A Solution Manual For

Selected problems from homeworks from different courses

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Contents

1 Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota

1.1 problem HW 1 problem 6(a)

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Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 1 problem 6(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y}{x \ln(x)} = 0}$$

1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{x \ln(x)} \end{aligned}$$

Where $f(x) = \frac{1}{x \ln(x)}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{x \ln(x)}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{x \ln(x)}\,dx \\
\ln(y) &= \ln(\ln(x)) + c_1 \\
y &= \mathrm{e}^{\ln(\ln(x)) + c_1} \\
&= c_1 \ln(x)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \ln(x) \tag{1}$$

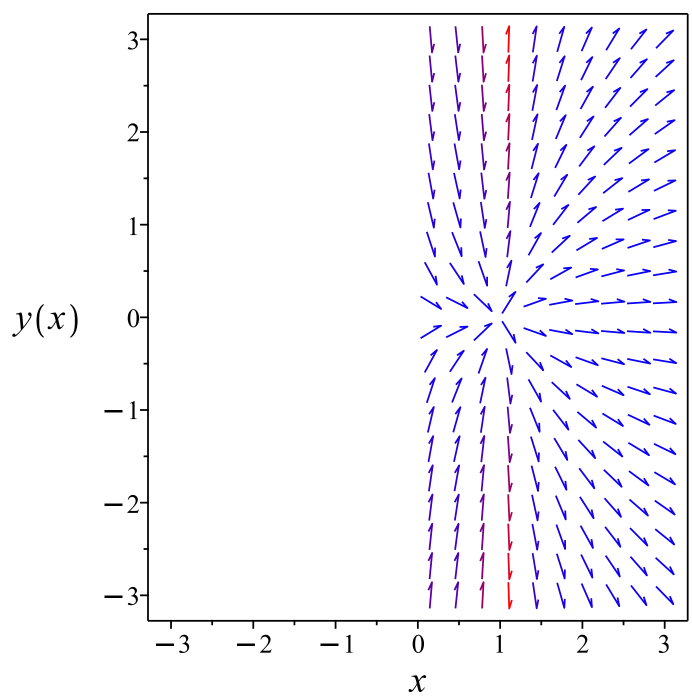


Figure 1: Slope field plot

Verification of solutions

$$y = c_1 \ln(x)$$

Verified OK.

1.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x \ln(x)} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x \ln(x)} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{x \ln(x)} dx} \\ &= \frac{1}{\ln(x)} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{y}{\ln(x)} \right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{\ln(x)} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\ln(x)}$ results in

$$y = c_1 \ln(x)$$

Summary
The solution(s) found are the following

$$y = c_1 \ln(x) \tag{1}$$

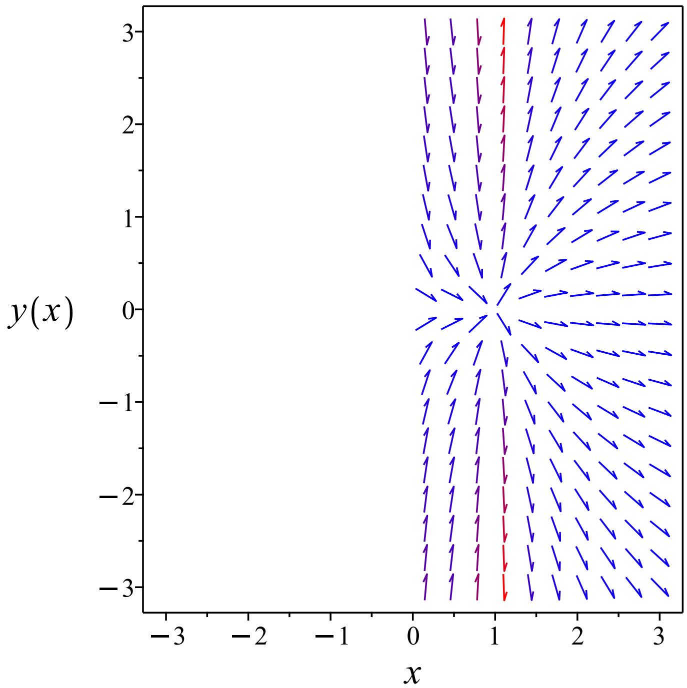


Figure 2: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \ln(x)$$

Verified OK.

1.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)}{\ln(x)} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(\ln(x)-1)}{x\ln(x)} \end{aligned}$$

Where $f(x) = -\frac{\ln(x)-1}{x\ln(x)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{\ln(x)-1}{x\ln(x)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{\ln(x)-1}{x\ln(x)}\,dx \\
\ln(u) &= -\ln(x) + \ln(\ln(x)) + c_2 \\
u &= \mathrm{e}^{-\ln(x)+\ln(\ln(x))+c_2} \\
&= c_2 \mathrm{e}^{-\ln(x)+\ln(\ln(x))}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2 \ln(x)}{x}$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= c_2 \ln(x)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 \ln(x) \tag{1}$$

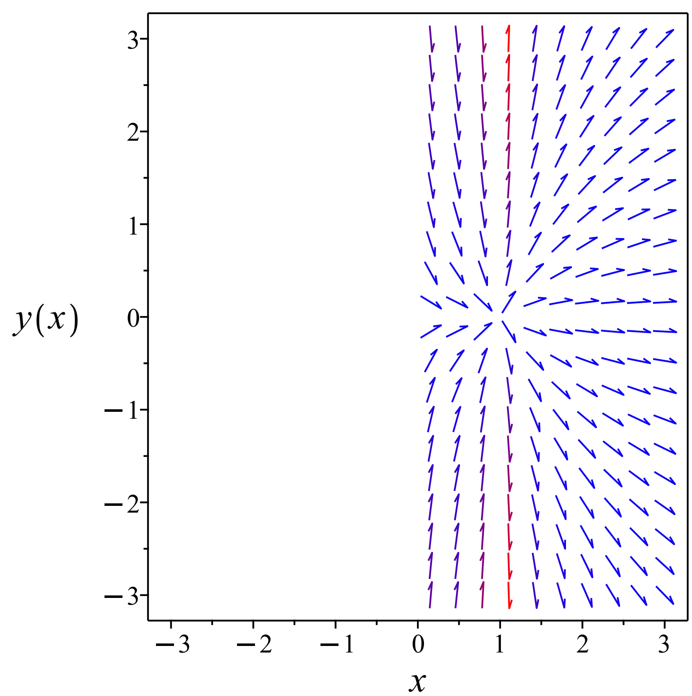


Figure 3: Slope field plot

Verification of solutions

$$
y = c_2 \ln(x)
$$

Verified OK.

1.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y}{x \ln(x)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \ln(x)
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\ln (x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\ln (x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y}{x \ln (x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{\ln (x)^2 x} \\ S_y &= \frac{1}{\ln (x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\ln (x)} = c_1$$

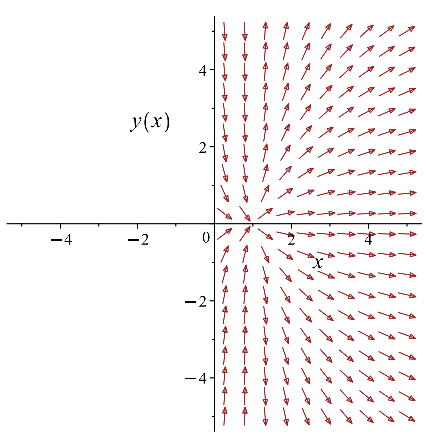
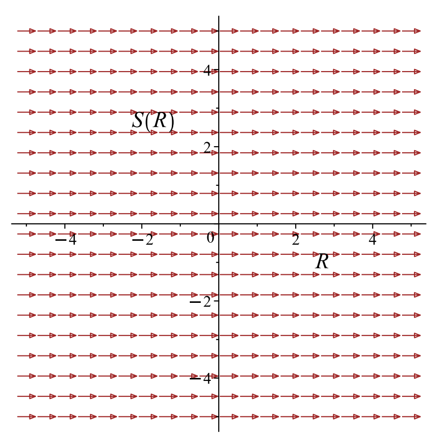
Which simplifies to

$$\frac{y}{\ln (x)} = c_1$$

Which gives

$$y = c_1 \ln (x)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x \ln(x)}$ 	$R = x$ $S = \frac{y}{\ln (x)}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 \ln (x) \tag{1}$$

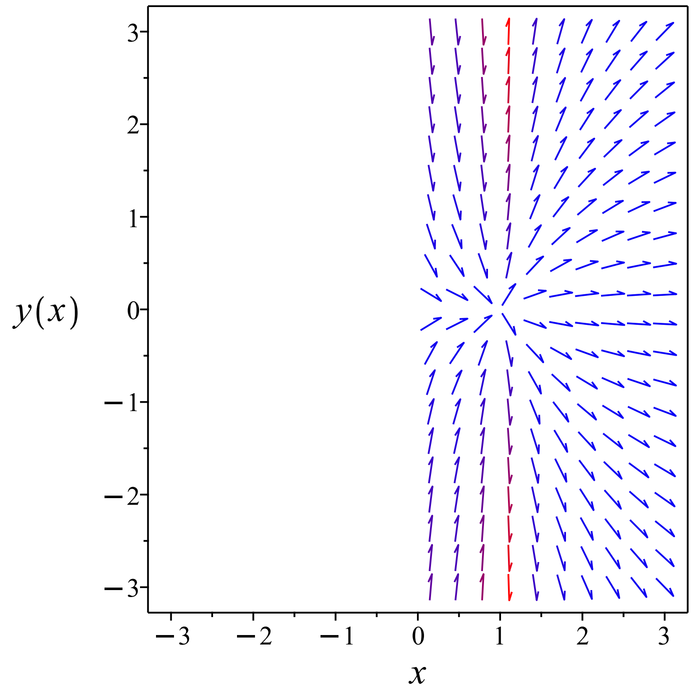


Figure 4: Slope field plot

Verification of solutions

$$y = c_1 \ln(x)$$

Verified OK.

1.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y}\right) \mathrm{d}y = \left(\frac{1}{x \ln(x)}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x \ln(x)}\right) \mathrm{d}x + \left(\frac{1}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x \ln(x)}$$
$$N(x,y) = \frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x \ln(x)}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{1}{x \ln(x)} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(\ln(x)) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(\ln(x)) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\ln(x)) + \ln(y)$$

The solution becomes

$$y = e^{c_1} \ln(x)$$

Summary

The solution(s) found are the following

$$y = e^{c_1} \ln(x) \tag{1}$$

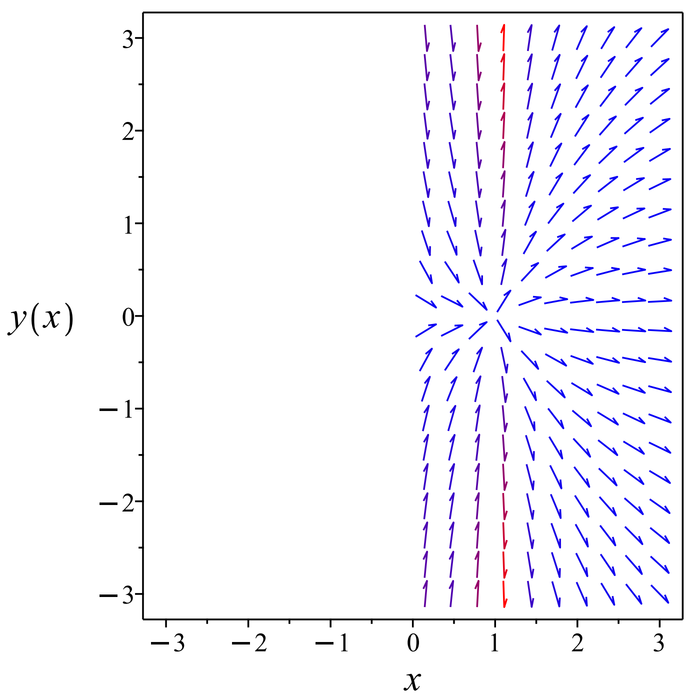


Figure 5: Slope field plot

Verification of solutions

$$y = e^{c_1} \ln(x)$$

Verified OK.

1.1.6 Maple step by step solution

Let's solve

$y' - \frac{y}{x \ln(x)} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x \ln(x)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x \ln(x)} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(\ln(x)) + c_1$

- Solve for y

  $y = e^{c_1} \ln(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(y(x),x)=y(x)/(x*ln(x)),y(x), singsol=all)
```

$$y(x) = c_1 \ln(x)$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 15

```
DSolve[y'[x]==y[x]/(x*Log[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \log(x)$$
$$y(x) \to 0$$

1.2 problem HW 1 problem 6(b)

Internal problem ID [7030]
Internal file name [`OUTPUT/6016_Sunday_June_05_2022_04_13_53_PM_47177539/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 1 problem 6(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x^2+1)\,y'+y^2=-1}$$

With initial conditions

$$[y(0)=1]$$

1.2.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{y^2+1}{x^2+1} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y^2+1}{x^2+1}\right) \\ &= -\frac{2y}{x^2+1}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.2.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{-y^2-1}{x^2+1}\end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(y) = -y^2 - 1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{-y^2-1}\,dy &= \frac{1}{x^2+1}\,dx \\ \int \frac{1}{-y^2-1}\,dy &= \int \frac{1}{x^2+1}\,dx \\ -\arctan(y) &= \arctan(x) + c_1\end{aligned}$$

Which results in

$$y = -\tan(\arctan(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\tan(c_1)$$

$$c_1 = -\frac{\pi}{4}$$

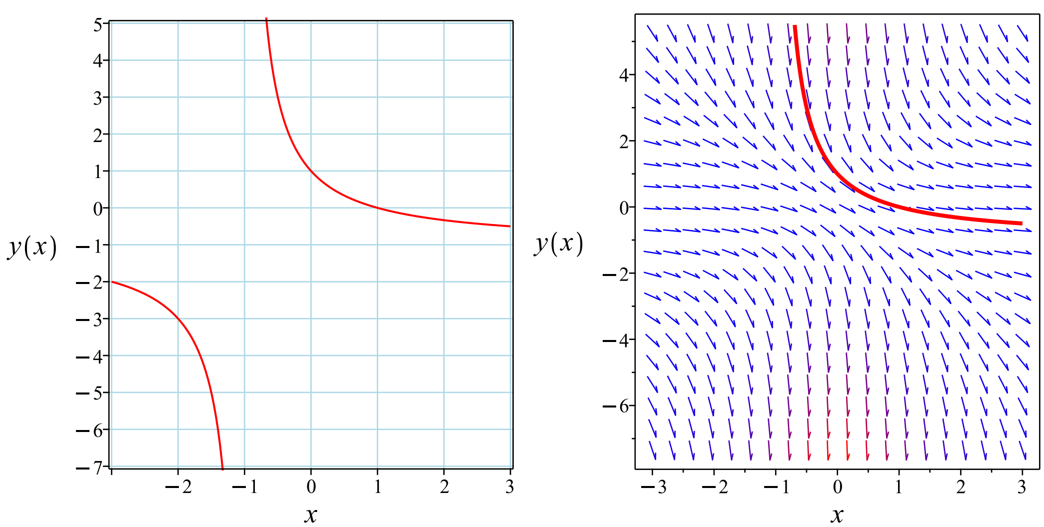
Substituting c_1 found above in the general solution gives

$$y = \frac{1-x}{1+x}$$

Summary

The solution(s) found are the following

$$y = \frac{1-x}{1+x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{1-x}{1+x}$$

Verified OK.

1.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y^2+1}{x^2+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= x^2+1 \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x^2+1} dx
\end{aligned}
$$

Which results in

$$
S = \arctan(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y^2+1}{x^2+1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x^2+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\arctan(x) = -\arctan(y) + c_1$$

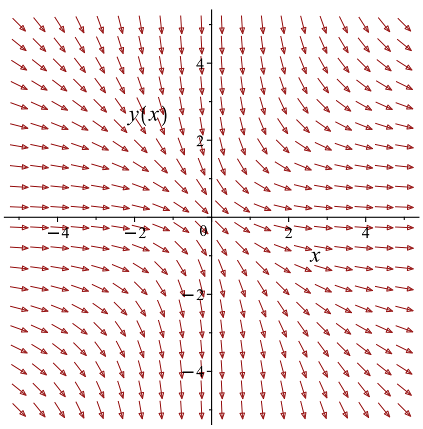
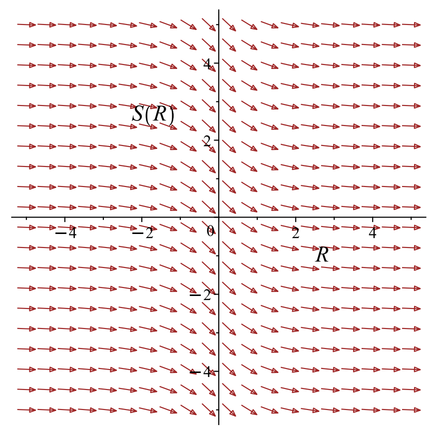
Which simplifies to

$$\arctan(x) = -\arctan(y) + c_1$$

Which gives

$$y = \tan(-\arctan(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2+1}{x^2+1}$ 	$R = y$ $S = \arctan(x)$	$\frac{dS}{dR} = -\frac{1}{R^2+1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \tan(c_1)$$

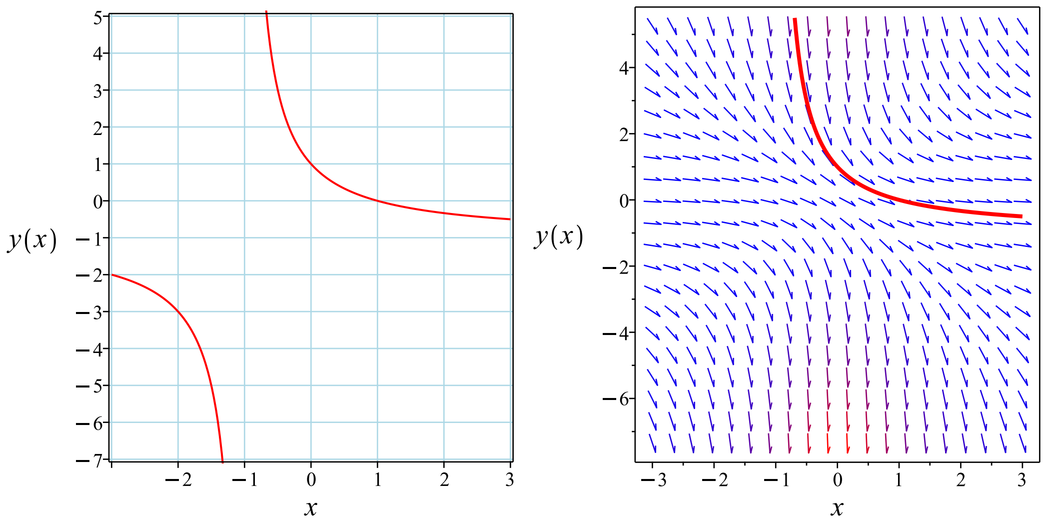
$$c_1 = \frac{\pi}{4}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{1-x}{1+x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{1-x}{1+x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1 - x}{1 + x}$$

Verified OK.

