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= -\frac{y^2 t + t + y}{t} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y}{t} - 1 - y^2$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = -1$, $f_1(t) = -\frac{1}{t}$ and $f_2(t) = -1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(t) - (f_2' + f_1f_2)\,u'(t) + f_2^2f_0u(t) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= 0 \\
f_1f_2 &= \frac{1}{t} \\
f_2^2f_0 &= -1
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(t) - \frac{u'(t)}{t} - u(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 \operatorname{BesselJ}(0, t) + c_2 \operatorname{BesselY}(0, t)$$

The above shows that

$$u'(t) = -c_1 \operatorname{BesselJ}(1, t) - c_2 \operatorname{BesselY}(1, t)$$

Using the above in (1) gives the solution

$$y = \frac{-c_1 \operatorname{BesselJ}(1, t) - c_2 \operatorname{BesselY}(1, t)}{c_1 \operatorname{BesselJ}(0, t) + c_2 \operatorname{BesselY}(0, t)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 \operatorname{BesselJ}(1, t) - \operatorname{BesselY}(1, t)}{c_3 \operatorname{BesselJ}(0, t) + \operatorname{BesselY}(0, t)}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{-c_3 \operatorname{BesselJ}(1, t) - \operatorname{BesselY}(1, t)}{c_3 \operatorname{BesselJ}(0, t) + \operatorname{BesselY}(0, t)} \tag{1}$$

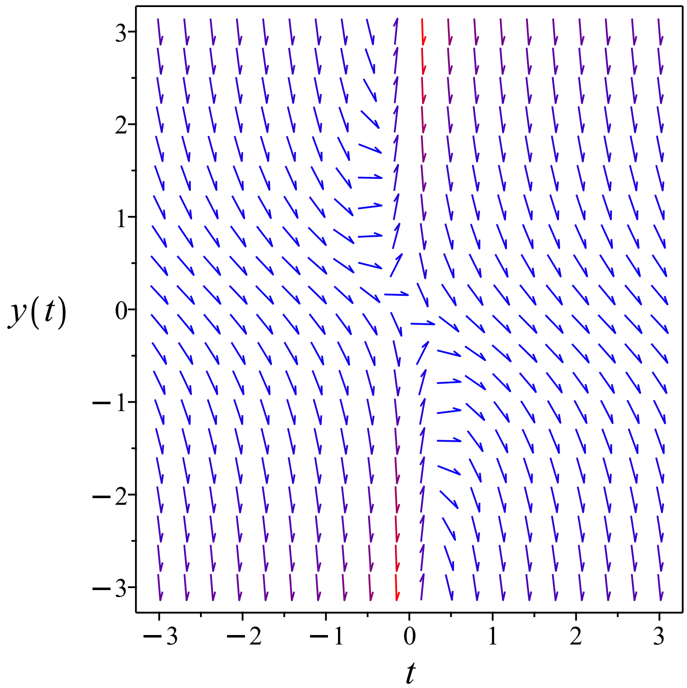


Figure 31: Slope field plot

Verification of solutions

$$y = \frac{-c_3 \operatorname{BesselJ}(1, t) - \operatorname{BesselY}(1, t)}{c_3 \operatorname{BesselJ}(0, t) + \operatorname{BesselY}(0, t)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
        -> searching for a solution in terms of Whittaker functions
        <- solution in terms of Whittaker functions successful
   <- Abel AIR successful: ODE belongs to the 1F1 2-parameter class`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 35

```
dsolve(diff(y(t),t)=-y(t)/t-1-y(t)^2,y(t), singsol=all)
```

$$y(t) = \frac{-i\,\mathrm{BesselK}\,(1, it)\,c_1 - \mathrm{BesselJ}\,(1, t)}{\mathrm{BesselK}\,(0, it)\,c_1 + \mathrm{BesselJ}\,(0, t)}$$

✓ Solution by Mathematica

Time used: 0.189 (sec). Leaf size: 43

```
DSolve[y'[t]==-y[t]/t-1-y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{\mathrm{BesselY}(1, t) + c_1\,\mathrm{BesselJ}(1, t)}{\mathrm{BesselY}(0, t) + c_1\,\mathrm{BesselJ}(0, t)}$$
$$y(t) \to -\frac{\mathrm{BesselJ}(1, t)}{\mathrm{BesselJ}(0, t)}$$

1.13 problem Example 3.14

Internal problem ID [5846]
Internal file name [`OUTPUT/5094_Sunday_June_05_2022_03_24_13_PM_25223300/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.14.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[_dAlembert]`

$$\boxed{y'y - ay'^2 = -x}$$

1.13.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$-a\,p^2 + py = -x$$

Solving for y from the above results in

$$y = -\frac{x}{p} + ap \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -\frac{1}{p}$$
$$g = ap$$

Hence (2) becomes

$$p + \frac{1}{p} = \left(\frac{x}{p^2} + a\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \frac{1}{p} = 0$$

Solving for p from the above gives

$$p = i$$
$$p = -i$$

Substituting these in (1A) gives

$$y = -ia - ix$$
$$y = ia + ix$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) + \frac{1}{p(x)}}{\frac{x}{p(x)^2} + a} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{\frac{x(p)}{p^2} + a}{p + \frac{1}{p}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$
\begin{aligned}
p(p) &= -\frac{1}{p\,(p^2+1)} \\
q(p) &= \frac{pa}{p^2+1}
\end{aligned}
$$

Hence the ode is

$$
\frac{d}{dp}x(p) - \frac{x(p)}{p\,(p^2+1)} = \frac{pa}{p^2+1}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{p\left(p^2+1\right)}dp} \\
&= \mathrm{e}^{\frac{\ln\left(p^2+1\right)}{2}-\ln(p)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{\sqrt{p^2+1}}{p}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(\frac{pa}{p^2+1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\sqrt{p^2+1}\,x}{p}\right) &= \left(\frac{\sqrt{p^2+1}}{p}\right)\left(\frac{pa}{p^2+1}\right) \\
\mathrm{d}\left(\frac{\sqrt{p^2+1}\,x}{p}\right) &= \left(\frac{a}{\sqrt{p^2+1}}\right)\,\mathrm{d}p
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\sqrt{p^2+1}\,x}{p} &= \int \frac{a}{\sqrt{p^2+1}}\,\mathrm{d}p \\
\frac{\sqrt{p^2+1}\,x}{p} &= a\ \mathrm{arcsinh}\,(p) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{p^2+1}}{p}$ results in

$$
x(p) = \frac{pa\ \mathrm{arcsinh}\,(p)}{\sqrt{p^2+1}} + \frac{c_1 p}{\sqrt{p^2+1}}
$$

which simplifies to

$$x(p) = \frac{p(a \,\mathrm{arcsinh}\,(p) + c_1)}{\sqrt{p^2+1}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned} p &= \frac{y+\sqrt{y^2+4ax}}{2a} \\ p &= -\frac{-y+\sqrt{y^2+4ax}}{2a} \end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned} x &= \frac{\left(y+\sqrt{y^2+4ax}\right)\left(a \,\mathrm{arcsinh}\left(\frac{y+\sqrt{y^2+4ax}}{2a}\right)+c_1\right)\sqrt{2}}{2\sqrt{\frac{y\sqrt{y^2+4ax}+2a^2+2ax+y^2}{a^2}}\, a} \\ x &= \frac{\left(-y+\sqrt{y^2+4ax}\right)\left(a \,\mathrm{arcsinh}\left(\frac{-y+\sqrt{y^2+4ax}}{2a}\right)-c_1\right)\sqrt{2}}{2\sqrt{\frac{2a^2+2ax-y\sqrt{y^2+4ax}+y^2}{a^2}}\, a} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -ia - ix \tag{1}$$

$$y = ia + ix \tag{2}$$

$$x = \frac{\left(y+\sqrt{y^2+4ax}\right)\left(a \,\mathrm{arcsinh}\left(\frac{y+\sqrt{y^2+4ax}}{2a}\right)+c_1\right)\sqrt{2}}{2\sqrt{\frac{y\sqrt{y^2+4ax}+2a^2+2ax+y^2}{a^2}}\, a} \tag{3}$$

$$x = \frac{\left(-y+\sqrt{y^2+4ax}\right)\left(a \,\mathrm{arcsinh}\left(\frac{-y+\sqrt{y^2+4ax}}{2a}\right)-c_1\right)\sqrt{2}}{2\sqrt{\frac{2a^2+2ax-y\sqrt{y^2+4ax}+y^2}{a^2}}\, a} \tag{4}$$

Verification of solutions

$$y = -ia - ix$$

Verified OK.

$$y = ia + ix$$

Verified OK.

$$x = \frac{\left(y + \sqrt{y^2 + 4ax}\right)\left(a \operatorname{arcsinh}\left(\frac{y+\sqrt{y^2+4ax}}{2a}\right) + c_1\right)\sqrt{2}}{2\sqrt{\frac{y\sqrt{y^2+4ax}+2a^2+2ax+y^2}{a^2}}\, a}$$

Verified OK.

$$x = \frac{\left(-y + \sqrt{y^2 + 4ax}\right)\left(a \operatorname{arcsinh}\left(\frac{-y+\sqrt{y^2+4ax}}{2a}\right) - c_1\right)\sqrt{2}}{2\sqrt{\frac{2a^2+2ax-y\sqrt{y^2+4ax}+y^2}{a^2}}\, a}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 246

```
dsolve(x+y(x)*diff(y(x),x)=a*(diff(y(x),x))^2,y(x), singsol=all)
```

$$\frac{-\frac{\sqrt{2}\left(y(x)+\sqrt{4ax+y(x)^2}\right)\operatorname{arcsinh}\left(\frac{y(x)+\sqrt{4ax+y(x)^2}}{2a}\right)}{2}+x\sqrt{\frac{y(x)\sqrt{4ax+y(x)^2}+2a^2+2ax+y(x)^2}{a^2}}+c_1y(x)+c_1\sqrt{4ax+y\left(x\right)^2}}{\sqrt{\frac{y(x)\sqrt{4ax+y(x)^2}+y(x)^2+2a(a+x)}{a^2}}}$$
$$=0$$

$$\frac{\sqrt{\frac{-2y(x)\sqrt{4ax+y(x)^2}+2y(x)^2+4a(a+x)}{a^2}}\,x-\left(y(x)-\sqrt{4ax+y\left(x\right)^2}\right)\left(-\operatorname{arcsinh}\left(\frac{-y(x)+\sqrt{4ax+y(x)^2}}{2a}\right)+c_1\right)}{\sqrt{\frac{-2y(x)\sqrt{4ax+y(x)^2}+2y(x)^2+4a(a+x)}{a^2}}}$$
$$=0$$

✓ Solution by Mathematica

Time used: 1.371 (sec). Leaf size: 71

```
DSolve[x+y[x]*y'[x]==a*(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\left\{x=-\frac{aK[1]\log\left(\sqrt{K[1]^2+1}-K[1]\right)}{\sqrt{K[1]^2+1}}+\frac{c_1K[1]}{\sqrt{K[1]^2+1}},y(x)=aK[1]-\frac{x}{K[1]}\right\},\{y(x),K[1]\}\right]$$

1.14 problem Example 3.15

Internal problem ID [5847]
Internal file name [`OUTPUT/5095_Sunday_June_05_2022_03_24_30_PM_71446160/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.15.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{{y'}^{2}-y^{2}a^{2}=0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = ya \tag{1}$$
$$y' = -ya \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{ya}dy = \int dx$$
$$\frac{\ln(y)}{a} = x + c_1$$

Raising both side to exponential gives

$$e^{\frac{\ln(y)}{a}} = e^{x+c_1}$$

Which simplifies to

$$y^{\frac{1}{a}} = c_2 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = (c_2 \mathrm{e}^{x})^{a} \tag{1}$$

Verification of solutions

$$y = (c_2 \mathrm{e}^{x})^{a}$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{ya} dy = \int dx$$
$$-\frac{\ln (y)}{a} = x + c_3$$

Raising both side to exponential gives

$$\mathrm{e}^{-\frac{\ln(y)}{a}} = \mathrm{e}^{x+c_3}$$

Which simplifies to

$$y^{-\frac{1}{a}} = c_4 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = (c_4 \mathrm{e}^{x})^{-a} \tag{1}$$

Verification of solutions

$$y = (c_4 \mathrm{e}^{x})^{-a}$$

Verified OK.

1.14.1 Maple step by step solution

Let's solve

$y'^2 - y^2a^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = a$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int a dx + c_1$

- Evaluate integral

  $\ln(y) = ax + c_1$

- Solve for y

  $y = e^{ax+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(diff(y(x),x)^2-a^2*y(x)^2=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{ax}$$
$$y(x) = c_1 e^{-ax}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 31

```
DSolve[(y'[x])^2-a^2*y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to c_1 e^{-ax} \\
y(x) &\to c_1 e^{ax} \\
y(x) &\to 0
\end{aligned}
$$

1.15 problem Example 3.16

Internal problem ID [5848]
Internal file name [`OUTPUT/5096_Sunday_June_05_2022_03_24_32_PM_70360323/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.2 FIRST ORDER ODE. Page 114
Problem number: Example 3.16.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y'^2 = 4x^2}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 2x \tag{1}$$
$$y' = -2x \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int 2x \, \mathrm{d}x \\ &= x^2 + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x^2 + c_1 \tag{1}$$

Verification of solutions

$$y = x^2 + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -2x \, \mathrm{d}x \\ &= -x^2 + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -x^2 + c_2 \tag{1}$$

Verification of solutions

$$y = -x^2 + c_2$$

Verified OK.

1.15.1 Maple step by step solution

Let's solve

$y'^2 = 4x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'^2 dx = \int 4x^2 dx + c_1$

- Cannot compute integral

  $\int y'^2 dx = \frac{4x^3}{3} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)^2=4*x^2,y(x), singsol=all)
```

$$y(x) = x^2 + c_1$$
$$y(x) = -x^2 + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 23

```
DSolve[(y'[x])^2==4*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^2 + c_1$$
$$y(x) \to x^2 + c_1$$

2 Chapter 3. Ordinary Differential Equations. Section 3.3 SECOND ORDER ODE. Page 147

2.1 problem Example 3.17

Internal problem ID [5849]
Internal file name [`OUTPUT/5097_Sunday_June_05_2022_03_24_34_PM_42517123/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.3 SECOND ORDER ODE. Page 147
Problem number: Example 3.17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' - 3y = 0}$$

2.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} - 3 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -3$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(-3)} \\
&= 1 \pm 2
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= 1 + 2 \\
\lambda_2 &= 1 - 2
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 3 \\
\lambda_2 &= -1
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(3)x} + c_2 e^{(-1)x}
\end{aligned}
$$

Or

$$
y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-x}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-x} \tag{1}
$$

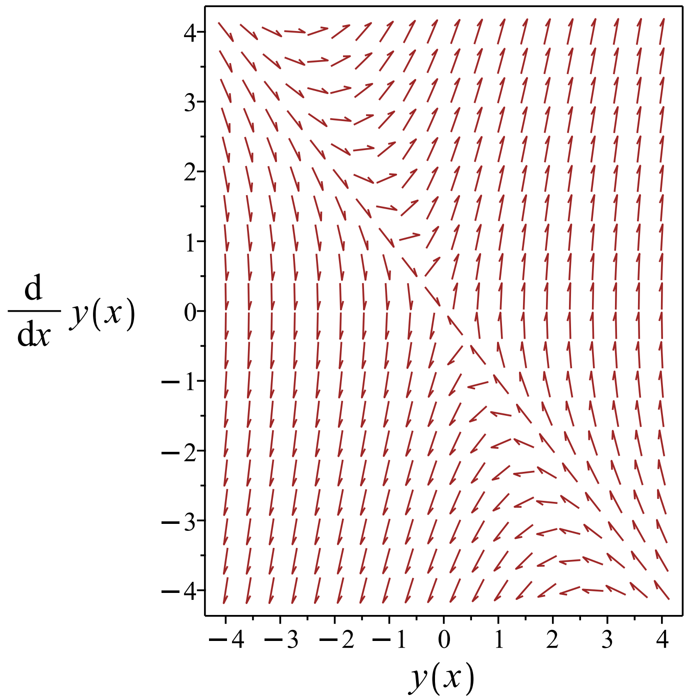


Figure 32: Slope field plot

Verification of solutions

$$y = e^{3x}c_1 + c_2e^{-x}$$

Verified OK.

2.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 20: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1 (\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{3x}}{4} \tag{1}
$$

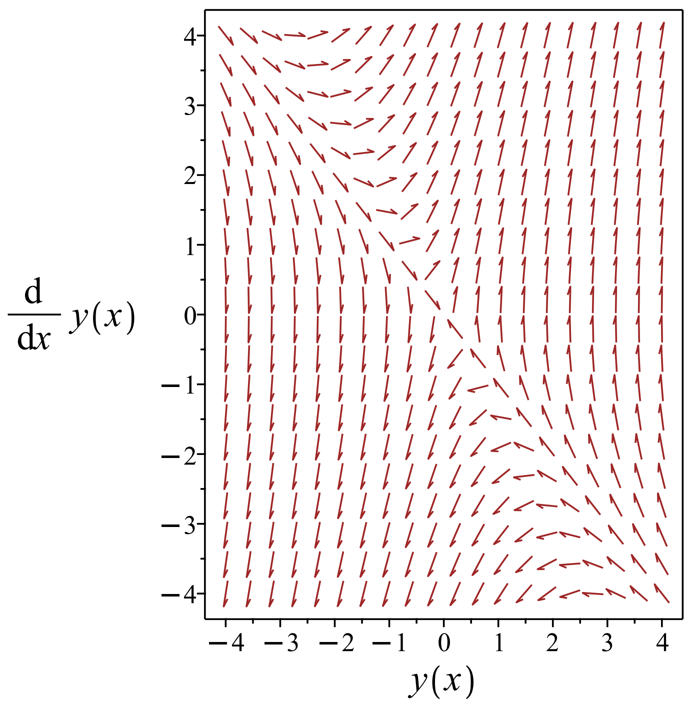


Figure 33: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{3x}}{4}
$$

Verified OK.

2.1.3 Maple step by step solution

Let's solve

$y'' - 2y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 3)$

- 1st solution of the ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-x} + c_2 e^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{3x}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 22

```
DSolve[y''[x]-2*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_2 e^{4x} + c_1\right)$$

2.2 problem Example 3.18

Internal problem ID [5850]
Internal file name [`OUTPUT/5098_Sunday_June_05_2022_03_24_35_PM_26496268/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.3 SECOND ORDER ODE. Page 147
Problem number: Example 3.18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{s'' + 2s' + s = 0}$$

With initial conditions

$$[s(0) = 4, s'(0) = -2]$$

2.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$s'' + p(t)s' + q(t)s = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$s'' + 2s' + s = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.2.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$As''(t) + Bs'(t) + Cs(t) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $s = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda \, e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)} \\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$s = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$s = e^{-t} c_1 + t\, e^{-t} c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $s = 4$ and $t = 0$ in the above gives

$$4 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$s' = -e^{-t} c_1 + c_2 e^{-t} - t\, e^{-t} c_2$$

substituting $s' = -2$ and $t = 0$ in the above gives

$$-2 = -c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}c_1 &= 4 \\ c_2 &= 2\end{aligned}$$

Substituting these values back in above solution results in

$$s = 2t\, e^{-t} + 4\, e^{-t}$$

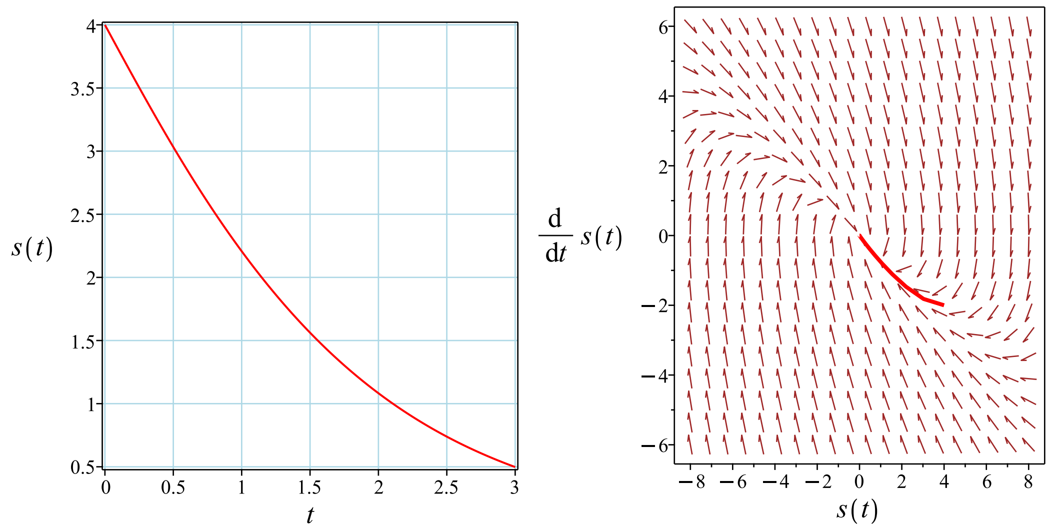
Which simplifies to

$$s = 2(t + 2)\, e^{-t}$$

Summary

The solution(s) found are the following

$$s = 2(t + 2)\, e^{-t} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$s = 2(t+2)\,e^{-t}$$

Verified OK.

2.2.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$s'' + p(t)\,s' + \frac{\left(p(t)^2 + p'(t)\right)s}{2} = f(t)$$

Where $p(t) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= e^{t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)s)'' &= 0 \\ \left(e^{t} s\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(e^t s\right)' = c_1$$

Integrating again gives

$$\left(e^t s\right) = c_1 t + c_2$$

Hence the solution is

$$s = \frac{c_1 t + c_2}{e^t}$$

Or

$$s = c_1 t \,e^{-t} + c_2 e^{-t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$s = c_1 t \,e^{-t} + c_2 e^{-t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $s = 4$ and $t = 0$ in the above gives

$$4 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$s' = e^{-t} c_1 - c_1 t \,e^{-t} - c_2 e^{-t}$$

substituting $s' = -2$ and $t = 0$ in the above gives

$$-2 = c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 4$$

Substituting these values back in above solution results in

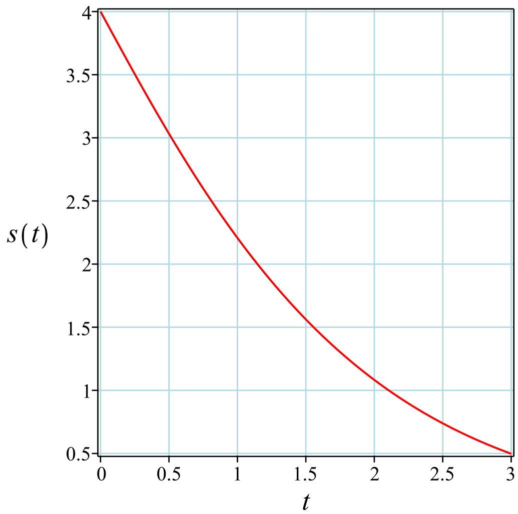
$$s = 2t \,e^{-t} + 4 \,e^{-t}$$

Which simplifies to

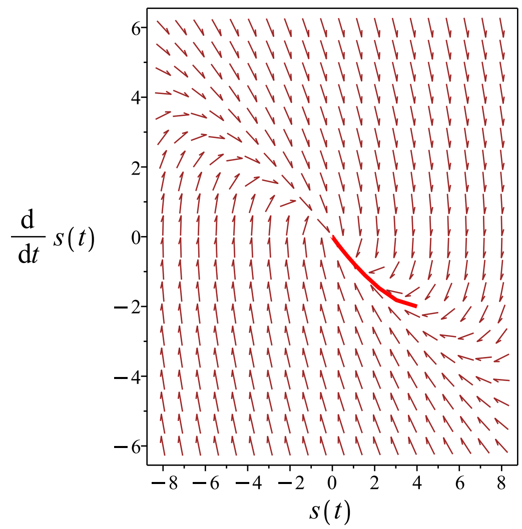
$$s = 2(t + 2) \,e^{-t}$$

Summary
The solution(s) found are the following

$$s = 2(t+2)\,\mathrm{e}^{-t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$s = 2(t+2)\,\mathrm{e}^{-t}$$

Verified OK.

2.2.4 Solving using Kovacic algorithm

Writing the ode as

$$s'' + 2s' + s = 0 \tag{1}$$
$$As'' + Bs' + Cs = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = s e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then s is found using the inverse transformation

$$s = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 22: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in s is found from

$$\begin{aligned} s_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\, dt} \\ &= z_1 e^{-t} \\ &= z_1 \left(\mathrm{e}^{-t}\right) \end{aligned}$$

Which simplifies to

$$s_1 = \mathrm{e}^{-t}$$

The second solution s_2 to the original ode is found using reduction of order

$$s_2 = s_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{s_1^2}\, dt$$

Substituting gives

$$\begin{aligned} s_2 &= s_1 \int \frac{e^{\int -\frac{2}{1}\, dt}}{(s_1)^2}\, dt \\ &= s_1 \int \frac{e^{-2t}}{(s_1)^2}\, dt \\ &= s_1(t) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
s &= c_1 s_1 + c_2 s_2 \\
&= c_1 \left(\mathrm{e}^{-t}\right) + c_2 \left(\mathrm{e}^{-t}(t)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
s = \mathrm{e}^{-t} c_1 + t\,\mathrm{e}^{-t} c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $s = 4$ and $t = 0$ in the above gives

$$
4 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
s' = -\mathrm{e}^{-t} c_1 + c_2 \mathrm{e}^{-t} - t\,\mathrm{e}^{-t} c_2
$$

substituting $s' = -2$ and $t = 0$ in the above gives

$$
-2 = -c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 4 \\
c_2 &= 2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
s = 2t\,\mathrm{e}^{-t} + 4\,\mathrm{e}^{-t}
$$

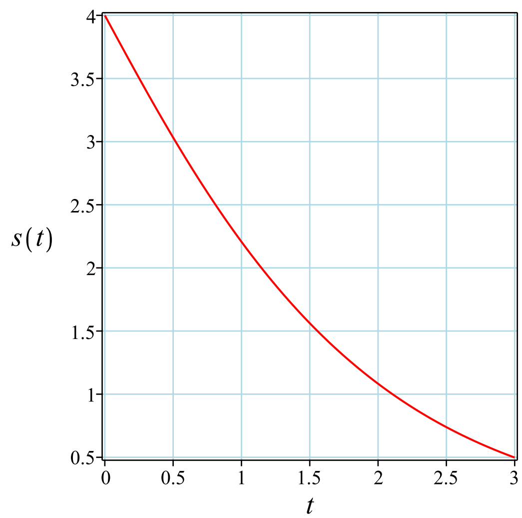
Which simplifies to

$$
s = 2(t+2)\,\mathrm{e}^{-t}
$$

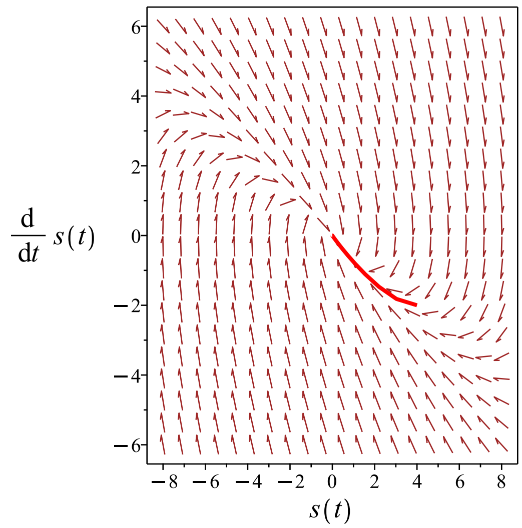
Summary

The solution(s) found are the following

$$
s = 2(t+2)\,\mathrm{e}^{-t} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$s = 2(t+2)\,e^{-t}$$

Verified OK.

2.2.5 Maple step by step solution

Let's solve

$$\left[s''+2s'+s=0, s(0)=4, s'\Big|_{\{t=0\}}=-2\right]$$

- Highest derivative means the order of the ODE is 2

  s''

- Characteristic polynomial of ODE

  $r^2+2r+1=0$

- Factor the characteristic polynomial

  $(r+1)^2=0$

- Root of the characteristic polynomial

  $r=-1$

- 1st solution of the ODE

$s_1(t) = \mathrm{e}^{-t}$

- Repeated root, multiply $s_1(t)$ by t to ensure linear independence

$s_2(t) = t\,\mathrm{e}^{-t}$

- General solution of the ODE

$s = c_1 s_1(t) + c_2 s_2(t)$

- Substitute in solutions

$s = \mathrm{e}^{-t} c_1 + t\,\mathrm{e}^{-t} c_2$

□ Check validity of solution $s = \mathrm{e}^{-t} c_1 + t\mathrm{e}^{-t} c_2$

  - Use initial condition $s(0) = 4$

  $4 = c_1$

  - Compute derivative of the solution

  $s' = -\mathrm{e}^{-t} c_1 + c_2 \mathrm{e}^{-t} - t\,\mathrm{e}^{-t} c_2$

  - Use the initial condition $s'\Big|_{\{t=0\}} = -2$

  $-2 = -c_1 + c_2$

  - Solve for c_1 and c_2

  $\{c_1 = 4, c_2 = 2\}$

  - Substitute constant values into general solution and simplify

  $s = 2(t+2)\,\mathrm{e}^{-t}$

- Solution to the IVP

$s = 2(t+2)\,\mathrm{e}^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve([diff(s(t),t$2)+2*diff(s(t),t)+s(t)=0,s(0) = 4, D(s)(0) = -2],s(t), singsol=all)
```

$$s(t) = 2\,\mathrm{e}^{-t}(t+2)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 15

```
DSolve[{s''[t]+2*s'[t]+s[t]==0,{s[0]==4,s'[0]==-2}},s[t],t,IncludeSingularSolutions -> True]
```

$$s(t) \to 2e^{-t}(t+2)$$

2.3 problem Example 3.19

Internal problem ID [5851]
Internal file name [`OUTPUT/5099_Sunday_June_05_2022_03_24_37_PM_94357119/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.3 SECOND ORDER ODE. Page 147
Problem number: Example 3.19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 5y = 0}$$

2.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + 5 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 5$ into the above gives

$$\lambda_{1,2} = \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(5)}$$
$$= 1 \pm 2i$$

Hence

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

Which simplifies to

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{x}(c_1 \cos(2x) + c_2 \sin(2x)) \tag{1}$$

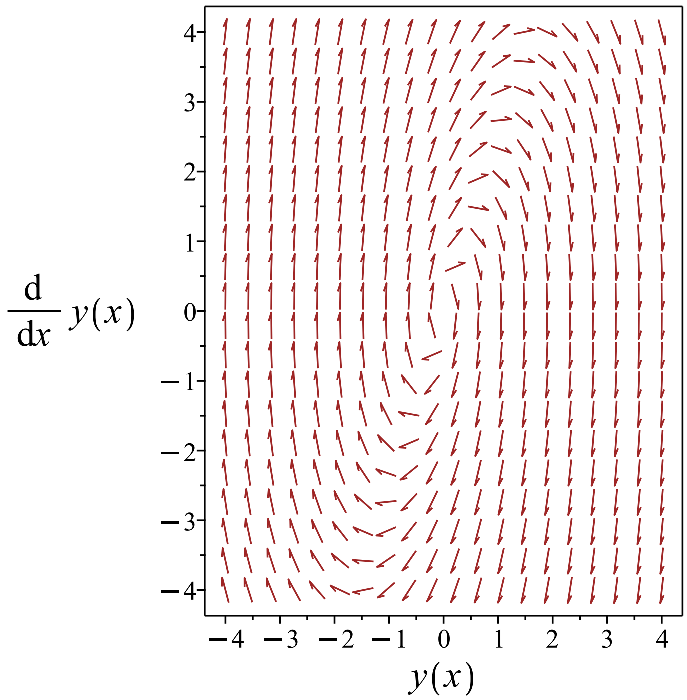


Figure 37: Slope field plot

Verification of solutions

$$y = e^x(c_1 \cos(2x) + c_2 \sin(2x))$$

Verified OK.

2.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 5y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 24: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1 (\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}\cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (e^x \cos(2x)) + c_2 \left(e^x \cos(2x) \left(\frac{\tan(2x)}{2} \right) \right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 e^x \cos(2x) + \frac{c_2 e^x \sin(2x)}{2} \tag{1}
$$

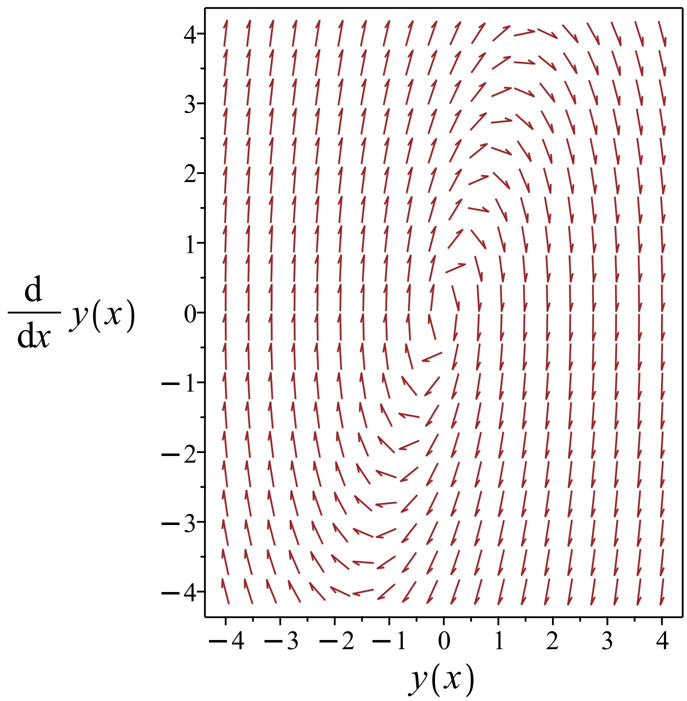


Figure 38: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 e^x \cos(2x) + \frac{c_2 e^x \sin(2x)}{2}
$$

Verified OK.

2.3.3 Maple step by step solution

Let's solve

$y'' - 2y' + 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - 2\,\mathrm{I}, 1 + 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^x \cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^x \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^x \cos(2x) + c_2 \mathrm{e}^x \sin(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+5*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{x}(c_1 \sin(2x) + c_2 \cos(2x))$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 24

```
DSolve[y''[x]-2*y'[x]+5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}(c_2 \cos(2x) + c_1 \sin(2x))$$

2.4 problem Example 3.21

Internal problem ID [5852]
Internal file name [`OUTPUT/5100_Sunday_June_05_2022_03_24_38_PM_53342084/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.3 SECOND ORDER ODE. Page 147
Problem number: Example 3.21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y' - 3y = 3x + 1}$$

2.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = -3, f(x) = 3x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' - 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda \,\mathrm{e}^{\lambda x} - 3\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(-3)}\\ &= 1 \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 2\\ \lambda_2 &= 1 - 2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(3)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_2 x - 3A_1 - 2A_2 = 3x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = -1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x + \frac{1}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x} c_1 + c_2 e^{-x}\right) + \left(-x + \frac{1}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{3x} c_1 + c_2 e^{-x} - x + \frac{1}{3} \tag{1}$$

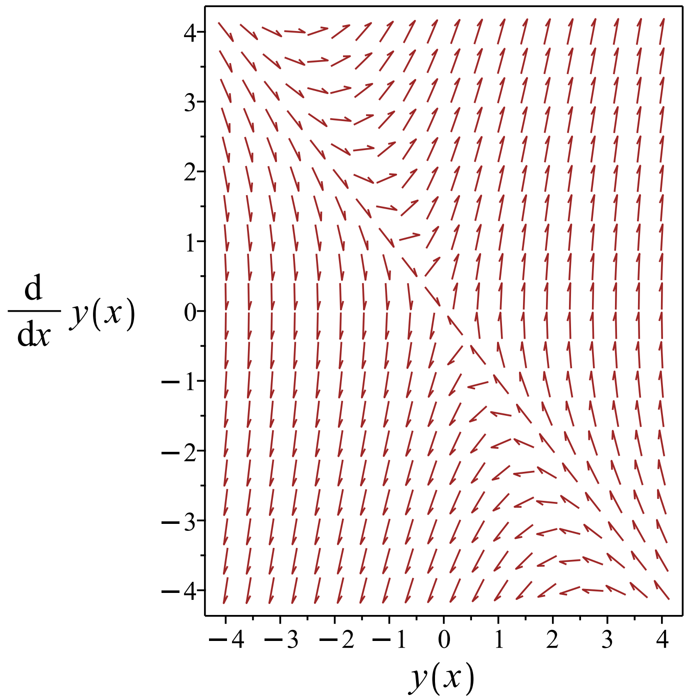


Figure 39: Slope field plot

Verification of solutions

$$y = e^{3x}c_1 + c_2e^{-x} - x + \frac{1}{3}$$

Verified OK.

2.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 26: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}\\ &= z_1 e^{x}\\ &= z_1(\mathrm{e}^{x})\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx\\ &= y_1 \left(\frac{\mathrm{e}^{4x}}{4}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' - 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{3x}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
1 + x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{3x}}{4}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_2 x + A_1
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_2x - 3A_1 - 2A_2 = 3x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = -1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x + \frac{1}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{3x}}{4}\right) + \left(-x + \frac{1}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{3x}}{4} - x + \frac{1}{3} \tag{1}$$

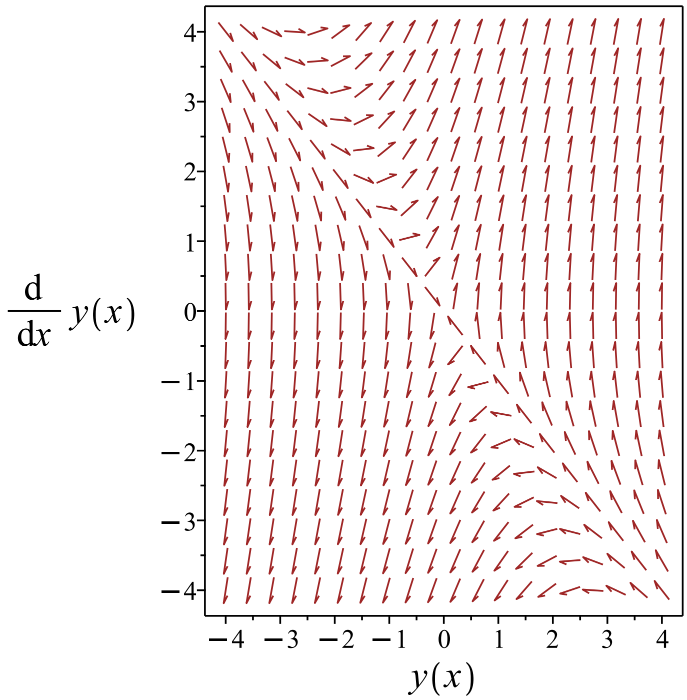


Figure 40: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{3x}}{4} - x + \frac{1}{3}$$

Verified OK.

2.4.3 Maple step by step solution

Let's solve

$y'' - 2y' - 3y = 3x + 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r + 1)(r - 3) = 0$

- Roots of the characteristic polynomial

$r = (-1, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 3x + 1\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3\,e^{3x} \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4\,e^{2x}$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{e^{-x}\left(\int (3x+1) e^{x} dx\right)}{4} + \frac{e^{3x}\left(\int e^{-3x}(3x+1) dx\right)}{4}$

  ○ Compute integrals

  $y_p(x) = -x + \frac{1}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 e^{3x} - x + \frac{1}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-3*y(x)=3*x+1,y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + e^{3x} c_1 - x + \frac{1}{3}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 28

```
DSolve[y''[x]-2*y'[x]-3*y[x]==3*x+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x + c_1 e^{-x} + c_2 e^{3x} + \frac{1}{3}$$

2.5 problem Example 3.22

Internal problem ID [5853]
Internal file name [`OUTPUT/5101_Sunday_June_05_2022_03_24_40_PM_47116087/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.3 SECOND ORDER ODE. Page 147
Problem number: Example 3.22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 3y' + 2y = e^{2x}x}$$

2.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = e^{2x}x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda\, \mathrm{e}^{\lambda x} + 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-3^2 - (4)(1)(2)} \\ &= \frac{3}{2} \pm \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= \frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= \frac{3}{2} - \frac{1}{2} \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 1 \end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned} y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(1)x} \end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}x, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{2x}, e^{2x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{2x} + A_2 e^{2x} x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{2x} + 2A_1 x\, e^{2x} + A_2 e^{2x} = e^{2x} x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 e^{2x}}{2} - e^{2x} x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{x}\right) + \left(\frac{x^2 e^{2x}}{2} - e^{2x} x\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{x} + \frac{x^2 e^{2x}}{2} - e^{2x} x \tag{1}$$

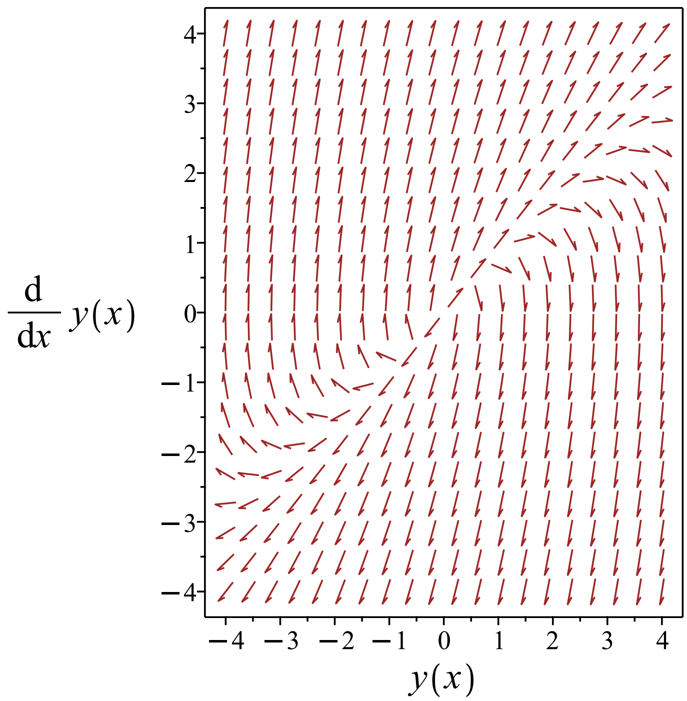


Figure 41: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{x} + \frac{x^2 e^{2x}}{2} - e^{2x} x$$

Verified OK.

2.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -3 \\
C &= 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 28: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2}\, dx \\ &= y_1(e^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2(e^x(e^x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^x + c_2 e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x} x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}x, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2e^{2x}, e^{2x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2e^{2x} + A_2e^{2x}x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1e^{2x} + 2A_1x\,e^{2x} + A_2e^{2x} = e^{2x}x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2e^{2x}}{2} - e^{2x}x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{x} + c_2e^{2x}\right) + \left(\frac{x^2e^{2x}}{2} - e^{2x}x\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1e^{x} + c_2e^{2x} + \frac{x^2e^{2x}}{2} - e^{2x}x \tag{1}$$

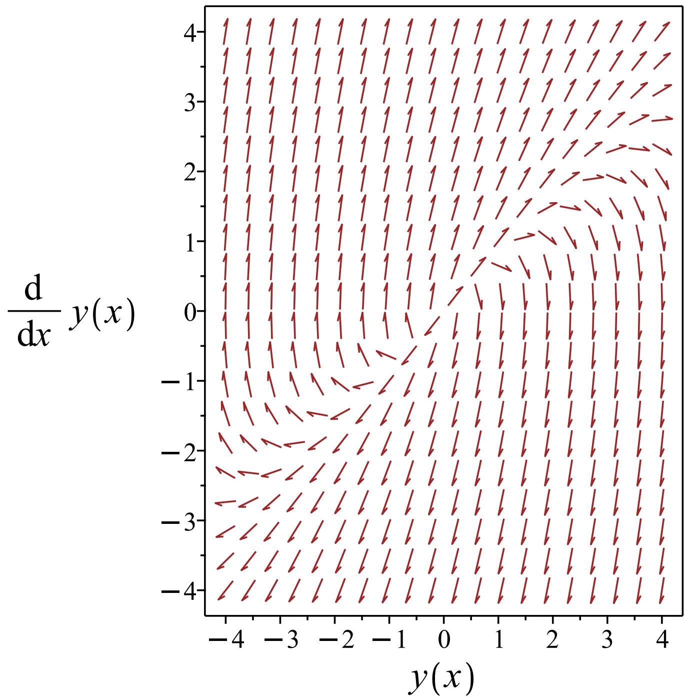


Figure 42: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^x + c_2 e^{2x} + \frac{x^2 e^{2x}}{2} - e^{2x} x$$

Verified OK.

2.5.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = e^{2x} x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r - 2) = 0$

- Roots of the characteristic polynomial

$r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^x + c_2 e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{2x} x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2\,e^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{3x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -e^x\left(\int x\, e^x dx\right) + e^{2x}\left(\int x dx\right)$

  - Compute integrals

    $y_p(x) = e^{2x}\left(1 + \frac{1}{2}x^2 - x\right)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^x + c_2 e^{2x} + e^{2x}\left(1 + \frac{1}{2}x^2 - x\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=x*exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{((x^2 + 2c_1 - 2x + 2)\,\mathrm{e}^x + 2c_2)\,\mathrm{e}^x}{2}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 33

```
DSolve[y''[x]-3*y'[x]+2*y[x]==x*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^x\left(e^x\left(x^2 - 2x + 2 + 2c_2\right) + 2c_1\right)$$

2.6 problem Example 3.23

Internal problem ID [5854]
Internal file name [`OUTPUT/5102_Sunday_June_05_2022_03_24_41_PM_35209794/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.3 SECOND ORDER ODE. Page 147
Problem number: Example 3.23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = 4\sin(x)}$$

2.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 4\sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = \mathrm{e}^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4 \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x) x, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x) x + A_2 \sin(x) x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = 4 \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2 \cos(x) x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x) c_1 + c_2 \sin(x)) + (-2 \cos(x) x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) - 2\cos(x) x \tag{1}$$

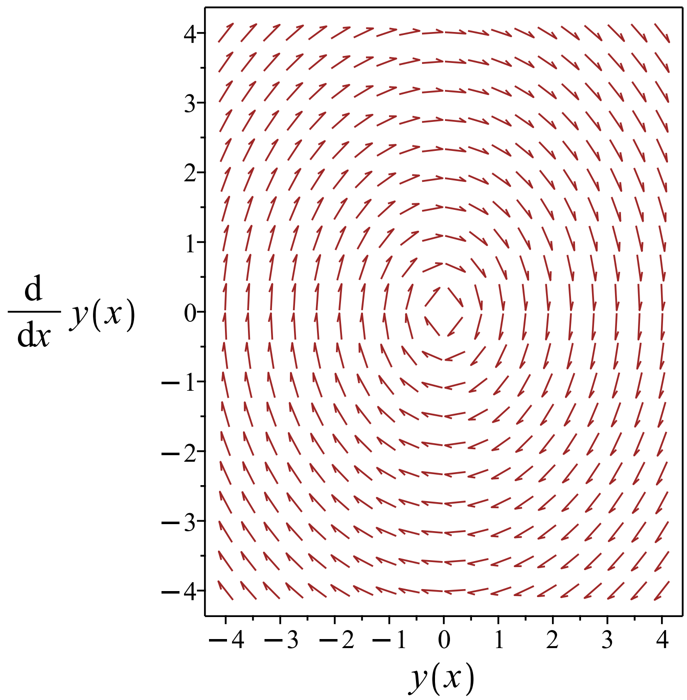


Figure 43: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - 2\cos(x) x$$

Verified OK.

2.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 30: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (\cos(x)) + c_2 (\cos(x) (\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4 \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = 4 \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2\cos(x)\, x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\, c_1 + c_2 \sin(x)) + (-2\cos(x)\, x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x)\, c_1 + c_2 \sin(x) - 2\cos(x)\, x \tag{1}$$

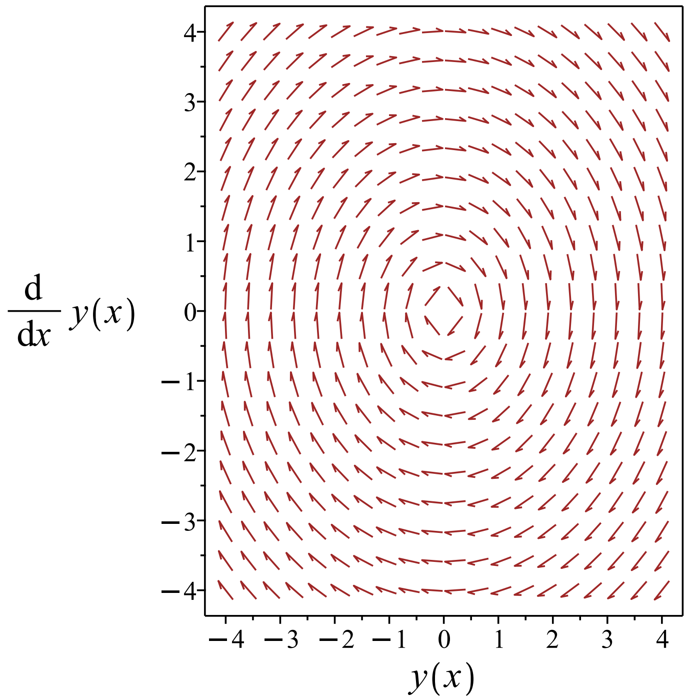


Figure 44: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - 2\cos(x) x$$

Verified OK.

2.6.3 Maple step by step solution

Let's solve

$y'' + y = 4\sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \cos(x)\, c_1 + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4\sin(x)\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$
- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -4\cos(x)\left(\int \sin(x)^2\, dx\right) + 2\sin(x)\left(\int \sin(2x)\, dx\right)$
- ○ Compute integrals

  $y_p(x) = \sin(x) - 2\cos(x)\, x$

- Substitute particular solution into general solution to ODE

  $y = \cos(x)\, c_1 + c_2 \sin(x) + \sin(x) - 2\cos(x)\, x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+y(x)=4*sin(x),y(x), singsol=all)
```

$$y(x) = (c_1 - 2x)\cos(x) + \sin(x)(c_2 + 2)$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 20

```
DSolve[y''[x]+y[x]==4*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-2x + c_1)\cos(x) + c_2\sin(x)$$

2.7 problem Example 3.24

Internal problem ID [5855]
Internal file name [`OUTPUT/5103_Sunday_June_05_2022_03_24_43_PM_39382539/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.3 SECOND ORDER ODE. Page 147
Problem number: Example 3.24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2x^2y' + \left(x^4 + 2x - 1\right) y = 0}$$

2.7.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= 2x^2 \\ q(x) &= x^4 + 2x - 1 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= x^4 + 2x - 1 - \frac{(2x^2)'}{2} - \frac{(2x^2)^2}{4} \\
&= x^4 + 2x - 1 - \frac{(4x)}{2} - \frac{(4x^4)}{4} \\
&= x^4 + 2x - 1 - (2x) - x^4 \\
&= -1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{2x^2}{2}} \\
&= \mathrm{e}^{-\frac{x^3}{3}}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$y = v(x)\, \mathrm{e}^{-\frac{x^3}{3}} \tag{4}$$

Applying this change of variable to the original ode results in

$$\mathrm{e}^{-\frac{x^3}{3}} (v''(x) - v(x)) = 0$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ v(x) &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$v(x) = c_1 e^{x} + c_2 e^{-x}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x)\\ &= \left(c_1 e^{x} + c_2 e^{-x}\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = e^{-\frac{x^3}{3}}$$

Hence (7) becomes

$$y = \left(c_1 e^{x} + c_2 e^{-x}\right) e^{-\frac{x^3}{3}}$$

Summary

The solution(s) found are the following

$$y = (c_1 e^{x} + c_2 e^{-x}) e^{-\frac{x^3}{3}} \tag{1}$$

Verification of solutions

$$y = (c_1 e^{x} + c_2 e^{-x}) e^{-\frac{x^3}{3}}$$

Verified OK.

2.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2x^2 y' + (x^4 + 2x - 1) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x^2 \\ C &= x^4 + 2x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 32: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2}{1}\,dx} \\
&= z_1 e^{-\frac{x^3}{3}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^3}{3}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2}{1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{2x^3}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}}\right) + c_2 \left(\mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}} + \frac{c_2 \mathrm{e}^{-\frac{x\left(x^2-3\right)}{3}}}{2} \tag{1}
$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}} + \frac{c_2 \mathrm{e}^{-\frac{x\left(x^2-3\right)}{3}}}{2}$$

Verified OK.