1.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{gathered}\left(\frac{1}{-y^2-1}\right)\mathrm{d}y=\left(\frac{1}{x^2+1}\right)\mathrm{d}x\\ \left(-\frac{1}{x^2+1}\right)\mathrm{d}x+\left(\frac{1}{-y^2-1}\right)\mathrm{d}y=0 \end{gathered}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{x^2+1}\\ N(x,y)&=\frac{1}{-y^2-1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{x^2+1}\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(\frac{1}{-y^2-1}\right)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{x^2+1} \, \mathrm{d}x$$
$$\phi = -\arctan(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-y^2-1}$. Therefore equation (4) becomes

$$\frac{1}{-y^2-1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{1}{y^2+1}\right) \mathrm{d}y$$
$$f(y) = -\arctan(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arctan(x) - \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arctan(x) - \arctan(y)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\pi}{4} = c_1$$

$$c_1 = -\frac{\pi}{4}$$

Substituting c_1 found above in the general solution gives

$$-\arctan(x) - \arctan(y) = -\frac{\pi}{4}$$

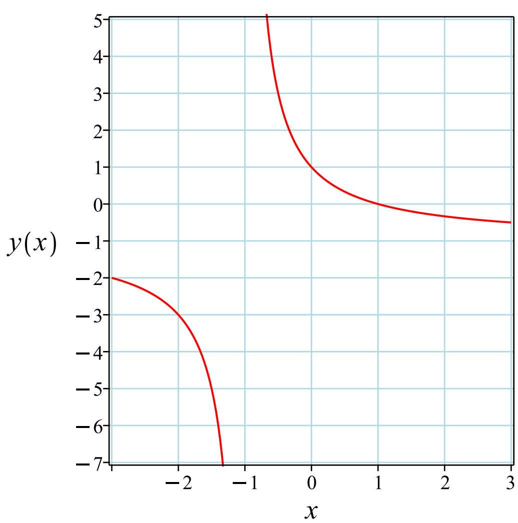
Solving for y from the above gives

$$y = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$$

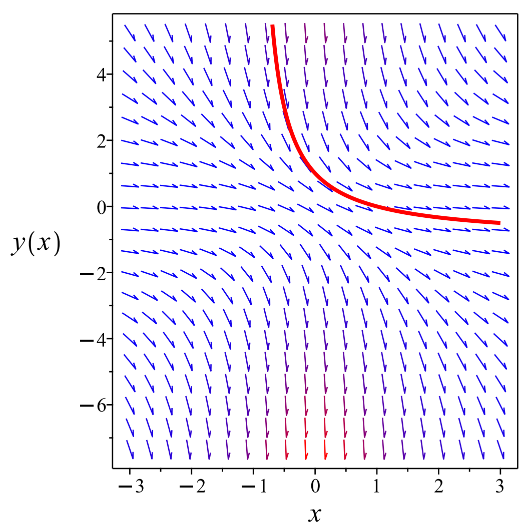
Summary

The solution(s) found are the following

$$y = \cot\left(\arctan(x) + \frac{\pi}{4}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$$

Verified OK.

1.2.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= -\frac{y^2+1}{x^2+1}
\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2}{x^2+1} - \frac{1}{x^2+1}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{1}{x^2+1}$, $f_1(x) = 0$ and $f_2(x) = -\frac{1}{x^2+1}$. Let

$$\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{u}{x^2+1}} \qquad (1)
\end{aligned}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)$$

But

$$\begin{aligned}
f_2' &= \frac{2x}{(x^2+1)^2} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= -\frac{1}{(x^2+1)^3}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x^2+1} - \frac{2xu'(x)}{(x^2+1)^2} - \frac{u(x)}{(x^2+1)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_1 x + c_2}{\sqrt{x^2 + 1}}$$

The above shows that

$$u'(x) = \frac{-c_2 x + c_1}{(x^2 + 1)^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$y = \frac{-c_2 x + c_1}{c_1 x + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 - x}{c_3 x + 1}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_3$$

$$c_3 = 1$$

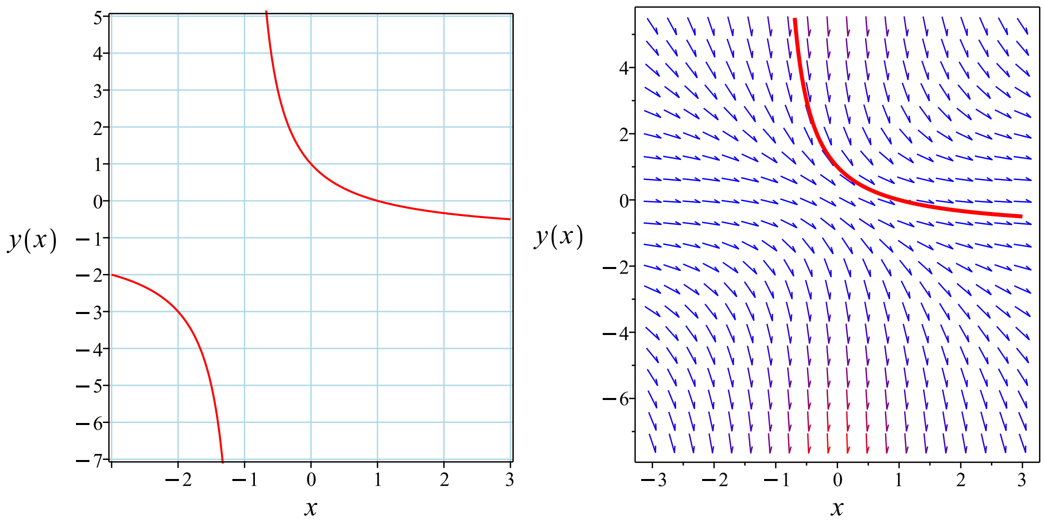
Substituting c_3 found above in the general solution gives

$$y = -\frac{x - 1}{1 + x}$$

Summary

The solution(s) found are the following

$$y = -\frac{x - 1}{1 + x} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{x-1}{1+x}$$

Verified OK.

1.2.6 Maple step by step solution

Let's solve

$\left[\left(x^2+1\right)y'+y^2=-1, y(0)=1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y^2-1}=\frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y^2-1}dx=\int \frac{1}{x^2+1}dx+c_1$

- Evaluate integral

  $-\arctan\left(y\right)=\arctan\left(x\right)+c_1$

- Solve for y

  $y=-\tan\left(\arctan\left(x\right)+c_1\right)$

- Use initial condition $y(0) = 1$

  $1 = -\tan(c_1)$
- Solve for c_1

  $c_1 = -\frac{\pi}{4}$
- Substitute $c_1 = -\frac{\pi}{4}$ into general solution and simplify

  $y = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$
- Solution to the IVP

  $y = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 11

```
dsolve([(x^2+1)*diff(y(x),x)+y(x)^2=-1,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.264 (sec). Leaf size: 14

```
DSolve[{(x^2+1)*y'[x]+y[x]^2==-1,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cot\left(\arctan(x) + \frac{\pi}{4}\right)$$

1.3 problem HW 1 problem 7(a)

Internal problem ID [7031]
Internal file name [`OUTPUT/6017_Sunday_June_05_2022_04_13_55_PM_42998421/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 1 problem 7(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{2y}{x} = 5x^2}$$

1.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = 5x^2$$

Hence the ode is

$$y' + \frac{2y}{x} = 5x^2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{2}{x}dx} \\
&= x^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(5x^2\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(y\,x^2\right) &= \left(x^2\right)\left(5x^2\right) \\
\mathrm{d}\left(y\,x^2\right) &= \left(5x^4\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,x^2 &= \int 5x^4\,\mathrm{d}x \\
y\,x^2 &= x^5 + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$
y = x^3 + \frac{c_1}{x^2}
$$

Summary

The solution(s) found are the following

$$
y = x^3 + \frac{c_1}{x^2} \tag{1}
$$

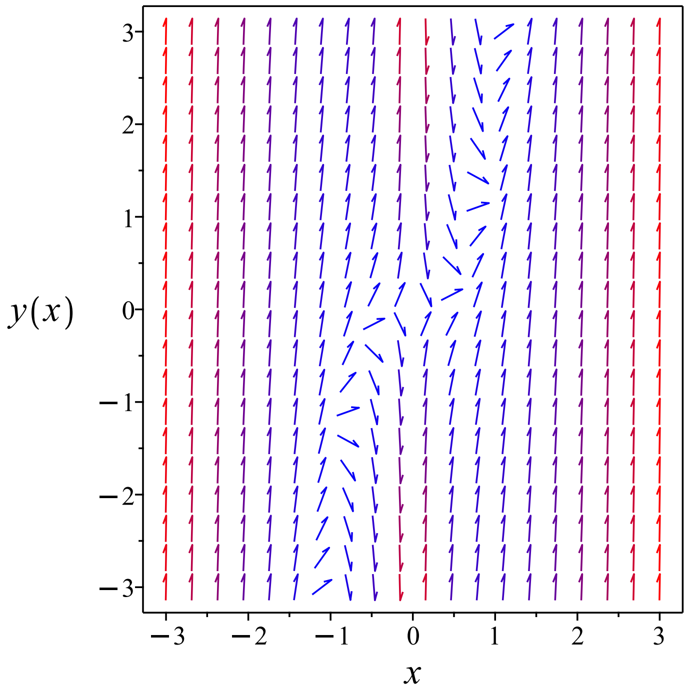


Figure 10: Slope field plot

Verification of solutions

$$y = x^3 + \frac{c_1}{x^2}$$

Verified OK.

1.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{-5x^3 + 2y}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 7: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy\end{aligned}$$

Which results in

$$S = y\, x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-5x^3 + 2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= 2xy \\ S_y &= x^2\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 5x^4 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 5R^4$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^5 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 y = x^5 + c_1$$

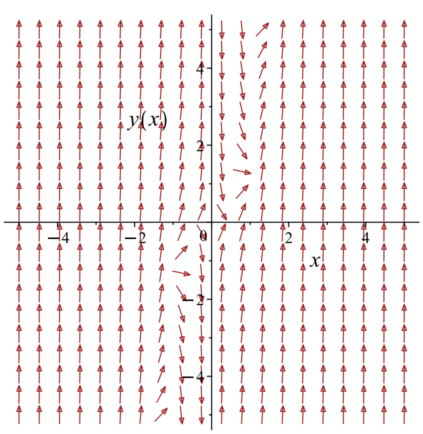
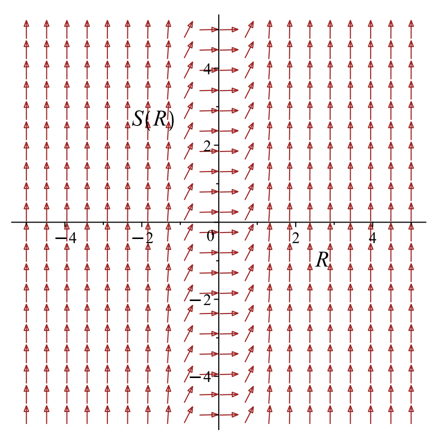
Which simplifies to

$$x^2 y = x^5 + c_1$$

Which gives

$$y = \frac{x^5 + c_1}{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-5x^3+2y}{x}$ 	$R = x$ $S = y\,x^2$	$\frac{dS}{dR} = 5R^4$ 

Summary
The solution(s) found are the following

$$y = \frac{x^5 + c_1}{x^2} \tag{1}$$

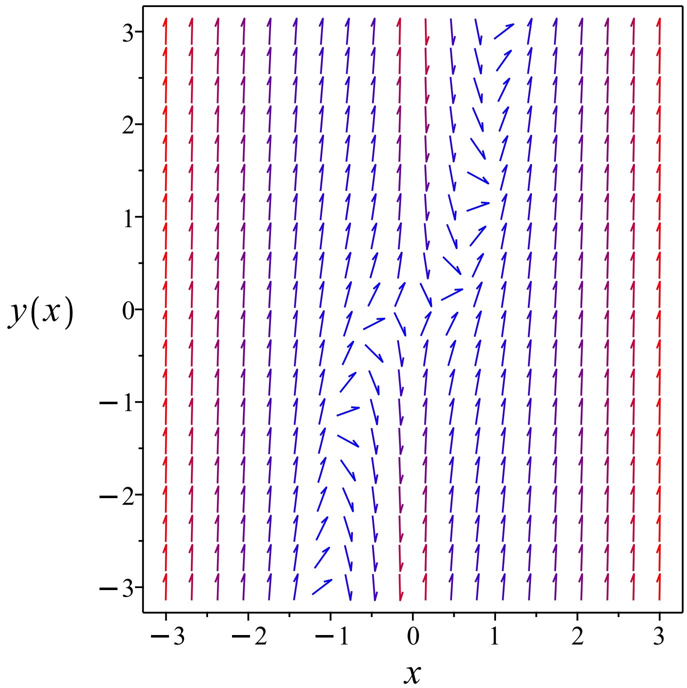


Figure 11: Slope field plot

Verification of solutions

$$y = \frac{x^5 + c_1}{x^2}$$

Verified OK.

1.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(-\frac{2y}{x} + 5x^2\right) \mathrm{d}x \\
\left(-5x^2 + \frac{2y}{x}\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2\mathrm{A})
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -5x^2 + \frac{2y}{x} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-5x^2 + \frac{2y}{x}\right) \\
&= \frac{2}{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{2}{x}\right) - (0)\right) \\
&= \frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2\ln(x)} \\
&= x^2
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x^2\left(-5x^2 + \frac{2y}{x}\right) \\
&= -5x^4 + 2xy
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x^2(1) \\
&= x^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-5x^4 + 2xy\right) + \left(x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -5x^4 + 2xy\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -x^5 + y\,x^2 + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$
x^2 = x^2 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = -x^5 + y\,x^2 + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^5 + y\,x^2$$

The solution becomes

$$y = \frac{x^5 + c_1}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^5 + c_1}{x^2} \tag{1}$$

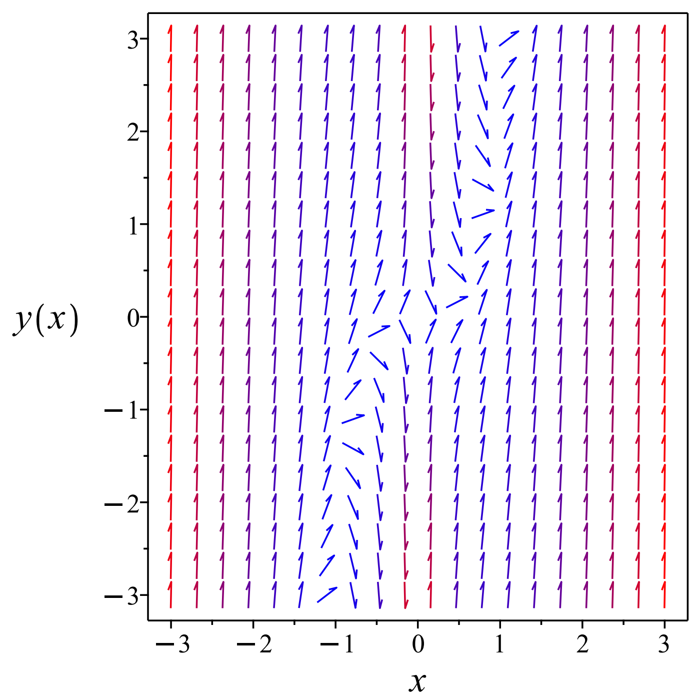


Figure 12: Slope field plot

Verification of solutions

$$y = \frac{x^5 + c_1}{x^2}$$

Verified OK.