2.7.3 Maple step by step solution

Let's solve

$y'' + 2x^2y' + (x^4 + 2x - 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..4$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  - Shift index using $k- >k - 1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k + 2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 - a_0 + (6a_3 - a_1 + 2a_0)\,x + (12a_4 - a_2 + 4a_1)\,x^2 + (20a_5 - a_3 + 6a_2)\,x^3 + \left(\sum_{k=4}^{\infty} (a_{k+2}(k+2)\right.$$

- The coefficients of each power of x must be 0

  $[2a_2 - a_0 = 0, 6a_3 - a_1 + 2a_0 = 0, 12a_4 - a_2 + 4a_1 = 0, 20a_5 - a_3 + 6a_2 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6} - \frac{a_0}{3}, a_4 = \frac{a_0}{24} - \frac{a_1}{3}, a_5 = \frac{a_1}{120} - \frac{a_0}{6}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\,a_{k+2} + 2a_{k-1}k - a_k + a_{k-4} = 0$

- Shift index using $k- >k + 4$

  $\left((k+4)^2 + 3k + 14\right) a_{k+6} + 2a_{k+3}(k+4) - a_{k+4} + a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{2ka_{k+3} + a_k + 8a_{k+3} - a_{k+4}}{k^2 + 11k + 30}, a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6} - \frac{a_0}{3}, a_4 = \frac{a_0}{24} - \frac{a_1}{3}, a_5 = \frac{a_1}{120} - \frac{a_0}{6}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+2*x^2*diff(y(x),x)+(x^4+2*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{x(x^2-3)}{3}} + c_2 e^{-\frac{x(x^2+3)}{3}}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 34

```
DSolve[y''[x]+2*x^2*y'[x]+(x^4+2*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-\frac{1}{3}x(x^2+3)} \left(c_2 e^{2x} + 2c_1\right)$$

2.8 problem Example 3.26

Internal problem ID [5856]
Internal file name [`OUTPUT/5104_Sunday_June_05_2022_03_24_44_PM_61124771/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.3 SECOND ORDER ODE. Page 147
Problem number: Example 3.26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{p\,x^2u'' + qxu' + ru = f(x)}$$

2.8.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Au''(x) + Bu'(x) + Cu(x) = f(x)$$

Where $A = x^2p, B = qx, C = r, f(x) = f(x)$. Let the solution be

$$u = u_h + u_p$$

Where u_h is the solution to the homogeneous ODE $Au''(x)+Bu'(x)+Cu(x) = 0$, and u_p is a particular solution to the non-homogeneous ODE $Au''(x)+Bu'(x)+Cu(x) = f(x)$. Solving for u_h from

$$p\,x^2u'' + qxu' + ru = 0$$

This is Euler second order ODE. Let the solution be $u = x^r$, then $u' = rx^{r-1}$ and $u'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2p(r(r-1))x^{r-2} + qxrx^{r-1} + r\,x^r = 0$$

Simplifying gives

$$pr(r-1)\,x^r + qr\,x^r + r\,x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$pr(r-1) + qr + r = 0$$

Or

$$p\,r^2 + (-p+q)\,r + r = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{p - q + \sqrt{p^2 - 2qp - 4pr + q^2}}{2p}$$
$$r_2 = -\frac{-p + q + \sqrt{p^2 - 2qp - 4pr + q^2}}{2p}$$

Since the roots are real and distinct, then the general solution is

$$u = c_1u_1 + c_2u_2$$

Where $u_1 = x^{r_1}$ and $u_2 = x^{r_2}$. Hence

$$u = c_1x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} + c_2x^{-\frac{-p+q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}$$

Next, we find the particular solution to the ODE

$$p\,x^2u'' + qxu' + ru = f(x)$$

The particular solution u_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$u_p(x) = u_1u_1 + u_2u_2 \tag{1}$$

Where u_1, u_2 to be determined, and u_1, u_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$u_1 = x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}$$

$$u_2 = x^{-\frac{-p+q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{u_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{u_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of u'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} & x^{-\frac{-p+q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} \\ \frac{d}{dx}\left(x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}\right) & \frac{d}{dx}\left(x^{-\frac{-p+q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} & x^{-\frac{-p+q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} \\ \frac{x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}\left(p-q+\sqrt{p^2-2qp-4pr+q^2}\right)}{2px} & -\frac{x^{-\frac{-p+q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}\left(-p+q+\sqrt{p^2-2qp-4pr+q^2}\right)}{2px} \end{vmatrix}$$

Therefore

$$W = \left(x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}\right)\left(-\frac{x^{-\frac{-p+q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}\left(-p+q+\sqrt{p^2-2qp-4pr+q^2}\right)}{2px}\right) - \left(x^{-\frac{-p+q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}\right)\left(\frac{x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}\left(p-q+\sqrt{p^2-2qp-4pr+q^2}\right)}{2px}\right)$$

Which simplifies to

$$W=-\frac{x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}x^{-\frac{-p+q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}\sqrt{p^2-2qp-4pr+q^2}}{px}$$

Which simplifies to

$$W=-\frac{x^{-\frac{q}{p}}\sqrt{p^2+(-2q-4r)\,p+q^2}}{p}$$

Therefore Eq. (2) becomes

$$u_1=-\int\frac{x^{-\frac{-p+q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}f(x)}{-x^2x^{-\frac{q}{p}}\sqrt{p^2+(-2q-4r)\,p+q^2}}\,dx$$

Which simplifies to

$$u_1=-\int-\frac{x^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}f(x)}{\sqrt{p^2+(-2q-4r)\,p+q^2}}dx$$

Hence

$$u_1=-\left(\int_0^x-\frac{\alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}f(\alpha)}{\sqrt{p^2+(-2q-4r)\,p+q^2}}d\alpha\right)$$

And Eq. (3) becomes

$$u_2=\int\frac{x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}f(x)}{-x^2x^{-\frac{q}{p}}\sqrt{p^2+(-2q-4r)\,p+q^2}}\,dx$$

Which simplifies to

$$u_2=\int-\frac{x^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}f(x)}{\sqrt{p^2+(-2q-4r)\,p+q^2}}dx$$

Hence

$$u_2=\int_0^x-\frac{\alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}f(\alpha)}{\sqrt{p^2+(-2q-4r)\,p+q^2}}d\alpha$$

Which simplifies to

$$u_1 = \frac{\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha}{\sqrt{p^2+(-2q-4r)\,p+q^2}}$$

$$u_2 = -\frac{\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha}{\sqrt{p^2+(-2q-4r)\,p+q^2}}$$

Therefore the particular solution, from equation (1) is

$$u_p(x) = \frac{\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}} - \frac{\left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{-\frac{-p+q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}}$$

Which simplifies to

$$u_p(x) = \frac{\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{\frac{p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} - \left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{-\frac{-p+q}{}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}}$$

Summary

The solution(s) found are the following

$$u = \frac{\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{\frac{p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} - \left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{-\frac{-p+q}{}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}} + c_1 x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} + c_2 x^{-\frac{-p+q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} \tag{1}$$

Verification of solutions

$$u = \frac{\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{\frac{p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} - \left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{-\frac{-p+q}{}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}} + c_1 x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} + c_2 x^{-\frac{-p+q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}$$

Verified OK.

2.8.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$u = u_h + u_p$$

Where u_h is the solution to the homogeneous ODE $Au''(x)+Bu'(x)+Cu(x) = 0$, and u_p is a particular solution to the non-homogeneous ODE $Au''(x)+Bu'(x)+Cu(x) = f(x)$. u_h is the solution to

$$p\,x^2u'' + qxu' + ru = 0$$

In normal form the ode

$$p\,x^2u'' + qxu' + ru = 0 \tag{1}$$

Becomes

$$u'' + p(x)\,u' + q(x)\,u = 0 \tag{2}$$

Where

$$p(x) = \frac{q}{xp}$$
$$q(x) = \frac{r}{p\,x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}u(\tau) + p_1\left(\frac{d}{d\tau}u(\tau)\right) + q_1u(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int \frac{q}{xp} dx\right)} dx \\
&= \int e^{-\frac{q\ln(x)}{p}}\, dx \\
&= \int x^{-\frac{q}{p}} dx \\
&= \frac{p\,x^{1-\frac{q}{p}}}{p-q} \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{r}{p\,x^2}}{x^{-\frac{2q}{p}}} \\
&= \frac{r\,x^{\frac{-2p+2q}{p}}}{p} \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}u(\tau) + q_1 u(\tau) = 0$$

$$\frac{d^2}{d\tau^2}u(\tau) + \frac{r\,x^{\frac{-2p+2q}{p}}\,u(\tau)}{p} = 0$$

But in terms of τ

$$\frac{r\,x^{\frac{-2p+2q}{p}}}{p} = \frac{pr}{(p-q)^2\,\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}u(\tau) + \frac{pru(\tau)}{(p-q)^2\,\tau^2} = 0$$

The above ode is now solved for $u(\tau)$. The ode can be written as

$$\left(\frac{d^{2}}{d\tau^{2}}u(\tau)\right)(p-q)^{2}\tau^{2}+pru(\tau)=0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $u(\tau)=\tau^{r}$, then $u'=r\tau^{r-1}$ and $u''=r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$(p-q)^{2}\tau^{2}(r(r-1))\tau^{r-2}+0r\tau^{r-1}+pr\,\tau^{r}=0$$

Simplifying gives

$$(p-q)^{2}r(r-1)\,\tau^{r}+0\,\tau^{r}+pr\,\tau^{r}=0$$

Since $\tau^{r}\neq 0$ then dividing throughout by τ^{r} gives

$$(p-q)^{2}r(r-1)+0+pr=0$$

Or

$$\left(p^{2}-2qp+q^{2}\right)r^{2}+\left(-p^{2}+2qp-q^{2}\right)r+pr=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_{1}=-\frac{-p+q+\sqrt{p^{2}-2qp-4pr+q^{2}}}{2(p-q)}$$
$$r_{2}=\frac{p-q+\sqrt{p^{2}-2qp-4pr+q^{2}}}{2p-2q}$$

Since the roots are real and distinct, then the general solution is

$$u(\tau)=c_{1}u_{1}+c_{2}u_{2}$$

Where $u_{1}=\tau^{r_{1}}$ and $u_{2}=\tau^{r_{2}}$. Hence

$$u(\tau)=c_{1}\tau^{-\frac{-p+q+\sqrt{p^{2}-2qp-4pr+q^{2}}}{2(p-q)}}+c_{2}\tau^{\frac{p-q+\sqrt{p^{2}-2qp-4pr+q^{2}}}{2p-2q}}$$

The above solution is now transformed back to u using (6) which results in

$$u=c_{2}\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q+\sqrt{p^{2}+(-2q-4r)p+q^{2}}}{2p-2q}}+c_{1}\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^{2}+(-2q-4r)p+q^{2}}}{2p-2q}}$$

Therefore the homogeneous solution u_h is

$$u_{h}=c_{2}\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q+\sqrt{p^{2}+(-2q-4r)p+q^{2}}}{2p-2q}}+c_{1}\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^{2}+(-2q-4r)p+q^{2}}}{2p-2q}}$$

The particular solution u_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$u_p(x) = u_1 u_1 + u_2 u_2 \tag{1}$$

Where u_1, u_2 to be determined, and u_1, u_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$u_1 = \left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}} \left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}} \left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p-2q}}$$

$$u_2 = \left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}} \left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}} \left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p-2q}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{u_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{u_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of u'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$. Hence

$$W = \left|\begin{array}{cc} \left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}} \left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}} \left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p-2q}} & \left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}} \left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}} \left(\cdots \right. \\ \frac{d}{dx}\left(\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}} \left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}} \left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p-2q}}\right) & \frac{d}{dx}\left(\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}} \left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}} \left(\cdots \right.\right. \end{array}\right.$$

Which gives

$$W = \begin{vmatrix} & \left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{}{2}} \\ \frac{\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}}\left(\frac{p\,x^{-\frac{q}{p}}}{p-q}-\frac{x^{-\frac{q}{p}}q}{p-q}\right)x^{\frac{q}{p}}(p-q)\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p-2q}}}{(2p-2q)x} - \frac{\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-}}}{} & \end{vmatrix}$$

Therefore

W

$$= \left(\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p-2q}}\right)\left(\frac{\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}}\left(\frac{p\,x^{-\frac{q}{p}}}{p-q}-\frac{x^{-\frac{q}{p}}q}{p-q}\right)x^{\frac{q}{p}}(p-}{(2p}\right.$$

$$-\frac{\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}}\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p-2q}}q\left(\frac{p\,x^{-\frac{q}{p}}}{p-q}-\frac{x^{-\frac{q}{p}}q}{p-q}\right)x^{\frac{q}{p}}(p-q)}{(2p-2q)\,px}$$

$$+\frac{\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}}\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p-2q}}\sqrt{p^2-2qp-4pr+q^2}\left(\frac{p\,x^{-\frac{q}{p}}}{p-q}-\frac{x^{-\frac{q}{p}}q}{p-q}\right)x^{\frac{q}{p}}(p-}{(2p-2q)\,px}$$

$$-\left(\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}}\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p-2q}}\right)\left(\frac{\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}}\left(\frac{p\,x^{-\frac{q}{p}}}{p-q}-\frac{x^{-\frac{q}{p}}q}{p-q}\right)x^{\frac{q}{p}}(p}{(2}\right.$$

$$-\frac{\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p-2q}}q\left(\frac{p\,x^{-\frac{q}{p}}}{p-q}-\frac{x^{-\frac{q}{p}}q}{p-q}\right)x^{\frac{q}{p}}(p-q)}{(2p-2q)\,px}$$

$$-\frac{\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p-2q}}\sqrt{p^2-2qp-4pr+q^2}\left(\frac{p\,x^{-\frac{q}{p}}}{p-q}-\frac{x^{-\frac{q}{p}}q}{p-q}\right)x^{\frac{q}{p}}(p-}{(2p-2q)\,px}$$

Which simplifies to

$$W = \frac{\sqrt{p^2-2qp-4pr+q^2}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{\frac{p}{p-q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{p-q}}}{xp}$$

Which simplifies to

$$W = \frac{x^{-\frac{q}{p}}\sqrt{p^2+(-2q-4r)\,p+q^2}}{p-q}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}}\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p-2q}} f(x)}{\frac{x^2p\,x^{-\frac{q}{p}}\sqrt{p^2+(-2q-4r)p+q^2}}{p-q}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{\frac{-2p+q}{p}} f(x)\,(p-q)\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}}}{p\sqrt{p^2+(-2q-4r)\,p+q^2}}dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{\alpha^{\frac{-2p+q}{p}} f(\alpha)\,(p-q)\left(\frac{p\,\alpha^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}}}{p\sqrt{p^2+(-2q-4r)\,p+q^2}}d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}}\left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p-2q}} f(x)}{\frac{x^2p\,x^{-\frac{q}{p}}\sqrt{p^2+(-2q-4r)p+q^2}}{p-q}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x^{\frac{-2p+q}{p}} f(x)\,(p-q) \left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}}}{p\sqrt{p^2+(-2q-4r)\,p+q^2}}dx$$

Hence

$$u_2 = \int_0^x \frac{\alpha^{\frac{-2p+q}{p}} f(\alpha)\,(p-q) \left(\frac{p\,\alpha^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}}}{p\sqrt{p^2+(-2q-4r)\,p+q^2}}d\alpha$$

Which simplifies to

$$u_1 = -\frac{(p-q)\left(\int_0^x \alpha^{\frac{-2p+q}{p}} f(\alpha) \left(\frac{p\,\alpha^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}} d\alpha\right)}{\sqrt{p^2+(-2q-4r)\,p+q^2}\,p}$$

$$u_2 = \frac{(p-q)\left(\int_0^x \alpha^{\frac{-2p+q}{p}} f(\alpha) \left(\frac{p\,\alpha^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}} d\alpha\right)}{\sqrt{p^2+(-2q-4r)\,p+q^2}\,p}$$

Therefore the particular solution, from equation (1) is

$$u_p(x) =$$

$$-\frac{(p-q)\left(\int_0^x \alpha^{\frac{-2p+q}{p}} f(\alpha) \left(\frac{p\,\alpha^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}} d\alpha\right) \left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}} \left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}} \left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{\sqrt{\cdots}}{\cdots}} \cdots}{\sqrt{p^2+(-2q-4r)\,p+q^2}\,p}$$

$$+\frac{(p-q)\left(\int_0^x \alpha^{\frac{-2p+q}{p}} f(\alpha) \left(\frac{p\,\alpha^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}} d\alpha\right) \left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{p}{2p-2q}} \left(\frac{p\,x^{1-\frac{q}{p}}}{p-q}\right)^{-\frac{q}{2p-2q}} \left(\frac{px\,x^{-\frac{q}{p}}}{p-q}\right)^{\frac{\sqrt{p^2 \cdots}}{\cdots}} \cdots}{\sqrt{p^2+(-2q-4r)\,p+q^2}\,p}$$

Which simplifies to

$$
u_p(x) = \frac{(p-q)\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}}\left(\left(\int_0^x \alpha^{\frac{-2p+q}{p}} f(\alpha)\left(\frac{p\,\alpha^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}} d\alpha\right)\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\sqrt{p^2+(-\cdots}}\cdots}{\sqrt{p^2+(-2q-4r)\,p+q^2}\,p\cdots}
$$

Therefore the general solution is

$$
\begin{aligned}
u &= u_h + u_p \\
&= \left(c_2\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}} + c_1\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}}\right) \\
&\quad + \left(\frac{(p-q)\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}}\left(\left(\int_0^x \alpha^{\frac{-2p+q}{p}} f(\alpha)\left(\frac{p\,\alpha^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}} d\alpha\right)\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)\cdots}{\sqrt{p^2+(-2q-4r)\,p+q^2}\,p\cdots}\right.
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
u &= c_2\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}} + c_1\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}} \\
&\quad + \frac{(p-q)\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}}\left(\left(\int_0^x \alpha^{\frac{-2p+q}{p}} f(\alpha)\left(\frac{p\,\alpha^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}} d\alpha\right)\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\sqrt{p^2\cdots}}\cdots}{\sqrt{p^2+(-2q-4r)\,p+q^2}\,p\cdots}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$u = c_2 \left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}} + c_1 \left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}}$$
$$+ \frac{(p-q)\left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}} \left(\left(\int_0^x \alpha^{\frac{-2p+q}{p}} f(\alpha) \left(\frac{p\,\alpha^{\frac{p-q}{p}}}{p-q}\right)^{\frac{p-q-\sqrt{p^2+(-2q-4r)p+q^2}}{2p-2q}} d\alpha\right) \left(\frac{p\,x^{\frac{p-q}{p}}}{p-q}\right)^{\sqrt{p^2 \cdots}} \cdots\right)}{\sqrt{p^2+(-2q-4r)\,p+q^2}\,p \cdots}$$

Verified OK.

2.8.3 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Au''(x) + Bu'(x) + Cu(x) = f(x)$$

Where $A = x^2p, B = qx, C = r, f(x) = f(x)$. Let the solution be

$$u = u_h + u_p$$

Where u_h is the solution to the homogeneous ODE $Au''(x)+Bu'(x)+Cu(x) = 0$, and u_p is a particular solution to the non-homogeneous ODE $Au''(x)+Bu'(x)+Cu(x) = f(x)$. Solving for u_h from

$$p\,x^2u'' + qxu' + ru = 0$$

In normal form the ode

$$p\,x^2u'' + qxu' + ru = 0 \tag{1}$$

Becomes

$$u'' + p(x)\,u' + q(x)\,u = 0 \tag{2}$$

Where

$$p(x) = \frac{q}{xp}$$
$$q(x) = \frac{r}{p\,x^2}$$

Applying change of variables on the depndent variable $u = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not u.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{nq}{x^2 p} + \frac{r}{p\,x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = \frac{p - q + \sqrt{p^2 - 2qp - 4pr + q^2}}{2p} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{p - q + \sqrt{p^2 - 2qp - 4pr + q^2}}{px} + \frac{q}{xp}\right) v'(x) &= 0 \\ v''(x) + \frac{\left(p + \sqrt{p^2 + (-2q - 4r)\,p + q^2}\right) v'(x)}{px} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{\left(p + \sqrt{p^2 + (-2q - 4r)\,p + q^2}\right) u(x)}{px} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{\left(p + \sqrt{p^2 + (-2q - 4r)\,p + q^2}\right) u}{px} \end{aligned}$$

Where $f(x) = -\frac{p+\sqrt{p^2+(-2q-4r)p+q^2}}{px}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{p+\sqrt{p^2+(-2q-4r)\,p+q^2}}{px}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{p+\sqrt{p^2+(-2q-4r)\,p+q^2}}{px}\,dx \\
\ln(u) &= -\ln(x) - \frac{\sqrt{p^2-2qp-4pr+q^2}\,\ln(x)}{p} + c_1 \\
u &= \mathrm{e}^{-\ln(x)-\frac{\sqrt{p^2-2qp-4pr+q^2}\,\ln(x)}{p}+c_1} \\
&= c_1 \mathrm{e}^{-\ln(x)-\frac{\sqrt{p^2-2qp-4pr+q^2}\,\ln(x)}{p}}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_1 x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}}{x}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1 p\, x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}}{\sqrt{p^2-2qp-4pr+q^2}} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
u &= v(x)\,x^n \\
&= \left(-\frac{c_1 p\, x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}}{\sqrt{p^2-2qp-4pr+q^2}} + c_2\right) x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} \\
&= \frac{x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} \left(c_2 \sqrt{p^2+(-2q-4r)\,p+q^2}\, x^{\frac{\sqrt{p^2+(-2q-4r)p+q^2}}{p}} - c_1 p\right)}{\sqrt{p^2+(-2q-4r)\,p+q^2}}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
p\,x^2 u'' + qxu' + ru = f(x)
$$

The particular solution u_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$u_p(x) = u_1 u_1 + u_2 u_2 \tag{1}$$

Where u_1, u_2 to be determined, and u_1, u_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$u_1 = \sqrt{x}\, x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}}$$

$$u_2 = \sqrt{x}\, x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}} x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{u_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{u_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of u'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x}\, x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}} & \sqrt{x}\, x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}} x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}} \\ \frac{d}{dx}\left(\sqrt{x}\, x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}}\right) & \frac{d}{dx}\left(\sqrt{x}\, x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}} x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x}\, x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}} & \\ \frac{x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}}}{2\sqrt{x}} - \frac{x^{-\frac{q}{2p}} q\, x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}}}{2\sqrt{x}\, p} + \frac{x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}} \sqrt{p^2-2qp-4pr+q^2}}{2\sqrt{x}\, p} & \frac{x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}} x^{-}}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x}\, x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}}\right) \left(\frac{x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}} x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}}{2\sqrt{x}} - \frac{x^{-\frac{q}{2p}} q\, x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}} x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}}{2\sqrt{x}\, p} - \frac{x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}} \sqrt{p^2-2qp-4pr+q^2}\, x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}}{2\sqrt{x}\, p}\right) - \left(\sqrt{x}\, x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}} x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}\right) \left(\frac{x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}}}{2\sqrt{x}} - \frac{x^{-\frac{q}{2p}} q\, x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}}}{2\sqrt{x}\, p} + \frac{x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}} \sqrt{p^2-2qp-4pr+q^2}}{2\sqrt{x}\, p}\right)$$

Which simplifies to

$$W = -\frac{x^{-\frac{q}{p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}} \sqrt{p^2-2qp-4pr+q^2}\, x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}}{p}$$

Which simplifies to

$$W = -\frac{x^{-\frac{q}{p}} \sqrt{p^2+(-2q-4r)\,p+q^2}}{p}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sqrt{x}\, x^{-\frac{q}{2p}} x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}} x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}} f(x)}{-x^2 x^{-\frac{q}{p}} \sqrt{p^2+(-2q-4r)\,p+q^2}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{x^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(x)}{\sqrt{p^2+(-2q-4r)\,p+q^2}} dx$$

Hence

$$u_1 = -\left(\int_0^x -\frac{\alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)}{\sqrt{p^2+(-2q-4r)\,p+q^2}}d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x}\,x^{-\frac{q}{2p}}x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}} f(x)}{-x^2 x^{-\frac{q}{p}}\sqrt{p^2+(-2q-4r)\,p+q^2}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{x^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(x)}{\sqrt{p^2+(-2q-4r)\,p+q^2}}dx$$

Hence

$$u_2 = \int_0^x -\frac{\alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)}{\sqrt{p^2+(-2q-4r)\,p+q^2}}d\alpha$$

Which simplifies to

$$u_1 = \frac{\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\,d\alpha}{\sqrt{p^2+(-2q-4r)\,p+q^2}}$$

$$u_2 = -\frac{\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\,d\alpha}{\sqrt{p^2+(-2q-4r)\,p+q^2}}$$

Therefore the particular solution, from equation (1) is

$$u_p(x) = \frac{\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\,d\alpha\right)\sqrt{x}\,x^{-\frac{q}{2p}}x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}} - \frac{\left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\,d\alpha\right)\sqrt{x}\,x^{-\frac{q}{2p}}x^{\frac{\sqrt{p^2-2qp-4pr+q^2}}{2p}}x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}}$$

Which simplifies to

$$u_p(x) = \frac{x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} \left(\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha \right) x^{\frac{\sqrt{p^2+(-2q-4r)p+q^2}}{p}} - \left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p}}{2p}} \right.\right.}{\sqrt{p^2+(-2q-4r)\,p+q^2}}$$

Therefore the general solution is

$$u = u_h + u_p$$

$$= \left(\left(-\frac{c_1 p\, x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}}{\sqrt{p^2-2qp-4pr+q^2}} + c_2 \right) x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} \right)$$

$$+ \left(\frac{x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} \left(\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha \right) x^{\frac{\sqrt{p^2+(-2q-4r)p+q^2}}{p}} - \left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-}}{2p}} \right.\right.}{\sqrt{p^2+(-2q-4r)\,p+q^2}} \right.$$

$$= \frac{x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} \left(\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha \right) x^{\frac{\sqrt{p^2+(-2q-4r)p+q^2}}{p}} - \left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r}}{2p}} \right.\right.}{\sqrt{p^2+(-2q-4r)\,p+q^2}}$$

$$+ \left(-\frac{c_1 p\, x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}}{\sqrt{p^2-2qp-4pr+q^2}} + c_2 \right) x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}$$

Summary

The solution(s) found are the following

$$u = \frac{x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} \left(\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha \right) x^{\frac{\sqrt{p^2+(-2q-4r)p+q^2}}{p}} - \left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p}}{2p}} \right.\right.}{\sqrt{p^2+(-2q-4r)\,p+q^2}} \tag{1}$$

$$+ \left(-\frac{c_1 p\, x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}}{\sqrt{p^2-2qp-4pr+q^2}} + c_2 \right) x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}$$

Verification of solutions

$$
\begin{aligned}
u &= \frac{x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}\left(\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\,d\alpha\right) x^{\frac{\sqrt{p^2+(-2q-4r)p+q^2}}{p}} - \left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p}}{2p}}\right.\right.}{\sqrt{p^2+(-2q-4r)\,p+q^2}} \\
&\quad + \left(-\frac{c_1 p\, x^{-\frac{\sqrt{p^2-2qp-4pr+q^2}}{p}}}{\sqrt{p^2-2qp-4pr+q^2}} + c_2\right) x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}
\end{aligned}
$$

Verified OK.

2.8.4 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
p\,x^2u'' + qxu' + ru &= 0 \tag{1}\\
Au'' + Bu' + Cu &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2p \\
B &= qx \qquad (3)\\
C &= r
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ue^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \qquad (5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-2qp - 4pr + q^2}{4x^2p^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -2qp - 4pr + q^2 \\
t &= 4x^2p^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-2qp - 4pr + q^2}{4x^2p^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$
u = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 34: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2p^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2qp + 4pr - q^2}{4x^2p^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{(-2q-4r)p+q^2}{4p^2}$. Hence

$$\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\
&= \left\{2, 2 - 2\sqrt{1 + \frac{(-2q-4r)\,p + q^2}{p^2}}, 2 + 2\sqrt{1 + \frac{(-2q-4r)\,p + q^2}{p^2}}\right\}
\end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-2qp - 4pr + q^2}{4x^2p^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{2}$. Hence

$$\begin{aligned}
E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\
&= \{2\}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\left\{2, 2 - 2\sqrt{1 + \frac{(-2q-4r)p+q^2}{p^2}}, 2 + 2\sqrt{1 + \frac{(-2q-4r)p+q^2}{p^2}}\right\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 2, \; e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (2)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{2}{(x-(0))}\right) \\ &= \frac{1}{x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{x} + \frac{(2q + 4r)\,p - q^2}{4x^2p^2} = 0$$

Solving for ω gives

$$\omega = \frac{p + \sqrt{p^2 - 2qp - 4pr + q^2}}{2px}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \frac{p+\sqrt{p^2-2qp-4pr+q^2}}{2px} dx} \\
&= x^{\frac{p+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}
\end{aligned}$$

The first solution to the original ode in u is found from

$$\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{qx}{x^2p}\, dx} \\
&= z_1 e^{-\frac{q\ln(x)}{2p}} \\
&= z_1 \left(x^{-\frac{q}{2p}}\right)
\end{aligned}$$

Which simplifies to

$$u_1 = x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{qx}{x^2 p} dx}}{(u_1)^2} dx \\
&= u_1 \int \frac{e^{-\frac{q \ln(x)}{p}}}{(u_1)^2} dx \\
&= u_1 \left(-\frac{p\, x^{-\frac{\sqrt{p^2+(-2q-4r)p+q^2}}{p}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} \right) + c_2 \left(x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} \left(-\frac{p\, x^{-\frac{\sqrt{p^2+(-2q-4r)p+q^2}}{p}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}} \right) \right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
u = u_h + u_p
$$

Where u_h is the solution to the homogeneous ODE $Au''(x)+Bu'(x)+Cu(x)=0$, and u_p is a particular solution to the nonhomogeneous ODE $Au''(x)+Bu'(x)+Cu(x)=f(x)$. u_h is the solution to

$$
p\,x^2 u'' + qxu' + ru = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
u_h = c_1 x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} - \frac{c_2 p\, x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}}
$$

The particular solution u_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
u_p(x) = u_1 u_1 + u_2 u_2 \tag{1}
$$

Where u_1, u_2 to be determined, and u_1, u_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$u_1 = x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}$$

$$u_2 = -\frac{p\,x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{u_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{u_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of u'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} & -\frac{p\,x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}}{\sqrt{p^2+(-2q-4r)p+q^2}} \\ \frac{d}{dx}\left(x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}\right) & \frac{d}{dx}\left(-\frac{p\,x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}}{\sqrt{p^2+(-2q-4r)p+q^2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} & -\frac{p\,x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}}{\sqrt{p^2+(-2q-4r)p+q^2}} \\ \frac{x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}\left(p-q+\sqrt{p^2-2qp-4pr+q^2}\right)}{2px} & \frac{x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}\left(-p+q+\sqrt{p^2+(-2q-4r)p+q^2}\right)}{2x\sqrt{p^2+(-2q-4r)p+q^2}} \end{vmatrix}$$

Therefore

$$W = \left(x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}\right) \left(\frac{x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} \left(-p+q+\sqrt{p^2+(-2q-4r)\,p+q^2}\right)}{2x\sqrt{p^2+(-2q-4r)\,p+q^2}}\right) - \left(-\frac{p\,x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}}\right) \left(\frac{x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} \left(p-q+\sqrt{p^2-2qp-4pr+q^2}\right)}{2px}\right)$$

Which simplifies to

$$W = \frac{x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} x^{-\frac{-p+q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}}{x}$$

Which simplifies to

$$W = x^{-\frac{q}{p}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{p\,x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(x)}{\sqrt{p^2+(-2q-4r)p+q^2}}}{x^2 p\,x^{-\frac{q}{p}}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{x^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(x)}{\sqrt{p^2+(-2q-4r)\,p+q^2}}\,dx$$

Hence

$$u_1 = -\left(\int_0^x -\frac{\alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)}{\sqrt{p^2+(-2q-4r)\,p+q^2}}\,d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} f(x)}{x^2 p\,x^{-\frac{q}{p}}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(x)}{p} dx$$

Hence

$$u_2 = \int_0^x \frac{\alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)}{p} d\alpha$$

Which simplifies to

$$u_1 = \frac{\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha}{\sqrt{p^2+(-2q-4r)\,p+q^2}}$$

$$u_2 = \frac{\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha}{p}$$

Therefore the particular solution, from equation (1) is

$$u_p(x) = \frac{\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}} - \frac{\left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}}$$

Which simplifies to

$$u_p(x) = \frac{\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{\frac{p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} - \left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{-\frac{-p+q}{}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}}$$

Therefore the general solution is

$$
\begin{aligned}
u &= u_h + u_p \\
&= \left(c_1 x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} - \frac{c_2 p\, x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}}\right) \\
&\quad + \left(\frac{\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{\frac{p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} - \left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{-\cdots}}{\sqrt{p^2+(-2q-4r)\,p+q^2}}\right.
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
u &= c_1 x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} - \frac{c_2 p\, x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}} \\
&\quad + \frac{\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{\frac{p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} - \left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{-\cdots}}{\sqrt{p^2+(-2q-4r)\,p+q^2}}
\end{aligned}
\tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
u &= c_1 x^{\frac{p-q+\sqrt{p^2-2qp-4pr+q^2}}{2p}} - \frac{c_2 p\, x^{-\frac{-p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}}}{\sqrt{p^2+(-2q-4r)\,p+q^2}} \\
&\quad + \frac{\left(\int_0^x \alpha^{-\frac{3p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{\frac{p-q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} - \left(\int_0^x \alpha^{\frac{-3p+q+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} f(\alpha)\, d\alpha\right) x^{-\cdots}}{\sqrt{p^2+(-2q-4r)\,p+q^2}}
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 259

```
dsolve(p*x^2*diff(u(x),x$2)+q*x*diff(u(x),x)+r*u(x)=f(x),u(x), singsol=all)
```

$$u(x) = \frac{x^{\frac{-q+p+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} c_2 \sqrt{p^2+(-2q-4r)\,p+q^2} + x^{-\frac{q-p+\sqrt{p^2+(-2q-4r)p+q^2}}{2p}} c_1 \sqrt{p^2+(-2q-4r)\,p+q^2} + x}{\sqrt{p^2+}}$$

✓ Solution by Mathematica

Time used: 1.17 (sec). Leaf size: 342

```
DSolve[p*x^2*u''[x]+q*x*u'[x]+r*u[x]==f[x],u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to x^{-\frac{\sqrt{p}\sqrt{r}\sqrt{\frac{p^2-2p(q+2r)+q^2}{pr}}-p+q}{2p}} \left(x^{\frac{\sqrt{r}\sqrt{\frac{p^2-2p(q+2r)+q^2}{pr}}}{\sqrt{p}}} \int_1^x \frac{f(K[2])K[2]^{\frac{-3p-\sqrt{r}\sqrt{\frac{p^2-2(q+2r)p+q^2}{pr}}\sqrt{p}+q}{2p}}}{\sqrt{p}\sqrt{r}\sqrt{\frac{p^2-2(q+2r)p+q^2}{pr}}} dK[2] + \int_1^x -\frac{f(K[1])K[1]^{\frac{-3p+\sqrt{r}\sqrt{\frac{p^2-2(q+2r)p+q^2}{pr}}\sqrt{p}+q}{2p}}}{\sqrt{p}\sqrt{r}\sqrt{\frac{p^2-2(q+2r)p+q^2}{pr}}} dK[1] + c_2 x^{\frac{\sqrt{r}\sqrt{\frac{p^2-2p(q+2r)+q^2}{pr}}}{\sqrt{p}}} + c_1 \right)$$

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3.1 problem Example 3.29

Internal problem ID [5857]
Internal file name [OUTPUT/5105_Sunday_June_05_2022_03_24_47_PM_25745770/index.tex]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{\sin(x)\,u'' + 2\cos(x)\,u' + \sin(x)\,u = 0}$$

3.1.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$u'' + p(x)\,u' + q(x)\,u = 0 \tag{2}$$

Where

$$p(x) = \frac{2\cos(x)}{\sin(x)}$$
$$q(x) = 1$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= 1 - \frac{\left(\frac{2\cos(x)}{\sin(x)}\right)'}{2} - \frac{\left(\frac{2\cos(x)}{\sin(x)}\right)^2}{4} \\
&= 1 - \frac{\left(-2 - \frac{2\cos(x)^2}{\sin(x)^2}\right)}{2} - \frac{\left(\frac{4\cos(x)^2}{\sin(x)^2}\right)}{4} \\
&= 1 - \left(-1 - \frac{\cos(x)^2}{\sin(x)^2}\right) - \frac{\cos(x)^2}{\sin(x)^2} \\
&= 2
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
u = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{2\cos(x)}{\sin(x)}}{2}} \\
&= \csc(x)
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
u = v(x) \csc(x) \tag{4}
$$

Applying this change of variable to the original ode results in

$$
2v(x) + v''(x) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(2)}\\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{2}\\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{2}\\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0\left(c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)\right)$$

Or

$$v(x) = c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)$$

Now that $v(x)$ is known, then

$$\begin{aligned} u &= v(x)\, z(x) \\ &= \left(c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)\right)(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \csc(x)$$

Hence (7) becomes

$$u = \left(c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)\right)\csc(x)$$

Summary
The solution(s) found are the following

$$u = \left(c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)\right)\csc(x) \tag{1}$$

Verification of solutions

$$u = \left(c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)\right)\csc(x)$$

Verified OK.

3.1.2 Solving using Kovacic algorithm

Writing the ode as

$$\sin(x)\, u'' + 2\cos(x)\, u' + \sin(x)\, u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= \sin(x) \\ B &= 2\cos(x) \\ C &= \sin(x) \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 35: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(x\sqrt{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned}u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2\cos(x)}{\sin(x)}\, dx} \\ &= z_1 e^{-\ln(\sin(x))} \\ &= z_1(\csc(x))\end{aligned}$$

Which simplifies to

$$u_1 = \cos\left(x\sqrt{2}\right)\csc(x)$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$\begin{aligned}u_2 &= u_1 \int \frac{e^{\int -\frac{2\cos(x)}{\sin(x)}\, dx}}{(u_1)^2}\, dx \\ &= u_1 \int \frac{e^{-2\ln(\sin(x))}}{(u_1)^2}\, dx \\ &= u_1\left(\frac{\sqrt{2}\,\tan\left(x\sqrt{2}\right)}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}u &= c_1 u_1 + c_2 u_2\\ &= c_1\left(\cos\left(x\sqrt{2}\right)\csc\left(x\right)\right) + c_2\left(\cos\left(x\sqrt{2}\right)\csc\left(x\right)\left(\frac{\sqrt{2}\,\tan\left(x\sqrt{2}\right)}{2}\right)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$u = c_1\cos\left(x\sqrt{2}\right)\csc\left(x\right) + \frac{c_2\sqrt{2}\,\sin\left(x\sqrt{2}\right)\csc\left(x\right)}{2} \tag{1}$$

Verification of solutions

$$u = c_1\cos\left(x\sqrt{2}\right)\csc\left(x\right) + \frac{c_2\sqrt{2}\,\sin\left(x\sqrt{2}\right)\csc\left(x\right)}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(sin(x)*diff(u(x),x$2)+2*cos(x)*diff(u(x),x)+sin(x)*u(x)=0,u(x), singsol=all)
```

$$u(x) = \csc\left(x\right)\left(c_1\sin\left(\sqrt{2}\,x\right) + c_2\cos\left(\sqrt{2}\,x\right)\right)$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 51

```
DSolve[Sin[x]*u''[x]+2*Cos[x]*u'[x]+Sin[x]*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{1}{4}e^{-i\sqrt{2}x}\left(4c_1 - i\sqrt{2}c_2e^{2i\sqrt{2}x}\right)\csc(x)$$

3.2 problem Example 3.30

Internal problem ID [5858]
Internal file name [`OUTPUT/5106_Sunday_June_05_2022_03_24_49_PM_35055999/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.30.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x], [_3rd_order, _missing_y], [
   _3rd_order, _with_exponential_symmetries], [_3rd_order,
   _with_linear_symmetries], [_3rd_order, _reducible, _mu_y2], [
   _3rd_order, _reducible, _mu_poly_yn]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying 3rd order ODE linearizable_by_differentiation
differential order: 3; trying a linearization to 4th order
trying differential order: 3; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(diff(_b(_a), _a), _a))*_b(_a)^2-(diff(_b(_a), _a))*_
   symmetry methods on request
`, `2nd order, trying reduction of order with given symmetries:`[0, _b^2], [1, 0], [0, _b]
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 38

```
dsolve(3*diff(y(x),x$2)^2-diff(y(x),x)*diff(y(x),x$3)-diff(y(x),x$2)*diff(y(x),x)^2=0,y(x),
```

$$y(x) = c_1$$
$$y(x) = \frac{\text{LambertW}\left(-\frac{e^{\frac{c_3+x}{c_1}}}{c_2 c_1}\right) c_1 - c_3 - x}{c_1}$$

✓ Solution by Mathematica

Time used: 3.622 (sec). Leaf size: 79

```
DSolve[3*(y''[x])^2-y'[x]*y'''[x]-y''[x]*(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \log\left(\text{InverseFunction}\left[-\frac{1}{\#1} - c_1\log(\#1) + c_1\log(1+\#1c_1)\&\right][x+c_2]\right) - \log\left(1 + c_1\text{InverseFunction}\left[-\frac{1}{\#1} - c_1\log(\#1) + c_1\log(1+\#1c_1)\&\right][x+c_2]\right) + c_3$$

3.3 problem Example 3.32

Internal problem ID [5859]
Internal file name [`OUTPUT/5107_Sunday_June_05_2022_03_24_51_PM_9413660/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[_Gegenbauer, [_2nd_order, _linear, `_with_symmetry_[0,F(x)]`]]
```

$$\boxed{y'' - \frac{xy'}{-x^2+1} + \frac{y}{-x^2+1} = 0}$$

3.3.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$\left(x^2-1\right)y''+xy'-y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=\frac{x}{x^2-1}\\ q(x)&=-\frac{1}{x^2-1}\end{aligned}$$

Applying change of variables $\tau=g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \tag{4}$$

$$q_1(\tau)=\frac{q(x)}{\tau'\left(x\right)^2} \tag{5}$$

Let $p_1=0$. Eq (4) simplifies to

$$\tau''(x)+p(x)\,\tau'(x)=0$$

This ode is solved resulting in

$$\begin{aligned}\tau&=\int \mathrm{e}^{-\left(\int p(x)dx\right)}dx\\ &=\int \mathrm{e}^{-\left(\int \frac{x}{x^2-1}dx\right)}dx\\ &=\int e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}}\,dx\\ &=\int \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}dx\\ &=\frac{\sqrt{(x-1)\,(1+x)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{1+x}}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{x^2-1}}{\frac{1}{(x-1)(1+x)}} \\
&= -1
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} - \mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\
&= \pm 1
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +1 \\
\lambda_2 &= -1
\end{aligned}$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}$$

Or

$$y(\tau) = c_1 e^{\tau} + c_2 e^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(x + \sqrt{x^2 - 1}\right)^{\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x + \sqrt{x^2 - 1}\right)^{-\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(x + \sqrt{x^2 - 1}\right)^{\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x + \sqrt{x^2 - 1}\right)^{-\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} \tag{1}$$

Verification of solutions

$$y = c_1 \left(x + \sqrt{x^2 - 1}\right)^{\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}} + c_2 \left(x + \sqrt{x^2 - 1}\right)^{-\frac{\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}}$$

Verified OK.

3.3.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$\left(x^2 - 1\right) y'' + x y' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{x}{x^2 - 1}$$
$$q(x) = -\frac{1}{x^2 - 1}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-\frac{1}{x^2-1}}}{c} \\ \tau'' &= \frac{x}{c\sqrt{-\frac{1}{x^2-1}}\,(x^2-1)^2} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{x}{c\sqrt{-\frac{1}{x^2-1}}\,(x^2-1)^2} + \frac{x}{x^2-1}\frac{\sqrt{-\frac{1}{x^2-1}}}{c}}{\left(\frac{\sqrt{-\frac{1}{x^2-1}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{-\frac{1}{x^2-1}}dx}{c} \\
&= \frac{\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
\begin{aligned}
y = c_1 \cos\left(\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \\
+ c_2 \sin\left(\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 \cos\left(\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \\
+ c_2 \sin\left(\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 \cos\left(\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right) \\
+ c_2 \sin\left(\sqrt{-\frac{1}{x^2-1}}\,\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned}
$$

Verified OK.

3.3.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$(x^2-1)\,y''+xy'-y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=\frac{x}{x^2-1}\\ q(x)&=-\frac{1}{x^2-1}\end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right)v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right)v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{n}{x^2-1}-\frac{1}{x^2-1}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(x)+\left(\frac{2}{x}+\frac{x}{x^2-1}\right)v'(x)&=0\\ v''(x)+\frac{(3x^2-2)\,v'(x)}{x^3-x}&=0\end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(3x^2 - 2)\, u(x)}{x^3 - x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u(3x^2 - 2)}{x\,(x^2 - 1)} \end{aligned}$$

Where $f(x) = -\frac{3x^2-2}{x(x^2-1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3x^2 - 2}{x\,(x^2 - 1)}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3x^2 - 2}{x\,(x^2 - 1)}\, dx \\ \ln(u) &= -\frac{\ln(1+x)}{2} - \frac{\ln(x-1)}{2} - 2\ln(x) + c_1 \\ u &= \mathrm{e}^{-\frac{\ln(1+x)}{2} - \frac{\ln(x-1)}{2} - 2\ln(x) + c_1} \\ &= c_1 \mathrm{e}^{-\frac{\ln(1+x)}{2} - \frac{\ln(x-1)}{2} - 2\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1}{\sqrt{1+x}\,\sqrt{x-1}\,x^2}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= \frac{\sqrt{x-1}\,\sqrt{1+x}\,c_1}{x} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(\frac{\sqrt{x-1}\,\sqrt{1+x}\,c_1}{x} + c_2\right) x \\ &= c_1\sqrt{x-1}\,\sqrt{1+x} + c_2 x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(\frac{\sqrt{x-1}\,\sqrt{1+x}\,c_1}{x} + c_2\right) x \tag{1}$$

Verification of solutions

$$y = \left(\frac{\sqrt{x-1}\,\sqrt{1+x}\,c_1}{x} + c_2\right) x$$

Verified OK.

3.3.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(x^2-1\right) y'' + xy' - y\right) dx = 0$$
$$-xy + \left(x^2-1\right) y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{x}{x^2-1}$$
$$q(x) = \frac{c_1}{x^2-1}$$

Hence the ode is

$$y' - \frac{xy}{x^2-1} = \frac{c_1}{x^2-1}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{x}{x^2-1}dx}$$
$$= e^{-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2}}$$

Which simplifies to

$$\mu = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x^2-1}\right)$$

$$\frac{d}{dx}\left(\frac{y}{\sqrt{x-1}\,\sqrt{1+x}}\right) = \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)\left(\frac{c_1}{x^2-1}\right)$$

$$d\left(\frac{y}{\sqrt{x-1}\,\sqrt{1+x}}\right) = \left(\frac{c_1}{(x^2-1)\sqrt{x-1}\,\sqrt{1+x}}\right) dx$$

Integrating gives

$$\frac{y}{\sqrt{x-1}\,\sqrt{1+x}} = \int \frac{c_1}{(x^2-1)\sqrt{x-1}\,\sqrt{1+x}}\, dx$$

$$\frac{y}{\sqrt{x-1}\,\sqrt{1+x}} = -\frac{\sqrt{x-1}\,\sqrt{1+x}\, x c_1}{x^2-1} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$ results in

$$y = -\frac{(x-1)(1+x)\, x c_1}{x^2-1} + c_2\sqrt{x-1}\,\sqrt{1+x}$$

which simplifies to

$$y = -c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x}$$

Summary

The solution(s) found are the following

$$y = -c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x} \tag{1}$$

Verification of solutions

$$y = -c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x}$$

Verified OK.

3.3.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right)v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right)u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x^2 - 1 \\
B &= x \\
C &= -1 \\
F &= 0
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(x^2 - 1\right)(0) + (x)(1) + (-1)(x) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
x^3 - xv'' + \left(3x^2 - 2\right)v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
\left(x^3 - x\right)u'(x) + \left(3x^2 - 2\right)u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(3x^2-2)}{x\,(x^2-1)} \end{aligned}$$

Where $f(x) = -\frac{3x^2-2}{x(x^2-1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{3x^2-2}{x\,(x^2-1)}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{3x^2-2}{x\,(x^2-1)}\,dx \\ \ln(u) &= -\frac{\ln(1+x)}{2} - \frac{\ln(x-1)}{2} - 2\ln(x) + c_1 \\ u &= \mathrm{e}^{-\frac{\ln(1+x)}{2} - \frac{\ln(x-1)}{2} - 2\ln(x) + c_1} \\ &= c_1 \mathrm{e}^{-\frac{\ln(1+x)}{2} - \frac{\ln(x-1)}{2} - 2\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1}{\sqrt{1+x}\,\sqrt{x-1}\,x^2}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{\sqrt{1+x}\,\sqrt{x-1}\,x^2} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{\sqrt{1+x}\,\sqrt{x-1}\,x^2}\,\mathrm{d}x \\ &= \frac{\sqrt{x-1}\,\sqrt{1+x}\,c_1}{x} + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (x)\left(\frac{\sqrt{x-1}\,\sqrt{1+x}\,c_1}{x} + c_2\right) \\ &= c_1\sqrt{x-1}\,\sqrt{1+x} + c_2 x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x-1}\,\sqrt{1+x}+c_2x \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x-1}\,\sqrt{1+x}+c_2x$$

Verified OK.

3.3.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$\left(x^2-1\right)y''+xy'-y=0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(x^2-1\right)y''+xy'-y\right)dx = 0$$
$$-xy+\left(x^2-1\right)y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$p(x) = -\frac{x}{x^2-1}$$
$$q(x) = \frac{c_1}{x^2-1}$$

Hence the ode is

$$y'-\frac{xy}{x^2-1} = \frac{c_1}{x^2-1}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{x}{x^2-1}dx}$$
$$= \mathrm{e}^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}}$$

Which simplifies to

$$\mu = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x^2-1}\right) \\
\frac{d}{dx}\left(\frac{y}{\sqrt{x-1}\,\sqrt{1+x}}\right) &= \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)\left(\frac{c_1}{x^2-1}\right) \\
d\left(\frac{y}{\sqrt{x-1}\,\sqrt{1+x}}\right) &= \left(\frac{c_1}{(x^2-1)\sqrt{x-1}\,\sqrt{1+x}}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{\sqrt{x-1}\,\sqrt{1+x}} &= \int \frac{c_1}{(x^2-1)\sqrt{x-1}\,\sqrt{1+x}}\, dx \\
\frac{y}{\sqrt{x-1}\,\sqrt{1+x}} &= -\frac{\sqrt{x-1}\,\sqrt{1+x}\, x c_1}{x^2-1} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$ results in

$$y = -\frac{(x-1)(1+x)\, x c_1}{x^2-1} + c_2\sqrt{x-1}\,\sqrt{1+x}$$

which simplifies to

$$y = -c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x}$$

Summary

The solution(s) found are the following

$$y = -c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x} \tag{1}$$

Verification of solutions

$$y = -c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x}$$

Verified OK.

3.3.7 Solving using Kovacic algorithm

Writing the ode as

$$\left(x^2-1\right) y''+x y'-y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2-1\\ B&=x\\ C&=-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2 A}\, d x}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2 A B'-2 B A'+B^2-4 A C}{4 A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{3 x^2-6}{4\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=3 x^2-6 \\ t&=4\left(x^2-1\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3 x^2-6}{4\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y=z(x) e^{-\int \frac{B}{2 A}\, d x}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 36: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(x-1)^2} + \frac{9}{16(x-1)} - \frac{9}{16(1+x)} - \frac{3}{16(1+x)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^2 - 6}{4\left(x^2 - 1\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^2 - 6}{4\left(x^2 - 1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+}\right) \\ &= \frac{3}{2} - \left(\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{3}{4(x-1)} + \frac{3}{4(1+x)} + (0) \\ &= \frac{3}{4(x-1)} + \frac{3}{4(1+x)} \\ &= \frac{3x}{2x^2-2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4(x-1)} + \frac{3}{4(1+x)}\right)(0) + \left(\left(-\frac{3}{4(x-1)^2} - \frac{3}{4(1+x)^2}\right) + \left(\frac{3}{4(x-1)} + \frac{3}{4(1+x)}\right)^2 - \left(\frac{3}{4(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{4(x-1)}+\frac{3}{4(1+x)}\right)dx} \\
&= (x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2-1} \, dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{4}-\frac{\ln(1+x)}{4}} \\
&= z_1 \left(\frac{1}{(x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x-1}\,\sqrt{1+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2-1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{x}{\sqrt{x-1}\,\sqrt{1+x}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{x-1}\,\sqrt{1+x}\right) + c_2 \left(\sqrt{x-1}\,\sqrt{1+x}\left(-\frac{x}{\sqrt{x-1}\,\sqrt{1+x}}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x-1}\,\sqrt{1+x} - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x-1}\,\sqrt{1+x} - c_2 x$$

Verified OK.

3.3.8 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 - 1 \\ q(x) &= x \\ r(x) &= -1 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 1 \end{aligned}$$

Therefore (1) becomes

$$2 - (1) + (-1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + \left(q(x) - p'(x)\right)y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + \left(q(x) - p'(x)\right)y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$-xy + (x^2 - 1)\, y' = c_1$$

We now have a first order ode to solve which is

$$-xy + (x^2 - 1)\, y' = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{x}{x^2 - 1} \\ q(x) &= \frac{c_1}{x^2 - 1} \end{aligned}$$

Hence the ode is

$$y' - \frac{xy}{x^2 - 1} = \frac{c_1}{x^2 - 1}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{x}{x^2-1}dx} \\ &= \mathrm{e}^{-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2}} \end{aligned}$$

Which simplifies to

$$\mu = \frac{1}{\sqrt{x - 1}\,\sqrt{1 + x}}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2 - 1}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{x-1}\,\sqrt{1+x}}\right) &= \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)\left(\frac{c_1}{x^2-1}\right) \\ \mathrm{d}\left(\frac{y}{\sqrt{x-1}\,\sqrt{1+x}}\right) &= \left(\frac{c_1}{(x^2-1)\,\sqrt{x-1}\,\sqrt{1+x}}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{\sqrt{x-1}\,\sqrt{1+x}} &= \int \frac{c_1}{(x^2-1)\,\sqrt{x-1}\,\sqrt{1+x}}\,\mathrm{d}x \\ \frac{y}{\sqrt{x-1}\,\sqrt{1+x}} &= -\frac{\sqrt{x-1}\,\sqrt{1+x}\,x c_1}{x^2-1} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x-1}\,\sqrt{1+x}}$ results in

$$y = -\frac{(x-1)(1+x)\,x c_1}{x^2-1} + c_2\sqrt{x-1}\,\sqrt{1+x}$$

which simplifies to

$$y = -c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x} \tag{1}$$

<u>Verification of solutions</u>

$$y = -c_1 x + c_2\sqrt{x-1}\,\sqrt{1+x}$$

Verified OK.