1.3.4 Maple step by step solution

Let's solve

$y' + \frac{2y}{x} = 5x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + 5x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = 5x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = 5\mu(x)\, x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 5\mu(x)\, x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 5\mu(x)\, x^2 dx + c_1$

- Solve for y

  $y = \frac{\int 5\mu(x)x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

  $y = \frac{\int 5x^4 dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{x^5 + c_1}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)+2/x*y(x)=5*x^2,y(x), singsol=all)
```

$$y(x) = \frac{x^5 + c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 15

```
DSolve[y'[x]+2/x*y[x]==5*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^5 + c_1}{x^2}$$

1.4 problem HW 1 problem 7(b)

Internal problem ID [7032]
Internal file name [`OUTPUT/6018_Sunday_June_05_2022_04_13_58_PM_29814816/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 1 problem 7(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{tx' + 2x = 4\,\mathrm{e}^{t}}$$

1.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = \frac{2}{t}$$
$$q(t) = \frac{4\,\mathrm{e}^{t}}{t}$$

Hence the ode is

$$x' + \frac{2x}{t} = \frac{4\,\mathrm{e}^{t}}{t}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{2}{t}dt} \\
&= t^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\frac{4\,\mathrm{e}^t}{t}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(t^2 x\right) &= \left(t^2\right)\left(\frac{4\,\mathrm{e}^t}{t}\right) \\
\mathrm{d}\left(t^2 x\right) &= \left(4\,\mathrm{e}^t t\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
t^2 x &= \int 4\,\mathrm{e}^t t\,\mathrm{d}t \\
t^2 x &= 4(t-1)\,\mathrm{e}^t + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$
x = \frac{4(t-1)\,\mathrm{e}^t}{t^2} + \frac{c_1}{t^2}
$$

which simplifies to

$$
x = \frac{(4t-4)\,\mathrm{e}^t + c_1}{t^2}
$$

Summary

The solution(s) found are the following

$$
x = \frac{(4t-4)\,\mathrm{e}^t + c_1}{t^2} \tag{1}
$$

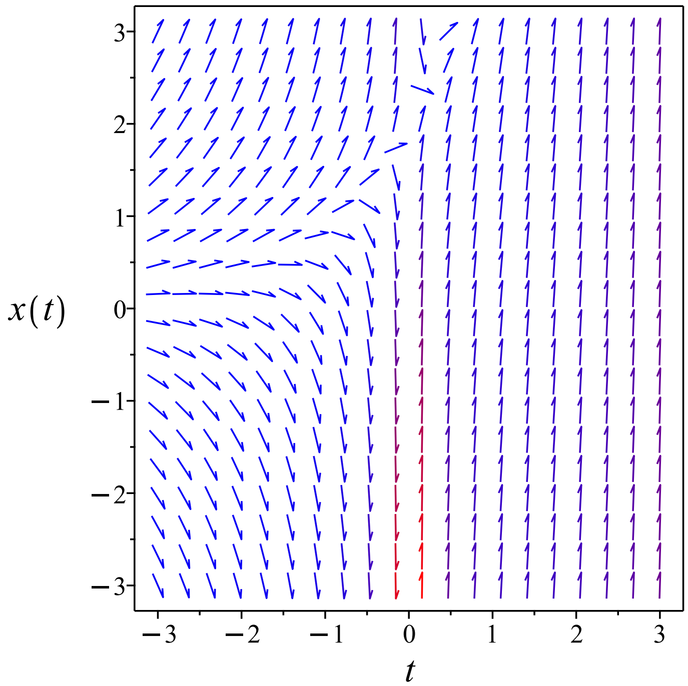


Figure 13: Slope field plot

Verification of solutions

$$x = \frac{(4t-4)\,\mathrm{e}^{t} + c_1}{t^2}$$

Verified OK.

1.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} x' &= \frac{-2x + 4\,\mathrm{e}^{t}}{t} \\ x' &= \omega(t, x) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 10: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \frac{1}{t^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{t^2}} dy \end{aligned}$$

Which results in

$$S = t^2 x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \frac{-2x + 4\,\mathrm{e}^t}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= 2tx \\ S_x &= t^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4\,\mathrm{e}^t t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4\,\mathrm{e}^R R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 4(R-1)\,\mathrm{e}^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$t^2 x = 4(t-1)\,\mathrm{e}^{t} + c_1$$

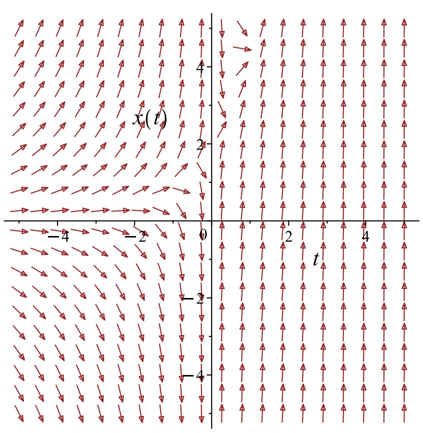
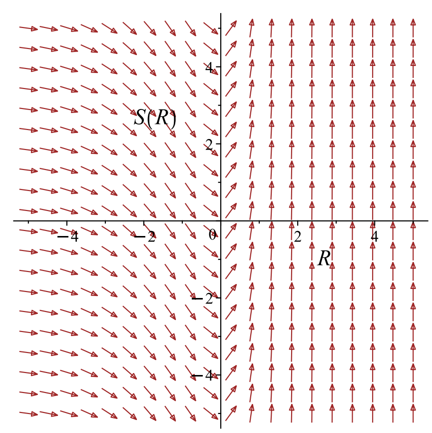
Which simplifies to

$$t^2 x = 4(t-1)\,\mathrm{e}^{t} + c_1$$

Which gives

$$x = \frac{4\,\mathrm{e}^{t} t - 4\,\mathrm{e}^{t} + c_1}{t^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \frac{-2x+4\,\mathrm{e}^{t}}{t}$ 	$R = t$ $S = t^2 x$	$\frac{dS}{dR} = 4\,\mathrm{e}^{R} R$ 

Summary

The solution(s) found are the following

$$x = \frac{4\,\mathrm{e}^{t} t - 4\,\mathrm{e}^{t} + c_1}{t^2} \tag{1}$$

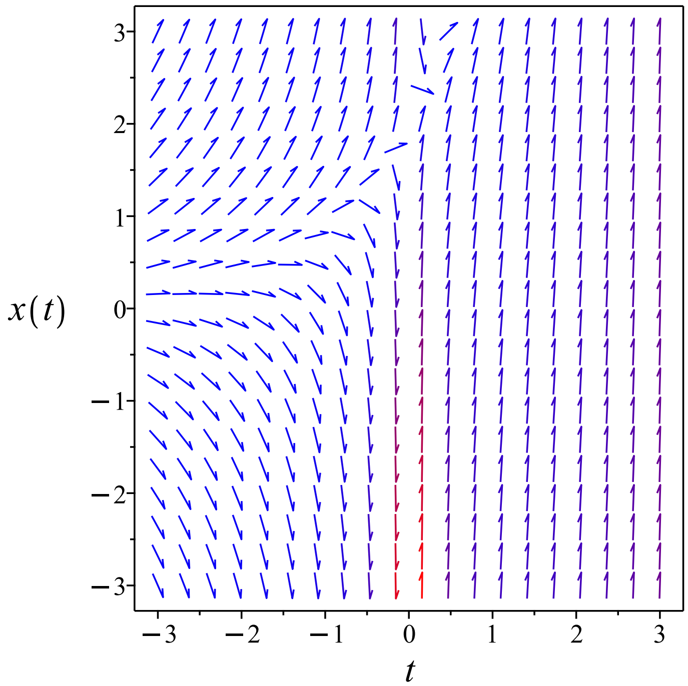


Figure 14: Slope field plot

<u>Verification of solutions</u>

$$x = \frac{4\,\mathrm{e}^{t}t - 4\,\mathrm{e}^{t} + c_1}{t^2}$$

Verified OK.

1.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$(t)\,\mathrm{d}x = \left(-2x + 4\,\mathrm{e}^{t}\right)\mathrm{d}t$$
$$\left(2x - 4\,\mathrm{e}^{t}\right)\mathrm{d}t + (t)\,\mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t,x) = 2x - 4\,\mathrm{e}^{t}$$
$$N(t,x) = t$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(2x - 4\,\mathrm{e}^{t}\right)\\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\ &= \frac{1}{t}((2) - (1)) \\ &= \frac{1}{t} \end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int \frac{1}{t} \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(t)} \\ &= t \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= t\left(2x - 4\,\mathrm{e}^{t}\right) \\ &= 2t\left(x - 2\,\mathrm{e}^{t}\right) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= t(t) \\ &= t^2 \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ \left(2t\left(x - 2\,\mathrm{e}^{t}\right)\right) + \left(t^2\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int 2t(x - 2\,\mathrm{e}^{t})\,\mathrm{d}t$$

$$\phi = (-4t+4)\,\mathrm{e}^{t} + t^2 x + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = t^2 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = t^2$. Therefore equation (4) becomes

$$t^2 = t^2 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = (-4t+4)\,\mathrm{e}^{t} + t^2 x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (-4t+4)\,\mathrm{e}^{t} + t^2 x$$

The solution becomes

$$x = \frac{4\,e^{t}t - 4\,e^{t} + c_1}{t^2}$$

Summary

The solution(s) found are the following

$$x = \frac{4\,e^{t}t - 4\,e^{t} + c_1}{t^2} \tag{1}$$

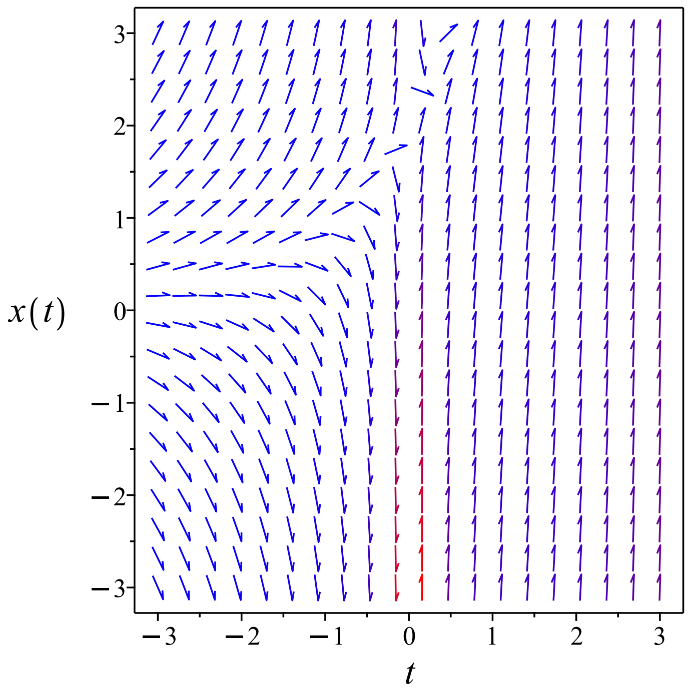


Figure 15: Slope field plot

Verification of solutions

$$x = \frac{4\,e^{t}t - 4\,e^{t} + c_1}{t^2}$$

Verified OK.

1.4.4 Maple step by step solution

Let's solve

$tx' + 2x = 4\,\mathrm{e}^{t}$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -\frac{2x}{t} + \frac{4\,\mathrm{e}^{t}}{t}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + \frac{2x}{t} = \frac{4\,\mathrm{e}^{t}}{t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x' + \frac{2x}{t}\right) = \frac{4\mu(t)\mathrm{e}^{t}}{t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

  $\mu(t)\left(x' + \frac{2x}{t}\right) = \mu'(t)\,x + \mu(t)\,x'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{2\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = t^{2}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,x)\right) dt = \int \frac{4\mu(t)\mathrm{e}^{t}}{t} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,x = \int \frac{4\mu(t)\mathrm{e}^{t}}{t} dt + c_1$

- Solve for x

  $x = \frac{\int \frac{4\mu(t)\mathrm{e}^{t}}{t} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = t^{2}$

  $x = \frac{\int 4\,\mathrm{e}^{t} t dt + c_1}{t^{2}}$

- Evaluate the integrals on the rhs

  $x = \frac{4(t-1)\mathrm{e}^{t} + c_1}{t^{2}}$

- Simplify

$$x = \frac{(4t-4)\,\mathrm{e}^{t}+c_1}{t^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(t*diff(x(t),t)+2*x(t)=4*exp(t),x(t), singsol=all)
```

$$x(t) = \frac{(4t-4)\,\mathrm{e}^{t}+c_1}{t^2}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 20

```
DSolve[t*x'[t]+2*x[t]==4*Exp[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{4e^{t}(t-1)+c_1}{t^2}$$

1.5 problem HW 1 problem 10

Internal problem ID [7033]
Internal file name [`OUTPUT/6019_Sunday_June_05_2022_04_14_00_PM_11805134/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 1 problem 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{2x - y}{x + 4y} = 0}$$

With initial conditions

$$[y(1) = 1]$$

1.5.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{-2x + y}{x + 4y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < -4 \vee -4 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\left\{y < -\frac{1}{4} \vee -\frac{1}{4} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{-2x + y}{x + 4y}\right) \\ &= -\frac{1}{x + 4y} + \frac{-8x + 4y}{(x + 4y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < -4 \vee -4 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\left\{y < -\frac{1}{4} \vee -\frac{1}{4} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.5.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - \frac{2x - u(x)\, x}{x + 4u(x)\, x} = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2(2u^2 + u - 1)}{x\, (4u + 1)}\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{2u^2+u-1}{4u+1}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+u-1}{4u+1}}\,du = -\frac{2}{x}\,dx$$
$$\int \frac{1}{\frac{2u^2+u-1}{4u+1}}\,du = \int -\frac{2}{x}\,dx$$
$$\ln\left(2u^2+u-1\right) = -2\ln\left(x\right)+c_2$$

Raising both side to exponential gives

$$2u^2+u-1 = \mathrm{e}^{-2\ln(x)+c_2}$$

Which simplifies to

$$2u^2+u-1 = \frac{c_3}{x^2}$$

Which simplifies to

$$2u(x)^2+u(x)-1 = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$

The solution is

$$2u(x)^2+u(x)-1 = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{2y^2}{x^2}+\frac{y}{x}-1 = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$
$$-\frac{(x+y)(x-2y)}{x^2} = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$

Which simplifies to

$$-(x+y)(x-2y) = c_3\mathrm{e}^{c_2}$$

Substituting initial conditions and solving for c_2 gives $c_2 = \ln\left(\frac{2}{c_3}\right)$. Hence the solution becomes

Summary

The solution(s) found are the following

$$-(x+y)(x-2y) = 2 \tag{1}$$

Verification of solutions

$$-(x+y)(x-2y) = 2$$

Verified OK.

1.5.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{2x - y}{x + 4y} \tag{1}$$

Which becomes

$$(4y)\,dy = (-x)\,dy + (2x - y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (2x - y)\,dx = d\left(x^2 - xy\right)$$

Hence (2) becomes

$$(4y)\,dy = d\left(x^2 - xy\right)$$

Integrating both sides gives gives these solutions

$$\begin{aligned} y &= -\frac{x}{4} + \frac{\sqrt{9x^2 + 8c_1}}{4} + c_1 \\ y &= -\frac{x}{4} - \frac{\sqrt{9x^2 + 8c_1}}{4} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{4} - \frac{\sqrt{9 + 8c_1}}{4} + c_1$$

$$c_1 = \frac{3}{2} + \frac{\sqrt{5}}{2}$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{x}{4} - \frac{\sqrt{9x^2 + 12 + 4\sqrt{5}}}{4} + \frac{3}{2} + \frac{\sqrt{5}}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{4} + \frac{\sqrt{9 + 8c_1}}{4} + c_1$$

$$c_1 = \frac{3}{2} - \frac{\sqrt{5}}{2}$$

Substituting c_1 found above in the general solution gives

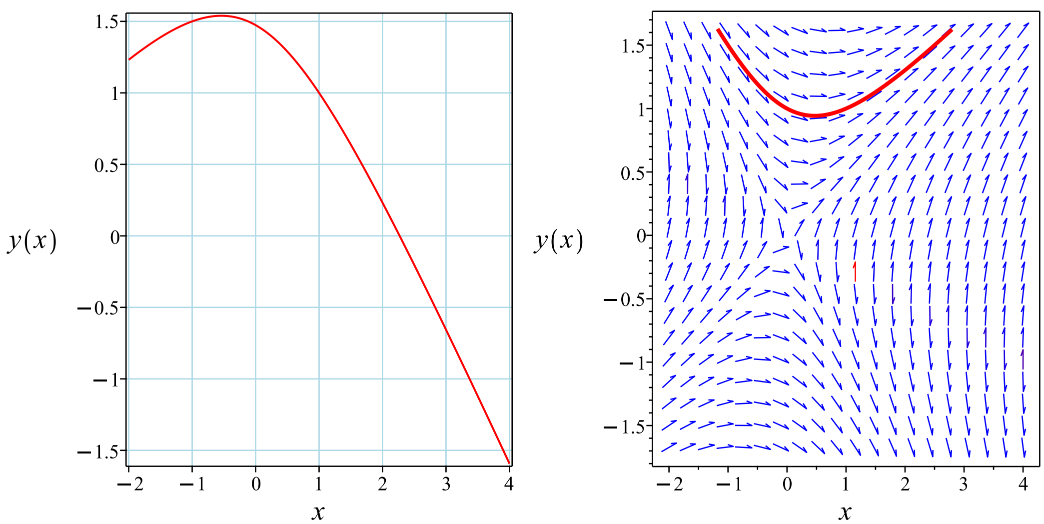
$$y = -\frac{x}{4} + \frac{\sqrt{9x^2 + 12 - 4\sqrt{5}}}{4} + \frac{3}{2} - \frac{\sqrt{5}}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{4} + \frac{\sqrt{9x^2 + 12 - 4\sqrt{5}}}{4} + \frac{3}{2} - \frac{\sqrt{5}}{2} \tag{1}$$

$$y = -\frac{x}{4} - \frac{\sqrt{9x^2 + 12 + 4\sqrt{5}}}{4} + \frac{3}{2} + \frac{\sqrt{5}}{2} \tag{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{x}{4} + \frac{\sqrt{9x^2 + 12 - 4\sqrt{5}}}{4} + \frac{3}{2} - \frac{\sqrt{5}}{2}$$

Verified OK.

$$y = -\frac{x}{4} - \frac{\sqrt{9x^2 + 12 + 4\sqrt{5}}}{4} + \frac{3}{2} + \frac{\sqrt{5}}{2}$$

Verified OK.