3.3.9 Maple step by step solution

Let's solve

$(x^2-1)\,y'' + xy' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{xy'}{x^2-1} + \frac{y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{xy'}{x^2-1} - \frac{y}{x^2-1} = 0$

- Multiply by denominators of ODE

  $(-x^2+1)\,y'' - xy' + y = 0$

- Make a change of variables

  $\theta = \arccos(x)$

- Calculate y' with change of variables

  $y' = \left(\frac{d}{d\theta}y(\theta)\right)\theta'(x)$

- Compute 1st derivative y'

$$y' = -\frac{\frac{d}{d\theta}y(\theta)}{\sqrt{-x^2+1}}$$

- Calculate y'' with change of variables

$$y'' = \left(\frac{d^2}{d\theta^2}y(\theta)\right)\theta'(x)^2 + \theta''(x)\left(\frac{d}{d\theta}y(\theta)\right)$$

- Compute 2nd derivative y''

$$y'' = \frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1} - \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}$$

- Apply the change of variables to the ODE

$$(-x^2+1)\left(\frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1} - \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}}\right) + \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{\sqrt{-x^2+1}} + y = 0$$

- Multiply through

$$-\frac{\left(\frac{d^2}{d\theta^2}y(\theta)\right)x^2}{-x^2+1} + \frac{\frac{d^2}{d\theta^2}y(\theta)}{-x^2+1} + \frac{x^3\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}} - \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{(-x^2+1)^{\frac{3}{2}}} + \frac{x\left(\frac{d}{d\theta}y(\theta)\right)}{\sqrt{-x^2+1}} + y = 0$$

- Simplify ODE

$$\frac{d^2}{d\theta^2}y(\theta) + y = 0$$

- ODE is that of a harmonic oscillator with given general solution

$$y(\theta) = c_1\sin(\theta) + c_2\cos(\theta)$$

- Revert back to x

$$y = c_1\sin(\arccos(x)) + c_2\cos(\arccos(x))$$

- Use trig identity to simplify $\sin(\arccos(x))$

$$\sin(\arccos(x)) = \sqrt{-x^2+1}$$

- Simplify solution to the ODE

$$y = c_1\sqrt{-x^2+1} + c_2 x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)-x/(1-x^2)*diff(y(x),x)+y(x)/(1-x^2)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 \sqrt{x-1}\,\sqrt{x+1}$$

✓ Solution by Mathematica

Time used: 0.177 (sec). Leaf size: 97

```
DSolve[y''[x]-x/(1-x^2)*y'[x]+y[x]/(1-x^2)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cosh\left(\frac{2\sqrt{1-x^2}\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right)}{\sqrt{x^2-1}}\right) - ic_2 \sinh\left(\frac{2\sqrt{1-x^2}\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right)}{\sqrt{x^2-1}}\right)$$

3.4 problem Example 3.33

Internal problem ID [5860]
Internal file name [`OUTPUT/5108_Sunday_June_05_2022_03_24_53_PM_20418152/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _reducible
   , _mu_y_y1], [_2nd_order, _reducible, _mu_xy]]
```

Unable to solve or complete the solution.

$$\boxed{x^2yy'' - x^2y'^2 + y^2 = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
<- quadrature successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve(x^2*y(x)*diff(y(x),x$2)=x^2*(diff(y(x),x))^2-y(x)^2,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_2 x \,\mathrm{e}^{-c_1 x + 1}$$

✓ Solution by Mathematica

Time used: 0.212 (sec). Leaf size: 15

```
DSolve[x^2*y[x]*y''[x]==x^2*(y'[x])^2-y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x e^{c_1 x}$$

3.5 problem Example 3.34

Internal problem ID [5861]
Internal file name [`OUTPUT/5109_Sunday_June_05_2022_03_24_54_PM_13785063/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.34.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' - 3y'' + 3y' - y = 4\,\mathrm{e}^{t}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 3y'' + 3y' - y = 0$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(t) = c_1 e^t + t\, e^t c_2 + t^2 e^t c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^t$$
$$y_2 = t\, e^t$$
$$y_3 = e^t t^2$$

Now the particular solution to the given ODE is found

$$y''' - 3y'' + 3y' - y = 4\, e^t$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\, e^t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{t\, e^t, e^t t^2, e^t\}$$

Since e^t is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\, e^t\}]$$

Since $t\, e^t$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{e^t t^2\}]$$

Since $e^t t^2$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{e^t t^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^t t^3$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^t = 4\,e^t$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{2\,e^t t^3}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^t + t\,e^t c_2 + t^2 e^t c_3\right) + \left(\frac{2\,e^t t^3}{3}\right) \end{aligned}$$

Which simplifies to

$$y = e^t\left(c_3 t^2 + c_2 t + c_1\right) + \frac{2\,e^t t^3}{3}$$

Summary

The solution(s) found are the following

$$y = e^t\left(c_3 t^2 + c_2 t + c_1\right) + \frac{2\,e^t t^3}{3} \tag{1}$$

Verification of solutions

$$y = e^t\left(c_3 t^2 + c_2 t + c_1\right) + \frac{2\,e^t t^3}{3}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(diff(y(t),t$3)-3*diff(y(t),t$2)+3*diff(y(t),t)-y(t)=4*exp(t),y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{t}\left(\frac{2}{3}t^{3} + c_{1} + tc_{2} + t^{2}c_{3}\right)$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 34

```
DSolve[y'''[t]-3*y''[t]+3*y'[t]-y[t]==4*Exp[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{3}e^{t}\left(2t^{3} + 3c_{3}t^{2} + 3c_{2}t + 3c_{1}\right)$$

3.6 problem Example 3.35

Internal problem ID [5862]
Internal file name [`OUTPUT/5110_Sunday_June_05_2022_03_24_56_PM_17758930/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.35.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 2y'' + y = 3\sin(t) - 5\cos(t)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y'' + y = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^2 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(t) = e^{-it}c_1 + t\,e^{-it}c_2 + e^{it}c_3 + t\,e^{it}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-it} \\ y_2 &= t\,e^{-it} \\ y_3 &= e^{it} \\ y_4 &= t\,e^{it} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 2y'' + y = 3\sin(t) - 5\cos(t)$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(t)$ are functions to be determined as follows

$$U_i = (-1)^{n-i}\int \frac{F(t)W_i(t)}{aW(t)}\,dt$$

Where $W(t)$ is the Wronskian and $W_i(t)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(t)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(t)$. This is given by

$$W(t) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{-it} & t\,e^{-it} & e^{it} & t\,e^{it} \\ -ie^{-it} & e^{-it}(-it+1) & ie^{it} & e^{it}(it+1) \\ -e^{-it} & e^{-it}(-2i-t) & -e^{it} & e^{it}(2i-t) \\ ie^{-it} & e^{-it}(it-3) & -ie^{it} & -e^{it}(it+3) \end{bmatrix}$$

$$|W| = 16\,e^{2it}e^{-2it}$$

The determinant simplifies to

$$|W| = 16$$

Now we determine W_i for each U_i.

$$W_1(t) = \det \begin{bmatrix} t\,\mathrm{e}^{-it} & \mathrm{e}^{it} & t\,\mathrm{e}^{it} \\ \mathrm{e}^{-it}(-it+1) & i\mathrm{e}^{it} & \mathrm{e}^{it}(it+1) \\ \mathrm{e}^{-it}(-2i-t) & -\mathrm{e}^{it} & \mathrm{e}^{it}(2i-t) \end{bmatrix}$$
$$= -4\,\mathrm{e}^{it}(i+t)$$

$$W_2(t) = \det \begin{bmatrix} \mathrm{e}^{-it} & \mathrm{e}^{it} & t\,\mathrm{e}^{it} \\ -i\mathrm{e}^{-it} & i\mathrm{e}^{it} & \mathrm{e}^{it}(it+1) \\ -\mathrm{e}^{-it} & -\mathrm{e}^{it} & \mathrm{e}^{it}(2i-t) \end{bmatrix}$$
$$= -4\,\mathrm{e}^{it}$$

$$W_3(t) = \det \begin{bmatrix} \mathrm{e}^{-it} & t\,\mathrm{e}^{-it} & t\,\mathrm{e}^{it} \\ -i\mathrm{e}^{-it} & \mathrm{e}^{-it}(-it+1) & \mathrm{e}^{it}(it+1) \\ -\mathrm{e}^{-it} & \mathrm{e}^{-it}(-2i-t) & \mathrm{e}^{it}(2i-t) \end{bmatrix}$$
$$= -4\,\mathrm{e}^{-it}(-i+t)$$

$$W_4(t) = \det \begin{bmatrix} \mathrm{e}^{-it} & t\,\mathrm{e}^{-it} & \mathrm{e}^{it} \\ -i\mathrm{e}^{-it} & \mathrm{e}^{-it}(-it+1) & i\mathrm{e}^{it} \\ -\mathrm{e}^{-it} & \mathrm{e}^{-it}(-2i-t) & -\mathrm{e}^{it} \end{bmatrix}$$
$$= -4\,\mathrm{e}^{-it}$$

Now we are ready to evaluate each $U_i(t)$.

$$\begin{aligned} U_1 &= (-1)^{4-1} \int \frac{F(t)W_1(t)}{aW(t)}\,dt \\ &= (-1)^3 \int \frac{(3\sin(t) - 5\cos(t))\,(-4\,\mathrm{e}^{it}(i+t))}{(1)\,(16)}\,dt \\ &= -\int \frac{-4(3\sin(t) - 5\cos(t))\,\mathrm{e}^{it}(i+t)}{16}\,dt \\ &= -\int \left(-\frac{(3\sin(t) - 5\cos(t))\,\mathrm{e}^{it}(i+t)}{4}\right)dt \\ &= -\left(\int -\frac{(3\sin(t) - 5\cos(t))\,\mathrm{e}^{it}(i+t)}{4}dt\right) \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(t)W_2(t)}{aW(t)}\,dt \\
&= (-1)^2 \int \frac{(3\sin(t) - 5\cos(t))\left(-4\,\mathrm{e}^{it}\right)}{(1)\,(16)}\,dt \\
&= \int \frac{-4(3\sin(t) - 5\cos(t))\,\mathrm{e}^{it}}{16}\,dt \\
&= \int \left(-\frac{(3\sin(t) - 5\cos(t))\,\mathrm{e}^{it}}{4}\right)dt \\
&= \frac{5t}{8} - \frac{3it}{8} + \frac{3\,\mathrm{e}^{2it}}{16} - \frac{5i\mathrm{e}^{2it}}{16}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(t)W_3(t)}{aW(t)}\,dt \\
&= (-1)^1 \int \frac{(3\sin(t) - 5\cos(t))\left(-4\,\mathrm{e}^{-it}(-i+t)\right)}{(1)\,(16)}\,dt \\
&= -\int \frac{-4(3\sin(t) - 5\cos(t))\,\mathrm{e}^{-it}(-i+t)}{16}\,dt \\
&= -\int \left(-\frac{(3\sin(t) - 5\cos(t))\,\mathrm{e}^{-it}(-i+t)}{4}\right)dt \\
&= -\left(\int -\frac{(3\sin(t) - 5\cos(t))\,\mathrm{e}^{-it}(-i+t)}{4}dt\right)
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(t)W_4(t)}{aW(t)}\,dt \\
&= (-1)^0 \int \frac{(3\sin(t) - 5\cos(t))\left(-4\,\mathrm{e}^{-it}\right)}{(1)\,(16)}\,dt \\
&= \int \frac{-4(3\sin(t) - 5\cos(t))\,\mathrm{e}^{-it}}{16}\,dt \\
&= \int \left(-\frac{(3\sin(t) - 5\cos(t))\,\mathrm{e}^{-it}}{4}\right)dt \\
&= \int -\frac{(3\sin(t) - 5\cos(t))\,\mathrm{e}^{-it}}{4}dt
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4
$$

Hence

$$y_p = \left(-\left(\int -\frac{(3\sin(t) - 5\cos(t))\,\mathrm{e}^{it}(i+t)}{4}dt\right)\right)\left(\mathrm{e}^{-it}\right) + \left(\frac{5t}{8} - \frac{3it}{8} + \frac{3\,\mathrm{e}^{2it}}{16} - \frac{5i\mathrm{e}^{2it}}{16}\right)\left(t\,\mathrm{e}^{-it}\right) + \left(-\left(\int -\frac{(3\sin(t) - 5\cos(t))\,\mathrm{e}^{-it}(-i+t)}{4}dt\right)\right)\left(\mathrm{e}^{it}\right) + \left(\int -\frac{(3\sin(t) - 5\cos(t))\,\mathrm{e}^{-it}}{4}dt\right)\left(t\,\mathrm{e}^{it}\right)$$

Therefore the particular solution is

$$y_p = \frac{\left(\int (-3\sin(t) + 5\cos(t))\,\mathrm{e}^{-it}(i-t)\,dt\right)\mathrm{e}^{it}}{4} - \frac{\left(\int (-3\sin(t) + 5\cos(t))\,\mathrm{e}^{it}(i+t)\,dt\right)\mathrm{e}^{-it}}{4} + \frac{3\left(\left(\frac{5}{3} - i\right)t\,\mathrm{e}\right.}{}$$

Which simplifies to

$$y_p = -\frac{\left(\int (-3\sin(t) + 5\cos(t))\,(-\sin(t) + \cos(t)\,t)\,dt\right)\cos(t)}{2} - \frac{\left(\int (-3\sin(t) + 5\cos(t))\,(\sin(t)\,t + \cos\right.}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-it}c_1 + t\,\mathrm{e}^{-it}c_2 + \mathrm{e}^{it}c_3 + t\,\mathrm{e}^{it}c_4\right) \\ &\quad + \left(-\frac{\left(\int (-3\sin(t) + 5\cos(t))\,(-\sin(t) + \cos(t)\,t)\,dt\right)\cos(t)}{2} - \frac{\left(\int (-3\sin(t) + 5\cos(t))\,(\sin(t)\,t + \cos(t))\,dt\right)\sin(t)}{2} - \frac{5t\left(\left(-\frac{3}{5} + i - 4t\right)\cos(t) + \frac{3\sin(t)(-5+i+4t)}{5}\right)}{16}\right) \end{aligned}$$

Which simplifies to

$$y = (c_4t + c_3)\,\mathrm{e}^{it} + (c_2t + c_1)\,\mathrm{e}^{-it} - \frac{\left(\int (-3\sin(t) + 5\cos(t))\,(-\sin(t) + \cos(t)\,t)\,dt\right)\cos(t)}{2} - \frac{\left(\int (-3\sin(t) + 5\cos(t))\,(\sin(t)\,t + \cos(t))\,dt\right)\sin(t)}{2} - \frac{5t\left(\left(-\frac{3}{5} + i - 4t\right)\cos(t) + \frac{3\sin(t)(-5+i+4t)}{5}\right)}{16}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & (c_4 t + c_3)\,\mathrm{e}^{it} + (c_2 t + c_1)\,\mathrm{e}^{-it} \\
& - \frac{\left(\int (-3\sin(t) + 5\cos(t))\,(-\sin(t) + \cos(t)\,t)\,dt\right)\cos(t)}{2} \\
& - \frac{\left(\int (-3\sin(t) + 5\cos(t))\,(\sin(t)\,t + \cos(t))\,dt\right)\sin(t)}{2} \\
& - \frac{5t\left(\left(-\frac{3}{5} + i - 4t\right)\cos(t) + \frac{3\sin(t)(-5+i+4t)}{5}\right)}{16}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & (c_4 t + c_3)\,\mathrm{e}^{it} + (c_2 t + c_1)\,\mathrm{e}^{-it} \\
& - \frac{\left(\int (-3\sin(t) + 5\cos(t))\,(-\sin(t) + \cos(t)\,t)\,dt\right)\cos(t)}{2} \\
& - \frac{\left(\int (-3\sin(t) + 5\cos(t))\,(\sin(t)\,t + \cos(t))\,dt\right)\sin(t)}{2} \\
& - \frac{5t\left(\left(-\frac{3}{5} + i - 4t\right)\cos(t) + \frac{3\sin(t)(-5+i+4t)}{5}\right)}{16}
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.625 (sec). Leaf size: 45

```
dsolve(diff(y(t),t$4)+2*diff(y(t),t$2)+y(t)=3*sin(t)-5*cos(t),y(t), singsol=all)
```

$$y(t) = \frac{(5t^2 + (8c_3 - 6)\,t + 8c_1 - 10)\cos(t)}{8} - \frac{3\sin(t)\left(t^2 + \left(-\frac{8c_4}{3} + \frac{10}{3}\right)t - \frac{8c_2}{3} - 2\right)}{8}$$

✓ Solution by Mathematica

Time used: 0.128 (sec). Leaf size: 56

```
DSolve[y''''[t]+2*y''[t]+y[t]==3*Sin[t]-5*Cos[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{16}\left(\left(10t^2 + 2(-3 + 8c_2)t - 25 + 16c_1\right)\cos(t) + \left(-6t^2 + 2(-15 + 8c_4)t + 3 + 16c_3\right)\sin(t)\right)$$

3.7 problem Example 3.36

Internal problem ID [5863]
Internal file name [`OUTPUT/5111_Sunday_June_05_2022_03_24_59_PM_9958800/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.36.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - y'' - y' + y = g(t)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' - y' + y = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(t) = \mathrm{e}^{-t} c_1 + c_2 \mathrm{e}^{t} + t\,\mathrm{e}^{t} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-t} \\ y_2 &= \mathrm{e}^{t} \\ y_3 &= t\,\mathrm{e}^{t} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - y'' - y' + y = g(t)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(t)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(t) W_i(t)}{a W(t)}\, dt$$

Where $W(t)$ is the Wronskian and $W_i(t)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(t)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(t)$. This is given by

$$W(t) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \mathrm{e}^{-t} & \mathrm{e}^{t} & t\,\mathrm{e}^{t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{t} & \mathrm{e}^{t}(t+1) \\ \mathrm{e}^{-t} & \mathrm{e}^{t} & \mathrm{e}^{t}(t+2) \end{bmatrix}$$

$$|W| = 4\,\mathrm{e}^{-t} \mathrm{e}^{2t}$$

The determinant simplifies to

$$|W| = 4\,\mathrm{e}^t$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(t) &= \det \begin{bmatrix} \mathrm{e}^t & t\,\mathrm{e}^t \\ \mathrm{e}^t & \mathrm{e}^t(t+1) \end{bmatrix} \\ &= \mathrm{e}^{2t} \end{aligned}$$

$$\begin{aligned} W_2(t) &= \det \begin{bmatrix} \mathrm{e}^{-t} & t\,\mathrm{e}^t \\ -\mathrm{e}^{-t} & \mathrm{e}^t(t+1) \end{bmatrix} \\ &= 2t+1 \end{aligned}$$

$$\begin{aligned} W_3(t) &= \det \begin{bmatrix} \mathrm{e}^{-t} & \mathrm{e}^t \\ -\mathrm{e}^{-t} & \mathrm{e}^t \end{bmatrix} \\ &= 2 \end{aligned}$$

Now we are ready to evaluate each $U_i(t)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(t) W_1(t)}{a W(t)}\, dt \\ &= (-1)^2 \int \frac{(g(t))\,(\mathrm{e}^{2t})}{(1)\,(4\,\mathrm{e}^t)}\, dt \\ &= \int \frac{g(t)\,\mathrm{e}^{2t}}{4\,\mathrm{e}^t}\, dt \\ &= \int \left(\frac{g(t)\,\mathrm{e}^t}{4} \right)\, dt \\ &= \int \frac{g(t)\,\mathrm{e}^t}{4} dt \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(t)W_2(t)}{aW(t)}\, dt \\
&= (-1)^1 \int \frac{(g(t))\,(2t+1)}{(1)\,(4\,\mathrm{e}^t)}\, dt \\
&= -\int \frac{g(t)\,(2t+1)}{4\,\mathrm{e}^t}\, dt \\
&= -\int \left(\frac{g(t)\,(2t+1)\,\mathrm{e}^{-t}}{4}\right) dt \\
&= -\left(\int \frac{g(t)\,(2t+1)\,\mathrm{e}^{-t}}{4} dt\right)
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(t)W_3(t)}{aW(t)}\, dt \\
&= (-1)^0 \int \frac{(g(t))\,(2)}{(1)\,(4\,\mathrm{e}^t)}\, dt \\
&= \int \frac{2g(t)}{4\,\mathrm{e}^t}\, dt \\
&= \int \left(\frac{g(t)\,\mathrm{e}^{-t}}{2}\right) dt \\
&= \int \frac{g(t)\,\mathrm{e}^{-t}}{2} dt
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\int \frac{g(t)\,\mathrm{e}^t}{4} dt\right) \left(\mathrm{e}^{-t}\right) \\
&+ \left(-\left(\int \frac{g(t)\,(2t+1)\,\mathrm{e}^{-t}}{4} dt\right)\right) \left(\mathrm{e}^t\right) \\
&+ \left(\int \frac{g(t)\,\mathrm{e}^{-t}}{2} dt\right) \left(t\,\mathrm{e}^t\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\left(\int g(t)\,\mathrm{e}^t dt\right)\mathrm{e}^{-t}}{4} - \frac{\left(\int g(t)\,(2t+1)\,\mathrm{e}^{-t} dt\right)\mathrm{e}^t}{4} + \frac{\left(\int g(t)\,\mathrm{e}^{-t} dt\right) t\,\mathrm{e}^t}{2}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{-t} c_1 + c_2 \mathrm{e}^{t} + t\,\mathrm{e}^{t} c_3\right) \\
&\quad + \left(\frac{\left(\int g(t)\,\mathrm{e}^{t} dt\right) \mathrm{e}^{-t}}{4} - \frac{\left(\int g(t)\,(2t+1)\,\mathrm{e}^{-t} dt\right) \mathrm{e}^{t}}{4} + \frac{\left(\int g(t)\,\mathrm{e}^{-t} dt\right) t\,\mathrm{e}^{t}}{2}\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{-t} c_1 + \mathrm{e}^{t}(c_3 t + c_2) + \frac{\left(\int g(t)\,\mathrm{e}^{t} dt\right) \mathrm{e}^{-t}}{4} - \frac{\left(\int g(t)\,(2t+1)\,\mathrm{e}^{-t} dt\right) \mathrm{e}^{t}}{4} + \frac{\left(\int g(t)\,\mathrm{e}^{-t} dt\right) t\,\mathrm{e}^{t}}{2}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = \mathrm{e}^{-t} c_1 + \mathrm{e}^{t}(c_3 t + c_2) + \frac{\left(\int g(t)\,\mathrm{e}^{t} dt\right) \mathrm{e}^{-t}}{4} \\
- \frac{\left(\int g(t)\,(2t+1)\,\mathrm{e}^{-t} dt\right) \mathrm{e}^{t}}{4} + \frac{\left(\int g(t)\,\mathrm{e}^{-t} dt\right) t\,\mathrm{e}^{t}}{2}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{-t} c_1 + \mathrm{e}^{t}(c_3 t + c_2) + \frac{\left(\int g(t)\,\mathrm{e}^{t} dt\right) \mathrm{e}^{-t}}{4} - \frac{\left(\int g(t)\,(2t+1)\,\mathrm{e}^{-t} dt\right) \mathrm{e}^{t}}{4} + \frac{\left(\int g(t)\,\mathrm{e}^{-t} dt\right) t\,\mathrm{e}^{t}}{2}
$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 64

```
dsolve(diff(y(t),t$3)-diff(y(t),t$2)-diff(y(t),t)+y(t)=g(t),y(t), singsol=all)
```

$$y(t) = -\frac{\left(\int (2t+1)\, g(t)\,\mathrm{e}^{-t} dt\right)\mathrm{e}^{t}}{4} + \frac{\left(\int \mathrm{e}^{-t} g(t)\, dt\right)\mathrm{e}^{t} t}{2} + \frac{\left(\int \mathrm{e}^{t} g(t)\, dt\right)\mathrm{e}^{-t}}{4} + c_2\mathrm{e}^{-t} + \mathrm{e}^{t}(c_3 t + c_1)$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 106

```
DSolve[y'''[t]-y''[t]-y'[t]+y[t]==g[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t}\int_1^t \frac{1}{4} e^{K[1]} g(K[1]) dK[1] + e^t t\int_1^t \frac{1}{2} e^{-K[3]} g(K[3]) dK[3] + e^t\int_1^t -\frac{1}{4} e^{-K[2]} g(K[2])(2K[2]+1) dK[2] + c_1 e^{-t} + c_2 e^t + c_3 e^t t$$

3.8 problem Example 3.37

Internal problem ID [5864]
Internal file name [`OUTPUT/5112_Sunday_June_05_2022_03_25_02_PM_40711304/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.37.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y^{(5)} - \frac{y''''}{t} = 0}$$

Since y is missing from the ode then we can use the substitution $y' = v(t)$ to reduce the order by one. The ODE becomes

$$-v''''(t)\, t + v'''(t) = 0$$

Since $v(t)$ is missing from the ode then we can use the substitution $v'(t) = w(t)$ to reduce the order by one. The ODE becomes

$$-w'''(t)\, t + w''(t) = 0$$

Since $w(t)$ is missing from the ode then we can use the substitution $w'(t) = r(t)$ to reduce the order by one. The ODE becomes

$$-r''(t)\, t + r'(t) = 0$$

Integrating both sides of the ODE w.r.t t gives

$$\int (-r''(t)\, t + r'(t))\, dt = 0$$
$$-r'(t)\, t + 2r(t) = c_1$$

Which is now solved for $r(t)$. In canonical form the ODE is

$$\begin{aligned} r' &= F(t, r) \\ &= f(t)g(r) \\ &= \frac{2r - c_1}{t} \end{aligned}$$

Where $f(t) = \frac{1}{t}$ and $g(r) = 2r - c_1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{2r - c_1}\, dr &= \frac{1}{t}\, dt \\ \int \frac{1}{2r - c_1}\, dr &= \int \frac{1}{t}\, dt \\ \frac{\ln(-2r + c_1)}{2} &= \ln(t) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{-2r + c_1} = e^{\ln(t) + c_2}$$

Which simplifies to

$$\sqrt{-2r + c_1} = c_3 t$$

But since $w'(t) = r(t)$ then we now need to solve the ode $w'(t) = -\frac{c_3^2 e^{2c_2} t^2}{2} + \frac{c_1}{2}$. Integrating both sides gives

$$\begin{aligned} w(t) &= \int -\frac{c_3^2 e^{2c_2} t^2}{2} + \frac{c_1}{2}\, dt \\ &= -\frac{c_3^2 e^{2c_2} t^3}{6} + \frac{c_1 t}{2} + c_4 \end{aligned}$$

But since $v'(t) = w(t)$ then we now need to solve the ode $v'(t) = -\frac{c_3^2 e^{2c_2} t^3}{6} + \frac{c_1 t}{2} + c_4$. Integrating both sides gives

$$\begin{aligned} v(t) &= \int -\frac{c_3^2 e^{2c_2} t^3}{6} + \frac{c_1 t}{2} + c_4\, dt \\ &= -\frac{c_3^2 e^{2c_2} t^4}{24} + \frac{c_1 t^2}{4} + c_4 t + c_5 \end{aligned}$$

But since $y' = v(t)$ then we now need to solve the ode $y' = -\frac{c_3^2 e^{2c_2} t^4}{24} + \frac{c_1 t^2}{4} + c_4 t + c_5$. Integrating both sides gives

$$
\begin{aligned}
y &= \int -\frac{c_3^2 e^{2c_2} t^4}{24} + \frac{c_1 t^2}{4} + c_4 t + c_5 \, \mathrm{d}t \\
&= -\frac{c_3^2 e^{2c_2} t^5}{120} + \frac{c_1 t^3}{12} + \frac{c_4 t^2}{2} + t c_5 + c_6
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{c_3^2 e^{2c_2} t^5}{120} + \frac{c_1 t^3}{12} + \frac{c_4 t^2}{2} + t c_5 + c_6 \tag{1}
$$

Verification of solutions

$$
y = -\frac{c_3^2 e^{2c_2} t^5}{120} + \frac{c_1 t^3}{12} + \frac{c_4 t^2}{2} + t c_5 + c_6
$$

Verified OK.

3.8.1 Maple step by step solution

Let's solve

$-y^{(5)} t + y'''' = 0$

- Highest derivative means the order of the ODE is 5

  $y^{(5)}$

- Isolate 5th derivative

  $y^{(5)} = \frac{y''''}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y^{(5)} - \frac{y''''}{t} = 0$

- Multiply by denominators of the ODE

  $y^{(5)} t - y'''' = 0$

- Make a change of variables

  $s = \ln(t)$

- □ Substitute the change of variables back into the ODE

  - ○ Calculate the 1st derivative of y with respect to t , using the chain rule

    $y' = \left(\frac{d}{ds} y(s)\right) s'(t)$

- Compute derivative

  $y' = \frac{\frac{d}{ds}y(s)}{t}$

- Calculate the 2nd derivative of y with respect to t , using the chain rule

  $y'' = \left(\frac{d^2}{ds^2}y(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}y(s)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

- Calculate the 3rd derivative of y with respect to t , using the chain rule

  $y''' = \left(\frac{d^3}{ds^3}y(s)\right) s'(t)^3 + 3s'(t)\, s''(t) \left(\frac{d^2}{ds^2}y(s)\right) + s'''(t) \left(\frac{d}{ds}y(s)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{ds^3}y(s)}{t^3} - \frac{3\left(\frac{d^2}{ds^2}y(s)\right)}{t^3} + \frac{2\left(\frac{d}{ds}y(s)\right)}{t^3}$

- Calculate the 4th derivative of y with respect to t , using the chain rule

  $y'''' = \left(\frac{d^4}{ds^4}y(s)\right) s'(t)^4 + 3s'(t)^2 s''(t) \left(\frac{d^3}{ds^3}y(s)\right) + 3s''(t)^2 \left(\frac{d^2}{ds^2}y(s)\right) + 3\left(s'''(t) \left(\frac{d^2}{ds^2}y(s)\right) + \left(\frac{d^3}{ds^3}y\right.\right.$

- Compute derivative

  $y'''' = \frac{\frac{d^4}{ds^4}y(s)}{t^4} - \frac{3\left(\frac{d^3}{ds^3}y(s)\right)}{t^4} + \frac{5\left(\frac{d^2}{ds^2}y(s)\right)}{t^4} + \frac{3\left(\frac{2\left(\frac{d^2}{ds^2}y(s)\right)}{t^3} - \frac{\frac{d^3}{ds^3}y(s)}{t^3}\right)}{t} - \frac{6\left(\frac{d}{ds}y(s)\right)}{t^4}$

- Calculate the 5th derivative of y with respect to t , using the chain rule

  $y^{(5)} = \left(\frac{d^5}{ds^5}y(s)\right) s'(t)^5 + 4s'(t)^3 s''(t) \left(\frac{d^4}{ds^4}y(s)\right) + 9s'(t)\, s''(t)^2 \left(\frac{d^3}{ds^3}y(s)\right) + 3\left(s'''(t) \left(\frac{d^3}{ds^3}y(s)\right) + \left(\right.\right.$

- Compute derivative

  $y^{(5)} = \frac{\frac{d^5}{ds^5}y(s)}{t^5} - \frac{4\left(\frac{d^4}{ds^4}y(s)\right)}{t^5} + \frac{11\left(\frac{d^3}{ds^3}y(s)\right)}{t^5} + \frac{3\left(\frac{2\left(\frac{d^3}{ds^3}y(s)\right)}{t^3} - \frac{\frac{d^4}{ds^4}y(s)}{t^3}\right)}{t^2} - \frac{26\left(\frac{d^2}{ds^2}y(s)\right)}{t^5} + \frac{3\left(-\frac{6\left(\frac{d^2}{ds^2}y(s)\right)}{t^4} + \frac{5\left(\frac{d^3}{ds^3}y\right.}{t^4}\right.}{t}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^5}{ds^5}y(s)}{t^5} - \frac{4\left(\frac{d^4}{ds^4}y(s)\right)}{t^5} + \frac{11\left(\frac{d^3}{ds^3}y(s)\right)}{t^5} + \frac{3\left(\frac{2\left(\frac{d^3}{ds^3}y(s)\right)}{t^3} - \frac{\frac{d^4}{ds^4}y(s)}{t^3}\right)}{t^2} - \frac{26\left(\frac{d^2}{ds^2}y(s)\right)}{t^5} + \frac{3\left(-\frac{6\left(\frac{d^2}{ds^2}y(s)\right)}{t^4} + \frac{5\left(\frac{d^3}{ds^3}y(s)\right)}{t^4} - \right.}{t}\right.$

- Simplify

  $\frac{\frac{d^5}{ds^5}y(s) - 11\frac{d^4}{ds^4}y(s) + 41\frac{d^3}{ds^3}y(s) - 61\frac{d^2}{ds^2}y(s) + 30\frac{d}{ds}y(s)}{t^4} = 0$

- Isolate 5th derivative

$\frac{d^5}{ds^5}y(s) = 11\frac{d^4}{ds^4}y(s) - 41\frac{d^3}{ds^3}y(s) + 61\frac{d^2}{ds^2}y(s) - 30\frac{d}{ds}y(s)$

- Group terms with $y(s)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^5}{ds^5}y(s) - 11\frac{d^4}{ds^4}y(s) + 41\frac{d^3}{ds^3}y(s) - 61\frac{d^2}{ds^2}y(s) + 30\frac{d}{ds}y(s) = 0$

☐ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(s)$

    $y_1(s) = y(s)$

  - Define new variable $y_2(s)$

    $y_2(s) = \frac{d}{ds}y(s)$

  - Define new variable $y_3(s)$

    $y_3(s) = \frac{d^2}{ds^2}y(s)$

  - Define new variable $y_4(s)$

    $y_4(s) = \frac{d^3}{ds^3}y(s)$

  - Define new variable $y_5(s)$

    $y_5(s) = \frac{d^4}{ds^4}y(s)$

  - Isolate for $\frac{d}{ds}y_5(s)$ using original ODE

    $\frac{d}{ds}y_5(s) = 11y_5(s) - 41y_4(s) + 61y_3(s) - 30y_2(s)$

    Convert linear ODE into a system of first order ODEs

    $[y_2(s) = \frac{d}{ds}y_1(s), y_3(s) = \frac{d}{ds}y_2(s), y_4(s) = \frac{d}{ds}y_3(s), y_5(s) = \frac{d}{ds}y_4(s), \frac{d}{ds}y_5(s) = 11y_5(s) - 41y_4(s)$

- Define vector

$$\vec{y}(s) = \begin{bmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \\ y_4(s) \\ y_5(s) \end{bmatrix}$$

- System to solve

$$\frac{d}{ds}\vec{y}(s) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -30 & 61 & -41 & 11 \end{bmatrix} \cdot \vec{y}(s)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -30 & 61 & -41 & 11 \end{bmatrix}$$

- Rewrite the system as

$\frac{d}{ds}\vec{y}(s) = A \cdot \vec{y}(s)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} \frac{1}{625} \\ \frac{1}{125} \\ \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^s \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2s} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{3s} \cdot \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[5, \begin{bmatrix} \frac{1}{625} \\ \frac{1}{125} \\ \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_5 = e^{5s} \cdot \begin{bmatrix} \frac{1}{625} \\ \frac{1}{125} \\ \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4 + c_5 \vec{y}_5$$

- Substitute solutions into the general solution

$$\vec{y} = c_2 e^{s} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{2s} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_4 e^{3s} \cdot \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + c_5 e^{5s} \cdot \begin{bmatrix} \frac{1}{625} \\ \frac{1}{125} \\ \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y(s) = c_2 e^{s} + \frac{c_3 e^{2s}}{16} + \frac{c_4 e^{3s}}{81} + \frac{c_5 e^{5s}}{625} + c_1$$

- Change variables back using $s = \ln(t)$

$$y = c_2 t + \frac{1}{16} c_3 t^2 + \frac{1}{81} c_4 t^3 + \frac{1}{625} c_5 t^5 + c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(diff(y(t),t$5)-1/t*diff(y(t),t$4)=0,y(t), singsol=all)
```

$$y(t) = c_3 t^5 + c_5 t^3 + c_2 t^2 + c_4 t + c_1$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 33

```
DSolve[y'''''[t]-1/t*y''''[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow \frac{c_1 t^5}{120} + c_5 t^3 + c_4 t^2 + c_3 t + c_2$$

3.9 problem Example 3.38

Internal problem ID [5865]
Internal file name [`OUTPUT/5113_Sunday_June_05_2022_03_25_03_PM_89277202/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{x x'' - x'^2 = 0}$$

3.9.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable x an independent variable. Using

$$x' = p(x)$$

Then

$$\begin{aligned} x'' &= \frac{dp}{dt} \\ &= \frac{dx}{dt}\frac{dp}{dx} \\ &= p\frac{dp}{dx} \end{aligned}$$

Hence the ode becomes

$$x p(x)\left(\frac{d}{dx}p(x)\right)-p(x)^2=0$$

Which is now solved as first order ode for $p(x)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(x,p)\\
&= f(x)g(p)\\
&= \frac{p}{x}
\end{aligned}$$

Where $f(x)=\frac{1}{x}$ and $g(p)=p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= \frac{1}{x}\,dx\\
\int \frac{1}{p}\,dp &= \int \frac{1}{x}\,dx\\
\ln(p) &= \ln(x)+c_1\\
p &= \mathrm{e}^{\ln(x)+c_1}\\
&= c_1 x
\end{aligned}$$

For solution (1) found earlier, since $p=x'$ then we now have a new first order ode to solve which is

$$x' = c_1 x$$

Integrating both sides gives

$$\begin{aligned}
\int \frac{1}{c_1 x}dx &= \int dt\\
\frac{\ln(x)}{c_1} &= t+c_2
\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\frac{\ln(x)}{c_1}} = \mathrm{e}^{t+c_2}$$

Which simplifies to

$$x^{\frac{1}{c_1}} = c_3\mathrm{e}^t$$

<u>Summary</u>

The solution(s) found are the following

$$x = \left(c_3\mathrm{e}^t\right)^{c_1} \tag{1}$$

<u>Verification of solutions</u>

$$x = \left(c_3\mathrm{e}^t\right)^{c_1}$$

Verified OK.

3.9.2 Maple step by step solution

Let's solve

$xx'' - x'^2 = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Define new dependent variable u

  $u(t) = x'$

- Compute x''

  $u'(t) = x''$

- Use chain rule on the lhs

  $x'\left(\frac{d}{dx}u(x)\right) = x''$

- Substitute in the definition of u

  $u(x)\left(\frac{d}{dx}u(x)\right) = x''$

- Make substitutions $x' = u(x)\,, x'' = u(x)\left(\frac{d}{dx}u(x)\right)$ to reduce order of ODE

  $xu(x)\left(\frac{d}{dx}u(x)\right) - u(x)^2 = 0$

- Separate variables

  $\frac{\frac{d}{dx}u(x)}{u(x)} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}u(x)}{u(x)}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $\ln\left(u(x)\right) = \ln\left(x\right) + c_1$

- Solve for $u(x)$

  $u(x) = x\,\mathrm{e}^{c_1}$

- Solve 1st ODE for $u(x)$

  $u(x) = x\,\mathrm{e}^{c_1}$

- Revert to original variables with substitution $u(x) = x', x = x$

  $x' = x\,\mathrm{e}^{c_1}$

- Separate variables

  $\frac{x'}{x} = \mathrm{e}^{c_1}$

- Integrate both sides with respect to t

  $\int \frac{x'}{x} dt = \int e^{c_1} dt + c_2$
- Evaluate integral

  $\ln(x) = t\,e^{c_1} + c_2$
- Solve for x

  $x = e^{t\,e^{c_1}+c_2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 14

```
dsolve(x(t)*diff(x(t),t$2)-diff(x(t),t)^2=0,x(t), singsol=all)
```

$$x(t) = 0$$
$$x(t) = e^{c_1 t} c_2$$

✓ Solution by Mathematica

Time used: 0.108 (sec). Leaf size: 14

```
DSolve[x[t]*x''[t]-(x'[t])^2==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_2 e^{c_1 t}$$

3.10 problem Example 3.39

Internal problem ID [5866]
Internal file name [`OUTPUT/5114_Sunday_June_05_2022_03_25_07_PM_84169406/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.39.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 4y''' + 3y'' - 4y' - 4y = f(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 4y''' + 3y'' - 4y' - 4y = 0$$

The characteristic equation is

$$\lambda^4 + 4\lambda^3 + 3\lambda^2 - 4\lambda - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = -2$$
$$\lambda_4 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x} + x\,e^{-2x} c_3 + c_4 e^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= e^{-x} \\
y_2 &= e^{-2x} \\
y_3 &= x\,e^{-2x} \\
y_4 &= e^{x}
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 4y''' + 3y'' - 4y' - 4y = f(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{-x} & e^{-2x} & x\,e^{-2x} & e^{x} \\ -e^{-x} & -2\,e^{-2x} & e^{-2x}(1-2x) & e^{x} \\ e^{-x} & 4\,e^{-2x} & 4\,e^{-2x}(x-1) & e^{x} \\ -e^{-x} & -8\,e^{-2x} & (-8x+12)\,e^{-2x} & e^{x} \end{bmatrix}$$

$$|W| = 18\,e^{-x} e^{-4x} e^{x}$$

The determinant simplifies to

$$|W| = 18\,e^{-4x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det\begin{bmatrix} e^{-2x} & x\,e^{-2x} & e^{x} \\ -2\,e^{-2x} & e^{-2x}(1-2x) & e^{x} \\ 4\,e^{-2x} & 4\,e^{-2x}(x-1) & e^{x} \end{bmatrix}$$
$$= 9\,e^{-3x}$$

$$W_2(x) = \det\begin{bmatrix} e^{-x} & x\,e^{-2x} & e^{x} \\ -e^{-x} & e^{-2x}(1-2x) & e^{x} \\ e^{-x} & 4\,e^{-2x}(x-1) & e^{x} \end{bmatrix}$$
$$= (-6x+8)\,e^{-2x}$$

$$W_3(x) = \det\begin{bmatrix} e^{-x} & e^{-2x} & e^{x} \\ -e^{-x} & -2\,e^{-2x} & e^{x} \\ e^{-x} & 4\,e^{-2x} & e^{x} \end{bmatrix}$$
$$= -6\,e^{-2x}$$

$$W_4(x) = \det\begin{bmatrix} e^{-x} & e^{-2x} & x\,e^{-2x} \\ -e^{-x} & -2\,e^{-2x} & e^{-2x}(1-2x) \\ e^{-x} & 4\,e^{-2x} & 4\,e^{-2x}(x-1) \end{bmatrix}$$
$$= e^{-5x}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\
&= (-1)^3 \int \frac{(f(x))\,(9\,\mathrm{e}^{-3x})}{(1)\,(18\,\mathrm{e}^{-4x})}\, dx \\
&= -\int \frac{9f(x)\,\mathrm{e}^{-3x}}{18\,\mathrm{e}^{-4x}}\, dx \\
&= -\int \left(\frac{f(x)\,\mathrm{e}^{x}}{2}\right) dx \\
&= -\left(\int \frac{f(x)\,\mathrm{e}^{x}}{2} dx\right) \\
&= -\left(\int \frac{f(x)\,\mathrm{e}^{x}}{2} dx\right)
\end{aligned}$$

$$\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\
&= (-1)^2 \int \frac{(f(x))\,((-6x+8)\,\mathrm{e}^{-2x})}{(1)\,(18\,\mathrm{e}^{-4x})}\, dx \\
&= \int \frac{f(x)\,(-6x+8)\,\mathrm{e}^{-2x}}{18\,\mathrm{e}^{-4x}}\, dx \\
&= \int \left(-\frac{(3x-4)\,f(x)\,\mathrm{e}^{2x}}{9}\right) dx \\
&= \int -\frac{(3x-4)\,f(x)\,\mathrm{e}^{2x}}{9} dx
\end{aligned}$$

$$\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^1 \int \frac{(f(x))\,(-6\,\mathrm{e}^{-2x})}{(1)\,(18\,\mathrm{e}^{-4x})}\, dx \\
&= -\int \frac{-6f(x)\,\mathrm{e}^{-2x}}{18\,\mathrm{e}^{-4x}}\, dx \\
&= -\int \left(-\frac{f(x)\,\mathrm{e}^{2x}}{3}\right) dx \\
&= -\left(\int -\frac{f(x)\,\mathrm{e}^{2x}}{3} dx\right) \\
&= -\left(\int -\frac{f(x)\,\mathrm{e}^{2x}}{3} dx\right)
\end{aligned}$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\, dx \\
&= (-1)^{0} \int \frac{(f(x))\left(\mathrm{e}^{-5x}\right)}{(1)\left(18\,\mathrm{e}^{-4x}\right)}\, dx \\
&= \int \frac{f(x)\,\mathrm{e}^{-5x}}{18\,\mathrm{e}^{-4x}}\, dx \\
&= \int \left(\frac{f(x)\,\mathrm{e}^{-x}}{18}\right) dx \\
&= \int \frac{f(x)\,\mathrm{e}^{-x}}{18} dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\left(\int \frac{f(x)\,\mathrm{e}^{x}}{2} dx\right)\right)\left(\mathrm{e}^{-x}\right) \\
&+ \left(\int -\frac{(3x-4)\, f(x)\,\mathrm{e}^{2x}}{9} dx\right)\left(\mathrm{e}^{-2x}\right) \\
&+ \left(-\left(\int -\frac{f(x)\,\mathrm{e}^{2x}}{3} dx\right)\right)\left(x\,\mathrm{e}^{-2x}\right) \\
&+ \left(\int \frac{f(x)\,\mathrm{e}^{-x}}{18} dx\right)\left(\mathrm{e}^{x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\left(\left(\int f(x)\,\mathrm{e}^{-x} dx\right)\mathrm{e}^{3x} - 9\left(\int f(x)\,\mathrm{e}^{x} dx\right)\mathrm{e}^{x} + 6x\left(\int f(x)\,\mathrm{e}^{2x} dx\right) - 2\left(\int (3x-4)\, f(x)\,\mathrm{e}^{2x} dx\right)\right)\mathrm{e}^{-2x}}{18}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_3 + c_4 \mathrm{e}^{x}\right) \\
&\quad + \left(\frac{\left(\left(\int f(x)\,\mathrm{e}^{-x} dx\right)\mathrm{e}^{3x} - 9\left(\int f(x)\,\mathrm{e}^{x} dx\right)\mathrm{e}^{x} + 6x\left(\int f(x)\,\mathrm{e}^{2x} dx\right) - 2\left(\int (3x-4)\, f(x)\,\mathrm{e}^{2x} dx\right)\right)\mathrm{e}^{-2x}}{18}\right)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y = &\left(c_4 \mathrm{e}^{3x} + c_1 \mathrm{e}^{x} + c_3 x + c_2\right)\mathrm{e}^{-2x} \\
&+ \frac{\left(\left(\int f(x)\,\mathrm{e}^{-x} dx\right)\mathrm{e}^{3x} - 9\left(\int f(x)\,\mathrm{e}^{x} dx\right)\mathrm{e}^{x} + 6x\left(\int f(x)\,\mathrm{e}^{2x} dx\right) - 2\left(\int (3x-4)\, f(x)\,\mathrm{e}^{2x} dx\right)\right)\mathrm{e}^{-2x}}{18}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = (c_4 e^{3x} + c_1 e^{x} + c_3 x + c_2) e^{-2x} + \frac{\left(\left(\int f(x) e^{-x} dx\right) e^{3x} - 9\left(\int f(x) e^{x} dx\right) e^{x} + 6x\left(\int f(x) e^{2x} dx\right) - 2\left(\int (3x-4) f(x) e^{2x} dx\right)\right) e^{-2x}}{18} \tag{1}$$

Verification of solutions

$$y = (c_4 e^{3x} + c_1 e^{x} + c_3 x + c_2) e^{-2x} + \frac{\left(\left(\int f(x) e^{-x} dx\right) e^{3x} - 9\left(\int f(x) e^{x} dx\right) e^{x} + 6x\left(\int f(x) e^{2x} dx\right) - 2\left(\int (3x-4) f(x) e^{2x} dx\right)\right) e^{-2x}}{18}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 83

```
dsolve(diff(y(x),x$4)+4*diff(y(x),x$3)+3*diff(y(x),x$2)-4*diff(y(x),x)-4*y(x)=f(x),y(x), sin
```

$$y(x) = \frac{\left(\left(\int f(x) e^{-x} dx\right) e^{3x} + 18 e^{3x} c_1 - 9\left(\int f(x) e^{x} dx\right) e^{x} + 6x\left(\int f(x) e^{2x} dx\right) + 18 c_3 e^{x} + 18 c_4 x - 2\left(\int f(x) (3\right.}{18}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 128

```
DSolve[y''''[x]+4*y'''[x]+3*y''[x]-4*y'[x]-4*y[x]==f[x],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{-2x}\left(x\int_1^x \frac{1}{3}e^{2K[2]}f(K[2])dK[2] + e^x\int_1^x -\frac{1}{2}e^{K[3]}f(K[3])dK[3] + e^{3x}\int_1^x \frac{1}{18}e^{-K[4]}f(K[4])dK[4] + \int_1^x -\frac{1}{9}e^{2K[1]}f(K[1])(3K[1]-4)dK[1] + c_2x + c_3e^x + c_4e^{3x} + c_1\right)$$

3.11 problem Example 3.40

Internal problem ID [5867]
Internal file name [`OUTPUT/5115_Sunday_June_05_2022_03_25_09_PM_21333835/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' - (1 + 2x) u' + (x^2 + x - 1) u = 0}$$

3.11.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$u'' + p(x) u' + q(x) u = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -1 - 2x \\ q(x) &= x^2 + x - 1 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= x^2 + x - 1 - \frac{(-1-2x)'}{2} - \frac{(-1-2x)^2}{4} \\
&= x^2 + x - 1 - \frac{(-2)}{2} - \frac{((-1-2x)^2)}{4} \\
&= x^2 + x - 1 - (-1) - \frac{(-1-2x)^2}{4} \\
&= -\frac{1}{4}
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
u = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-1-2x}{2}} \\
&= \mathrm{e}^{\frac{x(1+x)}{2}}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
u = v(x)\, \mathrm{e}^{\frac{x(1+x)}{2}} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
\mathrm{e}^{\frac{x(1+x)}{2}} (4v''(x) - v(x)) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 4, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
4\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
4\lambda^2 - 1 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{0^2 - (4)(4)(-1)} \\ &= \pm\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\frac{1}{2} \\ \lambda_2 &= -\frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{1}{2} \\ \lambda_2 &= -\frac{1}{2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ v(x) &= c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{\left(-\frac{1}{2}\right)x}\end{aligned}$$

Or

$$v(x) = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Now that $v(x)$ is known, then

$$\begin{aligned}u &= v(x)\, z(x) \\ &= \left(c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}\right)(z(x)) \qquad (7)\end{aligned}$$

But from (5)

$$z(x) = \mathrm{e}^{\frac{x(1+x)}{2}}$$

Hence (7) becomes

$$u = \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}}\right) e^{\frac{x(1+x)}{2}}$$

Summary

The solution(s) found are the following

$$u = \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}}\right) e^{\frac{x(1+x)}{2}} \tag{1}$$

Verification of solutions

$$u = \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}}\right) e^{\frac{x(1+x)}{2}}$$

Verified OK.

3.11.2 Solving using Kovacic algorithm

Writing the ode as

$$u'' + (-1-2x)\, u' + \left(x^2 + x - 1\right) u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1-2x \\ C &= x^2 + x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 40: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1-2x}{1}\,dx} \\
&= z_1 e^{\frac{1}{2}x+\frac{1}{2}x^2} \\
&= z_1\left(\mathrm{e}^{\frac{x(1+x)}{2}}\right)
\end{aligned}$$

Which simplifies to

$$u_1 = \mathrm{e}^{\frac{x^2}{2}}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{-1-2x}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{x^2+x}}{(u_1)^2}\,dx \\
&= u_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
u &= c_1u_1 + c_2u_2 \\
&= c_1\left(\mathrm{e}^{\frac{x^2}{2}}\right) + c_2\left(\mathrm{e}^{\frac{x^2}{2}}(\mathrm{e}^{x})\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$u = c_1\mathrm{e}^{\frac{x^2}{2}} + c_2\mathrm{e}^{\frac{x(x+2)}{2}} \tag{1}$$

Verification of solutions

$$u = c_1 \mathrm{e}^{\frac{x^2}{2}} + c_2 \mathrm{e}^{\frac{x(x+2)}{2}}$$

Verified OK.

3.11.3 Maple step by step solution

Let's solve

$u'' + (-1-2x)\, u' + (x^2+x-1)\, u = 0$

- Highest derivative means the order of the ODE is 2

  u''

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot u$ to series expansion for $m = 0..2$

  $x^m \cdot u = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- > k-m$

  $x^m \cdot u = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x^m \cdot u'$ to series expansion for $m = 0..1$

  $x^m \cdot u' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

- ○ Shift index using $k- > k+1-m$

  $x^m \cdot u' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

- ○ Convert u'' to series expansion

  $u'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- > k+2$

  $u'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 - a_1 - a_0 + (6a_3 - 2a_2 - 3a_1 + a_0)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - a_{k+1}(k+1) - a_k(2k+1$$

- The coefficients of each power of x must be 0

  $[2a_2 - a_1 - a_0 = 0, 6a_3 - 2a_2 - 3a_1 + a_0 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = \frac{a_1}{2} + \frac{a_0}{2}, a_3 = \frac{2a_1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $k^2 a_{k+2} + (-2a_k - a_{k+1} + 3a_{k+2})\, k - a_k + a_{k-2} + a_{k-1} - a_{k+1} + 2a_{k+2} = 0$

- Shift index using $k- >k+2$

  $(k+2)^2\, a_{k+4} + (-2a_{k+2} - a_{k+3} + 3a_{k+4})\,(k+2) - a_{k+2} + a_k + a_{k+1} - a_{k+3} + 2a_{k+4} = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2ka_{k+2} + ka_{k+3} - a_k - a_{k+1} + 5a_{k+2} + 3a_{k+3}}{k^2 + 7k + 12}, a_2 = \frac{a_1}{2} + \frac{a_0}{2}, a_3 = \frac{2a_1}{3}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(u(x),x$2)-(2*x+1)*diff(u(x),x)+(x^2+x-1)*u(x)=0,u(x), singsol=all)
```

$$u(x) = e^{\frac{x^2}{2}} c_1 + c_2 e^{\frac{x(x+2)}{2}}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 24

```
DSolve[u''[x]-(2*x+1)*u'[x]+(x^2+x-1)*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to e^{\frac{x^2}{2}}(c_2 e^x + c_1)$$

3.12 problem Example 3.41

Internal problem ID [5868]
Internal file name [`OUTPUT/5116_Sunday_June_05_2022_03_25_10_PM_46278530/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 6y' + 9y = 50\,\mathrm{e}^{2x}}$$

3.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 6, C = 9, f(x) = 50\,\mathrm{e}^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(6)^2 - (4)\,(1)\,(9)}\\ &= -3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$y = c_1 e^{-3x} + c_2 x\, e^{-3x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-3x} + x\, e^{-3x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$50\, e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-3x}, e^{-3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$25A_1 \mathrm{e}^{2x} = 50\,\mathrm{e}^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-3x} + x\,\mathrm{e}^{-3x} c_2\right) + \left(2\,\mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-3x}(c_2 x + c_1) + 2\,\mathrm{e}^{2x}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-3x}(c_2 x + c_1) + 2\,\mathrm{e}^{2x} \tag{1}$$

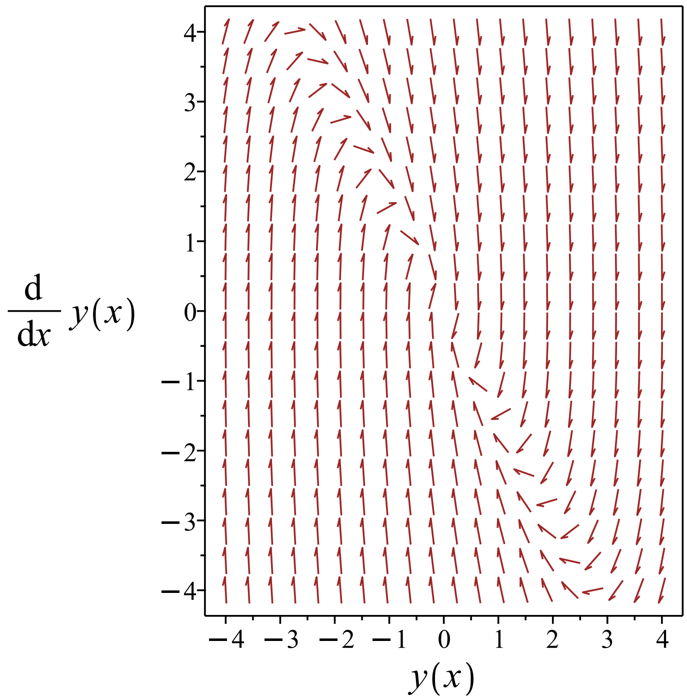


Figure 45: Slope field plot

<u>Verification of solutions</u>

$$y = e^{-3x}(c_2x + c_1) + 2\,e^{2x}$$

Verified OK.

3.12.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 6\,dx} \\ &= e^{3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 50\,e^{3x}e^{2x}$$
$$(e^{3x}y)'' = 50\,e^{3x}e^{2x}$$

Integrating once gives

$$(e^{3x}y)' = 10\,e^{5x} + c_1$$

Integrating again gives

$$(e^{3x}y) = c_1x + 2\,e^{5x} + c_2$$

Hence the solution is

$$y = \frac{c_1x + 2\,e^{5x} + c_2}{e^{3x}}$$

Or

$$y = 2\,e^{2x} + c_1x\,e^{-3x} + c_2e^{-3x}$$

Summary
The solution(s) found are the following

$$y = 2\,e^{2x} + c_1x\,e^{-3x} + c_2e^{-3x} \tag{1}$$

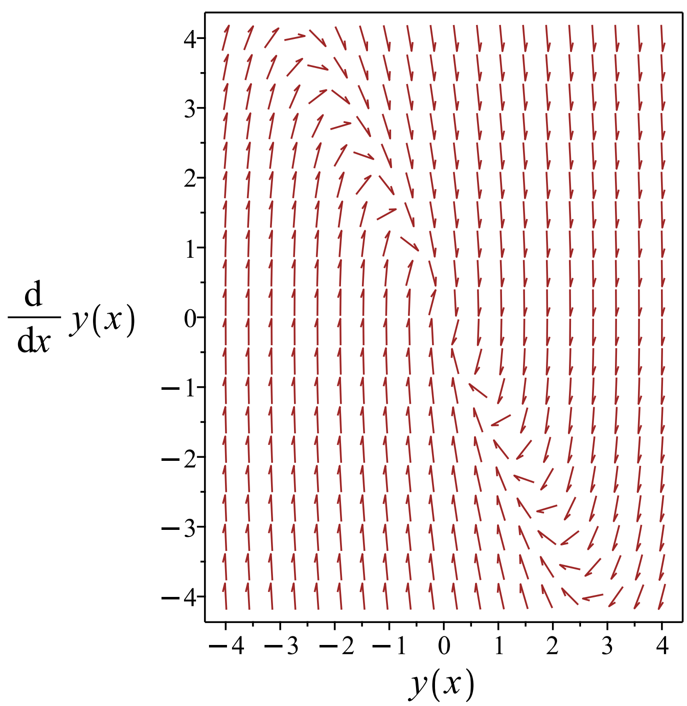


Figure 46: Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{2x} + c_1 x\,\mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x}$$

Verified OK.

3.12.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 42: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{6}{1} \, dx} \\
&= z_1 e^{-3x} \\
&= z_1 \left(\mathrm{e}^{-3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-6x}}{(y_1)^2} \, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-3x} + x\,e^{-3x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$50\,e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{-3x}, e^{-3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$25 A_1 e^{2x} = 50\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + x\,e^{-3x} c_2\right) + \left(2\,e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-3x}(c_2x + c_1) + 2e^{2x}$$

<u>Summary</u>

The solution(s) found are the following

$$y = e^{-3x}(c_2x + c_1) + 2e^{2x} \tag{1}$$

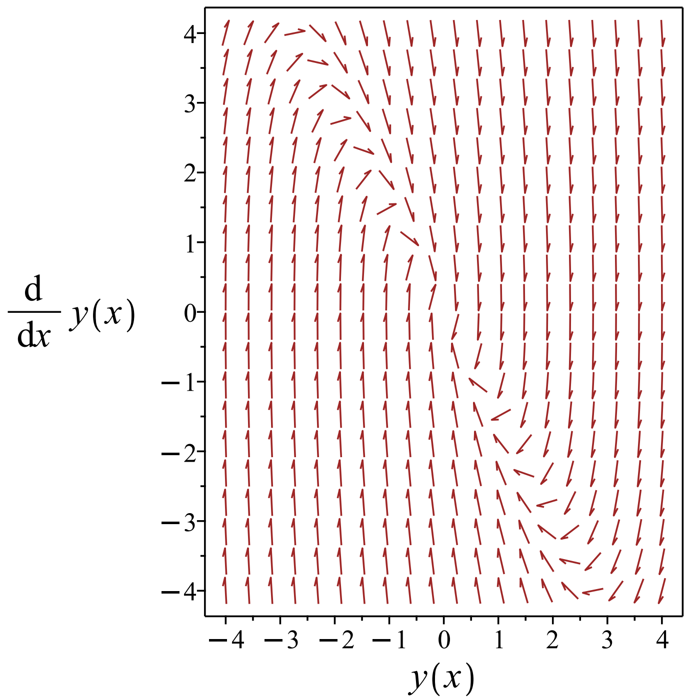


Figure 47: Slope field plot

<u>Verification of solutions</u>

$$y = e^{-3x}(c_2x + c_1) + 2e^{2x}$$

Verified OK.

3.12.4 Maple step by step solution

Let's solve

$y'' + 6y' + 9y = 50\,\mathrm{e}^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r+3)^2 = 0$

- Root of the characteristic polynomial

  $r = -3$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + x\,\mathrm{e}^{-3x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 50\,\mathrm{e}^{2x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & x\,\mathrm{e}^{-3x} \\ -3\,\mathrm{e}^{-3x} & \mathrm{e}^{-3x} - 3x\,\mathrm{e}^{-3x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-6x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -50\,\mathrm{e}^{-3x}\left(\int \mathrm{e}^{5x} x dx - \left(\int \mathrm{e}^{5x} dx\right) x\right)$

- Compute integrals

  $y_p(x) = 2\,e^{2x}$

- Substitute particular solution into general solution to ODE

  $y = x\,e^{-3x}c_2 + c_1 e^{-3x} + 2\,e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+9*y(x)=50*exp(2*x),y(x), singsol=all)
```

$$y(x) = \left(2\,e^{5x} + c_1 x + c_2\right) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 25

```
DSolve[y''[x]+6*y'[x]+9*y[x]==50*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(2e^{5x} + c_2 x + c_1\right)$$

3.13 problem Example 3.42

Internal problem ID [5869]
Internal file name [`OUTPUT/5117_Sunday_June_05_2022_03_25_12_PM_65165470/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4y' + 4y = 50\,e^{2x}}$$

3.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = 50\,e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-4)^2 - (4)(1)(4)}\\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 e^{2x} + c_2 e^{2x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + x\, e^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$50\, e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x} x, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x}x\}]$$

Since $e^{2x}x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1e^{2x} = 50\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 25]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 25x^2e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{2x} + x\,e^{2x}c_2\right) + \left(25x^2e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2x + c_1) + 25x^2e^{2x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{2x}(c_2x + c_1) + 25x^2e^{2x} \tag{1}$$

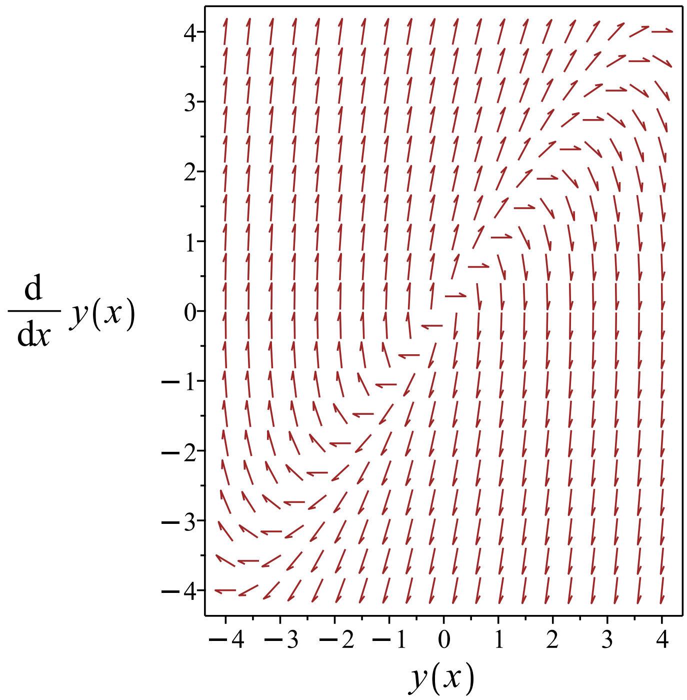


Figure 48: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2 x + c_1) + 25x^2 e^{2x}$$

Verified OK.