1.5.4 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{-2x+y}{x+4y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 &- \frac{(-2x+y)(b_3-a_2)}{x+4y} - \frac{(-2x+y)^2 a_3}{(x+4y)^2} \\
&- \left(\frac{2}{x+4y} + \frac{-2x+y}{(x+4y)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(-\frac{1}{x+4y} + \frac{-8x+4y}{(x+4y)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{2x^2a_2 + 4x^2a_3 - 10x^2b_2 - 2x^2b_3 + 16xya_2 - 4xya_3 - 8xyb_2 - 16xyb_3 - 4y^2a_2 + 10y^2a_3 - 16y^2b_2 + 4y^2}{(x+4y)^2}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-2x^2a_2 - 4x^2a_3 + 10x^2b_2 + 2x^2b_3 - 16xya_2 + 4xya_3 + 8xyb_2 \\
&+ 16xyb_3 + 4y^2a_2 - 10y^2a_3 + 16y^2b_2 - 4y^2b_3 + 9xb_1 - 9ya_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -2a_2v_1^2 - 16a_2v_1v_2 + 4a_2v_2^2 - 4a_3v_1^2 + 4a_3v_1v_2 - 10a_3v_2^2 + 10b_2v_1^2 \\ + 8b_2v_1v_2 + 16b_2v_2^2 + 2b_3v_1^2 + 16b_3v_1v_2 - 4b_3v_2^2 - 9a_1v_2 + 9b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} (-2a_2 - 4a_3 + 10b_2 + 2b_3)\, v_1^2 + (-16a_2 + 4a_3 + 8b_2 + 16b_3)\, v_1v_2 \\ + 9b_1v_1 + (4a_2 - 10a_3 + 16b_2 - 4b_3)\, v_2^2 - 9a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -9a_1 &= 0 \\ 9b_1 &= 0 \\ -16a_2 + 4a_3 + 8b_2 + 16b_3 &= 0 \\ -2a_2 - 4a_3 + 10b_2 + 2b_3 &= 0 \\ 4a_2 - 10a_3 + 16b_2 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_2 + b_3 \\ a_3 &= 2b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{-2x+y}{x+4y}\right)(x) \\
&= \frac{-2x^2+2xy+4y^2}{x+4y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2x^2+2xy+4y^2}{x+4y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(-x^2+xy+2y^2\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-2x + y}{x + 4y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x - y}{2(x + y)(x - 2y)} \\ S_y &= \frac{-x - 4y}{2(x + y)(x - 2y)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

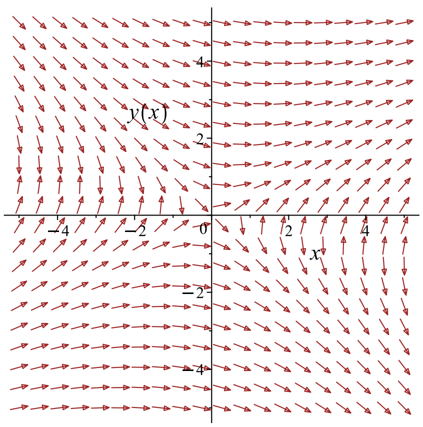
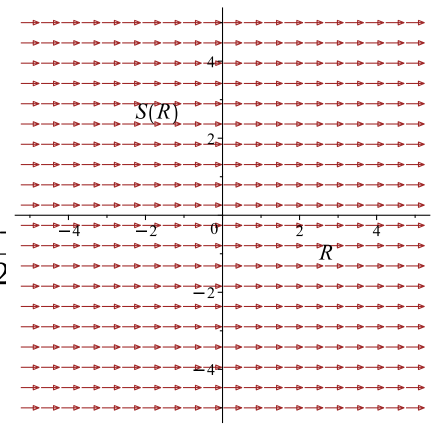
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(-x - y)}{2} + \frac{\ln(x - 2y)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(-x - y)}{2} + \frac{\ln(x - 2y)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-2x+y}{x+4y}$ 	$R = x$ $S = \frac{\ln(-x-y)}{2} + \frac{\ln (x -}{2}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\ln(2)}{2} + i\pi = c_1$$

$$c_1 = \frac{\ln(2)}{2} + i\pi$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(-x-y)}{2} + \frac{\ln(x-2y)}{2} = \frac{\ln(2)}{2} + i\pi$$

Summary

The solution(s) found are the following

$$\frac{\ln(-x-y)}{2} + \frac{\ln(x-2y)}{2} = \frac{\ln(2)}{2} + i\pi \tag{1}$$

Verification of solutions

$$\frac{\ln(-x-y)}{2} + \frac{\ln(x-2y)}{2} = \frac{\ln(2)}{2} + i\pi$$

Verified OK.

1.5.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x+4y)\,\mathrm{d}y&=(2x-y)\,\mathrm{d}x\\ (-2x+y)\,\mathrm{d}x+(x+4y)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-2x+y\\ N(x,y)&=x+4y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-2x+y)\\&=1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(x+4y)\\&=1\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$
$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int M\,\mathrm{d}x\\\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int -2x+y\,\mathrm{d}x\end{aligned}$$
$$\phi=-x(x-y)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=x+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=x+4y$. Therefore equation (4) becomes

$$x+4y=x+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (4y)\,\mathrm{d}y$$

$$f(y) = 2y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x(x - y) + 2y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x(x - y) + 2y^2$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$-x(x - y) + 2y^2 = 2$$

Summary
The solution(s) found are the following

$$-(x + y)(x - 2y) = 2 \tag{1}$$

Verification of solutions

$$-(x + y)(x - 2y) = 2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.157 (sec). Leaf size: 19

```
dsolve([diff(y(x),x)=(2*x-y(x))/(x+4*y(x)),y(1) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{x}{4} + \frac{\sqrt{9x^2 + 16}}{4}$$

✓ Solution by Mathematica

Time used: 0.482 (sec). Leaf size: 24

```
DSolve[{y'[x]==(2*x-y[x])/(x+4*y[x]),{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(\sqrt{9x^2 + 16} - x\right)$$

1.6 problem HW 1 problem 11

Internal problem ID [7034]
Internal file name [`OUTPUT/6020_Sunday_June_05_2022_04_14_06_PM_22645042/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 1 problem 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y' + \frac{2y}{x} - 6x^4y^2 = 0}$$

1.6.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{2y(3y\,x^5 - 1)}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 13: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= x^2y^2 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2 y^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{x^2 y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2y(3y\, x^5 - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{x^3 y} \\ S_y &= \frac{1}{x^2 y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 6x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 6R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 2R^3 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x^2 y} = 2x^3 + c_1$$

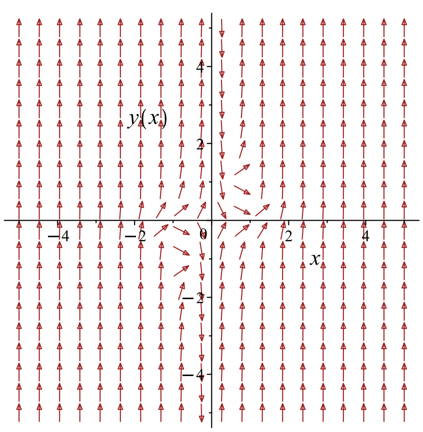
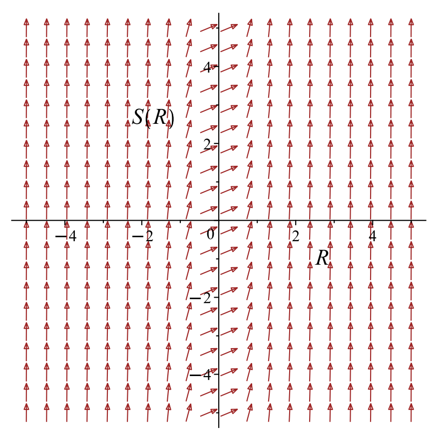
Which simplifies to

$$-\frac{1}{x^2 y} = 2x^3 + c_1$$

Which gives

$$y = -\frac{1}{x^2 (2x^3 + c_1)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y(3y\,x^5-1)}{x}$  	$R = x$ $S = -\frac{1}{x^2 y}$	$\frac{dS}{dR} = 6R^2$ 

Summary
The solution(s) found are the following

$$y = -\frac{1}{x^2 (2x^3 + c_1)} \tag{1}$$

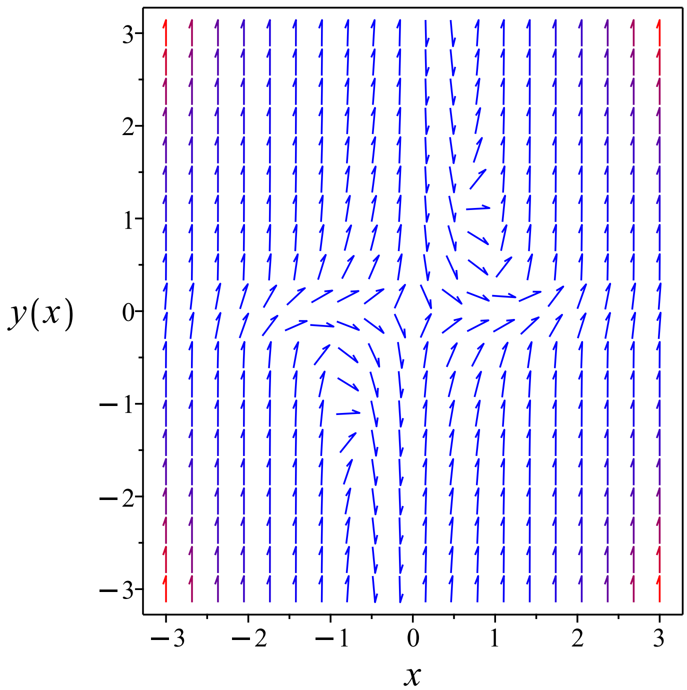


Figure 17: Slope field plot

Verification of solutions

$$y = -\frac{1}{x^2 (2x^3 + c_1)}$$

Verified OK.

1.6.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{2y(3y\,x^5 - 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2}{x}y + 6x^4y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{2}{x} \\ f_1(x) &= 6x^4 \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{2}{xy} + 6x^4 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{2w(x)}{x} + 6x^4 \\ w' &= \frac{2w}{x} - 6x^4 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= -6x^4 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = -6x^4$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-6x^4\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(-6x^4\right) \\ \mathrm{d}\left(\frac{w}{x^2}\right) &= \left(-6x^2\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{w}{x^2} &= \int -6x^2\,\mathrm{d}x \\ \frac{w}{x^2} &= -2x^3 + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = -2x^5 + c_1 x^2$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = -2x^5 + c_1 x^2$$

Or

$$y = \frac{1}{-2x^5 + c_1 x^2}$$

Which is simplified to

$$y = \frac{1}{x^2 \left(-2x^3 + c_1\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x^2 \left(-2x^3 + c_1\right)} \tag{1}$$

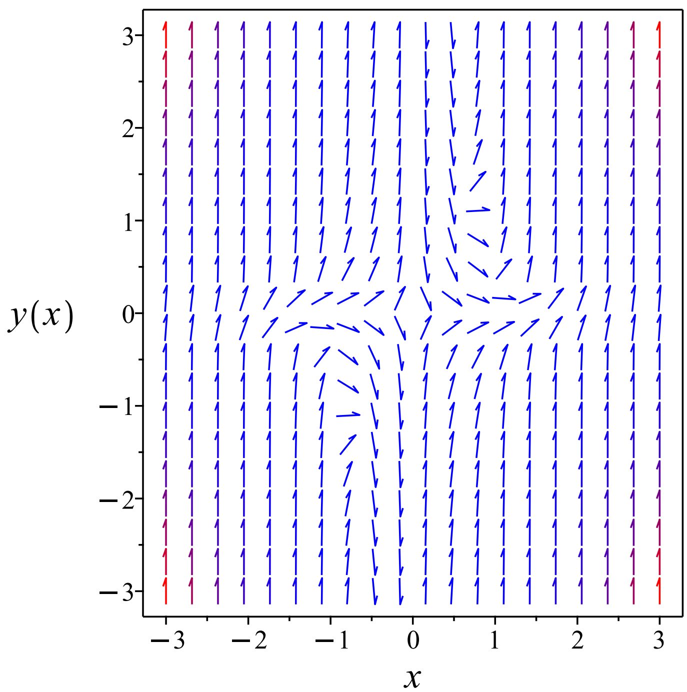


Figure 18: Slope field plot

Verification of solutions

$$y = \frac{1}{x^2 \left(-2x^3 + c_1\right)}$$

Verified OK.

1.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\frac{2y}{x} + 6x^4y^2\right)\mathrm{d}x$$

$$\left(\frac{2y}{x} - 6x^4y^2\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= \frac{2y}{x} - 6x^4y^2 \\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2y}{x} - 6x^4y^2\right) \\
&= \frac{2}{x} - 12y\,x^4
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{2}{x} - 12y\,x^4\right) - (0)\right) \\
&= \frac{2}{x} - 12y\,x^4
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{x}{6x^5y^2 - 2y}\left((0) - \left(\frac{2}{x} - 12y\,x^4\right)\right) \\
&= \frac{-6y\,x^5 + 1}{3x^5y^2 - y}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(0) - \left(\frac{2}{x} - 12y\,x^4\right)}{x\left(\frac{2y}{x} - 6x^4y^2\right) - y\,(1)} \\
&= -\frac{2}{xy}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R\,\mathrm{d}t} \\
&= e^{\int \left(-\frac{2}{t}\right)\mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^2y^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2y^2}\left(\frac{2y}{x} - 6x^4y^2\right) \\
&= \frac{-6y\,x^5 + 2}{y\,x^3}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2y^2}(1) \\
&= \frac{1}{x^2y^2}
\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-6y\,x^5+2}{y\,x^3}\right) + \left(\frac{1}{x^2y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-6y\,x^5+2}{y\,x^3}\,\mathrm{d}x
\end{aligned}$$

$$\phi = \frac{-2y\,x^5-1}{y\,x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{2x^3}{y} - \frac{-2y\,x^5-1}{y^2x^2} + f'(y) \tag{4}$$

$$= \frac{1}{x^2y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2y^2}$. Therefore equation (4) becomes

$$\frac{1}{x^2y^2} = \frac{1}{x^2y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-2y\,x^5 - 1}{y\,x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-2y\,x^5 - 1}{y\,x^2}$$

The solution becomes

$$y = -\frac{1}{x^2\,(2x^3 + c_1)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x^2\,(2x^3 + c_1)} \tag{1}$$

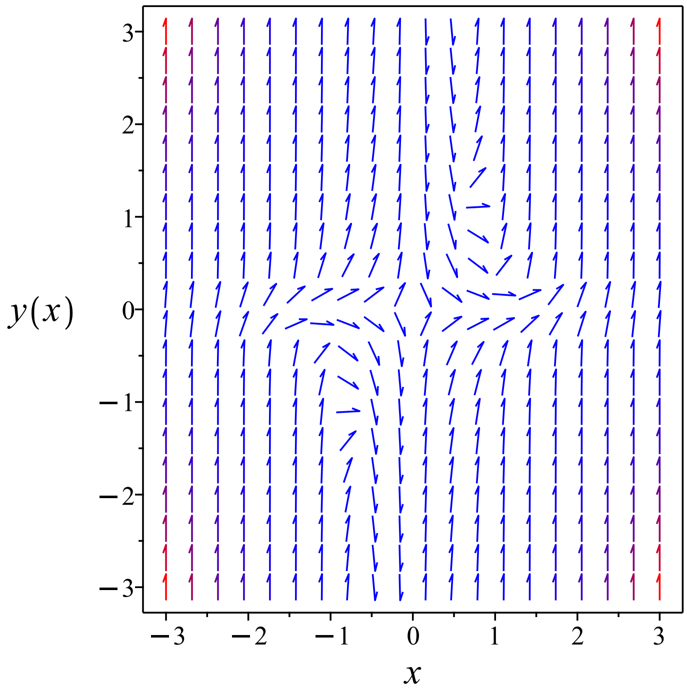


Figure 19: Slope field plot

Verification of solutions

$$y = -\frac{1}{x^2\left(2x^3+c_1\right)}$$

Verified OK.