3.13.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -4\,dx} \\ &= e^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 50\,e^{-2x}e^{2x}$$
$$\left(y\,e^{-2x}\right)'' = 50\,e^{-2x}e^{2x}$$

Integrating once gives

$$\left(y\,e^{-2x}\right)' = 50x + c_1$$

Integrating again gives

$$\left(y\,e^{-2x}\right) = x(c_1 + 25x) + c_2$$

Hence the solution is

$$y = \frac{x(c_1 + 25x) + c_2}{e^{-2x}}$$

Or

$$y = c_1 x\,e^{2x} + 25x^2e^{2x} + c_2e^{2x}$$

Summary
The solution(s) found are the following

$$y = c_1 x\,e^{2x} + 25x^2e^{2x} + c_2e^{2x} \tag{1}$$

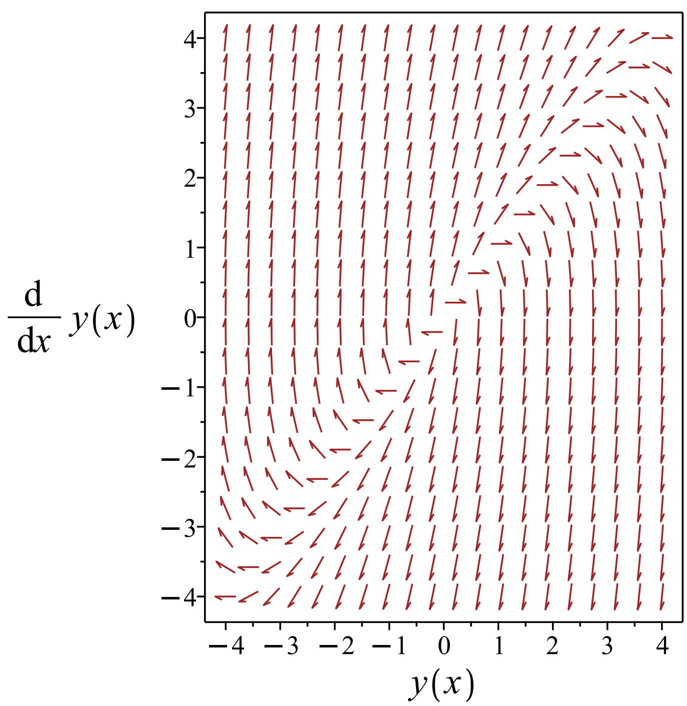


Figure 49: Slope field plot

Verification of solutions

$$y = c_1 x\, e^{2x} + 25x^2 e^{2x} + c_2 e^{2x}$$

Verified OK.

3.13.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 44: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{2x} + x\, e^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$50\, e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}x, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x}x\}]$$

Since $e^{2x}x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{2x} = 50\, e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 25]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 25x^2 e^{2x}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(c_1 e^{2x} + x\, e^{2x} c_2\right) + \left(25x^2 e^{2x}\right)$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) + 25x^2 e^{2x}$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) + 25x^2 e^{2x} \tag{1}$$

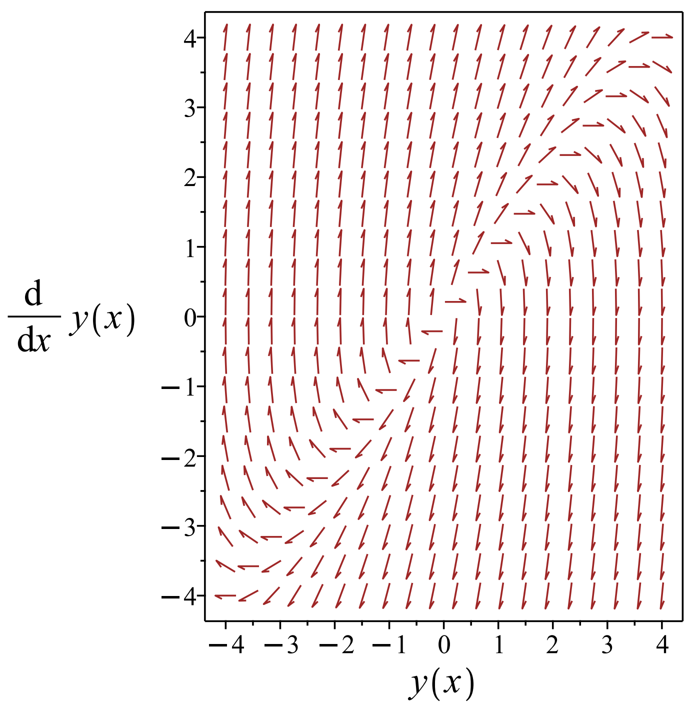


Figure 50: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2 x + c_1) + 25x^2 e^{2x}$$

Verified OK.

3.13.4 Maple step by step solution

Let's solve

$y'' - 4y' + 4y = 50\,\mathrm{e}^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = \mathrm{e}^{2x}x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\mathrm{e}^{2x} + x\,\mathrm{e}^{2x}c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 50\,\mathrm{e}^{2x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{2x} & \mathrm{e}^{2x}x \\ 2\,\mathrm{e}^{2x} & 2\,\mathrm{e}^{2x}x + \mathrm{e}^{2x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{4x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -50\,\mathrm{e}^{2x}\left(\int x dx - \left(\int 1 dx\right) x\right)$

- Compute integrals

  $y_p(x) = 25x^2 e^{2x}$

- Substitute particular solution into general solution to ODE

  $y = x\,e^{2x} c_2 + 25x^2 e^{2x} + c_1 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=50*exp(2*x),y(x), singsol=all)
```

$$y(x) = e^{2x}(c_1 x + 25x^2 + c_2)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 23

```
DSolve[y''[x]-4*y'[x]+4*y[x]==50*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(25x^2 + c_2 x + c_1)$$

3.14 problem Example 3.43

Internal problem ID [5870]
Internal file name [`OUTPUT/5118_Sunday_June_05_2022_03_25_14_PM_98685473/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.43.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = \cos(2x)}$$

3.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = \cos(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 3\lambda\, \mathrm{e}^{\lambda x} + 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \cos(2x) - 2A_2 \sin(2x) - 6A_1 \sin(2x) + 6A_2 \cos(2x) = \cos(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{20}, A_2 = \frac{3}{20}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left(-\frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20} \tag{1}$$

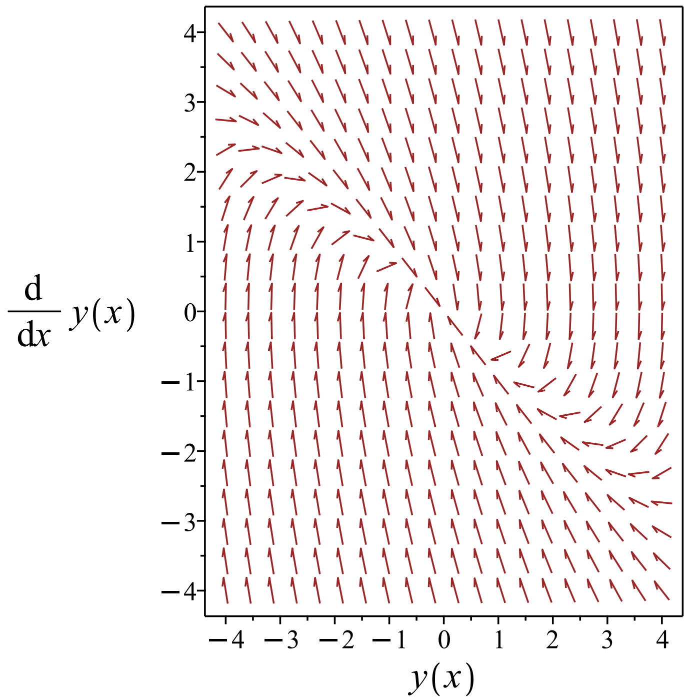


Figure 51: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20}$$

Verified OK.

3.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 46: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(\mathrm{e}^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \cos(2x) - 2A_2 \sin(2x) - 6A_1 \sin(2x) + 6A_2 \cos(2x) = \cos(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{20}, A_2 = \frac{3}{20}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{-x}\right) + \left(-\frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{-x} - \frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20} \tag{1}$$

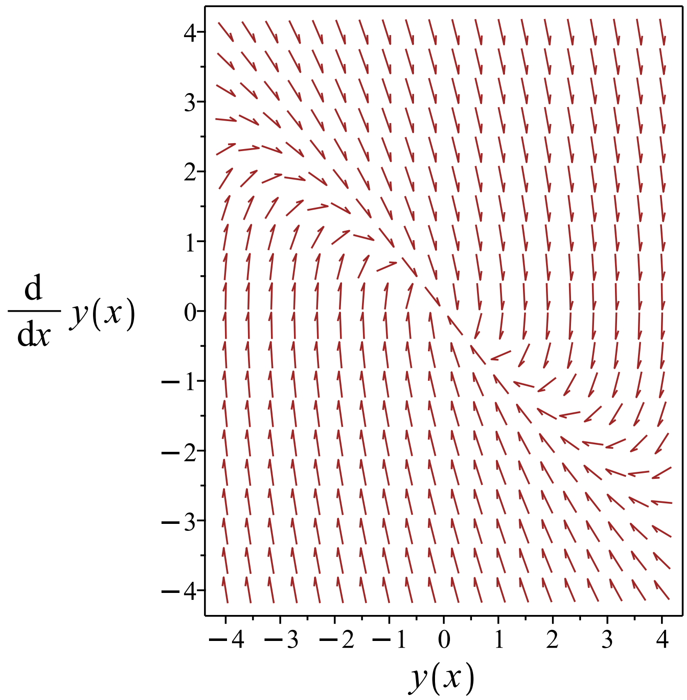


Figure 52: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-x} - \frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20}$$

Verified OK.

3.14.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = \cos(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

$r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cos(2x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-2x} & \mathrm{e}^{-x} \\ -2\,\mathrm{e}^{-2x} & -\mathrm{e}^{-x} \end{array}\right]$$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-3x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\mathrm{e}^{-2x} \left(\int \mathrm{e}^{2x} \cos(2x)\, dx\right) + \mathrm{e}^{-x} \left(\int \mathrm{e}^{x} \cos(2x)\, dx\right)$

- ○ Compute integrals

  $y_p(x) = -\frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} - \frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=cos(2*x),y(x), singsol=all)
```

$$y(x) = -e^{-2x}c_1 + c_2e^{-x} - \frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20}$$

✓ Solution by Mathematica

Time used: 0.119 (sec). Leaf size: 37

```
DSolve[y''[x]+3*y'[x]+2*y[x]==Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3}{20}\sin(2x) - \frac{1}{20}\cos(2x) + e^{-2x}(c_2e^x + c_1)$$

3.15 problem Example 3.44

Internal problem ID [5871]
Internal file name [`OUTPUT/5119_Sunday_June_05_2022_03_25_16_PM_20762596/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.44.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + 6y'' + 11y' + 6y = 2\sin(3x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 6y'' + 11y' + 6y = 0$$

The characteristic equation is

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = -3$$
$$\lambda_2 = -2$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x} + e^{-3x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{-2x}$$
$$y_3 = e^{-3x}$$

Now the particular solution to the given ODE is found

$$y''' + 6y'' + 11y' + 6y = 2\sin(3x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3x}, e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(3x) + A_2 \sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \sin(3x) + 6A_2 \cos(3x) - 48A_1 \cos(3x) - 48A_2 \sin(3x) = 2\sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{195}, A_2 = -\frac{8}{195}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(3x)}{195} - \frac{8\sin(3x)}{195}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x} + e^{-3x} c_3\right) + \left(-\frac{\cos(3x)}{195} - \frac{8\sin(3x)}{195}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-3x} c_3 - \frac{\cos(3x)}{195} - \frac{8\sin(3x)}{195} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-3x} c_3 - \frac{\cos(3x)}{195} - \frac{8\sin(3x)}{195}$$

Verified OK.

3.15.1 Maple step by step solution

Let's solve

$y''' + 6y'' + 11y' + 6y = 2\sin(3x)$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = 2\sin(3x) - 6y_3(x) - 11y_2(x) - 6y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 2\sin(3x) - 6y_3(x) - 11y_2(x) - 6y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 2\sin(3x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 2\sin(3x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \vec{y}_p(x)$$

□ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-3x}}{9} & \frac{e^{-2x}}{4} & e^{-x} \\ -\frac{e^{-3x}}{3} & -\frac{e^{-2x}}{2} & -e^{-x} \\ e^{-3x} & e^{-2x} & e^{-x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-3x}}{9} & \frac{e^{-2x}}{4} & e^{-x} \\ -\frac{e^{-3x}}{3} & -\frac{e^{-2x}}{2} & -e^{-x} \\ e^{-3x} & e^{-2x} & e^{-x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{9} & \frac{1}{4} & 1 \\ -\frac{1}{3} & -\frac{1}{2} & -1 \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} e^{-3x} - 3\,e^{-2x} + 3\,e^{-x} & \frac{3\,e^{-3x}}{2} - 4\,e^{-2x} + \frac{5\,e^{-x}}{2} & \frac{e^{-3x}}{2} - e^{-2x} + \frac{e^{-x}}{2} \\ -3\,e^{-3x} + 6\,e^{-2x} - 3\,e^{-x} & -\frac{9\,e^{-3x}}{2} + 8\,e^{-2x} - \frac{5\,e^{-x}}{2} & -\frac{3\,e^{-3x}}{2} + 2\,e^{-2x} - \frac{e^{-x}}{2} \\ 9\,e^{-3x} - 12\,e^{-2x} + 3\,e^{-x} & \frac{27\,e^{-3x}}{2} - 16\,e^{-2x} + \frac{5\,e^{-x}}{2} & \frac{9\,e^{-3x}}{2} - 4\,e^{-2x} + \frac{e^{-x}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} -\frac{\cos(3x)}{195} - \frac{8\sin(3x)}{195} + \frac{3\,\mathrm{e}^{-x}}{10} - \frac{6\,\mathrm{e}^{-2x}}{13} + \frac{\mathrm{e}^{-3x}}{6} \\ -\frac{8\cos(3x)}{65} + \frac{\sin(3x)}{65} - \frac{3\,\mathrm{e}^{-x}}{10} + \frac{12\,\mathrm{e}^{-2x}}{13} - \frac{\mathrm{e}^{-3x}}{2} \\ \frac{3\cos(3x)}{65} + \frac{24\sin(3x)}{65} + \frac{3\,\mathrm{e}^{-x}}{10} - \frac{24\,\mathrm{e}^{-2x}}{13} + \frac{3\,\mathrm{e}^{-3x}}{2} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \left[\begin{array}{c} -\frac{\cos(3x)}{195} - \frac{8\sin(3x)}{195} + \frac{3\,\mathrm{e}^{-x}}{10} - \frac{6\,\mathrm{e}^{-2x}}{13} + \frac{\mathrm{e}^{-3x}}{6} \\ -\frac{8\cos(3x)}{65} + \frac{\sin(3x)}{65} - \frac{3\,\mathrm{e}^{-x}}{10} + \frac{12\,\mathrm{e}^{-2x}}{13} - \frac{\mathrm{e}^{-3x}}{2} \\ \frac{3\cos(3x)}{65} + \frac{24\sin(3x)}{65} + \frac{3\,\mathrm{e}^{-x}}{10} - \frac{24\,\mathrm{e}^{-2x}}{13} + \frac{3\,\mathrm{e}^{-3x}}{2} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{(260c_1+390)\mathrm{e}^{-3x}}{2340} + \frac{(585c_2-1080)\mathrm{e}^{-2x}}{2340} + \frac{(2340c_3+702)\mathrm{e}^{-x}}{2340} - \frac{\cos(3x)}{195} - \frac{8\sin(3x)}{195}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$3)+6*diff(y(x),x$2)+11*diff(y(x),x)+6*y(x)=2*sin(3*x),y(x), singsol=all)
```

$$y(x) = -\frac{\cos(3x)}{195} - \frac{8\sin(3x)}{195} + c_1 e^{-3x} + c_2 e^{-2x} + c_3 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 44

```
DSolve[y'''[x]+6*y''[x]+11*y'[x]+6*y[x]==2*Sin[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{8}{195}\sin(3x) - \frac{1}{195}\cos(3x) + e^{-3x}(e^x(c_3 e^x + c_2) + c_1)$$

3.16 problem Example 3.45

Internal problem ID [5872]
Internal file name [`OUTPUT/5120_Sunday_June_05_2022_03_25_18_PM_76771421/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.45.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y = x^2}$$

3.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_3x^2 + 4A_2x + 4A_1 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = 0, A_3 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{4} - \frac{1}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{x^2}{4} - \frac{1}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x^2}{4} - \frac{1}{8} \tag{1}$$

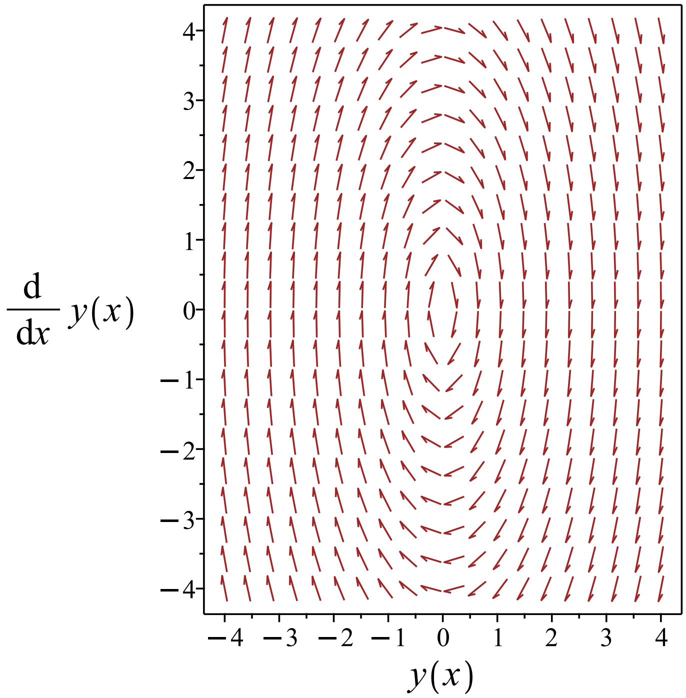


Figure 53: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x^2}{4} - \frac{1}{8}$$

Verified OK.

3.16.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 49: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_3x^2 + 4A_2x + 4A_1 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = 0, A_3 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{4} - \frac{1}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(2x) + \frac{c_2\sin(2x)}{2}\right) + \left(\frac{x^2}{4} - \frac{1}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + \frac{x^2}{4} - \frac{1}{8} \tag{1}$$

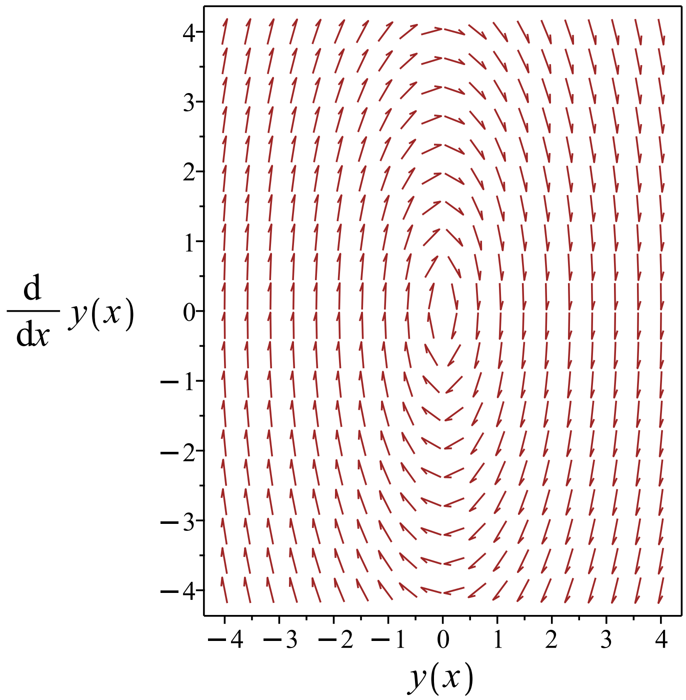


Figure 54: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{x^2}{4} - \frac{1}{8}$$

Verified OK.

3.16.3 Maple step by step solution

Let's solve

$y'' + 4y = x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\cos(2x)\left(\int x^2 \sin(2x) dx\right)}{2} + \frac{\sin(2x)\left(\int \cos(2x) x^2 dx\right)}{2}$

  - Compute integrals

    $y_p(x) = \frac{x^2}{4} - \frac{1}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x^2}{4} - \frac{1}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+4*y(x)=x^2,y(x), singsol=all)
```

$$y(x) = \sin(2x)\, c_2 + \cos(2x)\, c_1 + \frac{x^2}{4} - \frac{1}{8}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 30

```
DSolve[y''[x]+4*y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{4} + c_1 \cos(2x) + c_2 \sin(2x) - \frac{1}{8}$$

3.17 problem Example 3.46

Internal problem ID [5873]
Internal file name [`OUTPUT/5121_Sunday_June_05_2022_03_25_19_PM_74831686/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.46.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y' + 3y = x^3}$$

3.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 3, f(x) = x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\, \mathrm{e}^{\lambda x} + 3\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(3)}\\ &= 2 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 2 + 1\\ \lambda_2 &= 2 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= 1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(3)x} + c_2 e^{(1)x}\end{aligned}$$

Or

$$y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4 x^3 + A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_4 x^3 + 3A_3 x^2 - 12x^2 A_4 + 3A_2 x - 8x A_3 + 6x A_4 + 3A_1 - 4A_2 + 2A_3 = x^3$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{80}{27}, A_2 = \frac{26}{9}, A_3 = \frac{4}{3}, A_4 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{3}x^3 + \frac{4}{3}x^2 + \frac{26}{9}x + \frac{80}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x} c_1 + c_2 e^x\right) + \left(\frac{1}{3}x^3 + \frac{4}{3}x^2 + \frac{26}{9}x + \frac{80}{27}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{3x} c_1 + c_2 e^{x} + \frac{x^3}{3} + \frac{4x^2}{3} + \frac{26x}{9} + \frac{80}{27} \tag{1}$$

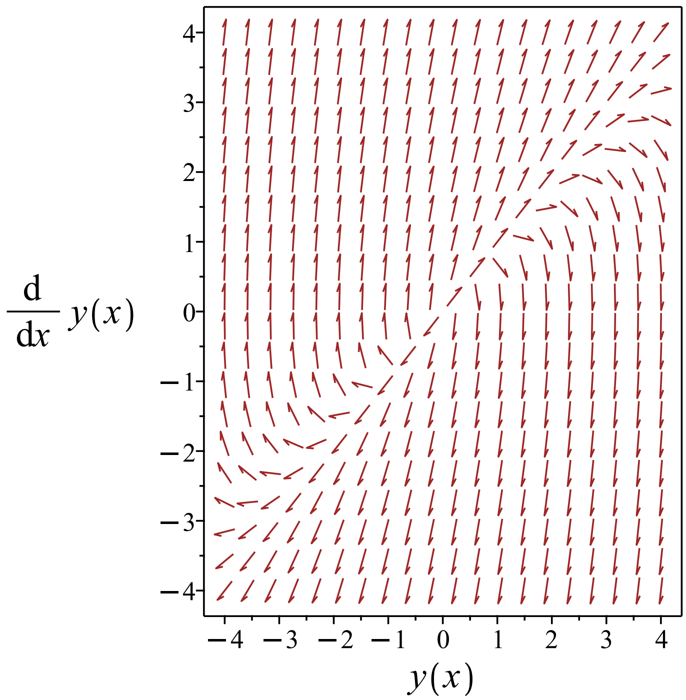


Figure 55: Slope field plot

Verification of solutions

$$y = e^{3x} c_1 + c_2 e^{x} + \frac{x^3}{3} + \frac{4x^2}{3} + \frac{26x}{9} + \frac{80}{27}$$

Verified OK.

3.17.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 51: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4}{1} \, dx} \\ &= z_1 e^{2x} \\ &= z_1 \left(\mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{\mathrm{e}^{2x}}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (\mathrm{e}^x) + c_2 \left(\mathrm{e}^x \left(\frac{\mathrm{e}^{2x}}{2} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^x + \frac{c_2 \mathrm{e}^{3x}}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{3x}}{2}, e^x\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_4x^3 + 3A_3x^2 - 12x^2A_4 + 3A_2x - 8xA_3 + 6xA_4 + 3A_1 - 4A_2 + 2A_3 = x^3$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{80}{27}, A_2 = \frac{26}{9}, A_3 = \frac{4}{3}, A_4 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{3}x^3 + \frac{4}{3}x^2 + \frac{26}{9}x + \frac{80}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^x + \frac{c_2e^{3x}}{2}\right) + \left(\frac{1}{3}x^3 + \frac{4}{3}x^2 + \frac{26}{9}x + \frac{80}{27}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^x + \frac{c_2e^{3x}}{2} + \frac{x^3}{3} + \frac{4x^2}{3} + \frac{26x}{9} + \frac{80}{27} \quad (1)$$

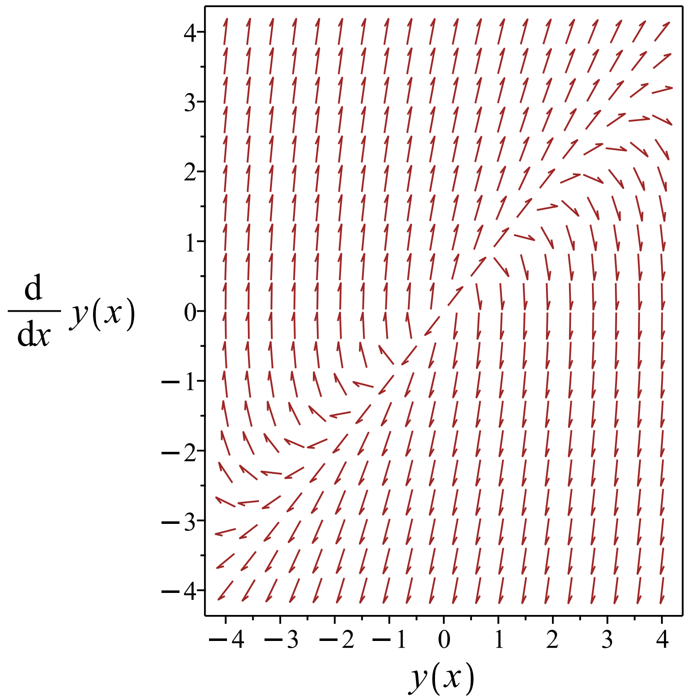


Figure 56: Slope field plot

Verification of solutions

$$y = c_1 e^x + \frac{c_2 e^{3x}}{2} + \frac{x^3}{3} + \frac{4x^2}{3} + \frac{26x}{9} + \frac{80}{27}$$

Verified OK.

3.17.3 Maple step by step solution

Let's solve

$y'' - 4y' + 3y = x^3$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r - 3) = 0$

- Roots of the characteristic polynomial

$r = (1, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^x + c_2 e^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^3\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{3x} \\ e^x & 3\,e^{3x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,e^{4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{e^x\left(\int x^3 e^{-x} dx\right)}{2} + \frac{e^{3x}\left(\int e^{-3x} x^3 dx\right)}{2}$

  - Compute integrals

    $y_p(x) = \frac{1}{3}x^3 + \frac{4}{3}x^2 + \frac{26}{9}x + \frac{80}{27}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^x + c_2 e^{3x} + \frac{x^3}{3} + \frac{4x^2}{3} + \frac{26x}{9} + \frac{80}{27}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+3*y(x)=x^3,y(x), singsol=all)
```

$$y(x) = e^{x} c_2 + e^{3x} c_1 + \frac{x^3}{3} + \frac{4x^2}{3} + \frac{26x}{9} + \frac{80}{27}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 39

```
DSolve[y''[x]-4*y'[x]+3*y[x]==x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{27}\left(9x^3 + 36x^2 + 78x + 80\right) + c_1 e^x + c_2 e^{3x}$$

3.18 problem Example 3.47

Internal problem ID [5874]
Internal file name [`OUTPUT/5122_Sunday_June_05_2022_03_25_21_PM_26947731/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.5 HIGHER ORDER ODE. Page 181
Problem number: Example 3.47.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y' + \left(1 + \frac{2}{(3x+1)^2}\right) y = 0}$$

3.18.1 Solving using Kovacic algorithm

Writing the ode as

$$9\left(x + \frac{1}{3}\right)^2 y'' + 18\left(x + \frac{1}{3}\right)^2 y' + \left(9x^2 + 6x + 3\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9\left(x + \frac{1}{3}\right)^2 \\ B &= 18\left(x + \frac{1}{3}\right)^2 \\ C &= 9x^2 + 6x + 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{(3x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= (3x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{(3x+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 53: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (3x + 1)^2$. There is a pole at $x = -\frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9\left(x + \frac{1}{3}\right)^2}$$

For the pole at $x = -\frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{2}{(3x+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{(3x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{3}$	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{3x+1} + (-)\,(0) \\
&= \frac{1}{3x+1} \\
&= \frac{1}{3x+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{3x+1}\right)(0) + \left(\left(-\frac{1}{3\left(x+\frac{1}{3}\right)^2} \right) + \left(\frac{1}{3x+1}\right)^2 - \left(-\frac{2}{(3x+1)^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1}{3x+1} dx} \\
&= (3x+1)^{\frac{1}{3}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{18\left(x+\frac{1}{3}\right)^2}{9\left(x+\frac{1}{3}\right)^2}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (3x+1)^{\frac{1}{3}}\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{18\left(x+\frac{1}{3}\right)^2}{9\left(x+\frac{1}{3}\right)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1 \left((3x+1)^{\frac{1}{3}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((3x+1)^{\frac{1}{3}}\,\mathrm{e}^{-x}\right) + c_2\left((3x+1)^{\frac{1}{3}}\,\mathrm{e}^{-x}\left((3x+1)^{\frac{1}{3}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(3x+1)^{\frac{1}{3}}\,\mathrm{e}^{-x} + c_2(3x+1)^{\frac{2}{3}}\,\mathrm{e}^{-x} \tag{1}
$$

Verification of solutions

$$
y = c_1(3x+1)^{\frac{1}{3}}\,\mathrm{e}^{-x} + c_2(3x+1)^{\frac{2}{3}}\,\mathrm{e}^{-x}
$$

Verified OK.

3.18.2 Maple step by step solution

Let's solve

$9\left(x+\frac{1}{3}\right)^{2} y''+18\left(x+\frac{1}{3}\right)^{2} y'+\left(9x^{2}+6x+3\right) y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3\left(3x^{2}+2x+1\right) y}{\left(3x+1\right)^{2}}-2y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+2y'+\frac{3\left(3x^{2}+2x+1\right) y}{\left(3x+1\right)^{2}}=0$

☐ Check to see if $x_{0}=-\frac{1}{3}$ is a regular singular point

  ○ Define functions

   $\left[P_{2}(x)=2, P_{3}(x)=\frac{3\left(3x^{2}+2x+1\right)}{\left(3x+1\right)^{2}}\right]$

  ○ $\left(x+\frac{1}{3}\right)\cdot P_{2}(x)$ is analytic at $x=-\frac{1}{3}$

   $\left(\left(x+\frac{1}{3}\right)\cdot P_{2}(x)\right)\bigg|_{x=-\frac{1}{3}}=0$

  ○ $\left(x+\frac{1}{3}\right)^{2}\cdot P_{3}(x)$ is analytic at $x=-\frac{1}{3}$

   $\left(\left(x+\frac{1}{3}\right)^{2}\cdot P_{3}(x)\right)\bigg|_{x=-\frac{1}{3}}=\frac{2}{9}$

  ○ $x=-\frac{1}{3}$is a regular singular point

  Check to see if $x_{0}=-\frac{1}{3}$ is a regular singular point

  $x_{0}=-\frac{1}{3}$

- Multiply by denominators

  $y''\left(3x+1\right)^{2}+2y'\left(3x+1\right)^{2}+\left(9x^{2}+6x+3\right) y=0$

- Change variables using $x=u-\frac{1}{3}$ so that the regular singular point is at $u=0$

  $9u^{2}\left(\frac{d^{2}}{du^{2}}y(u)\right)+18u^{2}\left(\frac{d}{du}y(u)\right)+\left(9u^{2}+2\right) y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_{k} u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
- Shift index using $k- >k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
- Convert $u^2 \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

  $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r+1}$
- Shift index using $k- >k-1$

  $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, u^{k+r}$
- Convert $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)\,(-2+3r)\, u^r + (a_1(2+3r)\,(1+3r) + 18a_0 r)\, u^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r-1)\,(3k+3r\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)\,(-2+3r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$
- Each term must be 0

  $a_1(2+3r)\,(1+3r) + 18a_0 r = 0$
- Solve for the dependent coefficient(s)

  $a_1 = -\frac{18a_0 r}{9r^2+9r+2}$
- Each term in the series must be 0, giving the recursion relation

  $9\left(k+r-\frac{2}{3}\right)\left(k+r-\frac{1}{3}\right) a_k + 18a_{k-1}k + 18a_{k-1}r + 9a_{k-2} - 18a_{k-1} = 0$
- Shift index using $k- >k+2$

  $9\left(k+\frac{4}{3}+r\right)\left(k+\frac{5}{3}+r\right) a_{k+2} + 18a_{k+1}(k+2) + 18a_{k+1}r + 9a_k - 18a_{k+1} = 0$
- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{9(2ka_{k+1}+2a_{k+1}r+a_k+2a_{k+1})}{(3k+4+3r)(3k+5+3r)}$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1 = -a_0\right]$$

- Revert the change of variables $u = x + \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{1}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1 = -a_0\right]$$

- Recursion relation for $r = \frac{2}{3}$

$$a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}$$

- Solution for $r = \frac{2}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{2}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}, a_1 = -a_0\right]$$

- Revert the change of variables $u = x + \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{2}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}, a_1 = -a_0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x+\frac{1}{3}\right)^{k+\frac{2}{3}}\right), a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1 = -a_0, b_{k+2} =$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+(1+2/(1+3*x)^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x}(3x+1)^{\frac{1}{3}}\left((3x+1)^{\frac{1}{3}} c_2 + c_1\right)$$

✓ Solution by Mathematica
Time used: 0.045 (sec). Leaf size: 35

```
DSolve[y''[x]+2*y'[x]+(1+2/(1+3*x)^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\sqrt[3]{3x+1}\left(c_2\sqrt[3]{3x+1}+c_1\right)$$

4 Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218

4.1 problem Problem 3.1

Internal problem ID [5875]
Internal file name [`OUTPUT/5123_Sunday_June_05_2022_03_25_23_PM_61504530/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y+\sqrt{x^2+y^2}-xy'=0}$$

4.1.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y+\sqrt{x^2+y^2}}{x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(y+\sqrt{x^2+y^2}\right)(b_3-a_2)}{x} - \frac{\left(y+\sqrt{x^2+y^2}\right)^2 a_3}{x^2} - \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{y+\sqrt{x^2+y^2}}{x^2}\right)(xa_2+ya_3+a_1) - \frac{\left(1+\frac{y}{\sqrt{x^2+y^2}}\right)(xb_2+yb_3+b_1)}{x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(x^2+y^2)^{\frac{3}{2}} a_3 + x^3 a_2 - x^3 b_3 + 2x^2 y a_3 + x^2 y b_2 + y^3 a_3 + \sqrt{x^2+y^2}\, x b_1 - \sqrt{x^2+y^2}\, y a_1 + x y b_1 - a_1 y^2}{\sqrt{x^2+y^2}\, x^2} = 0$$

Setting the numerator to zero gives

$$-(x^2+y^2)^{\frac{3}{2}} a_3 - x^3 a_2 + x^3 b_3 - 2x^2 y a_3 - x^2 y b_2 - y^3 a_3 - \sqrt{x^2+y^2}\, x b_1 + \sqrt{x^2+y^2}\, y a_1 - x y b_1 + a_1 y^2 = 0 \tag{6E}$$

Simplifying the above gives

$$-(x^2+y^2)^{\frac{3}{2}} a_3 + (x^2+y^2) x b_3 - (x^2+y^2) y a_3 - x^3 a_2 - x^2 y a_3 - x^2 y b_2 - x y^2 b_3 + (x^2+y^2) a_1 - \sqrt{x^2+y^2}\, x b_1 + \sqrt{x^2+y^2}\, y a_1 - x^2 a_1 - x y b_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-x^3 a_2 + x^3 b_3 - \sqrt{x^2+y^2}\, a_3 x^2 - 2x^2 y a_3 - x^2 y b_2 - \sqrt{x^2+y^2}\, a_3 y^2 - y^3 a_3 - \sqrt{x^2+y^2}\, x b_1 - x y b_1 + \sqrt{x^2+y^2}\, y a_1 + a_1 y^2 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2 + y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -v_1^3 a_2 - 2v_1^2 v_2 a_3 - v_3 a_3 v_1^2 - v_2^3 a_3 - v_3 a_3 v_2^2 - v_1^2 v_2 b_2 \\ & + v_1^3 b_3 + a_1 v_2^2 + v_3 v_2 a_1 - v_1 v_2 b_1 - v_3 v_1 b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} & (b_3 - a_2) v_1^3 + (-2a_3 - b_2) v_1^2 v_2 - v_3 a_3 v_1^2 - v_1 v_2 b_1 \\ & - v_3 v_1 b_1 - v_2^3 a_3 - v_3 a_3 v_2^2 + a_1 v_2^2 + v_3 v_2 a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -2a_3 - b_2 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y+\sqrt{x^2+y^2}}{x}\right)(x) \\
&= -\sqrt{x^2+y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{x^2+y^2}} dy
\end{aligned}$$

Which results in

$$S = -\ln\left(y+\sqrt{x^2+y^2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + \sqrt{x^2 + y^2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{x}{\sqrt{x^2 + y^2}\,\left(y + \sqrt{x^2 + y^2}\right)} \\
S_y &= -\frac{1}{\sqrt{x^2 + y^2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2\left(\sqrt{x^2 + y^2}\, y + x^2 + y^2\right)}{x\sqrt{x^2 + y^2}\,\left(y + \sqrt{x^2 + y^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(y + \sqrt{x^2 + y^2}\right) = -2\ln(x) + c_1$$

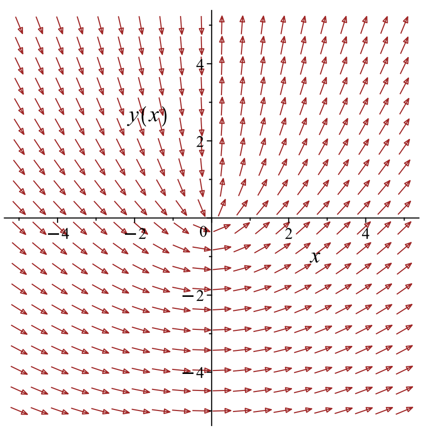
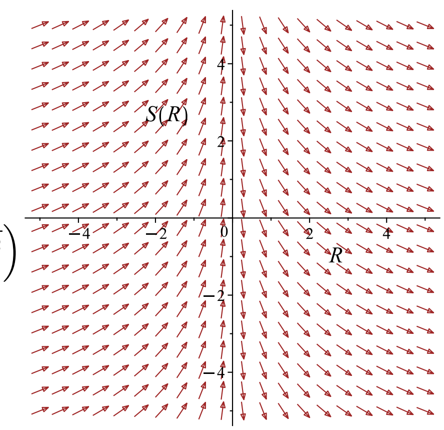
Which simplifies to

$$-\ln\left(y + \sqrt{x^2 + y^2}\right) = -2\ln(x) + c_1$$

Which gives

$$y = -\frac{e^{-c_1}\left(e^{2c_1} - x^2\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$ 	$R = x$ $S = -\ln\left(y + \sqrt{x^2 + y^2}\right)$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{e^{-c_1}\left(e^{2c_1} - x^2\right)}{2} \tag{1}$$

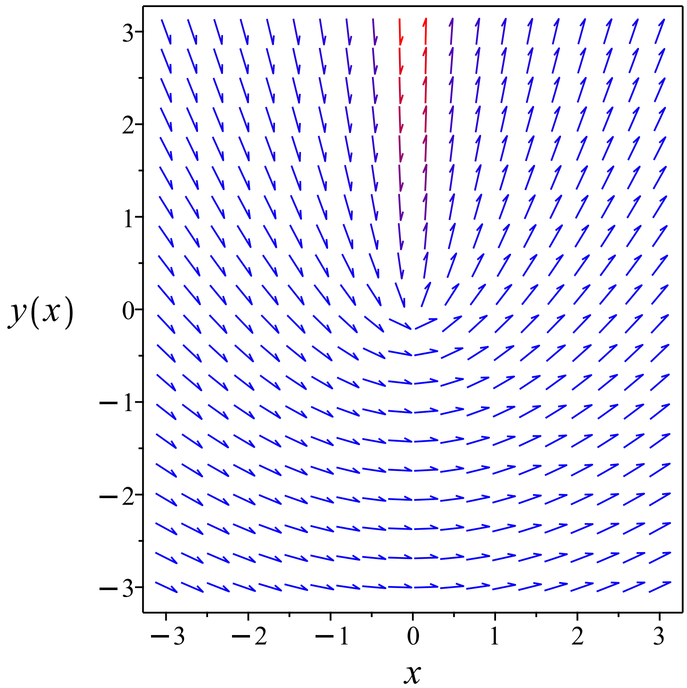


Figure 57: Slope field plot

Verification of solutions

$$y = -\frac{e^{-c_1}\left(e^{2c_1} - x^2\right)}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(y(x)+sqrt(x^2+y(x)^2)-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{-c_1x^2 + y(x) + \sqrt{x^2 + y(x)^2}}{x^2} = 0$$

✓ Solution by Mathematica

Time used: 0.347 (sec). Leaf size: 27

```
DSolve[y[x]+Sqrt[x^2+y[x]^2]-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-c_1}\left(-1 + e^{2c_1}x^2\right)$$

4.2 problem Problem 3.2

Internal problem ID [5876]
Internal file name [`OUTPUT/5124_Sunday_June_05_2022_03_25_26_PM_88849362/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.2.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y'^2 + y^2 = a^2}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{a^2 - y^2} \tag{1}$$
$$y' = -\sqrt{a^2 - y^2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{a^2 - y^2}} dy = \int dx$$
$$\arctan\left(\frac{y}{\sqrt{a^2 - y^2}}\right) = x + c_1$$

Summary
The solution(s) found are the following

$$\arctan\left(\frac{y}{\sqrt{a^2-y^2}}\right) = x + c_1 \tag{1}$$

Verification of solutions

$$\arctan\left(\frac{y}{\sqrt{a^2-y^2}}\right) = x + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{a^2-y^2}}dy = \int dx$$

$$-\arctan\left(\frac{y}{\sqrt{a^2-y^2}}\right) = c_2 + x$$

Summary
The solution(s) found are the following

$$-\arctan\left(\frac{y}{\sqrt{a^2-y^2}}\right) = c_2 + x \tag{1}$$

Verification of solutions

$$-\arctan\left(\frac{y}{\sqrt{a^2-y^2}}\right) = c_2 + x$$

Verified OK.

4.2.1 Maple step by step solution

Let's solve

$y'^2 + y^2 = a^2$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{a^2-y^2}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{a^2-y^2}} dx = \int 1 dx + c_1$

- Evaluate integral

  $\arctan\left(\frac{y}{\sqrt{a^2-y^2}}\right) = x + c_1$

- Solve for y

  $y = \tan(x+c_1)\sqrt{\frac{a^2}{\tan(x+c_1)^2+1}}$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   <- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 60

```
dsolve(diff(y(x),x)^2=a^2-y(x)^2,y(x), singsol=all)
```

$$y(x) = -a$$
$$y(x) = a$$
$$y(x) = -\tan(-x+c_1)\sqrt{\cos(-x+c_1)^2 a^2}$$
$$y(x) = \tan(-x+c_1)\sqrt{\cos(-x+c_1)^2 a^2}$$

✓ Solution by Mathematica

Time used: 3.336 (sec). Leaf size: 111

```
DSolve[(y'[x])^2==a^2-y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -\frac{a\tan(x-c_1)}{\sqrt{\sec^2(x-c_1)}} \\
y(x) &\to \frac{a\tan(x-c_1)}{\sqrt{\sec^2(x-c_1)}} \\
y(x) &\to -\frac{a\tan(x+c_1)}{\sqrt{\sec^2(x+c_1)}} \\
y(x) &\to \frac{a\tan(x+c_1)}{\sqrt{\sec^2(x+c_1)}} \\
y(x) &\to -a \\
y(x) &\to a
\end{aligned}
$$

4.3 problem Problem 3.3

Internal problem ID [5877]
Internal file name [`OUTPUT/5125_Sunday_June_05_2022_03_25_29_PM_26028336/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - 2xy' + y(x^2 + 2) = 0}$$

4.3.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{x^2 + 2}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{x^2+2}{x^2} - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\
&= \frac{x^2+2}{x^2} - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\
&= \frac{x^2+2}{x^2} - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= 1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{2}{x}}{2}} \\
&= x
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, x \tag{4}
$$

Applying this change of variable to the original ode results in

$$
x^3(v(x) + v''(x)) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
\lambda^2 + 1 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(\cos(x)\, c_1 + c_2 \sin(x))$$

Or

$$v(x) = \cos(x)\, c_1 + c_2 \sin(x)$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x)\\ &= (\cos(x)\, c_1 + c_2 \sin(x))\,(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = x$$

Hence (7) becomes

$$y = (\cos(x) c_1 + c_2 \sin(x)) x$$

Summary
The solution(s) found are the following

$$y = (\cos(x) c_1 + c_2 \sin(x)) x \tag{1}$$

Verification of solutions

$$y = (\cos(x) c_1 + c_2 \sin(x)) x$$

Verified OK.

4.3.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - 2xy' + y(x^2 + 2) = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + xy' + (-n^2 + x^2) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) xy' + (\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{3}{2}$$
$$\beta = 1$$
$$n = -\frac{1}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x \sqrt{2}\, \cos(x)}{\sqrt{\pi}} + \frac{c_2 x \sqrt{2}\, \sin(x)}{\sqrt{\pi}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1 x \sqrt{2}\, \cos(x)}{\sqrt{\pi}} + \frac{c_2 x \sqrt{2}\, \sin(x)}{\sqrt{\pi}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 x \sqrt{2}\, \cos(x)}{\sqrt{\pi}} + \frac{c_2 x \sqrt{2}\, \sin(x)}{\sqrt{\pi}}$$

Verified OK.

4.3.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 2xy' + y(x^2 + 2) = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 56: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\, x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)\, x) + c_2(\cos(x)\, x(\tan(x)))
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \cos(x) c_1 x + \sin(x) c_2 x \tag{1}$$

Verification of solutions

$$y = \cos(x) c_1 x + \sin(x) c_2 x$$

Verified OK.

4.3.4 Maple step by step solution

Let's solve

$x^2 y'' - 2xy' + y(x^2+2) = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - 2xy' + y(x^2+2) = 0$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$$x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  ◦ Shift index using $k->k-m$

$$x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  ◦ Convert $x\cdot y'$ to series expansion

$$x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$

  ◦ Convert $x^2\cdot y''$ to series expansion

$$x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(-1+r)\,(-2+r)\, x^r+a_1 r(-1+r)\, x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r-1)\,(k+r-2)+a_{k-2}\right) x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)\,(-2+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{1,2\}$$

- Each term must be 0

$$a_1 r(-1+r)=0$$

- Solve for the dependent coefficient(s)

$$a_1=0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-1)\,(k+r-2)+a_{k-2}=0$$

- Shift index using $k->k+2$

$$a_{k+2}(k+1+r)\,(k+r)+a_k=0$$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x(c_1 \sin(x) + \cos(x) c_2)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

4.4 problem Problem 3.4

Internal problem ID [5878]
Internal file name [`OUTPUT/5126_Sunday_June_05_2022_03_25_30_PM_56756824/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0}$$

4.4.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x(1+x)^2 y'' + 2y'(1+x)^2 - 2xy\right) dx = 0$$
$$\left(-x^2+1\right) y + \left(x^3+2x^2+x\right) y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{x-1}{x\,(1+x)} \\ q(x) &= \frac{c_1}{x\,(1+x)^2} \end{aligned}$$

Hence the ode is

$$y' - \frac{(x-1)\,y}{x\,(1+x)} = \frac{c_1}{x\,(1+x)^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{x-1}{x(1+x)}dx} \\ &= \mathrm{e}^{-2\ln(1+x)+\ln(x)} \end{aligned}$$

Which simplifies to

$$\mu = \frac{x}{(1+x)^2}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x\,(1+x)^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xy}{(1+x)^2}\right) &= \left(\frac{x}{(1+x)^2}\right)\left(\frac{c_1}{x\,(1+x)^2}\right) \\ \mathrm{d}\left(\frac{xy}{(1+x)^2}\right) &= \left(\frac{c_1}{(1+x)^4}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{xy}{(1+x)^2} &= \int \frac{c_1}{(1+x)^4}\,\mathrm{d}x \\ \frac{xy}{(1+x)^2} &= -\frac{c_1}{3\,(1+x)^3} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{x}{(1+x)^2}$ results in

$$y = -\frac{c_1}{3x\,(1+x)} + \frac{c_2(1+x)^2}{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{3x\,(1+x)} + \frac{c_2(1+x)^2}{x} \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{3x(1+x)} + \frac{c_2(1+x)^2}{x}$$

Verified OK.

4.4.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x(1+x)^2 y'' + 2y'(1+x)^2 - 2xy = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x(1+x)^2 y'' + 2y'(1+x)^2 - 2xy\right) dx = 0$$
$$\left(-x^2+1\right) y + \left(x^3+2x^2+x\right) y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{x-1}{x(1+x)}$$
$$q(x) = \frac{c_1}{x(1+x)^2}$$

Hence the ode is

$$y' - \frac{(x-1)\,y}{x(1+x)} = \frac{c_1}{x(1+x)^2}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{x-1}{x(1+x)}dx}$$
$$= \mathrm{e}^{-2\ln(1+x)+\ln(x)}$$

Which simplifies to

$$\mu = \frac{x}{(1+x)^2}$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x\left(1+x\right)^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xy}{\left(1+x\right)^2}\right) &= \left(\frac{x}{\left(1+x\right)^2}\right)\left(\frac{c_1}{x\left(1+x\right)^2}\right) \\
\mathrm{d}\left(\frac{xy}{\left(1+x\right)^2}\right) &= \left(\frac{c_1}{\left(1+x\right)^4}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{xy}{\left(1+x\right)^2} &= \int \frac{c_1}{\left(1+x\right)^4}\,\mathrm{d}x \\
\frac{xy}{\left(1+x\right)^2} &= -\frac{c_1}{3\left(1+x\right)^3} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{x}{(1+x)^2}$ results in

$$
y = -\frac{c_1}{3x\left(1+x\right)} + \frac{c_2(1+x)^2}{x}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{c_1}{3x\left(1+x\right)} + \frac{c_2(1+x)^2}{x} \tag{1}
$$

Verification of solutions

$$
y = -\frac{c_1}{3x\left(1+x\right)} + \frac{c_2(1+x)^2}{x}
$$

Verified OK.

4.4.3 Solving using Kovacic algorithm

Writing the ode as

$$
x(1+x)^2 y'' + 2y'(1+x)^2 - 2xy = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x(1+x)^2 \\
B &= 2(1+x)^2 \\
C &= -2x
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{(1+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= (1+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{(1+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 58: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1 + x)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(1 + x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{(1+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{(1+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{1+x} + (-)\,(0) \\
&= -\frac{1}{1+x} \\
&= -\frac{1}{1+x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{1+x}\right)(0) + \left(\left(\frac{1}{(1+x)^2}\right) + \left(-\frac{1}{1+x}\right)^2 - \left(\frac{2}{(1+x)^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{1+x} dx} \\
&= \frac{1}{1+x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2(1+x)^2}{x(1+x)^2}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^2+x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2(1+x)^2}{x(1+x)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(1+x)^3}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^2+x}\right) + c_2\left(\frac{1}{x^2+x}\left(\frac{(1+x)^3}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^2+x} + \frac{c_2(1+x)^2}{3x} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1}{x^2 + x} + \frac{c_2(1 + x)^2}{3x}$$

Verified OK.

4.4.4 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x(1 + x)^2 \\ q(x) &= 2(1 + x)^2 \\ r(x) &= -2x \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 4 + 6x \\ q'(x) &= 4x + 4 \end{aligned}$$

Therefore (1) becomes

$$4 + 6x - (4x + 4) + (-2x) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$x(1 + x)^2\, y' + \left((1 + x)^2 - 2x(1 + x)\right) y = c_1$$

We now have a first order ode to solve which is

$$x(1+x)^2 y' + \left((1+x)^2 - 2x(1+x)\right) y = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{x-1}{x(1+x)} \\ q(x) &= \frac{c_1}{x(1+x)^2} \end{aligned}$$

Hence the ode is

$$y' - \frac{(x-1)\,y}{x(1+x)} = \frac{c_1}{x(1+x)^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{x-1}{x(1+x)}dx} \\ &= e^{-2\ln(1+x)+\ln(x)} \end{aligned}$$

Which simplifies to

$$\mu = \frac{x}{(1+x)^2}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x(1+x)^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xy}{(1+x)^2}\right) &= \left(\frac{x}{(1+x)^2}\right)\left(\frac{c_1}{x(1+x)^2}\right) \\ \mathrm{d}\left(\frac{xy}{(1+x)^2}\right) &= \left(\frac{c_1}{(1+x)^4}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{xy}{(1+x)^2} &= \int \frac{c_1}{(1+x)^4}\,\mathrm{d}x \\ \frac{xy}{(1+x)^2} &= -\frac{c_1}{3(1+x)^3} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{x}{(1+x)^2}$ results in

$$y = -\frac{c_1}{3x(1+x)} + \frac{c_2(1+x)^2}{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{3x(1+x)} + \frac{c_2(1+x)^2}{x} \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{3x(1+x)} + \frac{c_2(1+x)^2}{x}$$

Verified OK.

4.4.5 Maple step by step solution

Let's solve

$x(1+x)^2 y'' + 2y'(1+x)^2 - 2xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} + \frac{2y}{(1+x)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -\frac{2}{(1+x)^2}\right]$

  ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = 0$

  ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

$$\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = -2$$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x(1+x)^2 y'' + 2y'(1+x)^2 - 2xy = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^3-u^2)\left(\frac{d^2}{du^2}y(u)\right) + 2u^2\left(\frac{d}{du}y(u)\right) + (-2u+2)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- >k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^2 \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

  $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r+1}$

- Shift index using $k- >k-1$

  $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$-a_0(1+r)(-2+r)u^r + \left(\sum_{k=1}^{\infty}(-a_k(k+r+1)(k+r-2)+a_{k-1}(k+r+1)(k+r-2))u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(1+r)(-2+r)=0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$-(k+r+1)(k+r-2)(a_k - a_{k-1}) = 0$$

- Shift index using $k->k+1$

$$-(k+r+2)(k-1+r)(a_{k+1} - a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = a_k$$

- Recursion relation for $r=-1$

$$a_{k+1} = a_k$$

- Solution for $r=-1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+1} = a_k\right]$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-1}, a_{k+1} = a_k\right]$$

- Recursion relation for $r=2$

$$a_{k+1} = a_k$$

- Solution for $r=2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = a_k\right]$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+2}, a_{k+1} = a_k\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+2}\right), a_{k+1} = a_k, b_{k+1} = b_k\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+2/x*diff(y(x),x)-2/(1+x)^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(x^3 + 3x^2 + 3x)\, c_2 + c_1}{x\,(x+1)}$$

✓ Solution by Mathematica
Time used: 0.03 (sec). Leaf size: 34

```
DSolve[y''[x]+2/x*y'[x]-2/(1+x)^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x(x^2 + 3x + 3) + 3c_1}{3x(x+1)}$$

4.5 problem Problem 3.6

Internal problem ID [5879]
Internal file name [OUTPUT/5127_Sunday_June_05_2022_03_25_32_PM_35783344/index.tex]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{\left(y^2x^2+1\right)y+\left(y^2x^2-1\right)xy'=0}$$

4.5.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{(y^2x^2+1)\,y}{(y^2x^2-1)\,x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2 - \frac{(y^2x^2+1)\, y(b_3-a_2)}{(y^2x^2-1)\, x} - \frac{(y^2x^2+1)^2 y^2 a_3}{(y^2x^2-1)^2 x^2} \\ &- \left(-\frac{2y^3}{y^2x^2-1} + \frac{2(y^2x^2+1)\, y^3}{(y^2x^2-1)^2} + \frac{(y^2x^2+1)\, y}{(y^2x^2-1)\, x^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(-\frac{2y^2x}{y^2x^2-1} - \frac{y^2x^2+1}{(y^2x^2-1)\, x} + \frac{2(y^2x^2+1)\, y^2x}{(y^2x^2-1)^2}\right)(xb_2 + yb_3 + b_1) = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^6y^4b_2 - 2x^4y^6a_3 + x^5y^4b_1 - x^4y^5a_1 - 6x^4y^2b_2 - 4x^3y^3a_2 - 4x^3y^3b_3 - 6x^2y^4a_3 - 4x^3y^2b_1 - 4x^2y^3a_1 - x}{(y^2x^2-1)^2 x^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned}&2x^6y^4b_2 - 2x^4y^6a_3 + x^5y^4b_1 - x^4y^5a_1 - 6x^4y^2b_2 - 4x^3y^3a_2 \\ &- 4x^3y^3b_3 - 6x^2y^4a_3 - 4x^3y^2b_1 - 4x^2y^3a_1 - xb_1 + ya_1 = 0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&-2a_3v_1^4v_2^6 + 2b_2v_1^6v_2^4 - a_1v_1^4v_2^5 + b_1v_1^5v_2^4 - 4a_2v_1^3v_2^3 - 6a_3v_1^2v_2^4 \\ &- 6b_2v_1^4v_2^2 - 4b_3v_1^3v_2^3 - 4a_1v_1^2v_2^3 - 4b_1v_1^3v_2^2 + a_1v_2 - b_1v_1 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$2b_2v_1^6v_2^4 + b_1v_1^5v_2^4 - 2a_3v_1^4v_2^6 - a_1v_1^4v_2^5 - 6b_2v_1^4v_2^2 + (-4a_2 - 4b_3)\,v_1^3v_2^3 \\ - 4b_1v_1^3v_2^2 - 6a_3v_1^2v_2^4 - 4a_1v_1^2v_2^3 - b_1v_1 + a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ -4a_1 &= 0 \\ -a_1 &= 0 \\ -6a_3 &= 0 \\ -2a_3 &= 0 \\ -4b_1 &= 0 \\ -b_1 &= 0 \\ -6b_2 &= 0 \\ 2b_2 &= 0 \\ -4a_2 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{(y^2x^2+1)\,y}{(y^2x^2-1)\,x}\right)(-x) \\
&= -\frac{2y}{y^2x^2-1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{2y}{y^2x^2-1}} dy
\end{aligned}$$

Which results in

$$S = -\frac{y^2x^2}{4} + \frac{\ln(y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{(y^2x^2+1)\,y}{(y^2x^2-1)\,x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y^2 x}{2} \\
S_y &= -\frac{y\, x^2}{2} + \frac{1}{2y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{2x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{2R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{\ln(R)}{2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{y^2 x^2}{4} + \frac{\ln(y)}{2} = \frac{\ln(x)}{2} + c_1
$$

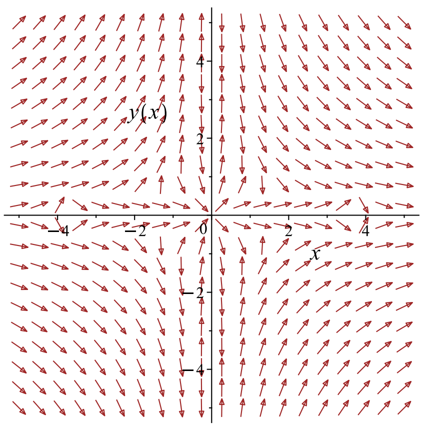
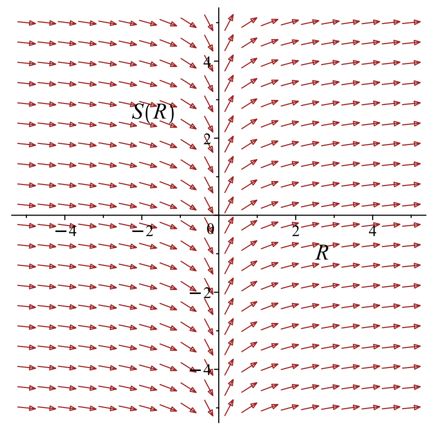
Which simplifies to

$$
-\frac{y^2 x^2}{4} + \frac{\ln(y)}{2} = \frac{\ln(x)}{2} + c_1
$$

Which gives

$$
y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-x^4 \mathrm{e}^{4c_1}\right)}{2} + 2c_1} x
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{(y^2x^2+1)y}{(y^2x^2-1)x}$ 	$R = x$ $S = -\frac{y^2x^2}{4} + \frac{\ln(y)}{2}$	$\frac{dS}{dR} = \frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-x^4 \mathrm{e}^{4c_1}\right)}{2}+2c_1} x \tag{1}$$

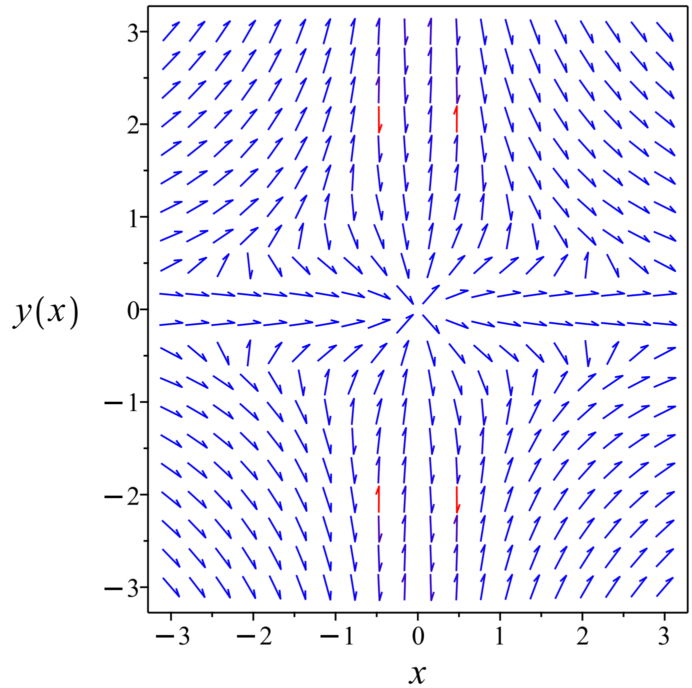


Figure 58: Slope field plot

Verification of solutions

$$y = e^{-\frac{\text{LambertW}\left(-x^4 e^{4c_1}\right)}{2}+2c_1} x$$

Verified OK.