1.6.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{2y(3y\,x^5-1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{2y}{x} + 6x^4y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{2}{x}$ and $f_2(x) = 6x^4$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{6x^4 u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 24x^3 \\ f_1 f_2 &= -12x^3 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$6x^4 u''(x) - 12x^3 u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 x^3 + c_1$$

The above shows that

$$u'(x) = 3c_2 x^2$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{2x^2 (c_2 x^3 + c_1)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{2x^2 (x^3 + c_3)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{2x^2 (x^3 + c_3)} \tag{1}$$

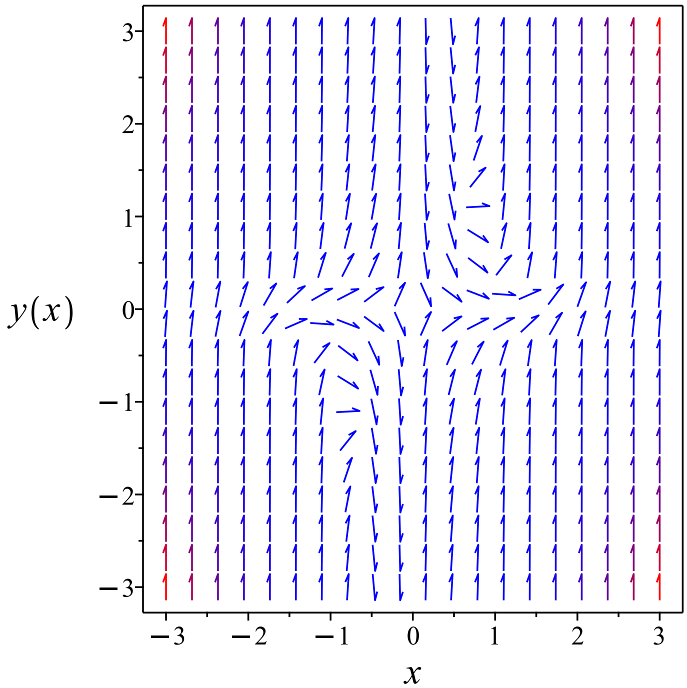


Figure 20: Slope field plot

Verification of solutions

$$y = -\frac{1}{2x^2 (x^3 + c_3)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)+2*y(x)/x=6*y(x)^2*x^4,y(x), singsol=all)
```

$$y(x) = \frac{1}{(-2x^3 + c_1)\, x^2}$$

✓ Solution by Mathematica

Time used: 0.153 (sec). Leaf size: 24

```
DSolve[y'[x]+2*y[x]/x==6*y[x]^2*x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{-2x^5 + c_1 x^2}$$
$$y(x) \to 0$$

1.7 problem HW 1 problem 13

Internal problem ID [7035]
Internal file name [`OUTPUT/6021_Sunday_June_05_2022_04_14_09_PM_89321347/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 1 problem 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{y^2 + (2yx + \sin(y))\, y' = -\cos(x)}$$

1.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy+\sin(y))\,\mathrm{d}y&=\left(-y^2-\cos(x)\right)\mathrm{d}x\\ \left(y^2+\cos(x)\right)\mathrm{d}x+(2xy+\sin(y))\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y^2+\cos(x)\\ N(x,y)&=2xy+\sin(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(y^2+\cos(x)\right)\\ &=2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(2xy+\sin(y))\\ &=2y\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathop{\mathrm{d}x} &= \int M \mathop{\mathrm{d}x} \\
\int \frac{\partial \phi}{\partial x} \mathop{\mathrm{d}x} &= \int y^2 + \cos(x) \mathop{\mathrm{d}x} \\
\phi &= x y^2 + \sin(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 2xy + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2xy + \sin(y)$. Therefore equation (4) becomes

$$
2xy + \sin(y) = 2xy + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \sin(y)
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathop{\mathrm{d}y} &= \int (\sin(y)) \mathop{\mathrm{d}y} \\
f(y) &= -\cos(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = x y^2 + \sin(x) - \cos(y) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = x y^2 + \sin(x) - \cos(y)
$$

Summary

The solution(s) found are the following

$$xy^2 + \sin(x) - \cos(y) = c_1 \tag{1}$$

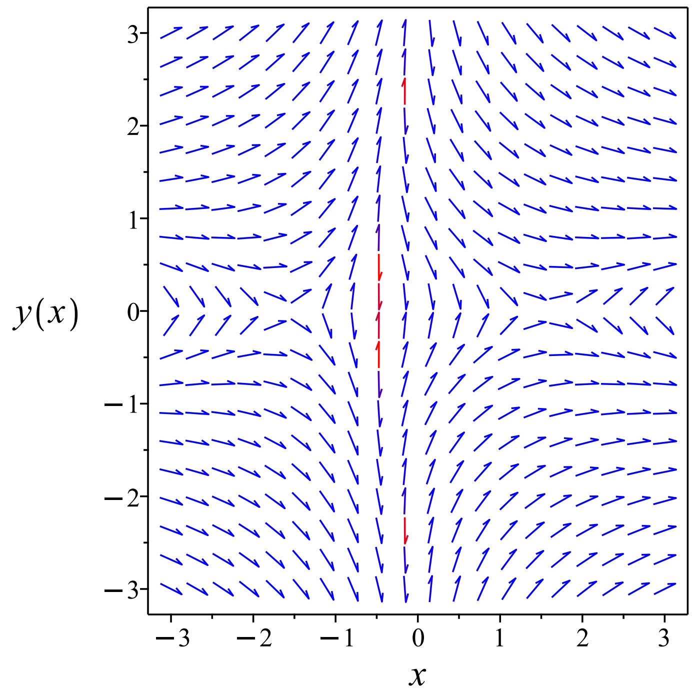


Figure 21: Slope field plot

Verification of solutions

$$xy^2 + \sin(x) - \cos(y) = c_1$$

Verified OK.

1.7.2 Maple step by step solution

Let's solve

$y^2 + (2yx + \sin(y))\, y' = -\cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $2y = 2y$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (y^2 + \cos(x))\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = x\, y^2 + \sin(x) + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $2xy + \sin(y) = 2xy + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = \sin(y)$

- Solve for $f_1(y)$

  $f_1(y) = -\cos(y)$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = x\, y^2 + \sin(x) - \cos(y)$

- Substitute $F(x, y)$ into the solution of the ODE

  $x\, y^2 + \sin(x) - \cos(y) = c_1$

- Solve for y

  $y = \mathit{RootOf}\left(-x\_Z^2 + c_1 + \cos(\_Z) - \sin(x)\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve((y(x)^2+cos(x))+(2*x*y(x)+sin(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$xy(x)^2 + \sin(x) - \cos(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.229 (sec). Leaf size: 20

```
DSolve[(y[x]^2+Cos[x])+(2*x*y[x]+Sin[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$\text{Solve}\left[xy(x)^2 - \cos(y(x)) + \sin(x) = c_1, y(x)\right]$$

1.8 problem HW 1 problem 14

Internal problem ID [7036]
Internal file name [`OUTPUT/6022_Sunday_June_05_2022_04_14_13_PM_14384862/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 1 problem 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{yx + x^2 y' = 1}$$

1.8.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x^2)\,\mathrm{d}y &= (-xy+1)\,\mathrm{d}x \\
(xy-1)\,\mathrm{d}x + (x^2)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= xy-1 \\
N(x,y) &= x^2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(xy-1) \\
&= x
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2) \\
&= 2x
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^2}((x) - (2x)) \\
&= -\frac{1}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(x)} \\ &= \frac{1}{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x}(xy - 1) \\ &= \frac{xy - 1}{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x}\left(x^2\right) \\ &= x\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{xy - 1}{x}\right) + (x) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{xy-1}{x} \mathrm{d}x$$

$$\phi = xy - \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = xy - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy - \ln(x)$$

The solution becomes

$$y = \frac{\ln(x) + c_1}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{\ln(x) + c_1}{x} \tag{1}$$

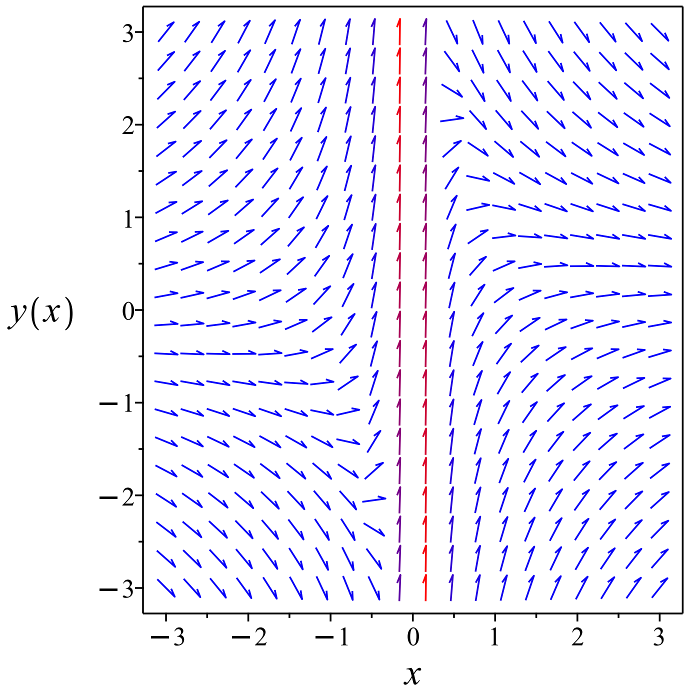


Figure 22: Slope field plot

Verification of solutions

$$y = \frac{\ln(x) + c_1}{x}$$

Verified OK.

1.8.2 Maple step by step solution

Let's solve

$yx + x^2 y' = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -\frac{y}{x} + \frac{1}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = \frac{1}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \frac{\mu(x)}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)}{x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)}{x^2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int \frac{1}{x} dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{\ln(x) + c_1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x*y(x)-1)+x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\ln(x) + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 14

```
DSolve[(x*y[x]-1)+x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\log(x) + c_1}{x}$$

1.9 problem HW 5 problem 1(a)

Internal problem ID [7037]
Internal file name [`OUTPUT/6023_Sunday_June_05_2022_04_14_15_PM_18084672/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 5 problem 1(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y' - 2y = 5\,\mathrm{e}^{2x}}$$

1.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -2, f(x) = 5\,\mathrm{e}^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda\, \mathrm{e}^{\lambda x} - 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(-2)} \\
&= \frac{1}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{3}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{3}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(2)x} + c_2 e^{(-1)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5\,e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{2x} = 5\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5x\,e^{2x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{-x}\right) + \left(\frac{5x\,e^{2x}}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{-x} + \frac{5x\,e^{2x}}{3} \tag{1}$$

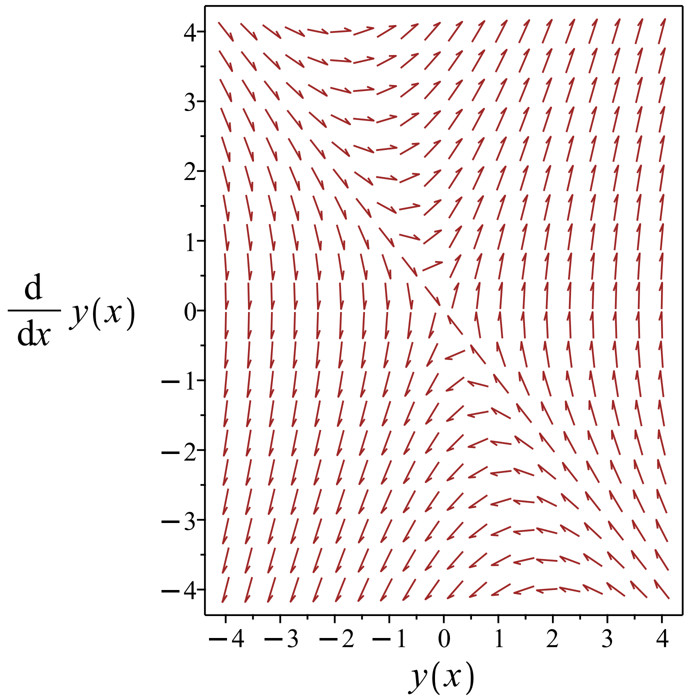


Figure 23: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-x} + \frac{5x\,e^{2x}}{3}$$

Verified OK.

1.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -1 \\
C &= -2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{9z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 17: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \frac{\mathrm{e}^{2x}}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{3} \\ \frac{d}{dx}(\mathrm{e}^{-x}) & \frac{d}{dx}\left(\frac{\mathrm{e}^{2x}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{3} \\ -\mathrm{e}^{-x} & \frac{2\,\mathrm{e}^{2x}}{3} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{-x})\left(\frac{2\,\mathrm{e}^{2x}}{3}\right) - \left(\frac{\mathrm{e}^{2x}}{3}\right)(-\mathrm{e}^{-x})$$

Which simplifies to

$$W = \mathrm{e}^{-x}\mathrm{e}^{2x}$$

Which simplifies to

$$W = e^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{5\,e^{4x}}{3}}{e^{x}} dx$$

Which simplifies to

$$u_1 = -\int \frac{5\,e^{3x}}{3} dx$$

Hence

$$u_1 = -\frac{5\,e^{3x}}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{5\,e^{-x} e^{2x}}{e^{x}} dx$$

Which simplifies to

$$u_2 = \int 5 dx$$

Hence

$$u_2 = 5x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{5\,e^{-x} e^{3x}}{9} + \frac{5x\,e^{2x}}{3}$$

Which simplifies to

$$y_p(x) = \frac{5\,e^{2x}(3x-1)}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{2x}}{3}\right) + \left(\frac{5\,e^{2x}(3x-1)}{9}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{3} + \frac{5\, e^{2x}(3x - 1)}{9} \tag{1}$$

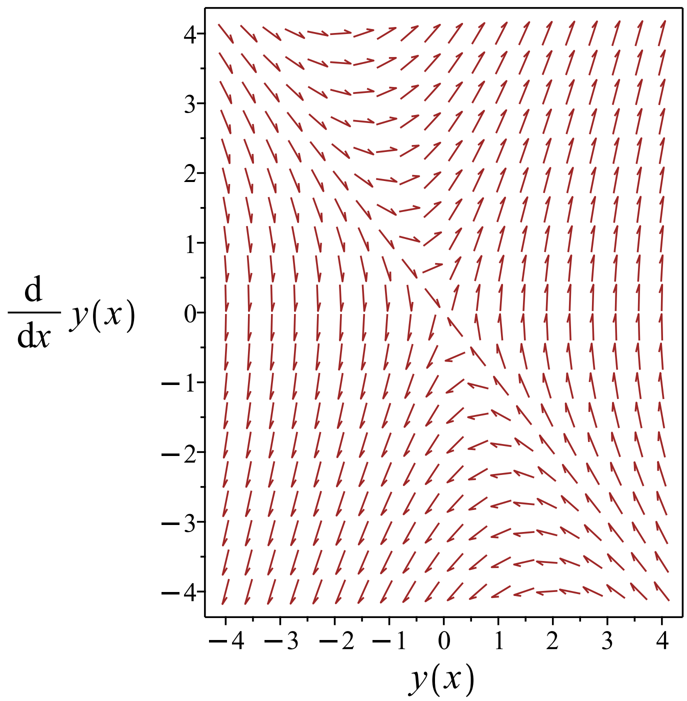


Figure 24: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{3} + \frac{5\, e^{2x}(3x - 1)}{9}$$

Verified OK.

1.9.3 Maple step by step solution

Let's solve

$y'' - y' - 2y = 5\, e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r + 1)(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 5\,e^{2x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2\,e^{2x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,e^{x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{5\,e^{-x}\left(\int e^{3x} dx\right)}{3} + \frac{5\,e^{2x}\left(\int 1 dx\right)}{3}$

  ◦ Compute integrals

  $y_p(x) = \frac{5\,e^{2x}(3x-1)}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 e^{2x} + \frac{5\,e^{2x}(3x-1)}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-diff(y(x),x)-2*y(x)=5*exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(5x + 3c_2)\,\mathrm{e}^{2x}}{3} + c_1\mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 31

```
DSolve[y''[x]-y'[x]-2*y[x]==5*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x} + e^{2x}\left(\frac{5x}{3} - \frac{5}{9} + c_2\right)$$

1.10 problem HW 5 problem 1(b)

Internal problem ID [7038]
Internal file name [`OUTPUT/6024_Sunday_June_05_2022_04_14_18_PM_11508899/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 5 problem 1(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 16y = 4\cos(x)}$$

1.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 16, f(x) = 4\cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 16y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 16$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 16\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 16$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(16)}\\ &= \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +4i\\ \lambda_2 &= -4i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4i\\ \lambda_2 &= -4i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(4x) + c_2 \sin(4x))$$

Or

$$y = c_1 \cos(4x) + c_2 \sin(4x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(4x) + c_2 \sin(4x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(4x), \sin(4x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$15A_1 \cos(x) + 15A_2 \sin(x) = 4\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{15}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{4\cos(x)}{15}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(4x) + c_2 \sin(4x)) + \left(\frac{4\cos(x)}{15}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(4x) + c_2 \sin(4x) + \frac{4\cos(x)}{15} \tag{1}$$

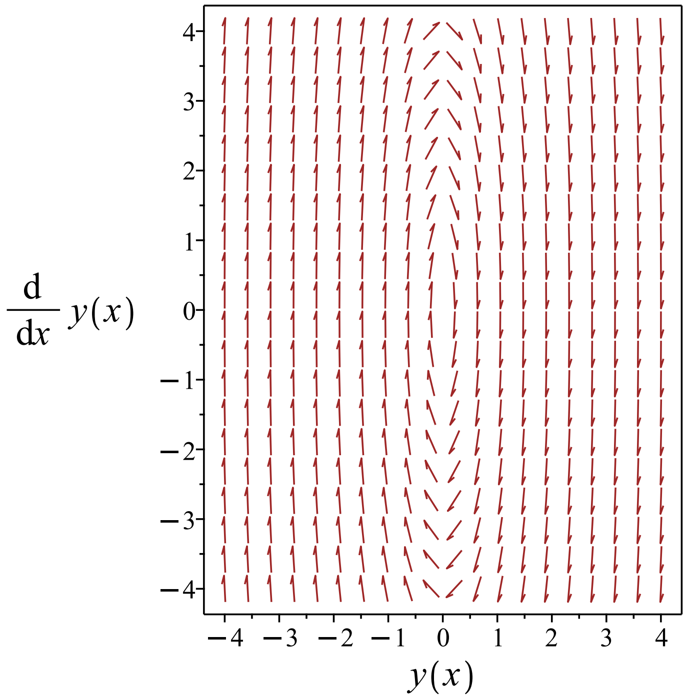


Figure 25: Slope field plot

Verification of solutions

$$y = c_1 \cos(4x) + c_2 \sin(4x) + \frac{4\cos(x)}{15}$$

Verified OK.

1.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 16y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 19: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(4x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(4x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(4x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(4x) \int \frac{1}{\cos(4x)^2}\,dx \\ &= \cos(4x)\left(\frac{\tan(4x)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(4x)) + c_2\left(\cos(4x)\left(\frac{\tan(4x)}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 16y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\cos(4x) + \frac{c_2\sin(4x)}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(4x)}{4}, \cos(4x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(x) + A_2\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$15A_1\cos(x) + 15A_2\sin(x) = 4\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{15}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{4\cos(x)}{15}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(4x) + \frac{c_2\sin(4x)}{4}\right) + \left(\frac{4\cos(x)}{15}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(4x) + \frac{c_2\sin(4x)}{4} + \frac{4\cos(x)}{15} \tag{1}$$

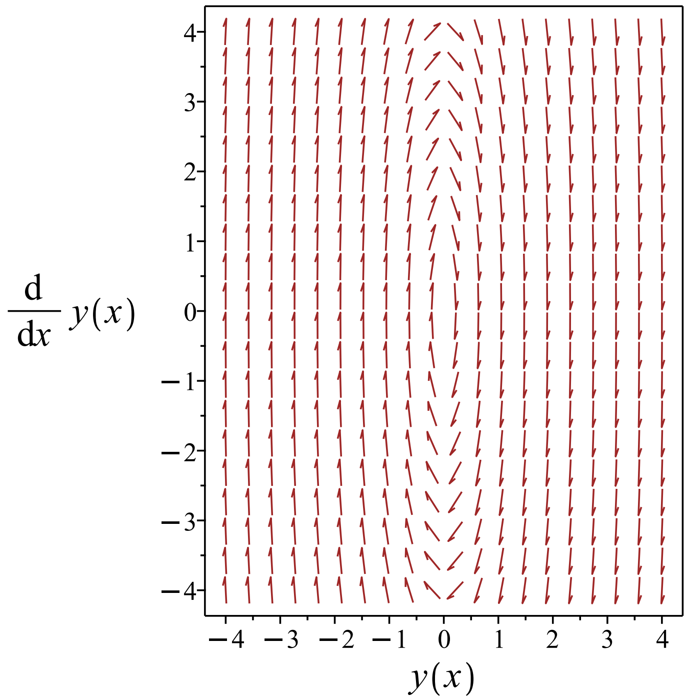


Figure 26: Slope field plot

Verification of solutions

$$y = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4} + \frac{4\cos(x)}{15}$$

Verified OK.

1.10.3 Maple step by step solution

Let's solve

$y'' + 16y = 4\cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 16 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

$r = (-4\,\mathrm{I}, 4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(4x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(4x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(4x) + c_2 \sin(4x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4\cos(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(4x) & \sin(4x) \\ -4\sin(4x) & 4\cos(4x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 4$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(4x)\left(\int \sin(4x)\cos(x)\, dx\right) + \sin(4x)\left(\int \cos(4x)\cos(x)\, dx\right)$

  - Compute integrals

    $y_p(x) = \frac{4\cos(x)}{15}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(4x) + c_2 \sin(4x) + \frac{4\cos(x)}{15}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+16*y(x)=4*cos(x),y(x), singsol=all)
```