4.5.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\left(y^2x^2 - 1\right)x\right) \mathrm{d}y = \left(-y\left(y^2x^2 + 1\right)\right) \mathrm{d}x$$
$$\left(y\left(y^2x^2 + 1\right)\right) \mathrm{d}x + \left(\left(y^2x^2 - 1\right)x\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y\left(y^2x^2 + 1\right)$$
$$N(x, y) = \left(y^2x^2 - 1\right)x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\left(y^2x^2 + 1\right)\right)\\ &= 3y^2x^2 + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\left(y^2x^2 - 1\right)x\right)\\ &= 3y^2x^2 - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{y^2x^3 - x}\left(\left(3y^2x^2 + 1\right) - \left(3y^2x^2 - 1\right)\right) \\ &= \frac{2}{y^2x^3 - x} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{y\left(y^2x^2 + 1\right)}\left(\left(3y^2x^2 - 1\right) - \left(3y^2x^2 + 1\right)\right) \\ &= -\frac{2}{y\left(y^2x^2 + 1\right)} \end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{\left(3y^2x^2 - 1\right) - \left(3y^2x^2 + 1\right)}{x\left(y\left(y^2x^2 + 1\right)\right) - y\left(\left(y^2x^2 - 1\right)x\right)} \\ &= -\frac{1}{yx} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{1}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \, \mathrm{d}t} \\ &= e^{\int \left(-\frac{1}{t}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(t)} \\ &= \frac{1}{t}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{yx}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{yx}\left(y\left(y^2x^2+1\right)\right) \\ &= \frac{y^2x^2+1}{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{yx}\left(\left(y^2x^2-1\right)x\right) \\ &= \frac{y^2x^2-1}{y}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{y^2x^2+1}{x}\right)+\left(\frac{y^2x^2-1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{y^2x^2+1}{x} \mathrm{d}x$$

$$\phi = \frac{y^2x^2}{2} + \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = y x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2x^2-1}{y}$. Therefore equation (4) becomes

$$\frac{y^2x^2-1}{y} = y x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(-\frac{1}{y}\right) \mathrm{d}y$$

$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y^2x^2}{2} + \ln(x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y^2x^2}{2} + \ln(x) - \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-\mathrm{e}^{-2c_1}x^4\right)}{2}-c_1} x$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-\mathrm{e}^{-2c_1}x^4\right)}{2}-c_1} x \tag{1}$$

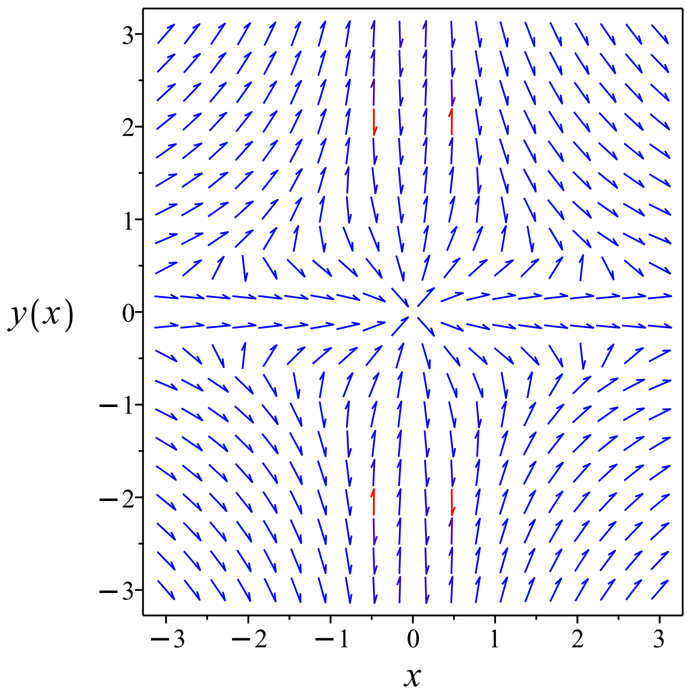


Figure 59: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-\mathrm{e}^{-2c_1}x^4\right)}{2}-c_1} x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 33

```
dsolve((x^2*y(x)^2+1)*y(x)+(x^2*y(x)^2-1)*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{-2c_1}x}{\sqrt{-\frac{x^4 e^{-4c_1}}{\text{LambertW}(-x^4 e^{-4c_1})}}}$$

✓ Solution by Mathematica

Time used: 6.032 (sec). Leaf size: 60

```
DSolve[(x^2*y[x]^2+1)*y[x]+(x^2*y[x]^2-1)*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\frac{i\sqrt{W(-e^{-2c_1}x^4)}}{x}$$
$$y(x) \to \frac{i\sqrt{W(-e^{-2c_1}x^4)}}{x}$$
$$y(x) \to 0$$

4.6 problem Problem 3.7

Internal problem ID [5880]
Internal file name [`OUTPUT/5128_Sunday_June_05_2022_03_25_35_PM_88074136/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{2x^3y^2 - y + (2y^3x^2 - x)\, y' = 0}$$

4.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2y^3x^2 - x)\,\mathrm{d}y &= (-2y^2x^3 + y)\,\mathrm{d}x\\ (2y^2x^3 - y)\,\mathrm{d}x + (2y^3x^2 - x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2y^2x^3 - y\\ N(x, y) &= 2y^3x^2 - x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y^2x^3 - y)\\ &= 4y\,x^3 - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2y^3x^2 - x)\\ &= 4x\,y^3 - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2y^3x^2 - x}\left((4y\,x^3 - 1) - (4x\,y^3 - 1)\right)\\ &= \frac{4y\,x^2 - 4y^3}{2x\,y^3 - 1}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= \frac{1}{2y^2x^3 - y}\left(\left(4x\,y^3 - 1\right) - \left(4y\,x^3 - 1\right)\right)\\ &= \frac{-4x^3 + 4y^2x}{2y\,x^3 - 1}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}\\ &= \frac{\left(4x\,y^3 - 1\right) - \left(4y\,x^3 - 1\right)}{x\left(2y^2x^3 - y\right) - y\left(2y^3x^2 - x\right)}\\ &= -\frac{2}{yx}\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}\mu &= e^{\int R\,\mathrm{d}t}\\ &= e^{\int \left(-\frac{2}{t}\right)\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(t)}\\ &= \frac{1}{t^2}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{y^2x^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^2 x^2}\left(2y^2 x^3 - y\right) \\
&= \frac{2y\,x^3 - 1}{y\,x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2 x^2}\left(2y^3 x^2 - x\right) \\
&= \frac{2x\,y^3 - 1}{x\,y^2}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2y\,x^3 - 1}{y\,x^2}\right) + \left(\frac{2x\,y^3 - 1}{x\,y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2y\,x^3 - 1}{y\,x^2}\,\mathrm{d}x \\
\phi &= \frac{y\,x^3 + 1}{xy} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{x^2}{y} - \frac{y\,x^3+1}{x\,y^2} + f'(y) \qquad (4)\\
&= -\frac{1}{y^2 x} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2x\,y^3-1}{x\,y^2}$. Therefore equation (4) becomes

$$
\frac{2x\,y^3-1}{x\,y^2} = -\frac{1}{y^2 x} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 2y
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int (2y)\,\mathrm{d}y\\
f(y) &= y^2 + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{y\,x^3+1}{xy} + y^2 + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{y\,x^3+1}{xy} + y^2
$$

<u>Summary</u>
The solution(s) found are the following

$$
\frac{yx^3+1}{xy} + y^2 = c_1 \qquad (1)
$$

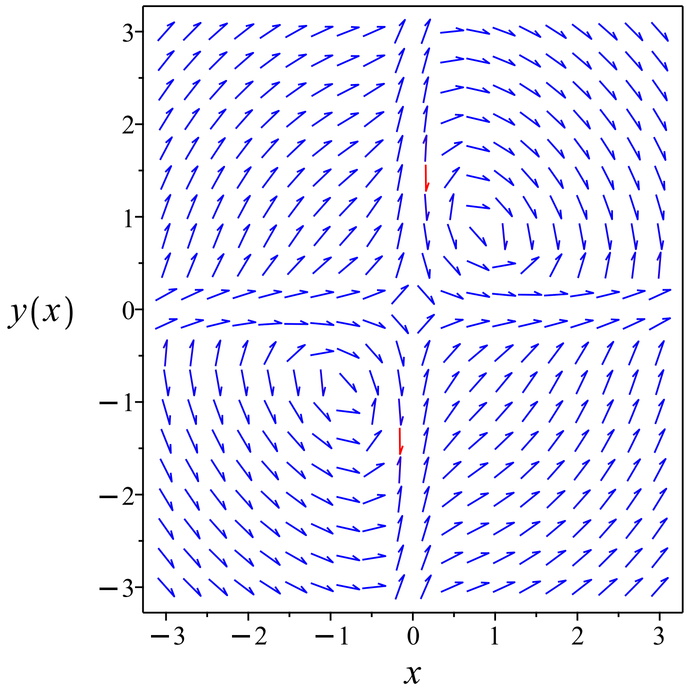


Figure 60: Slope field plot

Verification of solutions

$$\frac{yx^3+1}{xy}+y^2=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2`[0, y^2*x/(2*x*y^3-1)], [0, (x^4*y^2+x^2*y^4+x*y)
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 361

```
dsolve((2*x^3*y(x)^2-y(x))+(2*x^2*y(x)^3-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{12^{\frac{1}{3}}\left(-\left(\left(-9+\sqrt{12x^8-36c_1x^6+36c_1^2x^4-12c_1^3x^2+81}\right)x^2\right)^{\frac{2}{3}}+x^2 12^{\frac{1}{3}}(x^2-c_1)\right)}{6\left(\left(-9+\sqrt{12x^8-36c_1x^6+36c_1^2x^4-12c_1^3x^2+81}\right)x^2\right)^{\frac{1}{3}}x}$$

$$y(x) = -\frac{2^{\frac{2}{3}}3^{\frac{1}{3}}\left(\left(1+i\sqrt{3}\right)\left(\left(-9+\sqrt{12x^8-36c_1x^6+36c_1^2x^4-12c_1^3x^2+81}\right)x^2\right)^{\frac{2}{3}}+2^{\frac{2}{3}}x^2(x^2-c_1)\left(i3^{\frac{5}{6}}-3^{\frac{1}{3}}\right.\right.}{12\left(\left(-9+\sqrt{12x^8-36c_1x^6+36c_1^2x^4-12c_1^3x^2+81}\right)x^2\right)^{\frac{1}{3}}x}$$

$$y(x) = \frac{2^{\frac{2}{3}}3^{\frac{1}{3}}\left(\left(i\sqrt{3}-1\right)\left(\left(-9+\sqrt{12x^8-36c_1x^6+36c_1^2x^4-12c_1^3x^2+81}\right)x^2\right)^{\frac{2}{3}}+2^{\frac{2}{3}}x^2(x^2-c_1)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}\right)\right.}{12\left(\left(-9+\sqrt{12x^8-36c_1x^6+36c_1^2x^4-12c_1^3x^2+81}\right)x^2\right)^{\frac{1}{3}}x}$$

✓ Solution by Mathematica

Time used: 44.412 (sec). Leaf size: 358

```
DSolve[(2*x^3*y[x]^2-y[x])+(2*x^2*y[x]^3-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{\sqrt[3]{2}(-x^3+c_1x)}{\sqrt[3]{-27x^2+\sqrt{729x^4+108x^3\left(x^3-c_1x\right)^3}}} + \frac{\sqrt[3]{-27x^2+\sqrt{729x^4+108x^3\left(x^3-c_1x\right)^3}}}{3\sqrt[3]{2}x}$$

$$y(x) \to \frac{\left(1+i\sqrt{3}\right)\left(x^3-c_1x\right)}{2^{2/3}\sqrt[3]{-27x^2+\sqrt{729x^4+108x^3\left(x^3-c_1x\right)^3}}} - \frac{\left(1-i\sqrt{3}\right)\sqrt[3]{-27x^2+\sqrt{729x^4+108x^3\left(x^3-c_1x\right)^3}}}{6\sqrt[3]{2}x}$$

$$y(x) \to \frac{\left(1-i\sqrt{3}\right)\left(x^3-c_1x\right)}{2^{2/3}\sqrt[3]{-27x^2+\sqrt{729x^4+108x^3\left(x^3-c_1x\right)^3}}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{-27x^2+\sqrt{729x^4+108x^3\left(x^3-c_1x\right)^3}}}{6\sqrt[3]{2}x}$$

4.7 problem Problem 3.8

Internal problem ID [5881]
Internal file name [`OUTPUT/5129_Sunday_June_05_2022_03_25_37_PM_22330115/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`]]
```

$$\boxed{\frac{1}{y}+\sec\left(\frac{y}{x}\right)-\frac{xy'}{y^2}=0}$$

4.7.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\frac{1}{u(x)\,x}+\sec(u(x))-\frac{u'(x)\,x+u(x)}{xu(x)^2}=0$$

Integrating both sides gives

$$\int\frac{1}{u^2\sec(u)}du=\int dx$$
$$\int^{u(x)}\frac{1}{\_a^2\sec(\_a)}d\_a=c_2+x$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\int^{\frac{y}{x}}\frac{1}{\_a^2\sec(\_a)}d\_a=c_2+x$$
$$\int^{\frac{y}{x}}\frac{\cos(\_a)}{\_a^2}d\_a=c_2+x$$

Summary
The solution(s) found are the following

$$\int^{\frac{y}{x}} \frac{\cos(\_a)}{\_a^2} d\_a = c_2 + x \tag{1}$$

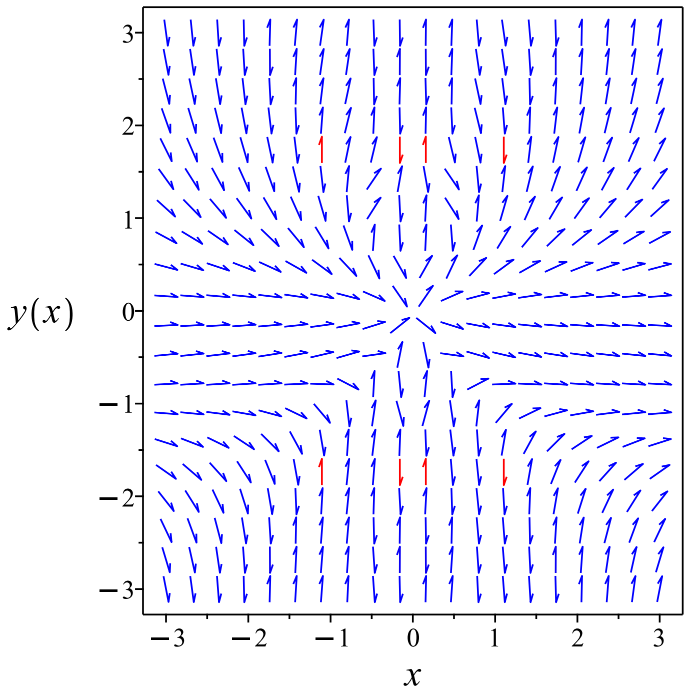


Figure 61: Slope field plot

Verification of solutions

$$\int^{\frac{y}{x}} \frac{\cos(\_a)}{\_a^2} d\_a = c_2 + x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 20

```
dsolve((1/y(x)+sec(y(x)/x))-x/y(x)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{RootOf}\left(\_Z \operatorname{Si}\left(\_Z\right) + \_Zc_1 + \_Zx + \cos\left(\_Z\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 32

```
DSolve[(1/y[x]+Sec[y[x]/x])-x/y[x]^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[-\operatorname{Si}\left(\frac{y(x)}{x}\right) - \frac{x\cos\left(\frac{y(x)}{x}\right)}{y(x)} = x + c_1, y(x)\right]$$

4.8 problem Problem 3.11

Internal problem ID [5882]
Internal file name [`OUTPUT/5130_Sunday_June_05_2022_03_25_39_PM_60515970/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{\phi' - \frac{\phi^2}{2} - \phi\cot(\theta) = 0}$$

4.8.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
\phi' &= \frac{\phi^2}{2} + \phi\cot(\theta) \\
\phi' &= \omega(\theta, \phi)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_\phi - \xi_\theta) - \omega^2\xi_\phi - \omega_\theta\xi - \omega_\phi\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 60: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(\theta, \phi) &= 0 \\
\eta(\theta, \phi) &= \frac{\phi^2}{\sin(\theta)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, \phi) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{d\theta}{\xi} = \frac{d\phi}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial \phi}\right) S(\theta, \phi) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = \theta$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{\phi^2}{\sin(\theta)}} dy \end{aligned}$$

Which results in

$$S = -\frac{\sin(\theta)}{\phi}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, \phi) S_\phi}{R_\theta + \omega(\theta, \phi) R_\phi} \tag{2}$$

Where in the above $R_\theta, R_\phi, S_\theta, S_\phi$ are all partial derivatives and $\omega(\theta, \phi)$ is the right hand side of the original ode given by

$$\omega(\theta, \phi) = \frac{\phi^2}{2} + \phi \cot(\theta)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_\theta &= 1 \\ R_\phi &= 0 \\ S_\theta &= -\frac{\cos(\theta)}{\phi} \\ S_\phi &= \frac{\sin(\theta)}{\phi^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sin(\theta)}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for θ, ϕ in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\sin(R)}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\cos(R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to θ, ϕ coordinates. This results in

$$-\frac{\sin(\theta)}{\phi} = -\frac{\cos(\theta)}{2} + c_1$$

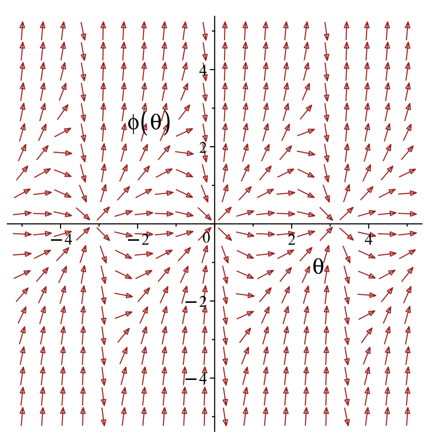
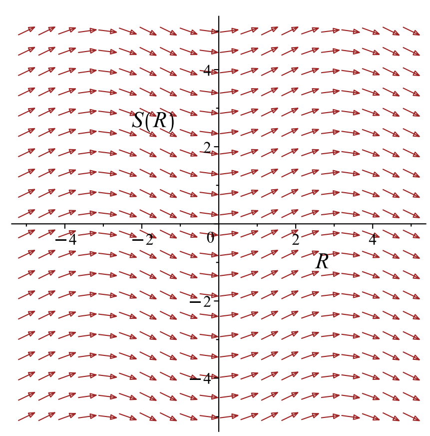
Which simplifies to

$$-\frac{\sin(\theta)}{\phi} = -\frac{\cos(\theta)}{2} + c_1$$

Which gives

$$\phi = \frac{2\sin(\theta)}{\cos(\theta) - 2c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in θ, ϕ coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{d\phi}{d\theta} = \frac{\phi^2}{2} + \phi\cot(\theta)$ 	$R = \theta$ $S = -\frac{\sin(\theta)}{\phi}$	$\frac{dS}{dR} = \frac{\sin(R)}{2}$ 

Summary
The solution(s) found are the following

$$\phi = \frac{2\sin(\theta)}{\cos(\theta) - 2c_1} \tag{1}$$

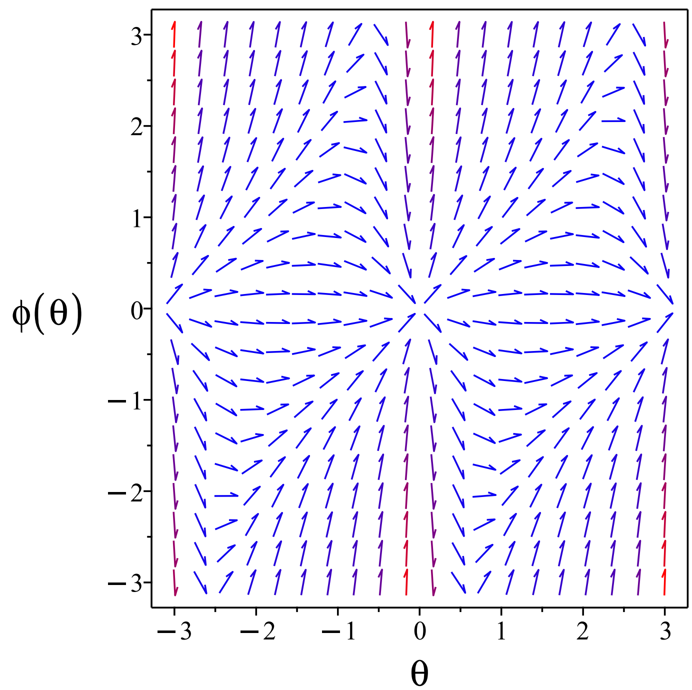


Figure 62: Slope field plot

Verification of solutions

$$\phi = \frac{2\sin(\theta)}{\cos(\theta) - 2c_1}$$

Verified OK.

4.8.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}\phi' &= F(\theta, \phi)\\ &= \frac{\phi^2}{2} + \phi\cot(\theta)\end{aligned}$$

This is a Bernoulli ODE.

$$\phi' = \cot(\theta)\,\phi + \frac{1}{2}\phi^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$\phi' = f_0(\theta)\phi + f_1(\theta)\phi^n \tag{2}$$

The first step is to divide the above equation by ϕ^n which gives

$$\frac{\phi'}{\phi^n} = f_0(\theta)\phi^{1-n} + f_1(\theta) \tag{3}$$

The next step is use the substitution $r = \phi^{1-n}$ in equation (3) which generates a new ODE in $r(\theta)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $\phi(\theta)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(\theta) &= \cot(\theta) \\ f_1(\theta) &= \frac{1}{2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $\phi^n = \phi^2$ gives

$$\phi' \frac{1}{\phi^2} = \frac{\cot(\theta)}{\phi} + \frac{1}{2} \tag{4}$$

Let

$$\begin{aligned} r &= \phi^{1-n} \\ &= \frac{1}{\phi} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t θ gives

$$r' = -\frac{1}{\phi^2}\phi' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -r'(\theta) &= \cot(\theta)\, r(\theta) + \frac{1}{2} \\ r' &= -\cot(\theta)\, r - \frac{1}{2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $r(\theta)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r'(\theta) + p(\theta)r(\theta) = q(\theta)$$

Where here

$$\begin{aligned}p(\theta) &= \cot(\theta)\\ q(\theta) &= -\frac{1}{2}\end{aligned}$$

Hence the ode is

$$r'(\theta) + \cot(\theta)\, r(\theta) = -\frac{1}{2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \cot(\theta)d\theta}\\ &= \sin(\theta)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{d\theta}(\mu r) &= (\mu)\left(-\frac{1}{2}\right)\\ \frac{d}{d\theta}(\sin(\theta)\, r) &= (\sin(\theta))\left(-\frac{1}{2}\right)\\ d(\sin(\theta)\, r) &= \left(-\frac{\sin(\theta)}{2}\right) d\theta\end{aligned}$$

Integrating gives

$$\begin{aligned}\sin(\theta)\, r &= \int -\frac{\sin(\theta)}{2}\, d\theta\\ \sin(\theta)\, r &= \frac{\cos(\theta)}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sin(\theta)$ results in

$$r(\theta) = \frac{\csc(\theta)\cos(\theta)}{2} + c_1 \csc(\theta)$$

which simplifies to

$$r(\theta) = \frac{\cot(\theta)}{2} + c_1 \csc(\theta)$$

Replacing r in the above by $\frac{1}{\phi}$ using equation (5) gives the final solution.

$$\frac{1}{\phi} = \frac{\cot(\theta)}{2} + c_1 \csc(\theta)$$

Or

$$\phi = \frac{1}{\frac{\cot(\theta)}{2} + c_1 \csc(\theta)}$$

Which is simplified to

$$\phi = \frac{2\sin(\theta)}{\cos(\theta) + 2c_1}$$

Summary

The solution(s) found are the following

$$\phi = \frac{2\sin(\theta)}{\cos(\theta) + 2c_1} \tag{1}$$

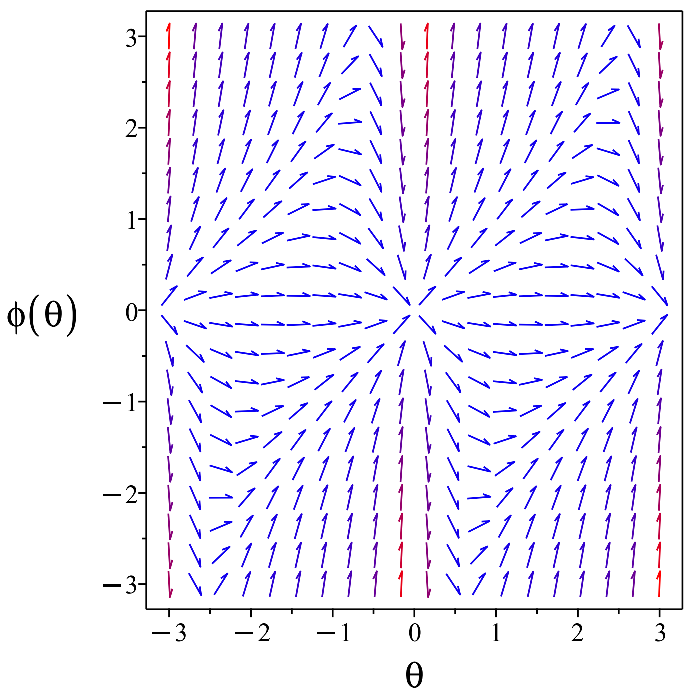


Figure 63: Slope field plot

Verification of solutions

$$\phi = \frac{2\sin(\theta)}{\cos(\theta) + 2c_1}$$

Verified OK.

4.8.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}\phi' &= F(\theta, \phi)\\ &= \frac{\phi^2}{2} + \phi\cot(\theta)\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$\phi' = \frac{\phi^2}{2} + \phi\cot(\theta)$$

With Riccati ODE standard form

$$\phi' = f_0(\theta) + f_1(\theta)\phi + f_2(\theta)\phi^2$$

Shows that $f_0(\theta) = 0$, $f_1(\theta) = \cot(\theta)$ and $f_2(\theta) = \frac{1}{2}$. Let

$$\begin{aligned}\phi &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{\frac{u}{2}}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(\theta) - (f_2' + f_1 f_2)\, u'(\theta) + f_2^2 f_0 u(\theta) = 0 \tag{2}$$

But

$$\begin{aligned}f_2' &= 0\\ f_1 f_2 &= \frac{\cot(\theta)}{2}\\ f_2^2 f_0 &= 0\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(\theta)}{2} - \frac{\cot(\theta)\, u'(\theta)}{2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(\theta) = c_1 + \cos(\theta)\, c_2$$

The above shows that

$$u'(\theta) = -\sin(\theta) c_2$$

Using the above in (1) gives the solution

$$\phi = \frac{2\sin(\theta) c_2}{c_1 + \cos(\theta) c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$\phi = \frac{2\sin(\theta)}{c_3 + \cos(\theta)}$$

Summary

The solution(s) found are the following

$$\phi = \frac{2\sin(\theta)}{c_3 + \cos(\theta)} \tag{1}$$

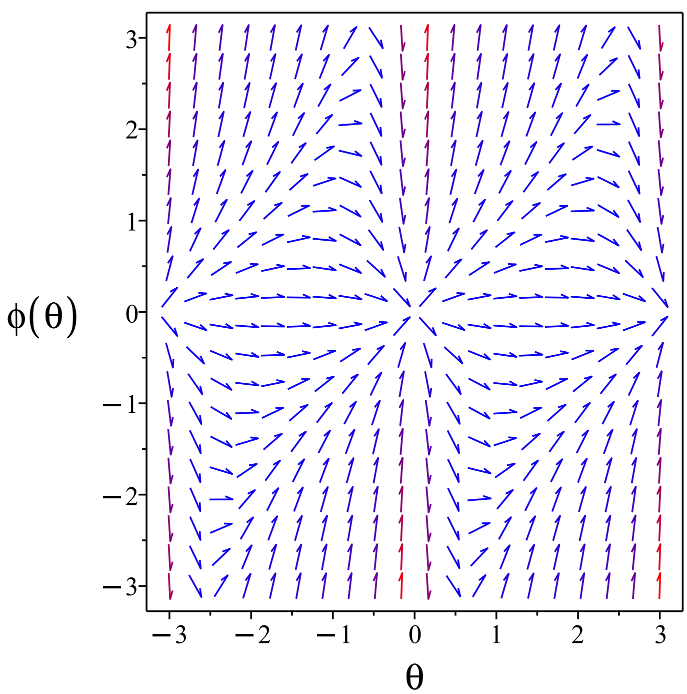


Figure 64: Slope field plot

Verification of solutions

$$\phi = \frac{2\sin(\theta)}{c_3 + \cos(\theta)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(phi(theta),theta)-1/2*phi(theta)^2-phi(theta)*cot(theta)=0,phi(theta), singsol=a
```

$$\phi(\theta) = \frac{2\sin(\theta)}{\cos(\theta) + 2c_1}$$

✓ Solution by Mathematica

Time used: 0.3 (sec). Leaf size: 23

```
DSolve[\[Phi]'[\[Theta]]-1/2*\[Phi][\[Theta]]^2-\[Phi][\[Theta]]*Cot[\[Theta]]==0,\[Phi][\[T
```

$$\phi(\theta) \rightarrow \frac{2\sin(\theta)}{\cos(\theta) + 2c_1}$$
$$\phi(\theta) \rightarrow 0$$

4.9 problem Problem 3.12

Internal problem ID [5883]
Internal file name [`OUTPUT/5131_Sunday_June_05_2022_03_25_41_PM_45774792/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{u'' - \cot(\theta)\, u' = 0}$$

4.9.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable u. Let

$$p(\theta) = u'$$

Then

$$p'(\theta) = u''$$

Hence the ode becomes

$$p'(\theta) - \cot(\theta)\, p(\theta) = 0$$

Which is now solve for $p(\theta)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(\theta, p) \\ &= f(\theta) g(p) \\ &= \cot(\theta)\, p \end{aligned}$$

Where $f(\theta) = \cot(\theta)$ and $g(p) = p$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{p}\,dp &= \cot(\theta)\ d\theta \\
\int \frac{1}{p}\,dp &= \int \cot(\theta)\ d\theta \\
\ln(p) &= \ln(\sin(\theta)) + c_1 \\
p &= \mathrm{e}^{\ln(\sin(\theta))+c_1} \\
&= c_1 \sin(\theta)
\end{aligned}
$$

Since $p = u'$ then the new first order ode to solve is

$$u' = c_1 \sin(\theta)$$

Integrating both sides gives

$$
\begin{aligned}
u &= \int c_1 \sin(\theta)\ \mathrm{d}\theta \\
&= -c_1 \cos(\theta) + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$u = -c_1 \cos(\theta) + c_2 \tag{1}$$

Verification of solutions

$$u = -c_1 \cos(\theta) + c_2$$

Verified OK.

4.9.2 Maple step by step solution

Let's solve

$u'' - \cot(\theta)\, u' = 0$

- Highest derivative means the order of the ODE is 2

  u''

- Make substitution $v = u'$ to reduce order of ODE

  $v'(\theta) - \cot(\theta)\, v(\theta) = 0$

- Separate variables

$\frac{v'(\theta)}{v(\theta)} = \cot(\theta)$

- Integrate both sides with respect to θ

  $\int \frac{v'(\theta)}{v(\theta)} d\theta = \int \cot(\theta)\, d\theta + c_1$

- Evaluate integral

  $\ln(v(\theta)) = \ln(\sin(\theta)) + c_1$

- Solve for $v(\theta)$

  $v(\theta) = e^{c_1} \sin(\theta)$

- Solve 1st ODE for $v(\theta)$

  $v(\theta) = e^{c_1} \sin(\theta)$

- Make substitution $v = u'$

  $u' = e^{c_1} \sin(\theta)$

- Integrate both sides to solve for u

  $\int u' d\theta = \int e^{c_1} \sin(\theta)\, d\theta + c_2$

- Compute integrals

  $u = -e^{c_1} \cos(\theta) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
<- LODE missing y successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(u(theta),theta$2)-cot(theta)*diff(u(theta),theta)=0,u(theta), singsol=all)
```

$$u(\theta) = c_1 + \cos(\theta)\, c_2$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 13

```
DSolve[u''[\[Theta]]-Cot[\[Theta]]*u'[\[Theta]]==0,u[\[Theta]],\[Theta],IncludeSingularSolut
```

$$u(\theta) \to c_2 \cos(\theta) + c_1$$

4.10 problem Problem 3.14

Internal problem ID [5884]
Internal file name [`OUTPUT/5132_Sunday_June_05_2022_03_25_43_PM_27953378/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$\boxed{\left(\phi' - \frac{\phi^2}{2}\right)\sin(\theta)^2 - \phi\cos(\theta)\sin(\theta) = \frac{\cos(2\theta)}{2} + 1}$$

4.10.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}\phi' &= F(\theta, \phi)\\ &= \frac{\phi^2\sin(\theta)^2 + 2\phi\cos(\theta)\sin(\theta) + \cos(2\theta) + 2}{2\sin(\theta)^2}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$\phi' = \frac{\phi^2}{2} + \frac{\phi\cos(\theta)}{\sin(\theta)} + \frac{\cos(\theta)^2}{\sin(\theta)^2} + \frac{1}{2\sin(\theta)^2}$$

With Riccati ODE standard form

$$\phi' = f_0(\theta) + f_1(\theta)\phi + f_2(\theta)\phi^2$$

Shows that $f_0(\theta) = \frac{2+\cos(2\theta)}{2\sin(\theta)^2}$, $f_1(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$ and $f_2(\theta) = \frac{1}{2}$. Let

$$\begin{aligned} \phi &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(\theta) - (f_2' + f_1 f_2)\, u'(\theta) + f_2^2 f_0 u(\theta) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \frac{\cos(\theta)}{2\sin(\theta)} \\ f_2^2 f_0 &= \frac{2+\cos(2\theta)}{8\sin(\theta)^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(\theta)}{2} - \frac{\cos(\theta)\, u'(\theta)}{2\sin(\theta)} + \frac{(2+\cos(2\theta))\, u(\theta)}{8\sin(\theta)^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(\theta) = \sqrt{\sin(\theta)}\left(c_1(\sin(\theta) + i\cos(\theta))^{\frac{i}{2}} + c_2(\sin(\theta) + i\cos(\theta))^{-\frac{i}{2}}\right)$$

The above shows that

$$u'(\theta) = \frac{(-\sin(\theta) + \cos(\theta))\, c_2(\sin(\theta) + i\cos(\theta))^{-\frac{i}{2}} + (\cos(\theta) + \sin(\theta))\, c_1(\sin(\theta) + i\cos(\theta))^{\frac{i}{2}}}{2\sqrt{\sin(\theta)}}$$

Using the above in (1) gives the solution

$$\phi = -\frac{(-\sin(\theta) + \cos(\theta))\, c_2(\sin(\theta) + i\cos(\theta))^{-\frac{i}{2}} + (\cos(\theta) + \sin(\theta))\, c_1(\sin(\theta) + i\cos(\theta))^{\frac{i}{2}}}{\sin(\theta)\left(c_1(\sin(\theta) + i\cos(\theta))^{\frac{i}{2}} + c_2(\sin(\theta) + i\cos(\theta))^{-\frac{i}{2}}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$\phi = \frac{(-\cot(\theta)+1)(\sin(\theta)+i\cos(\theta))^{-\frac{i}{2}} - c_3(\sin(\theta)+i\cos(\theta))^{\frac{i}{2}}(\cot(\theta)+1)}{c_3(\sin(\theta)+i\cos(\theta))^{\frac{i}{2}} + (\sin(\theta)+i\cos(\theta))^{-\frac{i}{2}}}$$

Summary

The solution(s) found are the following

$$\phi = \frac{(-\cot(\theta)+1)(\sin(\theta)+i\cos(\theta))^{-\frac{i}{2}} - c_3(\sin(\theta)+i\cos(\theta))^{\frac{i}{2}}(\cot(\theta)+1)}{c_3(\sin(\theta)+i\cos(\theta))^{\frac{i}{2}} + (\sin(\theta)+i\cos(\theta))^{-\frac{i}{2}}} \tag{1}$$

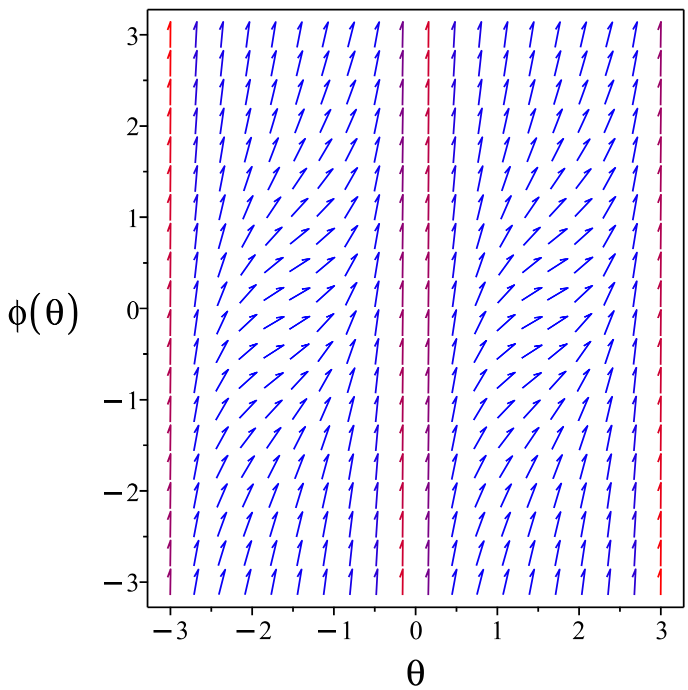


Figure 65: Slope field plot

Verification of solutions

$$\phi = \frac{(-\cot(\theta)+1)(\sin(\theta)+i\cos(\theta))^{-\frac{i}{2}} - c_3(\sin(\theta)+i\cos(\theta))^{\frac{i}{2}}(\cot(\theta)+1)}{c_3(\sin(\theta)+i\cos(\theta))^{\frac{i}{2}} + (\sin(\theta)+i\cos(\theta))^{-\frac{i}{2}}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = cos(x)*(diff(y(x), x))/sin(x)-
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Reducible group (found an exponential solution)
         Reducible group (found another exponential solution)
      <- Kovacics algorithm successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve((diff(phi(theta),theta)-1/2*phi(theta)^2)*sin(theta)^2-phi(theta)*sin(theta)*cos(thet
```

$$\phi(\theta) = \frac{-\sinh\left(\frac{\theta}{2}\right)c_1 - \cosh\left(\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{2}\right)c_1 + \sinh\left(\frac{\theta}{2}\right)} - \cot(\theta)$$

✓ Solution by Mathematica

Time used: 0.64 (sec). Leaf size: 36

```
DSolve[(\[Phi]'[\[Theta]]-1/2\[Phi][\[Theta]]^2)*Sin[\[Theta]]^2-\[Phi][\[Theta]]*Sin[\[Thet
```

$$\phi(\theta) \to -\cot(\theta) - \frac{2e^{\theta}}{e^{\theta} - 2c_1} + 1$$
$$\phi(\theta) \to 1 - \cot(\theta)$$

4.11 problem Problem 3.18

Internal problem ID [5885]
Internal file name [`OUTPUT/5133_Sunday_June_05_2022_03_25_46_PM_97727503/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.18.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x], [_3rd_order, _missing_y], [
   _3rd_order, _with_linear_symmetries], [_3rd_order, _reducible
   , _mu_y2]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying 3rd order ODE linearizable_by_differentiation
differential order: 3; trying a linearization to 4th order
trying differential order: 3; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (1+_b(_a)^2)^(1/2)/(_b(_a)*a), _b(_a),
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0]
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 175

```
dsolve(a*diff(y(x),x$2)*diff(y(x),x$3)=sqrt(1+ diff(y(x),x$2)^2),y(x), singsol=all)
```

$$y(x) = -\frac{1}{2}ix^2 + c_1x + c_2$$

$$y(x) = \frac{1}{2}ix^2 + c_1x + c_2$$

$$y(x) = \frac{(2a^2 + (x + c_1)^2)\sqrt{-a^2 + c_1^2 + 2c_1x + x^2} - 3\left(a(x + c_1)\ln\left(c_1 + x + \sqrt{(c_1 + a + x)(c_1 - a + x)}\right) - 2c\right.}{6a}$$

$$y(x) = \frac{(-2a^2 - (x + c_1)^2)\sqrt{-a^2 + c_1^2 + 2c_1x + x^2} + 3a\left(a(x + c_1)\ln\left(c_1 + x + \sqrt{(c_1 + a + x)(c_1 - a + x)}\right) +\right.}{6a}$$

✓ Solution by Mathematica

Time used: 11.484 (sec). Leaf size: 209

```
DSolve[a*y''[x]*y'''[x]==Sqrt[1+ y''[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{a^2(-1 + c_1{}^2) + 2ac_1x + x^2}(a^2(2 + c_1{}^2) + 2ac_1x + x^2)}{6a} - \frac{1}{2}a(x + ac_1)\log\left(\sqrt{a^2(-1 + c_1{}^2) + 2ac_1x + x^2} + ac_1 + x\right) + c_3x + c_2$$

$$y(x) \to -\frac{\sqrt{a^2(-1 + c_1{}^2) + 2ac_1x + x^2}(a^2(2 + c_1{}^2) + 2ac_1x + x^2)}{6a} + \frac{1}{2}a(x + ac_1)\log\left(\sqrt{a^2(-1 + c_1{}^2) + 2ac_1x + x^2} + ac_1 + x\right) + c_3x + c_2$$

4.12 problem Problem 3.19

Internal problem ID [5886]
Internal file name [`OUTPUT/5134_Sunday_June_05_2022_03_25_49_PM_79468662/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.19.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{a^2y'''' - y'' = 0}$$

The characteristic equation is

$$a^2\lambda^4 - \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = \frac{1}{a}$$
$$\lambda_4 = -\frac{1}{a}$$

Therefore the homogeneous solution is

$$y_h(x) = c_2x + c_1 + e^{\frac{x}{a}}c_3 + e^{-\frac{x}{a}}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = e^{\frac{x}{a}}$$
$$y_4 = e^{-\frac{x}{a}}$$

Summary
The solution(s) found are the following

$$y = c_2x + c_1 + e^{\frac{x}{a}}c_3 + e^{-\frac{x}{a}}c_4 \tag{1}$$

Verification of solutions

$$y = c_2x + c_1 + e^{\frac{x}{a}}c_3 + e^{-\frac{x}{a}}c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 26

```
dsolve(a^2*diff(y(x),x$4)=diff(y(x),x$2),y(x), singsol=all)
```

$$y(x) = c_1 + c_2x + c_3e^{\frac{x}{a}} + c_4e^{-\frac{x}{a}}$$

✓ Solution by Mathematica
Time used: 0.071 (sec). Leaf size: 38

```
DSolve[a^2*y''''[x]==y''[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to a^2e^{-\frac{x}{a}}\left(c_1e^{\frac{2x}{a}} + c_2\right) + c_4x + c_3$$

4.13 problem Problem 3.20

Internal problem ID [5887]
Internal file name [`OUTPUT/5135_Sunday_June_05_2022_03_25_50_PM_78809699/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y\,\mathrm{e}^{xy} + x\,\mathrm{e}^{xy} y' = 0}$$

4.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= -\frac{1}{x}\, dx \\
\int \frac{1}{y}\, dy &= \int -\frac{1}{x}\, dx \\
\ln(y) &= -\ln(x) + c_1 \\
y &= e^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} \tag{1}$$

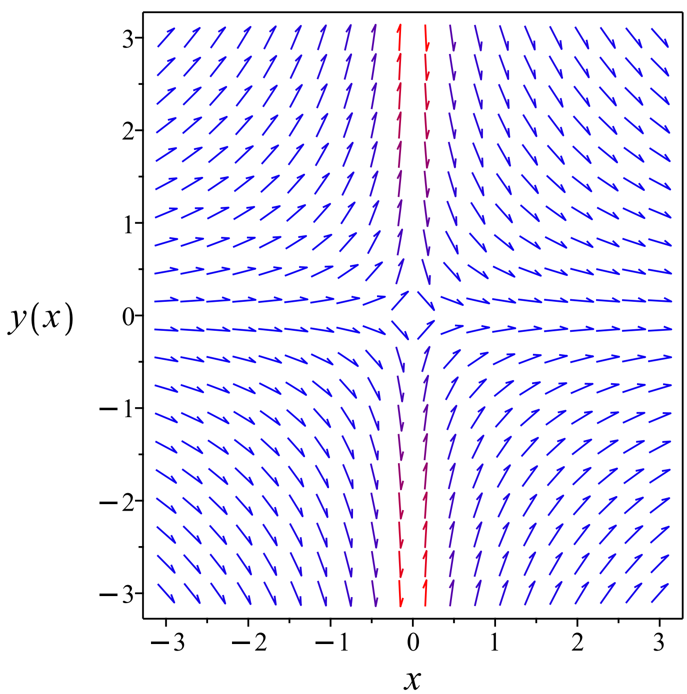


Figure 66: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x}$$

Verified OK.

4.13.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{1}{x} dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}(xy) &= 0 \end{aligned}$$

Integrating gives

$$xy = c_1$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{c_1}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} \tag{1}$$

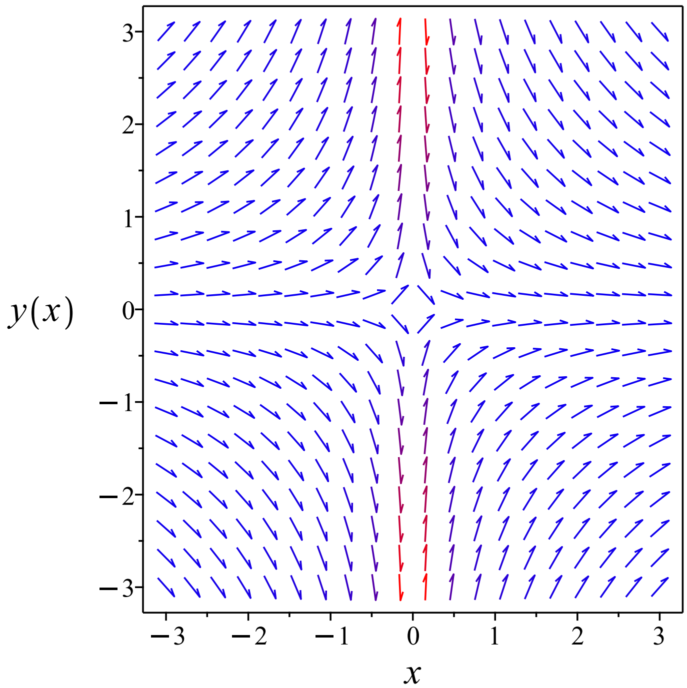


Figure 67: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x}$$

Verified OK.

4.13.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x\,\mathrm{e}^{x^2 u(x)} + x\,\mathrm{e}^{x^2 u(x)}(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{2}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{2}{x}\, dx \\
\ln(u) &= -2\ln(x) + c_2 \\
u &= \mathrm{e}^{-2\ln(x)+c_2} \\
&= \frac{c_2}{x^2}
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= \frac{c_2}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_2}{x} \tag{1}
$$

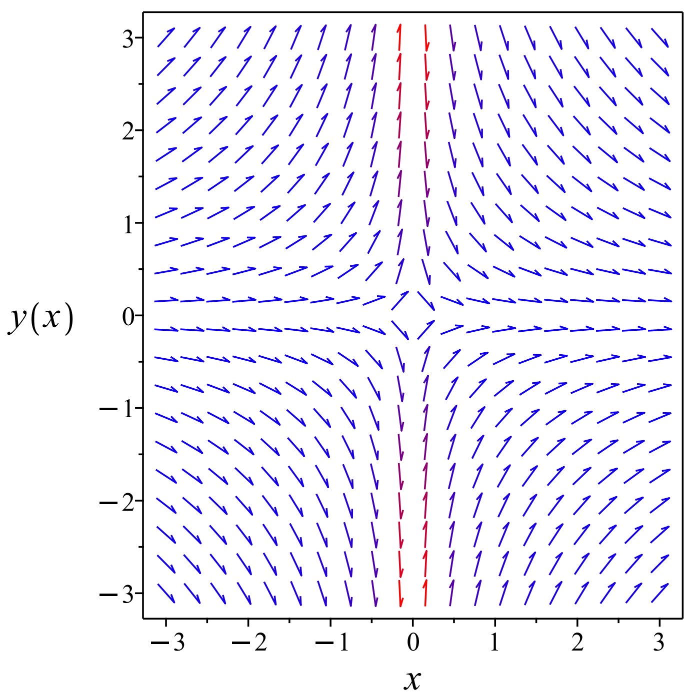


Figure 68: Slope field plot

Verification of solutions

$$y = \frac{c_2}{x}$$

Verified OK.

4.13.4 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{y}{x} \tag{1}$$

Which becomes

$$0 = (-x)\, dy + (-y)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + (-y)\, dx = d(-xy)$$

Hence (2) becomes

$$0 = d(-xy)$$

Integrating both sides gives gives these solutions

$$y = \frac{c_1}{x} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + c_1 \tag{1}$$

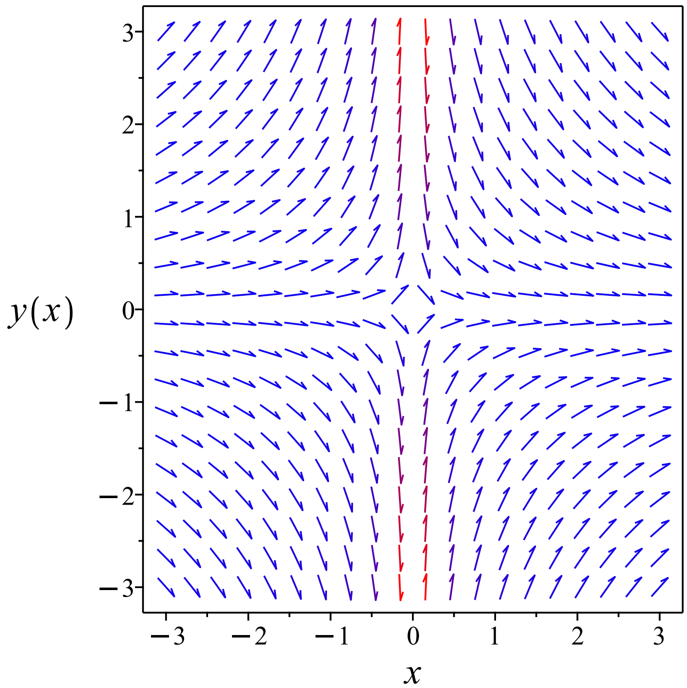


Figure 69: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x} + c_1$$

Verified OK.

4.13.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 63: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = xy$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$xy = c_1$$

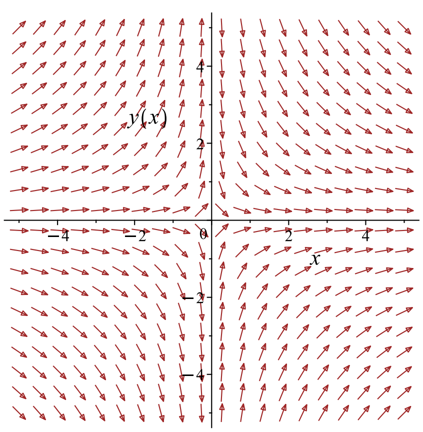
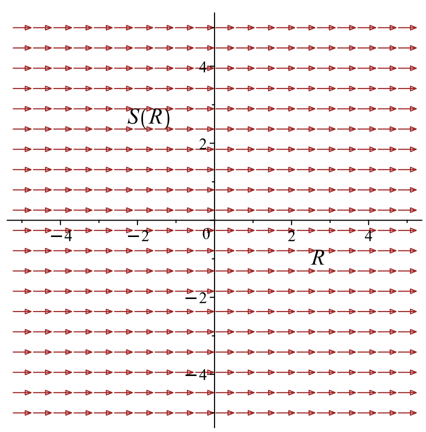
Which simplifies to

$$xy = c_1$$

Which gives

$$y = \frac{c_1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{x}$ 	$R = x$ $S = xy$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} \tag{1}$$

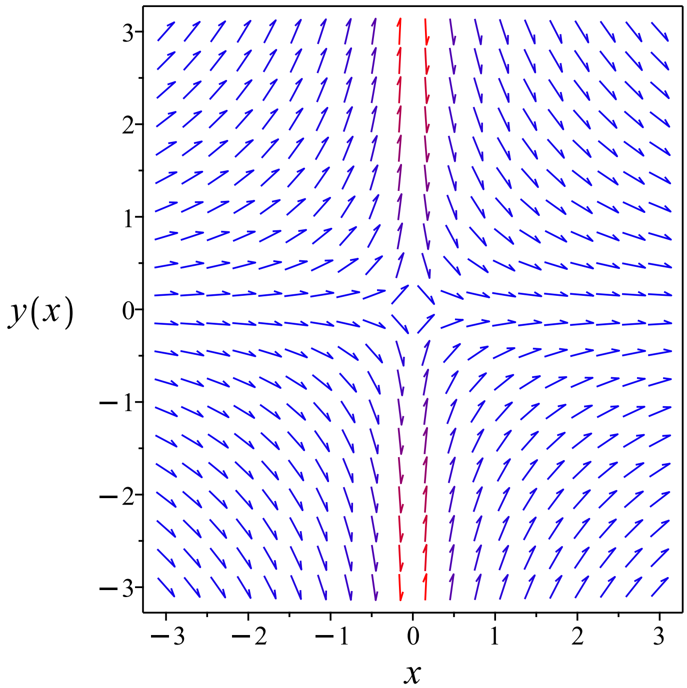


Figure 70: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{c_1}{x}$$

Verified OK.