$$y(x) = \sin(4x) c_2 + \cos(4x) c_1 + \frac{4\cos(x)}{15}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 26

```
DSolve[y''[x]+16*y[x]==4*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4\cos(x)}{15} + c_1\cos(4x) + c_2\sin(4x)$$

1.11 problem HW 5 problem 1(c)

Internal problem ID [7039]
Internal file name [`OUTPUT/6025_Sunday_June_05_2022_04_14_21_PM_78430375/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 5 problem 1(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4y' + 3y = 9x^2 + 4}$$

With initial conditions

$$[y(0) = 6, y'(0) = 8]$$

1.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= 3 \\ F &= 9x^2 + 4 \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 3y = 9x^2 + 4$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 9x^2 + 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.11.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 3, f(x) = 9x^2 + 4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\,\mathrm{e}^{\lambda x} + 3\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-4^2 - (4)(1)(3)} \\ &= 2 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 2 + 1 \\ \lambda_2 &= 2 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3 \\ \lambda_2 &= 1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(3)x} + c_2 e^{(1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_3x^2 + 3A_2x - 8xA_3 + 3A_1 - 4A_2 + 2A_3 = 9x^2 + 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 10, A_2 = 8, A_3 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x^2 + 8x + 10$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{3x} + c_2e^x\right) + \left(3x^2 + 8x + 10\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1e^{3x} + c_2e^x + 3x^2 + 8x + 10 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 6$ and $x = 0$ in the above gives

$$6 = c_1 + c_2 + 10 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1e^{3x} + c_2e^x + 6x + 8$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = 3c_1 + c_2 + 8 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -6$$

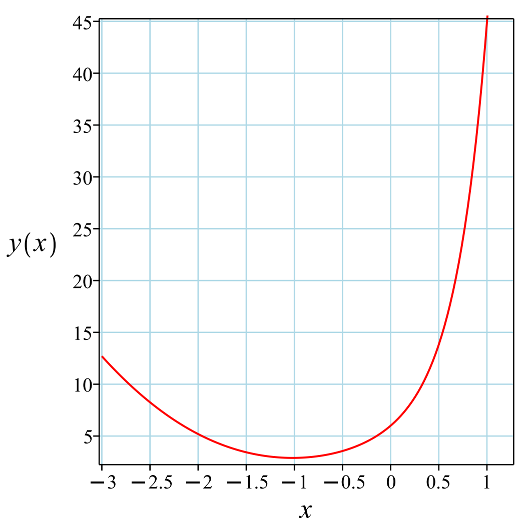
Substituting these values back in above solution results in

$$y = 10 + 3x^2 + 2\,e^{3x} - 6\,e^x + 8x$$

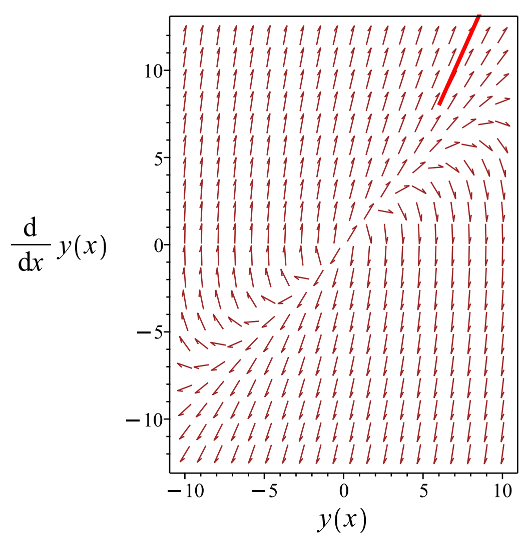
Summary

The solution(s) found are the following

$$y = 10 + 3x^2 + 2\,e^{3x} - 6\,e^x + 8x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 10 + 3x^2 + 2\,e^{3x} - 6\,e^x + 8x$$

Verified OK.

1.11.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 21: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx}
\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (\mathrm{e}^{x}) + c_2 \left(\mathrm{e}^{x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{x} + \frac{c_2 \mathrm{e}^{3x}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{3x}}{2}, e^x\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_3x^2 + 3A_2x - 8xA_3 + 3A_1 - 4A_2 + 2A_3 = 9x^2 + 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 10, A_2 = 8, A_3 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x^2 + 8x + 10$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + \frac{c_2 e^{3x}}{2}\right) + (3x^2 + 8x + 10) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{x} + \frac{c_2 e^{3x}}{2} + 3x^2 + 8x + 10 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 6$ and $x = 0$ in the above gives

$$6 = c_1 + \frac{c_2}{2} + 10 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^{x} + \frac{3c_2 e^{3x}}{2} + 6x + 8$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = c_1 + \frac{3c_2}{2} + 8 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -6 \\ c_2 &= 4 \end{aligned}$$

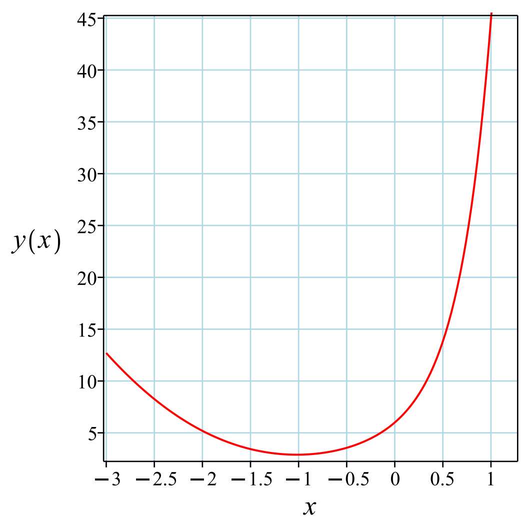
Substituting these values back in above solution results in

$$y = 10 + 3x^2 + 2\,e^{3x} - 6\,e^{x} + 8x$$

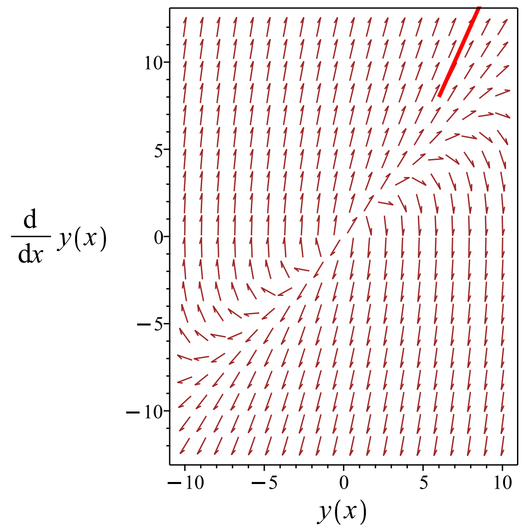
Summary

The solution(s) found are the following

$$y = 10 + 3x^2 + 2\,e^{3x} - 6\,e^{x} + 8x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 10 + 3x^2 + 2\,e^{3x} - 6\,e^{x} + 8x$$

Verified OK.

1.11.4 Maple step by step solution

Let's solve

$$\left[y'' - 4y' + 3y = 9x^2 + 4, y(0) = 6, y'\Big|_{\{x=0\}} = 8\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 - 4r + 3 = 0$
- Factor the characteristic polynomial
  $(r-1)(r-3) = 0$
- Roots of the characteristic polynomial
  $r = (1, 3)$
- 1st solution of the homogeneous ODE

$y_1(x) = e^x$

- 2nd solution of the homogeneous ODE

$y_2(x) = e^{3x}$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y = c_1 e^x + c_2 e^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 9x^2 + 4\right]$

- Wronskian of solutions of the homogeneous equation

$W(y_1(x), y_2(x)) = \left[\begin{array}{cc} e^x & e^{3x} \\ e^x & 3 e^{3x} \end{array}\right]$

- Compute Wronskian

$W(y_1(x), y_2(x)) = 2 e^{4x}$

- Substitute functions into equation for $y_p(x)$

$y_p(x) = -\frac{e^x \left(\int (9x^2+4) e^{-x} dx\right)}{2} + \frac{e^{3x} \left(\int e^{-3x} (9x^2+4) dx\right)}{2}$

- Compute integrals

$y_p(x) = 3x^2 + 8x + 10$

- Substitute particular solution into general solution to ODE

$y = c_1 e^x + c_2 e^{3x} + 3x^2 + 8x + 10$

☐ Check validity of solution $y = c_1 e^x + c_2 e^{3x} + 3x^2 + 8x + 10$

- Use initial condition $y(0) = 6$

$6 = c_1 + c_2 + 10$

- Compute derivative of the solution

$y' = c_1 e^x + 3c_2 e^{3x} + 6x + 8$

- Use the initial condition $y'\big|_{\{x=0\}} = 8$

$8 = c_1 + 3c_2 + 8$

- Solve for c_1 and c_2

  $\{c_1 = -6, c_2 = 2\}$

- Substitute constant values into general solution and simplify

  $y = 10 + 3x^2 + 2\,\mathrm{e}^{3x} - 6\,\mathrm{e}^{x} + 8x$

- Solution to the IVP

  $y = 10 + 3x^2 + 2\,\mathrm{e}^{3x} - 6\,\mathrm{e}^{x} + 8x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve([diff(y(x),x$2)-4*diff(y(x),x)+3*y(x)=9*x^2+4,y(0) = 6, D(y)(0) = 8],y(x), singsol=al
```

$$y(x) = 2\,\mathrm{e}^{3x} - 6\,\mathrm{e}^{x} + 3x^2 + 8x + 10$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 27

```
DSolve[{y''[x]-4*y'[x]+3*y[x]==9*x^2+4,{y[0]==6,y'[0]==8}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to 3x^2 + 8x - 6e^x + 2e^{3x} + 10$$

1.12 problem HW 5 problem 2

Internal problem ID [7040]
Internal file name [`OUTPUT/6026_Sunday_June_05_2022_04_14_24_PM_74506284/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 5 problem 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \tan(x)^2}$$

1.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \tan(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = \mathrm{e}^{0}(c_1 \cos{(x)} + c_2 \sin{(x)})$$

Or

$$y = c_1 \cos{(x)} + c_2 \sin{(x)}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(x) \\ y_2 &= \sin(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\tan(x)^2}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\tan(x)^2\,dx$$

Hence

$$u_1 = -\frac{\sin(x)^4}{\cos(x)} - \left(2 + \sin(x)^2\right)\cos(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\tan(x)^2}{1}\,dx$$

Which simplifies to

$$u_2 = \int \sin(x)\tan(x)\,dx$$

Hence

$$u_2 = -\sin(x) + \ln(\sec(x) + \tan(x))$$

Which simplifies to

$$u_1 = -\cos(x) - \sec(x)$$
$$u_2 = -\sin(x) + \ln(\sec(x) + \tan(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\cos(x) - \sec(x))\cos(x) + (-\sin(x) + \ln(\sec(x) + \tan(x)))\sin(x)$$

Which simplifies to

$$y_p(x) = -2 + \sin(x)\ln(\sec(x) + \tan(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (-2 + \sin(x)\ln(\sec(x) + \tan(x))) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - 2 + \sin(x)\ln(\sec(x) + \tan(x)) \tag{1}$$

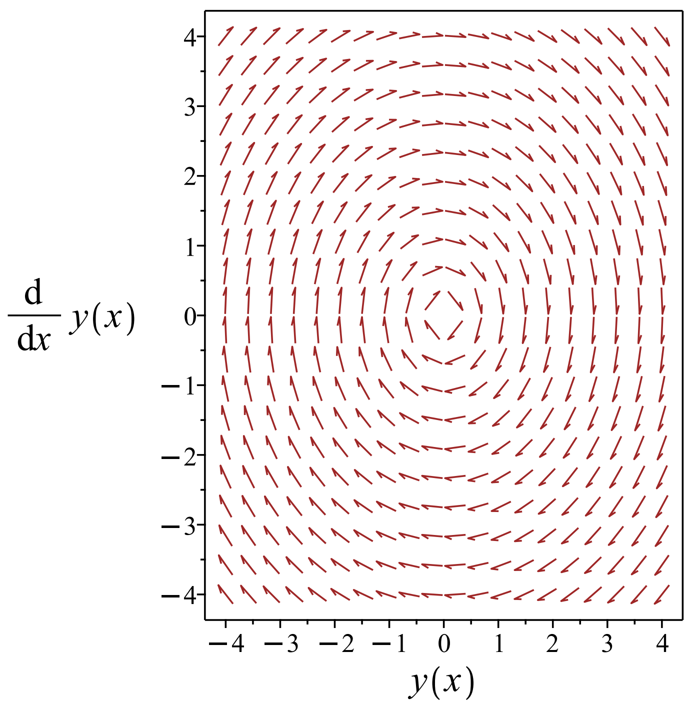


Figure 29: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - 2 + \sin(x)\ln(\sec(x) + \tan(x))$$

Verified OK.

1.12.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align}
y'' + y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align}
$$

Comparing (1) and (2) shows that

$$
\begin{align}
A &= 1 \\
B &= 0 \tag{3} \\
C &= 1
\end{align}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align}
s &= -1 \\
t &= 1
\end{align}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 23: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\tan(x)^2}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\tan(x)^2\,dx$$

Hence

$$u_1 = -\frac{\sin(x)^4}{\cos(x)} - \left(2+\sin(x)^2\right)\cos(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\tan(x)^2}{1}\,dx$$

Which simplifies to

$$u_2 = \int \sin(x)\tan(x)\,dx$$

Hence

$$u_2 = -\sin(x) + \ln(\sec(x)+\tan(x))$$

Which simplifies to

$$\begin{aligned} u_1 &= -\cos(x) - \sec(x) \\ u_2 &= -\sin(x) + \ln(\sec(x)+\tan(x)) \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\cos(x) - \sec(x))\cos(x) + (-\sin(x) + \ln(\sec(x)+\tan(x)))\sin(x)$$

Which simplifies to

$$y_p(x) = -2 + \sin(x)\ln(\sec(x)+\tan(x))$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + (-2 + \sin(x) \ln(\sec(x) + \tan(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) - 2 + \sin(x) \ln(\sec(x) + \tan(x)) \tag{1}
$$

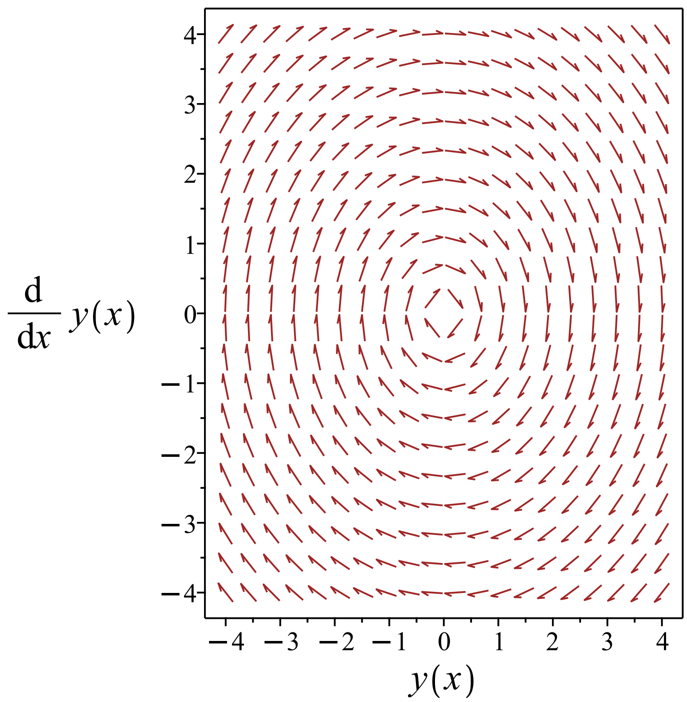


Figure 30: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) - 2 + \sin(x) \ln(\sec(x) + \tan(x))
$$

Verified OK.

1.12.3 Maple step by step solution

Let's solve

$y'' + y = \tan(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \tan(x)^2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(x)\left(\int \sin(x)\tan(x)^2 dx\right) + \sin(x)\left(\int \sin(x)\tan(x) dx\right)$$

- Compute integrals

$$y_p(x) = -2 + \sin(x)\ln(\sec(x) + \tan(x))$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(x) + c_2\sin(x) - 2 + \sin(x)\ln(\sec(x) + \tan(x))$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)=tan(x)^2,y(x), singsol=all)
```

$$y(x) = \sin(x)c_2 + \cos(x)c_1 - 2 + \sin(x)\ln(\sec(x) + \tan(x))$$

✓ Solution by Mathematica

Time used: 0.129 (sec). Leaf size: 23

```
DSolve[y''[x]+y[x]==Tan[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x)\text{arctanh}(\sin(x)) + c_1\cos(x) + c_2\sin(x) - 2$$

1.13 problem HW 5 problem 5

Internal problem ID [7041]
Internal file name [`OUTPUT/6027_Sunday_June_05_2022_04_14_26_PM_26900985/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 5 problem 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -2x(t) + 3y(t) \\ y'(t) &= -2x(t) + 5y(t) \end{aligned}$$

With initial conditions

$$[x(0) = -2, y(0) = 1]$$

1.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{6\,\mathrm{e}^{-t}}{5} - \frac{\mathrm{e}^{4t}}{5} & \frac{3\,\mathrm{e}^{4t}}{5} - \frac{3\,\mathrm{e}^{-t}}{5} \\ -\frac{2\,\mathrm{e}^{4t}}{5} + \frac{2\,\mathrm{e}^{-t}}{5} & -\frac{\mathrm{e}^{-t}}{5} + \frac{6\,\mathrm{e}^{4t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{6\,\mathrm{e}^{-t}}{5} - \frac{\mathrm{e}^{4t}}{5} & \frac{3\,\mathrm{e}^{4t}}{5} - \frac{3\,\mathrm{e}^{-t}}{5} \\ -\frac{2\,\mathrm{e}^{4t}}{5} + \frac{2\,\mathrm{e}^{-t}}{5} & -\frac{\mathrm{e}^{-t}}{5} + \frac{6\,\mathrm{e}^{4t}}{5} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -3\,\mathrm{e}^{-t} + \mathrm{e}^{4t} \\ 2\,\mathrm{e}^{4t} - \mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 3 \\ -2 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 3 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ -2 & 6 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -6 & 3 & 0 \\ -2 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} -6 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\
&= \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{4t} \\
&= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{4t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 3\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\mathrm{e}^{4t}}{2} \\ \mathrm{e}^{4t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3c_1 \mathrm{e}^{-t} + \frac{c_2 \mathrm{e}^{4t}}{2} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{4t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = -2 \\ y(0) = 1 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$
\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3c_1 + \frac{c_2}{2} \\ c_1 + c_2 \end{bmatrix}
$$

Solving for the constants of integrations gives

$$
\begin{bmatrix} c_1 = -1 \\ c_2 = 2 \end{bmatrix}
$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -3\,\mathrm{e}^{-t} + \mathrm{e}^{4t} \\ 2\,\mathrm{e}^{4t} - \mathrm{e}^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

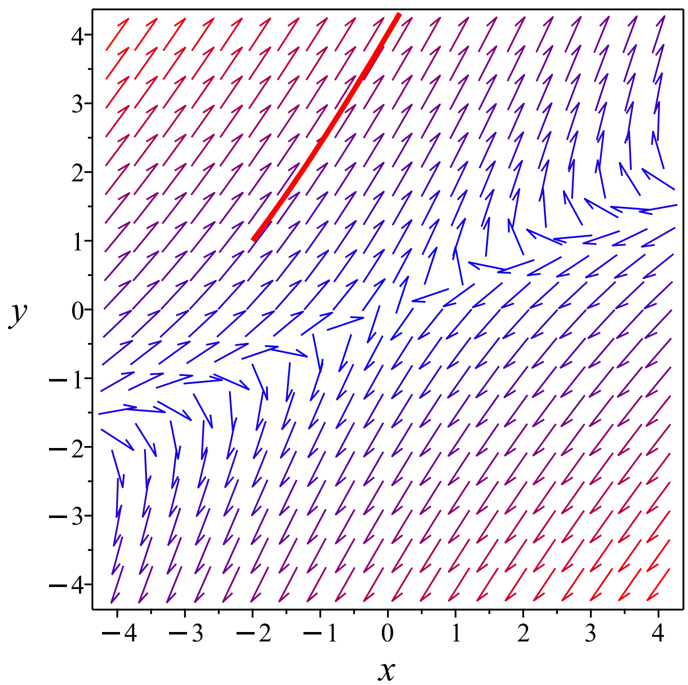
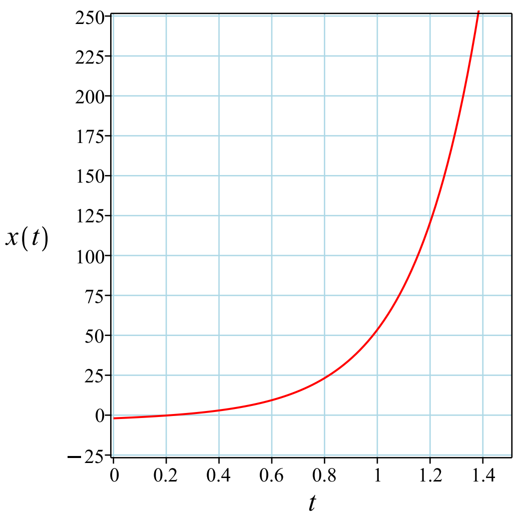
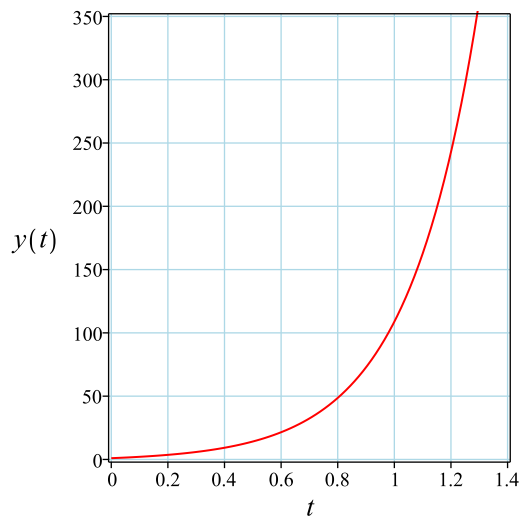


Figure 31: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve([diff(x(t),t) = -2*x(t)+3*y(t), diff(y(t),t) = -2*x(t)+5*y(t), x(0) = -2, y(0) = 1],
```

$$x(t) = -3\,\mathrm{e}^{-t} + \mathrm{e}^{4t}$$
$$y(t) = -\mathrm{e}^{-t} + 2\,\mathrm{e}^{4t}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 36

```
DSolve[{x'[t]==-2*x[t]+3*y[t],y'[t]==-2*x[t]+5*y[t]},{x[0]==-2,y[0]==1},{x[t],y[t]},t,Includ
```

$$x(t) \to e^{-t}\left(e^{5t} - 3\right)$$
$$y(t) \to e^{-t}\left(2e^{5t} - 1\right)$$

1.14 problem HW 5 problem 6

Internal problem ID [7042]
Internal file name [`OUTPUT/6028_Sunday_June_05_2022_04_14_28_PM_70078692/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 5 problem 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -x(t) + 4y(t) \\ y'(t) &= 2x(t) - 3y(t) \end{aligned}$$

With initial conditions

$$[x(0) = 3, y(0) = 0]$$

1.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(2\,\mathrm{e}^{6t}+1)\mathrm{e}^{-5t}}{3} & \frac{2(\mathrm{e}^{6t}-1)\mathrm{e}^{-5t}}{3} \\ \frac{(\mathrm{e}^{6t}-1)\mathrm{e}^{-5t}}{3} & \frac{(\mathrm{e}^{6t}+2)\mathrm{e}^{-5t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{(2\,\mathrm{e}^{6t}+1)\,\mathrm{e}^{-5t}}{3} & \frac{2(\mathrm{e}^{6t}-1)\,\mathrm{e}^{-5t}}{3} \\ \frac{(\mathrm{e}^{6t}-1)\,\mathrm{e}^{-5t}}{3} & \frac{(\mathrm{e}^{6t}+2)\,\mathrm{e}^{-5t}}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} (2\,\mathrm{e}^{6t}+1)\,\mathrm{e}^{-5t} \\ (\mathrm{e}^{6t}-1)\,\mathrm{e}^{-5t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

1.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -1-\lambda & 4 \\ 2 & -3-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 + 4\lambda - 5 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix} - (-5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \implies \left[\begin{array}{cc|c} -2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
-5	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t \end{aligned}$$

Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-5t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-5t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-5t} \\ \mathrm{e}^{-5t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (2c_1 \mathrm{e}^{6t} - c_2)\,\mathrm{e}^{-5t} \\ (c_1 \mathrm{e}^{6t} + c_2)\,\mathrm{e}^{-5t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 3 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2c_1 - c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 1 \\ c_2 = -1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (2\,\mathrm{e}^{6t} + 1)\,\mathrm{e}^{-5t} \\ (\mathrm{e}^{6t} - 1)\,\mathrm{e}^{-5t} \end{bmatrix}$$

The following is the phase plot of the system.

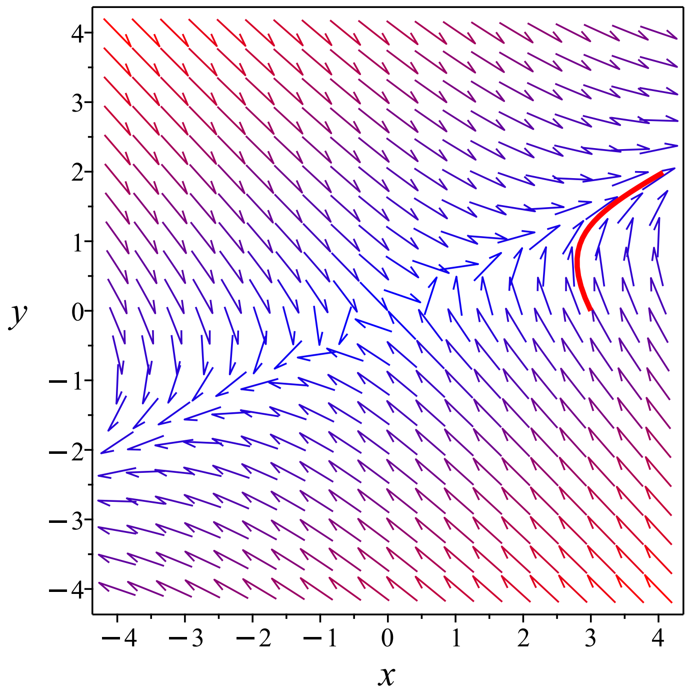
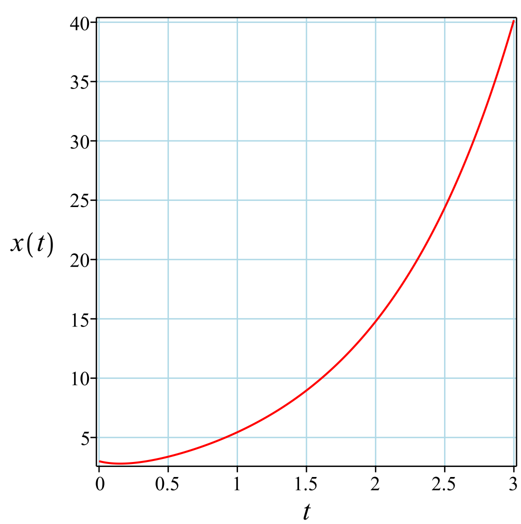
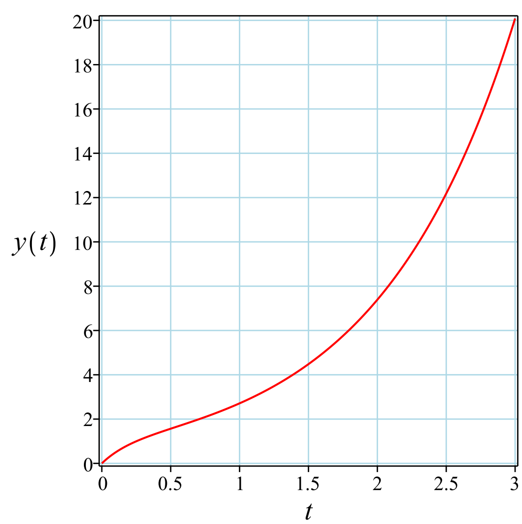


Figure 32: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve([diff(x(t),t) = -x(t)+4*y(t), diff(y(t),t) = 2*x(t)-3*y(t), x(0) = 3, y(0) = 0], sing
```

$$x(t) = 2\,\mathrm{e}^{t} + \mathrm{e}^{-5t}$$
$$y(t) = \mathrm{e}^{t} - \mathrm{e}^{-5t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 30

```
DSolve[{x'[t]==-x[t]+4*y[t],y'[t]==2*x[t]-3*y[t]},{x[0]==3,y[0]==0},{x[t],y[t]},t,IncludeSin
```

$$x(t) \to e^{-5t} + 2e^{t}$$
$$y(t) \to e^{t} - e^{-5t}$$

1.15 problem HW 5 problem 7

Internal problem ID [7043]
Internal file name [`OUTPUT/6029_Sunday_June_05_2022_04_14_30_PM_69169982/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: HW 5 problem 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) - y(t) \\
y'(t) &= -x(t) + 2y(t) + 4\,\mathrm{e}^{t}
\end{aligned}
$$

1.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 4\,\mathrm{e}^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^t}{2} + \frac{e^{3t}}{2} & -\frac{e^{3t}}{2} + \frac{e^t}{2} \\ -\frac{e^{3t}}{2} + \frac{e^t}{2} & \frac{e^t}{2} + \frac{e^{3t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^t}{2} + \frac{e^{3t}}{2} & -\frac{e^{3t}}{2} + \frac{e^t}{2} \\ -\frac{e^{3t}}{2} + \frac{e^t}{2} & \frac{e^t}{2} + \frac{e^{3t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^t}{2} + \frac{e^{3t}}{2}\right) c_1 + \left(-\frac{e^{3t}}{2} + \frac{e^t}{2}\right) c_2 \\ \left(-\frac{e^{3t}}{2} + \frac{e^t}{2}\right) c_1 + \left(\frac{e^t}{2} + \frac{e^{3t}}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1 - c_2)e^{3t}}{2} + \frac{e^t(c_1 + c_2)}{2} \\ \frac{(c_2 - c_1)e^{3t}}{2} + \frac{e^t(c_1 + c_2)}{2} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \frac{e^{-3t}(e^{2t}+1)}{2} & \frac{e^{-3t}(e^{2t}-1)}{2} \\ \frac{e^{-3t}(e^{2t}-1)}{2} & \frac{e^{-3t}(e^{2t}+1)}{2} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{e^t}{2} + \frac{e^{3t}}{2} & -\frac{e^{3t}}{2} + \frac{e^t}{2} \\ -\frac{e^{3t}}{2} + \frac{e^t}{2} & \frac{e^t}{2} + \frac{e^{3t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-3t}(e^{2t}+1)}{2} & \frac{e^{-3t}(e^{2t}-1)}{2} \\ \frac{e^{-3t}(e^{2t}-1)}{2} & \frac{e^{-3t}(e^{2t}+1)}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 4e^t \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{e^t}{2} + \frac{e^{3t}}{2} & -\frac{e^{3t}}{2} + \frac{e^t}{2} \\ -\frac{e^{3t}}{2} + \frac{e^t}{2} & \frac{e^t}{2} + \frac{e^{3t}}{2} \end{bmatrix} \begin{bmatrix} 2t + e^{-2t} \\ 2t - e^{-2t} \end{bmatrix} \\
&= \begin{bmatrix} e^t(2t+1) \\ e^t(-1+2t) \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \frac{(c_1-c_2)e^{3t}}{2} + 2\,e^t\left(t + \frac{c_1}{4} + \frac{c_2}{4} + \frac{1}{2}\right) \\ \frac{(c_2-c_1)e^{3t}}{2} + 2\,e^t\left(t + \frac{c_1}{4} + \frac{c_2}{4} - \frac{1}{2}\right) \end{bmatrix}
\end{aligned}
$$

1.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 4\,e^t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t \end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{3t}\\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{3t} \\ e^{3t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^{t} & -e^{3t} \\ e^{t} & e^{3t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{e^{-t}}{2} & \frac{e^{-t}}{2} \\ -\frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} \end{bmatrix}$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-t}}{2} & \frac{e^{-t}}{2} \\ -\frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 4\,e^t \end{bmatrix} dt \\
&= \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \int \begin{bmatrix} 2 \\ 2\,e^{-2t} \end{bmatrix} dt \\
&= \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \begin{bmatrix} 2t \\ -e^{-2t} \end{bmatrix} \\
&= \begin{bmatrix} e^t(2t+1) \\ e^t(-1+2t) \end{bmatrix}
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} c_1 e^t \\ c_1 e^t \end{bmatrix} + \begin{bmatrix} -c_2 e^{3t} \\ c_2 e^{3t} \end{bmatrix} + \begin{bmatrix} e^t(2t+1) \\ e^t(-1+2t) \end{bmatrix}
\end{aligned}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2\,e^t t + c_1 e^t - c_2 e^{3t} + e^t \\ c_2 e^{3t} + 2\,e^t\left(t + \frac{c_1}{2} - \frac{1}{2}\right) \end{bmatrix}
$$

1.15.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) - y(t)\,, y'(t) = -x(t) + 2y(t) + 4\,e^t]$

- Define vector

$$
\overrightarrow{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

- Convert system into a vector equation

$$
\overrightarrow{x}'(t) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 0 \\ 4\,e^t \end{bmatrix}
$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 4\,\mathrm{e}^{t} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 4\,\mathrm{e}^{t} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syste

  $$\phi(t) = \left[\begin{array}{cc} \mathrm{e}^{t} & -\mathrm{e}^{3t} \\ \mathrm{e}^{t} & \mathrm{e}^{3t} \end{array}\right]$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t) = \left[\begin{array}{cc} \mathrm{e}^{t} & -\mathrm{e}^{3t} \\ \mathrm{e}^{t} & \mathrm{e}^{3t} \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \left[\begin{array}{cc} \frac{\mathrm{e}^{t}}{2}+\frac{\mathrm{e}^{3t}}{2} & -\frac{\mathrm{e}^{3t}}{2}+\frac{\mathrm{e}^{t}}{2} \\ -\frac{\mathrm{e}^{3t}}{2}+\frac{\mathrm{e}^{t}}{2} & \frac{\mathrm{e}^{t}}{2}+\frac{\mathrm{e}^{3t}}{2} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

- Take the derivative of the particular solution

  $\overrightarrow{x}_p'(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} 2\,\mathrm{e}^t t - \mathrm{e}^{3t} + \mathrm{e}^t \\ \mathrm{e}^{3t} + \mathrm{e}^t(-1+2t) \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \begin{bmatrix} 2\,\mathrm{e}^t t - \mathrm{e}^{3t} + \mathrm{e}^t \\ \mathrm{e}^{3t} + \mathrm{e}^t(-1+2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (-c_2 - 1)\,\mathrm{e}^{3t} + 2\,\mathrm{e}^t\left(t + \frac{c_1}{2} + \frac{1}{2}\right) \\ (c_2 + 1)\,\mathrm{e}^{3t} + 2\,\mathrm{e}^t\left(t + \frac{c_1}{2} - \frac{1}{2}\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = (-c_2 - 1)\,\mathrm{e}^{3t} + 2\,\mathrm{e}^t\left(t + \frac{c_1}{2} + \frac{1}{2}\right), y(t) = (c_2 + 1)\,\mathrm{e}^{3t} + 2\,\mathrm{e}^t\left(t + \frac{c_1}{2} - \frac{1}{2}\right)\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 45