4.13.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right)\mathrm{d}y = \left(\frac{1}{x}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x}\right)\mathrm{d}x + \left(-\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x}$$
$$N(x,y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{1}{x}\, \mathrm{d}x\end{aligned}$$

$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\, \mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y\\ f(y) &= -\ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(y)$$

The solution becomes

$$y = \frac{e^{-c_1}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-c_1}}{x} \tag{1}$$

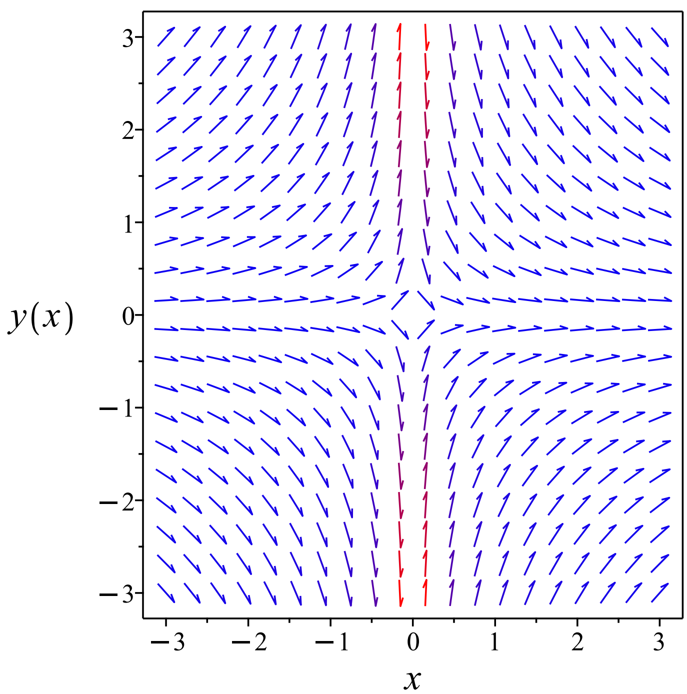


Figure 71: Slope field plot

Verification of solutions

$$y = \frac{e^{-c_1}}{x}$$

Verified OK.

4.13.7 Maple step by step solution

Let's solve

$y\,e^{xy} + x\,e^{xy}y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (y\,e^{xy} + x\,e^{xy}y')\,dx = \int 0dx + c_1$

- Evaluate integral

  $e^{xy} = c_1$

- Solve for y

  $y = \frac{\ln(c_1)}{x}$

Maple trace

```
`Classification methods on request
Methods to be used are: [exact]
----------------------------
* Tackling ODE using method: exact
--- Trying classification methods ---
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 9

```
dsolve(y(x)*exp(x*y(x))+x*exp(x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 16

```
DSolve[y[x]*Exp[x*y[x]]+x*Exp[x*y[x]]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{c_1}{x}$$
$$y(x)\to 0$$

4.14 problem Problem 3.21

Internal problem ID [5888]
Internal file name [`OUTPUT/5136_Sunday_June_05_2022_03_25_52_PM_1355447/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{-2xy + e^{y} + \left(y - x^{2} + x\,e^{y}\right) y' = -x}$$

4.14.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(y - x^2 + x\,\mathrm{e}^y\right)\mathrm{d}y &= (-x + 2xy - \mathrm{e}^y)\,\mathrm{d}x\\ (-2xy + \mathrm{e}^y + x)\,\mathrm{d}x + \left(y - x^2 + x\,\mathrm{e}^y\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -2xy + \mathrm{e}^y + x\\ N(x, y) &= y - x^2 + x\,\mathrm{e}^y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2xy + \mathrm{e}^y + x)\\ &= -2x + \mathrm{e}^y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y - x^2 + x\,\mathrm{e}^y\right)\\ &= -2x + \mathrm{e}^y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -2xy + \mathrm{e}^y + x \, \mathrm{d}x$$
$$\phi = x \, \mathrm{e}^y - \left(y - \frac{1}{2}\right) x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x \, \mathrm{e}^y - x^2 + f'(y) \tag{4}$$
$$= x(\mathrm{e}^y - x) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y - x^2 + x \, \mathrm{e}^y$. Therefore equation (4) becomes

$$y - x^2 + x \, \mathrm{e}^y = x(\mathrm{e}^y - x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y) \, \mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\,\mathrm{e}^{y} - \left(y - \frac{1}{2}\right)x^2 + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,\mathrm{e}^{y} - \left(y - \frac{1}{2}\right)x^2 + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$x\,\mathrm{e}^{y} - \left(y - \frac{1}{2}\right)x^2 + \frac{y^2}{2} = c_1 \tag{1}$$

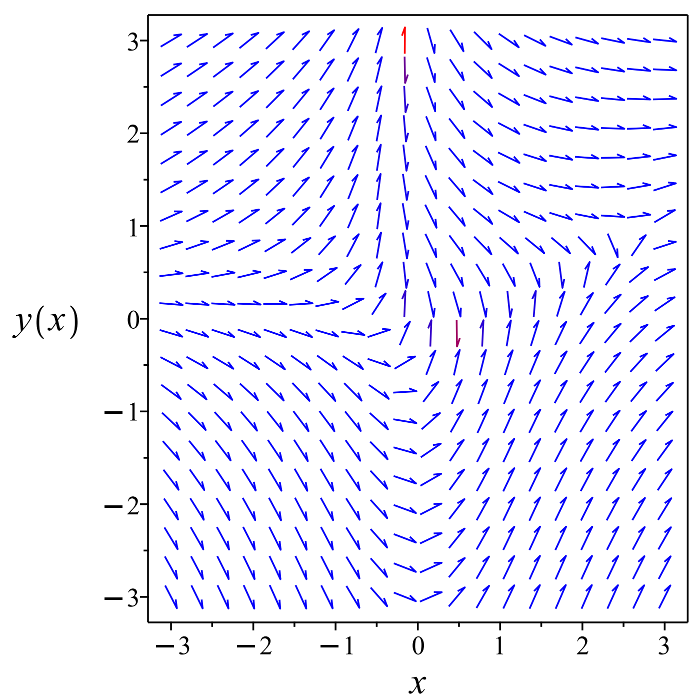


Figure 72: Slope field plot

Verification of solutions

$$x\,\mathrm{e}^{y} - \left(y - \frac{1}{2}\right)x^2 + \frac{y^2}{2} = c_1$$

Verified OK.

4.14.2 Maple step by step solution

Let's solve

$-2xy+\mathrm{e}^{y}+\left(y-x^{2}+x\,\mathrm{e}^{y}\right)y'=-x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$

  ◦ Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  ◦ Evaluate derivatives

  $-2x+\mathrm{e}^{y}=-2x+\mathrm{e}^{y}$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_{1},M(x,y)=F'(x,y),N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\left(-2xy+\mathrm{e}^{y}+x\right)dx+f_{1}(y)$

- Evaluate integral

  $F(x,y)=-y\,x^{2}+x\,\mathrm{e}^{y}+\frac{x^{2}}{2}+f_{1}(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $y-x^{2}+x\,\mathrm{e}^{y}=-x^{2}+x\,\mathrm{e}^{y}+\frac{d}{dy}f_{1}(y)$

- Isolate for $\frac{d}{dy}f_{1}(y)$

  $\frac{d}{dy}f_{1}(y)=y$

- Solve for $f_{1}(y)$

  $f_{1}(y)=\frac{y^{2}}{2}$

- Substitute $f_{1}(y)$ into equation for $F(x,y)$

$$F(x,y) = -y\,x^2 + x\,e^y + \frac{x^2}{2} + \frac{y^2}{2}$$

- Substitute $F(x,y)$ into the solution of the ODE

  $$-y\,x^2 + x\,e^y + \frac{x^2}{2} + \frac{y^2}{2} = c_1$$

- Solve for y

  $$y = RootOf\left(2x^2\_Z - 2\,e^{\_Z}x - \_Z^2 - x^2 + 2c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve((x-2*x*y(x)+exp(y(x)))+(y(x)-x^2+x*exp(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$-x^2y(x) + x\,e^{y(x)} + \frac{x^2}{2} + \frac{y(x)^2}{2} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.341 (sec). Leaf size: 35

```
DSolve[(x-2*x*y[x]+Exp[y[x]])+(y[x]-x^2+x*Exp[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolution
```

$$\text{Solve}\left[x^2(-y(x)) + \frac{x^2}{2} + xe^{y(x)} + \frac{y(x)^2}{2} = c_1, y(x)\right]$$

4.15 problem Problem 3.22

Internal problem ID [5889]
Internal file name [OUTPUT/5137_Sunday_June_05_2022_03_25_54_PM_22967781/index.tex]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - \frac{y'}{\sqrt{x}} + \frac{\left(x + \sqrt{x} - 8\right) y}{4x^2} = 0}$$

4.15.1 Solving using Kovacic algorithm

Writing the ode as

$$4y''x^{\frac{5}{2}} - 4x^2y' + \left(x^{\frac{3}{2}} + x - 8\sqrt{x}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^{\frac{5}{2}} \\ B &= -4x^2 \\ C &= x^{\frac{3}{2}} + x - 8\sqrt{x} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 67: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\
&= -1 - (-1)\\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty\\
&= -\frac{1}{x} + (-)(0)\\
&= -\frac{1}{x}\\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4x^2}{4x^{\frac{5}{2}}} \, dx} \\ &= z_1 e^{\sqrt{x}} \\ &= z_1 \left(\mathrm{e}^{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\sqrt{x}}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2}{4x^{\frac{5}{2}}} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2\sqrt{x}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\sqrt{x}}}{x}\right) + c_2 \left(\frac{e^{\sqrt{x}}}{x}\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 e^{\sqrt{x}}}{x} + \frac{c_2 x^2 e^{\sqrt{x}}}{3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 e^{\sqrt{x}}}{x} + \frac{c_2 x^2 e^{\sqrt{x}}}{3}
$$

Verified OK.
Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-1/x^(1/2)*diff(y(x),x)+1/(4*x^2)*(x+x^(1/2)-8)*y(x)=0,y(x), singsol=al
```

$$
y(x) = \frac{e^{\sqrt{x}}(c_2 x^3 + c_1)}{x}
$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 30

```
DSolve[y''[x]-1/x^(1/2)*y'[x]+1/(4*x^2)*(x+x^(1/2)-8)*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{e^{\sqrt{x}}(c_2x^3 + 3c_1)}{3x}$$

4.16 problem Problem 3.23

Internal problem ID [5890]
Internal file name [`OUTPUT/5138_Sunday_June_05_2022_03_25_56_PM_22904374/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{(-x^2+1)\,z''+(1-3x)\,z'+kz=0}$$

4.16.1 Maple step by step solution

Let's solve

$(-x^2+1)\,z''+(1-3x)\,z'+kz=0$

- Highest derivative means the order of the ODE is 2

  z''

- Isolate 2nd derivative

  $z''=-\frac{(3x-1)z'}{x^2-1}+\frac{kz}{x^2-1}$

- Group terms with z on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $z''+\frac{(3x-1)z'}{x^2-1}-\frac{kz}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{3x-1}{x^2-1}, P_3(x) = -\frac{k}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $((1+x) \cdot P_2(x))\Big|_{x=-1} = 2$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $((1+x)^2 \cdot P_3(x))\Big|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $z''(x^2-1) + (3x-1)z' - kz = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d^2}{du^2}z(u)\right) + (3u-4)\left(\frac{d}{du}z(u)\right) - kz(u) = 0$

- Assume series solution for $z(u)$

  $z(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}z(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}z(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}z(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}z(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}z(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}z(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$-2a_0r(1+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+r+1)\,(k+2+r)-a_k(-k^2-2kr-r^2+k-2k-2r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(1+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{-1,0\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2a_{k+1}(k+r+1)\,(k+2+r)-\left(-k^2+(-2r-2)\,k-r^2+k-2r\right)a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{(-k^2-2kr-r^2+k-2k-2r)\,a_k}{2(k+r+1)(k+2+r)}$$

- Recursion relation for $r=-1$

$$a_{k+1}=-\frac{(-k^2+k+1)\,a_k}{2k(k+1)}$$

- Solution for $r=-1$

$$\left[z(u)=\sum_{k=0}^{\infty}a_k u^{k-1},a_{k+1}=-\frac{(-k^2+k+1)\,a_k}{2k(k+1)}\right]$$

- Revert the change of variables $u=1+x$

$$\left[z=\sum_{k=0}^{\infty}a_k(1+x)^{k-1},a_{k+1}=-\frac{(-k^2+k+1)\,a_k}{2k(k+1)}\right]$$

- Recursion relation for $r=0$

$$a_{k+1}=-\frac{(-k^2+k-2k)\,a_k}{2(k+1)(k+2)}$$

- Solution for $r=0$

$$\left[z(u)=\sum_{k=0}^{\infty}a_k u^{k},a_{k+1}=-\frac{(-k^2+k-2k)\,a_k}{2(k+1)(k+2)}\right]$$

- Revert the change of variables $u=1+x$

$$\left[z=\sum_{k=0}^{\infty}a_k(1+x)^{k},a_{k+1}=-\frac{(-k^2+k-2k)\,a_k}{2(k+1)(k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[z=\left(\sum_{m=0}^{\infty}a_m(1+x)^{m-1}\right)+\left(\sum_{m=0}^{\infty}b_m(1+x)^{m}\right),a_{m+1}=-\frac{(-m^2+k+1)\,a_m}{2m(m+1)},b_{m+1}=-\frac{(-m^2+k-2m)\,b_m}{2(m+1)(m+2)}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
      -> solution has integrals; searching for one without integrals...
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric solution without integrals succesful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 99

```
dsolve((1-x^2)*diff(z(x),x$2)+(1-3*x)*diff(z(x),x)+k*z(x)=0,z(x), singsol=all)
```

$$z(x) = c_1(x+1)^{-1-\sqrt{k+1}}\,\mathrm{hypergeom}\left(\left[\sqrt{k+1}, 1+\sqrt{k+1}\right], \left[1+2\sqrt{k+1}\right], \frac{2}{x+1}\right) + c_2(x+1)^{-1+\sqrt{k+1}}\,\mathrm{hypergeom}\left(\left[-\sqrt{k+1}, 1-\sqrt{k+1}\right], \left[1-2\sqrt{k+1}\right], \frac{2}{x+1}\right)$$

✓ Solution by Mathematica

Time used: 0.407 (sec). Leaf size: 77

```
DSolve[(1-x^2)*z''[x]+(1-3*x)*z'[x]+k*z[x]==0,z[x],x,IncludeSingularSolutions -> True]
```

$$z(x) \to c_2 G_{2,2}^{2,0}\left(\frac{1-x}{2}\middle| \begin{array}{c} -\sqrt{k+1}, \sqrt{k+1} \\ 0,0 \end{array}\right) + c_1 \text{Hypergeometric2F1}\left(1-\sqrt{k+1}, \sqrt{k+1}+1, 1, \frac{1-x}{2}\right)$$

4.17 problem Problem 3.24

Internal problem ID [5891]
Internal file name [`OUTPUT/5139_Sunday_June_05_2022_03_25_59_PM_29405420/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{(-x^2+1)\,\eta'' - (1+x)\,\eta' + (k+1)\,\eta = 0}$$

4.17.1 Maple step by step solution

Let's solve

$(-x^2+1)\,\eta'' + (-1-x)\,\eta' + (k+1)\,\eta = 0$

- Highest derivative means the order of the ODE is 2

  η''

- Isolate 2nd derivative

  $\eta'' = \frac{(k+1)\eta}{x^2-1} - \frac{\eta'}{x-1}$

- Group terms with η on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\eta'' + \frac{\eta'}{x-1} - \frac{(k+1)\eta}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{1}{x-1}, P_3(x) = -\frac{k+1}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $((1+x) \cdot P_2(x))\bigg|_{x=-1} = 0$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $\eta''(x-1)(x^2-1) + \eta'(x^2-1) - (x-1)(k+1)\eta = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - 4u^2 + 4u)\left(\frac{d^2}{du^2}\eta(u)\right) + (u^2 - 2u)\left(\frac{d}{du}\eta(u)\right) + (-uk + 2k - u + 2)\eta(u) = 0$

- Assume series solution for $\eta(u)$

  $\eta(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \eta(u)$ to series expansion for $m = 0..1$

  $u^m \cdot \eta(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m \cdot \eta(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\eta(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d}{du}\eta(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}\eta(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}\eta(u)\right)$ to series expansion for $m = 1..3$

$$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}} \eta(u)\right) = \sum_{k=0}^{\infty} a_{k} (k+r) (k+r-1) u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}} \eta(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r) (k+1-m+r) u^{k+r}$$

Rewrite ODE with series expansions

$$4 a_{0} r(-1+r) u^{-1+r} + \left(4 a_{1}(1+r) r + 2 a_{0}\left(-2 r^{2}+k+r+1\right)\right) u^{r} + \left(\sum_{k=1}^{\infty} \left(4 a_{k+1}(k+1+r)(k+r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4 r(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0,1\}$$

- Each term must be 0

$$4 a_{1}(1+r) r + 2 a_{0}\left(-2 r^{2}+k+r+1\right) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$\left(-4 a_{k}+a_{k-1}+4 a_{k+1}\right) k^{2} + \left(\left(-8 a_{k}+2 a_{k-1}+8 a_{k+1}\right) r + 2 a_{k} - 2 a_{k-1} + 4 a_{k+1}\right) k + \left(-4 a_{k}+a_{k-}\right.$$

- Shift index using $k->k+1$

$$\left(-4 a_{k+1}+a_{k}+4 a_{k+2}\right)(k+1)^{2} + \left(\left(-8 a_{k+1}+2 a_{k}+8 a_{k+2}\right) r + 2 a_{k+1} - 2 a_{k} + 4 a_{k+2}\right)(k+1) + ($$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{-k^{2} a_{k}+4 k^{2} a_{k+1}-2 k r a_{k}+8 k r a_{k+1}-r^{2} a_{k}+4 r^{2} a_{k+1}+a_{k} k-2 k a_{k+1}+6 k a_{k+1}+6 r a_{k+1}+a_{k}}{4\left(k^{2}+2 k r+r^{2}+3 k+3 r+2\right)}$$

- Recursion relation for $r=0$

$$a_{k+2} = \frac{-k^{2} a_{k}+4 k^{2} a_{k+1}+a_{k} k-2 k a_{k+1}+6 k a_{k+1}+a_{k}}{4\left(k^{2}+3 k+2\right)}$$

- Solution for $r=0$

$$\left[\eta(u) = \sum_{k=0}^{\infty} a_{k} u^{k}, a_{k+2} = \frac{-k^{2} a_{k}+4 k^{2} a_{k+1}+a_{k} k-2 k a_{k+1}+6 k a_{k+1}+a_{k}}{4\left(k^{2}+3 k+2\right)}, 2 a_{0}(k+1) = 0\right]$$

- Revert the change of variables $u=1+x$

$$\left[\eta = \sum_{k=0}^{\infty} a_{k}(1+x)^{k}, a_{k+2} = \frac{-k^{2} a_{k}+4 k^{2} a_{k+1}+a_{k} k-2 k a_{k+1}+6 k a_{k+1}+a_{k}}{4\left(k^{2}+3 k+2\right)}, 2 a_{0}(k+1) = 0\right]$$

- Recursion relation for $r=1$

$$a_{k+2} = \frac{-k^{2} a_{k}+4 k^{2} a_{k+1}+a_{k} k-2 k a_{k+1}-2 k a_{k}+14 k a_{k+1}+10 a_{k+1}}{4\left(k^{2}+5 k+6\right)}$$

- Solution for $r=1$

$$\left[\eta(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = \frac{-k^2 a_k + 4k^2 a_{k+1} + a_k k - 2k a_{k+1} - 2k a_k + 14k a_{k+1} + 10 a_{k+1}}{4(k^2+5k+6)}, 2a_0 k + 8a_1 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[\eta = \sum_{k=0}^{\infty} a_k (1+x)^{k+1}, a_{k+2} = \frac{-k^2 a_k + 4k^2 a_{k+1} + a_k k - 2k a_{k+1} - 2k a_k + 14k a_{k+1} + 10 a_{k+1}}{4(k^2+5k+6)}, 2a_0 k + 8a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[\eta = \left(\sum_{m=0}^{\infty} a_m (1+x)^m\right) + \left(\sum_{m=0}^{\infty} b_m (1+x)^{m+1}\right), a_{m+2} = \frac{-m^2 a_m + 4m^2 a_{m+1} + k a_m - 2k a_{m+1} + 6m a_{m+1} + a_m}{4(m^2+3m+2)}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
      -> solution has integrals; searching for one without integrals...
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric solution without integrals succesful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 95

```
dsolve((1-x^2)*diff(eta(x),x$2)-(1+x)*diff(eta(x),x)+(k+1)*eta(x)=0,eta(x), singsol=all)
```

$$\eta(x) = c_1(x+1)^{\sqrt{k+1}} \operatorname{hypergeom}\left(\left[-\sqrt{k+1}, 1-\sqrt{k+1}\right], \left[1-2\sqrt{k+1}\right], \frac{2}{x+1}\right) + c_2(x+1)^{-\sqrt{k+1}} \operatorname{hypergeom}\left(\left[\sqrt{k+1}, 1+\sqrt{k+1}\right], \left[1+2\sqrt{k+1}\right], \frac{2}{x+1}\right)$$

✓ Solution by Mathematica

Time used: 0.283 (sec). Leaf size: 77

```
DSolve[(1-x^2)*z''[x]-(1+x)*z'[x]+(k+1)*z[x]==0,z[x],x,IncludeSingularSolutions -> True]
```

$$z(x) \to c_2 G_{2,2}^{2,0}\left(\frac{1-x}{2} \middle| \begin{array}{c} 1-\sqrt{k+1}, \sqrt{k+1}+1 \\ 0, 0 \end{array}\right) + c_1 \operatorname{Hypergeometric2F1}\left(-\sqrt{k+1}, \sqrt{k+1}, 1, \frac{1-x}{2}\right)$$

4.18 problem Problem 3.31

Internal problem ID [5892]
Internal file name [`OUTPUT/5140_Sunday_June_05_2022_03_26_03_PM_90099132/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 - 2xyy' = -x^2}$$

4.18.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 - 2x^2u(x)\,(u'(x)\,x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2-1}{2ux} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{u}}\, du = -\frac{1}{2x}\, dx$$

$$\int \frac{1}{\frac{u^2-1}{u}}\, du = \int -\frac{1}{2x}\, dx$$

$$\frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} = -\frac{\ln(x)}{2} + c_2$$

The above can be written as

$$\begin{aligned}
\left(\frac{1}{2}\right)(\ln(u-1) + \ln(u+1)) &= -\frac{\ln(x)}{2} + 2c_2 \\
\ln(u-1) + \ln(u+1) &= (2)\left(-\frac{\ln(x)}{2} + 2c_2\right) \\
&= -\ln(x) + 4c_2
\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u-1)+\ln(u+1)} = e^{-\ln(x)+2c_2}$$

Which simplifies to

$$\begin{aligned}
u^2 - 1 &= \frac{2c_2}{x} \\
&= \frac{c_3}{x}
\end{aligned}$$

The solution is

$$u(x)^2 - 1 = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x}$$

$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x}$$

Summary

The solution(s) found are the following

$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x} \tag{1}$$

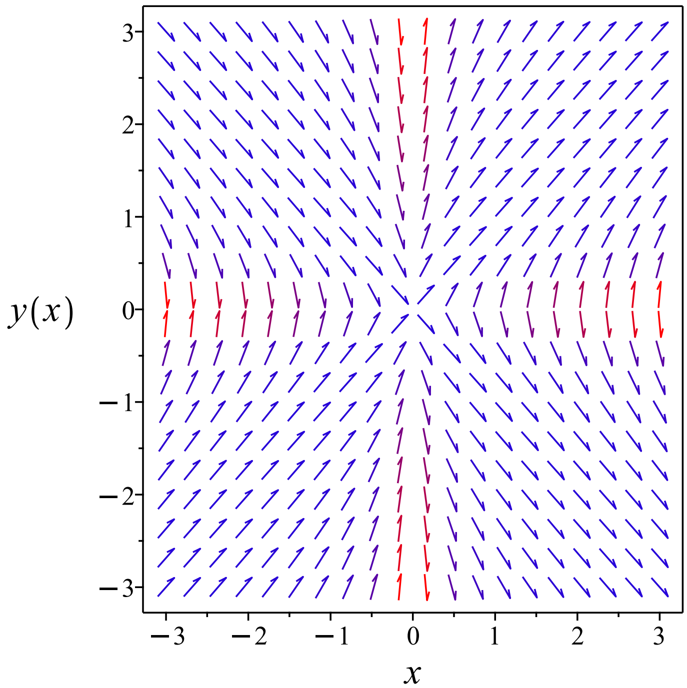


Figure 73: Slope field plot

Verification of solutions

$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x}$$

Verified OK.

4.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + y^2}{2xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 70: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x}{y}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 + y^2}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2}{2x^2} \\ S_y &= \frac{y}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \tag{4}$$

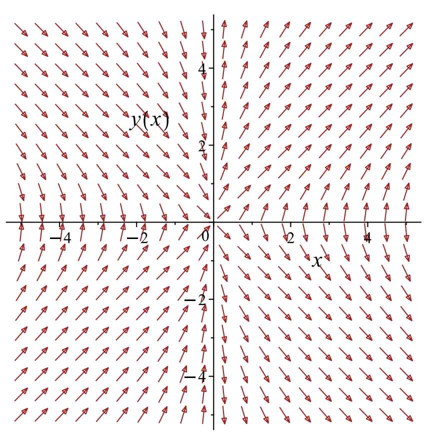
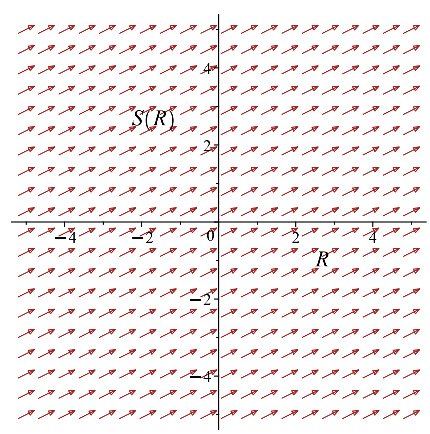
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x} = \frac{x}{2} + c_1$$

Which simplifies to

$$\frac{y^2}{2x} = \frac{x}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = \frac{1}{2}$ 

<u>Summary</u>
The solution(s) found are the following

$$\frac{y^2}{2x} = \frac{x}{2} + c_1 \tag{1}$$

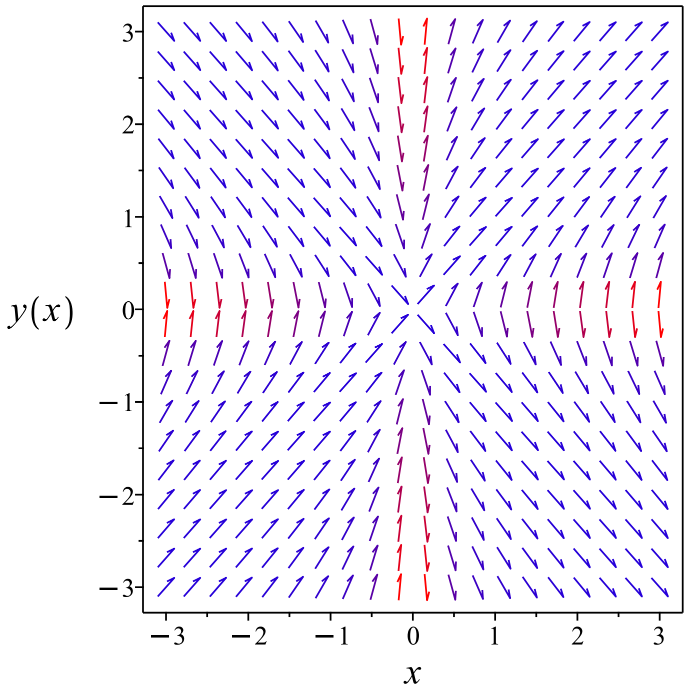


Figure 74: Slope field plot

Verification of solutions

$$\frac{y^2}{2x} = \frac{x}{2} + c_1$$

Verified OK.

4.18.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + y^2}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y + \frac{x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{2x} \\
f_1(x) &= \frac{x}{2} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = \frac{y^2}{2x} + \frac{x}{2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{w(x)}{2x} + \frac{x}{2} \\
w' &= \frac{w}{x} + x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= x
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(x) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(x) \\ d\left(\frac{w}{x}\right) &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int dx \\ \frac{w}{x} &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = c_1 x + x^2$$

which simplifies to

$$w(x) = x(x + c_1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x(x + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{x(x + c_1)} \\ y(x) &= -\sqrt{x(x + c_1)}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x(x + c_1)} \tag{1}$$
$$y = -\sqrt{x(x + c_1)} \tag{2}$$

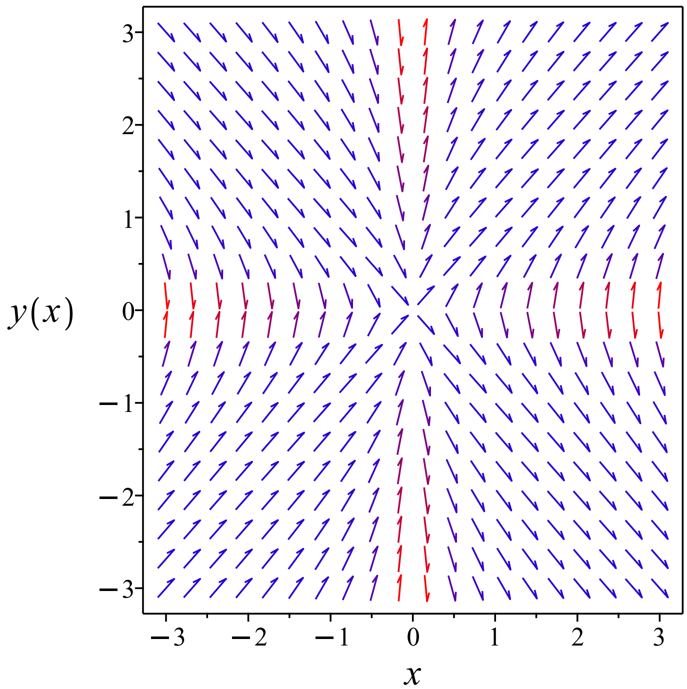


Figure 75: Slope field plot

Verification of solutions

$$y = \sqrt{x(x + c_1)}$$

Verified OK.

$$y = -\sqrt{x(x + c_1)}$$

Verified OK.

4.18.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2xy)\,\mathrm{d}y &= \left(-x^2 - y^2\right)\mathrm{d}x\\ \left(x^2 + y^2\right)\mathrm{d}x + (-2xy)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x^2 + y^2\\ N(x,y) &= -2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(x^{2}+y^{2}\right)\\&=2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(-2xy)\\&=-2y\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\&=-\frac{1}{2yx}((2y)-(-2y))\\&=-\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu&=e^{\int A\,\mathrm{d}x}\\&=e^{\int-\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu&=e^{-2\ln(x)}\\&=\frac{1}{x^{2}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M}&=\mu M\\&=\frac{1}{x^{2}}\left(x^{2}+y^{2}\right)\\&=\frac{x^{2}+y^{2}}{x^{2}}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(-2xy) \\
&= -\frac{2y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^2+y^2}{x^2}\right) + \left(-\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x^2+y^2}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x - \frac{y^2}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{2y}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{x}$. Therefore equation (4) becomes

$$
-\frac{2y}{x} = -\frac{2y}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x - \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x - \frac{y^2}{x}$$

Summary

The solution(s) found are the following

$$x - \frac{y^2}{x} = c_1 \tag{1}$$

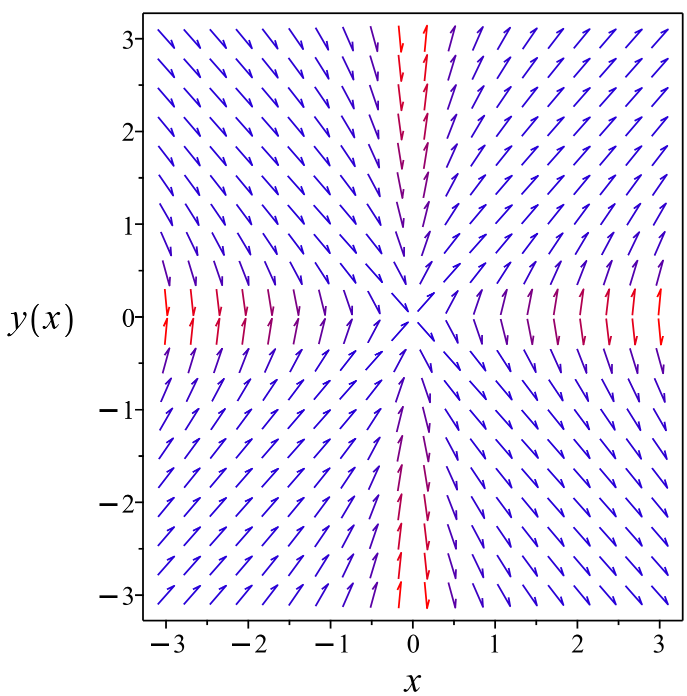


Figure 76: Slope field plot

Verification of solutions

$$x - \frac{y^2}{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve((x^2+y(x)^2)-2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{(x + c_1)\, x}$$
$$y(x) = -\sqrt{(x + c_1)\, x}$$

✓ Solution by Mathematica

Time used: 0.2 (sec). Leaf size: 38

```
DSolve[(x^2+y[x]^2)-2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x}\sqrt{x + c_1}$$
$$y(x) \to \sqrt{x}\sqrt{x + c_1}$$

4.19 problem Problem 3.32

Internal problem ID [5893]
Internal file name [`OUTPUT/5141_Sunday_June_05_2022_03_26_05_PM_40007959/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{-y^2 + 2xyy' = -x^2}$$

4.19.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-u(x)^2\,x^2 + 2x^2u(x)\,(u'(x)\,x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2+1}{2ux} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u}}\,du = -\frac{1}{2x}\,dx$$
$$\int \frac{1}{\frac{u^2+1}{u}}\,du = \int -\frac{1}{2x}\,dx$$
$$\frac{\ln(u^2+1)}{2} = -\frac{\ln(x)}{2} + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+1} = \mathrm{e}^{-\frac{\ln(x)}{2}+c_2}$$

Which simplifies to

$$\sqrt{u^2+1} = \frac{c_3}{\sqrt{x}}$$

Which simplifies to

$$\sqrt{u(x)^2+1} = \frac{c_3\mathrm{e}^{c_2}}{\sqrt{x}}$$

The solution is

$$\sqrt{u(x)^2+1} = \frac{c_3\mathrm{e}^{c_2}}{\sqrt{x}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}+1} = \frac{c_3\mathrm{e}^{c_2}}{\sqrt{x}}$$
$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3\mathrm{e}^{c_2}}{\sqrt{x}}$$

<u>Summary</u>

The solution(s) found are the following

$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3\mathrm{e}^{c_2}}{\sqrt{x}} \tag{1}$$

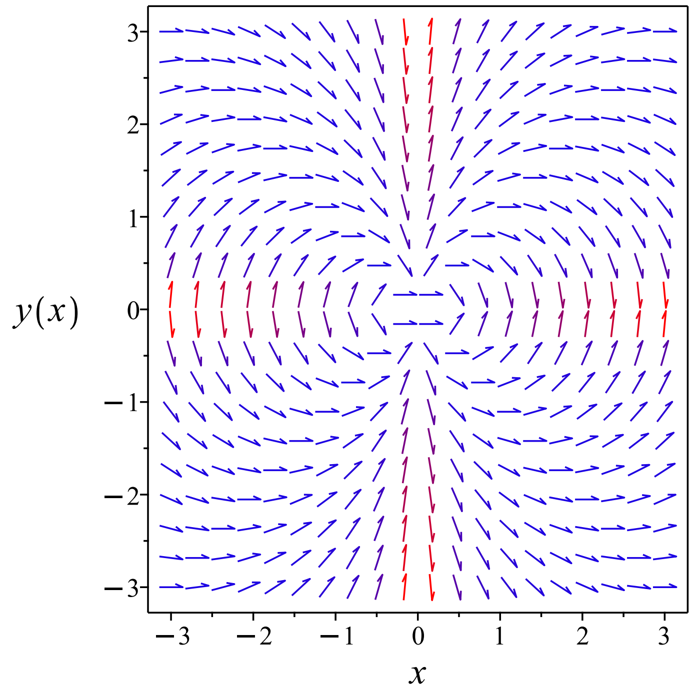


Figure 77: Slope field plot

Verification of solutions

$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}$$

Verified OK.

4.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-x^2+y^2}{2xy}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 72: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \frac{x}{y} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x^2 + y^2}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2}{2x^2} \\ S_y &= \frac{y}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{R}{2} + c_1 \tag{4}$$

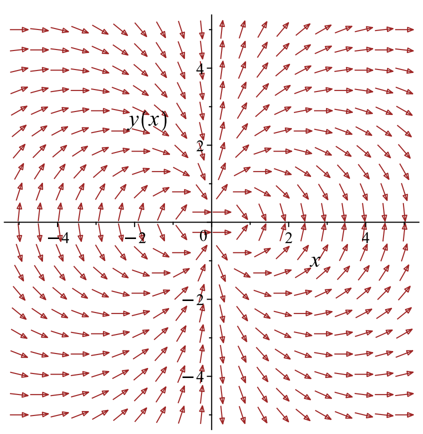
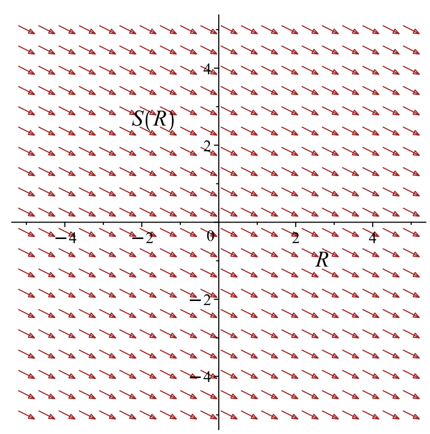
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x} = -\frac{x}{2} + c_1$$

Which simplifies to

$$\frac{y^2}{2x} = -\frac{x}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x^2+y^2}{2xy}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = -\frac{1}{2}$ 

Summary

The solution(s) found are the following

$$\frac{y^2}{2x} = -\frac{x}{2} + c_1 \tag{1}$$

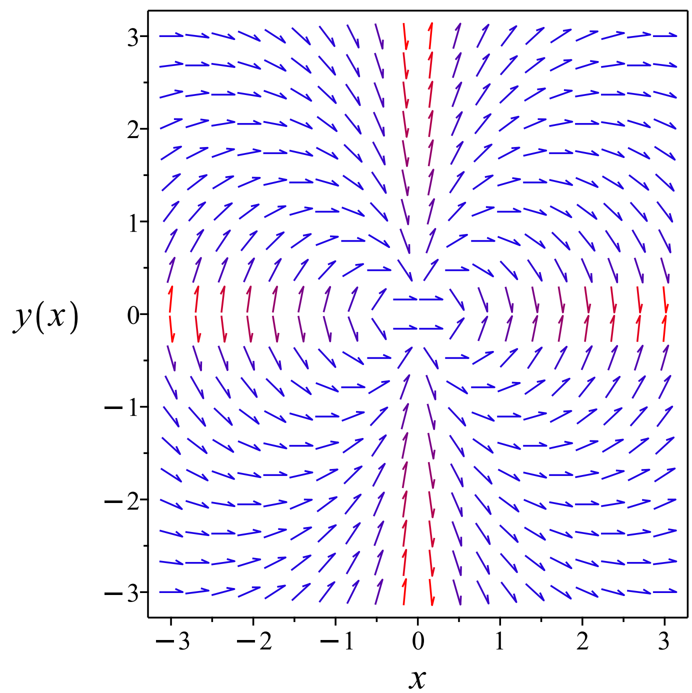


Figure 78: Slope field plot

Verification of solutions

$$\frac{y^2}{2x} = -\frac{x}{2} + c_1$$

Verified OK.

4.19.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{-x^2 + y^2}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y - \frac{x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{2x} \\ f_1(x) &= -\frac{x}{2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{y^2}{2x} - \frac{x}{2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{w(x)}{2x} - \frac{x}{2} \\ w' &= \frac{w}{x} - x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -x \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = -x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-x) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(-x) \\ d\left(\frac{w}{x}\right) &= -1\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int -1\,dx \\ \frac{w}{x} &= -x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = c_1 x - x^2$$

which simplifies to

$$w(x) = x(-x + c_1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x(-x + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{x\,(-x + c_1)} \\ y(x) &= -\sqrt{x\,(-x + c_1)}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x(-x + c_1)} \tag{1}$$
$$y = -\sqrt{x(-x + c_1)} \tag{2}$$

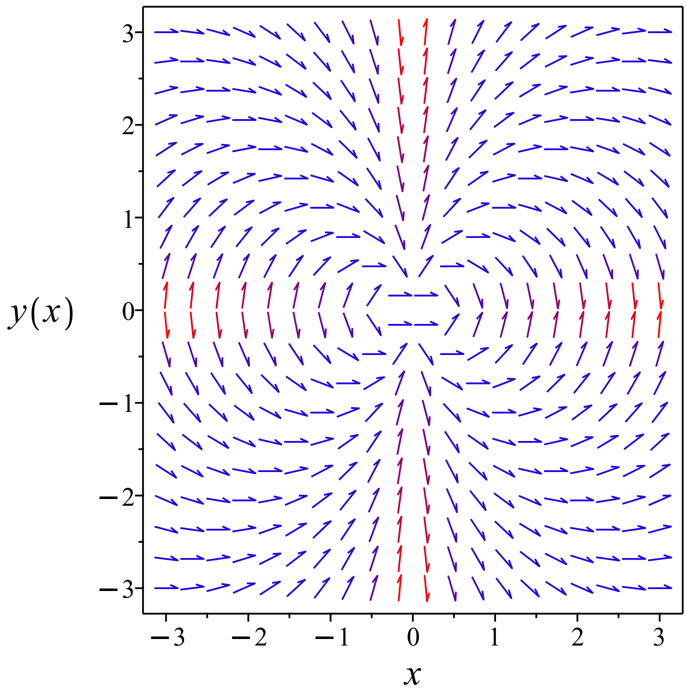


Figure 79: Slope field plot

Verification of solutions

$$y = \sqrt{x(-x + c_1)}$$

Verified OK.

$$y = -\sqrt{x(-x + c_1)}$$

Verified OK.

4.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(2xy)\,\mathrm{d}y = \left(-x^2 + y^2\right)\mathrm{d}x$$
$$\left(x^2 - y^2\right)\mathrm{d}x + (2xy)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = x^2 - y^2$$
$$N(x,y) = 2xy$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(x^2-y^2\right)\\&=-2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(2xy)\\&=2y\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\&=\frac{1}{2yx}((-2y)-(2y))\\&=-\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu&=e^{\int A\,\mathrm{d}x}\\&=e^{\int-\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu&=e^{-2\ln(x)}\\&=\frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M}&=\mu M\\&=\frac{1}{x^2}\left(x^2-y^2\right)\\&=\frac{x^2-y^2}{x^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(2xy) \\
&= \frac{2y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^2 - y^2}{x^2}\right) + \left(\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x^2 - y^2}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x + \frac{y^2}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{2y}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y}{x}$. Therefore equation (4) becomes

$$
\frac{2y}{x} = \frac{2y}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x + \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x + \frac{y^2}{x}$$

Summary

The solution(s) found are the following

$$x + \frac{y^2}{x} = c_1 \tag{1}$$

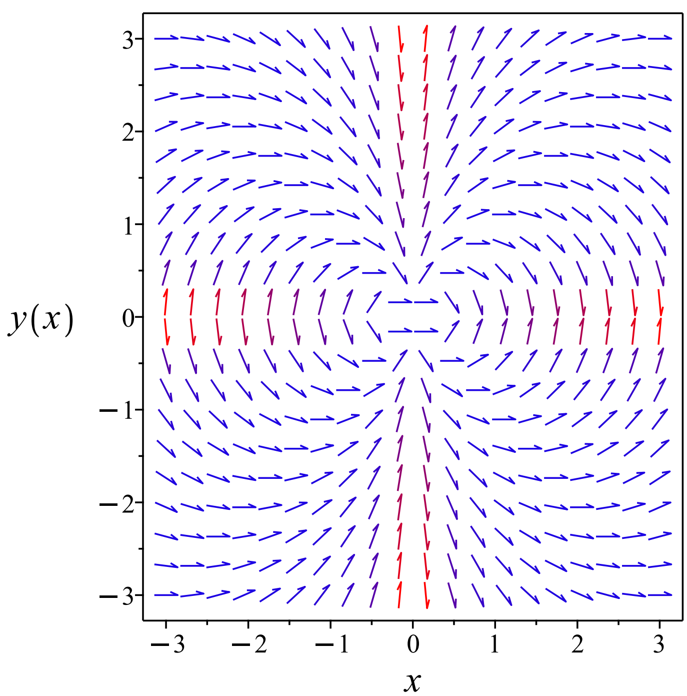


Figure 80: Slope field plot

Verification of solutions

$$x + \frac{y^2}{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve((x^2-y(x)^2)+2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{(-x + c_1)\,x}$$
$$y(x) = -\sqrt{(-x + c_1)\,x}$$

✓ Solution by Mathematica

Time used: 0.355 (sec). Leaf size: 37

```
DSolve[(x^2-y[x]^2)+2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x(x - c_1)}$$
$$y(x) \to \sqrt{-x(x - c_1)}$$

4.20 problem Problem 3.33

Internal problem ID [5894]
Internal file name [`OUTPUT/5142_Sunday_June_05_2022_03_26_08_PM_99395595/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "exactByInspection", "homogeneousTypeD2"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Riccati]
```

$$\boxed{-y + xy' - y^2 = x^2}$$

4.20.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$-u(x)\, x + x(u'(x)\, x + u(x)) - u(x)^2\, x^2 = x^2$$

Integrating both sides gives

$$\int \frac{1}{u^2+1} du = c_2 + x$$
$$\arctan(u) = c_2 + x$$

Solving for u gives these solutions

$$u_1 = \tan(c_2 + x)$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x \tan (c_2 + x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x \tan (c_2 + x) \tag{1}$$

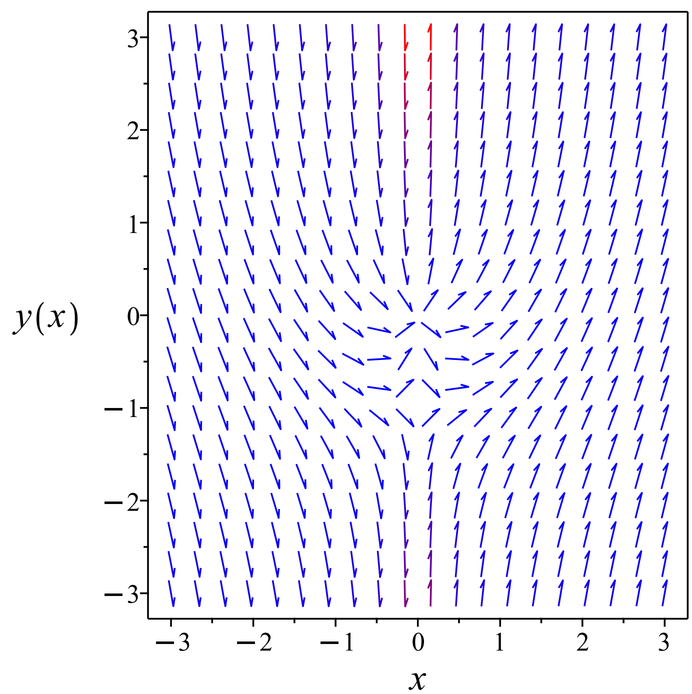


Figure 81: Slope field plot

Verification of solutions

$$y = x \tan (c_2 + x)$$

Verified OK.

4.20.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(x^2+y^2+y\right)\mathrm{d}x\\ \left(-x^2-y^2-y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x^2-y^2-y\\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 - y^2 - y\right) \\ &= -2y - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = -y^2 - x^2 - y$ and $N = x$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{-y^2 - x^2 - y}{x^2 + y^2} \\ N &= \frac{x}{x^2 + y^2}\end{aligned}$$

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{x}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{-x^2-y^2-y}{x^2+y^2}\right)\mathrm{d}x$$

$$\left(\frac{-x^2-y^2-y}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{x}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= \frac{-x^2-y^2-y}{x^2+y^2} \\ N(x, y) &= \frac{x}{x^2+y^2} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-x^2-y^2-y}{x^2+y^2}\right) \\ &= \frac{-x^2+y^2}{(x^2+y^2)^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) \\ &= \frac{-x^2+y^2}{(x^2+y^2)^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$

$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int \frac{-x^2-y^2-y}{x^2+y^2}\,\mathrm{d}x$$

$$\phi=-x-\arctan\left(\frac{x}{y}\right)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=\frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)}+f'(y) \tag{4}$$

$$=\frac{x}{x^2+y^2}+f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{x}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{x}{x^2+y^2}=\frac{x}{x^2+y^2}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=-x-\arctan\left(\frac{x}{y}\right)+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \arctan\left(\frac{x}{y}\right)$$

The solution becomes

$$y = -\frac{x}{\tan(x + c_1)}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{\tan(x + c_1)} \tag{1}$$

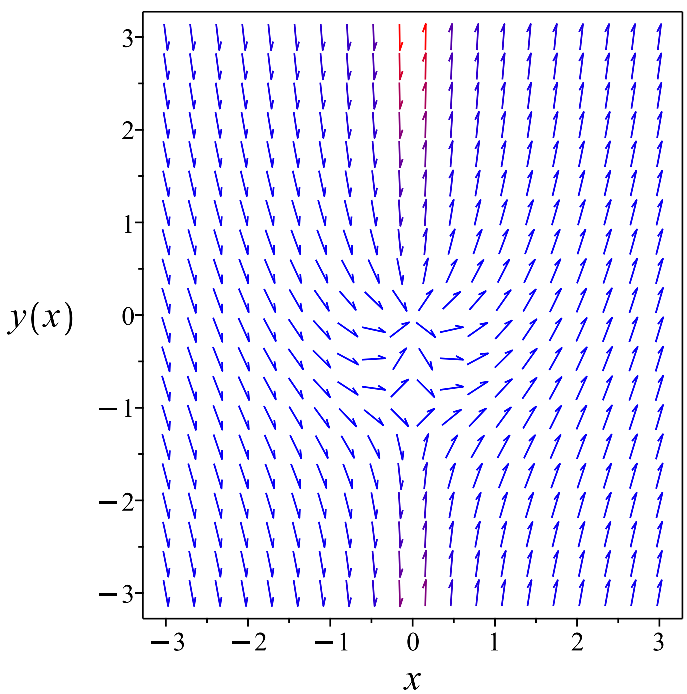


Figure 82: Slope field plot

Verification of solutions

$$y = -\frac{x}{\tan(x + c_1)}$$

Verified OK.

4.20.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^2+y^2+y}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x + \frac{y^2}{x} + \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{x^2} \\ f_1 f_2 &= \frac{1}{x^2} \\ f_2^2 f_0 &= \frac{1}{x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x} + \frac{u(x)}{x} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sin(x)\, c_1 + c_2 \cos(x)$$

The above shows that

$$u'(x) = \cos(x) c_1 - c_2 \sin(x)$$

Using the above in (1) gives the solution

$$y = -\frac{(\cos(x) c_1 - c_2 \sin(x)) x}{\sin(x) c_1 + c_2 \cos(x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x(-c_3 \cos(x) + \sin(x))}{c_3 \sin(x) + \cos(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{x(-c_3 \cos(x) + \sin(x))}{c_3 \sin(x) + \cos(x)} \tag{1}$$

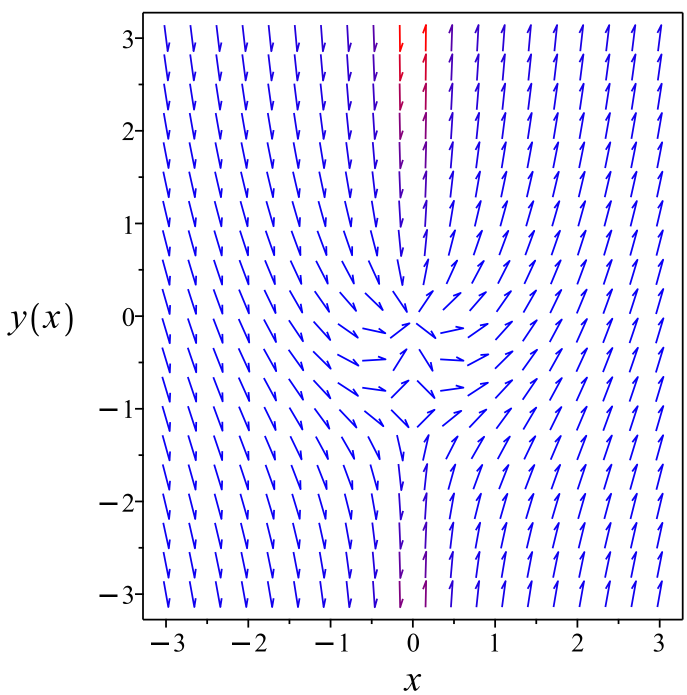


Figure 83: Slope field plot

Verification of solutions

$$y = \frac{x(-c_3 \cos(x) + \sin(x))}{c_3 \sin(x) + \cos(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(x*diff(y(x),x)-y(x)=(x^2+y(x)^2),y(x), singsol=all)
```

$$y(x) = \tan(x + c_1) x$$

✓ Solution by Mathematica

Time used: 0.18 (sec). Leaf size: 12

```
DSolve[x*y'[x]-y[x]==(x^2+y[x]^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow x \tan(x + c_1)$$

4.21 problem Problem 3.34

Internal problem ID [5895]
Internal file name [`OUTPUT/5143_Sunday_June_05_2022_03_26_10_PM_22258225/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$-y + xy' - x\sqrt{x^2 - y^2}\, y' = 0$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5`[0, (x^2-y^2)^(1/2)/((x^2-y^2)^(1/2)-1)]
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 27

```
dsolve(x*diff(y(x),x)-y(x)=x*sqrt(x^2-y(x)^2)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) - \arctan\left(\frac{y(x)}{\sqrt{x^2 - y(x)^2}}\right) - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.51 (sec). Leaf size: 29

```
DSolve[x*y'[x]-y[x]==x*Sqrt[x^2-y[x]^2]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\arctan\left(\frac{\sqrt{x^2 - y(x)^2}}{y(x)}\right) + y(x) = c_1, y(x)\right]$$

4.22 problem Problem 3.35

Internal problem ID [5896]
Internal file name [`OUTPUT/5144_Sunday_June_05_2022_03_26_12_PM_72686177/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y'y + y - xy' = -x}$$

4.22.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,u(x)\,x + u(x)\,x - x(u'(x)\,x + u(x)) = -x$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u^2+1}{x\,(u-1)}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2+1}{u-1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u^2+1)}{2} - \arctan(u) = -\ln(x) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2+1\right)}{2} - \arctan(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

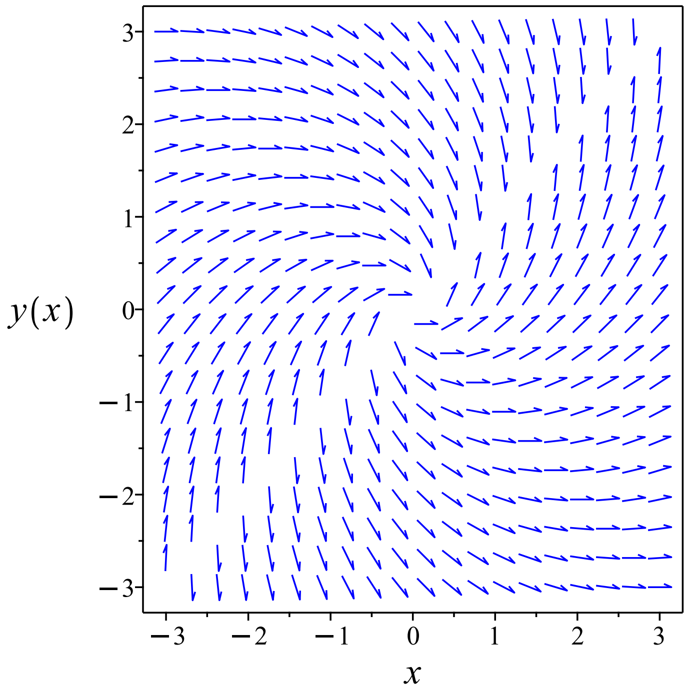


Figure 84: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2}-\arctan\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Verified OK.

4.22.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{x+y}{-x+y}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(x+y)(b_3-a_2)}{-x+y} - \frac{(x+y)^2 a_3}{(-x+y)^2} \\ & - \left(-\frac{1}{-x+y} - \frac{x+y}{(-x+y)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{1}{-x+y} + \frac{x+y}{(-x+y)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 + 2xyb_2 + 2xyb_3 - y^2a_2 - y^2a_3 - y^2b_2 + y^2b_3 + 2xb_1 - 2ya}{(x-y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -x^2a_2 - x^2a_3 - x^2b_2 + x^2b_3 + 2xya_2 - 2xya_3 - 2xyb_2 \\ & - 2xyb_3 + y^2a_2 + y^2a_3 + y^2b_2 - y^2b_3 - 2xb_1 + 2ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -a_2v_1^2 + 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 \\ & - 2b_2v_1v_2 + b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 - b_3v_2^2 + 2a_1v_2 - 2b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-a_2 - a_3 - b_2 + b_3) v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3) v_1 v_2 \\ - 2b_1 v_1 + (a_2 + a_3 + b_2 - b_3) v_2^2 + 2a_1 v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}2a_1 &= 0\\ -2b_1 &= 0\\ -a_2 - a_3 - b_2 + b_3 &= 0\\ a_2 + a_3 + b_2 - b_3 &= 0\\ 2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= b_3\\ a_3 &= -b_2\\ b_1 &= 0\\ b_2 &= b_2\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi\\ &= y - \left(-\frac{x+y}{-x+y}\right)(x)\\ &= \frac{-x^2 - y^2}{x - y}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2-y^2}{x-y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y}{-x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x + y}{x^2 + y^2} \\ S_y &= \frac{-x + y}{x^2 + y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

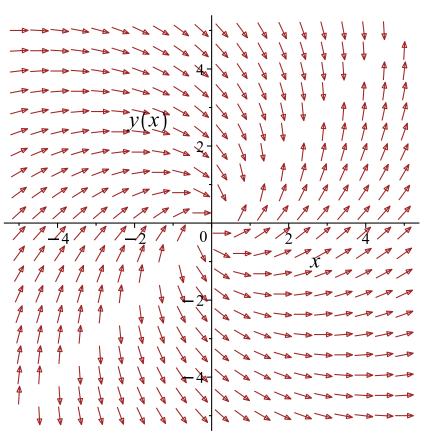
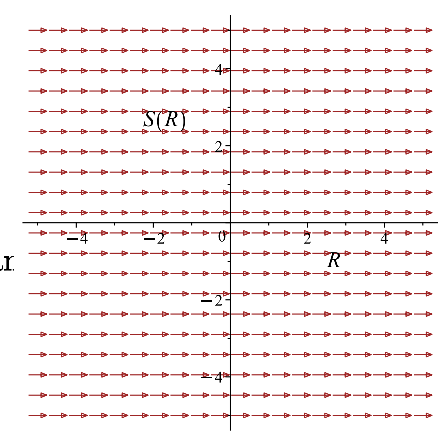
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Which simplifies to

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y}{-x+y}$ 	$R = x$ $S = \frac{\ln(x^2+y^2)}{2} - \arctan$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln \left(x^{2}+y^{2}\right)}{2}-\arctan \left(\frac{y}{x}\right)=c_{1} \tag{1}$$

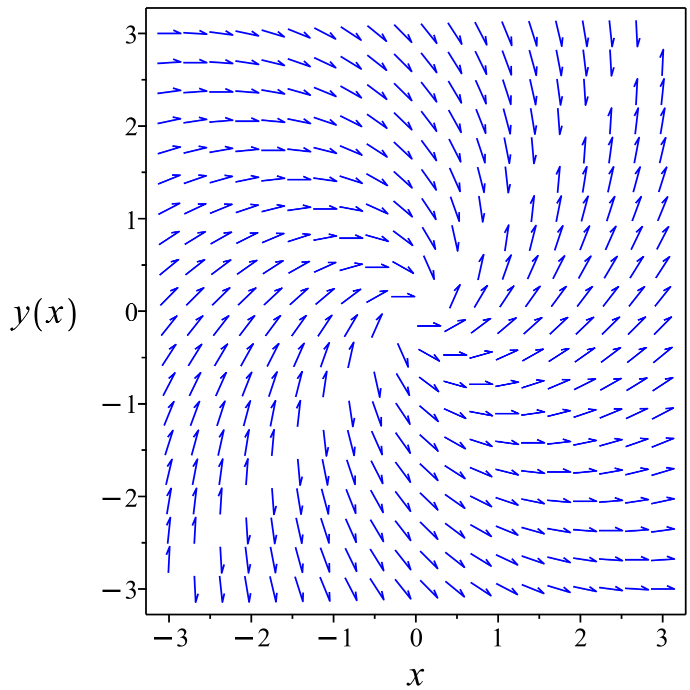


Figure 85: Slope field plot

Verification of solutions

$$\frac{\ln \left(x^{2}+y^{2}\right)}{2}-\arctan \left(\frac{y}{x}\right)=c_{1}$$

Verified OK.

4.22.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y)+N(x, y) \frac{d y}{d x}=0 \tag{A}$$

We assume there exists a function $\phi(x, y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{d x} \phi(x, y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x+y)\,\mathrm{d}y&=(-y-x)\,\mathrm{d}x\\ (x+y)\,\mathrm{d}x+(-x+y)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x+y\\ N(x,y)&=-x+y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(x+y)\\ &=1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x+y)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = x + y$ and $N = -x + y$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{x+y}{x^2+y^2}\\ N &= \frac{-x+y}{x^2+y^2}\end{aligned}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{-x+y}{x^2+y^2}\right)\mathrm{d}y &= \left(-\frac{x+y}{x^2+y^2}\right)\mathrm{d}x\\ \left(\frac{x+y}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{-x+y}{x^2+y^2}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \frac{x+y}{x^2+y^2}\\ N(x,y) &= \frac{-x+y}{x^2+y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x+y}{x^2+y^2}\right)\\ &= \frac{x^2-2xy-y^2}{(x^2+y^2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x+y}{x^2+y^2}\right)\\ &= \frac{x^2-2xy-y^2}{(x^2+y^2)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \frac{x+y}{x^2+y^2} \, \mathrm{d}x \\
\phi &= \frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right) + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{y}{x^2+y^2} - \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \qquad (4) \\
&= \frac{-x+y}{x^2+y^2} + f'(y)
\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+y}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{-x+y}{x^2+y^2} = \frac{-x+y}{x^2+y^2} + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right)$$

Summary

The solution(s) found are the following

$$\frac{\ln(x^2+y^2)}{2}+\arctan\left(\frac{x}{y}\right)=c_1 \tag{1}$$

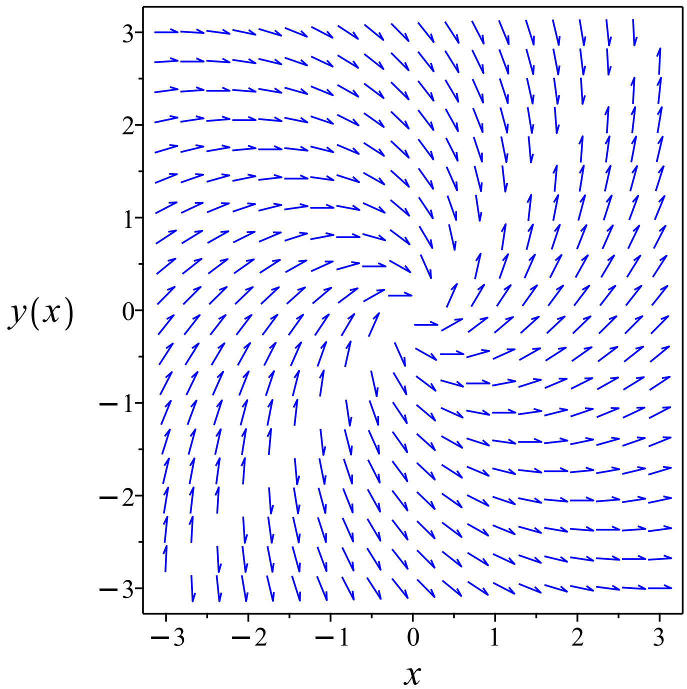


Figure 86: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+y^2)}{2}+\arctan\left(\frac{x}{y}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 24

```
dsolve(x+y(x)*diff(y(x),x)+y(x)-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan\left(\operatorname{RootOf}\left(-2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x\right) + 2c_1\right)\right)x$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 36

```
DSolve[x+y[x]*y'[x]+y[x]-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right) - \arctan\left(\frac{y(x)}{x}\right) = -\log(x) + c_1, y(x)\right]$$

4.23 problem Problem 3.38

Internal problem ID [5897]
Internal file name [`OUTPUT/5145_Sunday_June_05_2022_03_26_14_PM_80465364/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 3. Ordinary Differential Equations. Section 3.6 Summary and Problems. Page 218
Problem number: Problem 3.38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order,
   _with_potential_symmetries], [_2nd_order, _reducible, _mu_xy
   ]]
```

$$\boxed{yy'' - y'^2 - y'y^2 = 0}$$

4.23.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$y p(y) \left(\frac{d}{dy}p(y)\right)+\left(-p(y)-y^{2}\right) p(y) = 0$$

Which is now solved as first order ode for $p(y)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dy}p(y) + p(y)p(y) = q(y)$$

Where here

$$\begin{aligned} p(y) &= -\frac{1}{y} \\ q(y) &= y \end{aligned}$$

Hence the ode is

$$\frac{d}{dy}p(y) - \frac{p(y)}{y} = y$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{y}dy} \\ &= \frac{1}{y} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y}(\mu p) &= (\mu)(y) \\ \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{p}{y}\right) &= \left(\frac{1}{y}\right)(y) \\ \mathrm{d}\left(\frac{p}{y}\right) &= \mathrm{d}y \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{p}{y} &= \int \mathrm{d}y \\ \frac{p}{y} &= y + c_{1} \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{y}$ results in

$$p(y) = c_{1}y + y^{2}$$

which simplifies to

$$p(y) = y(y + c_1)$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = y(y + c_1)$$

Integrating both sides gives

$$\int \frac{1}{y (y + c_1)} dy = \int dx$$
$$\frac{\ln (y)}{c_1} - \frac{\ln (y + c_1)}{c_1} = c_2 + x$$

The above can be written as

$$\left(\frac{1}{c_1}\right) (\ln (y) - \ln (y + c_1)) = c_2 + x$$
$$\ln (y) - \ln (y + c_1) = (c_1) (c_2 + x)$$
$$= c_1(c_2 + x)$$

Raising both side to exponential gives

$$e^{\ln(y) - \ln(y + c_1)} = c_1 c_2 e^{c_1 x}$$

Which simplifies to

$$\frac{y}{y + c_1} = c_3 e^{c_1 x}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_3 e^{c_1 x} c_1}{-1 + c_3 e^{c_1 x}} \tag{1}$$

Verification of solutions

$$y = -\frac{c_3 e^{c_1 x} c_1}{-1 + c_3 e^{c_1 x}}$$

Verified OK.

4.23.2 Maple step by step solution

Let's solve

$yy'' + (-y' - y^2)\, y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $yu(y)\left(\frac{d}{dy}u(y)\right) + (-u(y) - y^2)\, u(y) = 0$

- Isolate the derivative

  $\frac{d}{dy}u(y) = \frac{u(y)}{y} + y$

- Group terms with $u(y)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{d}{dy}u(y) - \frac{u(y)}{y} = y$

- The ODE is linear; multiply by an integrating factor $\mu(y)$

  $\mu(y)\left(\frac{d}{dy}u(y) - \frac{u(y)}{y}\right) = \mu(y)\, y$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dy}(\mu(y)\, u(y))$

  $\mu(y)\left(\frac{d}{dy}u(y) - \frac{u(y)}{y}\right) = \left(\frac{d}{dy}\mu(y)\right) u(y) + \mu(y)\left(\frac{d}{dy}u(y)\right)$

- Isolate $\frac{d}{dy}\mu(y)$

  $\frac{d}{dy}\mu(y) = -\frac{\mu(y)}{y}$

- Solve to find the integrating factor

  $\mu(y) = \frac{1}{y}$

- Integrate both sides with respect to y

  $\int \left(\frac{d}{dy}(\mu(y)\, u(y))\right) dy = \int \mu(y)\, y dy + c_1$

- Evaluate the integral on the lhs

  $\mu(y)\, u(y) = \int \mu(y)\, y dy + c_1$

- Solve for $u(y)$

  $u(y) = \frac{\int \mu(y) y dy + c_1}{\mu(y)}$

- Substitute $\mu(y) = \frac{1}{y}$

  $u(y) = y\left(\int 1 dy + c_1\right)$

- Evaluate the integrals on the rhs

  $u(y) = y(y + c_1)$

- Solve 1st ODE for $u(y)$

  $u(y) = y(y + c_1)$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = y(y + c_1)$

- Separate variables

  $\frac{y'}{y(y+c_1)} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(y+c_1)} dx = \int 1 dx + c_2$

- Evaluate integral

  $\frac{\ln(y)}{c_1} - \frac{\ln(y+c_1)}{c_1} = c_2 + x$

- Solve for y

  $y = -\frac{c_1 e^{c_2 c_1 + c_1 x}}{-1 + e^{c_2 c_1 + c_1 x}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-_b(_a)*(_a^2+_b(_a))/_a = 0, _b(
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 2*_b]
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x$2)-(diff(y(x),x))^2-y(x)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = -\frac{c_1 \mathrm{e}^{(x+c_2)c_1}}{-1 + \mathrm{e}^{(x+c_2)c_1}}$$

✓ Solution by Mathematica

Time used: 1.53 (sec). Leaf size: 43

```
DSolve[y[x]*y''[x]-(y'[x])^2-y[x]^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{c_1 e^{c_1(x+c_2)}}{-1 + e^{c_1(x+c_2)}}$$
$$y(x) \to -\frac{1}{x + c_2}$$

5 Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360

5.1 problem Problem 5.1

Internal problem ID [5898]
Internal file name [`OUTPUT/5146_Sunday_June_05_2022_03_26_15_PM_67627155/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360
Problem number: Problem 5.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 3x_1(t) - 18x_2(t)$$
$$x_2'(t) = 2x_1(t) - 9x_2(t)$$

With initial conditions

$$[x_1(0) = 2, x_2(0) = 1]$$

5.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-3t}(1+6t) & -18t\,\mathrm{e}^{-3t} \\ 2t\,\mathrm{e}^{-3t} & \mathrm{e}^{-3t}(1-6t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-3t}(1+6t) & -18t\,e^{-3t} \\ 2t\,e^{-3t} & e^{-3t}(1-6t) \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2\,e^{-3t}(1+6t) - 18t\,e^{-3t} \\ 4t\,e^{-3t} + e^{-3t}(1-6t) \end{bmatrix} \\
&= \begin{bmatrix} (2-6t)\,e^{-3t} \\ e^{-3t}(1-2t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & -18 & 0 \\ 2 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 6 & -18 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -18 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	2	1	Yes	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

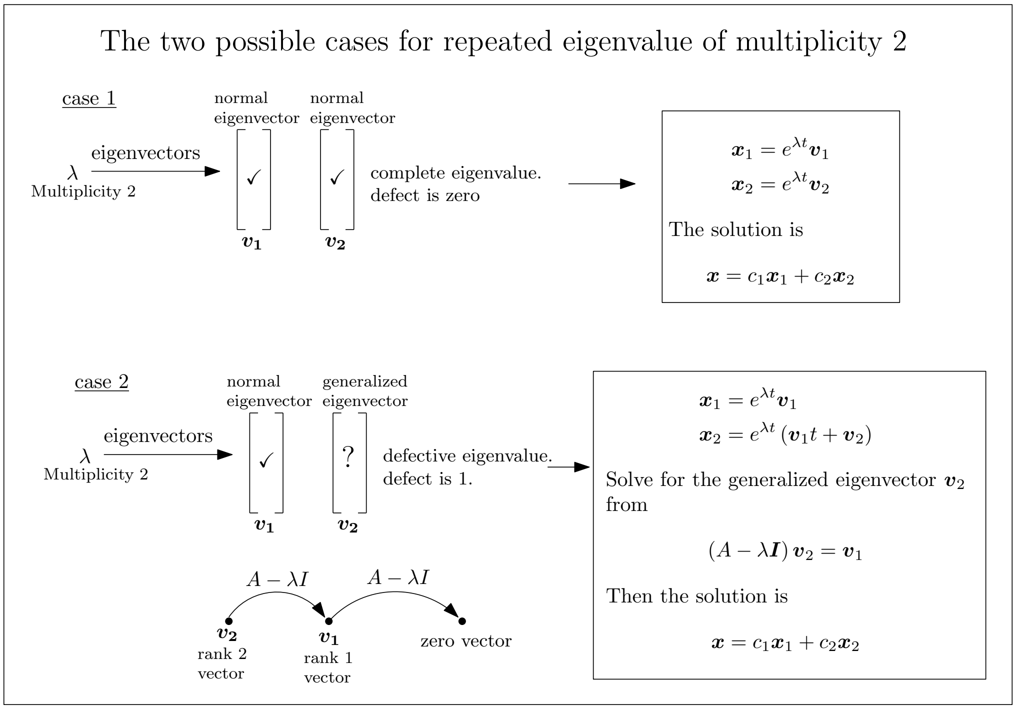


Figure 87: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{7}{2} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -3. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 3 \\ 1 \end{bmatrix} \mathrm{e}^{-3t} \\
&= \begin{bmatrix} 3\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{7}{2} \\ 1 \end{bmatrix} \right) \mathrm{e}^{-3t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-3t}(6t+7)}{2} \\ \mathrm{e}^{-3t}(t+1) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 3\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-3t}\left(3t + \frac{7}{2}\right) \\ \mathrm{e}^{-3t}(t+1) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-3t}\left(3c_1 + 3c_2 t + \frac{7}{2} c_2\right) \\ \mathrm{e}^{-3t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x_1(0) = 2 \\ x_2(0) = 1 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3c_1 + \frac{7c_2}{2} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 3 \\ c_2 = -2 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (2 - 6t)\, e^{-3t} \\ e^{-3t}(1 - 2t) \end{bmatrix}$$

The following is the phase plot of the system.

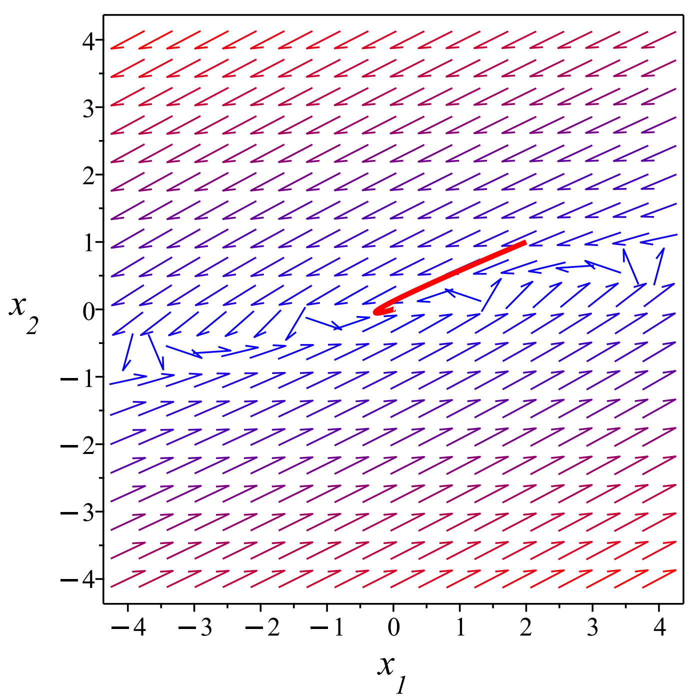
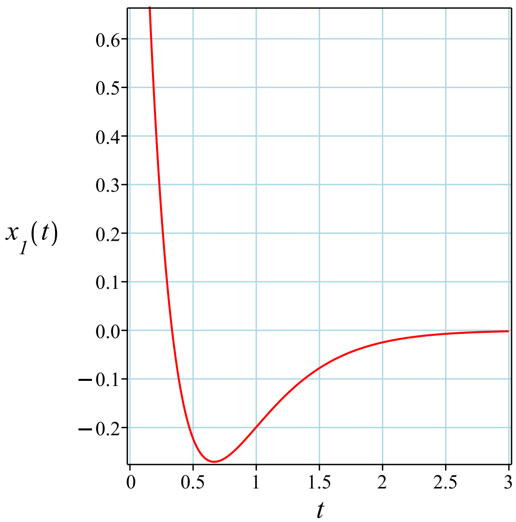
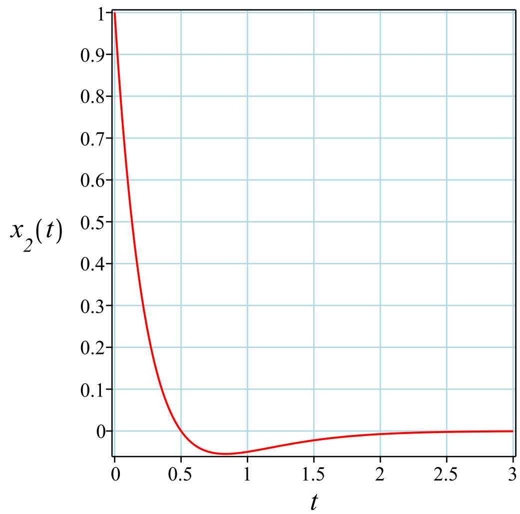


Figure 88: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve([diff(x__1(t),t) = 3*x__1(t)-18*x__2(t), diff(x__2(t),t) = 2*x__1(t)-9*x__2(t), x__1(
```

$$x_1(t) = e^{-3t}(-6t + 2)$$
$$x_2(t) = \frac{e^{-3t}(-36t + 18)}{18}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 30

```
DSolve[{x1'[t]==3*x1[t]-18*x2[t],x2'[t]==2*x1[t]-9*x2[t]},{x1[0]==2,x2[0]==1},{x1[t],x2[t]},
```

$$\text{x1}(t) \to e^{-3t}(2 - 6t)$$
$$\text{x2}(t) \to e^{-3t}(1 - 2t)$$

5.2 problem Problem 5.2

Internal problem ID [5899]
Internal file name [`OUTPUT/5147_Sunday_June_05_2022_03_26_16_PM_41729933/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360
Problem number: Problem 5.2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) + 3x_2(t)$$
$$x_2'(t) = 5x_1(t) + 3x_2(t)$$

5.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{5\,\mathrm{e}^{-2t}}{8} + \frac{3\,\mathrm{e}^{6t}}{8} & \frac{3\,\mathrm{e}^{6t}}{8} - \frac{3\,\mathrm{e}^{-2t}}{8} \\ \frac{5\,\mathrm{e}^{6t}}{8} - \frac{5\,\mathrm{e}^{-2t}}{8} & \frac{3\,\mathrm{e}^{-2t}}{8} + \frac{5\,\mathrm{e}^{6t}}{8} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{5\,e^{-2t}}{8} + \frac{3\,e^{6t}}{8} & \frac{3\,e^{6t}}{8} - \frac{3\,e^{-2t}}{8} \\ \frac{5\,e^{6t}}{8} - \frac{5\,e^{-2t}}{8} & \frac{3\,e^{-2t}}{8} + \frac{5\,e^{6t}}{8} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{5\,e^{-2t}}{8} + \frac{3\,e^{6t}}{8}\right) c_1 + \left(\frac{3\,e^{6t}}{8} - \frac{3\,e^{-2t}}{8}\right) c_2 \\ \left(\frac{5\,e^{6t}}{8} - \frac{5\,e^{-2t}}{8}\right) c_1 + \left(\frac{3\,e^{-2t}}{8} + \frac{5\,e^{6t}}{8}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(5c_1-3c_2)e^{-2t}}{8} + \frac{3\,e^{6t}(c_1+c_2)}{8} \\ \frac{(-5c_1+3c_2)e^{-2t}}{8} + \frac{5\,e^{6t}(c_1+c_2)}{8} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 3 \\ 5 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda - 12 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 6$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 5 & 5 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{5R_1}{3} \implies \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 3 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -5 & 3 & 0 \\ 5 & -3 & 0 \end{array} \right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -5 & 3 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
6	1	1	No	$\begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{6t} \\ &= \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} e^{6t}\end{aligned}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-2t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{3\,\mathrm{e}^{6t}}{5} \\ \mathrm{e}^{6t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{3c_1\mathrm{e}^{6t}}{5} - c_2\mathrm{e}^{-2t} \\ c_1\mathrm{e}^{6t} + c_2\mathrm{e}^{-2t} \end{bmatrix}$$

The following is the phase plot of the system.

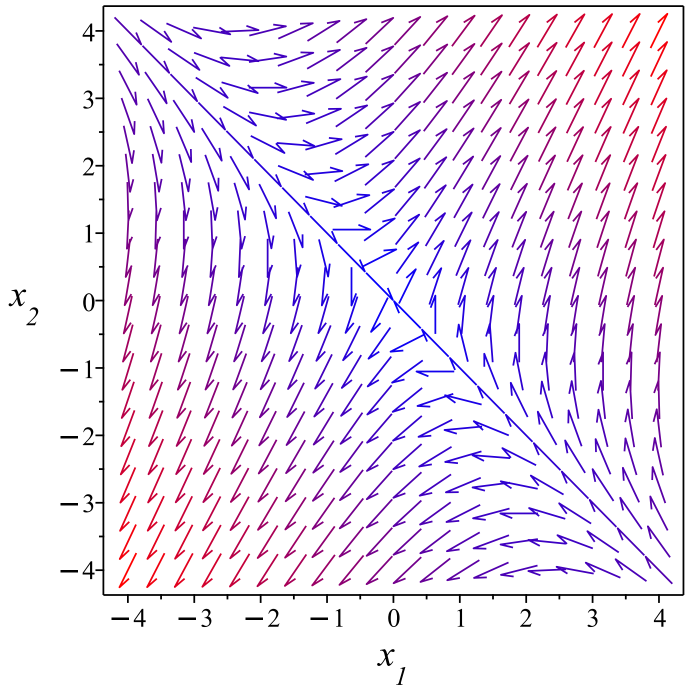


Figure 89: Phase plot

5.2.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) + 3x_2(t), x_2'(t) = 5x_1(t) + 3x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[6, \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = \mathrm{e}^{6t} \cdot \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 \mathrm{e}^{-2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{6t} \cdot \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = \left[\begin{array}{c} -c_1 \mathrm{e}^{-2t} + \frac{3c_2 \mathrm{e}^{6t}}{5} \\ c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{6t} \end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x_1(t) = -c_1 \mathrm{e}^{-2t} + \frac{3c_2 \mathrm{e}^{6t}}{5}, x_2(t) = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{6t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve([diff(x__1(t),t)=x__1(t)+3*x__2(t),diff(x__2(t),t)=5*x__1(t)+3*x__2(t)],singsol=all)
```

$$x_1(t) = \mathrm{e}^{6t} c_1 + c_2 \mathrm{e}^{-2t}$$
$$x_2(t) = \frac{5\,\mathrm{e}^{6t} c_1}{3} - c_2 \mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 74

```
DSolve[{x1'[t]==x1[t]+3*x2[t],x2'[t]==5*x1[t]+3*x2[t]},{x1[t],x2[t]},t,IncludeSingularSoluti
```

$$\mathrm{x1}(t) \to \frac{1}{8} e^{-2t} \left(c_1 \left(3e^{8t} + 5\right) + 3c_2 \left(e^{8t} - 1\right)\right)$$
$$\mathrm{x2}(t) \to \frac{1}{8} e^{-2t} \left(5c_1 \left(e^{8t} - 1\right) + c_2 \left(5e^{8t} + 3\right)\right)$$

5.3 problem Problem 5.3

Internal problem ID [5900]
Internal file name [OUTPUT/5148_Sunday_June_05_2022_03_26_18_PM_70103364/index.tex]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360
Problem number: Problem 5.3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -x_1(t) + 3x_2(t) \\
x_2'(t) &= -3x_1(t) + 5x_2(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 2]$$

5.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{2t}(1-3t) & 3t\,\mathrm{e}^{2t} \\ -3t\,\mathrm{e}^{2t} & \mathrm{e}^{2t}(1+3t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{2t}(1-3t) & 3t\,e^{2t} \\ -3t\,e^{2t} & e^{2t}(1+3t) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(1-3t) + 6t\,e^{2t} \\ -3t\,e^{2t} + 2\,e^{2t}(1+3t) \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(1+3t) \\ e^{2t}(2+3t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 3 \\ -3 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ -3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

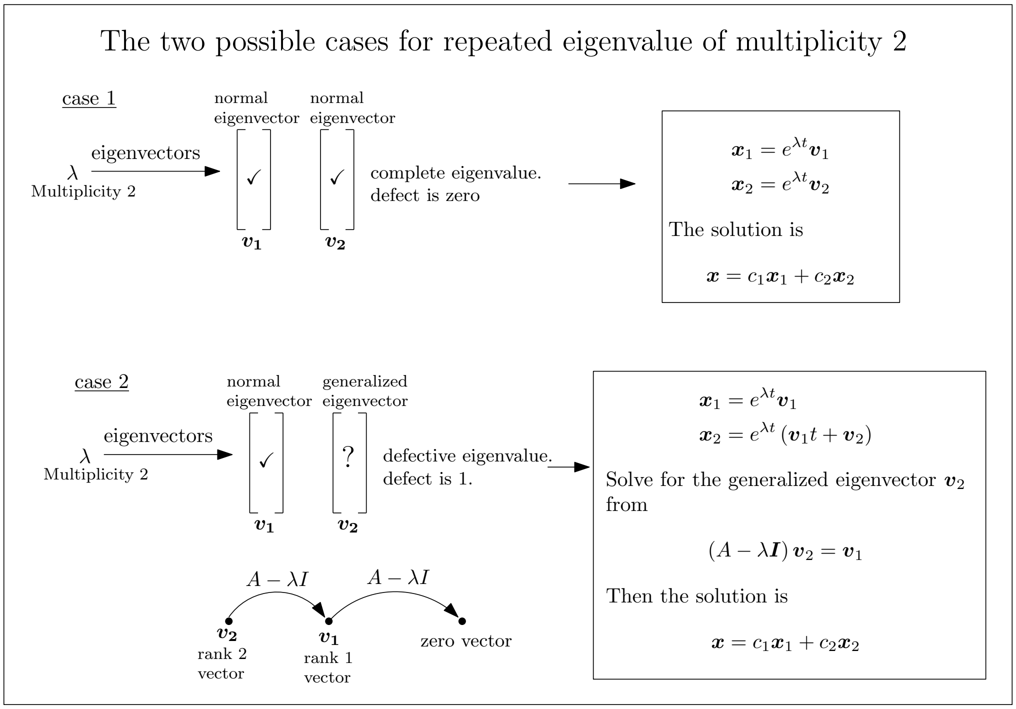


Figure 90: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{2t}\\ &= \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}\\ &= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right) \mathrm{e}^{2t}\\ &= \begin{bmatrix} \frac{\mathrm{e}^{2t}(2+3t)}{3} \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{2t}\left(t + \frac{2}{3}\right) \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{2t}\left(c_1 + c_2 t + \frac{2}{3} c_2\right) \\ \mathrm{e}^{2t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + \frac{2c_2}{3} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -1 \\ c_2 = 3 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{2t}(1 + 3t) \\ e^{2t}(2 + 3t) \end{bmatrix}$$

The following is the phase plot of the system.

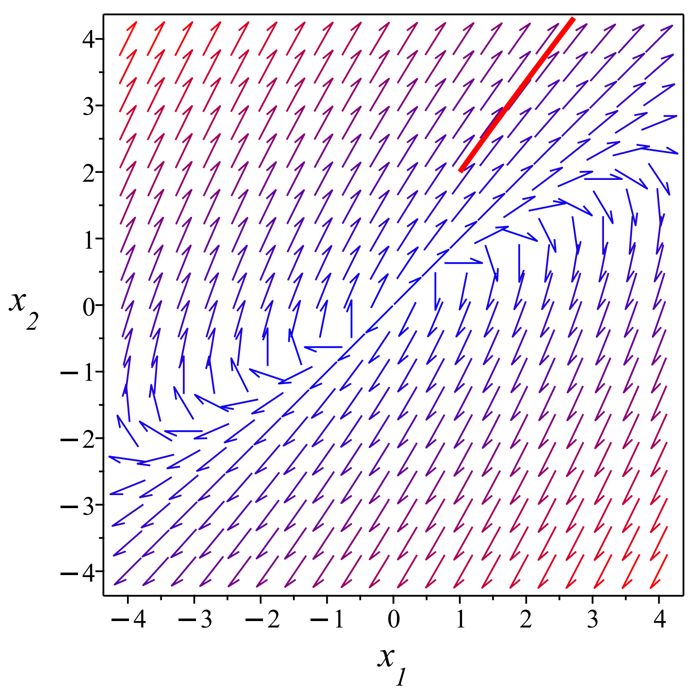
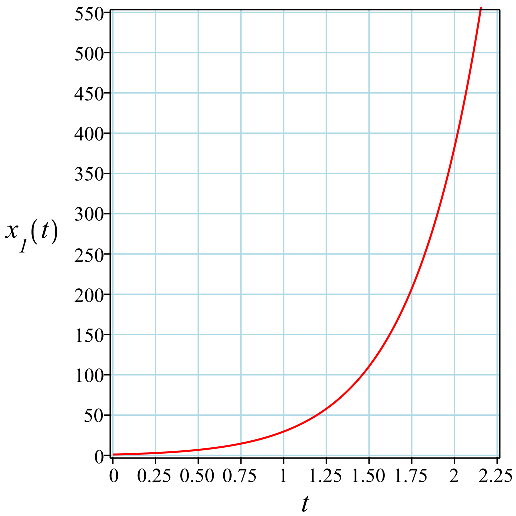
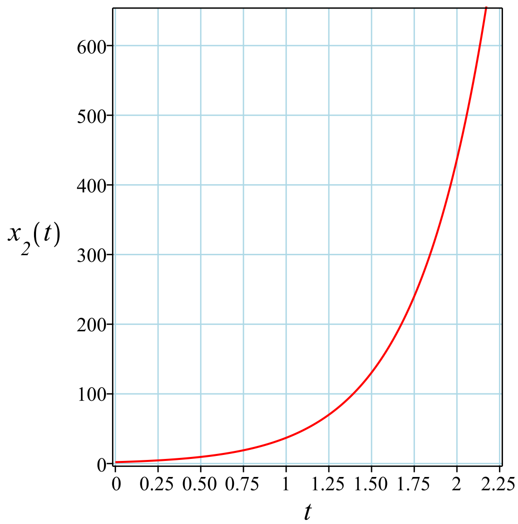


Figure 91: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve([diff(x__1(t),t) = -x__1(t)+3*x__2(t), diff(x__2(t),t) = -3*x__1(t)+5*x__2(t), x__1(0
```

$$x_1(t) = e^{2t}(3t+1)$$
$$x_2(t) = \frac{e^{2t}(9t+6)}{3}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
DSolve[{x1'[t]==-x1[t]+3*x2[t],x2'[t]==-3*x1[t]+5*x2[t]},{x1[0]==1,x2[0]==2},{x1[t],x2[t]},t
```

$$\text{x1}(t) \to e^{2t}(3t+1)$$
$$\text{x2}(t) \to e^{2t}(3t+2)$$

5.4 problem Problem 5.4

Internal problem ID [5901]
Internal file name [OUTPUT/5149_Sunday_June_05_2022_03_26_19_PM_55103563/index.tex]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360
Problem number: Problem 5.4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 4x_1(t) - x_2(t) \\
x_2'(t) &= 5x_1(t) + 2x_2(t)
\end{aligned}
$$

5.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{3t}\cos(2t) + \frac{e^{3t}\sin(2t)}{2} & -\frac{e^{3t}\sin(2t)}{2} \\ \frac{5\,e^{3t}\sin(2t)}{2} & e^{3t}\cos(2t) - \frac{e^{3t}\sin(2t)}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{3t}(2\cos(2t)+\sin(2t))}{2} & -\frac{e^{3t}\sin(2t)}{2} \\ \frac{5\,e^{3t}\sin(2t)}{2} & \frac{e^{3t}(2\cos(2t)-\sin(2t))}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{3t}(2\cos(2t)+\sin(2t))}{2} & -\frac{e^{3t}\sin(2t)}{2} \\ \frac{5\,e^{3t}\sin(2t)}{2} & \frac{e^{3t}(2\cos(2t)-\sin(2t))}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{3t}(2\cos(2t)+\sin(2t))c_1}{2} - \frac{e^{3t}\sin(2t)c_2}{2} \\ \frac{5\,e^{3t}\sin(2t)c_1}{2} + \frac{e^{3t}(2\cos(2t)-\sin(2t))c_2}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{3t}(-c_2+c_1)\sin(2t)}{2} + e^{3t}\cos(2t)\,c_1 \\ \frac{e^{3t}(5c_1-c_2)\sin(2t)}{2} + e^{3t}\cos(2t)\,c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 4-\lambda & -1 \\ 5 & 2-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 13 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3 + 2i$$
$$\lambda_2 = 3 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$3 - 2i$	1	complex eigenvalue
$3 + 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix} - (3 - 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 + 2i & -1 \\ 5 & -1 + 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 + 2i & -1 & 0 \\ 5 & -1 + 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 + 2i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 + 2i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 + 2i & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{5} - \frac{2i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{5} - \frac{2\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{5} - \frac{2i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{5} - \frac{2\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{5} - \frac{2i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{5} - \frac{2\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{5} - \frac{2i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{5} - \frac{2\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 - 2i \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix} - (3 + 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 - 2i & -1 \\ 5 & -1 - 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 - 2i & -1 & 0 \\ 5 & -1 - 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 - 2i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 - 2i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-2i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{5} + \frac{2i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{5} + \frac{2\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{5} + \frac{2i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{5} + \frac{2\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{5} + \frac{2i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{5} + \frac{2\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{5} + \frac{2i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{5} + \frac{2\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1+2i \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$3+2i$	1	1	No	$\begin{bmatrix} \frac{1}{5}+\frac{2i}{5} \\ 1 \end{bmatrix}$
$3-2i$	1	1	No	$\begin{bmatrix} \frac{1}{5}-\frac{2i}{5} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{5}+\frac{2i}{5}\right) \mathrm{e}^{(3+2i)t} \\ \mathrm{e}^{(3+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{5}-\frac{2i}{5}\right) \mathrm{e}^{(3-2i)t} \\ \mathrm{e}^{(3-2i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{5}+\frac{2i}{5}\right) c_1 \mathrm{e}^{(3+2i)t} + \left(\frac{1}{5}-\frac{2i}{5}\right) c_2 \mathrm{e}^{(3-2i)t} \\ c_1 \mathrm{e}^{(3+2i)t} + c_2 \mathrm{e}^{(3-2i)t} \end{bmatrix}$$

The following is the phase plot of the system.

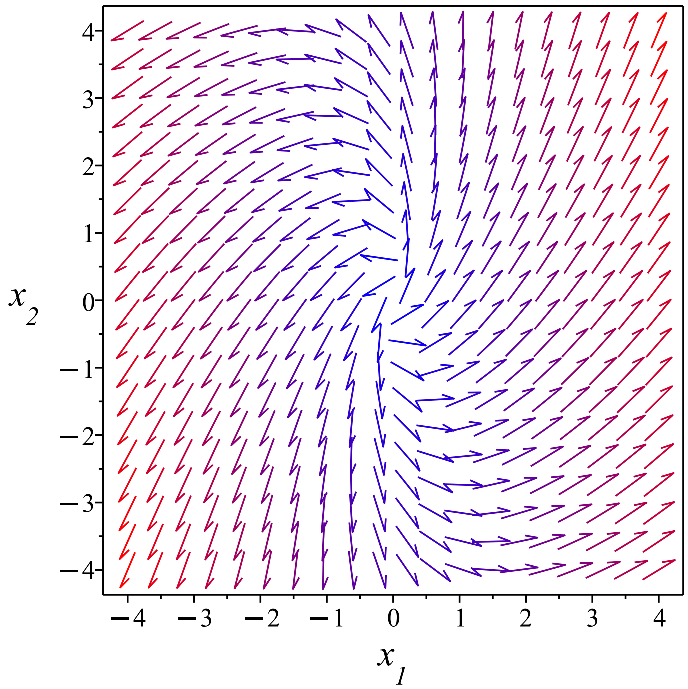


Figure 92: Phase plot

5.4.3 Maple step by step solution

Let's solve

$[x_1'(t) = 4x_1(t) - x_2(t), x_2'(t) = 5x_1(t) + 2x_2(t)]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 4 & -1 \\ 5 & 2 \end{array}\right]$$

- Rewrite the system as

$$\underline{x}^{\rightarrow \prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[3-2\,\mathrm{I}, \left[\begin{array}{c} \frac{1}{5}-\frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right], \left[3+2\,\mathrm{I}, \left[\begin{array}{c} \frac{1}{5}+\frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[3-2\,\mathrm{I}, \left[\begin{array}{c} \frac{1}{5}-\frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(3-2\,\mathrm{I})t} \cdot \left[\begin{array}{c} \frac{1}{5}-\frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{3t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \left[\begin{array}{c} \frac{1}{5}-\frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{3t} \cdot \left[\begin{array}{c} \left(\frac{1}{5}-\frac{2\,\mathrm{I}}{5}\right)(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}{}_{1}(t) = \mathrm{e}^{3t} \cdot \left[\begin{array}{c} \frac{\cos(2t)}{5} - \frac{2\sin(2t)}{5} \\ \cos(2t) \end{array}\right], \underline{x}^{\rightarrow}{}_{2}(t) = \mathrm{e}^{3t} \cdot \left[\begin{array}{c} -\frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5} \\ -\sin(2t) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_{1}(t) + c_2 \underline{x}^{\rightarrow}{}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 e^{3t} \cdot \begin{bmatrix} \frac{\cos(2t)}{5} - \frac{2\sin(2t)}{5} \\ \cos(2t) \end{bmatrix} + e^{3t} c_2 \cdot \begin{bmatrix} -\frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5} \\ -\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}\left((c_1-2c_2)\cos(2t)-2\sin(2t)\left(c_1+\frac{c_2}{2}\right)\right)}{5} \\ e^{3t}(c_1\cos(2t) - c_2\sin(2t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{e^{3t}\left((c_1-2c_2)\cos(2t)-2\sin(2t)\left(c_1+\frac{c_2}{2}\right)\right)}{5}, x_2(t) = e^{3t}(c_1\cos(2t) - c_2\sin(2t))\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 61

```
dsolve([diff(x__1(t),t)=4*x__1(t)-x__2(t),diff(x__2(t),t)=5*x__1(t)+2*x__2(t)],singsol=all)
```

$$x_1(t) = e^{3t}(c_1\sin(2t) + c_2\cos(2t))$$
$$x_2(t) = -e^{3t}(2c_1\cos(2t) - c_2\cos(2t) - c_1\sin(2t) - 2c_2\sin(2t))$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 70

```
DSolve[{x1'[t]==4*x1[t]-x2[t],x2'[t]==5*x1[t]+2*x2[t]},{x1[t],x2[t]},t,IncludeSingularSoluti
```

$$\text{x1}(t) \to \frac{1}{2}e^{3t}(2c_1\cos(2t) + (c_1 - c_2)\sin(2t))$$
$$\text{x2}(t) \to \frac{1}{2}e^{3t}(2c_2\cos(2t) + (5c_1 - c_2)\sin(2t))$$

5.5 problem Problem 5.6

Internal problem ID [5902]
Internal file name [`OUTPUT/5150_Sunday_June_05_2022_03_26_20_PM_27465064/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360
Problem number: Problem 5.6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x_1'(t) &= -2x_1(t) + x_2(t) \\ x_2'(t) &= x_1(t) - 2x_2(t) \end{aligned}$$

5.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-3t}}{2} + \frac{e^{-t}}{2}\right) c_1 + \left(\frac{e^{-t}}{2} - \frac{e^{-3t}}{2}\right) c_2 \\ \left(\frac{e^{-t}}{2} - \frac{e^{-3t}}{2}\right) c_1 + \left(\frac{e^{-3t}}{2} + \frac{e^{-t}}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(-c_2+c_1)e^{-3t}}{2} + \frac{e^{-t}(c_1+c_2)}{2} \\ \frac{(c_2-c_1)e^{-3t}}{2} + \frac{e^{-t}(c_1+c_2)}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4\lambda + 3 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= -3\\ \lambda_2 &= -1\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -3 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-3t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}\end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-t} \\ c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

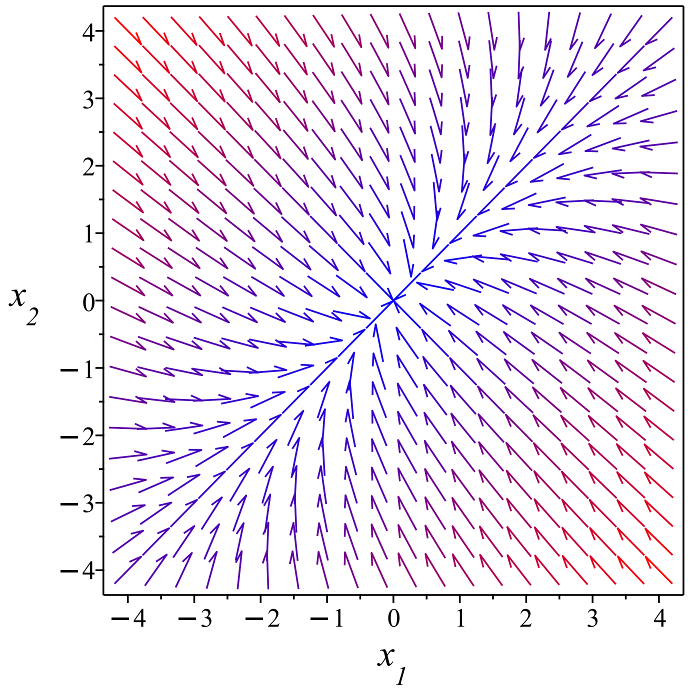


Figure 93: Phase plot

5.5.3 Maple step by step solution

Let's solve

$$[x_1'(t) = -2x_1(t) + x_2(t), x_2'(t) = x_1(t) - 2x_2(t)]$$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 \mathrm{e}^{-3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-t} \\ c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x_1(t) = -c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-t}, x_2(t) = c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-t}\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
dsolve([diff(x__1(t),t)=-2*x__1(t)+x__2(t),diff(x__2(t),t)=x__1(t)-2*x__2(t)],singsol=all)
```

$$x_1(t) = c_2 \mathrm{e}^{-t} + c_1 \mathrm{e}^{-3t}$$
$$x_2(t) = c_2 \mathrm{e}^{-t} - c_1 \mathrm{e}^{-3t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 68

```
DSolve[{x1'[t]==-2*x1[t]+x2[t],x2'[t]==x1[t]-2*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolutio
```

$$\text{x1}(t) \to \frac{1}{2} e^{-3t} \left(c_1 \left(e^{2t} + 1\right) + c_2 \left(e^{2t} - 1\right)\right)$$
$$\text{x2}(t) \to \frac{1}{2} e^{-3t} \left(c_1 \left(e^{2t} - 1\right) + c_2 \left(e^{2t} + 1\right)\right)$$

5.6 problem Problem 5.7

Internal problem ID [5903]
Internal file name [`OUTPUT/5151_Sunday_June_05_2022_03_26_21_PM_81970847/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360
Problem number: Problem 5.7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -2x_1(t) + x_2(t) + 2\,\mathrm{e}^{-t}$$
$$x_2'(t) = x_1(t) - 2x_2(t) + 3t$$

5.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2\,\mathrm{e}^{-t} \\ 3t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-3t}}{2} + \frac{e^{-t}}{2}\right) c_1 + \left(\frac{e^{-t}}{2} - \frac{e^{-3t}}{2}\right) c_2 \\ \left(\frac{e^{-t}}{2} - \frac{e^{-3t}}{2}\right) c_1 + \left(\frac{e^{-3t}}{2} + \frac{e^{-t}}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(-c_2+c_1)e^{-3t}}{2} + \frac{e^{-t}(c_1+c_2)}{2} \\ \frac{(c_2-c_1)e^{-3t}}{2} + \frac{e^{-t}(c_1+c_2)}{2} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t) \, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \frac{(e^{2t}+1)e^{t}}{2} & -\frac{(e^{2t}-1)e^{t}}{2} \\ -\frac{(e^{2t}-1)e^{t}}{2} & \frac{(e^{2t}+1)e^{t}}{2} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{(e^{2t}+1)e^{t}}{2} & -\frac{(e^{2t}-1)e^{t}}{2} \\ -\frac{(e^{2t}-1)e^{t}}{2} & \frac{(e^{2t}+1)e^{t}}{2} \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix} \begin{bmatrix} \frac{(1-3t)e^{3t}}{6} + \frac{e^{2t}}{2} + \frac{(9t-9)e^{t}}{6} + t \\ \frac{(-1+3t)e^{3t}}{6} - \frac{e^{2t}}{2} + \frac{(9t-9)e^{t}}{6} + t \end{bmatrix} \\
&= \begin{bmatrix} t - \frac{4}{3} + \frac{e^{-t}}{2} + t e^{-t} \\ t e^{-t} + 2t - \frac{5}{3} - \frac{e^{-t}}{2} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \frac{(6t+3c_1+3c_2+3)\mathrm{e}^{-t}}{6} + \frac{(3c_1-3c_2)\mathrm{e}^{-3t}}{6} + t - \frac{4}{3} \\ \frac{(6t+3c_1+3c_2-3)\mathrm{e}^{-t}}{6} + \frac{(-3c_1+3c_2)\mathrm{e}^{-3t}}{6} + 2t - \frac{5}{3} \end{bmatrix}
\end{aligned}
$$

5.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2\,\mathrm{e}^{-t} \\ 3t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4\lambda + 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-3t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \mathrm{e}^{-t} & -\mathrm{e}^{-3t} \\ \mathrm{e}^{-t} & \mathrm{e}^{-3t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{\mathrm{e}^{t}}{2} & \frac{\mathrm{e}^{t}}{2} \\ -\frac{\mathrm{e}^{3t}}{2} & \frac{\mathrm{e}^{3t}}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^{-t} & -e^{-3t} \\ e^{-t} & e^{-3t} \end{bmatrix} \int \begin{bmatrix} \frac{e^t}{2} & \frac{e^t}{2} \\ -\frac{e^{3t}}{2} & \frac{e^{3t}}{2} \end{bmatrix} \begin{bmatrix} 2\,e^{-t} \\ 3t \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{-t} & -e^{-3t} \\ e^{-t} & e^{-3t} \end{bmatrix} \int \begin{bmatrix} 1 + \frac{3t\,e^t}{2} \\ -e^{2t} + \frac{3t\,e^{3t}}{2} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{-t} & -e^{-3t} \\ e^{-t} & e^{-3t} \end{bmatrix} \begin{bmatrix} \frac{(3t-3)e^t}{2} + t \\ \frac{(-1+3t)e^{3t}}{6} - \frac{e^{2t}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} t - \frac{4}{3} + \frac{e^{-t}}{2} + t\,e^{-t} \\ t\,e^{-t} + 2t - \frac{5}{3} - \frac{e^{-t}}{2} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} \\ c_1 e^{-t} \end{bmatrix} + \begin{bmatrix} -c_2 e^{-3t} \\ c_2 e^{-3t} \end{bmatrix} + \begin{bmatrix} t - \frac{4}{3} + \frac{e^{-t}}{2} + t\,e^{-t} \\ t\,e^{-t} + 2t - \frac{5}{3} - \frac{e^{-t}}{2} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(6t+6c_1+3)e^{-t}}{6} - c_2 e^{-3t} + t - \frac{4}{3} \\ \frac{(6t+6c_1-3)e^{-t}}{6} + c_2 e^{-3t} + 2t - \frac{5}{3} \end{bmatrix}$$

5.6.3 Maple step by step solution

Let's solve

$$\left[x_1'(t) = -2x_1(t) + x_2(t) + \frac{2}{e^t}, x_2'(t) = x_1(t) - 2x_2(t) + 3t\right]$$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} \frac{x_2(t)e^t - 2x_1(t)e^t + 2}{e^t} + 2x_1(t) - x_2(t) \\ 3t \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 3t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 3t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}$

$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$\overrightarrow{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

$\overrightarrow{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\underline{\vec{x}}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\underline{\vec{x}}_p(t) = \begin{bmatrix} -\frac{e^{-3t}}{6} + t - \frac{4}{3} + \frac{3\,e^{-t}}{2} \\ \frac{3\,e^{-t}}{2} + 2t - \frac{5}{3} + \frac{e^{-3t}}{6} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\underline{\vec{x}}(t) = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2 + \begin{bmatrix} -\frac{e^{-3t}}{6} + t - \frac{4}{3} + \frac{3\,e^{-t}}{2} \\ \frac{3\,e^{-t}}{2} + 2t - \frac{5}{3} + \frac{e^{-3t}}{6} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(-6c_1-1)e^{-3t}}{6} + \frac{(6c_2+9)e^{-t}}{6} + t - \frac{4}{3} \\ \frac{(6c_1+1)e^{-3t}}{6} + \frac{(6c_2+9)e^{-t}}{6} + 2t - \frac{5}{3} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{(-6c_1-1)e^{-3t}}{6} + \frac{(6c_2+9)e^{-t}}{6} + t - \frac{4}{3}, x_2(t) = \frac{(6c_1+1)e^{-3t}}{6} + \frac{(6c_2+9)e^{-t}}{6} + 2t - \frac{5}{3}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 65

```
dsolve([diff(x__1(t),t)=-2*x__1(t)+x__2(t)+2*exp(-t),diff(x__2(t),t)=x__1(t)-2*x__2(t)+3*t],
```

$$x_1(t) = -c_2 e^{-3t} + e^{-t} c_1 + \frac{e^{-t}}{2} - \frac{4}{3} + t\,e^{-t} + t$$
$$x_2(t) = c_2 e^{-3t} + e^{-t} c_1 - \frac{e^{-t}}{2} - \frac{5}{3} + 2t + t\,e^{-t}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 93