```
dsolve([diff(x(t),t)=2*x(t)-y(t),diff(y(t),t)=-x(t)+2*y(t)+4*exp(t)],singsol=all)
```

$$x(t) = c_2 \mathrm{e}^t + c_1 \mathrm{e}^{3t} + 2\,\mathrm{e}^t t$$
$$y(t) = c_2 \mathrm{e}^t - c_1 \mathrm{e}^{3t} + 2\,\mathrm{e}^t t - 2\,\mathrm{e}^t$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 74

```
DSolve[{x'[t]==2*x[t]-y[t],y'[t]==-x[t]+2*y[t]+4*Exp[t]},{x[t],y[t]},t,IncludeSingularSoluti
```

$$x(t) \to \frac{1}{2}e^t\left(4t + c_1\left(e^{2t} + 1\right) - c_2e^{2t} + 2 + c_2\right)$$
$$y(t) \to \frac{1}{2}e^t\left(4t - c_1e^{2t} + c_2e^{2t} - 2 + c_1 + c_2\right)$$

1.16 problem Example 8.3.4 from Handout chapter 8.2

Internal problem ID [7044]
Internal file name [`OUTPUT/6030_Sunday_June_05_2022_04_14_32_PM_41096389/index.tex`]

Book: Selected problems from homeworks from different courses
Section: Math 2520, summer 2021. Differential Equations and Linear Algebra. Normandale college, Bloomington, Minnesota
Problem number: Example 8.3.4 from Handout chapter 8.2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 6x(t) - 7y(t) + 10 \\
y'(t) &= x(t) - 2y(t) - 2\,\mathrm{e}^{t}
\end{aligned}
$$

1.16.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 10 \\ -2\,\mathrm{e}^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{-t}}{6} + \frac{7e^{5t}}{6} & -\frac{7e^{5t}}{6} + \frac{7e^{-t}}{6} \\ \frac{e^{5t}}{6} - \frac{e^{-t}}{6} & \frac{7e^{-t}}{6} - \frac{e^{5t}}{6} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{e^{-t}}{6} + \frac{7e^{5t}}{6} & -\frac{7e^{5t}}{6} + \frac{7e^{-t}}{6} \\ \frac{e^{5t}}{6} - \frac{e^{-t}}{6} & \frac{7e^{-t}}{6} - \frac{e^{5t}}{6} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\frac{e^{-t}}{6} + \frac{7e^{5t}}{6}\right) c_1 + \left(-\frac{7e^{5t}}{6} + \frac{7e^{-t}}{6}\right) c_2 \\ \left(\frac{e^{5t}}{6} - \frac{e^{-t}}{6}\right) c_1 + \left(\frac{7e^{-t}}{6} - \frac{e^{5t}}{6}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(-c_1+7c_2)e^{-t}}{6} + \frac{7(c_1-c_2)e^{5t}}{6} \\ \frac{(-c_1+7c_2)e^{-t}}{6} + \frac{(c_1-c_2)e^{5t}}{6} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} -\frac{(e^{6t}-7)e^{-5t}}{6} & \frac{7(e^{6t}-1)e^{-5t}}{6} \\ -\frac{(e^{6t}-1)e^{-5t}}{6} & \frac{(7e^{6t}-1)e^{-5t}}{6} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\frac{e^{-t}}{6} + \frac{7e^{5t}}{6} & -\frac{7e^{5t}}{6} + \frac{7e^{-t}}{6} \\ \frac{e^{5t}}{6} - \frac{e^{-t}}{6} & \frac{7e^{-t}}{6} - \frac{e^{5t}}{6} \end{bmatrix} \int \begin{bmatrix} -\frac{(e^{6t}-7)e^{-5t}}{6} & \frac{7(e^{6t}-1)e^{-5t}}{6} \\ -\frac{(e^{6t}-1)e^{-5t}}{6} & \frac{(7e^{6t}-1)e^{-5t}}{6} \end{bmatrix} \begin{bmatrix} 10 \\ -2e^{t} \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{e^{-t}}{6} + \frac{7e^{5t}}{6} & -\frac{7e^{5t}}{6} + \frac{7e^{-t}}{6} \\ \frac{e^{5t}}{6} - \frac{e^{-t}}{6} & \frac{7e^{-t}}{6} - \frac{e^{5t}}{6} \end{bmatrix} \begin{bmatrix} -\frac{7e^{-5t}\left(2e^{7t} + \frac{20e^{6t}}{7} + e^{t} + 4\right)}{12} \\ -\frac{(14e^{7t} + 20e^{6t} + e^{t} + 4)e^{-5t}}{12} \end{bmatrix} \\
&= \begin{bmatrix} -4 - \frac{7e^{t}}{4} \\ -2 - \frac{5e^{t}}{4} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \left[\begin{array}{c} \frac{(-c_1+7c_2)\mathrm{e}^{-t}}{6} + \frac{7(c_1-c_2)\mathrm{e}^{5t}}{6} - 4 - \frac{7\,\mathrm{e}^{t}}{4} \\ \frac{(-c_1+7c_2)\mathrm{e}^{-t}}{6} + \frac{(c_1-c_2)\mathrm{e}^{5t}}{6} - 2 - \frac{5\,\mathrm{e}^{t}}{4} \end{array}\right]
\end{aligned}
$$

1.16.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\left[\begin{array}{c} x'(t) \\ y'(t) \end{array}\right] = \left[\begin{array}{cc} 6 & -7 \\ 1 & -2 \end{array}\right] \left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] + \left[\begin{array}{c} 10 \\ -2\,\mathrm{e}^{t} \end{array}\right]
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\left[\begin{array}{cc} 6 & -7 \\ 1 & -2 \end{array}\right] - \lambda \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right) = 0
$$

Therefore

$$
\det\left(\left[\begin{array}{cc} 6-\lambda & -7 \\ 1 & -2-\lambda \end{array}\right]\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 4\lambda - 5 = 0
$$

The roots of the above are the eigenvalues.

$$
\begin{aligned}
\lambda_1 &= -1 \\
\lambda_2 &= 5
\end{aligned}
$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$
\begin{aligned}
\left(\begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 7 & -7 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$
\left[\begin{array}{cc|c} 7 & -7 & 0 \\ 1 & -1 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{R_1}{7} \implies \left[\begin{array}{cc|c} 7 & -7 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 7 & -7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -7 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -7 & 0 \\ 1 & -7 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 7t\}$

Hence the solution is

$$\begin{bmatrix} 7t \\ t \end{bmatrix} = \begin{bmatrix} 7t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 7t \\ t \end{bmatrix} = t \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 7t \\ t \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
5	1	1	No	$\begin{bmatrix} 7 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{5t} \\
&= \begin{bmatrix} 7 \\ 1 \end{bmatrix} e^{5t}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 7\,\mathrm{e}^{5t} \\ \mathrm{e}^{5t} \end{bmatrix}
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \begin{bmatrix} \mathrm{e}^{-t} & 7\,\mathrm{e}^{5t} \\ \mathrm{e}^{-t} & \mathrm{e}^{5t} \end{bmatrix}
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt
$$

But

$$
\Phi^{-1} = \begin{bmatrix} -\frac{\mathrm{e}^{t}}{6} & \frac{7\,\mathrm{e}^{t}}{6} \\ \frac{\mathrm{e}^{-5t}}{6} & -\frac{\mathrm{e}^{-5t}}{6} \end{bmatrix}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \mathrm{e}^{-t} & 7\,\mathrm{e}^{5t} \\ \mathrm{e}^{-t} & \mathrm{e}^{5t} \end{bmatrix} \int \begin{bmatrix} -\frac{\mathrm{e}^{t}}{6} & \frac{7\,\mathrm{e}^{t}}{6} \\ \frac{\mathrm{e}^{-5t}}{6} & -\frac{\mathrm{e}^{-5t}}{6} \end{bmatrix} \begin{bmatrix} 10 \\ -2\,\mathrm{e}^{t} \end{bmatrix} dt \\
&= \begin{bmatrix} \mathrm{e}^{-t} & 7\,\mathrm{e}^{5t} \\ \mathrm{e}^{-t} & \mathrm{e}^{5t} \end{bmatrix} \int \begin{bmatrix} -\frac{5\,\mathrm{e}^{t}}{3} - \frac{7\,\mathrm{e}^{2t}}{3} \\ \frac{5\,\mathrm{e}^{-5t}}{3} + \frac{\mathrm{e}^{-4t}}{3} \end{bmatrix} dt \\
&= \begin{bmatrix} \mathrm{e}^{-t} & 7\,\mathrm{e}^{5t} \\ \mathrm{e}^{-t} & \mathrm{e}^{5t} \end{bmatrix} \begin{bmatrix} -\frac{7\,\mathrm{e}^{2t}}{6} - \frac{5\,\mathrm{e}^{t}}{3} \\ -\frac{\mathrm{e}^{-4t}}{12} - \frac{\mathrm{e}^{-5t}}{3} \end{bmatrix} \\
&= \begin{bmatrix} -4 - \frac{7\,\mathrm{e}^{t}}{4} \\ -2 - \frac{5\,\mathrm{e}^{t}}{4} \end{bmatrix}
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\begin{gathered}
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1\mathrm{e}^{-t} \\ c_1\mathrm{e}^{-t} \end{bmatrix} + \begin{bmatrix} 7c_2\mathrm{e}^{5t} \\ c_2\mathrm{e}^{5t} \end{bmatrix} + \begin{bmatrix} -4 - \frac{7\,\mathrm{e}^{t}}{4} \\ -2 - \frac{5\,\mathrm{e}^{t}}{4} \end{bmatrix}
\end{gathered}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1\mathrm{e}^{-t} + 7c_2\mathrm{e}^{5t} - 4 - \frac{7\,\mathrm{e}^{t}}{4} \\ c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{5t} - 2 - \frac{5\,\mathrm{e}^{t}}{4} \end{bmatrix}
$$

1.16.3 Maple step by step solution

Let's solve

$[x'(t) = 6x(t) - 7y(t) + 10, y'(t) = x(t) - 2y(t) - 2\,\mathrm{e}^{t}]$

- Define vector

$$
\overrightarrow{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

- Convert system into a vector equation

$$
\overrightarrow{x}'(t) = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 10 \\ -2\,\mathrm{e}^{t} \end{bmatrix}
$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 10 \\ -2\,\mathrm{e}^{t} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 10 \\ -2\,\mathrm{e}^{t} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} 7 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[5, \begin{bmatrix} 7 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{5t} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

  $\vec{x}(t)=c_1\vec{x}_1+c_2\vec{x}_2+\vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $\phi(t)=\begin{bmatrix} e^{-t} & 7e^{5t} \\ e^{-t} & e^{5t} \end{bmatrix}$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0)=I$ where I is th

  $\Phi(t)=\phi(t)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $\Phi(t)=\begin{bmatrix} e^{-t} & 7e^{5t} \\ e^{-t} & e^{5t} \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} 1 & 7 \\ 1 & 1 \end{bmatrix}}$

- Evaluate and simplify to get the fundamental matrix

  $\Phi(t)=\begin{bmatrix} -\frac{e^{-t}}{6}+\frac{7e^{5t}}{6} & -\frac{7e^{5t}}{6}+\frac{7e^{-t}}{6} \\ \frac{e^{5t}}{6}-\frac{e^{-t}}{6} & \frac{7e^{-t}}{6}-\frac{e^{5t}}{6} \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t)=\Phi(t)\cdot\vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{x}_p'(t)=\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$

- Cancel like terms

  $\Phi(t)\cdot\vec{v}'(t)=\vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \left[\begin{array}{c} -\frac{7\,\mathrm{e}^{t}}{4} - 4 + \frac{17\,\mathrm{e}^{-t}}{6} + \frac{35\,\mathrm{e}^{5t}}{12} \\ \frac{5\,\mathrm{e}^{5t}}{12} - \frac{5\,\mathrm{e}^{t}}{4} - 2 + \frac{17\,\mathrm{e}^{-t}}{6} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \left[\begin{array}{c} -\frac{7\,\mathrm{e}^{t}}{4} - 4 + \frac{17\,\mathrm{e}^{-t}}{6} + \frac{35\,\mathrm{e}^{5t}}{12} \\ \frac{5\,\mathrm{e}^{5t}}{12} - \frac{5\,\mathrm{e}^{t}}{4} - 2 + \frac{17\,\mathrm{e}^{-t}}{6} \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{(12c_1+34)\mathrm{e}^{-t}}{12} + \frac{(84c_2+35)\mathrm{e}^{5t}}{12} - \frac{7\,\mathrm{e}^{t}}{4} - 4 \\ \frac{(12c_1+34)\mathrm{e}^{-t}}{12} + \frac{(5+12c_2)\mathrm{e}^{5t}}{12} - \frac{5\,\mathrm{e}^{t}}{4} - 2 \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{(12c_1+34)\mathrm{e}^{-t}}{12} + \frac{(84c_2+35)\mathrm{e}^{5t}}{12} - \frac{7\,\mathrm{e}^{t}}{4} - 4, y(t) = \frac{(12c_1+34)\mathrm{e}^{-t}}{12} + \frac{(5+12c_2)\mathrm{e}^{5t}}{12} - \frac{5\,\mathrm{e}^{t}}{4} - 2\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 45

```
dsolve([diff(x(t),t)=6*x(t)-7*y(t)+10,diff(y(t),t)=x(t)-2*y(t)-2*exp(t)],singsol=all)
```

$$x(t) = c_2\mathrm{e}^{5t} + \mathrm{e}^{-t}c_1 - \frac{7\,\mathrm{e}^{t}}{4} - 4$$
$$y(t) = \frac{c_2\mathrm{e}^{5t}}{7} + \mathrm{e}^{-t}c_1 - \frac{5\,\mathrm{e}^{t}}{4} - 2$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 90

```
DSolve[{x'[t]==6*x[t]-7*y[t]+10,y'[t]==x[t]-2*y[t]-2*Exp[t]},{x[t],y[t]},t,IncludeSingularSo
```

$$
\begin{aligned}
x(t) &\to -\frac{7e^t}{4} - \frac{1}{6}(c_1 - 7c_2)e^{-t} + \frac{7}{6}(c_1 - c_2)e^{5t} - 4 \\
y(t) &\to -\frac{5e^t}{4} - \frac{1}{6}(c_1 - 7c_2)e^{-t} + \frac{1}{6}(c_1 - c_2)e^{5t} - 2
\end{aligned}
$$