```
DSolve[{x1'[t]==-2*x1[t]+x2[t]+2*Exp[-t],x2'[t]==x1[t]-2*x2[t]+3*t},{x1[t],x2[t]},t,IncludeS
```

$$
\begin{aligned}
\text{x1}(t) &\to \frac{1}{6}\left(6t + 3(c_1 - c_2)e^{-3t} + 3e^{-t}(2t + 1 + c_1 + c_2) - 8\right) \\
\text{x2}(t) &\to \frac{1}{6}e^{-3t}\left(2e^{3t}(6t - 5) + 3e^{2t}(2t - 1 + c_1 + c_2) - 3c_1 + 3c_2\right)
\end{aligned}
$$

5.7 problem Problem 5.8

Internal problem ID [5904]
Internal file name [`OUTPUT/5152_Sunday_June_05_2022_03_26_23_PM_18444826/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360
Problem number: Problem 5.8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 3x_1(t) - x_2(t) \\
x_2'(t) &= 16x_1(t) - 5x_2(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 1]$$

5.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 16 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t}(1+4t) & -t\,\mathrm{e}^{-t} \\ 16t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(1-4t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-t}(1+4t) & -t\,e^{-t} \\ 16t\,e^{-t} & e^{-t}(1-4t) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(1+4t) - t\,e^{-t} \\ 16t\,e^{-t} + e^{-t}(1-4t) \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(1+3t) \\ e^{-t}(1+12t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 16 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -1 \\ 16 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -1 \\ 16 & -5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -1 \\ 16 & -5 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 \\ 16 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -1 & 0 \\ 16 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - 4R_1 \Longrightarrow \left[\begin{array}{cc|c} 4 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \frac{t}{4}\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	2	1	Yes	$\begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

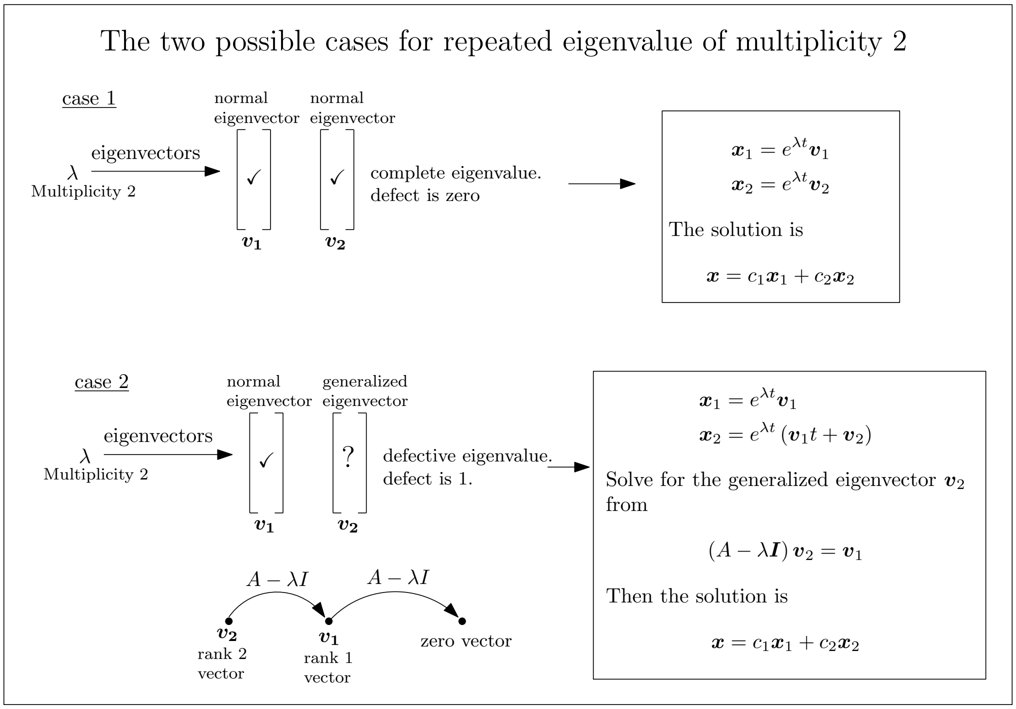


Figure 94: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\begin{aligned}\left(\begin{bmatrix} 3 & -1 \\ 16 & -5 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} \\ \begin{bmatrix} 4 & -1 \\ 16 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}\end{aligned}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ \frac{15}{4} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} \mathrm{e}^{-t} \\ &= \begin{bmatrix} \frac{\mathrm{e}^{-t}}{4} \\ \mathrm{e}^{-t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ \frac{15}{4} \end{bmatrix} \right) \mathrm{e}^{-t} \\ &= \begin{bmatrix} \frac{\mathrm{e}^{-t}(t+4)}{4} \\ \frac{\mathrm{e}^{-t}(4t+15)}{4} \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{\mathrm{e}^{-t}}{4} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t}\left(\frac{t}{4} + 1\right) \\ \mathrm{e}^{-t}\left(t + \frac{15}{4}\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{e}^{-t}((t+4)c_2+c_1)}{4} \\ \mathrm{e}^{-t}\left(c_1 + c_2 t + \frac{15}{4} c_2\right) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_2 + \frac{c_1}{4} \\ c_1 + \frac{15c_2}{4} \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -44 \\ c_2 = 12 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{-t}(12t+4)}{4} \\ e^{-t}(1 + 12t) \end{bmatrix}$$

The following is the phase plot of the system.

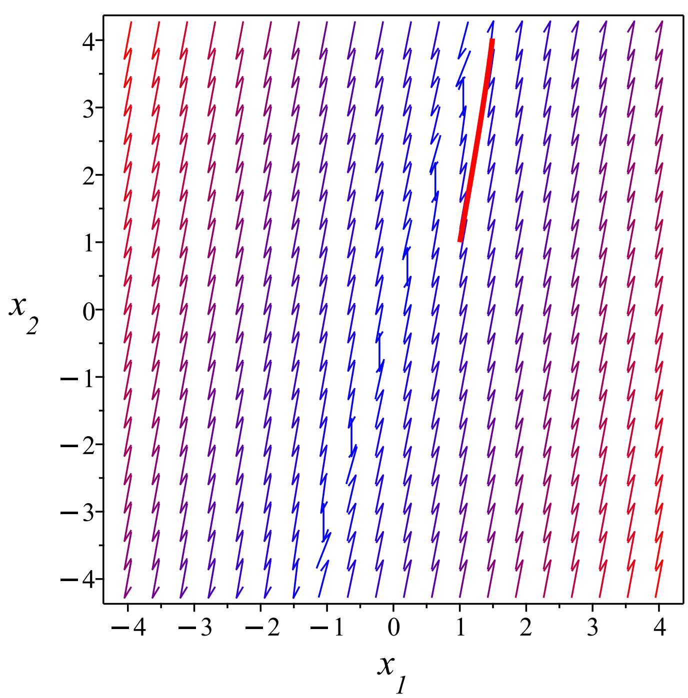
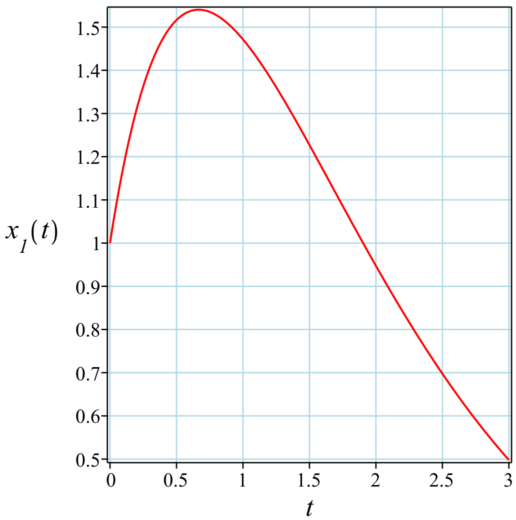
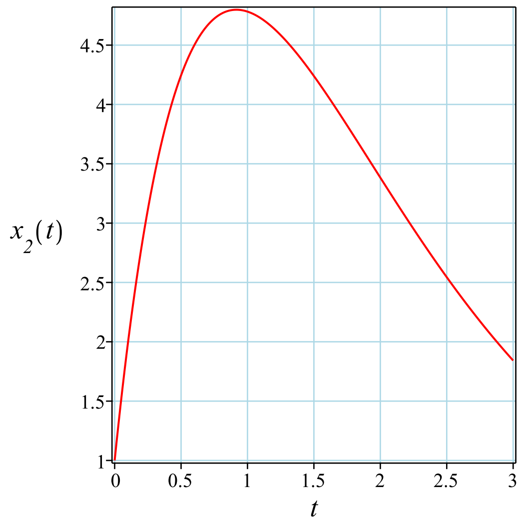


Figure 95: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve([diff(x__1(t),t) = 3*x__1(t)-x__2(t), diff(x__2(t),t) = 16*x__1(t)-5*x__2(t), x__1(0)
```

$$x_1(t) = e^{-t}(3t + 1)$$
$$x_2(t) = e^{-t}(12t + 1)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
DSolve[{x1'[t]==3*x1[t]-x2[t],x2'[t]==16*x1[t]-5*x2[t]},{x1[0]==1,x2[0]==1},{x1[t],x2[t]},t,
```

$$\text{x1}(t) \to e^{-t}(3t + 1)$$
$$\text{x2}(t) \to e^{-t}(12t + 1)$$

5.8 problem Problem 5.9

Internal problem ID [5905]
Internal file name [OUTPUT/5153_Sunday_June_05_2022_03_26_24_PM_8640362/index.tex]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360
Problem number: Problem 5.9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) - 2x_2(t)$$
$$x_2'(t) = 3x_1(t) - 4x_2(t)$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 0]$$

5.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -2\,\mathrm{e}^{-2t} + 3\,\mathrm{e}^{-t} & -2\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t} \\ 3\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{-2t} & 3\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} -2\,e^{-2t} + 3\,e^{-t} & -2\,e^{-t} + 2\,e^{-2t} \\ 3\,e^{-t} - 3\,e^{-2t} & 3\,e^{-2t} - 2\,e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -2\,e^{-2t} + 3\,e^{-t} \\ 3\,e^{-t} - 3\,e^{-2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 3\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{2} \implies \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-2t} \\ &= \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{2\,\mathrm{e}^{-2t}}{3} \\ \mathrm{e}^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{-t} + \frac{2c_2 \mathrm{e}^{-2t}}{3} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + \frac{2c_2}{3} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 3 \\ c_2 = -3 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2\,\mathrm{e}^{-2t} + 3\,\mathrm{e}^{-t} \\ 3\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{-2t} \end{bmatrix}
$$

The following is the phase plot of the system.

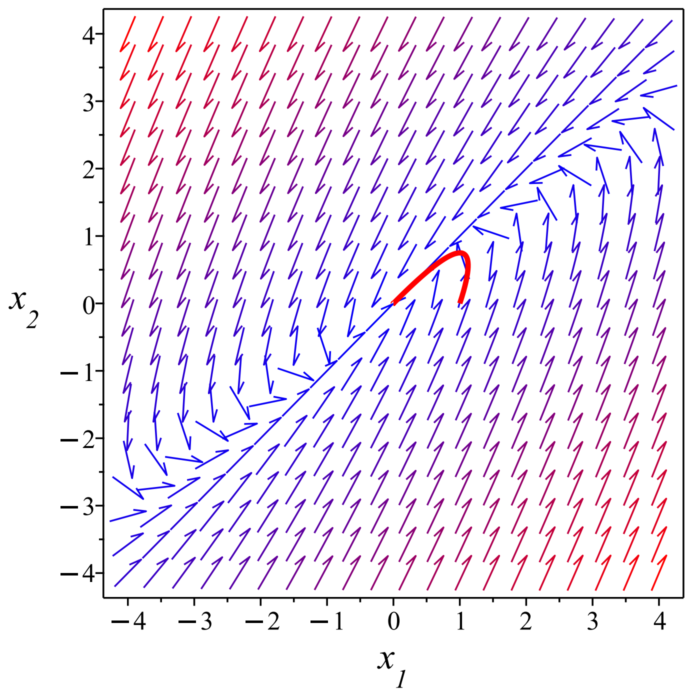
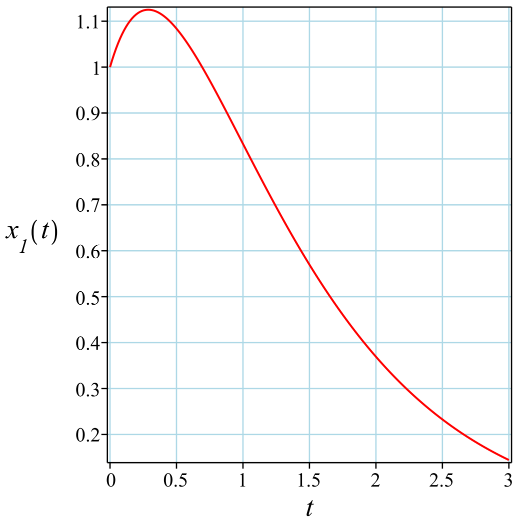
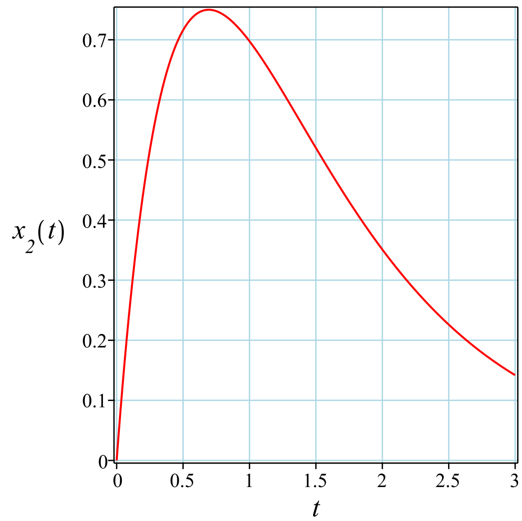


Figure 96: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
dsolve([diff(x__1(t),t) = x__1(t)-2*x__2(t), diff(x__2(t),t) = 3*x__1(t)-4*x__2(t), x__1(0)
```

$$x_1(t) = -2\,\mathrm{e}^{-2t} + 3\,\mathrm{e}^{-t}$$
$$x_2(t) = -3\,\mathrm{e}^{-2t} + 3\,\mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 33

```
DSolve[{x1'[t]==x1[t]-2*x2[t],x2'[t]==3*x1[t]-4*x2[t]},{x1[0]==1,x2[0]==0},{x1[t],x2[t]},t,I
```

$$\text{x1}(t) \to e^{-2t}(3e^t - 2)$$
$$\text{x2}(t) \to 3e^{-2t}(e^t - 1)$$

5.9 problem Problem 5.10

Internal problem ID [5906]
Internal file name [`OUTPUT/5154_Sunday_June_05_2022_03_26_25_PM_31343781/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360
Problem number: Problem 5.10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 3x_1(t) - 18x_2(t)$$
$$x_2'(t) = 2x_1(t) - 9x_2(t)$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 2]$$

5.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-3t}(1+6t) & -18t\,e^{-3t} \\ 2t\,e^{-3t} & e^{-3t}(1-6t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-3t}(1+6t) & -18t\, e^{-3t} \\ 2t\, e^{-3t} & e^{-3t}(1-6t) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-3t}(1+6t) - 36t\, e^{-3t} \\ 2t\, e^{-3t} + 2\, e^{-3t}(1-6t) \end{bmatrix} \\
&= \begin{bmatrix} e^{-3t}(1-30t) \\ (2-10t)\, e^{-3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & -18 & 0 \\ 2 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 6 & -18 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -18 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	2	1	Yes	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

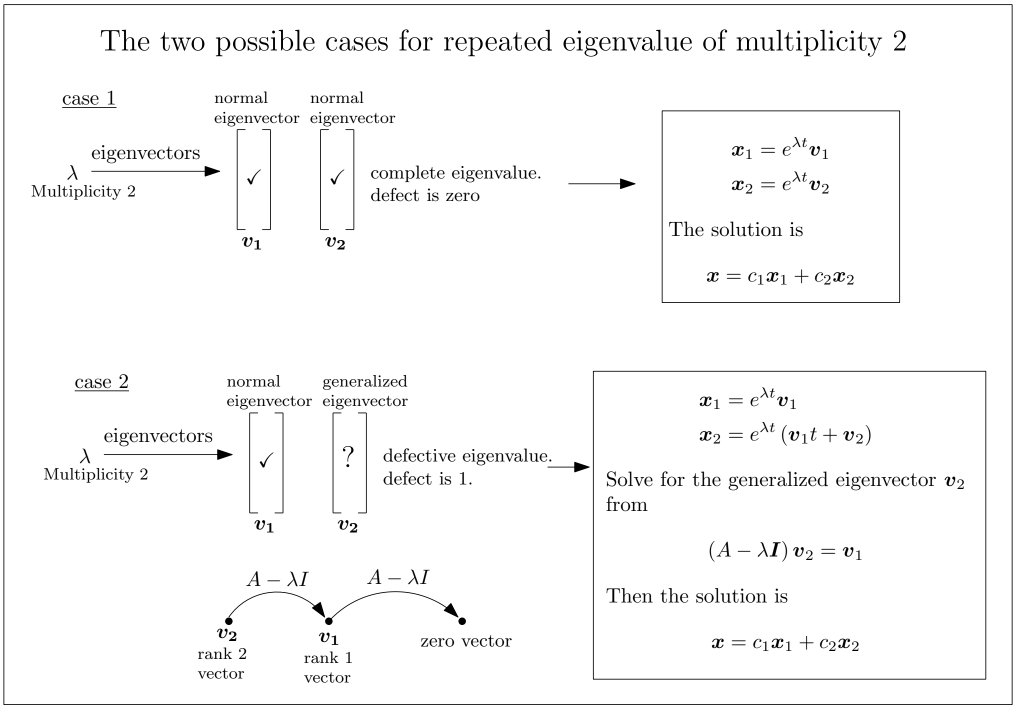


Figure 97: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{7}{2} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -3. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 3 \\ 1 \end{bmatrix} \mathrm{e}^{-3t} \\
&= \begin{bmatrix} 3\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{7}{2} \\ 1 \end{bmatrix} \right) \mathrm{e}^{-3t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-3t}(6t+7)}{2} \\ \mathrm{e}^{-3t}(t+1) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 3\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-3t}\left(3t + \frac{7}{2}\right) \\ \mathrm{e}^{-3t}(t+1) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-3t}\left(3c_1 + 3c_2 t + \frac{7}{2} c_2\right) \\ \mathrm{e}^{-3t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 2 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3c_1 + \frac{7c_2}{2} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 12 \\ c_2 = -10 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-3t}(1 - 30t) \\ (2 - 10t)\,\mathrm{e}^{-3t} \end{bmatrix}$$

The following is the phase plot of the system.

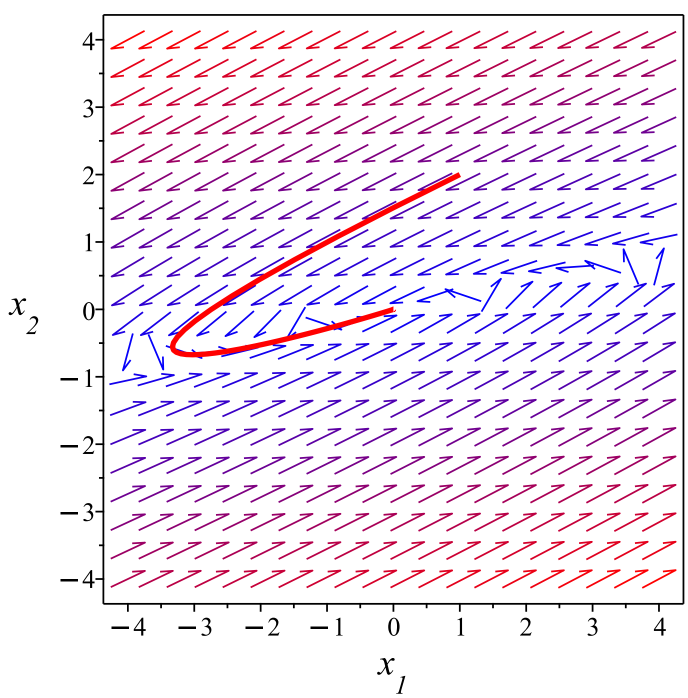
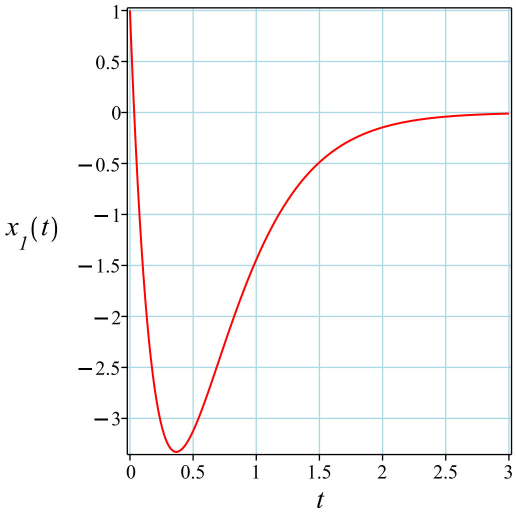
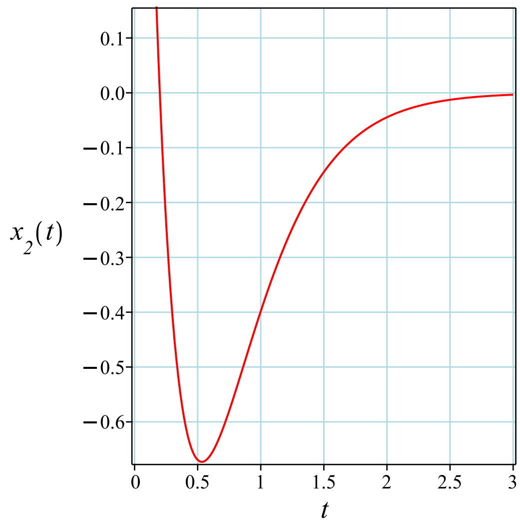


Figure 98: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve([diff(x__1(t),t) = 3*x__1(t)-18*x__2(t), diff(x__2(t),t) = 2*x__1(t)-9*x__2(t), x__1(
```

$$x_1(t) = e^{-3t}(-30t + 1)$$
$$x_2(t) = \frac{e^{-3t}(-180t + 36)}{18}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
DSolve[{x1'[t]==3*x1[t]-18*x2[t],x2'[t]==2*x1[t]-9*x2[t]},{x1[0]==1,x2[0]==2},{x1[t],x2[t]},
```

$$\text{x1}(t) \to e^{-3t}(1 - 30t)$$
$$\text{x2}(t) \to e^{-3t}(2 - 10t)$$

5.10 problem Problem 5.11

Internal problem ID [5907]
Internal file name [`OUTPUT/5155_Sunday_June_05_2022_03_26_26_PM_766659/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360
Problem number: Problem 5.11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -x_1(t) + 3x_2(t) \\
x_2'(t) &= -3x_1(t) + 5x_2(t)
\end{aligned}
$$

With initial conditions

$$
[x_1(0) = 1, x_2(0) = 2]
$$

5.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{2t}(1-3t) & 3t\,\mathrm{e}^{2t} \\ -3t\,\mathrm{e}^{2t} & \mathrm{e}^{2t}(1+3t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{2t}(1-3t) & 3t\,e^{2t} \\ -3t\,e^{2t} & e^{2t}(1+3t) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(1-3t) + 6t\,e^{2t} \\ -3t\,e^{2t} + 2\,e^{2t}(1+3t) \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(1+3t) \\ e^{2t}(2+3t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 3 \\ -3 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ -3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

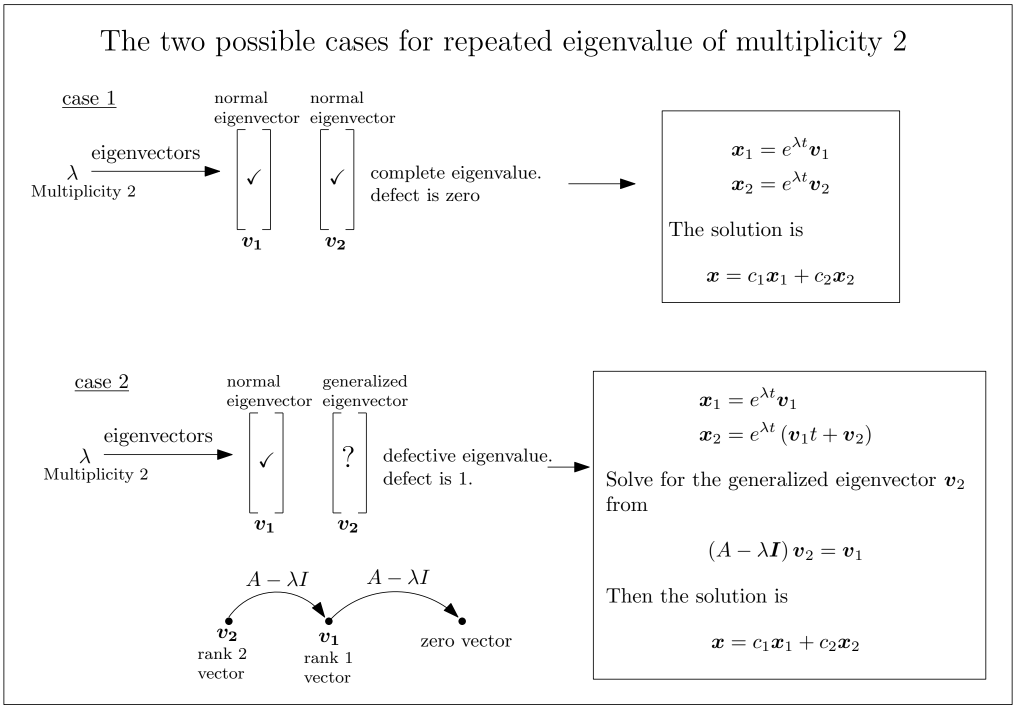


Figure 99: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\, \vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right) \mathrm{e}^{2t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{2t}(2+3t)}{3} \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{2t}\left(t + \frac{2}{3}\right) \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{2t}\left(c_1 + c_2 t + \frac{2}{3} c_2\right) \\ \mathrm{e}^{2t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 2 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + \frac{2c_2}{3} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -1 \\ c_2 = 3 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{2t}(1 + 3t) \\ e^{2t}(2 + 3t) \end{bmatrix}$$

The following is the phase plot of the system.

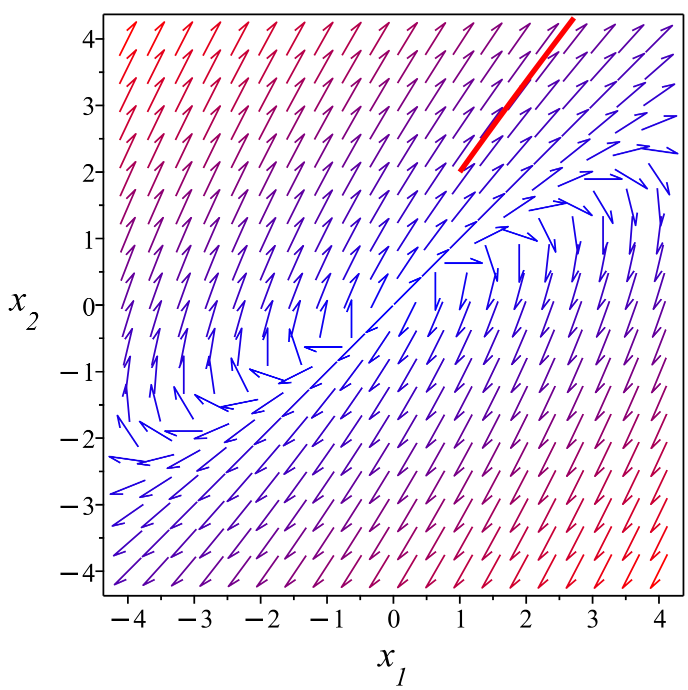
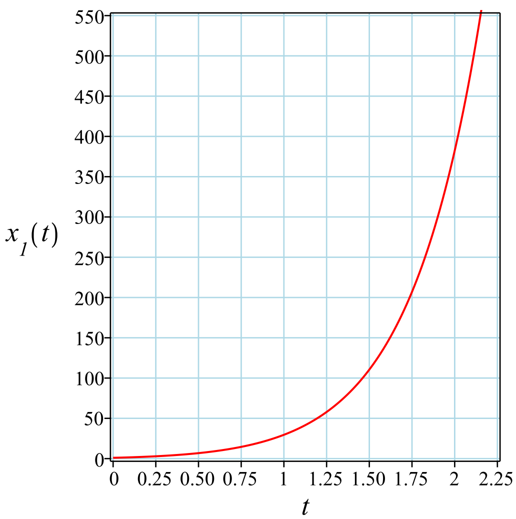
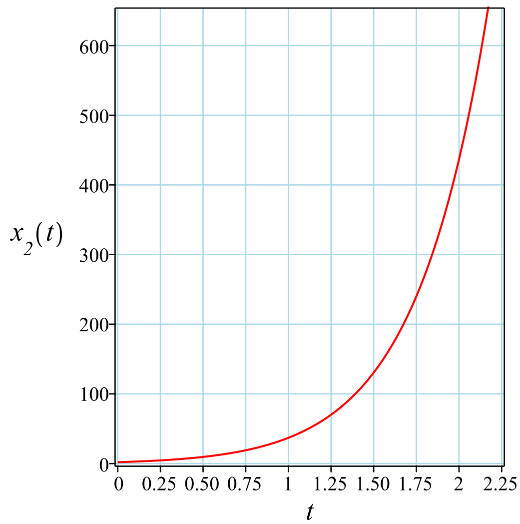


Figure 100: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve([diff(x__1(t),t) = -x__1(t)+3*x__2(t), diff(x__2(t),t) = -3*x__1(t)+5*x__2(t), x__1(0
```

$$x_1(t) = e^{2t}(3t+1)$$
$$x_2(t) = \frac{e^{2t}(9t+6)}{3}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
DSolve[{x1'[t]==-x1[t]+3*x2[t],x2'[t]==-3*x1[t]+5*x2[t]},{x1[0]==1,x2[0]==2},{x1[t],x2[t]},t
```

$$\text{x1}(t) \to e^{2t}(3t+1)$$
$$\text{x2}(t) \to e^{2t}(3t+2)$$

5.11 problem Problem 5.12

Internal problem ID [5908]
Internal file name [`OUTPUT/5156_Sunday_June_05_2022_03_26_27_PM_45427772/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360
Problem number: Problem 5.12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 3x_1(t) - 18x_2(t)$$
$$x_2'(t) = 2x_1(t) - 9x_2(t)$$

With initial conditions

$$[x_1(0) = 2, x_2(0) = 1]$$

5.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-3t}(1+6t) & -18t\,e^{-3t} \\ 2t\,e^{-3t} & e^{-3t}(1-6t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-3t}(1+6t) & -18t\, e^{-3t} \\ 2t\, e^{-3t} & e^{-3t}(1-6t) \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2\, e^{-3t}(1+6t) - 18t\, e^{-3t} \\ 4t\, e^{-3t} + e^{-3t}(1-6t) \end{bmatrix} \\
&= \begin{bmatrix} (2-6t)\, e^{-3t} \\ e^{-3t}(1-2t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & -18 & 0 \\ 2 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 6 & -18 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -18 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	2	1	Yes	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

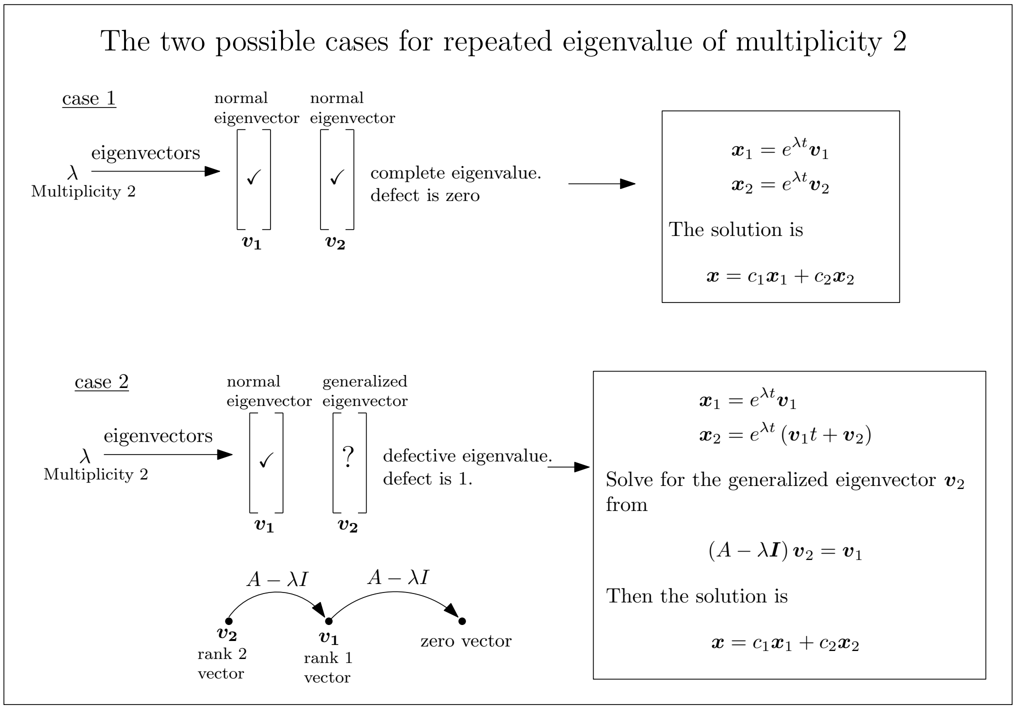


Figure 101: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{7}{2} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -3. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} \mathrm{e}^{-3t} \\ &= \begin{bmatrix} 3\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{7}{2} \\ 1 \end{bmatrix} \right) \mathrm{e}^{-3t} \\ &= \begin{bmatrix} \frac{\mathrm{e}^{-3t}(6t+7)}{2} \\ \mathrm{e}^{-3t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 3\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-3t}\left(3t + \frac{7}{2}\right) \\ \mathrm{e}^{-3t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-3t}\left(3c_1 + 3c_2 t + \frac{7}{2}c_2\right) \\ \mathrm{e}^{-3t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 2 \\ x_2(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3c_1 + \frac{7c_2}{2} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 3 \\ c_2 = -2 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (2 - 6t)\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t}(1 - 2t) \end{bmatrix}$$

The following is the phase plot of the system.

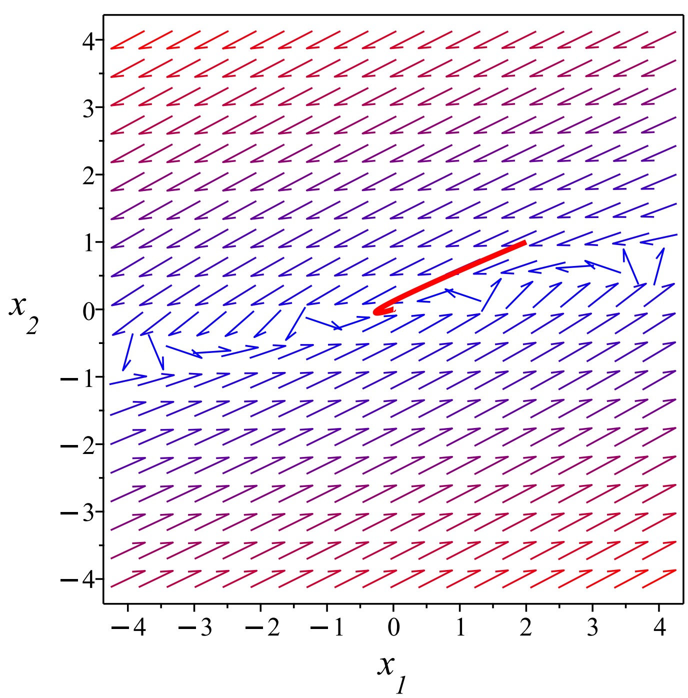
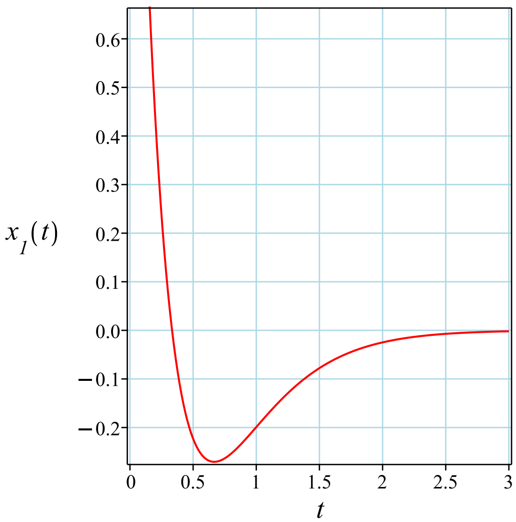
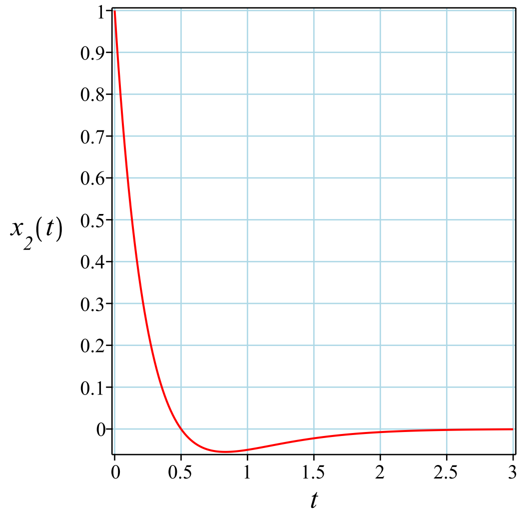


Figure 102: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve([diff(x__1(t),t) = 3*x__1(t)-18*x__2(t), diff(x__2(t),t) = 2*x__1(t)-9*x__2(t), x__1(
```

$$x_1(t) = e^{-3t}(-6t + 2)$$
$$x_2(t) = \frac{e^{-3t}(-36t + 18)}{18}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
DSolve[{x1'[t]==3*x1[t]-18*x2[t],x2'[t]==2*x1[t]-9*x2[t]},{x1[0]==2,x2[0]==1},{x1[t],x2[t]},
```

$$\text{x1}(t) \to e^{-3t}(2 - 6t)$$
$$\text{x2}(t) \to e^{-3t}(1 - 2t)$$

5.12 problem Problem 5.13

Internal problem ID [5909]
Internal file name [OUTPUT/5157_Sunday_June_05_2022_03_26_28_PM_4869892/index.tex]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360
Problem number: Problem 5.13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 3x_1(t) - x_2(t)$$
$$x_2'(t) = 4x_1(t) - 2x_2(t)$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 1]$$

5.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{-t}}{3} + \frac{4\,e^{2t}}{3} & -\frac{e^{2t}}{3} + \frac{e^{-t}}{3} \\ \frac{4\,e^{2t}}{3} - \frac{4\,e^{-t}}{3} & \frac{4\,e^{-t}}{3} - \frac{e^{2t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} -\frac{e^{-t}}{3} + \frac{4\,e^{2t}}{3} & -\frac{e^{2t}}{3} + \frac{e^{-t}}{3} \\ \frac{4\,e^{2t}}{3} - \frac{4\,e^{-t}}{3} & \frac{4\,e^{-t}}{3} - \frac{e^{2t}}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - \lambda - 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & -1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -1 & 0 \\ 4 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 4 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 4 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - 4R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{-t}}{4} \\ e^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} + \frac{c_2 e^{-t}}{4} \\ c_1 e^{2t} + c_2 e^{-t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + \frac{c_2}{4} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 1 \\ c_2 = 0 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

The following is the phase plot of the system.

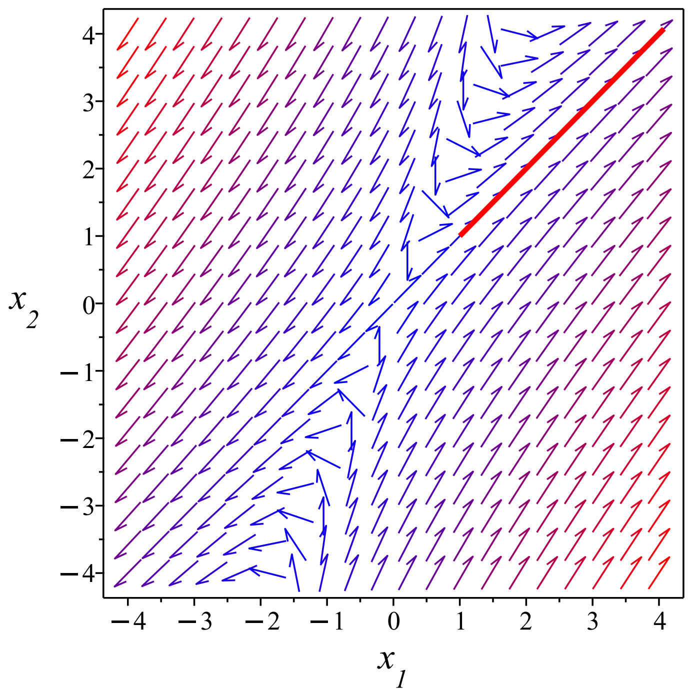
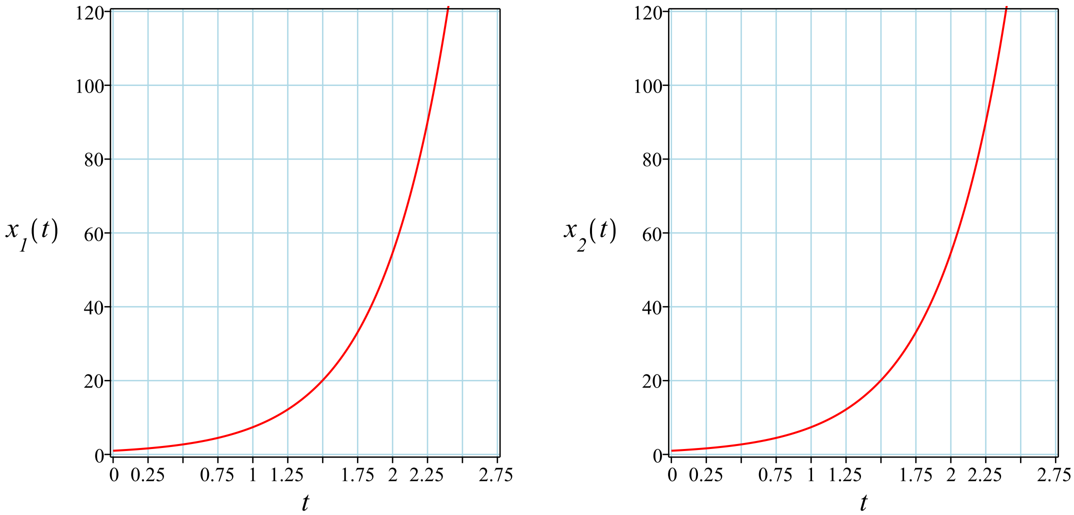


Figure 103: Phase plot

The following are plots of each solution.



✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(x__1(t),t) = 3*x__1(t)-x__2(t), diff(x__2(t),t) = 4*x__1(t)-2*x__2(t), x__1(0)
```

$$x_1(t) = e^{2t}$$
$$x_2(t) = e^{2t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 18

```
DSolve[{x1'[t]==3*x1[t]-x2[t],x2'[t]==4*x1[t]-2*x2[t]},{x1[0]==1,x2[0]==1},{x1[t],x2[t]},t,I
```

$$\text{x1}(t) \to e^{2t}$$
$$\text{x2}(t) \to e^{2t}$$

5.13 problem Problem 5.15 part 1

Internal problem ID [5910]
Internal file name [`OUTPUT/5158_Sunday_June_05_2022_03_26_30_PM_83297899/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360
Problem number: Problem 5.15 part 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) + x_2(t) - 8 \\
x_2'(t) &= x_1(t) + x_2(t) + 3
\end{aligned}
$$

5.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -8 \\ 3 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{1}{2} + \frac{e^{2t}}{2} & \frac{e^{2t}}{2} - \frac{1}{2} \\ \frac{e^{2t}}{2} - \frac{1}{2} & \frac{1}{2} + \frac{e^{2t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{1}{2} + \frac{e^{2t}}{2} & \frac{e^{2t}}{2} - \frac{1}{2} \\ \frac{e^{2t}}{2} - \frac{1}{2} & \frac{1}{2} + \frac{e^{2t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{1}{2} + \frac{e^{2t}}{2}\right) c_1 + \left(\frac{e^{2t}}{2} - \frac{1}{2}\right) c_2 \\ \left(\frac{e^{2t}}{2} - \frac{1}{2}\right) c_1 + \left(\frac{1}{2} + \frac{e^{2t}}{2}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(c_1+c_2)e^{2t}}{2} - \frac{c_2}{2} + \frac{c_1}{2} \\ \frac{(c_1+c_2)e^{2t}}{2} + \frac{c_2}{2} - \frac{c_1}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{1}{2} + \frac{e^{-2t}}{2} & -\frac{1}{2} + \frac{e^{-2t}}{2} \\ -\frac{1}{2} + \frac{e^{-2t}}{2} & \frac{1}{2} + \frac{e^{-2t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{1}{2} + \frac{e^{2t}}{2} & \frac{e^{2t}}{2} - \frac{1}{2} \\ \frac{e^{2t}}{2} - \frac{1}{2} & \frac{1}{2} + \frac{e^{2t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{1}{2} + \frac{e^{-2t}}{2} & -\frac{1}{2} + \frac{e^{-2t}}{2} \\ -\frac{1}{2} + \frac{e^{-2t}}{2} & \frac{1}{2} + \frac{e^{-2t}}{2} \end{bmatrix} \begin{bmatrix} -8 \\ 3 \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{1}{2} + \frac{e^{2t}}{2} & \frac{e^{2t}}{2} - \frac{1}{2} \\ \frac{e^{2t}}{2} - \frac{1}{2} & \frac{1}{2} + \frac{e^{2t}}{2} \end{bmatrix} \begin{bmatrix} -\frac{11t}{2} + \frac{5\,e^{-2t}}{4} \\ \frac{11t}{2} + \frac{5\,e^{-2t}}{4} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{11t}{2} + \frac{5}{4} \\ \frac{5}{4} + \frac{11t}{2} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \frac{(2c_1+2c_2)e^{2t}}{4} - \frac{11t}{2} + \frac{c_1}{2} - \frac{c_2}{2} + \frac{5}{4} \\ \frac{(2c_1+2c_2)e^{2t}}{4} + \frac{11t}{2} - \frac{c_1}{2} + \frac{c_2}{2} + \frac{5}{4} \end{bmatrix}
\end{aligned}
$$

5.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -8 \\ 3 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 2\lambda = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} \end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^0$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^0$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \mathrm{e}^{2t} & -1 \\ \mathrm{e}^{2t} & 1 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{\mathrm{e}^{-2t}}{2} & \frac{\mathrm{e}^{-2t}}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{2t} & -1 \\ e^{2t} & 1 \end{bmatrix} \int \begin{bmatrix} \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -8 \\ 3 \end{bmatrix} dt \\
&= \begin{bmatrix} e^{2t} & -1 \\ e^{2t} & 1 \end{bmatrix} \int \begin{bmatrix} -\frac{5\,e^{-2t}}{2} \\ \frac{11}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} e^{2t} & -1 \\ e^{2t} & 1 \end{bmatrix} \begin{bmatrix} \frac{5\,e^{-2t}}{4} \\ \frac{11t}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{11t}{2} + \frac{5}{4} \\ \frac{5}{4} + \frac{11t}{2} \end{bmatrix}
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} c_1 e^{2t} \\ c_1 e^{2t} \end{bmatrix} + \begin{bmatrix} -c_2 \\ c_2 \end{bmatrix} + \begin{bmatrix} -\frac{11t}{2} + \frac{5}{4} \\ \frac{5}{4} + \frac{11t}{2} \end{bmatrix}
\end{aligned}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} - c_2 - \frac{11t}{2} + \frac{5}{4} \\ c_1 e^{2t} + c_2 + \frac{5}{4} + \frac{11t}{2} \end{bmatrix}
$$

The following is the phase plot of the system.

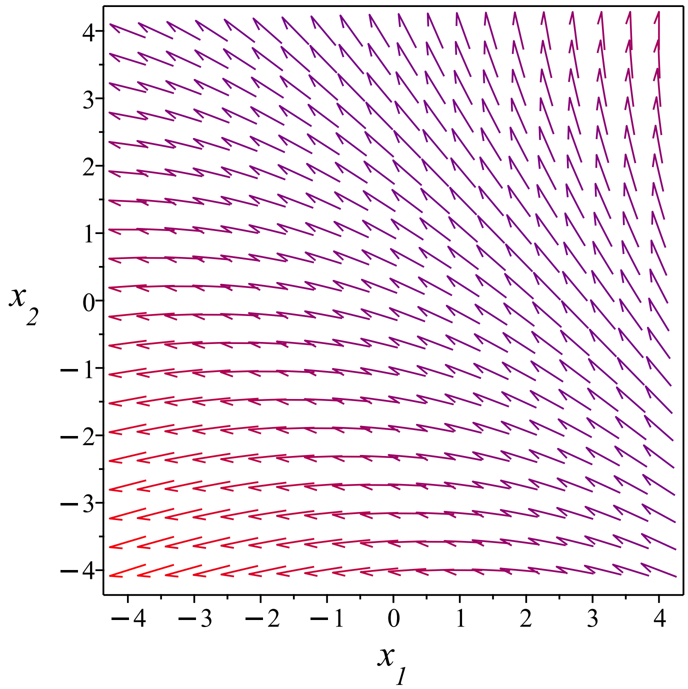


Figure 104: Phase plot

5.13.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) + x_2(t) - 8, x_2'(t) = x_1(t) + x_2(t) + 3]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} -8 \\ 3 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} -8 \\ 3 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} -8 \\ 3 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{2t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}$

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{x}_p(t)$$

☐ Fundamental matrix

  ◦ Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -1 & e^{2t} \\ 1 & e^{2t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t) = \begin{bmatrix} -1 & e^{2t} \\ 1 & e^{2t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}}$$
- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \begin{bmatrix} \frac{1}{2} + \frac{e^{2t}}{2} & \frac{e^{2t}}{2} - \frac{1}{2} \\ \frac{e^{2t}}{2} - \frac{1}{2} & \frac{1}{2} + \frac{e^{2t}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\underline{\vec{x}}_p(t) = \Phi(t) \cdot \vec{v}(t)$
- Take the derivative of the particular solution

  $\underline{\vec{x}}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$
- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\underset{\_\_\_}{\vec{x}}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\underset{\_\_\_}{\vec{x}}_p(t) = \begin{bmatrix} -\frac{11\ln(e^{2t})}{4} - \frac{5e^{2t}}{4} + \frac{5}{4} \\ -\frac{5e^{2t}}{4} + \frac{5}{4} + \frac{11\ln(e^{2t})}{4} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\underset{\_\_\_}{\vec{x}}(t) = c_1\underset{\_\_\_}{\vec{x}}_1 + c_2\underset{\_\_\_}{\vec{x}}_2 + \begin{bmatrix} -\frac{11\ln(e^{2t})}{4} - \frac{5e^{2t}}{4} + \frac{5}{4} \\ -\frac{5e^{2t}}{4} + \frac{5}{4} + \frac{11\ln(e^{2t})}{4} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_2e^{2t} - \frac{11\ln(e^{2t})}{4} - \frac{5e^{2t}}{4} + \frac{5}{4} - c_1 \\ c_2e^{2t} - \frac{5e^{2t}}{4} + \frac{5}{4} + \frac{11\ln(e^{2t})}{4} + c_1 \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = c_2e^{2t} - \frac{11\ln(e^{2t})}{4} - \frac{5e^{2t}}{4} + \frac{5}{4} - c_1, x_2(t) = c_2e^{2t} - \frac{5e^{2t}}{4} + \frac{5}{4} + \frac{11\ln(e^{2t})}{4} + c_1\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve([diff(x__1(t),t)=x__1(t)+x__2(t)-8,diff(x__2(t),t)=x__1(t)+x__2(t)+3],singsol=all)
```

$$x_1(t) = \frac{c_1e^{2t}}{2} - \frac{11t}{2} + c_2$$
$$x_2(t) = \frac{c_1e^{2t}}{2} + \frac{5}{2} + \frac{11t}{2} - c_2$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 74

```
DSolve[{x1'[t]==x1[t]+x2[t]-8,x2'[t]==x1[t]+x2[t]+3},{x1[t],x2[t]},t,IncludeSingularSolution
```

$$\text{x1}(t) \to \frac{1}{4}\left(-22t + 2c_1\left(e^{2t} + 1\right) + 2c_2e^{2t} + 5 - 2c_2\right)$$
$$\text{x2}(t) \to \frac{1}{4}\left(22t + 2c_1\left(e^{2t} - 1\right) + 2c_2e^{2t} + 5 + 2c_2\right)$$

5.14 problem Problem 5.15 part 3

Internal problem ID [5911]
Internal file name [`OUTPUT/5159_Sunday_June_05_2022_03_26_31_PM_4963010/index.tex`]

Book: THEORY OF DIFFERENTIAL EQUATIONS IN ENGINEERING AND MECHANICS. K.T. CHAU, CRC Press. Boca Raton, FL. 2018
Section: Chapter 5. Systems of First Order Differential Equations. Section 5.11 Problems. Page 360
Problem number: Problem 5.15 part 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) + x_2(t) - 8 \\
x_2'(t) &= x_1(t) + x_2(t) + 3
\end{aligned}
$$

With initial conditions

$$
[x_1(0) = 1, x_2(0) = 2]
$$

5.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -8 \\ 3 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{1}{2}+\frac{e^{2t}}{2} & \frac{e^{2t}}{2}-\frac{1}{2} \\ \frac{e^{2t}}{2}-\frac{1}{2} & \frac{1}{2}+\frac{e^{2t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{1}{2}+\frac{e^{2t}}{2} & \frac{e^{2t}}{2}-\frac{1}{2} \\ \frac{e^{2t}}{2}-\frac{1}{2} & \frac{1}{2}+\frac{e^{2t}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2}+\frac{3\,e^{2t}}{2} \\ \frac{3\,e^{2t}}{2}+\frac{1}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{1}{2}+\frac{e^{-2t}}{2} & -\frac{1}{2}+\frac{e^{-2t}}{2} \\ -\frac{1}{2}+\frac{e^{-2t}}{2} & \frac{1}{2}+\frac{e^{-2t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{1}{2}+\frac{e^{2t}}{2} & \frac{e^{2t}}{2}-\frac{1}{2} \\ \frac{e^{2t}}{2}-\frac{1}{2} & \frac{1}{2}+\frac{e^{2t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{1}{2}+\frac{e^{-2t}}{2} & -\frac{1}{2}+\frac{e^{-2t}}{2} \\ -\frac{1}{2}+\frac{e^{-2t}}{2} & \frac{1}{2}+\frac{e^{-2t}}{2} \end{bmatrix} \begin{bmatrix} -8 \\ 3 \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{1}{2}+\frac{e^{2t}}{2} & \frac{e^{2t}}{2}-\frac{1}{2} \\ \frac{e^{2t}}{2}-\frac{1}{2} & \frac{1}{2}+\frac{e^{2t}}{2} \end{bmatrix} \begin{bmatrix} -\frac{11t}{2}+\frac{5\,e^{-2t}}{4} \\ \frac{11t}{2}+\frac{5\,e^{-2t}}{4} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{11t}{2}+\frac{5}{4} \\ \frac{5}{4}+\frac{11t}{2} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{3}{4} + \frac{3\,e^{2t}}{2} - \frac{11t}{2} \\ \frac{3\,e^{2t}}{2} + \frac{7}{4} + \frac{11t}{2} \end{bmatrix}$$

5.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -8 \\ 3 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^0 \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^0 \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -1 & e^{2t} \\ 1 & e^{2t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} \end{bmatrix}$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -1 & e^{2t} \\ 1 & e^{2t} \end{bmatrix} \int \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} \end{bmatrix} \begin{bmatrix} -8 \\ 3 \end{bmatrix} dt \\
&= \begin{bmatrix} -1 & e^{2t} \\ 1 & e^{2t} \end{bmatrix} \int \begin{bmatrix} \frac{11}{2} \\ -\frac{5\,e^{-2t}}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} -1 & e^{2t} \\ 1 & e^{2t} \end{bmatrix} \begin{bmatrix} \frac{11t}{2} \\ \frac{5\,e^{-2t}}{4} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{11t}{2} + \frac{5}{4} \\ \frac{5}{4} + \frac{11t}{2} \end{bmatrix}
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} -c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2 e^{2t} \\ c_2 e^{2t} \end{bmatrix} + \begin{bmatrix} -\frac{11t}{2} + \frac{5}{4} \\ \frac{5}{4} + \frac{11t}{2} \end{bmatrix}
\end{aligned}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 e^{2t} - \frac{11t}{2} + \frac{5}{4} \\ c_1 + c_2 e^{2t} + \frac{5}{4} + \frac{11t}{2} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 2 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$
\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 + \frac{5}{4} \\ c_1 + c_2 + \frac{5}{4} \end{bmatrix}
$$

Solving for the constants of integrations gives

$$
\begin{bmatrix} c_1 = \frac{1}{2} \\ c_2 = \frac{1}{4} \end{bmatrix}
$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{3}{4} + \frac{e^{2t}}{4} - \frac{11t}{2} \\ \frac{7}{4} + \frac{e^{2t}}{4} + \frac{11t}{2} \end{bmatrix}$$

The following is the phase plot of the system.

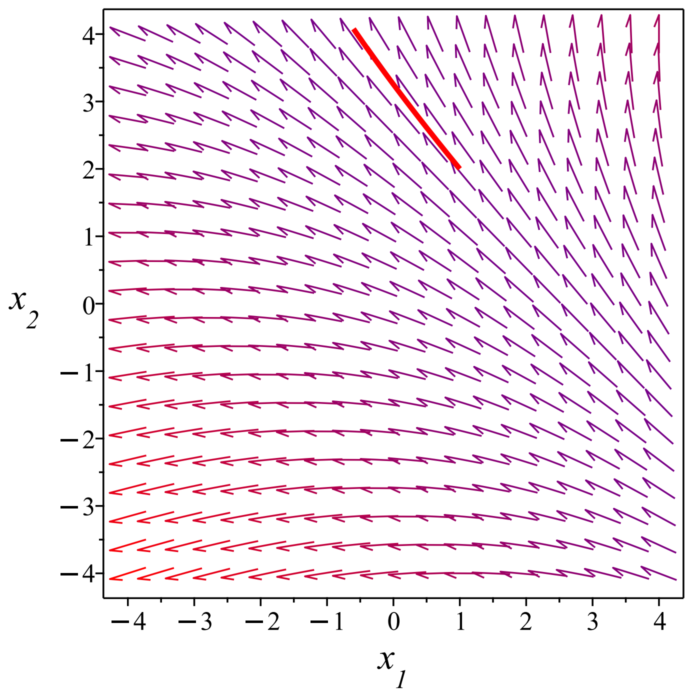
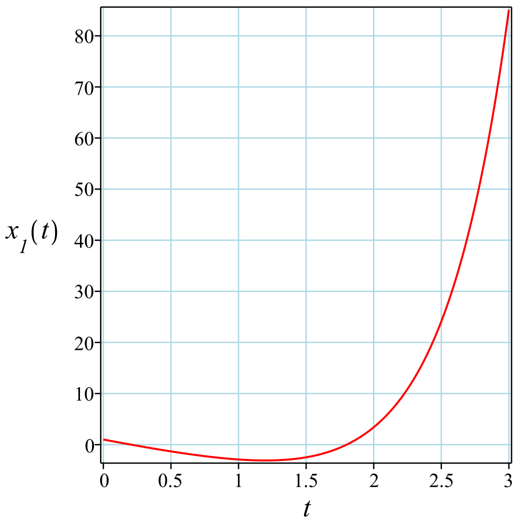
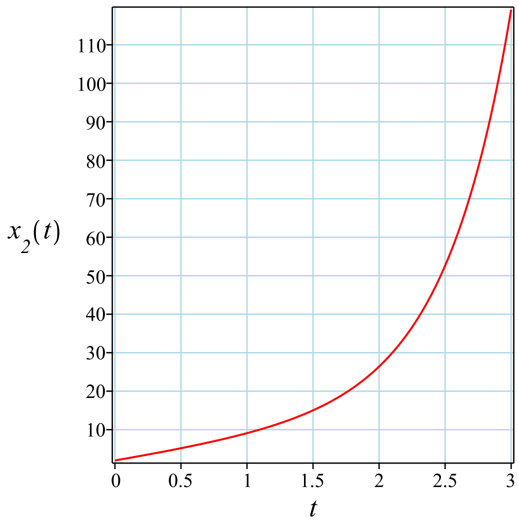


Figure 105: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve([diff(x__1(t),t) = x__1(t)+x__2(t)-8, diff(x__2(t),t) = x__1(t)+x__2(t)+3, x__1(0) =
```

$$x_1(t) = \frac{e^{2t}}{4} - \frac{11t}{2} + \frac{3}{4}$$
$$x_2(t) = \frac{e^{2t}}{4} + \frac{7}{4} + \frac{11t}{2}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 36

```
DSolve[{x1'[t]==x1[t]+x2[t]-8,x2'[t]==x1[t]+x2[t]+3},{x1[0]==1,x2[0]==2},{x1[t],x2[t]},t,Inc
```

$$\text{x1}(t) \to \frac{1}{4}\left(-22t + e^{2t} + 3\right)$$
$$\text{x2}(t) \to \frac{1}{4}\left(22t + e^{2t} + 7\right)$$